Truth Tables and Validity

2026-01-19

Background

What propositional logic is.

Associated Reading

This doesn't correspond to the book; it's going over what we're going to do in the book.

We start with one key assumption:

- Every sentence has precisely one of the two truth values: TRUE or FALSE.
- I will write these values as $\mathbb{T}$ and $\mathbb{F}$.
- I'm using these funny-shaped letters because they get used in *Boxes and Diamonds*, and because they make our lives easier when we start using trees.

1. There are just two truth values: $\mathbb{T}$ and $\mathbb{F}$.
2. Every sentence has one of them. There are no truth-value *gaps*.
3. No sentence has both of them. There are no truth-value *gluts*.

- Traditionally, classical logic is divided into two parts.
- We're just going to look at the first part here.
- The parts differ on what counts as a **structural** feature of a sentence.

The structural features are just five sentential connectives:

- And
- Or
- Not
- If
- If and only if; usually written iff.

The result is a very simple, but very weak, logic. It doesn't even tell us that the arguments about Skippy and Lucky are structurally valid.

Here we'll simplify further by not covering iff.

As well as those structural features, we add:

- The division of parts of sentences into names, variables, predicates, and logical terms.
- The addition of the logical terms All and Some.
- If this were a logic course, we'd do that as well.
- But for a formal methods course, we'll have enough to do with just the propositional part.

Tautology Checks

How to use truth tables to check for whether something is a **tautology**.

Associated Reading

forall x, chapter 12, especially section 12.1.

We are going to start with a particular kind of sentence, a **tautology**.

Definition

A tautology is a sentence that gets the value $\mathbb{T}$ in every row of its truth table.

What are some sentences that might fit the bill? We will do tables for four:

- $A \lor \neg A$
- $\neg(A \land \neg A)$
- $A \to A$
- $(A \to B) \lor (B \to A)$

Table 1: Truth table for excluded middle

A	A	$\vee$	$\neg$	A
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$

Table 2: Truth table for non-contradiction

A	¬	(A	∧	¬	A)
𝕋	𝕋	𝕋	𝔽	𝔽	𝕋
𝔽	𝕋	𝔽	𝔽	𝕋	𝔽

Table 3: Truth table for reflexive conditional

A	A	$\rightarrow$	A
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$

Table 4: Truth table for if A then B, or if B then A

A	B	(A	→	B)	∨	(B	→	A)
𝕋	𝕋	𝕋	𝕋	𝕋	𝕋	𝕋	𝕋	𝕋
𝕋	𝔽	𝕋	𝔽	𝔽	𝕋	𝔽	𝕋	𝕋
𝔽	𝕋	𝔽	𝕋	𝕋	𝕋	𝕋	𝔽	𝔽
𝔽	𝔽	𝔽	𝕋	𝔽	𝕋	𝔽	𝕋	𝔽

- All tautologies are logical truths.
- But the converse isn't true—some logical truths are not tautologies.
- E.g., If Brian is necessarily a human, then Brian is a human.

Validity

How to use truth tables to check for whether an argument is valid.

Associated Reading

forall x, chapter 12, especially section 12.4.

We can also use truth tables to check for properties of arguments, and in particular to check for validity.

- An argument is (truth-functionally) valid if (and only if) every line on the truth table where all the premises are $\mathbb{T}$, the conclusion is $\mathbb{T}$ as well.
- Equivalently, an argument is invalid if there is a line where the premises are $\mathbb{T}$ and the conclusion $\mathbb{F}$, and valid otherwise.

The argument A, therefore $A \wedge B$ is invalid because of the second line.

Table 5: The truth table for A therefore $A \wedge B$.

A	B	A	A	$\wedge$	B
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

Another Invalidity Example

Note that there are several lines with $\mathbb{T}$ premises and conclusion. But the argument $A \to B$, so $A \to C$ is invalid because of line 2.

Table 6: The truth table for $A \to B$ therefore $A \to C$.

A	B	C	A	$\to$	B	A	$\to$	C
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$

Hypothetical Syllogism

On the other hand, the argument from $A \rightarrow B$ and $B \rightarrow C$ to $A \rightarrow C$ is valid.

Table 7: The truth table for hypothetical syllogism.

A	B	C	A	$\rightarrow$	B	B	$\rightarrow$	C	A	$\rightarrow$	C
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$

Properties of Validity

Finishing our discussion of truth tables by looking at some properties validity has in the truth table system.

Associated Reading

forall x, chapter 12, sections 12.5-12.7.

- An argument is **invalid** if there is a row on the truth table where all the premises are true and the conclusion is false. (Roughly!)
- It is **valid** if all the rows where the premises are all true, the conclusion is true as well.

Is this argument valid?

$$A$$
$$\therefore \; B \vee \neg B$$

Yes!

- There is no line where the conclusion is false.
- So there are no lines where the premise is true and the conclusion false.
- So it is not invalid, i.e., it is valid.

Say a **valuation** is a function v from sentences to $\{\mathbb{T}, \mathbb{F}\}$ satisfying these constraints.

1. $v(\neg A) = \mathbb{T}$ if $v(A) = \mathbb{F}$, and $v(\neg A) = \mathbb{F}$ otherwise.
2. $v(A \vee B) = \mathbb{T}$ if $v(A) = \mathbb{T}$ or $v(B) = \mathbb{T}$, and $v(A \vee B) = \mathbb{F}$ otherwise.
3. $v(A \wedge B) = \mathbb{T}$ if $v(A) = \mathbb{T}$ and $v(B) = \mathbb{T}$, and $v(A \wedge B) = \mathbb{F}$ otherwise.
4. $v(A \to B) = \mathbb{T}$ if $v(A) = \mathbb{F}$ or $v(B) = \mathbb{T}$, and $v(A \to B) = \mathbb{F}$ otherwise.

- An argument is valid relative to a class of valuations V iff any valuation $v \in V$ that makes all the premises $\mathbb{T}$ also makes the conclusion $\mathbb{T}$.
- An argument is truth-functionally valid when the class V is the class of valuations satisfying the constraints on the previous slide.

- The symbol ⊨ (read as "entails" or "models") is used to express **semantic** validity.
- By that, I mean validity defined in terms of truth values rather than formal proofs.
- I'll use $\Gamma \vDash A$ to mean that the argument with premises Γ and conclusion A is valid in this sense—i.e., all valuations that make all of Γ come out $\mathbb{T}$ also make A come out $\mathbb{T}$.

- The double bar in $\vDash$ is to represent that this is a kind of validity defined in terms of valuations (or, as we'll start calling them, models), and not proofs.
- When we want to say there is a proof of A from premises Γ, we'll write $\Gamma \vdash A$, with a single bar.
- We won't spend *much* time in this course on proofs, but they will come up from time to time.

- If $\Gamma \vDash A$ and $\Gamma \vDash A \to B$, then $\Gamma \vDash B$.

Proof: Assume this is false. So assume that $\Gamma \nvDash B$. So there is a valuation function v that makes everything in Γ come out $\mathbb{T}$ and B come out $\mathbb{F}$.

Either $v(A) = \mathbb{T}$ or $v(A) = \mathbb{F}$.

If $v(A) = \mathbb{T}$, then $v(A \to B) = \mathbb{F}$, contradicting $\Gamma \vDash A \to B$.

If $v(A) = \mathbb{F}$, then v is a counterexample to $\Gamma \vDash A$, but we know $\Gamma \vDash A$ is true. Either way, such a v cannot exist, so $\Gamma \vDash B$ is true.

- If $\Gamma \vDash A$, and $\Gamma \subset \Delta$, then $\Delta \vDash A$.

That is, adding premises can't turn an argument from being valid to invalid.

- Assume that for all $B \in \Delta$, $v(B) = \mathbb{T}$.
- We need to prove that $v(A) = \mathbb{T}$.
- Assume $C \in \Gamma$.
- Then $C \in \Delta$, since $\Gamma \subset \Delta$.
- So by hypothesis, $v(C) = \mathbb{T}$, since everything in Δ is $\mathbb{T}$.
- So v is such that everything in Γ is $\mathbb{T}$.
- And since $\Gamma \vDash A$, that implies $v(A) = \mathbb{T}$, as required.

- This idea, that adding premises doesn't destroy validity, only works for logical arguments.
- It isn't true for good arguments in general.

Tweety is a bird.

∴ Tweety flies.

That's a perfectly good, though not logically valid, argument.

Tweety is a bird.

Tweety is black and white, lives in Antarctica, and lays large eggs.

∴ Tweety flies.

That's not a very good argument!

- If $\Gamma \vDash A$ and $\Delta \cup A \vDash B$, then $\Gamma \cup \Delta \vDash B$.

If some premises entail A, and some other premises plus A entail B, then the two sets of premises between them entail B.

This is crucial for being able to chain together lines of reasoning.

- Assume that for all $C \in \Gamma \cup \Delta$, $v(C) = \mathbb{T}$.
- We need to prove $v(B) = \mathbb{T}$.
- Since everything in Γ is $\mathbb{T}$ according to v, and $\Gamma \vDash A$, it follows that $v(A) = \mathbb{T}$.
- Since everything in Δ is $\mathbb{T}$ according to v, and A is $\mathbb{T}$ according to v, and $\Delta \cup A \vDash B$, it follows that $v(B) = \mathbb{T}$, as required.

This is why we define $\rightarrow$ the way we do.

- $\Gamma \vDash A \rightarrow B$ if and only if $\Gamma \cup A \vDash B$.

Note that there are two claims here—one for each direction. We need to prove each.

- Assume $\Gamma \vDash A \rightarrow B$, and prove $\Gamma \cup A \vDash B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and $v(A) = \mathbb{T}$, and aim to prove $v(B) = \mathbb{T}$.
- Since $\Gamma \vDash A \rightarrow B$ and $v(C) = \mathbb{T}$ for all $C \in \Gamma$, it follows that $v(A \rightarrow B) = \mathbb{T}$.
- Since $v(A \rightarrow B) = \mathbb{T}$ and $v(A) = \mathbb{T}$, it must be that $v(B) = \mathbb{T}$, since that's the only line on the truth table where $A \rightarrow B$ and A are both $\mathbb{T}$.

- Assume that $\Gamma \cup A \vDash B$, and prove $\Gamma \vDash A \rightarrow B$.
- So assume $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and prove $v(A \rightarrow B) = \mathbb{T}$.
- Either $v(A) = \mathbb{T}$ or $v(A) = \mathbb{F}$. Take each case in turn.
- If $v(A) = \mathbb{T}$, then since $v(C) = \mathbb{T}$ for all $C \in \Gamma$, and $\Gamma \cup A \vDash B$, it follows that $v(B) = \mathbb{T}$, so $v(A \rightarrow B) = \mathbb{T}$.
- If $v(A) = \mathbb{F}$, it follows directly that $v(A \rightarrow B) = \mathbb{T}$.
- Either way, $v(A \rightarrow B) = \mathbb{T}$ as required.

- This is a striking result.
- It shows that proving $A \rightarrow B$ is just the same as proving B, assuming you're allowed to add A as an extra assumption.
- And that's a good thing, intuitively. That is how we prove conditionals.
- But it only works if you have the (very odd-looking) truth table that we're using for $\rightarrow$.
- This is the main reason for thinking, despite its odd appearance, that this truth table is the right one for $\rightarrow$.

We will start working on a different way to analyze arguments: truth trees.