Truth Trees

2026-01-26

Introducing Trees

This section introduces a new way of testing for validity - truth trees.

Associated Reading

Boxes and Diamonds, section 2.1

What Tableaux Are

- A way for determining whether some combinations are logically possible.
- That can be used for determining whether some arguments are valid - if the truth of the premises and the falsity of the conclusion is not logically possible; then the argument is valid.

Structure

- Start out writing the things you care about.
- Each time one of those things implies that some other things must be the case, write those down too.
- For example, if you write down that $A \wedge B$ is true, also write down that A is true and that B is true.
- Each time there are multiple ways to make something you've written true, create multiple branches for those ways.
- For example, if you write down that $A \vee B$ is true, create a branch where A is true, and a branch where B is true.

Closing

- A branch of a tableau is closed if it contains that some particular claim has incompatible truth values.
- For now, this means that one sentence is both true and false.
- The whole tableau is closed if every branch is closed.

- If the tableau is closed, then the initial assumptions cannot be true together.
- If you are evaluating an argument, this means that the argument is valid.

If you start the tableau by just saying that one sentence is false, the closure of the tableau means that that sentence is a logical truth.

Tableaux have two big benefits over truth tables.

1. They don't grow exponentially when you increase the number of variables.
2. They can be generalised to things beyond propositional logic.

We are introducing them here because of point 2.

Here is something the book doesn't make a big deal of, but is kind of important.

- A closed tableau can show that an argument is valid.
- An open **and completed** tableau can show that an argument is invalid.

- The trick here is that it's hard to tell when a tableau is completed in the relevant sense.
- This will be easier to illustrate in practice than in theory, so let's start building tableau up.

Signs

The system we are using is what is called a **signed tableau** system.

- That means that every line consists of two parts.
- The bigger, second, part is a sentence.
- The first part is a **sign**, which for now is a truth value.
- That is, it is either $\mathbb{T}$ or $\mathbb{F}$.

So each line either says that a particular sentence is true, or says that it is false.

- The book for some reason includes the word ‘might’ here.
- That’s misleading; what they should say is that each line says what is true given (a) the starting assumptions and (b) the assumptions we made for branching purposes.

The system we’re using also has two other distinctive features.

1. We number each of the lines.
2. We list why we’re writing down each line to the right of the tree.

Both of these are somewhat idiosyncratic, though not abnormal. Unlike the truth tables, there just aren’t well defined conventions for how to write these things out.

Rules

This section introduces the rules we use for building up truth trees.

Associated Reading

Boxes and Diamonds, sections 2.2-2.3.

The rules tell you what new lines to write down given the lines you've already got.

- To some extent they simply have to be memorised.
- But hopefully they are all (except for the rules about $\rightarrow$) fairly intuitive.

Rules for $\neg$

1.	$\mathbb{T}\ \neg A$	
2.	$\mathbb{F}\ A$	$\neg\mathbb{T}\ 1$

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2.	$\mathbb{T}\ A$	$\neg\mathbb{F}\ 1$

Rules for $\neg$

- Note that the line numbers are just for illustration, and are arbitrary in two senses.
- First, you apply the rule wherever a sentence like $\neg A$ appears, not just at line 1.
- Second, you don't need to apply the rules immediately, so the successor line could come later than 2.

Rule for true ∧ sentence

1.	$\mathbb{T}\ A \wedge B$	
2.	$\mathbb{T}\ A$	$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ B$	$\wedge\mathbb{T}\ 1$

When you have a true $\wedge$ sentence, you can write down that the sentences either side of it are true.

Rule for true $\vee$ sentence

$$
\begin{array}{lccl}
1. & & \mathbb{T}\ A \vee B & \\
2. & \mathbb{T}\ A & \mathbb{T}\ B & \vee\mathbb{T}\ 1
\end{array}
$$

Rule for true $\vee$ sentence

- When you have a true $\vee$ sentence, you create two **branches**.
- The way to read the tree is that at least one of the branches must be all true.
- The 'trunk' above the branching (in this case just line 1), is part of both branches.
- Branches are inclusive; you are saying that at least one branch is true, not that precisely one is.

Rule for false ∧ sentence

1. $\mathbb{F}\ A \wedge B$

2. $\mathbb{F}\ A \qquad \mathbb{F}\ B \qquad \wedge\mathbb{F}\ 1$

Rule for false $\wedge$ sentence

- If an $\wedge$ sentence is false, then we know that one or other (or both) of the sides are false.
- So we create two branches, one where each side is false.

Rule for false $\vee$ sentence

1.	$\mathbb{F}\ A \vee B$	
2.	$\mathbb{F}\ A$	$\vee\mathbb{F}\ 1$
3.	$\mathbb{F}\ B$	$\vee\mathbb{F}\ 1$

Rule for false $\vee$ sentence

When you have a false $\vee$ sentence, you know that each side is false, so you write down that they are both false.

Justifying the rule for false ∨ sentences

Recall the truth table for ∨

Table 1: Truth table for ∨

A	B	A	∨	B
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

- The only line where the whole sentence is $\mathbb{F}$ is line 4.
- So if a ∨ sentence is $\mathbb{F}$, we know that we're on line 4.
- And on line 4, both A and B are false.

Rule for false $\rightarrow$ sentence

1.	$\mathbb{F}\ A \rightarrow B$	
2.	$\mathbb{T}\ A$	$\rightarrow\mathbb{F}\,1$
3.	$\mathbb{F}\ B$	$\rightarrow\mathbb{F}\,1$

Rule for false $\rightarrow$ sentence

When you have a false $\rightarrow$ sentence, you know that the left side is true and the right side is false, so you write those things down.

Justifying the rule for false $\to$ sentences

Recall the truth table for $\to$

Table 2: Truth table for $\to$

A	B	A	$\to$	B
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$

- The only line where the whole sentence is $\mathbb{F}$ is line 2.
- So if a $\to$ sentence is $\mathbb{F}$, we know that we're on line 2.
- And on line 2, A is true and B are false.

Rule for true $\rightarrow$ sentence

1. $\mathbb{T}\; A \rightarrow B$

2. $\mathbb{F}\; A \qquad \mathbb{T}\; B \qquad \rightarrow\mathbb{T}\, 1$

Rule for true $\rightarrow$ sentence

- When you have a true $\rightarrow$ sentence, you create two **branches**.
- On the first, A is false. That covers lines 3 and 4 of the truth table.
- On the second, B is true. That covers lines 1 and 3 of the truth table.
- Between them, they cover lines 1, 3 and 4 of the truth table.
- And those are the lines where $A \rightarrow B$ is true.

Examples

Overview

This section works through some examples of using tableaux.

Example 1: Showing a Tautology

Let's show that $(P \to Q) \vee (Q \to P)$ is a tautology.

- You might remember this from last time - we showed it was a tautology using a truth table.
- Now we'll use a tableau instead.

Strategy

To show a sentence is a tautology using tableaux:

- Start by assuming the sentence is **false**.
- If the tableau closes, then it's impossible for the sentence to be false.
- Remember, a tableau closes if every branch has at least one inconsistency.
- If the tableau closes, the sentence must be true - it's a tautology.

Building the Tree

We start by writing that $(P \to Q) \lor (Q \to P)$ is false.

1. $\mathbb{F}\ (P \to Q) \lor (Q \to P)$

Applying the False $\vee$ Rule

Since we have a false disjunction, we know both disjuncts must be false.

1.	$\mathbb{F}\ (P \to Q) \vee (Q \to P)$	
2.	$\mathbb{F}\ P \to Q$	$\vee\mathbb{F}\ 1$
3.	$\mathbb{F}\ Q \to P$	$\vee\mathbb{F}\ 1$

Applying the False $\rightarrow$ Rule (First Time)

When a conditional is false, the antecedent is true and the consequent is false.

So from $\mathbb{F}(P \rightarrow Q)$, we get $\mathbb{T}P$ and $\mathbb{F}Q$.

Applying the False $\to$ Rule (First Time)

1.	$\mathbb{F}\ (P \to Q) \vee (Q \to P)$	
2.	$\mathbb{F}\ P \to Q$	$\vee\mathbb{F}\ 1$
3.	$\mathbb{F}\ Q \to P$	$\vee\mathbb{F}\ 1$
4.	$\mathbb{T}\ P$	$\to\mathbb{F}\ 2$
5.	$\mathbb{F}\ Q$	$\to\mathbb{F}\ 2$

Applying the False $\rightarrow$ Rule (Second Time)

Now we apply the same rule to $\mathbb{F}(Q \rightarrow P)$ on line 3.

This gives us $\mathbb{T}Q$ and $\mathbb{F}P$.

Applying the False $\rightarrow$ Rule (Second Time)

1.	$\mathbb{F}\ (P \rightarrow Q) \vee (Q \rightarrow P)$	
2.	$\mathbb{F}\ P \rightarrow Q$	$\vee\mathbb{F}\,1$
3.	$\mathbb{F}\ Q \rightarrow P$	$\vee\mathbb{F}\,1$
4.	$\mathbb{T}\ P$	$\rightarrow\mathbb{F}\,2$
5.	$\mathbb{F}\ Q$	$\rightarrow\mathbb{F}\,2$
6.	$\mathbb{T}\ Q$	$\rightarrow\mathbb{F}\,3$
7.	$\mathbb{F}\ P$	$\rightarrow\mathbb{F}\,3$

Notice what we have now:

- Line 4: $\mathbb{T}P$
- Line 7: $\mathbb{F}P$
- Line 5: $\mathbb{F}Q$
- Line 6: $\mathbb{T}Q$

Both P and Q have incompatible truth values. The branch closes. (NB: We only need one inconsistency, even though we have two here.)

What This Means

- We started by assuming $(P \to Q) \vee (Q \to P)$ is false.
- Every possible way to make it false led to a contradiction.
- Therefore, it **cannot** be false.
- This means it's a tautology - it must always be true.

Comparison to Truth Tables

- With a truth table, we had to check 4 rows.
- With the tableau, we just followed the logical implications.
- For more complex sentences (with 3+ variables), tableaux can be much more efficient.
- They don't grow exponentially like truth tables do.

Let's show that $((P \vee Q) \wedge \neg P) \wedge \neg Q$ is a contradiction.

- A contradiction is a sentence that can never be true.
- We'll use a tableau to demonstrate this.

Strategy

To show a sentence is a contradiction using tableaux:

- Start by assuming the sentence is **true**.
- If the tableau closes, then it's impossible for the sentence to be true.
- That means the sentence must be false - it's a contradiction.

We start by writing that $((P \vee Q) \wedge \neg P) \wedge \neg Q$ is true.

$$1. \qquad \mathbb{T}\ ((P \vee Q) \wedge \neg P) \wedge \neg Q$$

Applying the True $\wedge$ Rule

Since we have a true conjunction, both conjuncts must be true.

So we get $\mathbb{T}((P \vee Q) \wedge \neg P)$ and $\mathbb{T}\neg Q$.

Applying the True $\wedge$ Rule

1.	$\mathbb{T}\ ((P \vee Q) \wedge \neg P) \wedge \neg Q$	
2.	$\mathbb{T}\ (P \vee Q) \wedge \neg P$	$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ \neg Q$	$\wedge\mathbb{T}\ 1$

Now we apply the rule to $\mathbb{T}((P \vee Q) \wedge \neg P)$ from line 2.

This gives us $\mathbb{T}(P \vee Q)$ and $\mathbb{T}\neg P$.

Applying the True $\wedge$ Rule Again

1.	$\mathbb{T}\ ((P \vee Q) \wedge \neg P) \wedge \neg Q$	
2.	$\mathbb{T}\ (P \vee Q) \wedge \neg P$	$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ \neg Q$	$\wedge\mathbb{T}\ 1$
4.	$\mathbb{T}\ P \vee Q$	$\wedge\mathbb{T}\ 2$
5.	$\mathbb{T}\ \neg P$	$\wedge\mathbb{T}\ 2$

From $\mathbb{T}\neg P$ we get $\mathbb{F}P$, and from $\mathbb{T}\neg Q$ we get $\mathbb{F}Q$.

1.	$\mathbb{T}\ ((P \lor Q) \land \neg P) \land \neg Q$	
2.	$\mathbb{T}\ (P \lor Q) \land \neg P$	$\land\mathbb{T}\ 1$
3.	$\mathbb{T}\ \neg Q$	$\land\mathbb{T}\ 1$
4.	$\mathbb{T}\ P \lor Q$	$\land\mathbb{T}\ 2$
5.	$\mathbb{T}\ \neg P$	$\land\mathbb{T}\ 2$
6.	$\mathbb{F}\ P$	$\neg\mathbb{T}\ 5$
7.	$\mathbb{F}\ Q$	$\neg\mathbb{T}\ 3$

Now we need to apply the rule for $\mathbb{T}(P \vee Q)$ from line 4.

This creates two branches - one where P is true, one where Q is true.

1.	$\mathbb{T}\ ((P \vee Q) \wedge \neg P) \wedge \neg Q$	
2.	$\mathbb{T}\ (P \vee Q) \wedge \neg P$	$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ \neg Q$	$\wedge\mathbb{T}\ 1$
4.	$\mathbb{T}\ P \vee Q$	$\wedge\mathbb{T}\ 2$
5.	$\mathbb{T}\ \neg P$	$\wedge\mathbb{T}\ 2$
6.	$\mathbb{F}\ P$	$\neg\mathbb{T}\ 5$
7.	$\mathbb{F}\ Q$	$\neg\mathbb{T}\ 3$
8.	$\mathbb{T}\ P \qquad \mathbb{T}\ Q$	$\vee\mathbb{T}\ 4$

Both Branches Close

Look at what we have on each branch:

Left branch: - Line 6: $\mathbb{F}P$ - Line 8: $\mathbb{T}P$

Right branch: - Line 7: $\mathbb{F}Q$ - Line 8: $\mathbb{T}Q$

Both branches have contradictions. The entire tableau closes.

What This Means

- We started by assuming $((P \vee Q) \wedge \neg P) \wedge \neg Q$ is true
- We tried every possible way to make it true (both branches)
- Every possibility led to a contradiction
- Therefore, the sentence **cannot** be true - it's a contradiction

Understanding the Result

This result makes intuitive sense:

- $(P \vee Q)$ says at least one of P or Q is true
- $\neg P$ says P is false
- $\neg Q$ says Q is false
- You can't have at least one true when both are false.

The tableau just systematically revealed this inconsistency.

Let's show that $P \wedge (P \to Q)$ is neither a tautology nor a contradiction.

- A contingent sentence is one that could be true or false.
- We'll use two tableaux to demonstrate this.

Strategy

To show a sentence is contingent:

- First, try to show it’s a tautology (start with it false)
- If that tableau stays open, it’s not a tautology
- Then, try to show it’s a contradiction (start with it true)
- If that tableau also stays open, it’s not a contradiction
- If both tableaux are open, the sentence is contingent

We start by assuming $P \wedge (P \to Q)$ is **false**.

$$1. \qquad \mathbb{F}\ P \wedge (P \to Q)$$

Applying the False $\wedge$ Rule

Since we have a false conjunction, at least one conjunct must be false.

This creates two branches.

Applying the False $\wedge$ Rule

1.	$\mathbb{F}\ P \wedge (P \to Q)$		
2.	$\mathbb{F}\ P$	$\mathbb{F}\ P \to Q$	$\wedge\mathbb{F}\ 1$

Left Branch: Already Complete

The left branch just has $\mathbb{F}P$.

- This doesn't contradict anything.
- We can't apply any more rules to it.
- So this branch is **complete and open**.

Right Branch: Applying False $\rightarrow$ Rule

On the right branch, we have $\mathbb{F}(P \rightarrow Q)$.

When a conditional is false, the antecedent is true and the consequent is false.

Right Branch: Applying False $\rightarrow$ Rule

1.	$\mathbb{F}\ P \wedge (P \rightarrow Q)$		
2.	$\mathbb{F}\ P$	$\mathbb{F}\ P \rightarrow Q$	$\wedge\mathbb{F}\,1$
3.		$\mathbb{T}\ P$	$\rightarrow\mathbb{F}\,2$
4.		$\mathbb{F}\ Q$	$\rightarrow\mathbb{F}\,2$

Right Branch: Also Complete and Open

The right branch has $\mathbb{F}(P \to Q)$, $\mathbb{T}P$, and $\mathbb{F}Q$.

- These don't contradict each other.
- We can't apply any more rules.
- So this branch is also **complete and open**.

What the First Tree Shows

- We assumed $P \wedge (P \to Q)$ is false.
- The tableau has open branches.
- This means it **is** possible for the sentence to be false.
- Therefore, it's **not a tautology**.

Now we start by assuming $P \wedge (P \to Q)$ is **true**.

$$1. \qquad \mathbb{T}\ P \wedge (P \to Q)$$

Since we have a true conjunction, both conjuncts must be true.

1.	$\mathbb{T}\ P \wedge (P \to Q)$	
2.	$\mathbb{T}\ P$	$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ P \to Q$	$\wedge\mathbb{T}\ 1$

Now we apply the rule for $\mathbb{T}(P \rightarrow Q)$ from line 3.

This creates two branches - one where P is false, one where Q is true.

Applying the True $\to$ Rule

$$
\begin{array}{lcl}
1. & \mathbb{T}\; P \wedge (P \to Q) & \\
2. & \mathbb{T}\; P & \wedge\mathbb{T}\,1 \\
3. & \mathbb{T}\; P \to Q & \wedge\mathbb{T}\,1 \\
4. & \mathbb{F}\; P \qquad \mathbb{T}\; Q & \to\mathbb{T}\,3
\end{array}
$$

Analyzing the Second Tree

Left branch:

- Line 2: $\mathbb{T}P$
- Line 4: $\mathbb{F}P$
- This branch **closes**.

Right branch:

- Line 2: $\mathbb{T}P$
- Line 4: $\mathbb{T}Q$
- These don't contradict anything.
- This branch is **complete and open**.

- We assumed $P \wedge (P \to Q)$ is true.
- The tableau has an open branch.
- This means it **is** possible for the sentence to be true.
- Therefore, it's **not a contradiction**.

Since $P \wedge (P \rightarrow Q)$:

- Is not a tautology (first tree has open branches),
- Is not a contradiction (second tree has an open branch),
- It must be **contingent** - sometimes true, sometimes false.

The open branches actually show us **when** it's true or false:

- It's false when P is false (first tree, left branch)
- It's true when both P and Q are true (second tree, right branch).

Comparison to Truth Tables

- A **tautology** has a T under the main connective in every row.
- A **contradiction** has an F under the main connective in every row.
- A **contingent** sentence has a mixture Ts and Fs under the main connective.

For Next Time

We'll use trees to test for validity.

Remember there is no lecture on Wednesday, but there is a discussion section.