Modal Logic

2026-02-04

Introduction

Overview

In this section we'll introduce what modal logic is, and why it's philosophically useful.

Associated Reading

- Boxes and Diamonds, section 3.1 and 3.2.

The logics of **must** and **might**

- Why plural? Because we do not assume that these words have a single determinate meaning.

1. If $x = 2 + 2$, then x must equal 4.
2. If something is a cat, then it must be a mammal.
3. If the gardener is innocent, then it must be the butler who did it.
4. You must drive under 70mph on I-94.
5. You must keep your promises.
6. If you set out a knife and fork, the fork must go on the left.

To my ears, 1 is **logical** necessity, 2 is **metaphysical** necessity, 3 is **epistemic** necessity, 4 is **legal** necessity, 5 is **moral** (or **deontic**) necessity and 6 is **etiquette** necessity.

Examples of May/Might

1. If x is prime, then x might be even.
2. If x is a cat, then x might be male.
3. It might be the butler or the gardener that did it.
4. You may drive at any speed below 30mph on State Street.
5. You may lie to save a friend's life.
6. You may use white napkins or red napkins.

To my ears, 1 is **logical** possibility, 2 is **metaphysical** possibility, 3 is **epistemic** possibility, 4 is **legal** possibility, 5 is **moral** (or **deontic**) possibility and 6 is **etiquette** possibility (though I'm not sure about any of these).

Consider this very general claim.

If something must be true, then it is true.

- That's true on the logical, epistemic and metaphysical interpretations of modality.
- Indeed, it's something like a logical truth of those domains.
- But it is very much not true on the legal, moral or etiquette interpretations.
- So we want some logics where it is a logical truth, and some where it is not.

We extend our language with two new operators: $\Box$ and $\Diamond$.

- If p is a sentence, so is $\Box p$ and so is $\Diamond p$.
- These mean, respectively, that p must be true, and that p might be true.
- We interpret these somewhat similar to negations; they just bind what they are immediately next to.
- So $\Box p \to q$ means $(\Box p) \to q$, not $\Box(p \to q)$.

Models

Overview

In this section, we'll go over what a **model** for modal logic is.

Associated Reading

- Boxes and Diamonds, section 3.3 and 3.4.

Worlds

We start with Leibniz's idea that necessity is truth in all possible worlds.

- Leibniz was interested in metaphysical necessity, so we'll have to qualify this a little, but it's a good idea.
- So instead of saying that each proposition simply has a truth value, we'll say that there are many **worlds**, and at each world each proposition has a truth value.
- But don't assume that propositions have the same truth value at each world.
- In one world I might be standing, and in another world I might be sitting.

We are well and truly not going to get into the metaphysics of worlds here.

- Indeed, they need not even be anything like possible worlds in the sense that metaphysicians usually care about.
- They might, for instance, be different times.
- All we care about is that they are things at which propositions can be true or false.

A valuation function tells us which worlds atomic sentences are true at.

- These can be completely arbitrary; we don't put any restrictions on them.

We want more generally a function that tells us whether a sentence is true at a particular world.

- For sentences built up using $\wedge, \vee, \rightarrow, \neg$, this is relatively easy.
- We just keep on using truth tables.
- So if at world w, A is true and B is false, then $A \wedge B$ is false and $A \vee B$ is true.

Modal Values

We also need values for these sentences:

- $\Box A$
- $\Diamond A$

It turns out these are more complicated - but not much more complicated.

The last part of our model is an **accessibility** relation between worlds.

- Again, this can be completely arbitrary.
- We don't yet put any restrictions on it.
- Notably, we don't assume that it is **reflexive**, **symmetric** or **transitive**.

Properties of Relations

- R is reflexive iff for all x, xRx.
- R is symmetric iff for all x, y, if xRy then yRx.
- R is transitive iff for all x, y, z if xRy and yRz then xRz.

A lot of relations we care about have one or more of these properties, but not all do. Consider, for example, **admires** as an example of a relation with none of them.

A sentence $\Box A$ is true at a world x just in case the following condition is met:

- For all worlds y such that xRy, A is true at world y.

A sentence $\Diamond A$ is true at a world x just in case the following condition is met:

- For some world y such that xRy, A is true at world y.

- Something is necessarily true iff it is true everywhere that is accessible.
- Something is possibly true iff it is true somewhere accessible.

We get back the Leibnizian idea that necessity is truth in all possible worlds if we assume the accessibility relation is the universal relation, i.e., xRy for all x, y.

On this Leibnizian model, where all worlds can access all worlds, iterated modalities are rather uninteresting. These three sentences are true in the same worlds/models.

1. $\Box A$
2. $\Box\Box A$
3. $\Diamond\Box A$

That's because if $\Box A$ is true at any world, then it is true at all worlds. In the general case, where we do not assume that R is universal, these are not equivalent.

Two Fundamental Results

In this section we'll explain two fundamental results in modal logic connecting possibility and necessity.

Associated Reading

- Boxes and Diamonds, section 3.4.

These two claims are equivalent.

1. $\Box A$
2. $\neg\Diamond\neg A$

From 1 to 2: If $\Box A$ is true at x, then A is true for all y such that xRy. That means there is no y such that xRy and A is not true. That means there is no y such that xRy and $\neg A$ is true. That means $\Diamond\neg A$ is not true at x. That means $\neg\Diamond\neg A$ is true at x.

Duality

These two claims are equivalent.

1. $\Box A$
2. $\neg\Diamond\neg A$

From 2 to 1: If $\neg\Diamond\neg A$ is true at x, then $\Diamond\neg A$ is not true at x. So there is no world y such that xRy and $\neg A$ is true at y. So at all worlds y such that xRy, $\neg A$ is not true. So at all worlds y such that xRy, A is true. So $\Box A$ is true at x.

These two claims are also equivalent.

1. $\Diamond A$
2. $\neg\Box\neg A$

The proof is similar: $\Diamond A$ is true at x iff there exists some y such that xRy and A is true at y. This is equivalent to saying it's not the case that $\neg A$ is true at all y such that xRy, which is exactly what $\neg\Box\neg A$ says.

These claims are both logically true.

1. $\Box\neg A \leftrightarrow \neg\Diamond A$
2. $\Diamond\neg A \leftrightarrow \neg\Box A$

To move a negation sign outside of a modal operator, either $\Box$ or $\Diamond$, you have to rotate the operator by 45 degrees.

This sentence is also true no matter what the model looks like, and no matter what sentence A is.

- $\Box(A \to B) \to (\Box A \to \Box B)$

$\Box(A \to B) \to (\Box A \to \Box B)$

- Assume it is false at w.
- So $\Box(A \to B)$ is true at w and $(\Box A \to \Box B)$ is false at w.
- So $\Box A$ is true at w and $\Box B$ is false at w.
- So at every world y such that wRy, A must be true (since $\Box A$ is true at w), and $A \to B$ must be true (since $\Box(A \to B)$ is true at w).
- If A and $A \to B$ are true at y, B must be true at y as well.
- But this implies that B is true at all y such that wRy, contradicting the assumption that $\Box B$ is false at w.

This principle has a very important role in the history of modal logics.

- $\Box(A \to B) \to (\Box A \to \Box B)$

Having this be an axiom is one of two conditions on what have come to be called **normal** modal logics.

For Next Time

We'll look in more detail at what happens when we adjust this accessibility relation, and how this creates logics that are philosophically useful.