

## Truth Tables and Validity

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## Background

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# Overview

What propositional logic is.

## Associated Reading

This doesn't correspond to the book; it's going over what we're going to do in the book.

## Key Assumption

We start with one key assumption:

- Every sentence has precisely one of the two truth values: TRUE or FALSE.
- I will write these values as  $\top$  and  $\mathbb{F}$ .
- I'm using these funny-shaped letters because they get used in *Boxes and Diamonds*, and because they make our lives easier when we start using trees.

## Unpacking the Assumption

1. There are just two truth values:  $\top$  and  $\perp$ .
2. Every sentence has one of them. There are no truth-value gaps.
3. No sentence has both of them. There are no truth-value gluts.

## Two Parts of Classical Logic

- Traditionally, classical logic is divided into two parts.
- We're just going to look at the first part here.
- The parts differ on what counts as a **structural** feature of a sentence.

# Classical Propositional Logic

The structural features are just five sentential connectives:

- And
- Or
- Not
- If
- If and only if; usually written iff.

The result is a very simple, but very weak, logic. It doesn't even tell us that the arguments about Skippy and Lucky are structurally valid.

Here we'll simplify further by not covering iff.

# Classical Predicate Logic

As well as those structural features, we add:

- The division of parts of sentences into names, variables, predicates, and logical terms.
- The addition of the logical terms **All** and **Some**.
- If this were a logic course, we'd do that as well.
- But for a formal methods course, we'll have enough to do with just the propositional part.

## Tautology Checks

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# Overview

How to use truth tables to check for whether something is a **tautology**.

## Associated Reading

forall x, chapter 12, especially section 12.1.

# Tautologies

We are going to start with a particular kind of sentence, a **tautology**.

## Definition

A tautology is a sentence that gets the value  $\top$  in every row of its truth table.

## Examples of Tautologies

What are some sentences that might fit the bill? We will do tables for four:

- $A \vee \neg A$
- $\neg(A \wedge \neg A)$
- $A \rightarrow A$
- $(A \rightarrow B) \vee (B \rightarrow A)$

# The Law of Excluded Middle

Table 1: Truth table for excluded middle

A	A	$\vee$	$\neg$	A
T	T	T	F	T
F	F	T	T	F

# The Law of Non-Contradiction

Table 2: Truth table for non-contradiction

A	$\neg$	(A	$\wedge$	$\neg$	A)
T	F	T	F	F	T
F	T	F	F	T	F

# Reflexive Conditionals

Table 3: Truth table for reflexive conditional

A	A	$\rightarrow$	A
T	T	T	T
F	F	T	F

# A Surprising One

Table 4: Truth table for if A then B, or if B then A

A	B	(A → B)	∨	(B → A)
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	F	F	F

## Tautologies and Logical Truth

- All tautologies are logical truths.
- But the converse isn't true—some logical truths are not tautologies.
- E.g., If Brian is necessarily a human, then Brian is a human.

## Validity

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# Overview

How to use truth tables to check for whether an argument is valid.

## Associated Reading

forall x, chapter 12, especially section 12.4.

## Validity

We can also use truth tables to check for properties of arguments, and in particular to check for validity.

## Truth Tables and Validity

- An argument is (truth-functionally) valid if (and only if) every line on the truth table where all the premises are  $\top$ , the conclusion is  $\top$  as well.
- Equivalently, an argument is invalid if there is a line where the premises are  $\top$  and the conclusion  $\perp$ , and valid otherwise.

## Example of Invalidity

The argument  $A$ , therefore  $A \wedge B$  is invalid because of the second line.

Table 5: The truth table for  $A$  therefore  $A \wedge B$ .

A	B	A	A	$\wedge$	B
T	T	T	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	F	F	F	F

## Another Invalidity Example

Note that there are several lines with  $\top$  premises and conclusion. But the argument  $A \rightarrow B$ , so  $A \rightarrow C$  is invalid because of line 2.

**Table 6:** The truth table for  $A \rightarrow B$  therefore  $A \rightarrow C$ .

A	B	C	A	$\rightarrow$	B	A	$\rightarrow$	C
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	F	F
T	F	T	T	F	F	T	T	T
T	F	F	T	F	F	T	F	F
F	T	T	F	T	T	F	T	T
F	T	F	F	T	T	F	T	F
F	F	T	F	T	F	F	T	T
F	F	F	F	T	F	F	T	F

## Hypothetical Syllogism

On the other hand, the argument from  $A \rightarrow B$  and  $B \rightarrow C$  to  $A \rightarrow C$  is valid.

Table 7: The truth table for hypothetical syllogism.

A	B	C	$A$	$\rightarrow$	B	$B$	$\rightarrow$	C	$A$	$\rightarrow$	C
T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	F	F	T	F	F
T	F	T	T	F	F	F	T	T	T	T	T
T	F	F	T	F	F	F	T	F	T	F	F
F	T	T	F	T	T	T	T	T	F	T	T
F	T	F	F	T	T	T	F	F	F	T	F
F	F	T	F	T	F	F	T	T	F	T	T
F	F	F	F	T	F	F	T	F	F	T	F

## Properties of Validity

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# Overview

Finishing our discussion of truth tables by looking at some properties validity has in the truth table system.

## Associated Reading

forall x, chapter 12, sections 12.5-12.7.

## The Rules

- An argument is **invalid** if there is a row on the truth table where all the premises are true and the conclusion is false. (Roughly!)
- It is **valid** if all the rows where the premises are all true, the conclusion is true as well.

## A Relevance Failure

Is this argument valid?

$$\begin{array}{c} A \\ \therefore B \vee \neg B \end{array}$$

Yes!

- There is no line where the conclusion is false.
- So there are no lines where the premise is true and the conclusion false.
- So it is not invalid, i.e., it is valid.

## Terminology

Say a **valuation** is a function  $v$  from sentences to  $\{\top, \perp\}$  satisfying these constraints.

1.  $v(\neg A) = \top$  if  $v(A) = \perp$ , and  $v(\neg A) = \perp$  otherwise.
2.  $v(A \vee B) = \top$  if  $v(A) = \top$  or  $v(B) = \top$ , and  $v(A \vee B) = \perp$  otherwise.
3.  $v(A \wedge B) = \top$  if  $v(A) = \top$  and  $v(B) = \top$ , and  $v(A \wedge B) = \perp$  otherwise.
4.  $v(A \rightarrow B) = \top$  if  $v(A) = \perp$  or  $v(B) = \top$ , and  $v(A \rightarrow B) = \perp$  otherwise.

## Restating

- An argument is valid relative to a class of valuations  $V$  iff any valuation  $v \in V$  that makes all the premises  $\mathbb{T}$  also makes the conclusion  $\mathbb{T}$ .
- An argument is truth-functionally valid when the class  $V$  is the class of valuations satisfying the constraints on the previous slide.

## Very Technical Terminology

- The symbol  $\models$  (read as “entails” or “models”) is used to express **semantic validity**.
- By that, I mean validity defined in terms of truth values rather than formal proofs.
- I’ll use  $\Gamma \models A$  to mean that the argument with premises  $\Gamma$  and conclusion  $A$  is valid in this sense—i.e., all valuations that make all of  $\Gamma$  come out  $\top$  also make  $A$  come out  $\top$ .

## Very Technical Terminology

- The double bar in  $\models$  is to represent that this is a kind of validity defined in terms of valuations (or, as we'll start calling them, models), and not proofs.
- When we want to say there is a proof of  $A$  from premises  $\Gamma$ , we'll write  $\Gamma \vdash A$ , with a single bar.
- We won't spend *much* time in this course on proofs, but they will come up from time to time.

## Closure

- If  $\Gamma \models A$  and  $\Gamma \models A \rightarrow B$ , then  $\Gamma \models B$ .

Proof: Assume this is false. So assume that  $\Gamma \not\models B$ . So there is a valuation function  $v$  that makes everything in  $\Gamma$  come out  $\mathbb{T}$  and  $B$  come out  $\mathbb{F}$ .

Either  $v(A) = \mathbb{T}$  or  $v(A) = \mathbb{F}$ .

If  $v(A) = \mathbb{T}$ , then  $v(A \rightarrow B) = \mathbb{F}$ , contradicting  $\Gamma \models A \rightarrow B$ .

If  $v(A) = \mathbb{F}$ , then  $v$  is a counterexample to  $\Gamma \models A$ , but we know  $\Gamma \models A$  is true. Either way, such a  $v$  cannot exist, so  $\Gamma \models B$  is true.

## Monotony

- If  $\Gamma \models A$ , and  $\Gamma \subset \Delta$ , then  $\Delta \models A$ .

That is, adding premises can't turn an argument from being valid to invalid.

## Monotony Proof

- Assume that for all  $B \in \Delta$ ,  $v(B) = \top$ .
- We need to prove that  $v(A) = \top$ .
- Assume  $C \in \Gamma$ .
- Then  $C \in \Delta$ , since  $\Gamma \subset \Delta$ .
- So by hypothesis,  $v(C) = \top$ , since everything in  $\Delta$  is  $\top$ .
- So  $v$  is such that everything in  $\Gamma$  is  $\top$ .
- And since  $\Gamma \vDash A$ , that implies  $v(A) = \top$ , as required.

## Monotony Commentary

- This idea, that adding premises doesn't destroy validity, only works for logical arguments.
- It isn't true for good arguments in general.

## Tweety the First

Tweety is a bird.

∴ Tweety flies.

That's a perfectly good, though not logically valid, argument.

## Tweety the Second

Tweety is a bird.

Tweety is black and white, lives in Antarctica, and lays large eggs.

∴ Tweety flies.

That's not a very good argument!

## Transitivity

- If  $\Gamma \models A$  and  $\Delta \cup A \models B$ , then  $\Gamma \cup \Delta \models B$ .

If some premises entail  $A$ , and some other premises plus  $A$  entail  $B$ , then the two sets of premises between them entail  $B$ .

This is crucial for being able to chain together lines of reasoning.

## Transitivity Proof

- Assume that for all  $C \in \Gamma \cup \Delta$ ,  $v(C) = \top$ .
- We need to prove  $v(B) = \top$ .
- Since everything in  $\Gamma$  is  $\top$  according to  $v$ , and  $\Gamma \vDash A$ , it follows that  $v(A) = \top$ .
- Since everything in  $\Delta$  is  $\top$  according to  $v$ , and  $A$  is  $\top$  according to  $v$ , and  $\Delta \cup A \vDash B$ , it follows that  $v(B) = \top$ , as required.

## Deduction Theorem

This is why we define  $\rightarrow$  the way we do.

- $\Gamma \vDash A \rightarrow B$  if and only if  $\Gamma \cup A \vDash B$ .

Note that there are two claims here—one for each direction. We need to prove each.

## Deduction Theorem Left-to-Right

- Assume  $\Gamma \vDash A \rightarrow B$ , and prove  $\Gamma \cup A \vDash B$ .
- So assume  $v(C) = \mathbb{T}$  for all  $C \in \Gamma$ , and  $v(A) = \mathbb{T}$ , and aim to prove  $v(B) = \mathbb{T}$ .
- Since  $\Gamma \vDash A \rightarrow B$  and  $v(C) = \mathbb{T}$  for all  $C \in \Gamma$ , it follows that  $v(A \rightarrow B) = \mathbb{T}$ .
- Since  $v(A \rightarrow B) = \mathbb{T}$  and  $v(A) = \mathbb{T}$ , it must be that  $v(B) = \mathbb{T}$ , since that's the only line on the truth table where  $A \rightarrow B$  and  $A$  are both  $\mathbb{T}$ .

## Deduction Theorem Right-to-Left

- Assume that  $\Gamma \cup A \vDash B$ , and prove  $\Gamma \vDash A \rightarrow B$ .
- So assume  $v(C) = \mathbb{T}$  for all  $C \in \Gamma$ , and prove  $v(A \rightarrow B) = \mathbb{T}$ .
- Either  $v(A) = \mathbb{T}$  or  $v(A) = \mathbb{F}$ . Take each case in turn.
- If  $v(A) = \mathbb{T}$ , then since  $v(C) = \mathbb{T}$  for all  $C \in \Gamma$ , and  $\Gamma \cup A \vDash B$ , it follows that  $v(B) = \mathbb{T}$ , so  $v(A \rightarrow B) = \mathbb{T}$ .
- If  $v(A) = \mathbb{F}$ , it follows directly that  $v(A \rightarrow B) = \mathbb{T}$ .
- Either way,  $v(A \rightarrow B) = \mathbb{T}$  as required.

## Deduction Theorem Comments

- This is a striking result.
- It shows that proving  $A \rightarrow B$  is just the same as proving  $B$ , assuming you're allowed to add  $A$  as an extra assumption.
- And that's a good thing, intuitively. That is how we prove conditionals.
- But it only works if you have the (very odd-looking) truth table that we're using for  $\rightarrow$ .
- This is the main reason for thinking, despite its odd appearance, that this truth table is the right one for  $\rightarrow$ .

## For Next Time

We will start working on a different way to analyze arguments: truth trees.