Trees and Validity

2026-02-02

Review: Tableaux Basics

Before we move on to using tableaux for validity testing, let's review the basics of them, including:

- What tableaux are;
- The rules for building them;
- What it means for a branch to close;
- What it means for a whole tableau to close

Core Idea

Tableaux systematically explore what has to be the case for some sentences to be true together.

So far we've only done this for a single sentence, today we'll expand this to multiple sentences.

- We start by writing down our initial sentences with truth values (signs). Call these our assumptions.
- We apply rules that tell us what else must be true given those assumptions.
- If every way of making the assumptions true leads to a contradiction, the assumptions cannot all be true together.

Each logical connective has rules for when it's true and when it's false.

Non-branching rules: When we know for certain what must follow.

Branching rules: When there are multiple ways to make something true.

1.	$\mathbb{T}\,\neg A$	
2.	$\mathbb{F}\,A$	$\neg\mathbb{T}\,1$

If a negation is true, the negated sentence is false.

1.	$\mathbb{F}\,\neg A$	
2.	$\mathbb{T}\,A$	$\neg\mathbb{F}\,1$

If a negation is false, the negated sentence is true.

The Rules: Conjunction

$$
\begin{array}{lcl}
1. & \mathbb{T}\, A \wedge B & \\
2. & \mathbb{T}\, A & \wedge\mathbb{T}\, 1 \\
3. & \mathbb{T}\, B & \wedge\mathbb{T}\, 1
\end{array}
$$

If a conjunction is true, both conjuncts are true.

$$
\begin{array}{lccl}
1. & \multicolumn{2}{c}{\mathbb{F}\, A \wedge B} & \\
2. & \mathbb{F}\, A & \mathbb{F}\, B & \wedge\mathbb{F}\, 1
\end{array}
$$

If a conjunction is false, at least one conjunct is false. This is our first instance of a **branching rule**.

1. $\mathbb{T}\, A \vee B$

2. $\mathbb{T}\, A \qquad \mathbb{T}\, B \qquad \vee\mathbb{T}\,1$

If a disjunction is true, at least one disjunct is true.

1. $\mathbb{F}\, A \vee B$
2. $\mathbb{F}\, A \qquad \vee\mathbb{F}\,1$
3. $\mathbb{F}\, B \qquad \vee\mathbb{F}\,1$

If a disjunction is false, both disjuncts are false.

$$\begin{array}{lccc} 1. & & \mathbb{T}\, A \to B & \\ 2. & \mathbb{F}\, A & \quad \mathbb{T}\, B & \to\mathbb{T}\, 1 \end{array}$$

If a conditional is true, either the antecedent is false or the consequent is true.

$$\begin{array}{lll} 1. & \mathbb{F}\, A \to B & \\ 2. & \quad \mathbb{T}\, A & \to\mathbb{F}\, 1 \\ 3. & \quad \mathbb{F}\, B & \to\mathbb{F}\, 1 \end{array}$$

If a conditional is false, the antecedent is true and the consequent is false.

A branch closes when it contains a contradiction.

For propositional logic, this means the branch has both $\mathbb{T}A$ and $\mathbb{F}A$ for some sentence A

In practice there will always be an **atomic** sentence, i.e., a sentence with just a single sentence letter, that triggers this. But it could be any arbitrarily long sentence.

- A closed branch represents an impossible combination.
- When a branch closes, we've shown that path cannot happen.

When a Tableau Closes

A tableau closes when **every** branch closes.

- This means every possible way of making our initial assumptions true leads to a contradiction.
- Therefore, the initial assumptions cannot all be true together.
- They are **logically inconsistent**.

The meaning of a closed tableau depends on what we started with.

- If we started with just $\mathbb{F}A$, then A is a **tautology**.
- If we started with just $\mathbb{T}A$, then A is a **contradiction**.
- If we started with premises and the negation of a conclusion, the argument is **valid**. We'll say much more on this over the lecture.

Open Branches

A branch that doesn't close represents a possible way to make the assumptions true.

- An open branch shows a scenario where all the claims on that branch hold together.
- A completed tableau with open branches shows the initial assumptions **can** be true together.
- It also shows **how** they are true together; by looking at the truth value of atomics on that branch we can see what it takes for the sentences to be true together.

Completion

Not all open tableaux are completed.

- A tableau is only completed when we've applied all applicable rules.
- An incomplete open tableau doesn't tell us anything yet.
- This distinction will matter more as tableaux get more complex.

Some More Notation

As the examples get more complicated, it will be helpful to keep track of what we've done, and what is left to do. So we'll introduce two more bits of notation.

- Check marks (✔), for lines that have been processed.
- Close marks ($\otimes$), for branches that are closed.

Tracking Progress in Tableaux

As we build tableaux, we need to track two things:

1. Which lines we've already applied rules to (so we don't repeat ourselves); and
2. Which branches have closed (so we know when we're done).

The Check Mark

When we apply a rule to a line, we mark it with a check (✔).

- This shows we've already used that line.
- The big benefit of this is that we can tell when we're finished. It's easy to check that we haven't missed any lines.
- When we're done, every line that has a logical symbol on it (and, or, not, or if) will have a check mark.

When a branch contains both $\mathbb{T}A$ and $\mathbb{F}A$ for some sentence, we close it with $\otimes$.

- This shows that branch represents an impossible scenario.
- We don't need to do any more work on that branch.

Let's work through: $\mathbb{T}(\neg P \wedge (P \vee Q))$

We'll see how to use checks and closes as we build the tableau.

Step 1: Starting Point

1.	$\mathbb{T}\ \neg P \wedge (P \vee Q)$	Assumption

We start with our assumption. We haven't applied any rules yet.

1.	$\mathbb{T}\ \neg P \wedge (P \vee Q)$ ✔	Assumption
2.	$\mathbb{T}\ \neg P$	$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ P \vee Q$	$\wedge\mathbb{T}\ 1$

We've applied the rule to line 1, so we check it off.

Step 3: Apply True ¬ Rule

1.	$\mathbb{T}\ \neg P \wedge (P \vee Q)$ ✔	Assumption
2.	$\mathbb{T}\ \neg P$ ✔	$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ P \vee Q$	$\wedge\mathbb{T}\ 1$
4.	$\mathbb{F}\ P$	$\neg\mathbb{T}\ 2$

We've applied the rule to line 2, so we check it off.

1.	$\mathbb{T}\ \neg P \wedge (P \vee Q)$ ✔	Assumption
2.	$\mathbb{T}\ \neg P$ ✔	$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ P \vee Q$ ✔	$\wedge\mathbb{T}\ 1$
4.	$\mathbb{F}\ P$	$\neg\mathbb{T}\ 2$
5.	$\mathbb{T}\ P \qquad \mathbb{T}\ Q$	$\vee\mathbb{T}\ 3$

We've applied the rule to line 3, creating two branches.

Step 5: Close the Left Branch

1.	$\mathbb{T}\ \neg P \wedge (P \vee Q)$ ✔	Assumption
2.	$\mathbb{T}\ \neg P$ ✔	$\wedge\mathbb{T}\,1$
3.	$\mathbb{T}\ P \vee Q$ ✔	$\wedge\mathbb{T}\,1$
4.	$\mathbb{F}\ P$	$\neg\mathbb{T}\,2$
5.	$\mathbb{T}\ P$ $\otimes$ | $\mathbb{T}\ Q$	$\vee\mathbb{T}\,3$

The left branch has both $\mathbb{F}P$ (line 4) and $\mathbb{T}P$ (line 5), so it closes.

1.	$\mathbb{T}\ \neg P \wedge (P \vee Q)$ ✔	Assumption
2.	$\mathbb{T}\ \neg P$ ✔	$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ P \vee Q$ ✔	$\wedge\mathbb{T}\ 1$
4.	$\mathbb{F}\ P$	$\neg\mathbb{T}\ 2$
5.	$\mathbb{T}\ P \quad \mathbb{T}\ Q$	$\vee\mathbb{T}\ 3$
	$\otimes$	

The right branch has $\mathbb{F}P$ and $\mathbb{T}Q$ - no contradiction. It stays open.

What This Tells Us

- The left branch closed, showing one way to make our assumptions doesn’t work.
- The right branch stayed open, showing there **is** a way to make them all true.
- Specifically: when P is false and Q is true.
- So $\neg P \wedge (P \vee Q)$ is **not** a contradiction - it’s satisfiable.

Check marks ():

- These are applied to lines after we are done with them.
- They help us track what is already done.

Close marks ():

- These are applied to branches that contain contradictions.
- They show which branches are impossible.

Branches in Branches

This is one of the more annoying parts of building these trees.

Once we branch, the rule for any sentence that hasn't been checked has to be applied with the results being put on both branches.

If that's a branching rule, that means we have branches within branches.

Branching Within Branches

Sometimes we need to apply branching rules multiple times in the same tableau.

- When this happens, we get branches splitting off from other branches.
- Each new branch explores a different combination of possibilities.
- We have to track each branch separately.

Example: $(P \vee Q) \wedge (\neg P \vee R)$

Let's build a tableau starting with $\mathbb{T}((P \vee Q) \wedge (\neg P \vee R))$.

This will require us to branch twice.

Step 1: Starting Point

1.	$\mathbb{T}\ (P \vee Q) \wedge (\neg P \vee R)$	Assumption

We start with our assumption.

1.	$\mathbb{T}\ (P \vee Q) \wedge (\neg P \vee R)$ ✔	Assumption
2.	$\mathbb{T}\ P \vee Q$	$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ \neg P \vee R$	$\wedge\mathbb{T}\ 1$

Both conjuncts must be true, so we write them both down.

Step 3: First Branch - True $\vee$ Rule

$$
\begin{array}{lcl}
1. & \mathbb{T}\ (P \vee Q) \wedge (\neg P \vee R)\ \checkmark & \text{Assumption} \\
2. & \mathbb{T}\ P \vee Q\ \checkmark & \wedge\mathbb{T}\ 1 \\
3. & \mathbb{T}\ \neg P \vee R & \wedge\mathbb{T}\ 1 \\
4. & \mathbb{T}\ P \qquad \mathbb{T}\ Q & \vee\mathbb{T}\ 2
\end{array}
$$

Line 2 gives us our first split: either P or Q is true.

Now we have two branches, and line 3 ($\mathbb{T}(\neg P \vee R)$) appears on both.

- We need to apply the rule for line 3 to **each branch separately**.
- This will create branches within branches.

Step 4: Apply Rule to Both Branches

1.	$\mathbb{T}\ (P \lor Q) \land (\neg P \lor R)$ ✔	Assumption
2.	$\mathbb{T}\ P \lor Q$ ✔	$\land\mathbb{T}\,1$
3.	$\mathbb{T}\ \neg P \lor R$ ✔	$\land\mathbb{T}\,1$
4.	$\mathbb{T}\ P$ $\qquad$ $\mathbb{T}\ Q$	$\lor\mathbb{T}\,2$
5.	$\mathbb{T}\ \neg P$ $\quad$ $\mathbb{T}\ R$ $\quad$ $\mathbb{T}\ \neg P$ $\quad$ $\mathbb{T}\ R$	$\lor\mathbb{T}\,3$

We apply line 3 to both main branches. Now we have **four** branches total.

Counting the Branches

Let's trace each complete branch from top to bottom:

1. Far left: $\mathbb{T}(P \vee Q), \mathbb{T}(\neg P \vee R), \mathbb{T}P, \mathbb{T}\neg P$
2. Middle left: $\mathbb{T}(P \vee Q), \mathbb{T}(\neg P \vee R), \mathbb{T}P, \mathbb{T}R$
3. Middle right: $\mathbb{T}(P \vee Q), \mathbb{T}(\neg P \vee R), \mathbb{T}Q, \mathbb{T}\neg P$
4. Far right: $\mathbb{T}(P \vee Q), \mathbb{T}(\neg P \vee R), \mathbb{T}Q, \mathbb{T}R$

Step 5: Expand Negations

1.	$\mathbb{T}\ (P \vee Q) \wedge (\neg P \vee R)$ ✔				Assumption
2.	$\mathbb{T}\ P \vee Q$ ✔				$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ \neg P \vee R$ ✔				$\wedge\mathbb{T}\ 1$
4.	$\mathbb{T}\ P$		$\mathbb{T}\ Q$		$\vee\mathbb{T}\ 2$
5.	$\mathbb{T}\ \neg P$ ✔	$\mathbb{T}\ R$	$\mathbb{T}\ \neg P$ ✔	$\mathbb{T}\ R$	$\vee\mathbb{T}\ 3$
6.	$\mathbb{F}\ P$				$\neg\mathbb{T}\ 5$
7.			$\mathbb{F}\ P$		$\neg\mathbb{T}\ 5$

We apply the negation rule to the $\mathbb{T}\neg P$ lines (5 and 6).

1.	$\mathbb{T}\ (P \lor Q) \land (\neg P \lor R)$ ✔				Assumption
2.	$\mathbb{T}\ P \lor Q$ ✔				$\land\mathbb{T}\ 1$
3.	$\mathbb{T}\ \neg P \lor R$ ✔				$\land\mathbb{T}\ 1$
4.	$\mathbb{T}\ P$		$\mathbb{T}\ Q$		$\lor\mathbb{T}\ 2$
5.	$\mathbb{T}\ \neg P$ ✔	$\mathbb{T}\ R$	$\mathbb{T}\ \neg P$ ✔	$\mathbb{T}\ R$	$\lor\mathbb{T}\ 3$
6.	$\mathbb{F}\ P$				$\neg\mathbb{T}\ 5$
7.	$\otimes$		$\mathbb{F}\ P$		$\neg\mathbb{T}\ 5$
			$\otimes$		

Two branches close: they have both $\mathbb{T}P$ and $\mathbb{F}P$.

1.	$\mathbb{T}\ (P \vee Q) \wedge (\neg P \vee R)$ ✔				Assumption
2.	$\mathbb{T}\ P \vee Q$ ✔				$\wedge\mathbb{T}\ 1$
3.	$\mathbb{T}\ \neg P \vee R$ ✔				$\wedge\mathbb{T}\ 1$
4.	$\mathbb{T}\ P$		$\mathbb{T}\ Q$		$\vee\mathbb{T}\ 2$
5.	$\mathbb{T}\ \neg P$ ✔	$\mathbb{T}\ R$	$\mathbb{T}\ \neg P$ ✔	$\mathbb{T}\ R$	$\vee\mathbb{T}\ 3$
6.	$\mathbb{F}\ P$				$\neg\mathbb{T}\ 5$
7.	$\otimes$		$\mathbb{F}\ P$		$\neg\mathbb{T}\ 6$
			$\otimes$		

Two branches remain open. The tableau doesn't close.

The two open branches show us when $(P \vee Q) \wedge (\neg P \vee R)$ can be true:

- **Branch 2:** When P is true and R is true (with Q either way)
- **Branch 4:** When Q is true and R is true (with P false)

In other words: the sentence is true when both Q or P holds, **and** R holds.

Summary

When you have multiple disjunctions (or other branching formulas):

- Each one creates its own split;
- Later splits happen **within** earlier branches;
- The total number of branches multiplies quickly; and
- You need to check each final branch separately for contradictions.

Testing for Validity

So far we've used tableaux to test whether sentences are tautologies, contradictions, or contingent.

Now we'll use them to test whether **arguments** are valid.

Key Point

An argument is valid if and only if it's impossible for the premises to be true and the conclusion false.

- We can test this with a tableau.
- We assume the premises are true and the conclusion is false.
- If the tableau **closes**, this combination is impossible, so the argument is **valid**.

Setting Up the Tableau

To test whether an argument is valid:

1. Write down each premise as **true** ($\mathbb{T}$);
2. Write down the conclusion as **false** ($\mathbb{F}$);
3. Apply the tableau rules; and
4. Check whether the tableau closes.

Reading the Result

If the tableau closes:

- Every branch contains a contradiction.
- So, it’s impossible for the premises to be true and conclusion false.
- So the argument is **valid**.

If the tableau stays open:

- At least one branch has no contradiction.
- So that branch shows a way the premises could be true and conclusion false.
- So the argument is invalid in propositional logic.

Why “In Propositional Logic”?

We add this qualification because we’re only using propositional logic rules.

- The argument might be valid for other reasons.
- For example, “This is a square; therefore this has four sides” has open branches.
- But it’s still a valid argument - just not in virtue of propositional logic alone.

An Important Caveat

The tableau must be **completed** before we can conclude invalidity.

- An incomplete open tableau doesn't tell us anything yet.
- We need to apply all applicable rules before declaring an argument invalid.
- For simple propositional arguments, this is usually straightforward.

Summary

- **Start with:** Premises true, conclusion false.
- **If it closes:** The argument is valid.
- **If it is open:** The argument is invalid in propositional logic.
- **Key insight:** We're testing whether premises can be true while the conclusion is false.

Example: A Valid Argument

Let's test this argument for validity:

$$P \to Q$$
$$\therefore (P \land R) \to Q$$

We'll see that this is valid.

Setup

To test validity, we assume:

- The premise is true: $\mathbb{T}(P \to Q)$.
- The conclusion is false: $\mathbb{F}((P \land R) \to Q)$.

Initial Tableau

1.	$\mathbb{T}\ P \to Q$	Assumption
2.	$\mathbb{F}\ (P \wedge R) \to Q$	Assumption

Now we need to apply rules to build out the tree.

We have two lines we could apply rules to. Which should we do first?

- If we apply the rules to line 1: $\mathbb{T}(P \to Q)$ we will **branch**.
- If we apply the rules to line 2: $\mathbb{F}((P \wedge R) \to Q)$ we will **not branch**.

The Golden Rule

Always apply non-branching rules before branching rules.

- Branching multiplies the number of paths through the tree.
- Every line you add before branching has to be added to each branch.
- So delay branching as long as possible.
- This keeps tableaux simpler and easier to work with.

Strictly speaking, either way will work. This is strategic advice, not logical advice.

But really, if you branch too early, you'll have twice as much work to do, which means twice as many chances to make a mistake.

Step 1: Apply Non-Branching Rule First

1.	$\mathbb{T}\ P \to Q$		Assumption
2.	$\mathbb{F}\ (P \land R) \to Q$	✔	Assumption
3.	$\mathbb{T}\ P \land R$		$\to\mathbb{F}\ 2$
4.	$\mathbb{F}\ Q$		$\to\mathbb{F}\ 2$

When a conditional is false, the antecedent is true and the consequent is false.

Step 2: Continue with Non-Branching Rules

1.	$\mathbb{T}\ P \to Q$	Assumption
2.	$\mathbb{F}\ (P \land R) \to Q$ ✔	Assumption
3.	$\mathbb{T}\ P \land R$ ✔	$\to\mathbb{F}\ 2$
4.	$\mathbb{F}\ Q$	$\to\mathbb{F}\ 2$
5.	$\mathbb{T}\ P$	$\land\mathbb{T}\ 3$
6.	$\mathbb{T}\ R$	$\land\mathbb{T}\ 3$

When a conjunction is true, both conjuncts are true.

Step 3: Now Apply the Branching Rule

1.	$\mathbb{T}\ P \to Q$ $\checkmark$	Assumption
2.	$\mathbb{F}\ (P \land R) \to Q$ $\checkmark$	Assumption
3.	$\mathbb{T}\ P \land R$ $\checkmark$	$\to\mathbb{F}\ 2$
4.	$\mathbb{F}\ Q$	$\to\mathbb{F}\ 2$
5.	$\mathbb{T}\ P$	$\land\mathbb{T}\ 3$
6.	$\mathbb{T}\ R$	$\land\mathbb{T}\ 3$
7.	$\mathbb{F}\ P \qquad \mathbb{T}\ Q$	$\to\mathbb{T}\ 1$

Now we apply the rule for $\mathbb{T}(P \to Q)$, which creates two branches.

1.	$\mathbb{T}\ P \to Q$ ✔	Assumption
2.	$\mathbb{F}\ (P \wedge R) \to Q$ ✔	Assumption
3.	$\mathbb{T}\ P \wedge R$ ✔	$\to\mathbb{F}\ 2$
4.	$\mathbb{F}\ Q$	$\to\mathbb{F}\ 2$
5.	$\mathbb{T}\ P$	$\wedge\mathbb{T}\ 3$
6.	$\mathbb{T}\ R$	$\wedge\mathbb{T}\ 3$
7.	$\mathbb{F}\ P$ $\mathbb{T}\ Q$	$\to\mathbb{T}\ 1$
	$\otimes$ $\otimes$	

Left branch: $\mathbb{T}P$ (line 5) and $\mathbb{F}P$ (line 7) - so it **closes**.

Right branch: $\mathbb{F}Q$ (line 4) and $\mathbb{T}Q$ (line 7) - so it **closes**.

Conclusion

Both branches close, so the entire tableau **closes**.

- This means it's **impossible** for the premise to be true and the conclusion false.
- Therefore, the argument is **valid**.
- The strategy of doing non-branching rules first kept the tree manageable.

If we had branched first at line 1, we would have had two branches from line 1.

- Then we'd have to apply line 2's rule to **each** branch.
- And then had to apply line 3's rule to **each** of those branches.
- That's a lot more work for the same result.

When building tableaux:

1. Look at which rules you could apply.
2. Identify which will branch and which won't.
3. **Always do non-branching rules first**.
4. Only branch when you have no other choice.

Example: An Invalid Argument

Let’s test this argument for validity:

$$P \to Q$$

$$\therefore (P \lor R) \to Q$$

Note: we’ve just changed $\land$ to $\lor$ from the previous example.

To test validity, we assume:

- The premise is true: $\mathbb{T}(P \to Q)$.
- The conclusion is false: $\mathbb{F}((P \lor R) \to Q)$.

1.	$\mathbb{T}\ P \to Q$	Assumption
2.	$\mathbb{F}\ (P \lor R) \to Q$	Assumption

As before, we need to decide which rule to apply first.

Apply Non-Branching Rule First

1.	$\mathbb{T}\ P \to Q$	Assumption
2.	$\mathbb{F}\ (P \vee R) \to Q$ ✔	Assumption
3.	$\mathbb{T}\ P \vee R$	$\to\mathbb{F}\ 2$
4.	$\mathbb{F}\ Q$	$\to\mathbb{F}\ 2$

When a conditional is false, the antecedent is true and the consequent is false.

At this point, both remaining unchecked lines require branching:

- Line 1: $\mathbb{T}(P \to Q)$ branches
- Line 3: $\mathbb{T}(P \vee R)$ branches

Either order works. Let's do line 3 first. (Actually it would be easier to do 1 first - can you see why? But there is more to show this way.)

Apply First Branching Rule

1.	$\mathbb{T}\ P \to Q$	Assumption
2.	$\mathbb{F}\ (P \vee R) \to Q$ ✔	Assumption
3.	$\mathbb{T}\ P \vee R$ ✔	$\to\mathbb{F}\ 2$
4.	$\mathbb{F}\ Q$	$\to\mathbb{F}\ 2$
5.	$\mathbb{T}\ P \qquad \mathbb{T}\ R$	$\vee\mathbb{T}\ 3$

When a disjunction is true, at least one disjunct is true. This creates two branches.

1.	$\mathbb{T}\ P \to Q$ ✔	Assumption
2.	$\mathbb{F}\ (P \vee R) \to Q$ ✔	Assumption
3.	$\mathbb{T}\ P \vee R$ ✔	$\to\mathbb{F}\,2$
4.	$\mathbb{F}\ Q$	$\to\mathbb{F}\,2$
5.	$\mathbb{T}\ P$ $\mathbb{T}\ R$	$\vee\mathbb{T}\,3$
6.	$\mathbb{F}\ P$ $\mathbb{T}\ Q$ $\mathbb{F}\ P$ $\mathbb{T}\ Q$	$\to\mathbb{T}\,1$

Now we apply the rule for line 1 to both branches.

1.	$\mathbb{T}\ P \to Q \checkmark$				Assumption
2.	$\mathbb{F}\ (P \vee R) \to Q \checkmark$				Assumption
3.	$\mathbb{T}\ P \vee R \checkmark$				$\to\mathbb{F}\ 2$
4.	$\mathbb{F}\ Q$				$\to\mathbb{F}\ 2$
5.	$\mathbb{T}\ P$		$\mathbb{T}\ R$		$\vee\mathbb{T}\ 3$
6.	$\mathbb{F}\ P$	$\mathbb{T}\ Q$	$\mathbb{F}\ P$	$\mathbb{T}\ Q$	$\to\mathbb{T}\ 1$
	$\otimes$	$\otimes$		$\otimes$	

Three branches close, but crucially one stays open.

Analyzing the Branches

Let's check each branch:

- Far left: $\mathbb{T}P$ and $\mathbb{F}P$ - closes
- Middle left: $\mathbb{T}P$, $\mathbb{F}Q$, and $\mathbb{T}Q$ - closes
- Middle right: $\mathbb{T}R$, $\mathbb{F}Q$, and $\mathbb{F}P$ - stays open
- Far right: $\mathbb{T}R$, $\mathbb{F}Q$, and $\mathbb{T}Q$ - closes

The middle right branch remains open with:

- $\mathbb{T}(P \to Q)$ (line 1)
- $\mathbb{F}Q$ (line 4)
- $\mathbb{T}R$ (line 6)
- $\mathbb{F}P$ (line 7)

These don't contradict each other, and we've applied all applicable rules.

The open branch shows a scenario where:

- P is false;
- Q is false; and
- R is true.
- In this scenario, $P \to Q$ is true (false antecedent);
- But $(P \vee R) \to Q$ is false ($P \vee R$ is true because R is true, but Q is false).

Since the tableau has an open branch it is possible for the premise to be true and conclusion false.

- Therefore, the argument is **invalid in propositional logic**.
- The open branch gives us a counterexample: P false, Q false, R true.

Valid: $P \to Q \therefore (P \wedge R) \to Q$

Invalid: $P \to Q \therefore (P \vee R) \to Q$

The difference:

- With $\wedge$: if $P \wedge R$ is true, then P must be true
- With $\vee$: if $P \vee R$ is true, P might be false (if R is true)
- This is exactly the case that leads to the open branch.

Let's test this argument:

$$P \to Q$$

$$Q \to R$$

$$\neg R$$

$$\therefore \neg P$$

This is a chain of modus tollens reasoning.

To test validity, we assume:

- The premises are true: $\mathbb{T}(P \to Q), \mathbb{T}(Q \to R), \mathbb{T}\neg R$
- The conclusion is false: $\mathbb{F}\neg P$

Initial Tableau

1.	$\mathbb{T}\ P \to Q$	Assumption
2.	$\mathbb{T}\ Q \to R$	Assumption
3.	$\mathbb{T}\ \neg R$	Assumption
4.	$\mathbb{F}\ \neg P$	Assumption

Now we need to decide our strategy.

Let's identify which rules will branch:

- Line 1: $\mathbb{T}(P \to Q)$ - **branches**
- Line 2: $\mathbb{T}(Q \to R)$ - **branches**
- Line 3: $\mathbb{T}\neg R$ - **does not branch**
- Line 4: $\mathbb{F}\neg P$ - **does not branch**

Strategy: Do lines 3 and 4 first!

Step 1: Apply Non-Branching Rules

1.	$\mathbb{T}\ P \to Q$	Assumption
2.	$\mathbb{T}\ Q \to R$	Assumption
3.	$\mathbb{T}\ \neg R$ ✔	Assumption
4.	$\mathbb{F}\ \neg P$ ✔	Assumption
5.	$\mathbb{F}\ R$	$\neg\mathbb{T}\ 3$
6.	$\mathbb{T}\ P$	$\neg\mathbb{F}\ 4$

We now have $\mathbb{F}R$ and $\mathbb{T}P$ on every branch we might create.

Now both remaining lines will branch. Which should we do first?

- Line 1: $\mathbb{T}(P \to Q)$ gives us $\mathbb{F}P$ or $\mathbb{T}Q$
- Line 2: $\mathbb{T}(Q \to R)$ gives us $\mathbb{F}Q$ or $\mathbb{T}R$

Think ahead: We have $\mathbb{F}R$ (line 5). If we do line 2 first, one branch will immediately close!

Step 2: Apply Line 2 First

1.	$\mathbb{T}\ P \to Q$	Assumption
2.	$\mathbb{T}\ Q \to R$ ✔	Assumption
3.	$\mathbb{T}\ \neg R$ ✔	Assumption
4.	$\mathbb{F}\ \neg P$ ✔	Assumption
5.	$\mathbb{F}\ R$	$\neg\mathbb{T}\,3$
6.	$\mathbb{T}\ P$	$\neg\mathbb{F}\,4$
7.	$\mathbb{F}\ Q \qquad \mathbb{T}\ R$	$\to\mathbb{T}\,2$

The right branch has $\mathbb{F}R$ (line 5) and $\mathbb{T}R$ (line 8).

1.	$\mathbb{T}\ P \to Q$	Assumption
2.	$\mathbb{T}\ Q \to R \checkmark$	Assumption
3.	$\mathbb{T}\ \neg R \checkmark$	Assumption
4.	$\mathbb{F}\ \neg P \checkmark$	Assumption
5.	$\mathbb{F}\ R$	$\neg\mathbb{T}\,3$
6.	$\mathbb{T}\ P$	$\neg\mathbb{F}\,4$
7.	$\mathbb{F}\ Q \qquad \mathbb{T}\ R$	$\to\mathbb{T}\,2$
	$\qquad\qquad \otimes$	

Now we only need to work on the left branch.

1.	$\mathbb{T}\ P \to Q$ ✔	Assumption
2.	$\mathbb{T}\ Q \to R$ ✔	Assumption
3.	$\mathbb{T}\ \neg R$ ✔	Assumption
4.	$\mathbb{F}\ \neg P$ ✔	Assumption
5.	$\mathbb{F}\ R$	$\neg\mathbb{T}\,3$
6.	$\mathbb{T}\ P$	$\neg\mathbb{F}\,4$
7.	$\mathbb{F}\ Q$ $\qquad$ $\mathbb{T}\ R$ $\otimes$	$\to\mathbb{T}\,2$
8.	$\mathbb{F}\ P$ $\qquad$ $\mathbb{T}\ Q$	$\to\mathbb{T}\,1$

Now we apply the rule for line 1 to the remaining open branch.

1.	$\mathbb{T}\ P \to Q$ ✔		Assumption
2.	$\mathbb{T}\ Q \to R$ ✔		Assumption
3.	$\mathbb{T}\ \neg R$ ✔		Assumption
4.	$\mathbb{F}\ \neg P$ ✔		Assumption
5.	$\mathbb{F}\ R$		$\neg\mathbb{T}\,3$
6.	$\mathbb{T}\ P$		$\neg\mathbb{F}\,4$
7.	$\mathbb{F}\ Q$	$\mathbb{T}\ R$ $\otimes$	$\to\mathbb{T}\,2$
8.	$\mathbb{F}\ P$ $\otimes$ $\mathbb{T}\ Q$ $\otimes$		$\to\mathbb{T}\,1$

Left: $\mathbb{T}P$ (line 6) and $\mathbb{F}P$ (line 8) - **closes**

Middle: $\mathbb{F}Q$ (line 7) and $\mathbb{T}Q$ (line 8) - **closes**

Right: Already closed

Every branch contains a contradiction.

- So the tableau is closed.
- So it's impossible for the premises to be true and conclusion false.
- So the argument is **valid**.

Notice what our strategy accomplished:

1. We did all non-branching rules first (lines 3 and 4).
2. We thought ahead about which branching rule to apply first.
3. Doing line 2 before line 1 meant one branch closed immediately.
4. This reduced our work significantly.

What If We'd Branched Differently?

If we'd applied line 1 before line 2:

- We'd get branches with $\mathbb{F}P$ or $\mathbb{T}Q$.
- Then we'd have to apply line 2 to **both** branches.
- We'd end up with more branches to track.
- Same result, but more work.

Sometimes thinking one step ahead pays off. But it will still work if you do it the other way, as we did in the previous example.

The Logical Content

This argument shows a chain of reasoning:

- If P then Q; and
- If Q then R; and
- But R is false,
- So Q must be false (otherwise R would be true),
- So P must be false (otherwise Q would be true).

The tableau mechanically verifies this reasoning is valid.

For Next Time

We'll start on modal logic, but all the material for the weekly assignment is now on the table.