Coherent sheaves and quantum Coulomb branches II: diagrams for quiver gauge theories

Ben Webster[1]
Department of Pure Mathematics, University of Waterloo &
Perimeter Institute for Theoretical Physics
Waterloo, ON
Email: ben.webster@uwaterloo.ca

Abstract. We continue our study of noncommutative resolutions of Coulomb branches in the case of quiver gauge theories. These resolutions are based on vortex line defects in quantum field theory, but have a precise mathematical description, which in the quiver case is a modification of the formalism of weighted KLR algebras. While best understood in a context which depends on the geometry of the affine Grassmannian and representation theory in characteristic p, we give a description of the Coulomb branches and their commutative and non-commutative resolutions which can be understood purely in terms of algebra.

Slodowy slices in type A and symmetric powers in $\mathbb{C}^2$ are special cases of these Coulomb branches, and in this case, we recover the noncommutative Springer resolution of Bezrukavnikov and those constructed using the Cherednik algebra by Bezrukavnikov, Finkelberg and Ginzburg.

Author's note: As this is a continuation of the first part of this paper [Weba], we will use notation and constructions from that paper without additional reference or comment.

5. (Re)introduction

In [Weba, Webb, Webc], we developed a general theory of Coulomb branches from an algebraic perspective. We showed that the Coulomb branch algebra itself, its extended category (of line operators) and various related algebras, such as the non-commutative resolution constructed through quantization by Bezrukavnikov and Kaledin [BK08, Kal08, Bez06] and the category controlling their Gelfand-Tsetlin modules all have explicit combinatorial descriptions based on the structure of their weight spaces. In this sequel to these papers, we focus in on understanding this construction in the quiver gauge case, especially on the non-commutative resolution and corresponding geometric constructions with coherent sheaves. Applications to the representation theory of quantum Coulomb branches in characteristic 0 have already been covered extensively in [KTW$^+$, Webb, Web19a], so we will only discuss these in passing.

At the root of this perspective is a description of the Coulomb branch algebra as paths in the space $T_{\mathbb{R}}/W$, the quotient of the compact torus of the gauge group G modulo the Weyl group W, modulo certain relations (see [Webc, (2.5a–c)]). In the case of GL_n, this space can be identified with the configuration of n points on the circle

[1]Supported by the NSF under Grant DMS-1151473 and the Alfred P. Sloan Foundation.

$\mathbb{R}/\mathbb{Z}$ (allowing collisions), and thus a path in this space with a diagram drawn on the cylinder.

Fix a quiver Γ with vertex set $\mathcal{V}(\Gamma)$, and dimension vectors $\mathbf{v}, \mathbf{w} \colon \mathcal{V}(\Gamma) \to \mathbb{Z}_{\geq 0}$ for this quiver. We should emphasize that we do allow edge loops. By a **quiver gauge theory** we mean the one attached to the gauge group and matter (G, V) given by:

$$G = \prod GL(\mathbb{C}^{v_i}) \qquad V = \Big(\bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j})\Big) \bigoplus \Big(\bigoplus_{i \in \mathcal{V}(\Gamma)} \mathrm{Hom}(\mathbb{C}^{w_i}, \mathbb{C}^{v_i})\Big), \tag{5.1}$$

As described above, we can think of a path in $T_{\mathbb{R}}/W$ as a path in a labeled configuration space where v_i points have label i, that is, as a string diagram on the cylinder where strands are labeled by points in the Dynkin diagram. When we translate the relations [Webc, (2.5a–c)] into this framework, they suddenly become very familiar to any one used to categorification: those of the KLR algebra, as presented in [KL11]. The author and his collaborators exploited this in [KTW$^+$] to study the representation theory of shfited Yangians, but by taking a more geometric perspective, we can apply the same idea to categories of coherent sheaves.

Thus our main result is a description of a tilting generator on the resolved Coulomb branch for affine type A gauge theories, i.e. those with cyclic or linear quivers. The most interesting examples of these are resolved Slodowy slices (or more generally, S3 varieties) for a linear quiver and the Hilbert scheme of points on $\mathbb{C}^2$ (for more generally, the resolved Kleinian singularity $\mathbb{C}^2/\mathbb{Z}/\ell\mathbb{Z}$).

Theorem D *If $\mathfrak{M}$ is the Coulomb branch of an affine type A quiver gauge theory, and $\tilde{\mathfrak{M}}$ a BFN resolution, then:*

1. *the homogeneous coordinate ring of $\tilde{\mathfrak{M}}$ is an algebra of twisted cylindrical KLR diagrams, module local relations.*
2. *The variety $\tilde{\mathfrak{M}}$ admits a tilting generator $\mathcal{T}$, described as an explicit module over the homogenous coordinate ring, such that $\mathrm{End}(\mathcal{T})$ is a cylindrical wKLR algebra.*
3. *The wall-crossing functors relating different tilting generators are given by tensor product with explicit bimodules, modeled on the braiding functors of* [Web17a, §6]*; more generally, the Schober connected to these functors can constructed using the representation theory of related algebras.*

Aside from making explicit a construction that otherwise requires some rather challenging techniques, it's generally believed that the algebras $\mathrm{End}(\mathcal{T})$ are Koszul; we hope that this can be proved geometrically, much as similar results have been proven for usual weighted wKLR algebras. Other possible applications include studying Bezrukavnikov and Okounkov's conjecture relating wall-crossing functors to quantum cohomology and generalizing Anno and Nandakumar's work on affine tangles in 2-block Springer fibers to more general actions of webs.

6. Diagrams for quiver gauge theories

First, we note some basic facts about quiver gauge theories. In this case, the group $H = N_{GL(V)}(G)$ is generated by G, the product $GL(\mathbb{C}^{w_i})$ acting by precomposition in the obvious way, and by $GL(\mathbb{C}^{\chi_{i,j}})$ where $\chi_{i,j}$ is the number of edges $i \to j$, acting by taking linear combinations of the maps along these edges, i.e. via the isomorphism

$$\bigoplus_{i\to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \cong \bigoplus_{(i,j)\in \mathcal{V}(\Gamma)} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \otimes \mathbb{C}^{\chi_{i,j}}.$$

For simplicity, we'll assume that $\delta = \frac{1}{2}$ (and so in pth root conventions, we have $\delta = \frac{1}{2p}$).

6.1. Unrolled diagrams.

In the case of a quiver gauge theory, the extended BFN category has a more graphical description. Let us first discuss a little how the constructions of [Weba] can be interpreted here.

The space $\mathfrak{t}_{\mathbb{R}}$ in this case is naturally isomorphic to $\bigoplus_{i\in\mathcal{V}(\Gamma)} \mathbb{R}^{v_i}$, using the usual coordinates on diagonal matrices. These coordinates are given by $z_{i,k} = \epsilon_{i,k}$ with $i \in \mathcal{V}(\Gamma), k = 1, \dots, v_i$ ranging over the weights of the natural representation of $GL(\mathbb{C}^{v_i})$.

In these terms, the unrolled arrangments defined in Section 2.2 are given by the unrolled root hyperplanes $\{\alpha(\eta) = n \mid n \in \mathbb{Z}\}$ of the form:

$$z_{i,k} - z_{i,m} = n \qquad \text{for all } k \neq m \in [1, v_i], n \in \mathbb{Z}, \tag{6.1a}$$

and the unrolled matter hyperplanes $\{\varphi_i^{\mathrm{mid}}(\eta) = n \mid n \in \mathbb{Z}\}$ of the form:

$$z_{j,k} - z_{i,m} = n - \frac{1}{2} \qquad \text{for all edges } i \to j, \text{ for all } k \in [1, v_j], m \in [1, v_i], n \in \mathbb{Z} \tag{6.1b}$$

$$z_{i,m} = n - \frac{1}{2} \qquad \text{for all } i \in \mathcal{V}(\Gamma), m \in [1, v_i], n \in \mathbb{Z} \tag{6.1c}$$

We can decompose a path $\pi\colon [0, 1] \to \mathfrak{t}$ into a v_i-tuple of paths $\pi_{i,k}$ for each $i \in \mathcal{V}(\Gamma)$. We can visualize this by superimposing the graphs of these paths (though we will use the opposite convention from calculus class, using the y-axis for the independent variable and the x-axis for the dependent). That is, we consider the path $t \mapsto (\pi_{i,k}(t), t)$ landing in $\mathbb{R} \times [0, 1]$.

We cross root hyperplanes when two of these paths for the same element of $\mathcal{V}(\Gamma)$ are integer distance from each other, and matter hyperplanes when either the x-value or the distance between two paths corresponding to adjacent nodes in $\mathcal{V}(\Gamma)$ differ by an element of $\mathbb{Z} - \frac{1}{2}$.

It's thus convenient to label the points $(\pi_{i,k}(t)+n, t)$ for $n \in \mathbb{Z}$ with "partner" strands and those for $n \in \mathbb{Z} - \frac{1}{2}$ with "ghost" strands so that we can see when these crossings occur. We'll label the strand corresponding to n with $i; n$. We'll draw partners as solid lines, and ghosts as dashed lines. We'll also draw in dashed lines at $x = n$ for $n \in \mathbb{Z} - \frac{1}{2}$, which we'll label with $\infty; n$.

There's an obvious action of the affine Weyl group on the set of such paths with the finite Weyl group acts on paths by permutation of the second indices in $\pi_{i,k}$, and the translations act by translations $\pi_{i.k}(t) \mapsto \pi_{i,k}(t) + n_{i,k}$ for integers $n_{i,k}$. This leaves the collections of the original curves and their partners unchanged, just changing the indices and which curves are partners, and which are originals. Thus, we can visualize the action of an affine Weyl group element by changing the labels accordingly at some fixed value of $y = a$. The equations (2.6e) and (2.6i) assure that the result does not depend on the value of a.

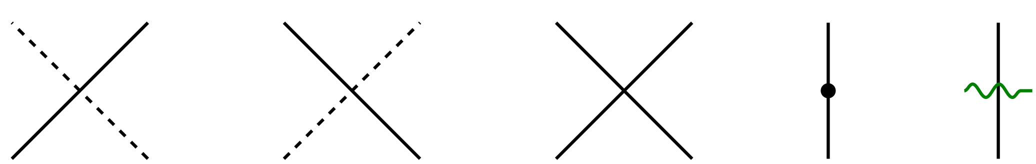
We can also visualize the action of S_h by identifying the weights $\epsilon_{i,k}$ with a dot on the corresponding path, at the top for multiplying at the left and at the bottom for multiplication on the right. It will be useful to also draw dots on partner strands with label $i;n$, which we will associate to $\epsilon_{i,k} + nh$. This likewise assures that inserting an element of the affine Weyl group above or below a dot will give the same answer, by equation (2.6a).

Thus, the morphisms in $\mathscr{B}$ can all be expressed as one of these diagrams. More precisely:

Definition 6.1 *An* **unrolled diagram** *is a collection of paths in* $\mathbb{R} \times [0,1]$ *of the form* $\{(\pi_{i,k}(t)+n, t) \mid t \in [0,1]\}$ *for* $n \in \mathbb{Z}$ *for some piecewise smooth map* $\pi_{i,k} \colon [0,1] \to \mathbb{R}$. *At a finite number of values* $t_1, \dots, t_k$, *we apply an elements* $w_1, \dots, w_k \in \widehat{W}$ *to the labels on the curves and their partners, which can create a discontinuity in the function* $\pi_{i,k}$, *but we assume that the resulting curves are still smooth and simply change labeling. We'll draw one of these changes as a squiggly green line.*

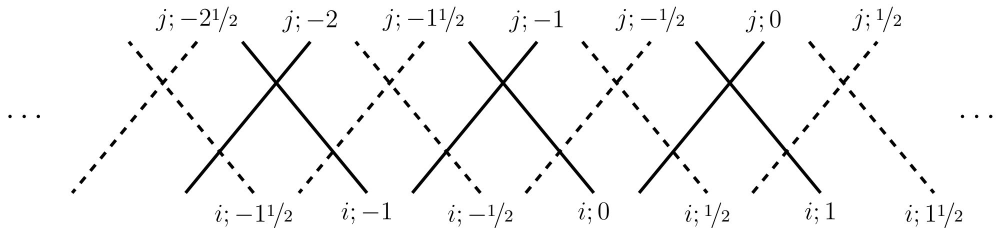
We also add ghost strands at $\{(\pi_{i,k}(t) + n - \frac{1}{2}, t) \mid t \in [0,1]\}$ *and at* $\{(n - \frac{1}{2}, t) \mid t \in [0,1]\}$ *for all* $n \in \mathbb{Z}$, *which we draw as dashed. Each curve is labeled with the element* $i \in \mathcal{V}(\Gamma)$ *and decorated with finitely many dots.*

The diagram must be locally of the form

(6.2) 

That is, there are no tangencies, triple crossings or dots on crossings. The curves (including ghosts) must meet the circles at $y = 0$ *and* $y = 1$ *at distinct points. We consider these diagrams up to isotopy preserving the conditions above.*

We'll draw an example below; lacking infinitely wide paper, we can only draw part of the diagram.

(6.3)


We can associate a morphism $\mathbb{r}_D$ to an unrolled diagram via the rules we have already discussed. The diagram D corresponds to a map $[0,1] \to \mathfrak{t}_{\mathbb{R}}$, and we can define π by adding the flavor ϕ to this path to obtain one in $\mathfrak{t}_{1,\mathbb{R}}$.

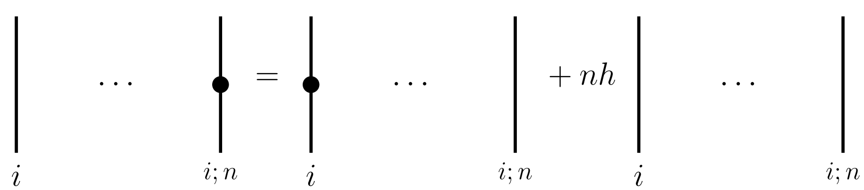
Definition 6.2

(1) Given an unrolled diagram D with no dots, let $\mathbb{r}_D$ be the morphism $\mathbb{r}_\pi$ for the corresponding path.
(2) Given an unrolled diagram with all $\pi_{i,k}$'s constant, and one dot on the strand at $(\pi_{i,k}+n,t)$, we let $\mathbb{r}_D$ be multiplication by $\epsilon_{i,k}+nh$.
(3) Given an unrolled diagram with no dots, and a single relabeling by $w \in \widehat{W}$, we let $\mathbb{r}_D = y_w$.

Any other diagram can be written as a composition of these, and we define $\mathbb{r}_D$ to be the composition of the corresponding morphisms.

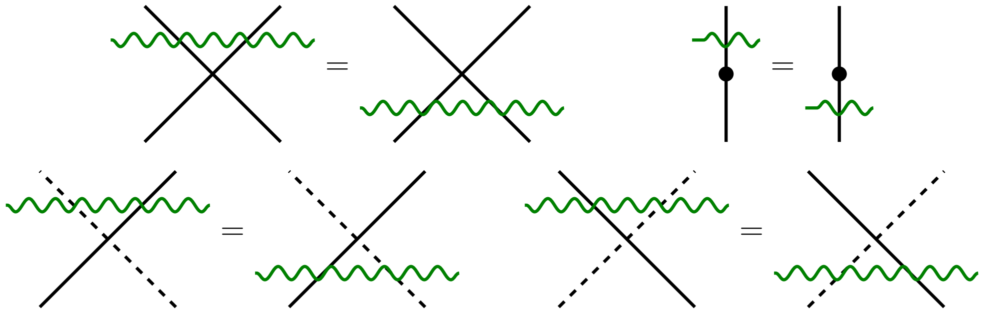
This associates a morphism to a diagram, and since we hit all the generators of the category $\mathscr{B}$, every morphism is a sum of $\mathbb{r}_D$'s. However, this description is redundant, since there are relations between these generators.

All our relations can be described locally in terms of unrolled diagrams. In order to write these relations in a consistent way, we need to adopt the "dot migration" convention for interpreting dots on partner strands:

(6.4a)


The fact that isotopy leaves $\mathbb{r}_D$ invariant is a combination of relations (2.6a) and the "boring" cases of (2.6c,2.6h,2.6j,2.6k) where the portions of diagrams commuting past each other are distant in $\mathbb{R}$. As discussed before, (2.6b,2.6e,2.6f,2.6i) imply that green lines can isotope past all crossings and dots, and (2.6d) that green lines can merge by multiplying their labels.

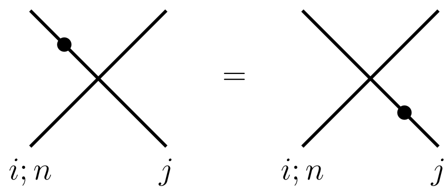
(6.4b)



These relations make the need for the rule (6.4a) clear.

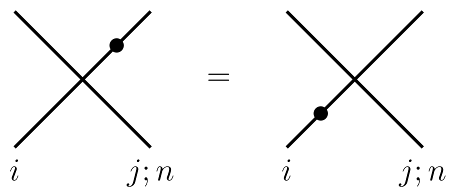
The relation (2.6a) implies that dots can commute past strands with a different label or their partners, or past all ghosts:

(6.4c)



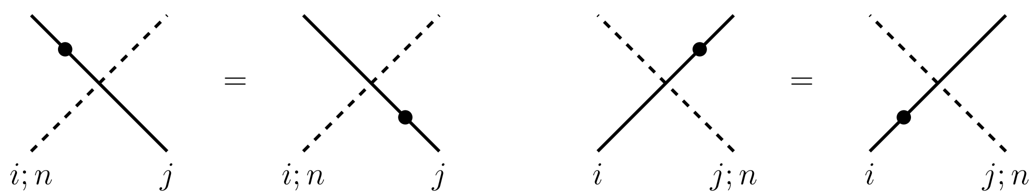
unless $i = j$

(6.4d)



unless $i = j$

(6.4e)

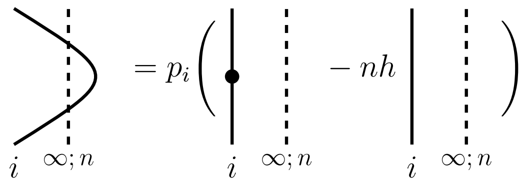


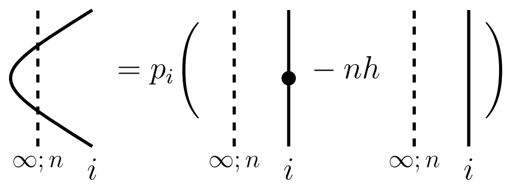
The flavor ϕ acts by automorphisms of the G-representation on V, and thus is induced by a cocharacter into $\prod_i GL(\mathbb{C}^{w_i}) \times \prod_{(i,j)\in\mathcal{V}(\Gamma)^2} GL(\mathbb{C}^{\chi_{i,j}})$. We can assume that this cocharacter lands in the usual torus of diagonal matrices; let $c_{i,1}, \dots, c_{i,w_i}$ be the weights of its components into $GL(\mathbb{C}^{w_i})$ and b_e for each edge e the weights of its components into $GL(\mathbb{C}^{\chi_{i,j}})$. Let

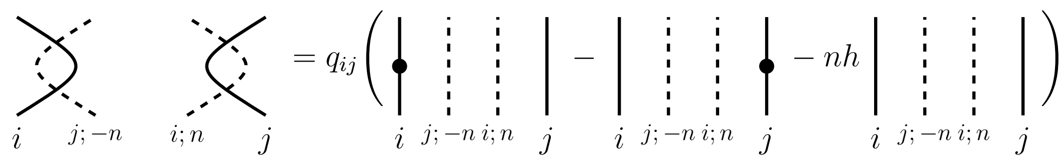
$$p_i(u) = \prod_{k=1}^{w_i} (u - c_{i,k} - \frac{h}{2}) \tag{6.4f}$$

$$q_{ij}(u) = \prod_{e\colon j\to i} (u + b_e + \frac{h}{2}) \cdot \prod_{e\colon i\to j} (-u + b_e + \frac{h}{2}). \tag{6.4g}$$

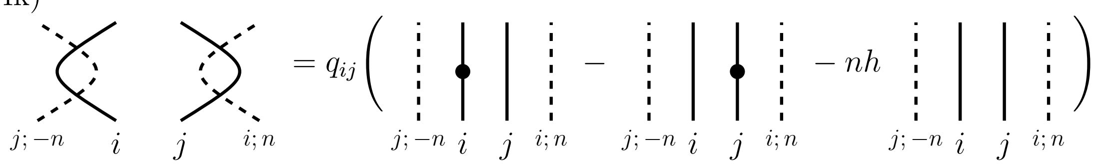
This is the point where we must incorporate that our path is shifted by ϕ. Thus, when we use the relation (2.6c) implies to switch the order of crossings on disjoint strands, we have:

(6.4h) 

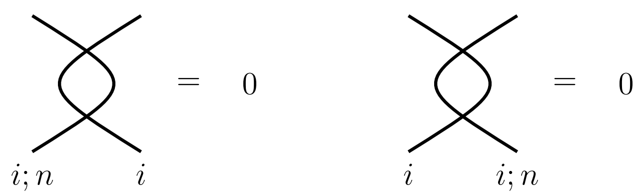
(6.4i) 

(6.4j) 

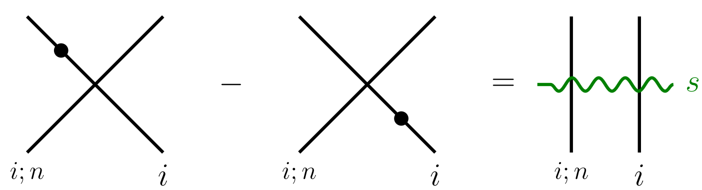
(6.4k)

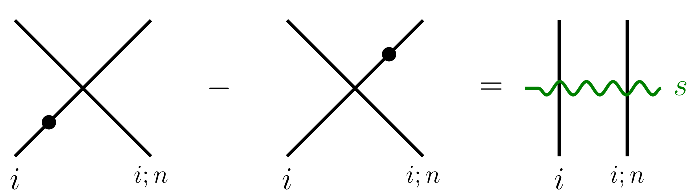


Note that the equations above are written assuming that $n > 0$ (and implicitly $n \in \mathbb{Z} - \frac{1}{2}$), but they are equally valid if $n < 0$, with the requisite reordering of strands. The relation (2.6g) implies that:

(6.4l) 

The relation (2.6j) is equivalent to isotopy and

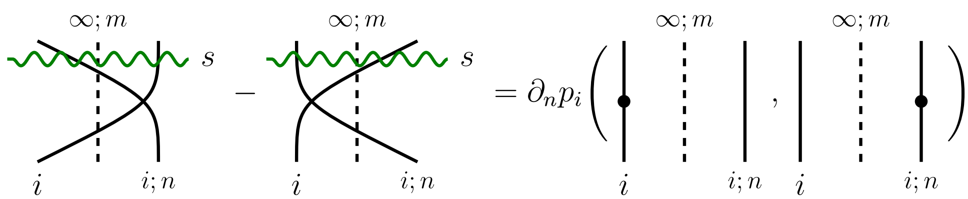
(6.4m) 

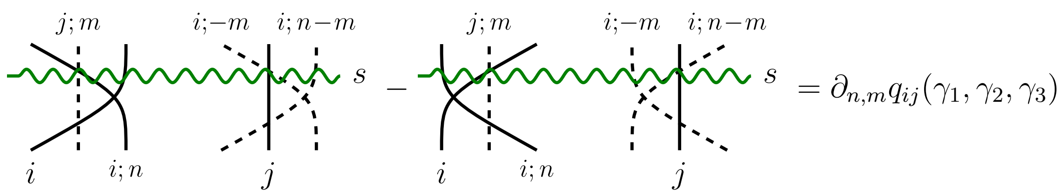
(6.4n) 

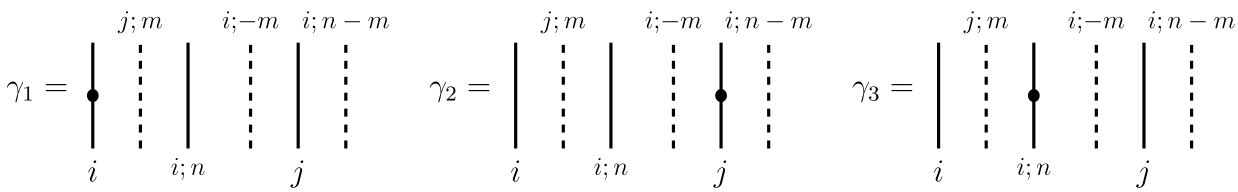
with s denoting the unique reflection in the affine Weyl group switching the top and bottom labels of the diagram.

Finally, the codimension 2 relations show how to relate the two resolutions of a triple point. The correct relation depends on the number of partners with the same label going through the triple point: if all three, then we use (2.6h), if two we use (2.6k) and if there is no such pair, then (2.6c). These imply we can isotope through any triple point unless it involves exactly two partners with the label $i \in \mathcal{V}(\Gamma)$. In order to cover this last case, let

$$\partial_m p(u_1, u_2) = \frac{p(u_1 - mh) - p(u_2 - mh)}{u_1 - u_2}$$
$$\partial_{n,m} q(u_1, v, u_2) = \frac{q(u_1 - v + mh) - q(u_2 - v + mh)}{u_1 - u_2}$$

(6.4o)


(6.4p)




with s denoting the unique reflection in the affine Weyl group making the top and bottom match. As before, using p and q accounts for our shift by ϕ.

Given this reinterpretation of our relations in terms of diagrams, this allows us to interpret the relations (6.4a–6.4p) as relations on unrolled diagrams, by setting $\sum a_i D_i = 0$ if we have that $\sum a_i \mathbb{r}_{D_i} = 0$. In this case, we have that:

Lemma 6.3 *Given $\eta, \eta' \in \mathfrak{t}_\tau$, the Hom space $\mathrm{Hom}_{\mathscr{B}}(\eta, \eta')$ is spanned by the morphisms $\mathbb{r}_D$ for unrolled diagrams D with top η' and bottom η, modulo the local relations (6.4a–6.4p).*

Proof. We have justified in each individual case why the relations (6.4a–6.4p) hold. Thus, we have a map from the formal span of unrolled diagrams modulo these relations to $\mathrm{Hom}_{\mathscr{B}}(\eta, \eta')$. This is surjective because the generating morphisms of the category $\mathscr{B}$ are given by the basic unrolled diagrams in (6.2). On the other hand, the relations (6.4a–6.4p) suffice to write any $\mathbb{r}_D$ as a sum of diagrams corresponding to a reduced word in $\widehat{W}$ with all dots and green lines at the bottom, and to relate any two reduced words for $w \in \widehat{W}$ modulo the diagrams for shorter elements of $\widehat{W}$. Thus, we find that the unrolled diagrams corresponding to the basis of [Webc, Cor. 3.12] are a spanning set of this quotient. This is only possible if the map is injective as well. □

6.2. **Antipodal diagrams.** The reader may have noticed that these diagrams are actually quite difficult to draw and interpret, but there is a symmetry that we have not exploited, the action of the extended affine Weyl group. The quotient of $\prod_i \mathbb{R}^{v_i}$ by the extended Weyl group is given by the space $\prod_i (\mathbb{R}/\mathbb{Z})^{v_i}/S_{v_i}$, which we can interpret as the moduli space of multisubsets of the circle $\mathbb{R}/\mathbb{Z}$ labeled with elements of $\mathcal{V}(\Gamma)$, such that v_i elements have label $i \in \mathcal{V}(\Gamma)$. Thus the path $[0,1] \to \prod_i \mathbb{R}^{v_i}$ composed with the projection $\prod_i \mathbb{R}^{v_i} \to \prod_i (\mathbb{R}/\mathbb{Z})^{v_i}/S_{v_i}$ can be thought of as a path in this moduli space.

We draw this by considering our diagrams in $\mathbb{R} \times [0,1]$, and considering the quotient of this plane by $\mathbb{Z}$ acting by addition to the x-coordinate. Note that this sends all the partners to a single curve in $\mathbb{R}/\mathbb{Z} \times [0,1]$, and all ghosts to a single curve $\frac{1}{2}$ units in either direction (since we are on a circle). We can describe the resulting diagrams as follows:

Definition 6.4 *An* **antipodal diagram** *is a collection of finitely many oriented curves in $\mathbb{R}/\mathbb{Z} \times [0,1]$ of the form $\{(\bar{\pi}(t), t) \mid t \in [0,1]\}$ for some path $\bar{\pi}\colon [0,1] \to \mathbb{R}/\mathbb{Z}$. Each curve is labeled with an element $i \in \mathcal{V}(\Gamma)$ and decorated with finitely many dots. For each curve, we draw a "ghost" curve at $\{(\bar{\pi}(t) - \frac{1}{2}, t) \mid t \in [0,1]\}$, as well as one at $\{(-\frac{1}{2}, t) \mid t \in [0,1]\}$. We draw these as dashed.*

The diagram must be locally of the form given in (6.2). That is, there are no tangencies, triple crossings or dots on crossings. The curves (including ghosts) must meet the circles at $y = 0$ and $y = 1$ at distinct points. We consider these diagrams up to isotopy preserving the conditions above.

We call a subset of $\mathbb{R}/\mathbb{Z}$ **generic** if the distance between no pair of elements is $\frac{1}{2}$. Our conditions above guarantee that for a fixed t, the elements $\bar{\pi}(t)$ for the different strands are distinct and form a generic subset if and only if no pair of strands or strand and ghost cross at height t.

Every antipodal diagram has a unique lift to a path $[0,1] \to \mathfrak{t}_\tau$ which starts in the fundamental region of $\widehat{W}$ where the coordinates $z_{i,k}$ satisfy

$$-\frac{1}{2} < z_{i,1} < z_{i,2} < \cdots < z_{i,v_i} < \frac{1}{2}. \tag{6.5}$$

That is, by the path lifting property of the universal cover, each of the curves $\bar{\pi}\colon [0,1] \to \mathbb{R}/\mathbb{Z}$ has a unique lift π with $-\frac{1}{2} < \pi(0) < \frac{1}{2}$, and we can number these so that

$$-\frac{1}{2} < \pi_{i,1}(0) < \cdots < \pi_{i,v_i}(0) < \frac{1}{2}. \tag{6.6}$$

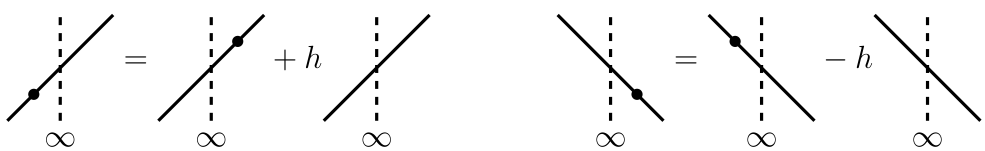
Definition 6.5 *Given an antipodal diagram D with no dots, let $\mathbb{r}_D$ be the morphism defined by the lifted path starting in the fundamental region, followed by the unique element of $\widehat{W}$ sending us back to the fundamental region.*

If the diagram does contain dots, then place these in the lifted diagram on the unique partner preimage which has x-value in $(-\frac{1}{2}, \frac{1}{2})$.

We have to be careful about lifting antipodal diagrams with dots, because if we do so in the most naive way, the result will not be compatible with composition, which the definition above is. We could accomplish the same effect if instead of applying a Weyl group element at the end, we applied one immediately whenever we left the fundamental region to move back into it.

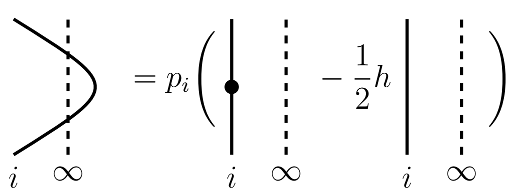
We can also modify the relations (6.4a–6.4p) to work as relations on antipodal diagrams, as before following the rule that $\sum a_i D_i = 0$ if we have that $\sum a_i \mathbb{r}_{D_i} = 0$. Some of these can interpreted locally exactly as they read above: (6.4c–6.4d) and (6.4l–6.4n) are of this type. On the other, if a dot on an antipodal diagram is slid over the half-integer ghost with label ∞, then it goes between lifting to the ghost just right of $x = -\frac{1}{2}$ to that just left of $x = \frac{1}{2}$. The effect of (6.4a) is thus the following relation on antipodal diagrams:

(6.7a)

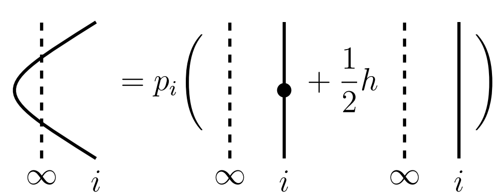


The other relations need to interpreted carefully to be compatible with lifting. For example, the relations (6.4h–6.4i) become

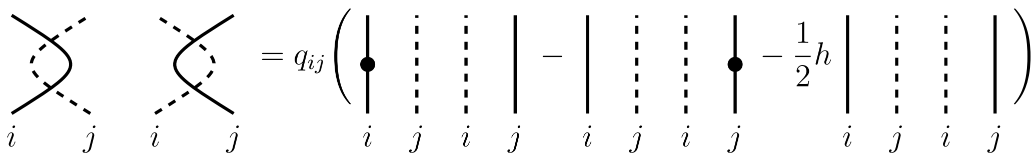
(6.7b)

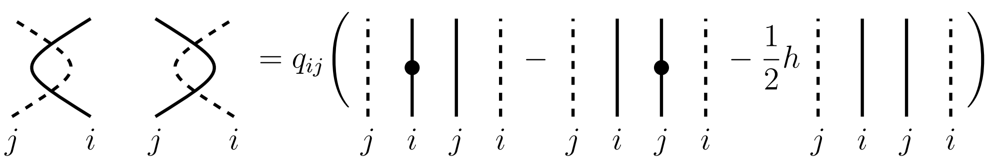


(6.7c)



Similarly, the relations (6.4j–6.4j) depend on the cyclic order of the strands i, j and the point $x = -\frac{1}{2}$ on the circle. Assuming these are cyclically ordered, we have:

(6.7d)


(6.7e)


Since we can switch i and j without loss of generality, we do not need a second version of these relations.

Definition 6.6 *The* **antipodal KLR category** *is the category whose*

- *objects are generic subsets of $\mathbb{R}/\mathbb{Z}$, labeled with elements of $\mathcal{V}(\Gamma)$, with v_i having label i.*
- *morphisms are the quotient of the formal span of antipodal diagrams over $\Bbbk[c_{*,*}, b_*]$ modulo the relations induced by (6.4a–6.4p), in particular by (6.7a–6.7e).*

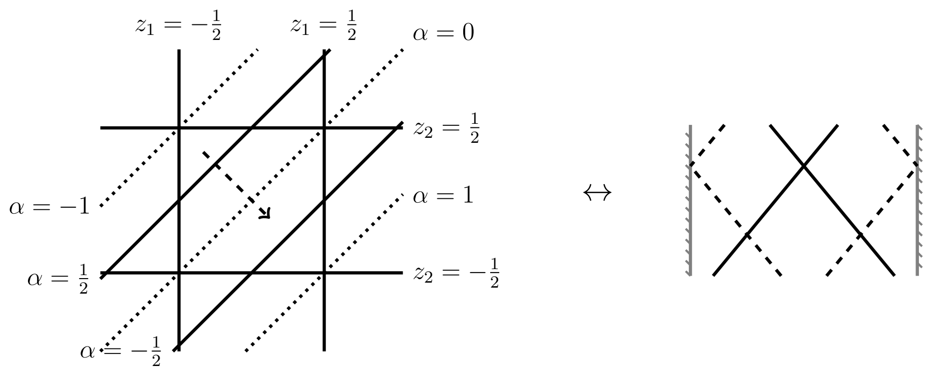
Some might prefer to think about the antipodal KLR algebra, which is the endomorphisms of the formal sum of all objects in this category. Lemma 6.3 can thus be rephrased as:

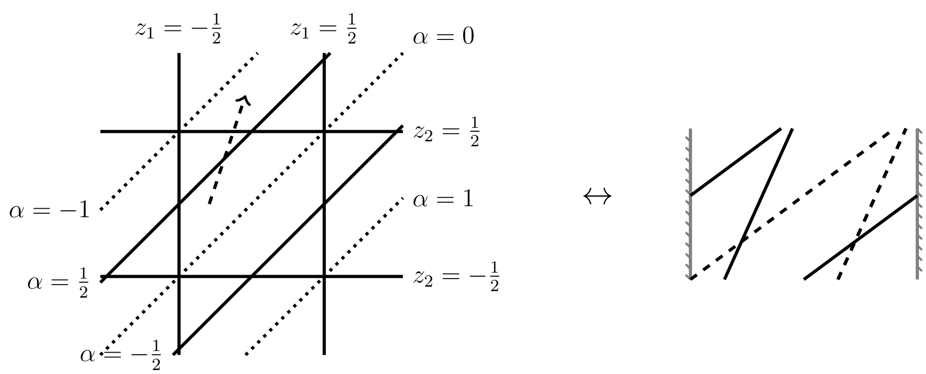
Proposition 6.7 *The antipodal KLR category is equivalent to the category $\mathscr{B}$.*

Note, this construction is closely related to the flag Yangian introduced in [KTW$^+$, Def. 4.12]. In that paper, we assumed that Γ was bipartite (with the sets of nodes called **even** and **odd**), that if $i \to j$ then i is even and j odd. Furthermore, the definition depended on a polynomial p_i. If $h = 2$, $b_e = 0$ and the scalars $c_{i,k}$ are the roots (with multiplicity) of $p_i(2u - 1)$, then the antipodal KLR category is closely related to the flag Yangian category, via the transformation of diagrams sending all odd strands to their antipodal ghosts. Since there are some minor differences between these categories, we will not make a precise statement about the relationship between them.

Let us give a simple example. Consider $G = GL(2)$ and $V \cong \mathbb{C}^2 \oplus \mathfrak{gl}_2$. We have a natural isomorphism $\mathfrak{t}_{\mathbb{R}} \cong \mathbb{R}^2$ with the coordinates given by z_1, z_2. The unrolled matter hyperplanes are $z_1, z_2, z_1 - z_2 \in \mathbb{Z} - \frac{1}{2}$ and the unrolled root hyperplanes are $\alpha = z_1 - z_2 \in \mathbb{Z}$.

With these conventions, we match morphisms of the extended category with antipodal diagrams. We'll draw these on a cylinder sliced open at $x = \frac{1}{2}$.





6.3. **Representations.** We can also naturally study the weight modules over the extended category in this framework.

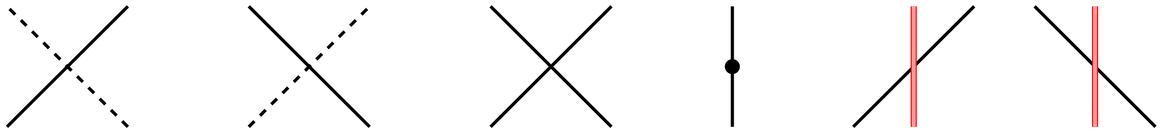
By Proposition 2.18 and Theorem 2.21, we can use this description to understand the category of $\mathcal{A}$-modules. We will assume again for simplicity that $\Bbbk = \mathbb{F}_p$. Thus, all roots and weights are relevant.

Now, we consider how our diagrams change under taking pth root conventions (as in Definition 2.12). The structure of our diagrams will depend on the flavor parameters $b_e, c_{i,k} \in \mathbb{Z}$. In particular, in our usual parameters, the unrolled root hyperplanes are unchanged and thus are the same as (6.1a) and the unrolled matter hyperplanes become:

(6.7f)
$$z_{j,k} - z_{i,m} = n - \frac{b_e}{p} - \frac{1}{2p} \qquad \text{for all edges } e\colon i \to j, k \in [1, v_j], m \in [1, v_i], n \in \mathbb{Z}$$

(6.7g)
$$z_{i,m} = n - \frac{c_{i,k}}{p} - \frac{1}{2p} \qquad \text{for all } i \in \mathcal{V}(\Gamma), m \in [1, v_i], k \in [1, w_i], n \in \mathbb{Z}$$

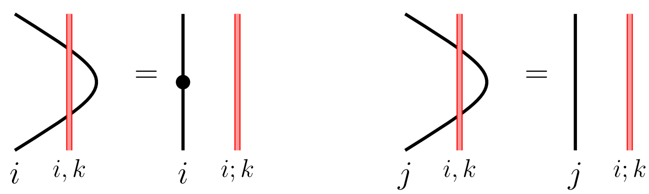
Definition 6.8 *A* **weighted antipodal** *diagram is a collection of finitely many oriented curves in $\mathbb{R}/\mathbb{Z} \times [0,1]$ of the form $\{(\bar{\pi}(t), t) \mid t \in [0,1]\}$ for some path $\bar{\pi}\colon [0,1] \to \mathbb{R}/\mathbb{Z}$. Just as before, each curve is labeled with an element $i \in \mathcal{V}(\Gamma)$ and decorated with finitely many dots. For each curve with label i, and each edge $i \to j$, we draw a "ghost"* *curve at* $\{(\bar{\pi}(t) - \frac{b_e}{p} - \frac{1}{2p}, t) \mid t \in [0,1]\}$, *which we draw as dashed. We also draw the ghosts of infinity at* $x = -\frac{c_{i,k}}{p} - \frac{1}{2p}$ *as red. These should satisfy the same genericity conditions as antipodal diagrams, in particular, the local possibilities all appear below:*



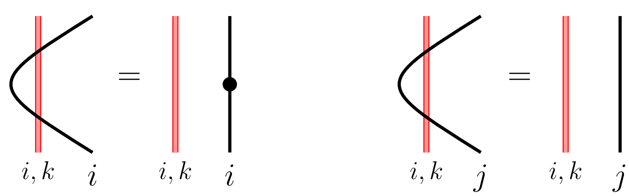
As in [Web19b], *we let a* **cylindrical loading** *be a map to* $\mathcal{V}(\Gamma)$ *from a finite subset of* $\mathbb{R}/\mathbb{Z}$ *which avoids* $x = -\frac{c_{i,k}}{p} - \frac{1}{2p}$ *and such that if there is an edge* $e\colon i \to j$, *then there is no pair of elements* x *and* y *mapping to* i *and* j *with* $x - y = \frac{b_e}{p} + \frac{1}{2p}$. *Note that the top and bottom of each weighted antipodal diagram gives a cylindrical loading.*

Note the similarity of these diagrams to **weighted KLR diagrams** as defined in [Web19b]. Just as antipodal diagrams define morphisms in $\mathscr{B}$, weighted antipodal diagrams define morphisms in B. The relations of B thus induce an appropriate modification of the relations (6.4a–6.4p), where we set $h = 0$ (which greatly simplifies them) and change of flavor means that we separate the contributions in (6.7b–6.7e) into the different ghost strands. That is, assuming that all b_e and $c_{i,k}$'s are different mod p, we have relations of the form:

(6.8a)

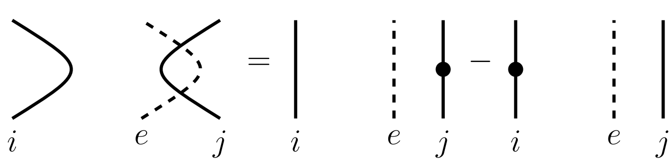


(6.8b)

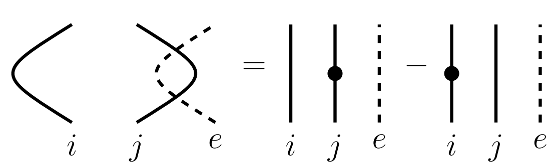


Given an edge $i \to j$, we have that:

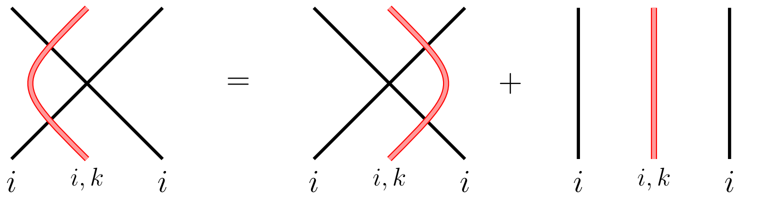
(6.8c)



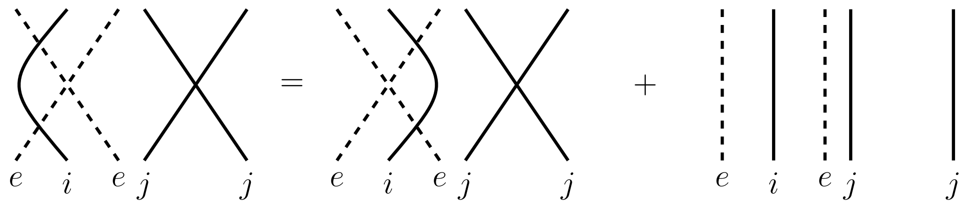
(6.8d)



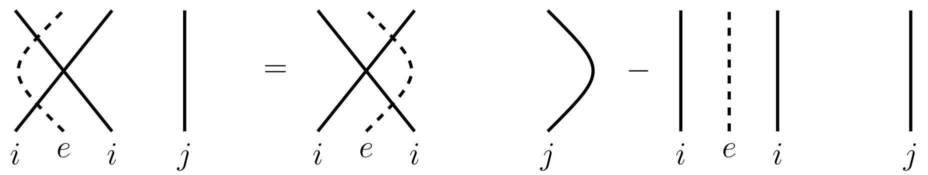
(6.8e)



(6.8f)



(6.8g)



As before, we'll draw these on the page in the rectangle $[0,1] \times [0,1]$ with seams on the left and right side of the diagram where we should glue to obtain the cylindrical diagram.

Definition 6.9 *The* **cylindrical wKLR algebra** $\mathring{R}$ *attached to these data is the quotient of the formal span of weighted antipodal diagrams by the local relations (6.4a–6.4e) and (6.4l–6.4n) with* $n = 0$ *and the relations (6.8a–6.8g) above.*

We let the **cylindrical wKLR category** *be the category whose*

- *objects are cylindrical loadings,*
- *morphisms are weighted antipodal diagrams with the source at bottom and target at top, modulo the relations already discussed.*

The algebra $\mathring{R}$ *is the endomorphisms of the sum of all objects in this category (considered up to isotopy).*

Applying Definition 6.5 with pth root data, we can associate a morphism in the category B for the corresponding quiver gauge theory to any weighted antipodal diagram, compatibly with composition and the relations of the cylindrical wKLR category. We find immediately that:

Theorem 6.10 *This map defines an equivalence between the cylindrical wKLR category and* B.

Remark 6.11. Note that in many earlier works, such as [Web17a, Web17b], we had an additional non-local relation setting a diagram to 0 if it had a black strand at far left of the diagram; we will not impose this relation since it corresponds to passing category $\mathcal{O}$, and here we consider all weight modules. It's not even clear how one could interpret this relation on the circle, which matches with the fact that category $\mathcal{O}$ doesn't make sense in characteristic p.

Given two objects in this category, we can describe the different morphisms joining them explicitly. Given a cylindrical loading with v_i elements mapping to $i \in \mathcal{V}(\Gamma)$, we have a unique way of lifting to real numbers $z_{i,1}, \dots, z_{i,v_i}$ in the fundamental region (that is, satisfying $-\frac{1}{2} < z_{i,1} < \cdots < z_{i,v_i} < \frac{1}{2}$), and the extended affine Weyl group $\widehat{W}$ acts freely transitively on the set of possible lifts. Having fixed two cylindrical loadings S and T, there is an unrolled diagram with a minimal number of crossings with the bottom given by this lift of S and the lift of T corresponding to $w \in \widehat{W}$; this diagram is not unique, but as usual, any two choices differ by the diagram for a shorter permutation by the relations (6.8e–6.8g). The image of this diagram D_w on the cylinder $\mathbb{R}/\mathbb{Z} \times [0,1]$ gives a weighted antipodal diagram.

From [Webc, Cor. 3.12], we find that:

Lemma 6.12 *The Hom space between two objects in the cylindrical wKLR category is a free module for the left action of polynomials in the dots, with basis D_w for $w \in \widehat{W}$.*

Example 6.i. Consider the case where $G = \mathbb{C}^*$ acting on $\mathbb{C}^2$ by scalars. In this case, the Coulomb branch is $T^*\mathbb{P}^1$. The corresponding cylindrical wKLR algebra has two red strands, and one black strand, all with the same label. There are two idempotents in this algebra, corresponding to the two cyclic orders of the 3 strands. Since the corresponding quiver has no edges, the black strand has no ghosts.

x $\quad$ x^* $\quad$ y $\quad$ y^*

These satisfy the quadratic relations

$$xx^* = yy^* \qquad x^*x = y^*y, \tag{6.9}$$

and it's easy to check that these are a complete set of relations. This algebra is Koszul and its Koszul/quadratic dual is easily seen to be defined by

$$xx^* = -yy^* \qquad x^*x = -y^*y \qquad y^*x = x^*y = yx^* = xy^* = 0. \tag{6.10}$$

This latter set of relations defines an 8-dimensional algebra studied by Nandakumar and Zhao in [NZ], which appears as the endomorphisms of a projective generator for exotic sheaves on $T^*\mathbb{P}^1$. ♣

We have defined this category so it would most directly reflect our general prescription, which makes it appear as though this category depends on p in a complicated way. In fact, its behavior is piecewise constant in the parameters $\frac{b_e}{p} + \frac{1}{2p}$ and $\frac{c_{i,k}}{p} + \frac{1}{2p}$. One can easily deduce by thinking about chambers of hyperplane arrangements that as long as the arrangement given by (6.7f)–(6.7g) is simple (i.e. any non-empty intersection of a subset of the hyperplanes is transverse), there will a neighborhood in the space of parameters around $\frac{b_e}{p}+\frac{1}{2p}$ and $\frac{c_{i,k}}{p}+\frac{1}{2p}$ where the category B is unchanged.

In particular, for any given real numbers $\beta_e, \gamma_{i,k}$, if the arrangement

$$z_{j,k}-z_{i,m}=n-\beta_e \qquad \text{for all edges } e\colon i\to j, k\in[1,v_j], m\in[1,v_i], n\in\mathbb{Z}$$
$$z_{i,m}=n-\gamma_{i,k} \qquad \text{for all } i\in\mathcal{V}(\Gamma), m\in[1,v_i], k\in[1,w_i], n\in\mathbb{Z}$$

is simple, then there is a real number ϵ such that the category B is independent of the choice of b_e and $c_{i,k}$ as long as

$$|\beta_e-\frac{b_e}{p}-\frac{1}{2p}|<\epsilon \qquad |\gamma_{i,k}-\frac{c_{i,k}}{p}+\frac{1}{2p}|<\epsilon.$$

This allows us to make the simplifying assumption that if $b_e=0$ then $\beta_e=0$. If Γ is a tree, then we can assume this without loss of generality.

Example 6.ii. The most important example of a case where weighting is useful is case of $\mathrm{Sym}^n(\mathbb{C}^2)$; in this case we have only a single node in our quiver, which carries a loop, equipped with the weight $\beta_e=\vartheta$, and a single red strand (of course, labeled with this node). This describes B when $\frac{b_e}{p}+\frac{1}{2p}\approx\theta$.

The objects in the cylindrical wKLR category are thus n-tuples of distinct points in $(0,1)$, where each point has a ghost ϑ units to its right, which the other points avoid. This information can be recorded by listing the order in which one encounters dots and ghosts; the set of possible configurations for a given ϑ corresponds to the set $\bar{\Lambda}$ discussed earlier.

Note that the set of possible configurations is locally constant, and will only change at values of θ where one has a non-simple hyperplane arrangment this can only be the case if there is a loop of equations

$$\begin{aligned} z_1-z_2 &\equiv \vartheta \pmod{\mathbb{Z}} \\ z_2-z_3 &\equiv \vartheta \pmod{\mathbb{Z}} \\ &\vdots \\ z_k-z_1 &\equiv \vartheta \pmod{\mathbb{Z}} \end{aligned}$$

for $k\le n$. This implies that $k\vartheta\in\mathbb{Z}$, i.e. that ϑ is rational with denominator $\le n$. Of course, this same set of values has shown up in the structure of Hilbert schemes and Cherednik algebras in other contexts. ♣

6.4. **Change of flavor.** Let ν be a cocharacter of the flavor torus T_F. Consider choices of flavor ϕ,ϕ' which differ by this cocharacter $\phi'=\nu+\phi$. Associated to ϕ we have a twisting bimodule ${}_{\phi+\nu}\mathscr{T}_\phi$ over the categories $\mathscr{B}_{\phi+\nu}$ and $\mathscr{B}_\phi$, discussed in Section 2.2. The elements of these bimodules are morphisms in the BFN category $\mathscr{B}^Q$ attached to the larger group Q acting on V. Thus, applying Proposition 2.18 to $\mathscr{B}_{\phi+\nu},\mathscr{B}_\phi$ and $\mathscr{B}^Q$, we obtain that the twisting bimodule ${}_{\phi+\nu}\mathscr{T}_\phi$ is intertwined with the corresponding a similar bimodule ${}_{\phi'_{1/p}}\mathsf{T}_{\phi_{1/p}}$ with pth root conventions. Since $\frac{1}{p}\nu$ might not be intergral, we cannot apply the definition of ${}_{\phi+\nu}\mathscr{T}_{\phi}$ directly, and we take the description above to be the definition, but let us say a few words about why the fact that $h = 0$ allows us to extend this definition to arbitrary cocharacters of $\mathfrak{t}_F$.

We will spare the reader the blizzard of notation required to say this carefully, but in brief $\mathsf{B}_{\phi_{1/p}}$ and $\mathsf{B}_{\phi'_{1/p}}$ can be realized as subcategories of B^Q modulo the action of polynomial morphisms $\mathfrak{t}_F^*$; note that this quotient is only well-defined because $h = 0$. We identify their object sets with orbits of $\mathfrak{t}$ in $\mathfrak{t}_{1;Q}$ which differ by $\frac{1}{p}\nu$, and note that morphisms in $\mathsf{B}_{\phi_{1/p}}$ and $\mathsf{B}_{\phi'_{1/p}}$ are precisely those in the larger category generated by paths, polynomials, and u_α and the extended affine Weyl group of G (as opposed to the affine Weyl group of Q, which has more translations). We can define ${}_{\phi'_{1/p}}\mathsf{T}_{\phi_{1/p}}$ as the space of morphisms in $\mathsf{B}^Q/(\mathfrak{t}_F^*)$ generated by paths, polynomials, and u_α and the extended affine Weyl group of G which begin in one coset and end in the other.

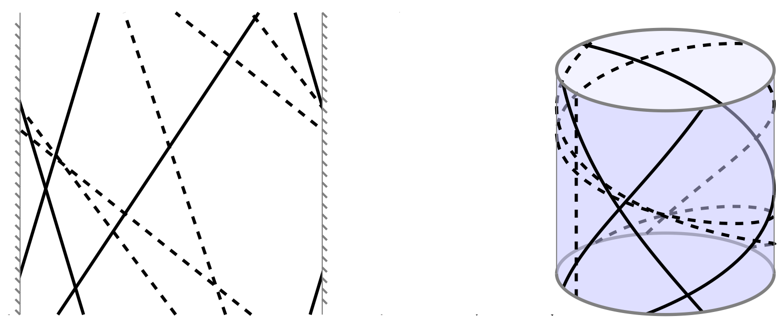
In the quiver case, we can also recover the twisting bimodules ${}_{\phi+\nu}\mathscr{T}_{\phi}$ and ${}_{\phi'_{1/p}}\mathsf{T}_{\phi_{1/p}}$ through the formalism discussed in this section. When we consider the matter hyperplanes for Q, we change the equations (6.1b–6.1c) to the hyperplanes

$$z_{j,k} - z_{i,m} + \vartheta_e = n - \frac{1}{2} \qquad \text{for all edges } e\colon i \to j, \text{ for all } k \in [1, v_j], m \in [1, v_i], n \in \mathbb{Z}$$
$$z_{i,m} + \vartheta_{i,k} = n - \frac{1}{2} \qquad \text{for all } i \in \mathcal{V}(\Gamma), m \in [1, v_i], k \in [1, w_i], n \in \mathbb{Z}$$

where $\vartheta_e, \vartheta_{i,k}$ are the weights of Q on the corresponding edge e or basis vector in $\mathbb{C}^{w_i}$. Since each element can be represented as a path in $\mathfrak{t}_Q$, the functions ϑ_e and $\vartheta_{i,k}$ will vary with t. The fact that we have fixed ν means that for elements of ${}_{\phi+\nu}\mathscr{T}_{\phi}$, we have that $\vartheta_e(0) = \vartheta_{i,k}(0) = 0$ and $\vartheta_e(1), \vartheta_{i,k}(1) \in \mathbb{Z}$ are the weights by which ν acts on the corresponding edge or basis vector in W_i.

If we incorporate this change into unrolled and antipodal diagrams corresponding to morphisms in $\mathscr{B}^Q$, then we must "subdivide" the ghosts we drew before into contributions from individual edges and basis vectors in $\mathbb{C}^{w_i}$ and make the distance between strands and ghosts the non-constant functions depending on $\vartheta_e(t)$ or $\vartheta_{i,k}(t)$. We refer to such diagrams as **ν-twisted.** More precisely, fix smooth functions $\vartheta_e\colon [0,1] \to \mathbb{R}, \vartheta_{i,k}\colon [0,1] \to \mathbb{R}$ for each edge e and $k \in [1, w_i]$ with $\vartheta_e(1) - \vartheta_e(0)$ and $\vartheta_{i,k}(1) - \vartheta_{i,k}(0)$ corresponding to ν as above.

Definition 6.13 *A* **ν-twisted unrolled diagram** *for ϑ_* is a collection of curves as before, but now given a strand with label i, for each edge $i \to j$, we draw a "ghost" curve at $\{(\bar{\pi}(t) + n - \vartheta_e(t), t) \mid t \in [0,1]\}$ for $n \in \mathbb{Z} + \frac{1}{2}$ and for each $k \in [1, w_i]$ we draw one at at $\{(n - \vartheta_{i,k}(t), t) \mid t \in [0,1]\}$, both of which we still draw as dashed. The genericity conditions are unchanged.*



We can still describe the relations of $\mathscr{B}^Q$ graphically as before, but several caveats are necessary; the first is the labelling of ghosts. When looking at a ghost with label $i;n$, this means that at $t=0$, its x-value is n. Accordingly, its x-value at $t=1$ is $n-\vartheta_*(1)$, where $*$ is corresponding edge or basis vector. In our conventions, b_e and $c_{i,k}$ are the parameters corresponding to ϕ, and $b'_e = b_e + \vartheta_e, c'_{i,k} = c_{i,k} + \vartheta_{i,k}$ are the parameters for ϕ'.

The relations (6.4a–6.4e, 6.4l–6.4n) should be applied as written. However, if we consider the relations (2.6c) and (2.6k) for $\mathscr{B}^Q$, we must make a small modification of (6.4h–6.4k) and (6.4o–6.4p) along the lines of (6.8a–6.8d): when we apply these equations with a ghost that only represents one edge $e\colon i \to j$, we replace the polynomials q_{ij} in these equations with the single factor $q_{ij}^{(e)}(u) = -u + b_e + \frac{h}{2}$ in the product (6.4g).

Note that these relations also be used to relate diagrams with different choices of ϑ_*, as long as $\vartheta_*(1)$ stays fixed. It's not hard to show that a diagram for arbitrary $\vartheta_*(t)$ can be written as sum of diagrams with ϑ_* linear (i.e. $\vartheta_*(t) = t\vartheta_*(1)$), but it can be useful to allow other choices.

There may seem to be peculiar privileging of ϕ over ϕ' in these conventions, but in fact we have done this twice in ways that cancel. If we instead decided to use $\vartheta'_*(t) = \vartheta_*(t) - \vartheta_*(1)$ and that we would use the parameters $b'_e, c'_{i,k}$, then we would leave the position of ghosts unchanged but change their labels to match the x-value at $t=1$ instead of $t=0$, which cancels with the change of parameters.

It is possible to do this same operation with antipodal diagrams, but it is a bit complicated, since in order to stay consistent with (6.7a), we have to actually change the parameters b_* and c_* as the distance between strands and ghosts change.

It is simpler to apply the same principle to weighted antipodal diagrams:

Definition 6.14 *A* ***ν-twisted weighted antipodal diagram*** *for ϑ_* is a collection of curves as before, but now given a strand with label i, for each edge $i \to j$, we draw a "ghost" curve at*

$$\left\{\left(\bar{\pi}(t) - \frac{1}{2p} - \frac{b_e + \vartheta_e(t)}{p}, t\right) \mid t \in [0,1]\right\}$$

and for each $k \in [1, w_i]$ we draw one at at

$$\left\{\Big(-\frac{1}{2p}-\frac{c_{i,k}+\vartheta_{i,k}(t)}{p},t\Big) \mid t\in[0,1]\}\right\},$$

both of which we still draw as dashed. The genericity conditions are unchanged.

No change is needed in the local relations for the weighted antipodal KLR category. Applying Propositions 2.18 and 6.7 to the theory with gauge group Q, we see that:

Lemma 6.15 *The equivalence of Proposition 6.7 sends ${}_{\phi+\nu}\mathscr{T}_{\phi}$ to the bimodule over the antipodal KLR category spanned by ν-twisted antipodal KLR diagrams for all modulo the local relations induced by (6.4a–6.4p) with (6.4h–6.4k) and (6.4o–6.4p) modified to account to separation of ghosts, in particular (6.7a) and the appropriate modification of (6.7b–6.7e).*

Applied with pth root conventions, we obtain an isomorphism of ${}_{\phi'_{1/p}}\mathsf{T}_{\phi_{1/p}}$ to the space of ν-twisted antipodal KLR diagrams modulo the relations (6.4a–6.4e) and (6.4l–6.4n) with $n = 0$ and the relations (6.8a–6.8g).

In both cases, the left and right actions of $\mathscr{B}_{\phi'}$ and $\mathscr{B}_{\phi}$ or $\mathsf{B}_{\phi'_{1/p}}$ and $\mathsf{B}_{\phi_{1/p}}$ correspond to attaching diagrams at the top or bottom respectively, in the usual style of KLR algebras.

We should emphasize that in the definition above, we are not fixing ϑ_* other than having fixed $\vartheta_*(1)$. In particular, the action of morphisms in the categories $\mathscr{B}_{\phi'}, \mathscr{B}_{\phi}, \mathsf{B}_{\phi'_{1/p}}, \mathsf{B}_{\phi_{1/p}}$ will change the functions ϑ_*.

7. Slodowy slices in type A

One particularly interesting special case of the constructions we have discussed are the **S3 varieties for $\mathfrak{sl}_n$**. These are resolutions of the intersections of Slodowy slices and nilpotent orbits. Every one of these varieties can be written as a Nakajima quiver variety and as an affine Grassmannian slice (both in type A). That is, they have a realization both as Higgs and as Coulomb branches of quiver gauge theories.

Let us remind the reader of the combinatorics underlying this realization. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ of N with n parts, we can consider λ as a (co)weight of $\mathfrak{sl}_n$, in the usual way. Given μ, another partition of N, we let

$$w_i = \lambda_i - \lambda_{i+1} \qquad v_i = \sum_{k=1}^{i} \lambda_k - \mu_k.$$

The significance of these are more easily seen from the familiar formulae

$$\lambda = \sum_{i=1}^{n} w_i\omega_i \qquad \mu = \lambda - \sum_{i=1}^{n} v_i\alpha_i.$$

Theorem 7.1 ([MV07, Th. 1.2], [BFN, Th. 5.6]) *The S3 variety $\mathfrak{X}^\lambda_\mu$ given by the slice to nilpotent matrices of Jordan type μ in the closure of those of Jordan type λ is isomorphic to the affine Grassmannian slice to Gr^μ inside $\mathrm{Gr}^{\bar{\lambda}}$, that is, to the Coulomb branch $\mathfrak{M}$ of the quiver gauge theory with dimension vectors $\mathbf{w}$ and $\mathbf{v}$. The resolution $\tilde{\mathfrak{M}}$ attached to a cocharacter $\xi\colon \mathbb{C}^* \to G$ is isomorphic to the convolution resolution of $\mathrm{Gr}^{\bar{\lambda}}_\mu$.*

On the other hand, the Higgs branch of this same gauge theory is the slice to nilpotent matrices of Jordan type λ^t inside the closure of those of Jordan type μ^t.

Thus, we find that these S3 varieties carry a tilting generator with a particularly nice structure:

Theorem 7.2 *For each choice of positions $\boldsymbol{\theta}$ of red strands, and labeling of red strands with fundamental weights, with ω_i appearing w_i times, the variety $\mathfrak{X}^\lambda_\mu$ carries a tilting generator whose endomorphism algebra is a cylindrical KLR algebra of type A with this positioning of red strands, and v_i black strands with label i.*

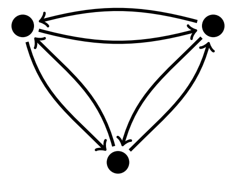
The action of wall-crossing functors on coherent sheaves matches the action on modules over this KLR algebra of derived tensor product with the bimodules $\mathring{\mathbb{B}}_s$.

In future work, we will study the structure of this algebra using the techniques of representation theory; in particular, using the techniques of web bimodules [MW18], we can define an action of $\widehat{\mathfrak{sl}}_\ell$ matching that discussed by Cautis and Koppensteiner [CK]. The wall-crossing functors are the action of Rickard complexes in this category, which leads us both to an explicit calculation of the decategorified action of wall-crossing (and thus a hands-on proof of Conjecture 4.17 in this case) and insight on the exotic t-structure Koszul dual to that induced by our tilting generator, recovering the work of Anno and Nandakumar [Ann, AN] on 2-row Springer fibers.

7.1. **Kleinian singularities.** The simplest special case is the Kleinian singularity $\mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z})$. This is isomorphic to the slice to the subregular orbit of $\mathfrak{sl}_\ell$ in the full nilcone, i.e. Jordan types $\lambda = (\ell, 0)$ and $\mu = (\ell-1, 1)$. Thus, this corresponds to the case where $w_1 = \ell$ and $v_1 = 1$. That is, we have ℓ red strands with the same label and a single black strand.

We have ℓ different idempotents depending on the position of the black strand, which we think of as positioned cyclically on a circle. The algebra of endomorphisms is generated by these idempotents, and by the degree 1 maps joining adjacent chambers by crossing the red strand:

Thus, this algebra can be written as a quotient of the path algebra of the quiver with ℓ cyclically ordered nodes and edges joining adjacent pairs of edges in both directions. The only relations needed are that the two length two paths starting and ending at a given node are equal: they are both multiplication by a single dot on the single black strand by (6.8a). Example 6.i covers the $n = 2$ case. In the $n = 3$ case, we have the quiver shown below, with the diagrams above corresponding to a single pair of edges (with the others coming from rotations of these diagrams).



7.2. **2-row Slodowy slices.** Another case which has attracted considerable attention is that of 2-row Slodowy slices. That is, for $k \leq \ell/2$, we consider the case $\lambda = (\ell, 0)$ and $\mu = (\ell - k, k)$. Thus, we have $w_1 = \ell, v_1 = k$. The result is a cylindrical version of the algebras $\tilde{T}^\ell_k$ defined in [Web16, Def. 2.3].

In addition to the wall-crossing functors, we can construct cylindrical versions of the cup and cap functors of [Web16, Def. 2.3]. These give an action of affine tangles on the categories of coherent sheaves on these varieties with $\ell - 2k$ held constant; we'll show in future work that this agrees with that of Anno and Nandakumar [AN].

7.3. **Cotangent bundles to projective spaces.** The example of $T^*\mathbb{P}^n$ corresponds to thinking of this as the S3 variety for the minimal orbit in type A, that is, for the Jordan types $\lambda = (2, 1, \dots, 1, 0)$ and $\mu = (1, \dots, 1)$. This corresponds to the quiver gauge theory attached to a linear quiver $n - 1$ nodes, and vectors $\mathbf{w} = (1, 0, \cdots, 0, 1)$ and $\mathbf{v} = (1, \dots, 1)$. One can easily check that the associated representation is that of $G = D \cap SL_n$, the diagonal matrices of determinant 1 acting on $V = \mathbb{C}^n$.

The lattice $\mathfrak{t}_\Bbbk$ is thus just the elements of $\mathbb{Z}^n$ whose entries sum to 0. We have an isomorphism between the Coulomb branch and the functions on this variety sending

$$\phi_i \mapsto x_i \frac{\partial}{\partial x_i} \qquad r_\nu \mapsto \prod x_i^{\max(\nu_i,0)} \Big(\frac{\partial}{\partial x_i}\Big)^{\max(-\nu_i,0)}.$$

Thus, the corresponding cylindrical KLR algebra has a single black strand labeled by each of $1, \dots, n-1$, the nodes of the quiver, and red strands labeled by 1 and $n-1$. All idempotents are isomorphic to one of the form $e_{n-k-1} = (1, 1, 2, 3, \dots, k-1, n-1, n-1, n-2, \dots, k)$ for $k = 1, 1, \dots, n$.

The choice of ϕ is just an assignment of ϕ_1 to the red strand with label 1 and ϕ_2 to the red strand with label $n - 1$. Given this choice of ϕ, each vector $\mu = (\mu_1, \dots, \mu_{n-1}) \in \Bbbk^{n-1}$ gives an associated line bundle $\mathcal{Q}_\mu$.

The different line bundles correspond to different chambers; these are easy to visualize if we change to the coordinates

$$(\nu_1 = \mu_1 - \phi_1, \nu_2 = \mu_2 - \mu_1, \dots, \nu_{n-1} = \mu_{n-1} - \mu_{n-2}).$$

In these coordinates, the hyperplanes separating chambers are given by $\nu_i = 0$ and

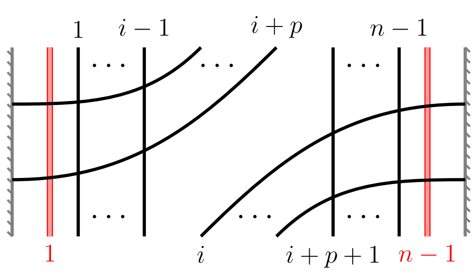
$$\mu_{n-1} - \phi_1 = \sum_{i=1}^{n-1} \nu_i = \phi_2 - \phi_1.$$

Thus, we can picture an $n-1$ dimensional cube, sliced by p hyperplanes separating it into chambers. If we choose the lifts of ν_i and $\phi_2 - \phi_1$ in $\{0, \dots, p-1\}$, then e_1 corresponds to elements with sum $< \phi_2 - \phi_1$, e_2 to elements with $\phi_2 - \phi_1 \leq \sum_{i=1}^{n-1} \nu_i < \phi_2 - \phi_1 + p$, and so on.

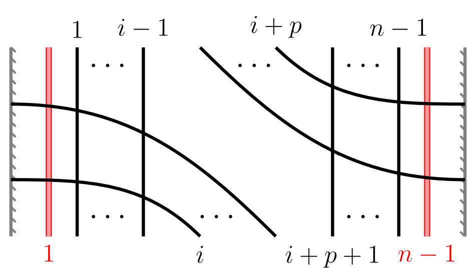
Consider $M_k = \oplus_m e_1({}_mT_0)e_k$; the elements of this are diagrams with e_k at the bottom, e_1 at the top, with the red line with label $n-1$ staying in place, and that with label 1 wrapping m times in the clockwise direction. We consider this as a graded module over the homogeneous coordinate ring, identified with $\oplus_m e_1({}_mT_0)e_1$.

This isomorphism with the coordinate ring of $T^*\mathbb{P}^{n-1}$ sends

- The diagram in ${}_0T_0$ wrapping the strands with label $i, i+1, \dots, i+p$ clockwise to $x_i \frac{\partial}{\partial x_{i+p}}$.
- That wrapping the same strands counterclockwise with $x_{i+p}\frac{\partial}{\partial x_i}$.
- The dot on the ith strand with $\sum_{m=1}^{i} x_m \frac{\partial}{\partial x_m}$.
- The diagram $e_{11}T_0e_1$ where the strands with labels $1, \dots, p-1$ stay straight (and thus cross the red strand with label $n-1$) and the strands with labels $p, \dots, n-1$ wrap around clockwise (and thus cross the red strand with label 1) correspond to the sections x_p of $\mathcal{O}(1)$.



$x_i \frac{\partial}{\partial x_{i+p}}$



$x_{i+p} \frac{\partial}{\partial x_i}$

Under this isomorphism, the module M_{k+1} corresponds to $\mathcal{O}(k)$. Thus, we have that the resulting tilting generator is of the form

$$\mathcal{Q}_\mu \cong \bigoplus_{k=0}^{n-1} \mathcal{O}(k)^{\oplus a_k},$$

where a_k is the number of elements of $\mathfrak{t}_{\mathbb{F}_p}$ in the corresponding chamber; it is well-known that this gives us a tilting generator if and only if all $a_k > 0$.

7.4. **The noncommutative Springer resolution.** One final variety of considerable interest that appears here is the cotangent bundle to the type A flag variety T^*GL_n/B. This arises from the dimension vectors $\mathbf{v} = (1, 2, 3, \dots, n-1)$ and $\mathbf{w} = (0, \cdots, 0, n)$. In this case, $\mu = \lambda^t = (1, \dots, 1)$ and $\lambda = \mu^t = (n)$. It is a well-known theorem of Nakajima that the Higgs branch of this theory is T^*G/B, and the equality $\mu = \lambda^t$ shows that this example is self-dual, and the Coulomb branch arises the same way.

It's also well-known that the quantum Coulomb branch that arises this way is essentially the universal enveloping algebra of $\mathfrak{gl}_n$; if we fix the flavors to numerical values, then this is the quotient of this ring by a maximal ideal of its center, but keeping the flavors as variables, it's easy to construct $U(\mathfrak{gl}_n)$ on the nose.

The construction of a tilting generator in Section 3 is thus just a rephrasing of the noncommutative Springer resolution as constructed by Bezrukavnikov, Mirković, Rumynin and Riche [BMRR08, Bez06]; the integrable system we consider is precisely the Gelfand-Tsetlin system as discussed in [WWY]. Thus, while we obtain a familiar object, we obtain a new perspective on it, since this KLR presentation is not at all obvious from the Lie theoretic perspective. Developing its consequences will have to wait for future work.

Glossary

Γ	The quiver used to define the quiver gauge theory as in (1.1)	2, 11, 16, 23
$\mathcal{V}(\Gamma)$	The vertex set of Γ	2–4, 8, 9, 11–13, 16, 17
$\mathbf{v}$	A dimension vector with components v_i for Γ that gives the ranks of the factors of the gauge group G	2, 3, 9, 11, 12, 15–17, 19–21, 23
$\mathbf{w}$	A dimension vector with components w_i for Γ that gives the ranks of the factors of the flavor group F	2, 3, 6, 12, 16, 17, 19–21, 23
G	The gauge group	2, 3, 6, 11, 15, 17, 23, 24
V	The matter representation	2, 11, 24
$\mathfrak{M}$	The Coulomb branch—the quotient of the convolution algebra of a modified affine Grassmannian as defined in [BFN18]	2, 20, 23, 24
$\tilde{\mathfrak{M}}$	The resolution of the Coulomb branch $\mathfrak{M}$ defined by taking symplectic redution with a GIT quotient of $\mathfrak{M}_Q$	2, 20
H	The normalizer $N^\circ_{GL(V)}(G)$	3, 24
δ	A parameter between the open interval $(0, 1) \subset \mathbb{R}$ used in the definition of $\mathscr{B}$	3

pth root	The conventions for the extended category adopted in Definition 2.12	3, 12, 14, 17, 24
$\mathfrak{t}_{*,\dagger}$	The Lie algebra of the torus T_* over $\dagger = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	3, 24
S_*	The symmetric algebra on $\tilde{\mathfrak{t}}^*$, that is, the ring of functions on the affine variety $\tilde{\mathfrak{t}}$, with the parameter h specialized at $h = *$	4
$\mathscr{B}$	The extended BFN category, defined in Definition 2.4	4, 5, 8, 11, 13, 16, 19, 23, 24
ϕ	The flavor: a fixed cocharacter $\phi\colon \mathbb{C}^* \to T_{\tilde{F}}$	5, 8
B	The extended BFN category with pth root conventions	13, 14, 16, 17, 19
T_*	A maximal torus of the group $*$	16, 24
F	The flavor group H/G	16, 23, 24
${}_{\phi+\nu}\mathscr{T}_{\phi}$	The $\mathscr{B}_{\phi+\nu}$ - $\mathscr{B}_{\phi}$ bimodule formed by the appropriate quotient of $\mathscr{T}(\nu)$, the morphisms of weight ν in $\mathscr{B}^Q$ as a bimodule over $\mathscr{B}^{\mathrm{def}}$	16, 17, 19
$\mathscr{B}^Q$	The extended BFN category attached to the pair (Q, V), defined in Definition 2.4	16
Q	The preimage in the normalizer H of a fixed torus T_F in the flavor group F	16, 24
$\mathfrak{t}_{1,\dagger}$	The elements of the Lie algebra $\mathfrak{t}_{\tilde{H},\dagger}$ which map to $d\phi$ under the map $\mathfrak{t}_{\tilde{H},\dagger} \to \mathfrak{t}_{\tilde{F},\dagger}$	17
$\mathfrak{M}_Q$	The Coulomb branch $\mathfrak{M}$ attached to the group Q acting on V with its usual action	23
$\mathscr{B}^{\mathrm{def}}$	The deformed extended BFN category, defined in [Webc, Def. 3.14]	24
$\tilde{H}$	The product $H \times \mathbb{C}^*$, acting on T^*V with the second factor acting by the cocharacter τ	24
$\tilde{F}$	The quotient $\tilde{H}/G$	24
τ	The cocharacter $\tau\colon \mathbb{C}^* \to GL(T^*V)$ that acts trivially on V and weight -1 on V^*	24

References

[AN] Rina Anno and Vinoth Nandakumar, *Exotic t-structures for two-block springer fibers*, `arXiv:1602.00768`.

[Ann] R. Anno, *Affine tangles and irreducible exotic sheaves*, `arXiv:0802.1070`.

[Bez06] R. Bezrukavnikov, *Noncommutative counterparts of the Springer resolution*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 1119–1144.

[BFN] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima, *Line bundles over coulomb branches*, `arXiv:1805.11826`.

[BFN18] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima, *Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II*, Adv. Theor. Math. Phys. **22** (2018), no. 5, 1071–1147. MR 3952347

[BK08] R. Bezrukavnikov and D. Kaledin, *Fedosov quantization in positive characteristic*, J. Amer. Math. Soc. **21** (2008), no. 2, 409–438. MR 2373355

[BMRR08] Roman Bezrukavnikov, Ivan Mirković, Dmitriy Rumynin, and Simon Riche, *Localization of modules for a semisimple lie algebra in prime characteristic*, Annals of Mathematics (2008), 945–991.

[CK] Sabin Cautis and Clemens Koppensteiner, *Exotic t-structures and actions of quantum affine algebras*, `arXiv:1611.02777`.

[Kal08] Dmitry Kaledin, *Derived equivalences by quantization*, Geom. Funct. Anal. **17** (2008), no. 6, 1968–2004.

[KL11] Mikhail Khovanov and Aaron D. Lauda, *A diagrammatic approach to categorification of quantum groups II*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2685–2700. MR 2763732 (2012a:17021)

[KTW$^+$] Joel Kamnitzer, Peter Tingley, Ben Webster, Alex Weekes, and Oded Yacobi, *On category $\mathcal{O}$ for affine Grassmannian slices and categorified tensor products*, `arXiv:1806.07519`.

[MV07] I. Mirković and M. Vybornov, *Quiver varieties and Beilinson-Drinfeld Grassmannians of type A*, 2007, `arXiv:0712.4160`.

[MW18] Marco Mackaay and Ben Webster, *Categorified skew Howe duality and comparison of knot homologies*, 2018, pp. 876–945.

[NZ] Vinoth Nandakumar and Gufang Zhao, *Categorification via blocks of modular representations for $\mathfrak{sl}(n)$*, `arXiv:1612.06941`.

[Weba] Ben Webster, *Coherent sheaves and quantum Coulomb branches I: tilting bundles from integrable systems*, `arXiv:1905.04623`.

[Webb] ———, *Gelfand-Tsetlin modules in the Coulomb context*, `arXiv:1904.05415`.

[Webc] ———, *Koszul duality between Higgs and Coulomb categories $\mathcal{O}$*, `arXiv:1611.06541`.

[Web16] ———, *Tensor product algebras, Grassmannians and Khovanov homology*, Physics and mathematics of link homology, Contemp. Math., vol. 680, Amer. Math. Soc., Providence, RI, 2016, pp. 23–58. MR 3591642

[Web17a] ———, *Knot invariants and higher representation theory*, Mem. Amer. Math. Soc. **250** (2017), no. 1191, 141.

[Web17b] ———, *Rouquier's conjecture and diagrammatic algebra*, Forum Math. Sigma **5** (2017), e27, 71. MR 3732238

[Web19a] ———, *Representation theory of the cyclotomic Cherednik algebra via the Dunkl-Opdam subalgebra*, New York J. Math. **25** (2019), 1017–1047. MR 4017214

[Web19b] ———, *Weighted Khovanov-Lauda-Rouquier algebras*, Doc. Math. **24** (2019), 209–250. MR 3946709

[WWY] Ben Webster, Alex Weekes, and Oded Yacobi, *A quantum Mirković–Vybornov isomorphism*, `arXiv:1706.03841`.