Truth Tables

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Understanding Truth Tables

We're going to discuss how to read and understand a truth table.

Associated Reading

forall x, chapters 9-11

- Note that I'm introducing things in a different order than the book.
- We will end up in the same spot, but the aim here is not to simply repeat what's in the book.

- Think about all the combinations of truth values for the atomic sentences in a longer sentence or in an argument.

Conceptual Idea

- Think about all the combinations of truth values for the atomic sentences in a longer sentence or in an argument.
- For each possible combination, evaluate the truth of every part of every sentence in an argument.

- Think about all the combinations of truth values for the atomic sentences in a longer sentence or in an argument.
- For each possible combination, evaluate the truth of every part of every sentence in an argument.
- See if it is possible for the premises to be true and the conclusion false.

- If an argument is invalid, there will be one combination of values where the premises are true and the conclusion false.

Truth Tables and Validity

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- If there is no such combination, mark the argument valid.

Truth Tables and Validity

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- If an argument is invalid, there will be one combination of values where the premises are true and the conclusion false.
- If there is no such combination, mark the argument valid.
- If there is such a combination, tentatively mark the argument invalid.
- We'll come back to why 'tentatively'.

- We list each of the combinations in separate rows.
- In each column we list the truth value of the sentence such that the symbol at the top of that column is the main connective.
- That's, I think, a lot easier to understand in practice than in theory, so let's start with some examples.

A Truth Table

Table 1: Complete truth table for (P → Q) $\boxed{?}$ (Q → P)

P	Q	(	P	→	Q	)	$\boxed{?}$	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

One Sentence

Table 2: Truth table for a single sentence

P	Q	(	P	$\rightarrow$	Q	)	$\boxed{?}$	(	Q	$\rightarrow$	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

This is a truth table for a single sentence, not an argument. We'll get to arguments in a bit.

Understanding

Table 3: Reading the main connective column

P	Q	(	P	$\rightarrow$	Q	)	$\boxed{?}$	(	Q	$\rightarrow$	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

We will also get (even sooner) to how to build these monsters. What I first want to talk about is how to read them.

Four Rows

Table 4: Highlighting one row of the truth table

P	Q	(	P	→	Q	)	?	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

Each of the four rows represents a way things could be. For instance, the second row (in blue here) represents how things are if P is true and Q is false.

Four Rows

Table 5: Why there are four rows

P	Q	(	P	→	Q	)	$\boxed{?}$	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

There are four rows because there are 2 sentence letters—P and Q—each of which could take 2 values, so there are $2 \times 2 = 4$ combinations of values.

More Rows!

Table 6: Number of rows for different numbers of variables

P	Q	(	P	$\rightarrow$	Q	)	$\boxed{?}$	(	Q	$\rightarrow$	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

- If there had been three sentence letters, there would be eight rows.
- Four sentence letters would mean 16 rows, etc.

The Columns

Table 7: Copying atomic sentence values

P	Q	(	P	→	Q	)	$\boxed{?}$	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

- The columns under the letters reflect the value of the atomic sentences in each row.
- As you can see, they are just copied and pasted from the left-hand side.

Table 8: Values for atomic sentence P

P	Q	(	P	→	Q	)	?	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

I’ve put in blue all the truth values for P, which as you can see were just copied and pasted from the columns on the far left.

Intermediate Sentences

Table 9: Values for P → Q

P	Q	(	P	→	Q	)	?	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

The surprising thing (or at least the thing that surprised me as a student) was what we mean by the column under the $\longrightarrow$, which I've put in blue.

Intermediate Sentences

Table 10: Understanding the conditional column

P	Q	(	P	→	Q	)	$\boxed{?}$	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

Each letter here is giving the truth value of the sentence that has that first $\rightarrow$ as its main connective. That is, $P \rightarrow Q$.

Table 11: Values for Q → P

P	Q	(	P	→	Q	)	$\boxed{?}$	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

And this column gives the truth values for $Q \to P$.

Table 12: Complete truth table showing all intermediate steps

P	Q	(	P	→	Q	)	$\boxed{?}$	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

Don’t worry for now about why we write those letters down; we’ll get to that in the next lecture. For now I just want to go over how to read these tables.

Table 13: The main connective values

P	Q	(	P	→	Q	)	$\boxed{?}$	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

- The column that I've put in red gives the truth value of the sentence whose main connective is ∨.
- That is, in this case, the whole sentence.

The Big Red Column

Table 14: The main connective is what matters

P	Q	(	P	→	Q	)	?	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

Ultimately the red column is all we really care about—the others are essentially scaffolding.

Logical Truth

Table 15: A logical truth (all T in main column)

P	Q	(	P	→	Q	)	?	(	Q	→	P	)
T	T		T	T	T		T		T	T	T	
T	F		T	F	F		T		F	T	T	
F	T		F	T	T		T		T	F	F	
F	F		F	T	F		T		F	T	F	

- There is something distinctive about this table—the red column is all T.
- That means the sentence is a logical truth.
- We'll have more to say about this presently in future lectures.

Basic Truth Tables

Overview

This lecture covers the truth tables for the basic connectives.

Associated Reading

- We're still working through forall x chapters 9-11.
- This is primarily about chapter 9.
- We're not going to cover biconditionals here (or elsewhere in this course).

- Building truth tables requires, unfortunately, a small amount of memorization.
- In particular, you just have to memorize the truth tables for each of the connectives.
- Equally unfortunately, justifying yourself using truth tables requires justifying these basic tables.
- And as we'll see, that's not trivial.
- But that's for much further down the line—let's learn how to use these first, then we'll get to justifying them.

Table 16: Truth table for negation

A	¬	A
T	F	T
F	T	F

You should read it as saying that if A is T then $\neg A$ is F, and if A is F, then $\neg A$ is T.

The Conjunction Table

Table 17: Truth table for conjunction

A	B	A	$\boxed{?}$	B
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	F	F

Conjunction in Words

- A conjunction is T if both conjuncts are T, and is F otherwise.

The Disjunction Table

Table 18: Truth table for disjunction

A	B	A	$\boxed{?}$	B
T	T	T	T	T
T	F	T	T	F
F	T	F	T	T
F	F	F	F	F

Disjunction in Words

- A disjunction is T if either disjunct is T, and is F otherwise.

The Conditional Table

Table 19: Truth table for the conditional

A	B	A	$\to$	B
T	T	T	T	T
T	F	T	F	F
F	T	F	T	T
F	F	F	T	F

Note that these three sentences have exactly the same table.

Table 20: Three equivalent formulations

A	B	A	→	B	¬	A	?	B	¬	(	A	?	¬	B	)
T	T	T	T	T	F	T	T	T	T		T	F	F	T	
T	F	T	F	F	F	T	F	F	F		T	T	T	F	
F	T	F	T	T	T	F	T	T	T		F	F	F	T	
F	F	F	T	F	T	F	T	F	T		F	F	T	F	

This conditional is sometimes called **material implication**.

Oddities

It is certainly an odd interpretation of ‘if’ that makes these sentences turn out true:

- If I am 200 years old, then Michigan is part of Canada.
- If I am in Los Angeles, then I am in Ann Arbor.

But they are both true on this table.

- It turns out that interpreting the conditional this way makes the most sense of the role of conditionals in certain arguments, in particular those involving disjunctive syllogism.
- There is an allusion to this at the end of chapter 1 of *Boxes and Diamonds*.

Arguments

The big advantage of thinking of 'if' this way is that it guarantees that for any value of A, B, C, these two arguments agree on validity—that is, they are either both valid or both invalid.

- $A, B \vdash C$
- $A \vdash B \to C$

And plausibly those should be the same. A suffices for $B \to C$ just in case A and B together suffice for C.

Complicated Truth Tables

This lecture is about how to build more complicated truth tables than we have looked at so far.

Associated Reading

forall x, chapters 10 and 11.

We are going to work out the truth table for this sentence:

- $(A \vee \neg B) \rightarrow (B \rightarrow (A \wedge C))$

- How many rows should there be in the truth table?
- There are three (3) atomic sentences, so there should be $2^3 = 8$ rows.

Laying Out the Rows

- The convention for these is a bit odd.
- Here's one way to think about it.
- For the left-most column you fill the first half of the rows with T and then the second half of the rows with F.

First Column

Table 21: Setting up the first column

A	B	C	(A	$\boxed{?}$	$\neg$	B)	$\rightarrow$	(B	$\rightarrow$	(A	$\boxed{?}$	C))
T												
T												
T												
T												
F												
F												
F												
F												

Second Column

- Then the second column has one quarter T, followed by one quarter F, followed by one quarter T, followed by one quarter F.
- In this case that means we alternate every two rows.

Second Column

Table 22: Adding the second column

A	B	C	(A	$\boxed{?}$	¬	B)	→	(B	→	(A	$\boxed{?}$	C))
T	T											
T	T											
T	F											
T	F											
F	T											
F	T											
F	F											
F	F											

Third Column

- From now on you do half as many rows between changes.
- In this table we did 4 rows with one value then 4 of another for column 1, 2 with one value then 2 with another for column 2, and now alternate every row for column 3.
- It's helpful to know the full algorithm in case you ever have to do this with 5 or more variables.
- But I won't do that in this course.

Third Column

Table 23: Completing all three columns

A	B	C	(A	$\boxed{?}$	$\neg$	B)	$\to$	(B	$\to$	(A	$\boxed{?}$	C))
T	T	T										
T	T	F										
T	F	T										
T	F	F										
F	T	T										
F	T	F										
F	F	T										
F	F	F										

Now we need to go back to our sentence.

- $(A \vee \neg B) \to (B \to (A \wedge C))$

What is its **main connective**?

- It's the first $\to$. The sentence is of the form $D \to E$, where D is $(A \vee \neg B)$ and E is $(B \to (A \wedge C))$

Building Up

So eventually, we will have the truth value for the whole sentence under the first $\rightarrow$.

But before we do that, we have to evaluate D and E.

Let's start with D, i.e., $(A \vee \neg B)$.

D, Step 1

Table 24: Filling in values for A

A	B	C	(A	$\boxed{?}$	¬	B)	→	(B	→	(A	$\boxed{?}$	C))
T	T	T	T			T						
T	T	F	T			T						
T	F	T	T			F						
T	F	F	T			F						
F	T	T	F			T						
F	T	F	F			T						
F	F	T	F			F						
F	F	F	F			F						

D, Step 2

Table 25: Filling in values for ¬B

A	B	C	(A	$\boxed{?}$	¬	B)	→	(B	→	(A	$\boxed{?}$	C))
T	T	T	T		F	T						
T	T	F	T		F	T						
T	F	T	T		T	F						
T	F	F	T		T	F						
F	T	T	F		F	T						
F	T	F	F		F	T						
F	F	T	F		T	F						
F	F	F	F		T	F						

Table 26: Completing A $\boxed{?}$ ¬B

A	B	C	(A	$\boxed{?}$	¬	B)	→	(B	→	(A	$\boxed{?}$	C))
T	T	T	T	T	F	T						
T	T	F	T	T	F	T						
T	F	T	T	T	T	F						
T	F	F	T	T	T	F						
F	T	T	F	F	F	T						
F	T	F	F	F	F	T						
F	F	T	F	T	T	F						
F	F	F	F	T	T	F						

Now let's look at E, i.e., $(B \to (A \wedge C))$.

Table 27: Filling in values for B, A, C

A	B	C	(A	$\boxed{?}$	¬	B)	→	(B	→	(A	$\boxed{?}$	C))
T	T	T	T	T	F	T		T		T		T
T	T	F	T	T	F	T		T		T		F
T	F	T	T	T	T	F		F		T		T
T	F	F	T	T	T	F		F		T		F
F	T	T	F	F	F	T		T		F		T
F	T	F	F	F	F	T		T		F		F
F	F	T	F	T	T	F		F		F		T
F	F	F	F	T	T	F		F		F		F

Table 28: Computing A $\boxed{?}$ C

A	B	C	(A	$\boxed{?}$	¬	B)	→	(B	→	(A	$\boxed{?}$	C))
T	T	T	T	T	F	T		T		T	T	T
T	T	F	T	T	F	T		T		T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

E, Step 3

Table 29: Highlighting A ? C values

A	B	C	(A	?	¬	B)	→	(B	→	(A	?	C))
T	T	T	T	T	F	T		T		T	T	T
T	T	F	T	T	F	T		T		T	F	F
T	F	T	T	T	T	F		F		T	T	T
T	F	F	T	T	T	F		F		T	F	F
F	T	T	F	F	F	T		T		F	F	T
F	T	F	F	F	F	T		T		F	F	F
F	F	T	F	T	T	F		F		F	F	T
F	F	F	F	T	T	F		F		F	F	F

Table 30: Computing B → (A $\boxed{?}$ C)

A	B	C	(A	$\boxed{?}$	¬	B)	→	(B	→	(A	$\boxed{?}$	C))
T	T	T	T	T	F	T		T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T	F	F	F	F
F	F	T	F	T	T	F		F	T	F	F	T
F	F	F	F	T	T	F		F	T	F	F	F

Now, finally, we're ready to look at the main connective.

That's T $\longrightarrow$ T, i.e., T.

Table 31: Evaluating the main connective for row 1

A	B	C	(A	?	¬	B)	→	(B	→	(A	?	C))
T	T	T	T	T	F	T	T	T	T	T	T	T
T	T	F	T	T	F	T		T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T	F	F	F	F
F	F	T	F	T	T	F		F	T	F	F	T
F	F	F	F	T	T	F		F	T	F	F	F

That’s T $\longrightarrow$ F, i.e., F.

Table 32: Evaluating the main connective for row 2

A	B	C	(A	$\boxed{?}$	¬	B)	→	(B	→	(A	$\boxed{?}$	C))
T	T	T	T	T	F	T	T	T	T	T	T	T
T	T	F	T	T	F	T	F	T	F	T	F	F
T	F	T	T	T	T	F		F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T	F	F	F	F
F	F	T	F	T	T	F		F	T	F	F	T
F	F	F	F	T	T	F		F	T	F	F	F

Row 3

That’s T $\longrightarrow$ T, i.e., T.

Table 33: Evaluating the main connective for row 3

A	B	C	(A	?	¬	B)	→	(B	→	(A	?	C))
T	T	T	T	T	F	T	T	T	T	T	T	T
T	T	F	T	T	F	T	F	T	F	T	F	F
T	F	T	T	T	T	F	T	F	T	T	T	T
T	F	F	T	T	T	F		F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T	F	F	F	F
F	F	T	F	T	T	F		F	T	F	F	T
F	F	F	F	T	T	F		F	T	F	F	F

Row 4

That's also T $\longrightarrow$ T, i.e., T.

Table 34: Evaluating the main connective for row 4

A	B	C	(A	$\boxed{?}$	¬	B)	→	(B	→	(A	$\boxed{?}$	C))
T	T	T	T	T	F	T	T	T	T	T	T	T
T	T	F	T	T	F	T	F	T	F	T	F	F
T	F	T	T	T	T	F	T	F	T	T	T	T
T	F	F	T	T	T	F	T	F	T	T	F	F
F	T	T	F	F	F	T		T	F	F	F	T
F	T	F	F	F	F	T		T	F	F	F	F
F	F	T	F	T	T	F		F	T	F	F	T
F	F	F	F	T	T	F		F	T	F	F	F

Rows 5 and 6

That's F $\longrightarrow$ F, i.e., T.

Table 35: Evaluating the main connective for rows 5 and 6

A	B	C	(A	?	¬	B)	→	(B	→	(A	?	C))
T	T	T	T	T	F	T	T	T	T	T	T	T
T	T	F	T	T	F	T	F	T	F	T	F	F
T	F	T	T	T	T	F	T	F	T	T	T	T
T	F	F	T	T	T	F	T	F	T	T	F	F
F	T	T	F	F	F	T	T	T	F	F	F	T
F	T	F	F	F	F	T	T	T	F	F	F	F
F	F	T	F	T	T	F		F	T	F	F	T
F	F	F	F	T	T	F		F	T	F	F	F

Rows 7 and 8

That's T $\longrightarrow$ T, i.e., T.

Table 36: Evaluating the main connective for rows 7 and 8

A	B	C	(A	?	¬	B)	→	(B	→	(A	?	C))
T	T	T	T	T	F	T	T	T	T	T	T	T
T	T	F	T	T	F	T	F	T	F	T	F	F
T	F	T	T	T	T	F	T	F	T	T	T	T
T	F	F	T	T	T	F	T	F	T	T	F	F
F	T	T	F	F	F	T	T	T	F	F	F	T
F	T	F	F	F	F	T	T	T	F	F	F	F
F	F	T	F	T	T	F	T	F	T	F	F	T
F	F	F	F	T	T	F	T	F	T	F	F	F

Summing Up

It's true everywhere except when A, B are both T, and C is F.

Table 37: Complete truth table with all values

A	B	C	(A	$\boxed{?}$	¬	B)	→	(B	→	(A	$\boxed{?}$	C))
T	T	T	T	T	F	T	T	T	T	T	T	T
T	T	F	T	T	F	T	F	T	F	T	F	F
T	F	T	T	T	T	F	T	F	T	T	T	T
T	F	F	T	T	T	F	T	F	T	T	F	F
F	T	T	F	F	F	T	T	T	F	F	F	T
F	T	F	F	F	F	T	T	T	F	F	F	F
F	F	T	F	F	T	F	T	F	T	F	F	T
F	F	F	F	F	T	F	T	F	T	F	F	F

For Next Week

- We’ll start on analyzing arguments using truth tables.
- Remember to do the assignment by Thursday 5pm.