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Ellery Eells & William L. Harper

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RATIFIABILITY, GAME THEORY, AND THE PRINCIPLE OF INDEPENDENCE OF IRRELEVANT ALTERNATIVES

Ellery Eells and William L. Harper

I. Introduction

A recent point of agreement among many philosophers concerned with the foundations of rational decision theory is the idea that a rationally chosen option in a decision problem should be, in some sense of the recently coined word, 'ratifiable'. Richard Jeffrey [13] introduced the term to stand for the idea that, if a decision maker has chosen (at least tentatively) an option, and if this has been a *rational* choice, then the very information that that option has been chosen should not itself constitute evidence to the decision maker that some *other* alternative option is in fact *better*. Intuitively, if an act is *unratifiable*, then as soon as you commit yourself to it, you give yourself evidence that the act is irrational. A little more precisely, an option A is *ratifiable* if and only if the expected utility of A is not less than the expected utility of any other option, when expected utility is calculated relative to (subjective) probabilities *conditional on the proposition that it is A that has been chosen.* Of course, for different expected utility formulas, there will be different kinds of ratifiability.

Jeffrey's thought at the time was to combine his new idea of ratifiability with the kind of expected utility that he advocated in the first edition of his *Logic of Decision* [12]. In the first edition of his *The Logic of Decision*, the kind of expected utility advocated was supposed to provide a measure of comparative choiceworthiness of options *without* the complication of ratificationism. The introduction of ratificationism in 1981 was an attempt to respond to a difficulty that has shaped the recent development of the philosophy of rational decision.

The kind of expected utility Jeffrey originally advocated is sometimes called *conditional* expected utility, since it is calculated in terms of probabilities *conditional* on the act being evaluated. It has also come to be known as 'evidential' expected utility, since conditional subjective probabilities measure the *evidential,* as conceptually distinct from the (believed) *causal,* bearing of the conditioning proposition on the conditioned proposition. Jeffrey's hope was that the combination of ratifiability and evidential expected utility would be a decision theory that is essentially evidential in nature and yet also immune to the charges of 'causal' decision theorists that evidential decision theory gives wrong answers in Newcomb like decision problems. In Newcomb problems, the evidential bearings of acts on outcomes are different from their (believed) causal bearings. By the second edition of *The Logic of Decision* [14], however, Jeffrey had become convinced that the ratifiability idea could not save the evidential paradigm.

We will not discuss the debates between evidential and causal decision theorists, nor will we delve into the details of how Newcomb problems should be evaluated, of how ratifiability was supposed to work in the 'rescue' of evidential decision theory, or of just how Jeffrey became convinced that his idea of ratifiability failed in the attempted rescue.[1] Ratifiability is of interest independently of the merits of its combination with evidential expected utility. It is now widely accepted, and intuitively plausible, that ratifiability is an important component of the idea of choiceworthiness (or rationality, or rational decision), no matter whether choiceworthiness is explicated on the causal paradigm or the evidential paradigm.

The ratifiability concepts we will be discussing in what follows will be various ways in which the ratifiability idea can be combined with *causal* expectation; they will be versions of *causal ratificationism.* These principles will be given precise formulations below. Henceforth, we shall refer to '*causal* ratificationism' simply as 'ratificationism'.

Before formally stating these ratifiability principles, we remind you of a quite different rationality concept. Since the 1950's at least, an idea that has played an important role in the development of formal utility and decision theory has been what has come to be called 'the principle of independence of irrelevant alternatives'. We do not have in mind here Arrow's [1] condition of the same name, which is a condition on the relationship between individual and social preferences. What we have in mind has been succinctly formulated by A. K. Sen [26] as two rationality conditions on *individual choice functions.*[2] If X is a nonempty set of options, then an *individual choice function on* X is any function C, defined on the nonempty subsets of X, such that for any such subset S of X, $C(S)$ is a nonempty subset of S. For a nonempty set S, $C(S)$ is called the *choice set* of S. Intuitively, $C(S)$ is the set of best options in S.

Sen defines:

Property alpha : If $a \in S_1$ and $S_1 \subseteq S_2$ and $a \in C(S_2)$, then $a \in C(S_1)$;

and

[1] See, for example, Gibbard and Harper [8], Skyrms [27], Lewis [19], Jeffrey [13, 14], Eells [3]. Specifically, see Jeffrey [14] for his reasons for abandoning his evidential ratificationist approach. And see Eells and Sober [7] for a defence of evidential decision theory that specifically addresses what Jeffrey [14] takes to be a problem for the ratificationist evidential theory. Also, Eells [5], Harper [9] and Seidenfeld [25].

[2] These ideas have also been discussed by Nash [21], Chernoff [2], Radner and Marschak [24], Luce and Raiffa [15]. We choose Sen's formulation for the convenient labelling, for his clear demarcation of the two kinds of principles of independence of irrelevant alternatives, and for the purposes of showing just which ratification principles conflict with just which parts of the 'principles of independence of irrelevant alternatives'.

Property beta : If $a \in C(S_1)$ and $b \in C(S_1)$ and $S_1 \subseteq S_2$, then $a \in C(S_2)$ if and only if $b \in C(S_2)$.

Alpha says that if a is among the best in a set, then a remains among the best when alternatives other than a are removed from the set. In other words, if a is choiceworthy in some set of available alternatives, then its choiceworthiness would not be affected by (its choiceworthiness is independent of) the possible removal of alternatives other than a. Beta says that if two options a and b are among the best in a set, then the addition of new alternatives cannot leave one of them, and not the other, among the best. In other words, intuitively, if a and b are among the alternatives tied for best, then the tie between a and b cannot be broken by (is independent of whether or not there is) any addition of new alternatives.

Alpha and beta can be viewed as instances of a more general idea: namely, that one's preference between any two available alternatives should not be affected by whether or not *other* alternatives are added to, or deleted from, the set of available alternatives. So it is natural to say that alpha and beta are two versions or instances of this *principle of independence of irrelevant alternatives.* In any case, alpha is quite widely accepted, though beta has occasioned more controversy.[3]

Our main result in this paper is that ratifiability conflicts with independence of irrelevant alternatives. We have described *two* independence of irrelevant alternatives principles. We will describe *four* principles of ratifiability. Two of them will be inconsistent with *both* alpha *and* beta; the other two will be inconsistent just with beta. We will describe decision problems in which the various versions of ratificationism imply choices that conflict with the requirements of alpha or beta. We will also maintain that in these test cases, *the prescriptions of ratificationism are correct,* so that *alpha and beta are in error.* These points will be covered in sections II-V, corresponding to the four kinds of ratifiability we describe. Finally, in Section VI, we turn to game theory and argue that our examples show that the Nash equilibrium concept is in conflict with alpha and beta.

II. Basic Ratifiability

Instead of calculating expected utility in terms of probabilities conditional on the act, causal decision theory uses (in some versions) probabilities of 'causal' conditionals with act antecedents.[4] Where a is an option and the possible states are $\{B_i\}_i$, the *causal expected utility*, U, of a is given by:

$$U(a) = \mathrm{SUM}_i \; Pr\,(a \,\Box\!\!\rightarrow B_i)\; u\,(a, B_i).$$

The value $u\,(a, B_i)$ is the utility of the outcome of a in state B_i. Of course, the states have to form an 'appropriate partition'; the B_i's have to be 'specific'

[3] See, for example, Luce and Raiffa [15] and Sen [26] for discussion.

[4] For example, the version of Gibbard and Harper [8] and Lewis [19]. By 'causal conditional', we mean 'nonbacktracking counterfactual conditional'; see the references just cited.

enough that they, together with an act, determine something that can count as an 'outcome'.[5] To define basic ratifiability, we need also the idea of causal expected utility *conditional* on an option. Where the possible states are as before, and a and b are options, the causal expectation of *a conditional on b*, $U_b(a)$, is the same as $U(a)$, except that the probabilities are taken conditional on b:

$$U_b(a) = \mathrm{SUM}_i \ Pr(a \,\Box\!\!\rightarrow B_i \,/\, b)\, u(a, B_i).$$

Let S be a set of options. Then an option a is *basically ratifiable relative to* S if and only if a is in S and $U_a(a) \geq U_a(b)$, for all b in S. Intuitively, a is ratifiable if and only if, if the decision maker were to learn that a will be taken, then a's causal expectation would be at least as great as that of any of the other options. We define the choice rule, or function, BR (for basic ratifiability) as follows:

$$BR(S) = \{a : a \text{ is basically ratifiable relative to } S\}.$$

To illustrate this rule, we now describe a decision problem that shows that ratifiability makes a difference in causal decision theory. It is an example in which 'straight' causal decision theory (maximise unconditional causal expectation) gives an answer that conflicts with BR. In the problem, BR gives the right answer, which shows that causal decision theory needs to be supplemented with some kind of ratifiability principle.

There are three available options, A, B, and C. As in standard Newcomb problems, a predictor makes a prediction about what the decision maker will do, and the decision maker believes that the predictor is very accurate. In this case, the predictor simply makes a prediction about whether or not C will be chosen. The two relevant states are: *PredC* (C predicted) and *Pred~C* (~C predicted). Here is the utility matrix:

	PredC	*Pred~C*
A	5	1
B	2	3
C	4	2

We now make three assumptions about the decision maker's subjective probabilities. *First*, we suppose that extreme confidence in the predictor's accuracy is reflected in the decision maker's subjective probabilities as follows:

$$Pr(PredC/C) \approx Pr(Pred{\sim}C/A) \approx Pr(Pred{\sim}C/B) \approx 1.$$

That is, conditional on any of the available acts, the agent is nearly certain that the predictor is correct. *Second*, we suppose that the agent believes that, no matter which option is chosen, the predictor's prediction is causally independent of the options. And we suppose that this belief is reflected in

[5] For discussion, see Gibbard and Harper [8], Lewis [19, 20], Levi [16, 17, 18], Eells [4], and Sobel [29, 30, 31].

subjective probabilities as follows: $Pr\ (a \mathbin{\Box\!\!\rightarrow} PredC /\ b\ ) = Pr\ (PredC\ /b)$, for each of the available acts $a, b \in \{A, B, C\}$; and the same, of course, for $Pred^{\sim}C$. And *third*, we suppose that the agent's initial subjective probability for the state *PredC* is greater than 1/3: $Pr\ (PredC) > 1/3$.

Let $x = Pr\ (PredC)$. Then we have:

$$\begin{aligned} U(A) &= 5x + 1(1-x) = 4x + 1; \\ U(B) &= 2x + 3(1-x) = -x + 3; \\ U(C) &= 4x + 2(1-x) = 2x + 2. \end{aligned}$$

Then,

$$U(C) > U(A) \text{ if and only if } x < 1/2.$$

Also,

$$U(C) > U(B) \text{ if and only if } x > 1/3.$$

So, given that $x > 1/3$, straight causal decision theory will prescribe A or C, depending on whether $x \geq 1/2$ or $x \leq 1/2$.

However, if the decision maker decides on C, and after this reevaluates the acts using causal expectation, then act A would beat C: $U_C\ (A) \approx 5 > 4 \approx U_C\ (C)$. Also, if the decision maker decided on A, then a reevaluation with causal expectation would show that B beats A: $U_A\ (B) \approx 3 > 1\ U_A\ (A)$. Only a decision in favour of B would look best after it has been chosen.

This shows, of course, that B is the unique ratifiable act, which can be checked by examining the array below.

$U_A\ (A) \approx 1$	$U_A\ (B) \approx 3$	$U_A\ (C) \approx 2$
$U_B\ (A) \approx 1$	$U_B\ (B) \approx 3$	$U_B\ (C) \approx 2$
$U_C\ (A) \approx 5$	$U_C\ (B) \approx 2$	$U_C\ (C) \approx 4$

The only entry on the main diagonal that is greater than or equal to all entries in its row is $U_B\ (B) \approx 3$. Hence B is the unique basically ratifiable act: $BR\ (\ \{A, B, C,\}\ ) = \{B\}$. Thus, straight causal decision theory and basic ratifiability conflict. In our opinion, basic ratifiability gives the right answer.

We now show that basic ratifiability conflicts with property beta. Consider the decision problem which is exactly the same as the one just described except that the only available acts now are B and C. Here is the utility matrix:[6]

	PredC	*Pred~C*
B	2	3
C	4	2

[6] We assume throughout that deleting rows corresponds to decision problems with subsets of acts, and that conditional causal expectations remain the same when acts are added or deleted; and we will make the corresponding assumptions when we turn to game theory. After giving our results, we will briefly discuss the suggestion that these assumptions may fail.

In this decision problem, both acts are ratifiable, as can easily be seen from the following array.

$$\begin{array}{ll} U_B(B) \approx 3 & U_B(C) \approx 2 \\ U_C(B) \approx 2 & U_C(C) \approx 4 \end{array}$$

Each entry on the main diagonal is greater than the other entry in its row; so B and C are both basically ratifiable.

So, $BR(\{B, C\}) = \{B, C\}$. And we have seen that $BR(\{A, B, C, \}) = \{B\}$. B and C are 'tied' for best in $\{B, C\}$. However, when A is added as an option, only B remains as a best option.

Conclusion: Basic ratifiability conflicts with beta.

On the other hand, *basic ratifiability cannot violate alpha.* If a is a member of a set S_2, and if $U_a(a) \geq U_a(b)$ for all $b \in S_2$ (so that a is ratifiable in S_2), then $U_a(a)$ must be $\geq U_a(b)$ for all b in any *subset* $S_1 \subseteq S_2$ (so a will be basically ratifiable in any subset of S_2 to which it belongs).

III. Maximisation Ratifiability

The principle of maximisation ratifiability is simply the idea that the choiceworthy options are just those *among the basically ratifiable options* that have *maximal (unconditional) causal expectation.* This can be formulated as a choice rule MR (for maximisation ratifiability) as follows:

$$MR(S) = \{ a : a \in BR(S), \text{ and } U(a) \geq U(b) \text{ for all } b \in BR(S) \}.$$

The rationale for the move from *basic* ratificationism to *maximisation* ratificationism is two-fold. First, there are decision problems in which there is *more than one* basically ratifiable option (such as the decision problem between B and C above), and it would be nice if a theory of choice could provide further guidance in such cases. Second, if ratifiability is at least a *necessary* condition of choiceworthiness, then this, together with the *comparisons* that expected utility calculations are able to provide, suggests that we weigh the basically ratifiable options against each other using the best expected utility conception available.[7] We now show that the natural idea of maximisation ratifiability violates both alpha and beta.

To demonstrate the conflict with alpha, we use the same decision problems described above. We assume again that $Pr(PredC) > 1/3$. First, note that $MR(\{A, B, C,\}) = \{B\}$, since B is the *only* (and hence the *maximal*) *basically* ratifiable act in the decision problem between A, B, and C. As to $MR(\{B, C\})$, recall that in the decision problem between B and C, both B and C were basically ratifiable. And the calculations given above, used to demonstrate the conflict between straight causal decision theory and basic ratifiability, showed that $U(C) > U(B)$ if $Pr(PredC) > 1/3$. So, given our assumption that $Pr(PredC) > 1/3$, C is the causally maximal option between

[7] For a defence of the idea that one should maximise among ratifiable options, see Harper [10]. For other discussion, see Rabinowicz [23], Weirich [34], and Sobel [32].

the two basically ratifiable options B and C, in the problem between B and C. So $MR(\{B, C\}) = \{C\}$. The combination of $MR(\{B, C\}) = \{C\}$ with $MR(\{A, B, C\}) = \{B\}$ violates alpha: alpha implies that if $B \in MR(\{A, B, C\})$, then $B \in MR(\{B, C\})$.

Conclusion: Maximisation ratifiability conflicts with alpha.

To show the conflict with beta, we need to consider a new decision problem. The acts are A', B' and C'. The states are *PredC'* (the predictor predicted C') and *Pred~C'* (the predictor predicted $\sim C'$). And the utility matrix is:

	PredC'	*Pred~C'*
A'	5	1
B'	4	4
C'	4	4

Again we assume, in the way these two assumptions were explained above, that the decision maker has extreme confidence in the predictor's accuracy, and that the decision maker believes that the prediction is causally independent of the act chosen. Then the conditional causal expectations are as follows:

$$\begin{array}{lll} U_{A}'(A') \approx 1 & U_{A}'(B') \approx 4 & U_{A}'(C') \approx 4 \\ U_{B}'(A') \approx 1 & U_{B}'(B') \approx 4 & U_{B}'(C') \approx 4 \\ U_{C}'(A') \approx 5 & U_{C}'(B') \approx 4 & U_{C}'(C') \approx 4 \end{array}$$

Since $U_B'(B')$ is the only entry on the main diagonal that is not less than any other entries in its row, B' is the *only basically* ratifiable act in the decision problem between A', B' and C'. It follows that $MR(\{A', B', C'\}) = \{B'\}$.

Consider now the subarray for the problem between B' and C':

$$\begin{array}{ll} U_{B}'(B') \approx 4 & U_{B}'(C') \approx 4 \\ U_{C}'(B') \approx 4 & U_{C}'(C') \approx 4 \end{array}$$

Each element of the main diagonal is at least as great as the other elements in its row. So each of B' and C' is *basically* ratifiable in the decision problem between B' and C'. And these two acts each has *maximal* (since equal) causal expectation among the basically ratifiable acts. So $MR(\{B', C'\}) = \{B', C'\}$.

According to maximisation ratifiability, each of B' and C' is best in the problem between B' and C', but when you add A' to the problem, only B' remains best, in violation of beta.

Conclusion: Maximisation ratifiability conflicts with beta.

IV. Basic Ratifiability with Mixed Strategies

There are decision problems in which none of the "pure" acts is ratifiable, and others in which the ratifiable pure acts are very unfortunate. However, if we deploy *mixed* strategies—choose *probability mixtures* of the pure options—these difficulties can disappear (see also Harper [10]). So it is worth explaining these ideas here and considering ratifiability when mixed acts are included among the options. We will show that when mixed acts are included, the two versions of ratifiability described above still conflict with alpha and beta, in the same ways as they did without mixed strategies.

Consider this version of the 'Death in Damascus' problem (Gibbard and Harper [8]). The options are G (go to Aleppo) and $\sim G$ (stay in Damascus). The states are *PredG* (Death has predicted G and will seek you in Aleppo) and *Pred*$\sim G$ (Death has predicted $\sim G$ and will seek you in Damascus). If you are where Death seeks you, you die (utility = -100); if not, you get a reprieve (utility = 0). The utility matrix for the problem is this:

	PredG	*Pred*$\sim G$
G	-100	0
$\sim G$	0	-100

We make the same assumptions about the predictor's believed accuracy and the believed causal independence of predictions from acts as before. The following array gives the conditional causal expectations for this problem.

$$U_G(G) \approx -100 \qquad U_G(\sim G) \approx 0$$
$$U_{\sim G}(G) \approx 0 \qquad U_{\sim G}(\sim G) \approx -100$$

None of the entries on the main diagonal is greater than or equal to all the others in its row. So neither act is ratifiable.

Now consider mixed strategies. Let $< xG; (1-x)\sim G >$ be the mixed act of doing G with probability x and $\sim G$ with probability $1-x$. One way to extend and adapt the assumptions about the believed causal independence of the prediction from acts and the predictor's believed accuracy is this:

For all probabilities p and x,

$$Pr(< pG; (1-p)\sim G > \Box\!\!\rightarrow PredG \,/ < xG; (1-x)\sim G >)$$
$$= Pr(PredG \,/ < xG; (1-x)\sim G >)$$
$$= x,$$

and,

$$Pr(< pG; (1-p)\sim G > \Box\!\!\rightarrow Pred\sim G \,/ < xG; (1-x)\sim G >)$$
$$= Pr(Pred\sim G \,/ < xG; (1-x)\sim G >)$$
$$= 1-x.$$

On this understanding of the problem, it is assumed that Death can predict your mixed strategy with certainty, but not your eventual pure act (unless you opt for a pure act), and it is assumed that Death will himself choose the mixed strategy corresponding to yours, mixing between seeking you in Aleppo and seeking you in Damascus as you mix between G and $\sim G$.

Alternatively, Death could seek you where he thinks it is more probable you will be, mixing if you choose $< .5G; .5\sim G >$. Of course, which kind of strategy it is most natural to assume Death chooses depends on Death's utilities. Either way, in this problem, it is easy to see that $< .5G; .5{\sim}G >$ is the ***unique basically ratifiable mixture*** as well as the ***unique maximisation ratifiable mixture.***

The main calculation is this:

$$\begin{aligned}
&U_{<xG;\,(1-x)\,{\sim}G>}(<yG;\,(1-y)\,{\sim}G>)\\
&= x\,[y\,(-100) + (1-y)\,(0)] + (1-x)\,[y(0) + (1-y)\,(-100)\,]\\
&= -200xy + 100x + 100y - 100.
\end{aligned}$$

From this it is easy to see that,

$$\begin{aligned}
&U_{<.5G;\,.5{\sim}G>}(<.5G;\,.5{\sim}G>) = -50\\
&= U_{<.5G;\,.5{\sim}G>}(<yG;\,(1-y)\,{\sim}G>), \text{ for all } y,
\end{aligned}$$

so that $< .5G; .5{\sim}G >$ is basically ratifiable; and when $x \neq .5$,

$$\begin{aligned}
&U_{<xG;\,(1-x)\,{\sim}G>}(<xG;\,(1-x)\,{\sim}G>)\\
&= -200x^2 + 200x - 100\\
&< -50\\
&= U_{<xG;\,(1-x)\,{\sim}G>}(<.5G;\,.5{\sim}G>),
\end{aligned}$$

so that $< .5G; .5{\sim}G >$ is the *only* basically ratifiable act, as well as the unique maximisation ratifiable act. So a mixed act can be ratifiable when none of the pure acts are.

There are decision problems in which there *are* ratifiable pure acts but in which they may seem irrational, or at least unfortunate; and the introduction of mixed acts can help here as well. Brian Skyrms [28] has described this problem. You are to choose one of three shells, and will receive what is under it. A very good predictor has predicted your choice. If he predicts you take shell 1 (A_1) he puts 10¢ under it and nothing under the others; if he predicts you pick shell 2 (A_2) he puts \$10 under it and \$100 under shell 3; if he predicts shell 3 (A_3) he puts \$20 under it and \$200 under shell 2. In the latter cases he puts nothing under shell 1. Here is the utility matrix:

	PredA_1	*Pred*A_2	*Pred*A_3
A_1	.10	0	0
A_2	0	10	200
A_3	0	100	20

If we assume that the decision maker assumes that the predictor is perfect, and if we assume believed causal independence of predictions from acts as before, then the conditional causal expectation table is the transpose the utility matrix. Clearly, A_1 is the unique basically ratifiable act, and the unique maximisation ratifiable act. But when the assignment of probabilities to states is not very extreme, the unconditional causal expectation of A_1 is terrible, compared to those of A_2 and A_3. In fact, unless Pr ($PredA_l$) is greater than (about) .9986, then at least one of A_2 and A_3 will have an unconditional causal expectation greater than that of A_1.

This decision problem is like the first one described above, in that it shows ratifiability to be in conflict with straight causal decision theory. In the earlier problem, it seemed clear that the ratifiable act was correct. For Skyrms' problem, it seems that the opposite is true; our first impression, at least, is that ratificationism is giving us an irrational act.

This difference in initial intuitions across the two problems *may* arise just because the *differences* between the *magnitudes* of the utilities of the possible outcomes are so extremely different in the two problems: the differences in these magnitudes are more extreme in Skyrms' problem. To test this diagnosis, consider the variation of our first problem defined by changing only the outcomes under *Pred~C* and the outcome of *B* in *PredC*, by dividing these by 1000:

	PredC	*Pred~C*
A	5	.001
B	.002	.003
C	4	.002

Nothing has changed as far as ratifiability is concerned: *B* is still the unique ratifiable act (for both *BR* and *MR*). But unless *Pr* (*Pred~C*) is very large (greater than .9996), straight causal decision theory will rank both *A* and *C* above *B*. And when the assignment of probabilities to states is not very extreme, the unconditional causal expectation of *B* is terrible, compared to those of *A* and *C*.

We are unsure about the force, against ratificationism, of Skyrms' problem. If the difference in our intuitions between Skyrms' problem and the one considered earlier *does* arise just because of a difference in *magnitude* in the comparison of utilities, then it is hard to see how ratificationism could be wrong in Skyrms' problem when it is clearly correct in the problem considered earlier. If our diagnosis is correct, then it seems that ratificationism gives the correct answer in one of these problems if and only if it does so in the other.

In any case, when mixed acts are introduced, ratifiability ceases to be in this extreme kind of conflict with unconditional causal expected utility. In Skyrms' example, our assumptions for mixed acts about the causal independence of predictions from acts and the predictor's accuracy come to this (but see below):

For all mixed acts, $< p_1\, A_1; p_2\, A_2; p_3\, A_3 >$ and

$< x_1\, A_1; x_2\, A_2; x_3\, A_3 >$ and i = 1, 2, 3,
$Pr\,(< p_1 A_1; p_2 A_2; p_3 A_3 > \Box\!\!\rightarrow PredA_i / < x_1 A_1; x_2 A_2; x_3 A_3 >)$

$$= Pr\,(PredA_i / < x_1A_1; x_2A_2; x_3A_3 >)$$
$$= x_i$$

Then ratifiability will admit, in addition to A_1, a 'good' mixture of A_2 *and* A_3, regardless of what our subjective probabilities of the states are. Without giving calculations, we simply note that, among all the mixed acts, the basically ratifiable acts will include, as well as A_1, the mixture $<(2/3)A_2; (1/3)A_3>$, which is the unique maximisation ratifiable act (there are other basically ratifiable mixtures). Also, assuming equal prior probabilities for the three states, the unconditional causal expectation of $<(2/3)A_2;(1/3)A_3>$ is 60, which compares to (about) .03 for the ratifiable A_1, and to 70 and 40 for the unratifiable pure acts A_2 and A_3. And the mixture $<(2/3)A_2;(1/3)A_3>$ looks like the intuitively correct act.

The assumption above about 'predictive accuracy' is actually the assumption that the predictor mixes as the agent does. The idea of predictive accuracy does not force this. As in Death in Damascus, the conditional probabilities depend on how we picture the predictor's utilities. Various things can happen depending on how we assign utilities to the predictor, but (still assuming the predictor is perfect for the pure acts) no way of doing this can make either of the unratifiable pure strategies reasonable (Harper [10]).

For the other problem just described, the assumptions about causal independence and predictive accuracy are this:

For all mixed acts $<pA; qB; rC>$ and $<xA; yB; zC>$,

$$Pr\,(<pA; qB; rC> \,\Box\!\!\rightarrow PredC / <xA; yB; zC>)$$
$$= Pr\,(PredC / <xA; yB; zC>)$$
$$= z,$$

and

$$Pr\,(<pA; qB; rC> \,\Box\!\!\rightarrow Pred{\sim}C / <xA; yB; zC>)$$
$$= Pr\,(Pred{\sim}C / <xA; yB; zC>)$$
$$= 1 - z \quad (= x+y).$$

And ratifiability will admit, in addition to B, the mixture $<(1000/1001)A; (1/1001)C>$, which is the unique maximisation ratifiable act (there are other basically ratifiable mixtures). Assuming equal prior probabilities for the states, the unconditional causal expectation of this mixture is (about) 2.500001, which compares to .0025 for the ratifiable pure act B, and 2.5005 and 2.001 for the unratifiable pure acts A and C.

Thus, when mixed strategies are available, various problems for ratifiability disappear. It has been suggested, however, that mixed strategies may not always be available, or that it could be built into a decision problem that mixing is penalised. In such cases, there may be no ratifiable acts, or ratifiability may prescribe a pure strategy with inferior unconditional causal expectation. We make no commitments in this paper about how such cases should be handled.

We turn now to the question of the compatibility of alpha and beta with ratifiability with mixed strategies. Where S is a set of options, let MS be the probability mixture closure of S:

$$MS = \{< p_1 a_1; \ldots ; p_n a_n > : a_1, \ldots, a_n \in S,$$
$$0 \leq p_1, \ldots, p_n \leq 1, \text{and } p_1 + \ldots + p_n = 1\}.$$

Then basic and maximisation ratifiability for mixed strategies relative to a set S are simply the applications of basic and maximisation ratifiability to the set MS: $BR(MS)$ and $MR(MS)$.

We show first that *basic ratifiability with mixed strategies violates beta.* We again use the decision problem described above involving the pure strategies A, B, and C, and now make available all mixtures of A, B, and C. The causal independence and 'predictive accuracy' assumptions are the same as stated above for the 'extreme' version of this problem.

Consider first the decision problem between just B, C, and mixtures of B and C. In the appendix, we prove that

$$BR(M\{B, C\}) = \{B, C, < (2/3)\, B; (1/3)\, C >\}.$$

When mixtures of B and C are included, B and C remain basically ratifiable; and there is a mixture, $< (2/3)\ B;\ (1/3)\ C >$, that is ratifiable as well. Turning to the decision problem between mixtures of A, B, and C, we also prove in the appendix that

$$BR(M\{A, B, C\}) = \{B, < (1/2)\, A; (1/2)\, C >, < (2/3)\, B; (1/3)\, C >\}.$$

Of course, $M\ \{B, C,\} \subseteq M\ \{A, B, C\}$. So beta implies that any two options that are in $BR\ (M\ \{B, C\})$ are either both or neither in $BR\ (M\ \{A, B, C\})$. Since both B and C are in $BR\ (M\ \{B, C\})$, but of these only B is in $BR\ (M\ \{A, B, C\})$, BR again violates beta.

Conclusion: Basic ratifiability with mixed strategies conflicts with beta.

Like basic ratifiability with just pure strategies, however, BR still cannot violate alpha—and for the same reason.

V. Maximisation Ratifiability with Mixed Strategies

Turning now to maximisation ratifiability in the context of mixed strategies, we will show that, for the same decision problem just used above, there is a value of $Pr\ (PredC)$ such that

$$(1) \quad MR(M\{A, B, C\}) = \{ < (2/3) B; (1/3)\, C > \},$$

and

$$(2) \quad MR(M\{B, C\}) = \{C\}.$$

Since $M\{B,\ C\} \subseteq M\{A,\ B,\ C\}$, and $<(2/3)B;\ (1/3)C> \in MR(M\ \{A,\ B,\ C\})$, and $<\ (2/3)B;\ (1/3)C>\ \in\ M\{B,\ C\}$, but $<(2/3)B;\ (1/3)C> \notin MR\ (M\{B, C\})$, we again have a violation of alpha by MR.

We will show that there is a value of $Pr\ (PredC)$ such that

(3) $U(C) > U\ (2/3)\,B; (1/3)\,C>)$

(4) $U(<(2/3)\,B; (1/3)\,C>) > U(B)$

(5) $U(<(2/3)B; (1/3)C>) > U(<(1/2)A; (1/2)C>)$.

Then (3) and (4) imply that $U(C)$ is greater than each of $<(2/3)B; (1/3)C>$ and $U(B)$, giving us (2). And (4) and (5) give us (1).

Let $x = Pr\,(PredC)$. Then,

$$
\begin{aligned}
U(C) &= x4 + (1-x)2 \\
&= 2x + 2; \\
U(<(2/3)B; (1/3)\,C>) &= x\,[(1/3)\,4 + (2/3)\,2] \\
&\quad + (1-x)\,[(1/3)\,2 + (2/3)\,3] \\
&= 8/3; \\
U(B) &= x2 + (1-x)\,3 \\
&= 3 - x;
\end{aligned}
$$

and,

$$
\begin{aligned}
U(<(1/2)A;(1/2)C>) &= x\,[(1/2)\,5 + (1/2)\,4] \\
&\quad + (1-x)\,[(1/2)\,1 + (1/2)\,2] \\
&= 3x + 3/2.
\end{aligned}
$$

Then (3), (4), and (5) require that $x > 1/3$, $x > 1/3$, and $x < 7/18$. So any x such that $1/3 < x < 7/18$ gives us our result.

Conclusion: Maximisation ratifiability with mixed strategies conflicts with alpha.

Finally, we show that maximisation ratifiability with mixed strategies violates beta as well. We use the same problem used to show that maximisation ratifiability with just pure strategies conflicts with beta, except we now make all mixed acts available. The assumption about predictive accuracy and causal independence of predictions from acts is parallel to the ones in the previous examples. Recall that $MR\,(\{A', B', C', \}) = \{B'\}$ and $MR\,(\{B', C'\}) = \{B', C'\}$. The reader may verify that

$$MR\,(M\{A', B', C'\}) = \{B'\}$$

and

$$MR\,(M\,\{B', C'\}) = \{<xB'; (1-x)\,C'> : 0 \leq x \leq 1\}.$$

Conclusion: Maximisation ratifiability with mixed strategies conflicts with beta.

VI Game Theory

Perhaps it should have come as no surprise that alpha and beta would be inappropriate for decisions which involve strategic considerations such as those introduced by our predictor. We shall show that Nash's famous solution concept [22] conflicts with alpha. We shall also show that (what we shall call) the weak Nash concept conflicts with beta.

Two strategies are in equilibrium if each is in optimal reply to the other. A two person game has a solution in Nash's sense just in case it has a *unique* equilibrium pair, or it has more than one such pair but they are *all 'equivalent' and comprised of 'interchangeable' equilibrium strategies* (so that it would not matter to either player which pair was selected and some such pair would be selected no matter which constituents of equilibrium strategies are played). When a game has a solution in this sense, it makes sense to say that *game theory* recommends that rational players choose equilibrium strategies.

There are many games where the Nash solution concept fails to apply, because there are multiple non-equivalent equilibria. For such games it is not so clear what game theory recommends. There is fairly wide agreement, however, that solutions ought to be restricted to equilibria. We shall call this restriction to equilibria the *weak Nash* solution concept.

Consider now this game, which is the result of combining our first example with an assignment of appropriate utilities to the predictor (1 for being right, 0 for being wrong):

	PredC	Pred~*C*
A	(5,0)	(1,1)
B	(2,0)	(3,1)
C	(4,1)	(2,0)

The set, call it S_2, of the row chooser's strategy options is $\{A, B, C\}$. This game has a unique pure strategy equilibrium pair, namely, (*B*, *Pred*~*C*). So it seems reasonalbe to say that the Nash solution concept applies, and constrains row chooser's choice from the set of her pure strategies. Thus, where N is the choice rule corresponding to the Nash solution concept, we have $N(S_2) = \{B\}$. This is because B is the only strategy in S_2 that is a constituent in the unique pure strategy equilibrium pair. This implies, of course, that, $B \in N(S_2)$.

Now let us consider the game resulting from deleting A from row chooser's options, so that her set of options is $S_1 = \{B, C\}$:

	PredC	*Pred*~*C*
B	(2,0)	(3,1)
C	(4,1)	(2,0)

This game has *two* pure strategy equilibrium pairs: (*B*, *Pred*~*C*), and (*C*, *PredC*). These are *not equivalent*, since *B* is not an optimal response to *PredC*, nor is *C* an optimal response to *Pred*~*C*. There is no Nash solution to this game, so that $N(S_1)$ is empty, and thus $B \notin N(S_1)$. Here we have a violation of condition alpha: $S_1 \subseteq S_2$, $B \in S_1$, $B \in N(S_2)$, but $B \notin N(S_1)$.

The same pair of games shows that the set of weak Nash options, construed as choice sets (*WN*), violates beta. Both *B* and *C* are weak Nash options for $S_1 = \{B, C\}$, since each is a constituent of an equilibrium strategy. But *C* is not a weak Nash option for S_2, since it is not a constituent of an equilibrium strategy in S_2. Thus $B \in WN(S_1)$, $C \in WN(S_1)$, $S_1 = \{B, C\} \subseteq S_2 = \{A, B, C\}$, $B \in WN(S_2)$, but $C \notin WN(S_2)$, so that beta fails.

When strategic factors are relevant to a decision, there may be some difficulty in saying what should count as the decision problems corresponding to different subsets of the available acts. We have used the game matrixes resulting from deleting rows to represent the decision problems corresponding to different subsets of row chooser's acts. In our earlier predictor examples as well, we represented these problems by deleting rows, while otherwise keeping the problem specification the same. These assumptions lead to examples where conditions alpha and beta are inappropriate. Perhaps, however, there are other ways of settling what should count as the decision problems corresponding to various subsets of the available acts. We doubt, however, that there is any reasonable way of settling this that will make alpha and beta apply to constrain appropriate choices in these problems. Indeed, the very difficulties in saying what problems correspond to various subsets of alternatives in situations where strategic reasoning is relevant makes it look like strategic reasoning is just *beyond the scope* of the kinds of reasoning about choice that conditions alpha and beta can successfully illuminate.

Ratifiability, on the other hand, seems quite a natural constraint on game theoretic reasoning. For example, in Harper [10], best-reply reasoning in zero-sum games was used to support the idea that admissible choices ought to be ratifiable, and in Harper [11], ratifiability was used to explicate the classic von Neumann-Morgenstern [33, p. 148] indirect argument for restricting solutions to equilibria.

The examples and arguments we have presented provide new reasons for reassessing conditions alpha and beta.[8]

University of Wisconsin-Madison
University of Western Ontario

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Appendix

Recall the utility matrix:

	PredC	*Pred~C*
A	5	1
B	2	3
C	4	2

and the assumption:

For all mixed acts $<pA; qB; rC>$ and $<xA; yB; zC>$,

$$\begin{aligned} & Pr(<pA; qB; rC> \Box\!\!\rightarrow PredC / <xA; yB; zC>) \\ &= Pr(PredC / <xA; yB; zC>) \\ &= z, \end{aligned}$$

and,

$$\begin{aligned} & Pr(<pA; qB; rC> \Box\!\!\rightarrow Pred\!\sim\!C / <xA; yB; zC>) \\ &= Pr(Pred\!\sim\!C / <xA; yB; zC>) \\ &= 1-z \quad (=x+y). \end{aligned}$$

We will establish:

(I) $BR(M\{B, C\}) = \{B, C, <(2/3)B; (1/3)C>\}$,

and

(II) $BR(M\{A, B, C\}) = \{B, <(1/2)A; (1/2)C>, <(2/3)B; (1/3)C>\}$.

Proof of (I): Where x and y are probability values, we must find all values of x such that for all y,

$$\begin{aligned} & U_{<xB;(1-x)C>}(<xB; (1-x)C>) \\ & \geq U_{<xB;(1-x)C>}(<yB; (1-y)C>). \end{aligned}$$

We will show that these values of x are exactly 1, 0, and 2/3. This will establish (I).

By the assumption, it follows that for any x,

$$\begin{aligned} U_{<xB;(1-x)C>}(<xB; (1-x)C>) &= (1-x)[x2 + (1-x)4] \\ &\quad + x[x3 + (1-x)2] \\ &= 3x^2 - 4x + 4; \end{aligned}$$

and for any x and y,

$$\begin{aligned} U_{<xB;(1-x)C>}(<yB; (1-y)C>) &= (1-x)[y2 + (1-y)4] \\ &\quad + x(y3 + (1-y)2] \\ &= 3xy - 2x - 2y + 4. \end{aligned}$$

So the problem is to find those x's such that for all y's,

$$3x^2-4x+4 \geq 3xy-2x-2y+4.$$

This simplifies to finding the x's such that for all y's,

(*) $(x-y)(3x-2) \geq 0.$

Clearly,

(1) $x = 0$ is such a value of x, for in this case, (*) simplifies to $2y \geq 0$;

(2) $x = 1$ is such a value of x, for in this case, (*) simplifies to $1-y \geq 0$;

and,

(3) $x = 2/3$ is such a value of x, for in this case (*) simplifies to $0 \geq 0$.

It remains to be shown that

(4) there are no such values of x such that $0 < x < 2/3$;

and

(5) there are no such values of x such that $2/3 < x < 1$.

If x is strictly between 0 and 2/3, then let y be $x/2$. In this case, (*), simplifies to $x((3/2)x-1) \geq 0$, which is false for any x strictly between 0 and 2/3. This establishes (4).

If x is strictly between 2/3 and 1, then let y be $(x+1)/2$. In this case, (*) simplifies to $(3x-2)(x-1) \geq 0$, which is false for any x strictly between 2/3 and 1. This establishes (5).

So (I) is established.

Proof of (II): Where x, y, and z are probabilities that sum to 1 and p, q and r are probabilities that sum to 1, we must find all those $<x, y, z>$ triples such that for all $<p, q, r>$ triples,

$$U_{<xA;\, yB;\, zC>}(<xA;\, yB;\, zC>)$$
$$\geq U_{<xA;\, yB;\, zC>}(<pA;\, qB;\, rC>).$$

We will show that these values of $< x, y, z >$ are exactly $< 0, 1, 0 >$, $< 1/2, 0, 1/2 >$, and $<0, 2/3, 1/3 >$. This will establish (II).

It follows from our assumption that for any triple $< x, y, z >$,

$$U_{<xA;\, yB;\, zC>}(<xA;\, yB;\, zC>) = z\,(5x + 2y + 4z) + (x + y)(x + 3y + 2z).$$

Setting $z = 1-x-y$, this is equivalent to:

$$U_{<xA;\, yB;\, zC>}(<xA;\, yB;\, zC>) = -2x^2 + 3y^2 - x - 4y + xy + 4.$$

Also by our assumption, for any triples $<x, y, z>$ and $<p, q, r>$,

$$U_{<xA;\, yB;\, zC>}(<pA;\, qB;\, rC>) = z\,(5p + 2q + 4r) + (x{+}y)\,(p + 3q + 2r).$$

Setting $z = 1 - x - y$ again, and $r = 1 - p - q$, this is equivalent to:

$$U_{<xA;\, yB;\, zC>}(<pA;\, qB;\, rC>) = -2xp + 3yq - 2x + p - 2y - 2q + 3xq - 2yp + 4.$$

With some algebraic manipulation, our problem simplifies, somewhat, to the problem of finding those $<x, y>$'s such that for all $<p, q>$'s,

(**) $$(y{-}q)\,(3\,(x + y) - 2) - (x{-}p)\,(2(x{+}y) - 1) \geq 0.$$

Clearly,

(a) $<0, 1>$ is such an $<x, y>$ pair, for in this case (**) simplifies to $1-q+p \geq 0$;

(b) $<1/2, 0>$ is such an $<x, y>$ pair, for in this case (**) simplifies to $y/2 \geq 0$;

and,

(c) $<0, 2/3>$ is such an $<x, y>$ pair, for in this case (**) simplifies to $x/3 \geq 0$.

It only remains to be shown that the $<x, y>$ pairs mentioned in (a)—(c) are the *only* pairs that, together with any $<p, q>$ pair, satisfy (**).

There are seven possibilities for the sum $x + y$. It can be either $(P1) = 0$, $(P2) = 1$, $(P3) = 1/2$, $(P4) = 2/3$, $(P5)$ in $(0, 1/2)$, $(P6)$ in $(1/2, 2/3)$, or $(P7)$ in $(2/3, 1)$. Corresponding to these seven possibilities, we prove seven claims:

($C1$) No such $<x, y>$ satisfies (**);
($C2$) The only such $<x, y>$ that satisfies (**) is $<0, 1>$;
($C3$) The only such $<x, y>$ that satisfies (**) is $<1/2, 0>$;
($C4$) The only such $<x, y>$ that satisfies (**) is $<0, 2/3>$;
($C5$) No such $<x, y>$ satisfies (**);
($C6$) No such $<x, y>$ satisfies (**);
($C7$) No such $<x, y>$ satisfies (**);

Of course, ($C1$)—($C7$) suffice to establish (II).

We now establish ($C1$)—($C7$) in turn.

Proof of $C1$: There are p and q that falsify $2q \geq p$.
Proof of $C2$: If $y \neq 1$, let $q = 1$.
Proof of $C3$: If $y \neq 0$, then there will of course be q such that $q > y$.
Proof of $C4$: If $x \neq 0$, then there will of course be p such that $p < x$.
Proof of $C5$: Let $p = 1$ and $q = 0$, to falsify (**).
Proof of $C6$: Let $p = q = 0$, to falsify (**).
Proof of $C7$: Let $p = 0$ and $q = 1$, to falsify (**).

This establishes (II).

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