Coherent sheaves and quantum Coulomb branches I: tilting bundles from integrable systems

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Abstract. In this paper, we consider how the approach of Bezrukavnikov and Kaledin to understanding the categories of coherent sheaves on symplectic resolutions can be applied to the Coulomb branches introduced by Braverman, Finkelberg and Nakajima. In particular, we construct tilting generators on resolved Coulomb branches, and give explicit quiver presentations of categories of coherent sheaves on these varieties, with the wall-crossing functors described by natural bimodules.

1. Introduction

Let V be a complex vector space, and let G be a connected reductive algebraic group with a fixed faithful linear action on V. Attached to this data, we have a symplectic variety $\mathfrak{M}$ called the **Coulomb branch**, defined by Braverman, Finkelberg and Nakajima [BFN18], based on proposals in the physics literature. Many interesting varieties appear this way, including quiver varieties in finite and affine type A, hypertoric varieties and slices between Schubert cells in affine Grassmannians. Part of the BFN construction is the construction of a number of partial resolutions $\tilde{\mathfrak{M}}$ of $\mathfrak{M}$; we'll call one of these a **BFN resolution** if it is a resolution of singularities.

Bezrukavnikov and Kaledin have developed a general theory of quantizations of algebraic varieties in arbitrary characteristic [BK04, BK08] and Kaledin showed that this theory can be applied to construct tilting generators on symplectic resolutions of singularities [Kal08]. Kaledin's theory is very powerful, but not very concrete from the perspective of a representation theorist. In particular, this work shows that the category of coherent sheaves on a conic symplectic resolution is derived equivalent to the category of modules over an algebra A (actually to many different algebras, one for each choice of a quantization parameter), but in any particular case, this algebra is quite challenging to calculate. Our goal in this paper is to develop Kaledin's theory as explicitly as possible in the case of Coulomb branches and in particular to describe this algebra A. We will show:

Theorem A *Any BFN Coulomb branch with a BFN resolution has an explicit combinatorially presented non-commutative resolution of singularities A. The category*

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$D^b(A\text{-mod})$ *is equivalent to the derived category of coherent sheaves on any BFN resolution via an explicit tilting generator.*

For readers who prefer to live in characteristic 0 to characteristic p, we should emphasize that the construction of this non-commutative resolution A and its tilting generator have a construction which is characteristic free (that is, over $\mathbb{Z}$); however, we use reduction to characteristic p and comparison to the Bezrukavnikov-Kaledin method to confirm Theorem A.

Of course, this theorem is only of interest to a reader who knows some examples of Coulomb branches with BFN resolutions. Most interesting come from a quiver gauge theory, that is, the case which leads to Nakajima quiver varieties as Higgs branches. Following Nakajima's notation, for a quiver Γ with vertex set $\mathcal{V}(\Gamma)$, consider dimension vectors $\mathbf{v}, \mathbf{w}\colon \mathcal{V}(\Gamma) \to \mathbb{Z}_{\geq 0}$, and the group and representation

$$G = \prod GL(\mathbb{C}^{v_i}) \qquad V = \Big(\bigoplus_{i\to j} \operatorname{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j})\Big) \bigoplus \Big(\bigoplus_{i\in\mathcal{V}(\Gamma)} \operatorname{Hom}(\mathbb{C}^{w_i}, \mathbb{C}^{v_i})\Big), \tag{1.1}$$

with the obvious induced action. In this case, the algebra A is a version of a KLR algebra drawn on a cylinder, as we will show in the second part of this paper [Weba]. Note that:

1. When the underlying quiver is of type A, then the resulting Coulomb branch is the Slodowy slice to one nilpotent orbit inside another in a type A nilcone. The BFN resolutions in this case are exactly those which arise from taking the preimage under a resolution of the larger orbit closure by $T^*(SL_n/P)$ for a parabolic P.
2. When the underlying quiver is of type D or E, the Coulomb branch is isomorphic to an affine Grassmannian slice, as shown in [**?**, App. B].
3. When the underlying quiver is a loop, the resulting Coulomb branch is the v-fold symmetric product of the singular surface $S = \mathbb{C}^2/(\mathbb{Z}/w\mathbb{Z})$. In particular, one of the BFN resolutions we obtain is the Hilbert scheme of v points on the crepant resolution $\tilde{S}$.
4. When the underlying graph is an n-cycle, we obtain a Nakajima quiver variety (or more generally a bow variety) for a cycle of size $w = w_i$ whose dimension vectors are related to $\mathbf{v}, \mathbf{w}$ by a version of rank-level duality [NT17], including both the results above as special cases.

The algebra A, which appears as endomorphisms of this tilting generator, can be interpreted in three very interesting ways:

1. it can be described algebraically as a finitely generated algebra constructed directly from the combinatorics of the group G and representation V.
2. it can also be described as a convolution algebra in the extended BFN category of [Webc], with adjusted flavor and $h = 0$ (we call these "pth root conventions.").

(3) it is the endomorphism algebra of a finite sum of line defects in the corresponding $\mathcal{N}=4$ supersymmetric $3d$ gauge theory.

The equivalence of these descriptions is discussed in [Webc]: the equivalence of (1) and (2) is [Webc, Th. 3.12] and of (2) and (3) is [Webc, Rem. 3.6]; the latter is a motivational statement rather than a theorem since we are not working with a precise definition of the category of line defects.

1.1. **Motivation from line defects.** Before getting bogged down in details, let us try to give a general sketch of our approach. In this section, we play a little fast and loose with the existence of certain geometric categories (not to mention aspects of quantum field theory); we promise to the reader that no such chicanery will appear in the rest of the paper.

Having fixed the vector space V and gauge group G, we consider the quotient space of $\mathbb{C}((t))$-points $\mathcal{L}=V((t))/G((t))$. In the world of derived algebraic geometry, we think of this as the loop space of the stack quotient V/G. We will consider the category of D-modules on this space.

For the utility of both readers who wish to read not-entirely-rigorous physics motivation and for those who wish to avoid it, the author will put such motivating paragraphs in "Physics Motivation" environments going forward.

Physics Motivation 1.1 *As discussed in* [DGGH, §1.1], *this category is natural to consider in this context because it should give the category of line defects in the A-twisted of a 3-dimension $\mathcal{N}=4$ supersymmetric field theory defined by the σ-model into the* 2*-shifted stacky cotangent bundle $T^*[2](V/G)$, turned into a 3-dimensional field theory using the AKSZ formalism. In particular, the local operators on a point should appear in this category as operators from the trivial line defect to itself.*

Obviously, there are many technical issues involved in doing this, and, with apologies to the reader, we will make no attempt to resolve them. We simply ask the reader to accept the existence of this category as a black box with one basic property:

(1) Given a reasonable map $p\colon \mathcal{M}\to\mathcal{L}$, we have the pushforward of the function D-module $\mathscr{O}_p=p_*\mathcal{O}_{\mathcal{M}}$. If we are given a second such map $p'\colon \mathcal{M}'\to\mathcal{L}$ then

$$\operatorname{Ext}^\bullet(\mathscr{O}_p,\mathscr{O}_{p'})=H_*^{BM}(\mathcal{M}\times_{\mathcal{L}}\mathcal{M}')$$

with composition induced by convolution.

The definition of Braverman-Finkelberg-Nakajima is that the functions on the Coulomb branch of the gauge theory of (G,V) arise when we take $\mathcal{M}=\mathcal{M}'=V[[t]]/G[[t]]$, the arc space of the quotient V/G; the natural quantization of this ring appears when we consider this same construction $\mathbb{C}^*$-equivariantly for the loop action (where t has weight 1).

Physics Motivation 1.2 *As mentioned above, the arc space should give the trivial line, so the BFN construction gives the local operators in the A-twist of the σ-model discussed, with $\mathbb{C}^*$-equivariance giving the Ω-deformed version of these operators (see* [Nak16] *and* [BBZB$^+$, §1.3] *for more details).*

However, the utility of this perspective does not stop when we have constructed the Coulomb branch; given any $\mathbb{C}^*$-action $\nu\colon \mathbb{C}^* \to \mathrm{Aut}_G(V)$ on V commuting with G, we can consider the image $\nu(t^k) \cdot V[[t]]/G[[t]]$ and the map

$$p^{(k)}\colon \nu(t^k) \cdot V[[t]]/G[[t]] \to V((t))/G((t)).$$

The BFN resolution $\tilde{\mathfrak{M}}$ constructed by Braverman-Finkelberg-Nakajima in [BFN] can be defined the property that

$$\Gamma(\tilde{\mathfrak{M}}; \mathcal{O}(k)) \cong \mathrm{Ext}^\bullet(\mathscr{O}_{p^{(m)}}, \mathscr{O}_{p^{(m+k)}})$$

with multiplication in the projective coordinate ring given by Yoneda product. Loop equivariantly, these spaces are not isomorphic, but instead give a $\mathbb{Z}$-algebra in the sense of Gordon and Stafford [GS05]; as discussed in [BPW16, §5.2], this is the quantum homogeneous coordinate ring of a quantization of $\tilde{M}$.

Thus, given a map $q\colon \mathcal{M} \to \mathcal{L}$, we can define a coherent sheaf $\mathcal{Q}_q$ on the BFN resolution. This is the construction we require for our tilting generators and noncommutative resolution.

Theorem B *There is a space $\mathcal{M}$ and map $q\colon \mathcal{M} \to \mathcal{L}$ such that $\mathcal{Q}_q$ is the tilting generator of Theorem A and*

$$A = \mathrm{Ext}^\bullet(\mathscr{O}_q, \mathscr{O}_q).$$

Physics Motivation 1.3 *The physics reader should view this as a version of S-duality for line defects: as a coherent sheaf on the Coulomb branch, $\mathcal{Q}_q$ is a B-twist line defect for the S-dual theory to our gauge theory, whereas $\mathscr{O}_q$ is an A-twist line defect in the original theory.*

We should note that there is not just one such space, but in fact there are several of them, which give rise to different noncommutative resolutions and different tilting generators. These different choices are related by wall-crossing functors (as defined, for example, in [Los, §2.5.1]). These functors also have a geometric realization:

Theorem C *The derived equivalence of coherent sheaves to A-modules intertwines wall-crossing functors with tensor product with the bimodules $\mathrm{Ext}^\bullet(\mathscr{O}_q, \mathscr{O}_{q'})$ for different spaces q, q' both giving resolutions. These actions define a Schober in the sense of* [KS]*, that is, a perverse sheaf of categories, on a particular subtorus arrangement in a complex torus.*

Physics Motivation 1.4 *All of these other D-modules can be interpreted naturally in physics in terms of natural modifications of the fields of the theory near the line defect. We leave a more detailed discussion of this point to future work and the interested reader.*

Unfortunately, the author knows no good physics explanation of which combinations of line operators give non-commutative resolutions, and which do not. Obviously, this would be an interesting question from a quantum field theory perspective.

1.2. **Summary of approach.** Proving these results depends on comparison with the characteristic p approach of Bezrukavnikov and Kaledin [BK08, Kal08]. That is, we come to understand by considering quantizations in characteristic p and applying the approach of [Webc] (based in turn on [DFO94, MV98]) in positive characteristic. The focus is on the action of a large polynomial subalgebra of the quantum Coulomb branch, and analyzing the representations of this algebra in terms of their weights for this subalgebra. In the perspective of Stadnik [Sta13] to resolving this problem for hypertoric varieties, this polynomial subalgebra played a key role in constructing the requisite étale cover where the Azumaya algebra constructed from a quantization splits.

This approach extends to the case of a general BFN Coulomb branch. Whenever we have a BFN resolution $\tilde{\mathfrak{M}}$, we obtain an explicit tilting generator for $\Bbbk$ either of large positive characteristic or characteristic 0, which has a natural description in terms of the BFN construction. The sections of its summands (and their twists by ample line bundles) are given by the homology of spaces on which the convolution realization of the BFN algebra acts. This is an extension of work of McBreen and the author [MW], which shows the same result in the abelian case.

As mentioned above, we'll cover the case of quiver gauge theories in considerably greater detail in a companion paper [Weba]. Beyond this, there are several interesting possibilities for extension of this work. The work of McBreen and author in the abelian case [MW] can be used to show one version of homological mirror symmetry for multiplicative hypertoric varieties, and it would be very interesting to relate the presentations of A appearing here with the Fukaya category of multiplicative Coulomb branches (i.e. the algebraic varieties obtained by the K-theoretic BFN construction).

The tilting bundles that appear also have a natural interpretation in terms of line operators in the corresponding 3-dimensional gauge theory, and one could hope that other perspectives on these line operators, such as the vertex algebra perspective suggested in Costello, Creuzig and Gaiotto [CCG19], will also see the same combinatorial constructions appear, hopefully eventually leading to a theory of S-duality where coherent sheaves on Coulomb branches can be described as a natural object on the Higgs side as well.

2. Quantum Coulomb branches

2.1. **Background.** Let us recall the construction of quantum Coulomb branches from [Webc]. As before, let G be a connected reductive algebraic group over $\mathbb{C}$, with $G((t)), G[[t]]$ its points over $\mathbb{C}((t)), \mathbb{C}[[t]]$. For a fixed Borel $B \subset G$, we let Iwa be the associated Iwahori subgroup

$$\mathrm{Iwa} = \{g(t) \in G[[t]] \mid g(0) \in B\} \subset G[[t]].$$

The **affine flag variety** $\mathcal{F} = G((t))/\mathrm{Iwa}$ is just the quotient by this Iwahori.

Let V be the G-representation fixed in the previous section, $H = N_{GL(V)}(G)$ be the normalizer of G in $GL(V)$, and let $F = H/G$ be the flavor quotient and T_H, T_F be compatible maximal tori of these groups. Of course, we use $\mathfrak{t}_H$, etc. for the Lie algebra of this torus, $\mathfrak{t}_{H,S}$ for the subset where integral weights have values in a subring $S \subset \mathbb{C}$, with the most important cases being $S = \mathbb{R}$ and $S = \mathbb{Z}$. It's also useful to consider Q, the preimage of T_F in H and $\tilde{Q} = Q \times \mathbb{C}^*$. This latter groups acts on $V((t))$ such that vt^a has weight a under the second factor and the obvious action of Q.

Fix a flavor $\phi\colon \mathbb{C}^* \to T_F$, and let

$$\tilde{G} = \{(g, s) \in Q \times \mathbb{C}^* \mid \phi(s) = g \pmod{G}\} \qquad \nu(g, s) = s,$$

with its induced action on $V((t))$. That is, $\tilde{G}$ is the pullback of the diagram $\mathbb{C}^* \to T_F \leftarrow Q$. Let $\tilde{T}$ be the induced torus of this group, and $\tilde{\mathfrak{t}}$ its Lie algebra.

Fix a subspace $U \subset V((t))$ invariant under Iwa. Let $\mathcal{X}_U := (G((t)) \times U)/\mathrm{Iwa}$. Note that we have a natural $G((t))$-equivariant projection map $\mathcal{X}_U \to V((t))$. Let $\widetilde{G((t))}$ be the subgroup of $H((t)) \times \mathbb{C}^*$ generated by $G((t))$ and the image of $\tilde{G} \hookrightarrow \tilde{G} \rtimes \mathbb{C}^*$ included via the identity times ν.

Definition 2.1 *The BFN Steinberg algebra $\mathcal{A}$ is the equivariant Borel-Moore homology group*

$$\mathcal{A} = H_*^{BM,\widetilde{G((t))}}(\mathcal{X}_{V[[t]]} \times_{V((t))} \mathcal{X}_{V[[t]]}; \Bbbk)$$

endowed with the convolution multiplication.

As discussed in Section 1.1, this algebra is intended to match an Ext algebra in the category of D-modules on $\mathcal{L}$. We can avoid any technicalities about the nature of this category by considering this homology space instead, and interpreting the equivariant homology $H_*^{BM,\widetilde{G((t))}}(\mathcal{X}_{V[[t]]} \times_{V((t))} \mathcal{X}_{V[[t]]}; \Bbbk)$ using the techniques in [BFN18, §2(ii)]. As usual, we let h be the equivariant parameter corresponding to the character ν, and $S_h = H^*(B\tilde{T}; \Bbbk) = \Bbbk[\tilde{\mathfrak{t}}]$, which is naturally a subalgebra of $\mathcal{A}$ under the identification $H_*^{BM,\widetilde{G((t))}}(\mathcal{X}_{V[[t]]}) \cong S_h$. When we specialize $h = 0, 1$, we will write S_0, S_1, etc.

The original BFN algebra $\mathcal{A}^{\mathrm{sph}}$ is defined in essentially the same way, using $\mathcal{Y}_{V[[t]]} := (G((t)) \times V[[t]])/G[[t]]$. The algebras $\mathcal{A}^{\mathrm{sph}}$ and $\mathcal{A}$ are Morita equivalent by [Webc, Lemma 3.3], with $e_{\mathrm{sph}}\mathcal{A}e_{\mathrm{sph}} = \mathcal{A}^{\mathrm{sph}}$ for an idempotent $e_{\mathrm{sph}} \in \mathcal{A}$.

Physics Motivation 2.2 *In the parlance of quantum field theory, $\mathcal{A}^{\mathrm{sph}}$ is the algebra of local operators on a trivial line defect, and $\mathcal{A}$ the algebra of local operators on the defect that comes from coupling to super quantum mechanics on the flag variety G/B; this precisely the "abelianizing" line denoted $\mathbb{V}_{\mathcal{I}}$ in* [DGGH, §7]. *The equivariant parameter h corresponds Ω-background for the circle rotating around this line in $\mathbb{R}^3$, as discussed in* [BBZB$^+$, §6].

Definition 2.3 *The* **Coulomb branch** $\mathfrak{M}$ *for (V, G) is the spectrum of the algebra $\mathcal{A}^{\mathrm{sph}}$ after specialization at $h = 0$ (at which point it becomes commutative). The* **quantum Coulomb branch** *is the specialization of this algebra at $h = 1$.*

Of course, Q still acts on V, and thus has an associated Coulomb branch $\mathfrak{M}_Q$. As discussed in [BFN18, §3] and [Webc, §3.3], this Coulomb branch has a Hamiltonian action of $K = T_F^\vee$, the Langlands dual of the dual of the torus of the flavor group F with moment map given by $\mathfrak{t}_F^* \to H_Q^*(pt)$, and $\mathfrak{M}$ is the categorical quotient of the zero-level of the moment map on $\mathfrak{M}_Q$, the Coulomb branch for Q. For a given cocharacter of T_F (considered as a character of K), we can instead take the associated GIT quotient of $\mathfrak{M}_Q$, which gives a variety $\tilde{\mathfrak{M}}$ which maps projectively to $\mathfrak{M}$. As mentioned in the introduction, if $\tilde{\mathfrak{M}} \to \mathfrak{M}$ is a resolution of singularities (or equivalently, if $\tilde{\mathfrak{M}}$ is smooth) then we call it a **BFN resolution**.

2.2. **The extended category.** The quantization of the Coulomb branch attached to (G, V) appears as an endomorphism algebra in a larger category, building on the geometric definition of this algebra by Braverman, Finkelberg and Nakajima [Nak16, BFN18]. This category is not unique; there are actually many variations on it one could choose, and it will be convenient for us to incorporate a parameter $\delta \in (0, 1) \subset \mathbb{R}$ into its definition; in [Webc], we assumed that $\delta = 1/2$, but this played no important role in the results of that paper (in fact, some results become simpler if we choose δ generic instead).

Let $\mathfrak{t}_{1,Q,\mathbb{R}} \subset \tilde{\mathfrak{t}}_{Q,\mathbb{R}}$ be the preimage of 1 under projection to $\mathbb{C} = \mathrm{Lie}(\mathbb{C}^*)$, and let $\mathfrak{t}_{1,\mathbb{R}} = \mathfrak{t}_{1,Q,\mathbb{R}} \cap \tilde{\mathfrak{t}}$, be the space of real lifts of the cocharacter ϕ.

As in [Webc], we let $\{\varphi_i\}$ be the multiset of weights of V (considered as functions on $\tilde{\mathfrak{t}}_Q$) and we let

$$\varphi_i^+ = \varphi_i \qquad \varphi_i^{\mathrm{mid}} = (1-\delta)\varphi_i^+ - \delta\varphi_i^- = \varphi_i + \delta\nu \qquad \varphi_i^- = -\varphi_i - \nu.$$

Given any $\eta \in \mathfrak{t}_{1,Q,\mathbb{R}}$, we can consider the induced action on the vector space $V((t))$.

- Let Iwa_η be the subgroup whose Lie algebra is the sum of positive weight spaces for the adjoint action of η. This only depends on the alcove in which η lies, i.e. which chamber of the arrangment given by the hyperplanes $\{\alpha(\eta) = n \mid \alpha \in \Delta, n \in \mathbb{Z}\}$ contains η; the subgroup Iwa_η is an Iwahori if η does not lie on any of these hyperplanes.

- Let $U_\eta \subset V((t))$ be the subspace of elements of weight $\geq -\delta$ under η. This subspace is closed under the action of Iwa_η. This only depends on the vector $\mathbf{a}$ such that

$$\eta \in C_{\mathbf{a}} = \{\xi \in \mathfrak{t}_{1,Q,\mathbb{R}} \mid a_i < \varphi_i^{\mathrm{mid}}(\xi) < a_i + 1 \text{ for all } i\}. \tag{2.1}$$

We call η **unexceptional** if does not lie on the unrolled matter hyperplanes $\{\varphi_i^{\mathrm{mid}}(\eta) = n \mid n \in \mathbb{Z}\}$ and **generic** if it is unexceptional and does not lie on any of the unrolled root hyperplanes $\{\alpha(\eta) = n \mid n \in \mathbb{Z}\}$. We'll call the hyperplanes generic points avoid the **unrolled hyperplane arrangment**.

For any $\eta \in \mathfrak{t}_{1,Q,\mathbb{R}}$, we can consider $\mathfrak{X}_\eta := \mathfrak{X}_{U_\eta} := G((t)) \times_{\mathrm{Iwa}_\eta} U_\eta$, the associated vector bundle. The space $\mathfrak{t}_{1,Q,\mathbb{R}}$ has a natural adjoint action of $\widehat{W} = N_{\widetilde{G((t))}}(T)/T$, and of course, $U_{w\cdot\eta} = w \cdot U_\eta$.

We let

$$_\eta\mathfrak{X}_{\eta'} = \{(g, v(t)) \in G((t)) \times U_\eta \mid g \cdot v(t) \in U_{\eta'}\} /\mathrm{Iwa}_\eta. \tag{2.2}$$

Definition 2.4 *Let the* **extended BFN category** $\mathscr{B}^Q$ *be the category whose objects are unexceptional cocharacters* $\eta \in \mathfrak{t}_{1,Q,\mathbb{R}}$*, with morphisms given by:*

$$\mathrm{Hom}(\eta, \eta') = H_*^{BM,\widetilde{G((t))}}(\mathfrak{X}_\eta \times_{V((t))} \mathfrak{X}_{\eta'}; \Bbbk) \cong H_*^{BM,\tilde{T}}\left({}_\eta\mathfrak{X}_{\eta'}; \Bbbk\right).$$

Let $\mathscr{B}$ *be the subcategory whose objects are given by* $\mathfrak{t}_{1,\mathbb{R}}$*.*

As before, this homology is defined using the techniques in [BFN18, §2(ii)].

Physics Motivation 2.5 *As discussed in Section 1.1, the objects in this category can be interpreted as D-modules on the loop space of V/G (for those inclined toward stacks) or as line defects in a σ-model (for those inclined toward quantum field theory). In the notation of* [DGGH, §4.3]*, these correspond to the Lagrangian $\mathcal{L}_0$ given by the conormal to U_η and the subgroup $\mathcal{G}_0$. The author has no especially good explanation from either of these perspectives why this is the "right" subcategory of line operators to consider when there are many others available, but it does get the job done.*

Note that by assumption, the cocharacter τ is unexceptional, but not generic and $U_\tau = V[[t]]$. Given any unexceptional point η, it has a neighborhood in the classical topology, which necessarily contains a generic point, on which $U_{\eta'} = U_\eta$. Thus we can find a generic element o of the fundamental alcove such that $U_o = V[[t]]$. In this case, we have that $\mathrm{Iwa}_\tau = G[[t]]$ and Iwa_o is the standard Iwahori so

$$\mathcal{A}^{\mathrm{sph}} = \mathrm{Hom}_{\mathscr{B}}(\tau, \tau) \qquad \mathcal{A} = \mathrm{Hom}_{\mathscr{B}}(o, o) \tag{2.3}$$

Thus, this extended category encodes the structure of $\mathcal{A}$.

Definition 2.6 *Let $\Phi(\eta, \eta')$ be the product of the terms $\varphi_i^+ - nh$ over pairs $(i, n) \in [1, d] \times \mathbb{Z}$ such that we have the inequalities*

$$\varphi_i^{\mathrm{mid}}(\eta) > n \qquad \varphi_i^{\mathrm{mid}}(\eta') < n$$

hold. Let $\Phi(\eta, \eta', \eta'')$ be the product of the terms $\varphi_i^+ - nh$ over pairs $(i, n) \in [1, d] \times \mathbb{Z}$ such that we have the inequalities

$$\varphi_i^{\mathrm{mid}}(\eta'') > n \qquad \varphi_i^{\mathrm{mid}}(\eta') < n \qquad \varphi_i^{\mathrm{mid}}(\eta) > n \tag{2.4a}$$

or the inequalities

$$\varphi_i^{\mathrm{mid}}(\eta'') < n \qquad \varphi_i^{\mathrm{mid}}(\eta') > n \qquad \varphi_i^{\mathrm{mid}}(\eta) < n. \tag{2.4b}$$

These terms correspond to the hyperplanes that a path $\eta \to \eta' \to \eta''$ must cross twice.

Recall from [Webc, Thm. 3.11] that we have:

Theorem 2.7 *The morphisms in the extended BFN category are generated by*

(1) y_w for $w \in \widehat{W}$, the graph of a lift of w:

$$y_w = [\{(w, v(t)) \mid v(t) \in U_\eta\}]/\mathrm{Iwa}_\eta;$$

(2) $r(\eta, \eta')$ for $\eta, \eta' \in \mathfrak{t}_{1,Q,\mathbb{R}}$ generic

$$r(\eta, \eta') = [\{(e, v(t)) \in T((t)) \times U_{\eta'} \mid v(t) \in U_\eta\}/T[[t]]] \in \mathrm{Hom}_{\mathrm{ab}}(\eta', \eta);$$

(3) $u_{\alpha'}(\eta) = u_{\alpha - n\delta}(\eta)$ for $\eta_\pm$ affine chambers adjacent across $\alpha'(\eta) = 0$ for $\alpha' \in \widehat{\Delta}$ an affine root (i.e. $\alpha' = \alpha - n\delta$ for some finite root α')

$$u_{\alpha'}(\eta) = [\{(gv(t), g \cdot \mathrm{Iwa}_\pm, g \cdot \mathrm{Iwa}_\mp) \in \mathfrak{X}_{\eta^\pm} \times_{V((t))} \mathfrak{X}_{\eta^\mp} \mid g \in G((t)), v(t) \in U_\eta\}];$$

(4) the polynomials in S_h.

This category has a polynomial representation where each object η is assigned to $H_*^{BM,\widetilde{G((t))}}(\mathfrak{X}_\eta) \cong S_h \cdot [\mathfrak{X}_\eta]$, and the generators above act by:

$$r(\eta, \eta') \cdot f[\mathfrak{X}_{\eta'}] = \Phi(\eta, \eta') f \cdot [\mathfrak{X}_\eta] \tag{2.5a}$$

$$u_\alpha \cdot f[\mathfrak{X}_{\eta\pm}] = \partial_\alpha(f) \cdot [\mathfrak{X}_{\eta\mp}] \tag{2.5b}$$

$$y_w \cdot f[\mathfrak{X}_\eta] = (w \cdot f)[\mathfrak{X}_{w\cdot\eta}] \tag{2.5c}$$

$$\mu \cdot f[\mathfrak{X}_\eta] = \mu f \cdot [\mathfrak{X}_\eta] \tag{2.5d}$$

The relations between these operators are given by:

$$\mu \cdot r(\eta,\eta') = r(\eta,\eta')\cdot \mu \tag{2.6a}$$

$$y_\zeta \cdot \mu \cdot y_{-\zeta} = \mu + h\langle \zeta,\mu\rangle \tag{2.6b}$$

$$r(\eta,\eta')r(\eta'',\eta''') = \delta_{\eta',\eta''}\Phi(\eta,\eta',\eta''')r(\eta,\eta''') \tag{2.6c}$$

$$y_w \cdot y_{w'} = y_{ww'} \tag{2.6d}$$

$$y_w r(\eta',\eta)y_w^{-1} = r(w\cdot\eta', w\cdot\eta) \tag{2.6e}$$

$$y_w \mu y_w^{-1} = w\cdot \mu \tag{2.6f}$$

$$u_\alpha^2 = 0 \tag{2.6g}$$

$$\underbrace{u_\alpha u_{s_\alpha\beta} u_{s_\alpha s_\beta \alpha}\cdots}_{m_{\alpha\beta}} = \underbrace{u_\beta u_{s_\beta\alpha} u_{s_\beta s_\alpha \beta}\cdots}_{m_{\alpha\beta}} \tag{2.6h}$$

$$y_w u_\alpha y_{w^{-1}} = u_{w\cdot\alpha} \tag{2.6i}$$

$$u_\alpha\mu - (s_\alpha\cdot\mu)u_\alpha = r(\eta_\mp,\eta_\pm)\partial_\alpha(\mu) \tag{2.6j}$$

whenever these morphisms are well-defined and finally, if $\eta'_\pm$ and $\eta''_\pm$ are two pairs of chambers opposite across $\alpha(\eta)=0$ on opposite sides of an intersection of affine root and flavor hyperplanes as shown below, and η,η'' differ by a 180° rotation around the corresponding codimension 2 subspace:

$$\begin{aligned} r(\eta''',\eta'_-)u_\alpha r(\eta'_+,\eta) - r(\eta''',\eta''_-)u_\alpha r(\eta''_+,\eta)& \\ = \partial_\alpha\left(\Phi(\eta'_+,\eta)\cdot s_\alpha\Phi(\eta,\eta'_-)\right) r(\eta''',s_\alpha\eta)s_\alpha.& \end{aligned} \tag{2.6k}$$

One important change in the characteristic p case is that the representation defined by (2.5a–2.5d) is no longer faithful, since the same is true of the corresponding representation of $\widehat{W}$: translations by cocharacters divisible by p act trivially.

It is possible to fix this, though it is somewhat less pleasant to think about. Fix $h = g \in \mathbb{C}$ (we will of course be primarily interested in the cases $g = 0,1$). Let K be the fraction field of S_g, and consider the induced action by convolution on $K \otimes_{S_g} H_*^{BM,T}(\mathfrak{X}_\eta^T) \cong \oplus_{\lambda\in X_*(T)} K\cdot [t^\lambda]$. We will not explicitly check that the action we define below arises from convolution due to fact the complications of localization in equivariant cohomology for loop groups, but it is worth pointing to as our source of inspiration.

Lemma 2.8 *There is a faithful action of $\mathscr{B}^Q$ that sends every object to $K_X := \oplus_{\lambda\in X_*(T)} K\cdot[t^\lambda]$ given by the formulas*

$$r(\eta,\eta')\cdot f[t^\lambda] = \Phi(\eta,\eta')f\cdot[t^\lambda] \tag{2.7a}$$

$$u_\alpha\cdot f[t^\lambda] = \frac{s_\alpha f}{\alpha}[t^{s_\alpha\lambda}] - \frac{f}{\alpha}[t^\lambda] \tag{2.7b}$$

$$y_w\cdot f[t^\lambda] = (w\cdot f)[t^{w\lambda}] \tag{2.7c}$$

$$\mu\cdot f[t^\lambda] = \mu f\cdot [t^\lambda]. \tag{2.7d}$$

Now, consider the case where $\hbar = 0$. In this case, $\mathrm{End}_{\mathscr{B}}(\tau,\tau) \cong \Bbbk[\mathfrak{M}]$ is the space of functions on the Coulomb branch. The different non-isomorphic objects of $\mathscr{B}$ define interesting modules over $\Bbbk[\mathfrak{M}]$, considering $Z_\eta = \mathrm{Hom}_{\mathscr{B}}(o,\eta)$ as a right module under composition.

Lemma 2.9 *The module Z_η, considered as a coherent sheaf on $\Bbbk[\mathfrak{M}]$, is generically free of rank $\#W$.*

Proof. As a module over $\mathrm{End}_{\mathscr{B}}(o,o)$ the module $\mathrm{End}_{\mathscr{B}}(o,\eta)$ is generically free of rank 1, since o and η become isomorphic after inverting all weights and roots. Since at $h = 0$, the algebra $\mathrm{End}_{\mathscr{B}}(o,o)$ is Azumaya over $\Bbbk[\mathfrak{M}]$ with degree $\#W$, this implies that Z_η generically has the correct rank. $\square$

One other construction we'll need to connect to wall-crossing functors is the twisting bimodules ${}_{\phi+\nu}\mathscr{T}_\phi$ and ${}_{\phi+\nu}T_\phi = {}_{\phi+\nu}\mathscr{T}_\phi(o,o)$ relating two flavors ϕ and $\phi+\nu$ that differ by $\nu \in \mathfrak{t}_{\mathbb{Z},F}$ defined in [Webc, Def. 3.16]. These are constructed much as the Hom spaces in Definition 2.4: let ${}_\eta\mathfrak{X}^{(\nu)}_{\eta'}$ be the component of the space ${}_\eta\mathfrak{X}^Q_{\eta'}$ as defined in (2.2) lying above t^ν in the affine Grassmannian of t^ν, and we have:

$$ {}_{\phi+\nu}\mathscr{T}_\phi(\eta,\eta') = H_*^{BM,\tilde{T}}\left({}_\eta\mathfrak{X}^{(\nu)}_{\eta'};\Bbbk\right). \tag{2.8}$$

2.3. **Representations.** Using this presentation, we can analyze the structure of this category of representations in characteristic p, just as we did in characteristic 0 in [Webc, §3.3]. It is worth noting that the group G, representation V and its associated objects are unchanged; we simply consider their homology over $\Bbbk$, a field of characteristic p. To save ourselves heart-burn, we assume that p is not a torsion prime for the group G. This is not a problematic restriction, since we will typically assume that $p \gg 0$. Throughout this subsection, we specialize $h = 1$.

Let M be a finite dimensional representation of the category $\mathscr{B}$ (which we will also call $\mathscr{B}$-modules), that is, a functor from $\mathscr{B}$ to the category $\Bbbk$-Vect of finite dimensional $\Bbbk$-vector spaces.

These are closely related to the theory of $\mathcal{A}$-modules since the finite dimensional vector space $N := M(o)$ has an induced $\mathcal{A}$-module structure. Furthermore, since $\mathrm{Hom}(\eta,o)$ and $\mathrm{Hom}(o,\eta)$ are finitely generated as $\mathcal{A}$-modules, this is in fact a quotient functor, with left adjoint given by

$$N \mapsto \mathscr{B} \otimes_{\mathcal{A}} N(\eta) := \mathrm{Hom}(\eta,o) \otimes_{\mathcal{A}} N.$$

Now, let us return to the theory of $\mathscr{B}$-modules. Of course, if we restrict the action on $M(\eta)$ to the subalgebra S_h, then this vector space breaks up as a sum of **weight spaces**:

$$\mathcal{W}_{\upsilon,\eta}(M) = \{m \in M(\eta) \mid \mathfrak{m}_\upsilon^N m = 0 \text{ for } N \gg 0\}, \tag{2.9}$$

We can think of this as an exact functor $\mathcal{W}_{\upsilon,\eta}\colon \mathscr{B}\text{-fdmod} \to \Bbbk\text{-Vect}$. In [Webc], we employed these weight functors to probe the category of representations of $\mathscr{B}$. Versions of this construction have appeared a number of places in the literature, including work of Musson and van der Bergh [MV98] and Drozd, Futorny and Ovsienko [DFO94].

Definition 2.10 *Let $\widehat{\mathscr{B}}$ be the category whose objects are the set $\mathfrak{J}$ of pairs of generic $\eta \in \mathfrak{t}_{\mathbb{R}} + \tau$ and any $\upsilon \in \mathfrak{t}_{1,\Bbbk}$, such that*

$$\operatorname{Hom}_{\widehat{\mathscr{B}}}((\eta', \upsilon'), (\eta, \upsilon)) = \varprojlim \operatorname{Hom}_{\mathscr{B}}(\eta', \eta)/(\mathfrak{m}_{\upsilon}^{N} \operatorname{Hom}_{\mathscr{B}}(\eta', \eta) + \operatorname{Hom}_{\mathscr{B}}(\eta', \eta)\mathfrak{m}_{\upsilon'}^{N}).$$

We can apply [Webb, Theorem B] here to get a sense of the size of this algebra: the endomorphism algebra of any object in this category is again a Galois order in a skew group algebra, but the group is now the stabilizer of η' in the affine Weyl group. Since we are now in characteristic p, this contains all translations that are p-divisible, and so this stabilizer is the semi-direct product of a parabolic subgroup in the finite Weyl group with this p-scaled group of translations.

Note that since $\Bbbk$ is of characteristic p, the set $\mathfrak{t}_{1,\Bbbk}$ has an action of $\mathfrak{t}_{\mathbb{Z}/p\mathbb{Z}}$ by addition. We let $\widehat{\mathscr{B}}_{\upsilon'}$ be the subcategory where we only allow objects with $\upsilon \in \upsilon' + \mathfrak{t}_{\mathbb{Z}/p\mathbb{Z}}$ and let $\widehat{\mathscr{A}}_{\upsilon'}$ be the subcategory with the objects of the form (o, υ) for $\upsilon \in \upsilon' + \mathfrak{t}_{\mathbb{Z}/p\mathbb{Z}}$.

This category is useful in that it lets us organize how the weight spaces of different values of η relate. For any $\mathscr{B}$-module M, the functor $(\upsilon, \eta) \mapsto W_{\upsilon,\eta}(M)$ defines a representation of $\widehat{\mathscr{B}}$. We have an analogue in this situation of [Webc, Lemma 3.22], which is a special case of a more general result of Drozd-Futorny-Ovsienko [DFO94, Th. 17]:

Lemma 2.11 *The functor above defines an equivalence of the category $\mathscr{B}\text{-mod}_{\upsilon'}$ of finite dimensional $\mathscr{B}$-modules with weights in $\widehat{W} \cdot \upsilon'$ to the category of representations of $\widehat{\mathscr{B}}_{\upsilon'}$ in $\Bbbk$-Vect.*

The analogous functor defines an equivalence of the category $\mathcal{A}\text{-mod}_{\upsilon'}$ of finite dimensional $\mathcal{A}$-modules with weights in $\widehat{W} \cdot \upsilon'$ to the category of representations of $\widehat{\mathscr{A}}_{\upsilon'}$ in $\Bbbk$-Vect.

Note that an important difference between the characteristic p and characteristic 0 cases: a module in $\mathcal{A}\text{-mod}_{\upsilon'}$ with $\Bbbk$ of characteristic p with finite dimensional weight spaces is necessarily finite dimensional (since the affine Weyl group orbit of any weight is finite) while it is typically not if $\Bbbk$ has characteristic 0. This is why here we only study finite dimensional modules, while in [Webc], we study the category of all weight modules.

2.4. **The homogeneous presentation.** We wish to give a homogeneous presentation of the categories $\widehat{\mathscr{A}}_{\upsilon'}$ and $\widehat{\mathscr{B}}_{\upsilon'}$, as in [Webc, §4]. For simplicity, we assume that $\Bbbk = \mathbb{F}_p$ for p not dividing $\#W$ and we are still specializing $h = 1$.

Recall that we have fixed $\phi \in \mathfrak{t}_{F,\mathbb{Z}}$; we can without loss of generality choose a lift $\tilde{\phi}$ which is fixed by the action of W to $\mathfrak{t}_{1,Q,\mathbb{Z}[\frac{1}{\#W}]}$ over the ring $\mathbb{Z}[\frac{1}{\#W}]$ of integers with the order of the group W inverted (note that this might not be possible over $\mathbb{Z}$). This has a unique reduction to $\mathfrak{t}_{1,\mathbb{F}_p}$, which is again W-invariant. Since $\mathfrak{t}_{1,\mathbb{F}_p}$ is a torsor for $\mathfrak{t}_{\mathbb{F}_p}$, we can assume $\upsilon' = \tilde{\phi} \pmod{p}$.

The stabilizer $\widehat{W}_{\upsilon'}$ of υ' in the affine Weyl group is generated by s_{α_i} for all i, and translations by $p\mathfrak{t}_{\mathbb{Z}}$. Note that the map

$$(\cdot)_p \colon \widehat{W} \to \widehat{W}_{\upsilon'} \qquad w_p(x) = p \cdot w\Big(\frac{1}{p}x\Big)$$

is an isomorphism between these groups. Furthermore, for reflection in an affine root α, we have that $(s_\alpha)_p = s_{\alpha^{(p)}}$ for some possibly different root $\alpha^{(p)}$.

We can also understand this homomorphism in terms of the Frobenius map $\mathsf{f}_V \colon V((t)) \to V((t))$ given by $\mathsf{f}_V(v(t)) = v(t^p)$: the element w_p is the unique one satisfying $w_p \circ \mathsf{f}_V = \mathsf{f}_V \circ w$. Similarly, we will want to understand the interaction between U_η and this map. Consider the $\tilde{G} \times \mathbb{C}^*$ action on $V((t))$ as usual (that is, with $\mathbb{C}^*$ acting by loop rotation). The map f_V intertwines the action of $G((t))$ with the action twisted by the endomorphism $\mathsf{f}_G(g(t)) = g(t^p)$.

Note, we cannot extend this automorphism to the semi-direct product incorporating the loop scaling, since we would need to act on the loop $\mathbb{C}^*$ by $s \mapsto s^{1/p}$. The corresponding automorphism on the level of Lie algebras is well-defined however, and we denote it by $\mathsf{f}_{\tilde{\mathfrak{g}}}$. Note that this does not preserve the subalgebra $\{(X, d\nu(X)) \mid X \in \tilde{\mathfrak{g}}\}$, and thus does not preserve $\widetilde{\mathfrak{g}((t))}$.

This shows that we need to have a different flavor in order to write $\mathsf{f}_V^{-1}(U_\eta)$ as the same sort of subspace.

Definition 2.12 *The* **"pth root"** *conventions for the extended category are taking:*

- *the gauge group G and representation V;*
- *the (rational) flavor $\phi_{1/p}(t) = (\phi_0(t^{1/p}), t)$ in place of ϕ;*
- *the constant δ/p in place of δ;*
- *specialize the equivariant parameter $h = 0$.*

Throughout, we will use sans-serif letters to denote objects defined in the pth root conventions. In particular, we write $\mathsf{t}_{1,\mathbb{R}}$ for $\mathfrak{t}_{1,\mathbb{R}}$, U_η for U_η. We let B for $\mathscr{B}$ when we use this modified flavor and constant.

Definition 2.13 *Given $\eta \in \mathsf{t}_{1,Q,\mathbb{R}}$, let $\eta_p \in \mathfrak{t}_{1,Q,\mathbb{R}}$ be the element $\eta_p = p \cdot \mathsf{f}_{\tilde{\mathfrak{g}}}(\eta)$. Given $\xi \in \mathfrak{t}_{1,Q,\mathbb{R}}$, let $\xi_{1/p} \in \mathsf{t}_{1,Q,\mathbb{R}}$ be the unique solution to $\xi = (\xi_{1/p})_p$, that is, the inverse map.*

Note that for $w \in \widehat{W}$, we have $(w\eta)_p = w_p\eta_p$, and that

$$\varphi_i^{\mathrm{mid}}(\eta_p) = p\varphi_i^{\mathrm{mid}}(\eta) \tag{2.10}$$

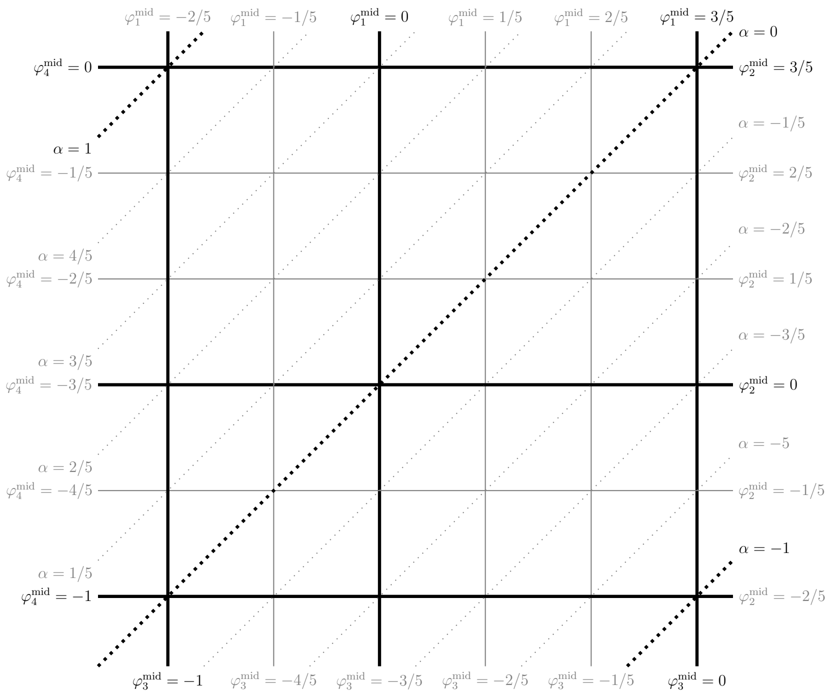


FIGURE 1. The effect of pth root conventions on matter hyperplanes

(where the second is calculated using pth root conventions).

Lemma 2.14 $\mathsf{f}_V^{-1}(U_{\eta_p}) = \mathsf{U}_\eta$.

Remark 2.15. The map $\xi \mapsto \xi_{1/p}$ has the effect of shrinking the space $\mathfrak{t}_{1,Q,\mathbb{R}}$ by a factor of $\frac{1}{p}$, and the unrolled matter hyperplanes, which are defined by $\varphi_i^{\mathrm{mid}}(\xi)$ taking an integral value, become the hyperplanes where this same function (with the pth root conventions) takes on a value in $\frac{1}{p}\mathbb{Z}$. Thus we only keep every pth one of these hyperplanes as a matter hyperplane.

Each unrolled matter hyperplane separates the locus of η such that a given weight vector in $V((t))$ lies in U_η from the locus where is does not; the hyperplanes we keep in the pth root conventions are those that correspond to vectors in $V((t^p))$.

Example 2.i. Let us consider the running example from [Webc]: the gauge group $G = GL(2)$ acting on $V = \mathbb{C}^2 \oplus \mathbb{C}^2$. The flavor group F is isomorphic to $PGL(2)$, so choosing a flavor is choosing a cocharacter into this group, fixing the difference between the weights of this cocharacter on the two copies of $\mathbb{C}^2$. Let's consider $p = 5$, and choose the flavor so that the difference is $\varphi_1^{\mathrm{mid}} - \varphi_3^{\mathrm{mid}} = 3$. In Figure 2.i, we draw the images under $\eta \mapsto \eta_{1/p}$ of all unrolled matter hyperplanes, but draw those which do not remain as matter hyperplanes for the pth root data in gray and with thinner weight. ♣

We'll want to consider the affine chambers for $\mathbf{a} \in \mathbb{Z}^d$ as in Section 2.1:

$$\mathsf{C}_{\mathbf{a}} = \{\xi \in \mathfrak{t}_{1,\mathbb{R}} \mid a_i < \varphi_i^{\mathrm{mid}}(\xi) < a_i + 1\}$$
$$\mathscr{C} = \{\mathbf{a} \in \mathbb{Z}^d \mid \mathsf{C}_{\mathbf{a}} \neq 0\}.$$

The reader should note that we use the pth root conventions here. Consider $\widehat{W}$, the extended affine Weyl group, acting on $\mathfrak{t}_{1,Q,\mathbb{R}}$ via the usual level 1 action. Note that if $w \in \widehat{W}$ and $\mathsf{C}_{\mathbf{a}} \neq 0$ then $w \cdot \mathsf{C}_{\mathbf{a}} = \mathsf{C}_{w\cdot\mathbf{a}}$ for a unique $w \cdot \mathbf{a}$, so this defines a $\widehat{W}$ action on $\mathscr{C}$.

First, we note how the polynomial changes when we complete it to match $\widehat{\mathscr{B}}_{\upsilon'}$. Given υ, we let $\widehat{S}_1^{(\upsilon)} = \varprojlim S_1/\mathfrak{m}_\upsilon^N$.

Proposition 2.16 *The category $\widehat{\mathscr{B}}_{\upsilon'}$ has a representation $\mathscr{P}$ sending $(\eta, \upsilon) \mapsto \widehat{S}_1^{(\upsilon)}$ defined by the formulas (2.5a–2.5d), and one $\mathscr{F}$ sending*

$$(\eta, \upsilon) \mapsto \bigoplus \mathrm{Frac}(\widehat{S}_1^{(\upsilon)}) \cdot [t^\lambda]$$

where Frac$(-)$ *denotes the fraction field of a commutative ring, with morphisms acting as in (2.7a–2.7d).*

Since $\widehat{S}_1^{(\upsilon)}$ is a profinite dimensional algebra, we can decompose any element of it into its semi-simple and nilpotent parts, by doing so in each quotient $S_1/\mathfrak{m}_\upsilon^N$. The grading we seek on a dense subcategory of $\widehat{\mathscr{B}}_{\upsilon'}$ is uniquely fixed a small list of requirements:

(i) For $\mu \in \mathfrak{t}^*$, the nilpotent part $\mu - \langle \mu, \upsilon\rangle$ of μ acting on $\widehat{S}_1^{(\upsilon)}$ is homogeneous of degree 2.

(ii) The action of $\widehat{W}$ is homogeneous of degree 0.

(iii) If $\eta \in \mathsf{C}_{\mathbf{a}}$ and $\eta' \in \mathsf{C}_{\mathbf{b}}$ then the obvious isomorphism $\mathscr{P}(\eta_p, \upsilon) \cong \widehat{S}_1^{(\upsilon)} \cong \mathscr{P}(\eta'_p, \upsilon)$ is homogeneous of degree $\sum_{i=1}^d a_i - b_i$.

The principles (i-iii) fix a grading on $\mathscr{P}(\eta, \upsilon)$ for all $\eta \in \mathfrak{t}_{1,Q,\mathbb{R}}$ and $\upsilon \in \hat{W} \cdot \upsilon'$ up to a global shift.

We would like to use this to fix a notion of what it means for a morphism in $\widehat{\mathscr{B}}_{\upsilon'}$ to be homogeneous, however this is slightly complicated by the fact that $\mathscr{P}$ is not faithful. However, it can still be a useful guide to the choice of an appropriate grading.

Let $\hat{\Phi}_0(\eta, \eta', \upsilon')$ be the product of the terms $\varphi_i^+ - n$ over pairs $(i, n) \in [1, d] \times \mathbb{Z}$ such that we have the inequalities

$$\varphi_i^{\mathrm{mid}}(\eta) > n \qquad \varphi_i^{\mathrm{mid}}(\eta') < n \qquad \langle \varphi_i^+, \upsilon'\rangle \not\equiv n \pmod{p}.$$

Note that these are precisely the factors in the product $\Phi(\eta, \eta')$ which remain invertible after reduction modulo p. Consider the morphisms:

$$\mathbb{w}(\eta, \eta') = \frac{1}{\hat{\Phi}_0(\eta, \eta', \upsilon')} r(\eta, \eta)$$

We'll check below that these morphisms are homogeneous, and together with a few other obvious homogeneous morphisms, they generate a dense subspace inside morphisms.

Consider the extended BFN category B with pth root conventions. Since $h = 0$, this category is graded with

$$\deg r(\eta, \eta') = \deg \Phi(\eta, \eta') + \deg \Phi(\eta', \eta) \tag{2.11a}$$

$$\deg w = 0 \qquad \deg u_\alpha(\mathbf{a}) = -2 \qquad \deg \mu = 2. \tag{2.11b}$$

Note that here $\deg \Phi(\eta, \eta')$ should be interpreted in the grading on S_0 where $\mathfrak{t}^*$ is concentrated in degree 1. We define a functor $\gamma_{\mathsf{B}} \colon \mathsf{B} \to \widehat{\mathscr{B}}_{\upsilon'}$ by sending $\eta \mapsto (\eta_p + \upsilon', \upsilon')$, and acting on morphisms by

$$\gamma_{\mathsf{B}}(r(\eta, \eta')) = \mathbb{w}(\eta_p + \upsilon', \eta'_p + \upsilon') = \frac{1}{\hat{\Phi}_0(\eta_p + \upsilon', \eta'_p + \upsilon', \upsilon')} r(\eta_p + \upsilon', \eta'_p + \upsilon') \tag{2.12a}$$

$$\gamma_{\mathsf{B}}(w) = w_p \tag{2.12b}$$

$$\gamma_{\mathsf{B}}(u_\alpha) = u_{\alpha^{(p)}} \tag{2.12c}$$

$$\gamma_{\mathsf{B}}(\mu) = \mu - \langle \mu, \upsilon' \rangle \tag{2.12d}$$

Note that since $w_p \eta_p = (w\eta)_p$ and $w_p \cdot \upsilon' = \upsilon'$, these morphisms go between the correct objects.

Remark 2.17. Just as discussed in [Webc], this isomorphism has a natural geometric interpretation as localization to the fixed points of a group action. In the characteristic zero case, we analyze the space corresponding to a weight in terms of the fixed points of the corresponding character; a version of this is explained in [Webb], generalizing work of Varagnolo-Vasserot [VV10, §2]. In characteristic p, we only obtain an isomorphism after completion with the fixed points of the p-torsion subgroup of this cocharacter. Of course, these fixed points again give versions of the spaces appearing in the extended BFN category. While this is a beautiful perspective, we think it will be clearer, especially for the reader unused to the geometry of the affine Grassmannian, to give an algebraic proof.

Proposition 2.18 *The functor $\gamma_{\mathsf{B}} \colon \mathsf{B} \to \widehat{\mathscr{B}}_{\upsilon'}$ is faithful, topologically full and essentially surjective; that is, it induces an equivalence $\widehat{\mathsf{B}} \cong \widehat{\mathscr{B}}_{\upsilon'}$*

Proof. Consider the representation $\mathscr{F}$ of $\widehat{\mathscr{B}}_{\upsilon'}$. In order to confirm the result, we must show that the images above under γ_{B} satisfy the relations of B and define a faithful representation on $\mathscr{F}$.

The formula 2.12d identifies each summand of $\mathscr{F}$ with the fraction field of the completion of $\widehat{S}_0^{(0)}$ at the origin of S_0, and thus $\mathscr{F}(\eta_p, \upsilon') \cong \bigoplus \widehat{S}_0^{(0)} \cdot [t^\lambda]$. Consider the images of the RHSs of (2.12a–2.12d) under transport of structure. We wish to show that this agrees with the representation K of B defined in Lemma 2.8.

For the morphism $\gamma_{\mathsf{B}}(\mu)$, this is automatic. For $\gamma_{\mathsf{B}}(w)$, this is an immediate consequence of the definition.

Now, consider $r(\eta,\eta')$. In its polynomial representation, this element acts by $\Phi(\eta,\eta')$, which is the product of φ_i^+ raised to number of integers n satisfying $\varphi_i^{\mathrm{mid}}(\eta) < n < \varphi_i^{\mathrm{mid}}(\eta')$. This is sent under γ_{B} to the shift $\varphi_i^+ - \langle \varphi^+, \upsilon'\rangle$, leaving the overall structure of the product the same.

Now note that by (2.10), we have that

$$\varphi_i^{\mathrm{mid}}(\eta_p+\upsilon') = p\varphi_i^{\mathrm{mid}}(\eta)+\langle\varphi^+,\upsilon'\rangle \qquad \langle\varphi^+,\eta_p+\upsilon'\rangle = p\langle\varphi^+,\eta\rangle+\langle\varphi^+,\upsilon'\rangle.$$

Thus, we have that

$$\varphi_i^{\mathrm{mid}}(\eta_p+\upsilon') > n \qquad \varphi_i^{\mathrm{mid}}(\eta'_p+\upsilon') < n \qquad \langle\varphi_i^+,\upsilon'\rangle \equiv n \pmod p \tag{2.13}$$

if and only if we have that $n_{1/p} = \frac{1}{p}(n-\langle\varphi_i^+,\upsilon'\rangle)\in\mathbb{Z}$ and

$$\varphi_i^{\mathrm{mid}}(\eta) > n_{1/p} \qquad \varphi_i^{\mathrm{mid}}(\eta') < n_{1/p} \tag{2.14}$$

Thus, we have an equality

$$\Phi(\eta,\eta') = \frac{\Phi(\eta_p-\upsilon',\eta'_p-\upsilon')}{\hat{\Phi}_0(\eta_p-\upsilon',\eta_p-\upsilon',\upsilon'')}$$

since the factors of the RHS are given by (2.13) and of the LHS by (2.14). Note the difference between Φ and Φ in the equation above, denoting the use of pth root conventions on the left hand side. Thus, indeed, in $\bigoplus \widehat{S}_0^{(0)}$, as desired we have that $\gamma_{\mathsf{B}}(r(\eta,\eta')) = \mathbb{w}(\eta_p,\eta'_p)$ acts by multiplication by $\Phi(\eta,\eta')$. Finally, $\gamma_{\mathsf{B}}(u_\alpha)$ must have the desired image because it can be written as $\frac{1}{\alpha}(s_\alpha - r(s_\alpha\eta,\eta))$ when it is well-defined and α is invertible in $\mathrm{Frac}(\widehat{S}_0^{(0)})$. This shows that we have recovered F.

Since F is a faithful representation, this shows that γ_{B} is well-defined and faithful.

Any object $(\eta, w\cdot\upsilon')$ is in the essential image of the functor, since it was isomorphic to $(w^{-1}\cdot\eta,\upsilon')$, and the map $(\cdot)_p$ is a bijection.

Finally, we need to show that the image of the functor is dense. Note that:

$$\begin{aligned}
r(\eta,\eta') &= \hat{\Phi}_0(\eta,\eta',\upsilon')\mathbb{w}(\eta,\eta') = \hat{\Phi}_0(\eta,\eta',\upsilon')\gamma_{\mathsf{B}}(r((\eta-\upsilon')_{1/p},(\eta'-\upsilon')_{1/p}))\\
w &= \gamma_{\mathsf{B}}(w_{1/p})\\
u_{\alpha^{(p)}} &= \gamma_{\mathsf{B}}(u_\alpha)\\
\mu &= \gamma_{\mathsf{B}}(\mu+\langle\mu,\upsilon'\rangle).
\end{aligned}$$

We are only left the task of showing that u_α is in the closure of the image of γ_{B} for $s_\alpha\notin\widehat{W}_{\upsilon'}$. In this case, α thought of as an element of S_1 will act invertibly, so $1/\alpha$ lies in the closure of the image. Thus, we can use the formula $\frac{1}{\alpha}(s_\alpha - r(s_\alpha\eta,\eta))$ as before. This shows the density. □

Remark 2.19. We should emphasize that the functor γ_{B} is only well-defined over a field of characteristic p, but the category B makes sense with coefficients in an arbitrary commutative ring (in particular, $\mathbb{Z}$). We'll write $\mathsf{B}(\Bbbk)$ when we wish to emphasize the choice of base field.

Definition 2.20 *We let $\mathsf{A}_p(\Bbbk)$ be the category whose objects are the elements of $\upsilon' + \mathfrak{t}_{1,\mathbb{Z}}$, with morphisms $\mathrm{Hom}_{\mathsf{A}_p(\Bbbk)}(\xi,\xi') \cong \mathrm{Hom}_{\mathsf{B}(\Bbbk)}((o-\xi)_{1/p},(o-\xi')_{1/p})$, and $\widehat{\mathsf{A}}_p(\Bbbk)$ its completion with respect to the grading.*

Note that we have broken a bit from our convention of using sans-serifizing (which would suggest that A should be the ring $\mathcal{A}_0$); the reason for this will be clearer below.

Since objects in $\mathsf{A}_p(\Bbbk)$ that differ by the level p action of $\widehat{W}$ are isomorphic, we could also consider only the elements of $-\upsilon' + \mathfrak{t}_{1,\mathbb{Z}}$ in a fundamental region for the action of this group. However, it's more convenient to describe the elements $\mathbb{w}$ in the unrolled picture.

Note that in $\widehat{\mathscr{B}}_{\upsilon'}$, the translation by $\upsilon'-\upsilon$ induces an isomorphism $(o,\upsilon) \cong (o-\upsilon+\upsilon',\upsilon')$.

Conjugating the functor γ_{B} by this isomorphism induces a functor $\gamma\colon \widehat{\mathsf{A}}_p \to \widehat{\mathscr{A}_{\upsilon'}}$ sending $\gamma(\upsilon) = (o,\upsilon)$. Proposition 2.18 immediately implies that:

Theorem 2.21 *The categories $\widehat{\mathscr{A}_{\upsilon'}}$ and $\widehat{\mathsf{A}}_p$ are equivalent via the functors*

$$\begin{tikzcd}
\widehat{\mathsf{A}}_p \arrow[r, "\upsilon \mapsto (o-\upsilon)_{1/p}"] \arrow[d, "\gamma"'] & \widehat{\mathsf{B}} \arrow[d, "\gamma_{\mathsf{B}}"] \\
\widehat{\mathscr{A}_{\upsilon'}} \arrow[r] & \widehat{\mathscr{B}}_{\upsilon'}
\end{tikzcd}$$

Note that the square commutes when applied to objects since

$$\begin{tikzcd}
\upsilon \arrow[r] \arrow[d] & (o-\upsilon)_{1/p} \arrow[d] \\
(o,\upsilon) \arrow[r] & (o,\upsilon) \cong (o-\upsilon+\upsilon',\upsilon')
\end{tikzcd}$$

2.5. **Consequences for representation theory.** Theorem 2.21 on its own has a quite interesting consequence for the behavior of the finite dimensional representations of $\mathcal{A}_\phi$ for different primes p with ϕ/p "held constant." For simplicity in this section, we only consider the case where $\Bbbk = \mathbb{F}_p$, though the results could be generalized without must difficulty. As we discussed in Remark 2.19, the category B has relations which are independent of p. This allows us to compare the representations of $\mathcal{A}_\phi$, by matching them with the representations of B.

Definition 2.22 *Let $\Lambda \subset \mathbb{Z}^d$ be the vectors such that $\mathsf{C}_{\mathbf{a}}$ contains $(o-\xi)_{1/p}$ for $\xi \in \phi + \mathfrak{t}_{1,\mathbb{Z}}$. Let $\bar{\Lambda}$ be the quotient of this set by the action of $\widehat{W}$ on Λ.*

Note that if we wish to compare different primes, it is perhaps more natural to consider the dependence of this set on $\psi = \phi_{1/p}$. The sets $\Lambda, \bar{\Lambda}$ make sense for an arbitrary $\psi \in \mathbb{R}^d$.

The set $\bar{\Lambda}$ is finite; its size is bounded above by the number of collections of weights φ_i which form a basis of $\mathfrak{t}^*$. We can then divide up choices of ϕ according to what the corresponding set $\bar{\Lambda}$ is. Given $\mathbf{a} \in \mathbb{Z}^d$, we let $\bar{\mathbf{a}}$ be its $\widehat{W}$-orbit.

Definition 2.23 *For a fixed $\bar{\Lambda}$, we let $\mathsf{B}^{\bar{\Lambda}}(\Bbbk)$ for any commutative ring $\Bbbk$ be the category with object set $\bar{\Lambda}$ such that $\operatorname{Hom}_{\mathsf{B}^{\bar{\Lambda}}(\Bbbk)}(\bar{\mathbf{a}}, \bar{\mathbf{b}}) = \operatorname{Hom}_{\mathsf{B}(\Bbbk)}(\eta_{\mathbf{a}}, \eta_{\mathbf{b}})$, for $\eta_{\mathbf{a}}$ an arbitrary element of the chamber $\mathsf{C}_{\mathbf{a}}$.*

We can also encapsulate this in the ring

$$A(\Bbbk) = \bigoplus_{\bar{\mathbf{a}}, \bar{\mathbf{b}} \in \bar{\Lambda}^{\mathbb{R}}} \operatorname{Hom}_{\mathsf{B}^{\bar{\Lambda}^{\mathbb{R}}}(\Bbbk)}(\bar{\mathbf{a}}, \bar{\mathbf{b}}), \tag{2.15}$$

Note that this is equivalent to $\mathsf{A}_p(\Bbbk)$. Since the center $Z(A(\Bbbk))$ is of finite codimension, the algebra $A(\Bbbk)$ has finitely many graded simple modules, all of which are finite dimensional. Each such simple for $\Bbbk = \mathbb{Q}$ has a $\mathbb{Z}$-form, which remains irreducible mod p for all but finitely many p. That is:

Theorem 2.24 *For a fixed $\bar{\Lambda}$, and $p \gg 0$, there is a bijection between simples L over $\mathsf{B}^{\bar{\Lambda}}(\mathbb{Q})$ and simples $L(p)$ over $\mathcal{A}_\phi$ for all $\phi \in \mathbb{Z}^d$ with $\bar{\Lambda}(\phi) = \bar{\Lambda}$. Under this bijection, each weight space of $L(p)$ for a weight υ with $\upsilon_{1/p} \in \mathsf{C}_{\mathbf{a}}$ is the same dimension as the $\mathbb{Q}$-vector space $L(\mathbf{a})$.*

Proof. As discussed above, for $p \gg 0$, the simple graded representations of $\mathsf{B}^{\bar{\Lambda}}(\mathbb{F}_p)$ are given by reductions mod p of an arbitrary invariant lattice of the simples L of $\mathsf{B}^{\bar{\Lambda}}(\mathbb{Q})$. This clearly preserves the dimension of the vector space assigned to an object $\mathbf{a}$. Under the equivalence of Theorem 2.21, the υ weight space of a $\mathcal{A}_\phi$-module matches the vector space assigned $\mathbf{a}$ defined as before in the $\mathsf{B}^{\bar{\Lambda}}(\mathbb{F}_p)$-module. $\square$

Thus, the dimension of $L(p)$ only depends on the number of weights of $\mathcal{A}_\phi$ with $\upsilon_{1/p} \in \mathsf{C}_{\mathbf{a}}$: it is the sum of the dimensions of the spaces $L(\mathbf{a})$ weighted by this count of integral points in a polytope. By the usual quasi-polynomiality of Erhart polynomials, we have that:

Corollary 2.25 *Fix $\bar{\Lambda}$ and let p and ϕ vary over values where $\bar{\Lambda}(\phi) = \bar{\Lambda}$. For $p \gg 0$, the dimension of $L(p)$ is a quasi-polynomial function of $\psi = \phi_{1/p}$ and p.*

3. Relation to geometry

Now, we turn to relating this approach to the study of coherent sheaves on resolved Coulomb branches. Throughout this section, we'll take the convention that $\widehat{\mathscr{A}_*}$ and $\mathcal{A}_*$

with $* \in \{h, 0, 1\}$ denote the category $\mathscr{A}_\phi$ or algebra $\mathcal{A}_\phi$ with ϕ left implicit, and h left as a formal variable, or specialized to be 0 or 1 (depending on the subscript).

3.1. **Frobenius constant quantization.** Recall that a quantization R_h of a $\Bbbk$-algebra R_0 is called **Frobenius constant** if there is a multiplicative map $\sigma\colon R_0 \to R_h$ congruent to the Frobenius map modulo h^{p-1}.

In the case of the quantum Coulomb branch, the Frobenius constancy of the quantization was recently proven by Lonergan.

Theorem 3.1 ([Lon, Thm. 1.1]) *There is a ring homomorphism $\sigma\colon \mathcal{A}_0^{\mathrm{sph}} \to \mathcal{A}_h^{\mathrm{sph}}$ making $\mathcal{A}_h^{\mathrm{sph}}$ into a FCQ for $\mathcal{A}_0^{\mathrm{sph}}$.*

Since Lonergan's construction is quite technical, it's worth reviewing the actual map that results. If G is abelian, then we can write this morphism very explicitly: in this case, we consider $\mathcal{A}_h^{\mathrm{sph}}$ as $\mathrm{End}_{\mathscr{B}}(\tau)$, and this space is spanned over S_h by the elements $r_\nu = y_\nu r(-\nu, 0)$ and [Lon, §3.15(3)] shows that it is induced by

$$\varphi \mapsto \mathrm{AS}(\varphi) = \varphi^p - h^{p-1}\varphi \tag{3.1}$$

$$r_\nu \mapsto r_{p\nu} \tag{3.2}$$

We can rewrite the action of the polynomial $\Phi(\eta+p\gamma, \eta)$ for $\gamma \in \mathfrak{t}_{\mathbb{Z}}$ using this map: this is a product of consecutive factors $\varphi_i^+ - kh$ for $k \in \mathbb{F}_p$, and must range over a number of these factors divisible by p. Furthermore, the number of such factors is $\varphi_i(\gamma)p$ if $\varphi_i(\gamma) \geq 0$ and 0 otherwise. That is,

$$\Phi(\eta + p\gamma, \eta) = \prod_{i=1}^{d} \mathrm{AS}(\varphi_i^+)^{\max(\varphi_i(\gamma),0)}$$

Having noted this, it is a straightforward calculation that this is a ring homomorphism.

If G is non-abelian, then this homomorphism is induced by the inclusion of $\mathcal{A}_0^{\mathrm{sph}}$ and $\mathcal{A}_h^{\mathrm{sph}}$ into the localization of the Coulomb branch algebras for the maximal torus T by inverting α for all affine roots α, since Steenrod operations commute with pushforward from the T-fixed locus, as discussed in [Lon, §3.15(4)].

A natural property to consider for varieties in characteristic p is whether they are Frobenius split. For the abelian case, it's easy to construct a splitting. Let $\kappa_0\colon S_0 \to S_0$ be any homogeneous Frobenius splitting.

Proposition 3.2 *The map*

$$\kappa(f \cdot r_\lambda) = \begin{cases} \kappa_0(f) r_{\lambda/p} & \lambda/p \in \mathfrak{t}_{\mathbb{Z}} \\ 0 & \lambda/p \notin \mathfrak{t}_{\mathbb{Z}} \end{cases}$$

is a Frobenius splitting for the ring $\Bbbk[\mathfrak{M}]$ when G is abelian.

Proof. This map is obviously a homomorphism of abelian groups sending 1 to 1, so we need only show that $\kappa(a^p b) = a\kappa(b)$ in the case where a and b are both of the form $a = f \cdot r_\lambda$ and $b = g \cdot r_\mu$. This is easy to see, since $r_\lambda^p = r_{p\lambda}$ and

$$\kappa(f^p r_{p\lambda} \cdot g r_\mu) = \kappa(f^p g \Phi(-p\lambda - \mu, -\mu, 0) \cdot r_{p\lambda+\mu})$$

If μ is not p-divisible, then this expression is 0, as is $f r_\lambda \kappa(g r_\mu) = 0$, so the result holds. On the other hand, if $\mu/p \in \mathfrak{t}_{\mathbb{Z}}$, then

$$\kappa(f^p g \Phi(-p\lambda - \mu, -\mu, 0) \cdot r_{p\lambda+\mu}) = f\kappa_0(g)\Phi(-\lambda - \frac{\mu}{p}, \frac{\mu}{p}, 0) r_{\lambda + \frac{\mu}{p}} = f r_\lambda \kappa(g r_\mu)$$

as desired. □

Now, assume that G is non-abelian and that the map κ_0 is equivariant for the group W; as usual this is possible because the average of the W-conjugates of a Frobenius splitting is again a splitting. Recall from [Webc, Def. 3.11] that we have an element $\mathbb{r}_\pi$ for any path π; let us write $\mathbb{r}(\eta, \eta')$ for the straight line path from η' to η.

By [Webc, Prop. 3.14], the algebra $\mathcal{A}_0^{\mathrm{sph}}$ has a basis given by the dressed monopole operators: the elements

$$\mathbb{m}_\lambda(f) = y_\lambda \tilde{\mathbb{r}}(-\lambda, -\lambda\epsilon) f \tilde{\mathbb{r}}(-\lambda\epsilon, 0)$$

for $\epsilon > 0$ a very small real number, λ running over dominant coweights of G and f over a basis of $S_0^{W_\lambda}$; we only need dominant coweights because

$$\mathbb{m}_\lambda(f) = y_{w\lambda} \tilde{\mathbb{r}}(-w\lambda, -w\lambda\epsilon) f^w \tilde{\mathbb{r}}(-w\lambda\epsilon, 0)$$

for any $w \in W$.

Proposition 3.3 *There is a Frobenius splitting $\kappa \colon \mathbb{k}[\mathfrak{M}] \to \mathbb{k}[\mathfrak{M}]$ such that*

$$\kappa(\mathbb{m}_\lambda(f)) = \begin{cases} \mathbb{m}_{\lambda/p}(\kappa_0(f)) & \lambda/p \in \mathfrak{t}_{\mathbb{Z}} \\ 0 & \lambda/p \notin \mathfrak{t}_{\mathbb{Z}} \end{cases} \tag{3.3}$$

Proof. Consider the usual inclusion $\mathbb{k}[\mathfrak{M}] \to \mathbb{k}[\mathfrak{M}_{\mathrm{ab}}^0]^W$ where $\mathfrak{M}_{\mathrm{ab}}^0$ is the open subset of $\mathfrak{M}_{\mathrm{ab}}$ where the root functions are non-vanishing. The former is Frobenius split by Proposition 3.2, and the restriction of the splitting map to $\mathbb{k}[\mathfrak{M}]$ acts by (3.3). In particular, it preserves the subring $\mathbb{k}[\mathfrak{M}]$ and thus gives a Frobenius splitting.

In order to do this calculation, it is useful to note that $(\mathbb{m}_\lambda(1))^p = \mathbb{m}_{p\lambda}(1)$, so this shows the result when $f = 1$. There are elements $h_w \in \mathbb{k}[\mathfrak{M}_{\mathrm{ab}}^0]^W$ such that

$$\mathbb{m}_\lambda(f) = \sum_w h_w (w \cdot f) \qquad \mathbb{m}_{p\lambda}(f) = \sum_w h_w^p (w \cdot f).$$

Thus, we have that

$$\kappa(\mathbb{m}_{p\lambda}(f)) = \sum_w h_w \kappa_0(w \cdot f) = \mathbb{m}_{\lambda/p}(\kappa_0(f)).$$ □

If G has non-trivial π_1, then this splitting is obviously equivariant for the induced action of the Pontryagin dual of π_1, and thus descends to the GIT quotient. Since any (partial) BFN resolution is a GIT quotient of this form, we thus also have that:

Corollary 3.4 *Any partial BFN resolution is Frobenius split.*

There are two natural ways to view $\mathcal{A}_h^{\mathrm{sph}}$ as a sheaf of algebras on $\mathfrak{M} = \operatorname{Spec} \mathcal{A}_0^{\mathrm{sph}}$:

(1) The first is the usual microlocalization W of $\mathcal{A}_h^{\mathrm{sph}}$. The sections $W(U_f)$ on the open set U_f where f is non-vanishing are given by $\mathcal{A}_h^{\mathrm{sph}}$ with every element congruent to f mod h inverted. This construction is discussed, for example, in [BPW16, §4.1]. This is a quantization in the usual sense of [BK08], and thus *not* a coherent sheaf.

(2) On the other hand, we can use σ to view $\mathcal{A}_h^{\mathrm{sph}}$ as a finite $\mathcal{A}_0^{\mathrm{sph}}[h]$-algebra, by the finiteness of the Frobenius map. We'll typically consider the specialization at $h = 1$, which realizes $\mathcal{A}_1^{\mathrm{sph}}$ as a finitely generated $\mathcal{A}_0^{\mathrm{sph}}$-module. Let $\mathscr{W}$ be the corresponding coherent sheaf on $\mathfrak{M} = \operatorname{Spec} \mathcal{A}_0^{\mathrm{sph}}$. This is essentially the pushforward of the usual microlocalization by the Frobenius map, specialized at $h = 1$.

The sheaf of algebras $\mathscr{W}$ is an Azumaya algebra on the smooth locus of $\mathfrak{M}$ of degree $p^{\mathrm{rank}(G)}$ by [BK08, Lemma 3.2]. We can also localize the algebra $\mathcal{A}_1$ using the map σ, and obtain an algebra $\mathscr{H}$ which on the smooth locus is Azumaya of degree $p^{\mathrm{rank}(G)} \cdot \#W$; the spherical idempotent in $\mathcal{A}_1$ induces a Morita equivalence between the Azumaya algebras $\mathscr{W}$ and $\mathscr{H}$.

Note that up to this point we have only obtained coherent sheaves on the affine variety $\mathfrak{M}$, but we will be more interested in considering the resolution $\tilde{\mathfrak{M}}$. By assumption, this resolution is the Hamiltonian reduction of the Coulomb branch $\mathfrak{M}_Q$ of Q by $K = T_F^\vee$. This Hamiltonian action of K is quantized by a non-commutative moment map $U(\mathfrak{k}) \to \mathcal{A}_{1,Q}^{\mathrm{sph}}$. Let

$$\mathcal{Q}_h = \mathcal{A}_{h,Q}^{\mathrm{sph}} / \mathfrak{k} \cdot (\mathcal{A}_{h,Q}^{\mathrm{sph}});$$

by [BFN18, 3(vii)(d)] and [Webc, Lem. 3.15], we then have that

$$\mathcal{A}_h^{\mathrm{sph}} = \operatorname{End}_{\mathcal{A}_{h,Q}^{\mathrm{sph}}}(\mathcal{Q}_h)^K \cong \mathcal{Q}_h^K. \tag{3.4}$$

Thus, we can follow the usual yoga for constructing quantizations of Hamiltonian reductions (see [Sta13, 4.3] for a discussion of doing this reduction for a torus in characteristic p, and [KR08, §2.5] for a more general discussion in characteristic 0) to obtain a Frobenius constant quantization of the resolved Coulomb branch $\tilde{\mathfrak{M}}$. We'll give an alternate construction of this quantization below using $\mathbb{Z}$-algebras.

Pushing forward by the Frobenius map and specializing $h = 1$ as above, we obtain a coherent sheaf of algebras, also denoted by $\mathscr{W}$ which is Azumaya on the smooth locus of $\tilde{\mathfrak{M}}$. We can perform the analogous operation with $\mathcal{A}_h^{\mathrm{sph}}$ replaced by $\mathcal{A}_h$. As before, we denote this by $\mathscr{H}$. In particular:

Lemma 3.5 *If $\tilde{\mathfrak{M}}$ is smooth, then $\mathscr{W}$ is an Azumaya algebra of degree $p^{\mathrm{rank}(G)}$ and $\mathscr{H}$ is Azumaya of degree $p^{\mathrm{rank}(G)} \cdot \#W$.*

3.2. **Homogeneous coordinate rings.** While this discussion is quite abstract, we can make it much more concrete by thinking about $\tilde{\mathfrak{M}}$ in terms of its homogeneous coordinate ring.

The variety $\tilde{\mathfrak{M}}$ is a GIT quotient of the moment map level $\mu^{-1}(0)$ with respect to some character $\chi\colon K \to \mathbb{G}_m$. Note that in our notation, we have that

$$\begin{aligned}
\mathbb{F}_p[\mathfrak{M}_Q] &= \mathcal{A}^{\mathrm{sph}}_{0,Q} \\
\mathbb{F}_p[\mu^{-1}(0)] &= \mathcal{Q}_0 = \mathcal{A}^{\mathrm{sph}}_{0,Q}\Big/\Big(\mu^*(\mathfrak{k}) \cdot (\mathcal{A}^{\mathrm{sph}}_{0,Q})\Big) \\
\mathbb{F}_p[\mathfrak{M}] &= \mathcal{Q}_0^K = (\mathcal{A}^{\mathrm{sph}}_{0,Q})^K\Big/\Big(\mu^*(\mathfrak{k}) \cdot (\mathcal{A}^{\mathrm{sph}}_{0,Q})^K\Big)
\end{aligned}$$

where $\mathfrak{k}$ is thought of as the space of linear functions on $\mathfrak{k}^*$, and μ^* is pullback by the moment map. By definition, we have that the section space of powers of the canonical ample bundle on the GIT quotient is given by the semi-invariants for χ^n:

$$\Gamma(\tilde{\mathfrak{M}}; \mathcal{O}(n)) \cong \mathcal{Q}_0^{\chi^n} = \{q \in \mathcal{Q}_0 \mid a^*(q) = \chi^n(k)q\}$$

for $a\colon K \times \mu^{-1}(0) \to \mu^{-1}(0)$ the action map. Since we are working in characteristic p, we need to phrase semi-invariance in terms of pullback of functions; it is necessary but not sufficient to check that $k \cdot q = \chi^n(k)q$ for points of the group K. Of course, we have, by definition, that

$$T \cong \bigoplus_{m \geq 0} \Gamma(\tilde{\mathfrak{M}}; \mathcal{O}(m)) \cong \bigoplus_{m \geq 0} \mathcal{Q}_0^{\chi^m} \qquad \tilde{\mathfrak{M}} = \mathrm{Proj}(T). \tag{3.5}$$

Let us describe the quantum version of this structure. It is tempting to simply change $h = 0$ in (3.5) to $h = 1$; unfortunately, this doesn't result in an algebra or a module over the projective coordinate ring. Instead, $\mathcal{Q}_1^{\chi^m} = {}_{\phi+m\nu}T_\phi^{\mathrm{sph}}$ is the twisting bimodule associated to the derivative $\nu = d\chi \in \mathfrak{k}^*_{\mathbb{Z}} \cong \mathfrak{t}_{\mathbb{Z}}$. With a bit more care, we could modify this structure to a $\mathbb{Z}$-algebra as discussed in [BPW16, §5.5].

However, being in characteristic p and having a Frobenius map gives us a second option. The quantum Frobenius map σ sends χ-semi-invariants to χ^p-semi-invariants, and thus induces a graded T-module structure on the graded algebra

$$\mathcal{T}^{\mathrm{sph}} := \bigoplus_{m \geq 0} \mathcal{Q}_1^{\chi^{pm}} = \bigoplus_{m \geq 0} {}_{\phi+pm\nu}T_\phi^{\mathrm{sph}}.$$

It's easy to see that the associated graded of this non-commutative algebra is

$$\bigoplus_{m \geq 0} \Gamma(\tilde{\mathfrak{M}}; \mathcal{O}(pm)),$$

with T acting by the Frobenius. In particular, $\mathcal{T}^{\mathrm{sph}}$ is finitely generated over T by the finiteness of the Frobenius map. This allows us to give our more "hands-on" definition of $\mathscr{W}$.

Definition 3.6 *Let $\mathscr{W}$ be the coherent sheaf of algebras on $\tilde{\mathfrak{M}}$ induced by $\mathcal{T}^{\mathrm{sph}}$. That is, $\mathscr{W} = \mathcal{Q}_1^{\chi^{pN}} \otimes_{\mathcal{A}_0^{\mathrm{sph}}} \mathcal{O}(-N)$ for $N \gg 0$.*

This sheaf stabilizes for N sufficiently large because of the finite generation of $\mathcal{T}^{\mathrm{sph}}$; thus multiplication is induced by the graded multiplication on $\mathcal{T}^{\mathrm{sph}}$ and on T. It follows immediately from standard results on projective coordinate rings that:

Corollary 3.7 *The functor $\mathcal{F} \mapsto \bigoplus_{m\geq 0} \Gamma(\tilde{\mathfrak{M}}, \mathcal{F}(m))$ induces an equivalence between the category of coherent $\mathscr{W}$-modules and the category of graded finitely generated $\mathcal{T}^{\mathrm{sph}}$-modules modulo those of bounded degree.*

As with the other structures we have considered, we can remove the superscripts of sph. This can be done from first principles, reconstructing all the objects defined above, but we ultimately know that the result will be Morita equivalent to the spherical version, so we can more define it quickly. Consider the tensor product $\mathcal{T}^{\mathrm{sph}} \otimes_{\mathcal{A}_1^{\mathrm{sph}}} e_{\mathrm{sph}}\mathcal{A}_1$, which is just a free module of rank $\#W$, and let $\mathcal{T}$ be the endomorphism algebra of this module. We let $\mathscr{H}$ be the corresponding algebra of coherent sheaves.

3.3. **Infinitesimal splittings.** Assume now that $\tilde{\mathfrak{M}}$ is smooth and a resolution of $\mathfrak{M}$. Recall that we have a map $\tilde{\mathfrak{M}} \to \mathfrak{t}/W$ induced by the inclusion of S_0^W into $\mathcal{A}_0^{\mathrm{sph}}$.

Definition 3.8 *We let $\hat{\mathfrak{M}}$ be the formal neighborhood of the fiber over the origin in $\mathfrak{t}/W$.*

Let $\hat{\mathscr{W}}_\phi$ be the corresponding pullback of $\mathscr{W}_\phi$, let $\hat{\mathscr{H}}_\phi$ be the corresponding pullback of $\mathscr{H}_\phi$ and similarly, $\hat{\mathsf{A}}_\phi$ the corresponding completion of A_ϕ.

The algebra $\hat{\mathscr{H}}_\phi$ can be written as the inverse limit

$$\hat{\mathscr{H}}_\phi = \varprojlim \mathscr{H}_\phi / \mathscr{H}_\phi \mathfrak{m}^N$$

for $\mathfrak{m} \subset S_0^W$ the maximal ideal corresponding to the origin. Of course, $\hat{\mathscr{H}}_\phi$ contains the larger commutative subalgebra $\hat{S}_1 = \varprojlim S_1/S_1\mathfrak{m}^N$ so we can consider how this profinite-dimensional algebra acts on $\mathscr{H}_\phi/\mathscr{H}_\phi\mathfrak{m}^N$.

As is well-known, an element $a \in \Bbbk$ satisfies $a^p - a = 0$ if and only if $a \in \mathbb{F}_p$. This extends to show that in S_1, the ideal $\mathfrak{m}S_1$ has radical given by the intersection of the maximal ideals $\mathfrak{m}_\mu$ defined by the points in $\mu \in \mathfrak{t}_{1,\mathbb{F}_p}$. Thus, $\hat{S}_1$ breaks up as the sum of the completions at these individual maximal ideals. For a given $\mu \in \mathfrak{t}_{1,\mathbb{F}_p}$ let e_μ be the idempotent that acts by 1 in the formal neighborhood of μ and vanishes everywhere else. Thus, $e_\mu\hat{\mathscr{H}}_\phi = \varprojlim \mathscr{H}_\phi/\mathscr{H}_\phi\mathfrak{m}_\mu^N$. Standard calculations show:

$$\mathrm{Hom}_{\hat{\mathscr{H}}_\phi}(e_\mu\hat{\mathscr{H}}_\phi, e_{\mu'}\hat{\mathscr{H}}_\phi) \cong \Gamma(\hat{\mathfrak{M}}, e_{\mu'}\hat{\mathscr{H}}_\phi e_\mu). \tag{3.6}$$

Of course, the reader should recognize this analysis as almost precisely the analysis of the functors of taking weight spaces discussed in Section 2.3 and in particular that of the category $\widehat{\mathscr{A}}$ defined in that section. We wish to consider the subcategory $\widehat{\mathscr{A}_{\mathbb{F}_p}}$ of objects of the form (o, μ) with $\mu \in \mathfrak{t}_{1,\mathbb{F}_p}$; for simplicity, we'll just denote this object by μ. In the notation introduced in that section, this subcategory would be $\widehat{\mathscr{A}_0}$, but we think that too likely to generate confusion with our convention of using this denote objects with $h = 0$.

Lemma 3.9 *There is a fully faithful functor from $\widehat{\mathscr{A}_{\mathbb{F}_p}}$ to the category of right $\hat{\mathscr{H}}_\phi$ modules sending $\mu \mapsto e_\mu \hat{\mathscr{H}}_\phi$.*

Proof. Note that the isomorphism $\mathcal{A}_1 \cong \Gamma(\tilde{\mathfrak{M}}, \mathscr{H}_\phi)$ induces a map

$$\mathcal{A}_1/(\mathfrak{m}_\mu^N \mathcal{A}_1 + \mathcal{A}_1 \mathfrak{m}_{\mu'}^N) \to \Gamma(\tilde{\mathfrak{M}}, \mathscr{H}_\phi/(\mathfrak{m}_\mu^N \mathscr{H}_\phi + \mathscr{H}_\phi \mathfrak{m}_{\mu'}^N))$$

It's not clear if this map is an isomorphism since sections are not right exact as a functor, but the theorem on formal functions [Sta, Theorem 02OC] shows that after completion, we obtain an isomorphism

$$\varprojlim \mathcal{A}_1/(\mathfrak{m}_\mu^N \mathcal{A}_1 + \mathcal{A}_1 \mathfrak{m}_{\mu'}^N) \to \Gamma(\tilde{\mathfrak{M}}, e_{\mu'} \hat{\mathscr{H}}_\phi e_\mu)$$

By (3.6), this shows that we have the desired fully-faithful functor. □

In particular, this means that in the case of $\mu = \tau$, this weight space has an additional action of the nilHecke algebra of W, so e_τ is the sum of $\#W$ isomorphic idempotents which are primitive in this subalgebra. We let $e_{0,\tau}$ be such an idempotent; since we assume p does not divide the order of $\#W$, we can assume that this is the symmetrizing idempotent for the W-action on the weight space.

Lemma 3.10 *For each μ, the algebra $e_\mu \hat{\mathscr{H}}_\phi e_\mu$ is Azumaya of degree $\#W$ over $\mathfrak{M}$, and split by the natural action on the vector bundle $\hat{\mathcal{Q}}_\mu := e_\mu \hat{\mathscr{H}}_\phi e_{0,\tau}$.*

Note that [BK08, Prop. 1.24] implies that these algebras must be split, but it is at least more satisfying to have a concrete splitting bundle.

Proof. Note first that for any idempotent e in an Azumaya algebra A, the centralizer eAe is again Azumaya. Thus, these algebras must all be Azumaya.

If $\tilde{\mathfrak{M}}$ is smooth, then $\hat{\mathcal{Q}}_\mu$ is a vector bundle since it is a summand of an Azumaya algebra. By Lemma 2.9, it is thus of rank $\#W$.

Since these algebras are Azumaya, this shows that their degree is no more than $\#W$, and if this bound is achieved, then they split. Since e_μ give $p^{\mathrm{rank}(G)}$ idempotents summing to the identity, and the total degree is $\#W \cdot p^{\mathrm{rank}(G)}$, this is only possible if the degree of each algebra is $\#W$. This shows the desired splitting. □

Corollary 3.11 *The vector bundle $\hat{\mathcal{Q}} \cong \bigoplus \hat{\mathcal{Q}}_\mu$ is a splitting bundle for the Azumaya algebra $\hat{\mathscr{H}}_\phi$.*

There is a fully faithful functor from $\widehat{\mathscr{A}_{\mathbb{F}_p}}$ *to the category of* $\mathrm{Coh}^{\ell f}(\hat{\mathfrak{M}})$ *of locally free coherent sheaves on* $\hat{\mathfrak{M}}$ *sending* $\mu \mapsto \hat{\mathcal{Q}}_\mu$.

Note that the bundle $e_{\mathrm{sph}}\hat{\mathcal{Q}}$ consequently is a splitting bundle for $\hat{\mathscr{W}}_\phi$; this summand can also be realized as the invariants of a W-action on $\hat{\mathcal{Q}}$. If W acts freely on the orbit of μ, then $\hat{\mathcal{Q}}_\mu$ is a summand of this bundle, but otherwise, we only obtain the invariants of the stabilizer of μ in W acting on this bundle. However, since e_{sph} induces a Morita equivalence, these bundles satisfy $\hat{\mathcal{Q}} \cong (e_{\mathrm{sph}}\hat{\mathcal{Q}})^{\oplus \# W}$.

3.4. **Lifting to characteristic 0.** Recall from Theorem 2.21 that we have an equivalence $\widehat{\mathscr{A}_{\mathbb{F}_p}} \cong \widehat{\mathsf{B}}(\mathbb{F}_p)$. Given $\mu \in \mathfrak{t}_{1,\mathbb{F}_p}$, let $\tilde{\mu} \in \mathfrak{t}_{1,\mathbb{Z}}$ be a lift. Combining this with Corollary 3.11, we that that:

Lemma 3.12 *There is a fully-faithful functor* $\mathsf{Q}\colon \mathsf{B} \to \mathrm{Coh}^{\ell f}(\hat{\mathfrak{M}})$ *sending* $-\tilde{\mu}_{1/p} + o \mapsto \hat{\mathcal{Q}}_\mu$.

Note that since o is isomorphic to the direct sum of $\# W$ copies of the object τ in B, we thus have that this functor sends $\tau = \tau_{1/p} \mapsto \mathcal{O}_{\hat{\mathfrak{M}}} = e_{0,\tau}\hat{\mathscr{H}}_\phi e_{0,\tau}$. This means that:

Lemma 3.13 *The functor* Q *when combined with quantum Frobenius* σ *or the functor* $\gamma\colon \widehat{\mathsf{B}} \to \widehat{\mathscr{B}}$ *induce two different isomorphisms*

$$\mathrm{End}_{\widehat{\mathscr{B}}}((\tau,\tau),(\tau,\tau)) \cong \mathcal{A}_0^{\mathrm{sph}}.$$

The resulting module structures on $\mathrm{Hom}_{\widehat{\mathscr{B}}}((\tau,\tau),(\eta,\mu))$ *are isomorphic.*

Proof. Using the action of $\widehat{W}$, we can assume that $\mu = \tau$. The module $\mathrm{Hom}_{\widehat{\mathscr{B}}}((\tau,\tau),(\eta,\tau))$ is spanned as a module over the dots by a basis consisting of the elements $y_w \mathbb{r}_\pi$for $w \in \widehat{W}$ such that $w \cdot \tau = \tau$ and a minimal length path τ to $w \cdot \eta$. The same is true of $\mathrm{Hom}_{\widehat{\mathsf{B}}}(\tau, \eta_{1/p})$.

We define an isomorphism

$$\ell\colon \mathrm{Hom}_{\widehat{\mathsf{B}}}(\tau,\eta_{1/p}) \to \mathrm{Hom}_{\widehat{\mathscr{B}}}((\tau,\tau),(\eta,\tau))$$

by the formulas

$$\ell(\lambda) = \lambda^p - \lambda \qquad \ell(w) = w_p \qquad \ell(u_\alpha) = \frac{u_{\alpha^{(p)}}}{(\alpha^{(p)})^{p-1} - 1}$$

$$\ell(r(\eta,\eta')) = r(\eta_p,\eta'_p).$$

This defines an isomorphism since the polynomials $\hat{\Phi}_0(\eta,\eta',\tau)$ and $\alpha^{p-1} - 1$ are invertible. It's important to note that this does not define an equivalence of categories, but only of $\mathcal{A}_0^{\mathrm{sph}}$-modules. $\square$

We wish to extend this result to the coherent sheaves $\hat{\mathcal{Q}}_\mu$. In order to do this, it's useful to consider the completed category $\widehat{\mathscr{B}}^Q$ attached to the gauge group Q. We have a functor from this category to $\mathrm{Coh}^K(\hat{\mathfrak{M}}_Q)$, the category of K-equivariant coherent sheaves on the corresponding completion of the Coulomb branch $\mathfrak{M}_Q$. This functor is given by considering $\mathrm{Hom}_{\widehat{\mathscr{B}}^Q}((\tau,\tau),(\eta,\mu))$ as a module over $\mathcal{A}_0^Q = \mathrm{End}_{\widehat{\mathscr{B}}^Q}((\tau,\tau),(\tau,\tau))$, where the isomorphism is via the quantum Frobenius.

This inherits a K-action from the category $\widehat{\mathscr{B}}^Q$ itself. If we change $\eta \mapsto \eta + p\nu$ for $\nu \in \mathfrak{t}_{Q,\mathbb{Z}}$, this has the effect of twisting the equivariant structure by the corresponding character of K induced by exponentiating γ. In particular, as an equivariant sheaf, this only depends on the image of ν in $\mathfrak{t}_{F,\mathbb{Z}}$, so if $\gamma \in \mathfrak{t}_{\mathbb{Z}}$, the resulting sheaf is K-equivariantly isomorphic.

By definition, the module $\hat{\mathcal{Q}}_\mu$ is the reduction of the coherent sheaf

$$\hat{\mathcal{R}}_\mu = \mathrm{Hom}_{\widehat{\mathscr{B}}^Q}((\tau,\tau),(o,\mu)),$$

thought of as a $\mathcal{A}_0^{Q;\mathrm{sph}}$-module via the quantum Frobenius σ.

Of course, we can apply the functor of Proposition 2.18 with the gauge group Q; this gives us an identification of $\hat{\mathcal{R}}_\mu$ with $\hat{\mathsf{R}}_\mu = \mathrm{Hom}_{\widehat{\mathsf{B}}^Q}(\tau, -\tilde{\mu}_{1/p} + o)$. This is a module over $\mathcal{A}_0^{Q;\mathrm{sph}} \cong \mathrm{Hom}_{\widehat{\mathsf{B}}^Q}(\tau,\tau)$, and the two possible module structures are isomorphic by Lemma 3.12.

Note that using this presentation has enormous advantages: we can consider the induced module $\mathsf{R}_\mu = \mathrm{Hom}_{\mathsf{B}^Q}(\tau, \mu_{1/p})$ in the uncompleted category B^Q; localizing, this gives a $K \times \mathbb{G}_m$-equivariant module on $\mathfrak{M}_Q$. Furthermore, whereas all of the geometry discussed earlier in this category required us to consider $\mathfrak{M}$ over a base field of characteristic p, the category $\mathsf{B}^Q(\mathbb{k})$ is well-defined over $\mathbb{Z}$ and thus over any commutative base ring $\mathbb{k}$.

Definition 3.14 *Let $\mathcal{Q}_\mu^{\mathbb{k}}$ be the $\mathbb{G}_m$-equivariant coherent sheaf on $\tilde{\mathfrak{M}}$ given by Hamiltonian reduction of $\mathsf{R}_\mu(\mathbb{k}) = \mathrm{Hom}_{\mathscr{B}^Q(\mathbb{k})}(\tau, -\tilde{\mu}_{1/p} + o)$.*

3.5. **Derived localization.** For now, let us specialize back to the case where $\mathbb{k} = \mathbb{F}_p$. By Grauert and Riemenschneider for Frobenius split varieties ([MvdK92]) and the splitting of Proposition 3.3, we have that:

Corollary 3.15 *For any prime p, we have the higher cohomology vanishing $H^i(\tilde{\mathfrak{M}};\mathcal{O}) = 0$ for all $i > 0$.*

As discussed in [Kal08], this means that the derived functor of localization $\mathbb{L}\mathrm{Loc}$ is right inverse to the functor $\mathbb{R}\Gamma_{\mathbb{S}}$ of derived sections for modules over $\mathscr{W}_\phi$. Recall that we have chosen χ such that $\tilde{\mathfrak{M}}$ is smooth. We can conclude from [Kal08, Thm. 4.2] that:

Lemma 3.16 *There is an integer N, such that for any p, and any line parallel to χ in $\mathfrak{t}_{1,\mathbb{F}_p}$, there are at most N values of ϕ for which $\mathbb{L}\mathrm{Loc}$ and $\mathbb{R}\Gamma_{\mathbb{S}}$ are* not *inverse equivalences.*

Remark 3.17. It seems likely that this result also holds when $\tilde{\mathfrak{M}}$ is not smooth, at least for the quantizations we have constructed, but let us leave this point unresolved for the time being.

Lemma 3.18 *The vector bundle* $\mathcal{Q}^{\mathbb{F}_p} = \bigoplus_\mu \mathcal{Q}_\mu^{\mathbb{F}_p}$ *is a tilting generator for* $\mathrm{Coh}(\mathfrak{M})$ *if and only if derived localization holds for* $\hat{\mathscr{W}}_\phi$.

Proof. First note that by semi-continuity, it's enough to show this for $\hat{\mathcal{Q}}^{\mathbb{F}_p}$ on $\hat{\mathfrak{M}}$. We know that on $\hat{\mathfrak{M}}$, we have an isomorphism $\hat{\mathscr{W}} \cong \mathscr{H}om_{\mathcal{O}_{\hat{\mathfrak{M}}}}(\hat{\mathcal{Q}}^{\mathbb{F}_p}, \hat{\mathcal{Q}}^{\mathbb{F}_p})$. Since the higher cohomology of $\hat{\mathscr{W}}$ vanishes, this shows that $\hat{\mathcal{Q}}^{\mathbb{F}_p}$ is a tilting bundle.

The $\hat{\mathscr{W}}$-modules are precisely the sheaves of the form $\mathscr{H}om_{\mathcal{O}_{\hat{\mathfrak{M}}}}(\hat{\mathcal{Q}}^{\mathbb{F}_p}, \mathcal{F})$ for a coherent sheaf $\mathcal{F}$. Since $\hat{\mathcal{Q}}^{\mathbb{F}_p}$ is a vector bundle, we have that

$$H^i(\mathfrak{M}; \mathscr{H}om_{\mathcal{O}_{\hat{\mathfrak{M}}}}(\hat{\mathcal{Q}}^{\mathbb{F}_p}, \mathcal{F})) \cong \mathrm{Ext}^i_{\mathcal{O}_{\hat{\mathfrak{M}}}}(\hat{\mathcal{Q}}^{\mathbb{F}_p}, \mathcal{F}).$$

Thus, $\mathcal{Q}^{\mathbb{F}_p}$ is a generator if and only if no module over $\hat{\mathscr{W}}$ has all cohomology groups trivial. □

Corollary 3.19 *If derived localization holds at* ϕ*, then the fully faithful functor* $\mathsf{Q}\colon \mathsf{B}(\mathbb{F}_p) \to \mathrm{Coh}(\tilde{\mathfrak{M}})$ *induces an equivalence of derived categories* $D^b(\mathsf{B}(\mathbb{F}_p)\text{-mod}) \cong D^b(\mathrm{Coh}(\tilde{\mathfrak{M}}))$.

Proof. If derived localization holds at ϕ, then the induced derived functor is essentially surjective, since $\mathcal{Q}$ is a generator of the derived category. Thus, this derived functor is an equivalence. □

Let $\Lambda, \bar{\Lambda}$ be as defined in Definition 2.22. As noted before, the set $\bar{\Lambda}$ is finite.

Definition 3.20 *We call a choice of* $\psi = \phi_{1/p}$ **generic** *if the number of elements of* $\bar{\Lambda}$ *is maximal amongst all choices of* $\psi \in \mathfrak{t}_{1,F,\mathbb{R}}$.

Note that for a given p, there may be no generic choices of ψ in $\mathfrak{t}_{1,F,\frac{1}{p}\mathbb{Z}}$, but since real numbers can be arbitrarily well approximated by fractions with prime denominators, there are generic ψ with $\phi \in \mathfrak{t}_{1,F,\mathbb{Z}}$ for all sufficiently large p. In fact, we can divide $\mathfrak{t}_{1,F,\mathbb{R}}$ up into regions $R_{\bar{\Lambda}'}$ according to what the set $\bar{\Lambda}'$ attached to ψ is. Having a maximal number of such non-empty chambers is a open dense property (it is the complement of the integral translates of finitely many hyperplanes). Simple geometry shows that:

Lemma 3.21 *For a fixed* $\bar{\Lambda}$ *with* $R_{\bar{\Lambda}}$ *open and non-empty and a fixed integer* N*, there is a constant* M *such that if* $p > M$ *then there is a choice* $\phi \in \mathfrak{t}_{1,\mathbb{Z}}$ *such that* $\phi, \phi + \chi, \phi + 2\chi, \dots, \phi + N\chi$ *are generic and*

$$R_{\bar{\Lambda}} \supset \left\{ (\phi + k\chi)_{1/p} \,\middle|\, k \in \mathbb{R}, 0 \leq k \leq N \right\}.$$

Recall that as we mentioned earlier that there is a constant N such that for a fixed ϕ, localization can only fail at N values of the form $\phi + k\chi$ for $k \in \mathbb{Z}/p\mathbb{Z}$. Fix $\bar{\Lambda}$ with $R_{\bar{\Lambda}}$ open and non-empty and let M be the associated constant in Lemma 3.21.

Theorem 3.22 *If ϕ is a generic parameter with $\bar{\Lambda}$ as fixed above, and $p > M$, then derived localization holds for ϕ, and so the associated $\mathcal{Q}^{\mathbb{F}_p}$ is a tilting generator.*

Proof. First note that it is enough to replace ϕ by any other generic parameter with the same set $\bar{\Lambda}$. In this case, tensor product with the bimodule ${}_{\phi}T_{\phi'}$ sends any object in $C_{\mathbf{a}}$ in the preimage of ϕ to one in $C_{\mathbf{a}}$ in the preimage of ϕ' (see (2.1) for the definition of $C_{\mathbf{a}}$). Thus the categories $\mathsf{A}_p(\mathbb{F}_p)$ are naturally equivalent via tensor product with bimodule ${}_{\phi}T_{\phi'}$ connecting them.

Thus, we can assume that ϕ is as in Lemma 3.21. If derived localization fails at ϕ, then it also fails at $\phi + \chi, \phi + 2\chi, \dots, \phi + N\chi$. This is impossible by our upper bound on the number of points where it fails from Lemma 3.16. $\square$

This is certainly too crude to give a sharp characterization of when derived localization holds. We expect that we will instead find that:

Conjecture 3.23 *If ϕ is a generic parameter, then derived localization holds for ϕ. Equivalently, if ϕ and ϕ' are generic, then derived tensor product with ${}_{\phi}T_{\phi'}$ is an equivalence between $D^b(\mathcal{A}_{\phi}\text{-mod})$ and $D^b(\mathcal{A}_{\phi'}\text{-mod})$.*

These results have consequences for the case where $\Bbbk$ is an arbitrary commutative ring. Note that by construction B, and thus $\mathcal{Q}^{\Bbbk}$, depends on a choice of ϕ and ultimately a prime p, but for fixed $\Bbbk$, this dependence is very weak.

Lemma 3.24 *The vector bundle $\mathcal{Q}^{\Bbbk}_{\mu}$ only depends on which element of $\bar{\Lambda}$ corresponds to the chamber $\mathsf{C}_{\mathbf{a}}$ containing μ. Consequently, the vector bundles that appear this way for a fixed ϕ only depends on the set $\bar{\Lambda}$.*

Proof. If μ_1 and μ_2 both lie in $\mathsf{C}_{\mathbf{a}}$ then we obtain an isomorphism $\mathcal{Q}^{\Bbbk}_{\mu_1} \cong \mathcal{Q}^{\Bbbk}_{\mu_2}$. $\square$

As we change ϕ and p while keeping $\bar{\Lambda}$ fixed, the number of integral points in each chamber $\mathsf{C}_{\mathbf{a}}$ will increase and decrease, so the vector bundle $\mathcal{Q}^{\Bbbk}$ will change, but only by changing the number of times different summands appear; that is, the vector bundles $\mathcal{Q}^{\Bbbk}$ for different ϕ are **equiconstituted**. Which summands appear at least once will only change when we change $\bar{\Lambda}$.

We obtain the cleanest statement if we pass to $\mathbb{Q}$, which as we mentioned before is essentially the case of p is infinitely large. In this case, it is convenient to fix a parameter $\psi \in \mathfrak{t}_{1,F,\mathbb{R}}$, defining a real flavor, and consider the set $\Lambda^{\mathbb{R}}$ of vectors with $\mathsf{C}_{\mathbf{a}}$ non-empty and $\bar{\Lambda}^{\mathbb{R}}$ its quotient by $\widehat{W}$; as before, we call ψ generic if the set $\bar{\Lambda}^{\mathbb{R}}$ has maximal size. We let $\mathcal{Q}^{\mathbb{Q}}_{\phi}$ be the sum of the vector bundles under $\mathcal{Q}^{\mathbb{Q}}_{\mu}$ for representatives μ of chamber in $\bar{\Lambda}^{\mathbb{R}}$. This analogous to the construction of the category $\mathsf{B}^{\bar{\Lambda}^{\mathbb{R}}}(\mathbb{Q})$ discussed in Definition 2.23.

Theorem 3.25 *If ψ is a generic parameter, the vector bundle $\mathcal{Q}_{\phi}^{\mathbb{Q}}$ on $\tilde{\mathfrak{M}}_{\mathbb{Q}}$ is a tilting generator and induces an equivalence $D^b(\mathsf{B}^{\bar{\Lambda}^{\mathbb{R}}}(\mathbb{Q})) \cong D^b(\mathrm{Coh}(\tilde{\mathfrak{M}}_{\mathbb{Q}}))$.*

Proof. Being a vector bundle and a tilting generator after base change to a point of Spec $\mathbb{Z}$ is an open property, so if the set of primes where this holds is non-empty, it must be so over $\mathbb{Q}$ as well. Thus, we need only show that $\mathcal{Q}^{\mathbb{F}_p}$ is a tilting generator for some prime p. By Lemma 3.24, this fact only depends on the corresponding $\bar{\Lambda}$. By Theorem 3.22, for $p \gg 0$, there is a ϕ which gives $\bar{\Lambda}$ as the set of chambers with integral points such that derived localization holds at ϕ. Thus, by Lemma 3.18, the associated sheaf $\mathcal{Q}_{\phi}^{\mathbb{F}_p}$ is a tilting generator, which establishes the result. □

3.6. **Non-commutative crepant resolutions.** Recall the notion of a **non-commutative crepant resolution** of the affine variety $\mathfrak{M}$, originally defined in [vdB04a]: this is an algebra $A = \mathrm{End}(M)$, for some reflexive coherent sheaf M on $\mathfrak{M}$, such that A is a Cohen-Macaulay as a coherent sheaf and the global dimension of A is equal to $\dim \mathfrak{M}$. A **D-equivalence** between a commutative resolution $\tilde{\mathfrak{M}}$ and a non-commutative resolution A is an equivalence of dg-categories $D^b(\mathrm{Coh}(\tilde{\mathfrak{M}})) \cong D^b(A\text{-mod})$.

The following is a corollary of [vdB04b, Lem. 3.2.9 & Prop. 3.2.10]:

Lemma 3.26 *Suppose $\mathcal{T}$ is a tilting generator on a resolution $\tilde{\mathfrak{M}}$ such that the structure sheaf $\mathcal{O}_{\tilde{\mathfrak{M}}}$ is a summand of $\mathcal{T}$, and let $M = \Gamma(\tilde{\mathfrak{M}}; \mathcal{T})$. Then $A = \mathrm{End}_{\mathrm{Coh}(\tilde{\mathfrak{M}})}(\mathcal{T}) \cong \mathrm{End}_R(M)$ is a non-commutative crepant resolution of singularities, canonically D-equivalent to Y.*

Assume that the flavor ϕ is chosen so that $\mathbf{0} \in \Lambda^{\mathbb{R}}$; this means that the structure sheaf $\mathscr{O}_{\tilde{\mathfrak{M}}}$ is a summand of $\mathcal{Q}_{\phi}^{\mathbb{Q}}$. By the equivalence of Theorem 3.25 and the definition (2.15), we have an isomorphism

$$A = \mathrm{End}_{\mathrm{Coh}(\tilde{\mathfrak{M}})}(\mathcal{Q}_{\phi}^{\mathbb{Q}}),$$

and we have an idempotent $e_0 \in A$ projecting to the structure sheaf. Then, applying Lemma 3.26, we can see that:

Corollary 3.27 *The ring A is a non-commutative crepant resolution of the Coulomb branch $\mathfrak{M}$.*

As mentioned earlier, we can give very explicit computations of the algebras in question when $\mathfrak{M}$ is a quiver gauge theory, which we will discuss in much greater detail in [Weba]. This is also true in the hypertoric case, as discussed in [MW, Prop. 3.35] and [GMW, §4.1].

3.7. **Presentations.** For the sanity of the reader, let us try to give a more explicit description of the resulting algebra A which gives our non-commutative resolution of singularities. For our gauge group G, consider the fundamental alcove ∇ in the Cartan of $\mathfrak{g}$ mod the action of the group $\widehat{W}_0$ of length 0 elements in the extended affine Weyl group (which are by definition, the elements sending the fundamental alcove to itself). That is, ∇ is the subset of positive Weyl chamber in $\mathfrak{t}$ which is not separated from the origin by the zeros of any affine root. For fixed flavor ϕ, every point in this space gives an object in the extended category B, but there are only finitely many isomorphism types, given by the set $\bar{\Lambda}$.

First of all, we divide this fundamental alcove by considering the hyperplanes defined by $\varphi_i(\eta) \equiv -\phi_i/p \pmod{\mathbb{Z}}$ for ϕ_i the weight of the flavor ϕ on the weight space V_i; these are the unrolled matter hyperplanes. Unrolled root hyperplanes only appear on the boundaries of the alcove and only ones corresponding to simple roots of the affinization of G are relevant. Also, note that the objects corresponding to the walls of the fundamental alcove are summands of the nearby generic objects, so up to isomorphism or inclusion of summands, we can take the algebra A to be the endomorphisms of a sum of representatives of the chambers cut out by the unrolled hyperplanes. By [Webc, Cor. 3.13], we have a basis of these endomorphisms which can visualize as straight (or small perturbation of straight) paths in $\mathfrak{t}$, folded using reflections to fit in the fundamental alcove. Of course, having chosen representatives of each chamber, we can factor this path to pass through the representative of each chamber it passes through, and thus factor it into shorter segments that either:

(1) join chambers which are adjacent across an unrolled matter hyperplane
(2) "bounce" off a root hyperplane within a chamber bounding it.

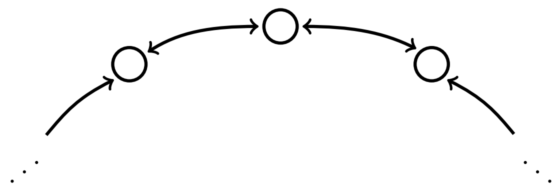
We can thus, we have that

Proposition 3.28 *The algebra A is a quotient of the path algebra of the quiver where:*

(1) nodes are given by $\bar{\Lambda}$, the chambers in this arrangement,
(2) we add as endomorphisms to each node the semi-direct product of S_h with the stabilizer of the corresponding chamber in $\widehat{W}_0$
(3) we add an opposing pair of edges for every pair of chambers adjacent across a matter hyperplane
(4) we add a self-loop for each adjacency of a chamber to a root hyperplane.

The relations that we need arise from (2.6a–2.6k). One simply takes the pictures [Webc, (2.5a–c)], replaces chambers with nodes in the quiver and hyperplanes with arrows, and interprets in the path as one in the quiver. These are a bit tedious to write out in full generality, so we leave this as an exercise to the reader.

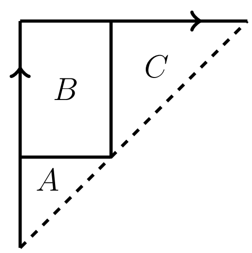
Example 3.ii. One valuable example to consider is when $\mathbb{C}^*$ acts on $\mathbb{C}^n$ with weight 1. In this case, the fundamental alcove is all of $\mathfrak{t}_{\mathbb{R}}$ and the extended affine Weyl group the coweight lattice, so the quotient is the maximal compact of the torus $T \subset G$. The flavor ϕ has n components $(\phi_1, \ldots, \phi_n)$, and the unrolled matter hyperplane arrangement is given by removing the points $x = -\phi_i/p$ from the circle. Thus, we have n chambers arranged in a circle. For simplicity, we draw each pair of arrows from a matter hyperplane as a double-headed arrow, so the structure we see is:



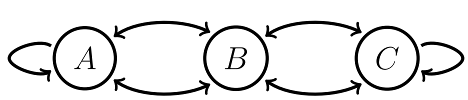
In fact, A is the preprojective algebra of this quiver, which is well-known to give the desired non-commutative resolution. ♣

Example 3.iii. In our usual running example, with $G = GL(2)$, the fundamental alcove is the region $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x - y \leq 1$, and the length 0 elements of the affine Weyl group act by the integer powers of the glide reflection $(y + 1, x)$. The quotient is thus a Möbius band, which we can identify with the configuration space of pairs of points on a circle.

We take matter representation $V = \mathbb{C}^2 \oplus \mathbb{C}^2$ and thus obtain a chamber structure in Figure 2.i. That is, we have a geometry like



where the solid lines are matter hyperplanes, dashed lines are root hyperplanes, and the lines with arrows indicate gluing to obtain a Möbius strip with dashed boundary. Thus, we have between A and B two adjacencies and thus two *pairs* of arrows, and similarly with B and C, with A and C both having self-loops corresponding to the adjacent root hyperplane.



♣

4. Schobers and wall-crossing

Our final section will concern the theory of **twisting functors** (also called **wall-crossing functors**), and in particular, their connection to the theory of Schobers. These functors are discussed for general symplectic singularities in [Los, §2.5.1]. Schobers constructed from categories of coherent sheaves and variation of GIT have already appeared in work of Donovan [Don] and Halpern-Leistner and Shipman [HLS16]. These works have mostly focused on a single wall-crossing, rather than a more complicated hyperplane arrangement, but the simplicity of Coulomb branches compared to other symplectic singularities gives us a tighter control over the structures appearing.

We will first give some preliminary results on Morita contexts. These are, of course, standard objects of study in non-commutative geometry and algebra, but their connections to spherical functors and thus to Schobers seem to have mostly escaped notice. Then, we turn to the construction of a Schober and thus a π_1-action from the algebraic and geometric objects considered earlier in the paper. We'll note here that essentially identical arguments will construct Schobers in many similar contexts where actions of fundamental groups have been constructed, in particular for the twisting and shuffling functors in characteristic 0 considered in [BPW16, BLPW16].

We'll also note that it seems quite likely that this argument proceeds essentially identically for all symplectic resolutions of singularities. However, both for reasons of notational convenience, and avoiding certain technical difficulties (in particular, proving the analogue of Lemma 4.6), we will restrict ourselves to the case of Coulomb branches.

4.1. **Morita contexts and spherical functors.** Recall that a **Morita** context (called "pre-equivalence data" in [Bas68]) is a category with 2 objects $\{+,-\}$. The endomorphism algebras of the two objects give two rings R_+ and R_-, and the Hom spaces give $R_\pm$-$R_\mp$ bimodules ${}_\pm R_\mp$. Let $I_\pm = {}_\pm R_\mp \cdot {}_\mp R_\pm$ be the two-sided ideal of morphisms factoring through $\mp$, and $Q_\pm = R_\pm / I_\pm$. For simplicity, we assume that R_+ and R_- have finite global dimension. Modules over this category are the equivalent to modules over the "matrix" ring

$$R = \begin{bmatrix} R_+ & {}_+R_- \\ {}_-R_+ & R_- \end{bmatrix}$$

Let e_+, e_- be the identities on the 2-objects. For any context, we have quotient functors $q_\pm \colon R\text{-mod} \to R_\pm\text{-mod}$ with $q_\pm(M) = e_\pm M = e_\pm R \otimes_R M = \mathrm{Hom}_R(Re_\pm, M)$. This functor has left and right adjoints

$${}^*q_\pm(N) = Re_\pm \otimes_{R_\pm} N \qquad q_\pm^*(N) = \mathrm{Hom}_{R_\pm}(e_\pm R, N).$$

Of course, both of these functors are fully faithful. The images of their derived functors thus give two copies of $\mathcal{E}_\pm := D^b(R_\pm\text{-mod})$ in $\mathcal{E}_0 = D^b(R\text{-mod})$ which are the left and right perpendiculars of $\mathcal{F}_\pm$, the subcategory of the derived category of $D^b(R\text{-mod})$ which become acyclic after applying $e_\pm$. This can be identified with modules over the dg-algebra $F_\pm = \mathrm{Ext}^\bullet_R(Q_\mp, Q_\mp)$. The inclusion $\xi_\pm$ of this subcategory can then be identified with $Q_\mp \overset{L}{\otimes}_{F_\pm} -$. Thus, left and right adjoints of this functor are given by

$${}^*\xi_\pm(M) = Q_\mp \overset{L}{\otimes}_R - \qquad \xi_\pm^*(M) = \mathrm{RHom}_R(Q_\mp, M).$$

The inclusions $j_\pm = q^*_\pm$ and $\xi_\pm$ thus fit in the setup of [KS, §3.C]. Consider the composition $S = q_\pm \circ \xi_\mp$. This has left and right adjoints

$$L = {}^*\xi_\mp \circ {}^*q_\pm = Q_\pm \overset{L}{\otimes}_{R_\pm} - \qquad R = \xi^*_\mp \circ q^*_\pm = \mathrm{RHom}_{R_\pm}(Q_\pm, -).$$

Consider the functors

$${}^*j_\pm \circ j_\mp = q_\pm \circ q^*_\mp = {}_\pm R_\mp \overset{L}{\otimes}_{R_\mp} - \colon \mathcal{E}_\mp \to \mathcal{E}_\pm$$

Lemma 4.1 *If ${}^*j_\pm \circ j_\mp$ and ${}^*j_\mp \circ j_\pm$ are equivalences of derived categories, then the data above define a spherical pair in the sense of Kapranov and Schechtman* [KS, §3.C], *and the functor S is spherical.*

Proof. In addition to our hypotheses, we need to prove that $\xi^*_\mp \circ \xi_\pm$ are equivalences of derived categories. If ${}_\mp R_\pm \overset{L}{\otimes}_{R_\pm} -$ is an equivalence, then its inverse is its adjoint $\mathrm{Hom}_{R_\mp}({}_\mp R_\pm, -)$. Thus, $N' = {}^{**}j_\mp({}_\mp R_\pm \overset{L}{\otimes}_{R_\pm} N)$ is an R-module such that ${}^*j_\pm(N') \cong N$. This shows we have a natural map $j_\pm(N) \to N'$, which is a quasi-isomorphism after applying $e_\pm$ (by the observation we just made) and a quasi-isomorphism after applying $e_\mp$, by the isomorphism of ${}^*j_\mp{}^{**}j_\mp$ to the identity.

Thus, $j_\pm$ and ${}^{**}j_\mp$ have the same image. Obviously, $\mathcal{F}_\pm$ is the left orthogonal to this image, and $\mathcal{F}_\mp$ its right orthogonal. Thus, $\xi^*_\mp \circ \xi_\pm$ is the mutation with respect to these dual semi-orthogonal decompositions. Note that this is a special case of [HLS16, Thm. 3.11], with the ambient dg-category being the derived category of R-modules, the category $\mathcal{A}$ being the image of $j_\pm$ and ${}^{**}j_\mp$, and $\mathcal{A}'$ the image of $j_\mp$ and ${}^{**}j_\pm$. $\square$

4.2. **Wall-crossing functors.** For different choices of flavor ϕ, we obtain different quantizations of the structure sheaf of $\mathfrak{M}$. Quantized line bundles give canonical equivalences of categories between the categories of modules over these sheaves, as in [BPW16]. Note that the isomorphism type of the underlying sheaf only depends on ϕ considered modulo p, but for different elements of the same coset, there is still a non-trivial autoequivalence, induced by tensoring with the quantizations of pth power line bundles. Similarly, for each element of the Weyl group W_F, there's an isomorphism between the section algebras of $\mathcal{A}_\phi$ and $\mathcal{A}_{w\cdot\phi}$; together, these give us such a morphism for every $w \in \widehat{W}_F$, affine Weyl group of F. We thus can consider the twisting bimodule ${}_{w\phi'}T_\phi$ discussed earlier, turned into a $\mathcal{A}_{\phi'}$-$\mathcal{A}_\phi$-bimodule using the isomorphism above to twist the left action.

Definition 4.2 *Given flavors ϕ and ϕ', and $w \in \widehat{W}_F$, we define the* **twisting** *or* **wall-crossing functor** $\Phi^{\phi',\phi}_w \colon D^b(\mathcal{A}_\phi\text{-mod}) \to D^b(\mathcal{A}_{\phi'}\text{-mod})$ *to be the derived tensor product with ${}_{w\phi'}T_\phi$.*

One can think of this functor as measuring the different sets Λ, Λ' attached to the parameters ϕ', ϕ. In particular:

Lemma 4.3 *If ϕ, ϕ' are generic and $\Lambda = \Lambda'$, then ${}_{\phi'}T_{\phi}$ induces a Morita equivalence and $\Phi_1^{\phi',\phi}$ is an exact functor.*

Proof. Of course, we have natural maps ${}_{\phi'}T_{\phi} \otimes {}_{\phi}T_{\phi'} \to \mathcal{A}_{\phi'}$ and similarly with ϕ, ϕ' reversed. This gives a Morita context, as discussed above, and by [Bas68, II.3.4], we will obtain the desired Morita equivalence if we prove both of these maps are surjective.

If this map is not surjective, then its image is a proper 2-sided ideal (sometimes called the trace of the Morita context). Since $\mathcal{A}_{\phi'}$ is finitely generated over its center, the quotient by this ideal has the same property, so it has at least one finite dimensional simple module L, which thus satisfies ${}_{\phi}T_{\phi'} \otimes_{\mathcal{A}_{\phi'}} L = 0$. Thus, any chamber that appears in the support of L must lie in Λ' but not Λ, which is impossible since these sets coincide. In fact, it's clear from Theorem 2.21 that ${}_{\phi}T_{\phi'} \otimes_{\mathcal{A}_{\phi'}} -$ induces an equivalence on the category of finite dimensional representations. Thus, we must have that ${}_{\phi'}T_{\phi} \otimes {}_{\phi}T_{\phi'} \to \mathcal{A}_{\phi'}$ is surjective, and similarly with ϕ, ϕ' reversed. □

Recall that $\tilde{\mathfrak{M}}$ depends on a choice of $\chi \in \mathfrak{t}_{F,\mathbb{Z}}$. This dependence is rather crude, though. By the usual theory of variation of variation of GIT [DH98], the space $\mathfrak{t}_{F,\mathbb{R}}$ is cut into a finite number of convex cones, such that $\tilde{\mathfrak{M}}$ is smooth when χ lies in the interior of one of these cones, called "chambers" in [DH98]. An element χ' will give an ample line bundle on $\tilde{\mathfrak{M}}$ if it is in the chamber as χ (since their stable loci coincide), or a semi-ample bundle if it lies in the boundary of the cone (since the semi-stable locus becomes strictly larger by the Hilbert-Mumford criterion). Since by Corollary 3.4, the variety $\mathfrak{M}$ is Frobenius split, [BK05, Thm. 1.4.8] shows that the corresponding line bundle induced by χ' has vanishing cohomology for all χ' in the closure of the chamber containing χ.

Lemma 4.4 *If $\chi' = w \cdot \phi' - \phi$ lies in the closure of the chamber containing χ, then we have a natural isomorphism*

$$\Phi_w^{\phi',\phi}(M) \cong \mathbb{R}\Gamma_{\mathbb{S}}({}_{w\phi'}\mathcal{L}_{\phi} \otimes \mathbb{L}\mathrm{Loc}(M))$$

where the action on the RHS is twisted by the isomorphism $\mathcal{A}_{\phi'} \cong \mathcal{A}_{w \cdot \phi'}$.

Proof. It's enough to check this on the algebra $\mathcal{A}_{\phi}$ itself. Thus, we need to show that $H^i(\mathfrak{M}; {}_{w\phi'}\mathcal{L}_{\phi}) = 0$ for $i > 0$. This is clear since this is a quantization of the line bundle induced by χ', which has trivial cohomology as discussed above. □

Corollary 4.5 *If derived localization holds at ϕ' and ϕ, then the functor $\Phi_w^{\phi',\phi}$ is an equivalence of categories.*

Corresponding to a flavor ϕ, we have a set $\Lambda^{\mathbb{R}}$ as defined as the vectors in $\mathbb{Z}^d$ such that $\mathsf{C}_{\mathbf{a}} \neq 0$; this agrees with Λ for p sufficiently large. The set $\Lambda^{\mathbb{R}}$ is locally constant, and only changes when $\psi = \phi_{1/p}$ lies on a hyperplanes in $\mathfrak{t}_{1,F,\mathbb{R}}$ defined by a circuit in the unrolled matter hyperplanes. We can thus cut the set $\mathfrak{t}_{1,F,\mathbb{Z}}$ into chambers according to what the set $\Lambda^{\mathbb{R}}$ is; these are chambers induced by the hyperplane arrangement defined by the circuits of the unrolled matter hyperplanes. We will use repeatedly that by choosing p sufficiently large, we make sure that any non-empty chamber in $\mathfrak{t}_{1,F,\mathbb{R}}$ contains a point of the form $\phi_{1/p}$ and in fact, any point in $\mathfrak{t}_{1,F,\mathbb{R}}$ can be approximated arbitrarily well by points satisfying this property. Combining Lemmata 4.3 and 4.4, we see an important compatibility for the twisting functors:

Lemma 4.6 *For p sufficiently large, if no hyperplane H_α separates both ϕ and ϕ'' from ϕ', then ${}_{\phi''}T_{\phi'} \overset{L}{\otimes}_{\mathcal{A}_{\phi'}} {}_{\phi'}T_{\phi} \cong {}_{\phi''}T_{\phi}$.*

Proof. We induct on the number m of hyperplanes separating ϕ and ϕ''. If $m = 1$, then this is trivial by Lemma 4.3, since ϕ' must be in the same chamber as one the endpoints. Let ϕ_1 be a point in the first chamber that the line segment joining ϕ to ϕ' passes through. Given that p is sufficiently large, we can assume that there is a point in this chamber such that $\phi' - \phi_1$ and $\phi_1 - \phi$ lie in the same GIT chamber, so we have ${}_{\phi'}T_{\phi_1} \overset{L}{\otimes}_{\mathcal{A}_{\phi_1}} {}_{\phi_1}T_{\phi} \cong {}_{\phi'}T_{\phi}$. By induction, ${}_{\phi}T_{\phi'} \overset{L}{\otimes}_{\mathcal{A}_{\phi'}} {}_{\phi'}T_{\phi_1} \cong {}_{\phi}T_{\phi_1}$. Thus, it suffices to prove that ${}_{\phi''}T_{\phi_1} \overset{L}{\otimes}_{\mathcal{A}_{\phi_1}} {}_{\phi_1}T_{\phi} \cong {}_{\phi''}T_{\phi}$. By replacing ϕ by another point in its chamber (again, we use that p is sufficiently large), we can assume that the straight line from ϕ to ϕ'' passes through the chamber of ϕ_1. This completes the proof. $\square$

As usual, we'll want to think of this action in a way such that p becomes large and then can be forgotten. Thus, we will want to take as our basic parameter $\psi = \phi_{1/p} \in \mathfrak{t}_{1,F,\mathbb{R}}$ which we can continuously vary. Note that the bad locus in $\mathfrak{t}_{1,F,\mathbb{R}}$ where the set $\Lambda^{\mathbb{R}}$ changes is closed under the action of the affine Weyl group $\widehat{W}_F$. We let $\mathring{\mathfrak{t}}_{1,F}$ denote the complement of the complexifications of these hyperplanes in $\mathfrak{t}_{1,F} = \mathfrak{t}_{1,F,\mathbb{C}}$, and $\mathring{T}_{1,F}$ the image of this locus under the isomorphism $T_{1,F} \cong \mathfrak{t}_{1,F}/\mathfrak{t}_{F,\mathbb{Z}}$.

Consider the fundamental group $\pi = \pi_1(\mathring{T}_{1,F}/W_F, \psi) = \pi_1(\mathring{\mathfrak{t}}_{1,F}/\widehat{W}_F, \psi)$. For each fixed p, we can consider the subgroupoid $\pi^{(p)}$ of the fundamental group with objects $\psi = \phi_{1/p}$ given by generic $\phi \in \mathfrak{t}_{1,F,\mathbb{Z}}$ (that is, the values of ϕ where derived equivalence holds).

It is a fact that seems to well-known to experts, though the author has not found any particularly satisfactory reference (this is stated as a conjecture in [Oko18, §3.2.8]), that:

Proposition 4.7 *For p sufficiently large, the functors $\Phi_w^{\phi',\phi''}$ define an action of the groupoid $\pi^{(p)}$ that induces an action of π on $D^b(\mathcal{A}_\phi\text{-mod})$.*

This should not be a special fact about Coulomb branches, but is expected to be a general fact about symplectic resolutions. A version of it is proven in [BR12] for the case of the Springer resolution and in the case of a Higgs branch by Halpern-Leistener and Sam in [HLS].

4.3. **Schobers.** We'll give a proof of Proposition 4.7 below, and in fact, show that this action is part of a more complicated structure: a *perverse Schober*, a notion proposed by Kapranov and Schechtman [KS]. Perverse schobers are not, in fact, a structure which has been defined in full generality, but for the complement of a subtorus arrangement, they can be defined using the presentation of the perverse sheaves on a complex vector space stratified by a complexified hyperplane arrangement given by the same authors in [KS16].

Definition 4.8 *Let Z be a finite-dimensional $\mathbb{R}$-affine space, and let $\{H_\gamma\}$ for γ running over a (possible infinite) index set be a locally finite hyperplane arrangement. Let ∇ be the poset of faces of this arrangement. A* **perverse Schober** *on the space $Z \otimes_{\mathbb{R}} \mathbb{C}$ stratified by the intersections of the hyperplanes $\{H_\gamma\}$ is an assignment of a dg-category $\mathcal{E}_C$ for each $C \in \nabla$, and to every pair of faces where $C' \leq C$, an assignment of* **generalization functors** *$\gamma_{CC'} : \mathcal{E}_{C'} \to \mathcal{E}_C$ and their left adjoints, the* **specialization functors** *$\delta_{C'C} \colon \mathcal{E}_C \to \mathcal{E}_{C'}$. These combine to give* **transition functors** *$\phi_{CC''} = \gamma_{CC'}\delta_{C'C''}$ whenever $\bar{C} \cap \bar{C}'' \neq \emptyset$, and C' is the unique open face in this intersection.*

1. *We have isomorphisms of functors $\gamma_{CC'}\gamma_{C'C''} \cong \gamma_{CC''}$ for a triple $C'' \leq C' \leq C$ with the usual associativity for a quadruple.*
2. *If $C' \leq C$, the unit of the adjoint pair $(\delta_{CC'}, \gamma_{C'C})$ is an isomorphism of $\gamma_{C'C}\delta_{CC'}$ to the identity. This gives a canonical isomorphism between $\phi_{CC''}$ and $\gamma_{CC'}\delta_{C'C''}$ for C' any face in the intersection $\bar{C} \cap \bar{C}''$.*
3. *If (C, C', C'') is colinear, then we have isomorphisms $\phi_{CC'}\phi_{C'C''} \cong \phi_{CC''}$ again with associativity for a colinear quadruple (C_1, C_2, C_3, C_4). This means we can define the functor $\phi_{CC''}$ for any pair of faces (C, C') by taking a generic line segment between these faces, and composing the functors $\phi_{CC_1}\phi_{C_1C_2} \cdots \phi_{C_nC'}$ for $C_1, \dots, C_n$ the full list of faces this line passes through.*
4. *If C and C' have the same dimension, span the same subspace, and are adjacent across a face with codimension 1 in C and C', then $\phi_{CC'}$ is an equivalence.*

Remark 4.9. For reasons of convenience here, we have departed a little from the framework of Kapranov and Schechtman. It would be more consistent with their definition of a Schober on a disk [KS], to assume that the equivalence $\phi_{CC'}$ will be the twist equivalence of a spherical functor, while it is more convenient for us to present it as the cotwist, as Lemma 4.1 shows, and the definition of a spherical functor is not totally symmetric. This seems to be a general feature of equivalences arising from Morita contexts.

A Schober on a complex torus T that is smooth on the faces of a subtorus arrangement is just a Schober on the preimage in the universal cover $\mathfrak{t}$, together with an action of $\pi_1(T)$ compatible with all the data above.

The case we'll be interested in the case where $Z = \mathfrak{t}_{1,F,\mathbb{R}}$ and H_α the hyperplanes defined by the circuits in unrolled matter hyperplanes. Thus, the faces are the sets on which Λ is constant. This collection of hyperplanes is invariant under the action of $\widehat{W}_F$. Thus, we can define a Schober on the quotient $\mathring{T}_{1,F}/W_F \cong \mathring{\mathfrak{t}}_{1,F}/\widehat{W}_F$ by defining a $\widehat{W}_F$-equivariant Schober on $\mathring{\mathfrak{t}}_{1,F}$, which we will do below.

This might concern some readers, since there are infinitely many hyperplanes in this arrangement, and thus infinitely many Schober relations to check. However, under the action of the affine Weyl group $\widehat{W}_F$, there are only finitely many orbits of faces, hyperplanes, etc., and thus finitely many Schober relations to check, once we have proven the obvious commutations with elements of the affine Weyl group. In particular, in the section below, we will give a proof where checking each Schober relation might require enlarging the prime p. Since we will only need to this once for each orbit of $\widehat{W}_F$, we can safely enlarge p as much as necessary at each step of the proof, and still have a finite p at the end.

4.4. **The Schober of quantized modules.** There are two natural ways to define a Schober based on a Coulomb branch. Let us first describe the quantum route, based on the representation theory of the algebras $\mathcal{A}$ and the wall-crossing functors of Section 4.2. Accordingly, this Schober is only defined over a positive base field. Now, choose a disjoint collection of open subsets $U_C \subset Z$ for each face C, contained in the star of this face, and having non-trivial intersection with each face in this star. Let u_C be the set of points $\phi \in \mathfrak{t}_{1,F;\mathbb{Z}}$ such that derived localization holds at ϕ and we have that $\phi_{1/p} \in U_C$. If $\mathsf{u}_C = \{\phi_1, \dots, \phi_k\}$ then we let $\mathcal{A}_C$ be the matrix algebra where the (i, j) entry is an element of ${}_{\phi_i}T_{\phi_j}$, that is

$$\mathcal{A}_C = \begin{bmatrix} \mathcal{A}_{\phi_1} & {}_{\phi_1}T_{\phi_2} & \cdots & {}_{\phi_1}T_{\phi_k} \\ {}_{\phi_2}T_{\phi_1} & \mathcal{A}_{\phi_2} & \cdots & {}_{\phi_2}T_{\phi_k} \\ \vdots & \vdots & \ddots & \vdots \\ {}_{\phi_k}T_{\phi_1} & {}_{\phi_k}T_{\phi_2} & \cdots & \mathcal{A}_{\phi_k} \end{bmatrix}$$

with the obvious multiplication. Any pair C and C' has a similarly defined bimodule where $\mathsf{u}_{C'} = \{\psi_1, \dots, \psi_h\}$ given by

$$T_{C,C'} = \begin{bmatrix} {}_{\phi_1}T_{\psi_k} & {}_{\phi_1}T_{\psi_2} & \cdots & {}_{\phi_1}T_{\psi_k} \\ {}_{\phi_2}T_{\psi_1} & {}_{\phi_2}T_{\psi_2} & \cdots & {}_{\phi_2}T_{\psi_k} \\ \vdots & \vdots & \ddots & \vdots \\ {}_{\phi_h}T_{\psi_1} & {}_{\phi_h}T_{\psi_2} & \cdots & {}_{\phi_h}T_{\psi_k} \end{bmatrix}$$

Of course, we can define this bimodule $T_{\mathsf{u},\mathsf{v}}$ for any pair $\mathsf{u}, \mathsf{v} \subset \mathfrak{t}_{1,F,\mathbb{Z}}$; if u or v is a singleton, then we omit brackets and just write the single element. It's easy to check using Lemma 4.3 that:

Lemma 4.10 *If we replace $U_C, U_{C'}$ by open sets $U'_C, U'_{C'}$ satisfying the same conditions, then the resulting algebras $\mathcal{A}_C$ and $\mathcal{A}'_C$ are Morita equivalent via the bimodules $T_{\mathsf{u}_C,\mathsf{u}'_C}$*

and $T_{\mathsf{u}'_C,\mathsf{u}_C}$, with this Morita equivalence preserving the bimodules $T'_{C,C'} \cong T_{\mathsf{u}'_{C'},\mathsf{u}_{C'}} \overset{L}{\otimes}_{\mathcal{A}_{C'}} T_{C,C'} \overset{L}{\otimes}_{\mathcal{A}_C} T_{\mathsf{u}_A,\mathsf{u}'_C}$

Thus the category $\mathcal{E}_C^{(p)} \cong D^-(\mathcal{A}_C\text{-mod})$ is independent of the choice of U_C, and only depends on C.

Theorem 4.11 *The assignment $\mathcal{E}_C^{(p)} \cong D^-(\mathcal{A}_C\text{-mod})$ for all $C \in \nabla$ and $\phi_{CC'} = T_{C,C'} \overset{L}{\otimes}_{\mathcal{A}_{C'}} -$ defines a Schober on $\mathfrak{t}_{1,F,\mathbb{R}}$ which is equivariant for the action $\widehat{W}_F$.*

Proof. First, we note that if $C' \leq C$, then the star of C lies in the star of C', so for any element of u_C, there is an element of $\mathsf{u}_{C'}$ Morita equivalent by the twisting bimodule. Thus, $\mathcal{A}_{C'}$ is Morita equivalent to the algebra obtained by taking the union of the sets $\mathsf{u}_C \cup \mathsf{u}_{C'}$. Now, let us check the conditions of a Schober each in turn:

(1) As discussed, if $C'' \leq C' \leq C$, then $\mathcal{A}_{C''}$ is Morita equivalent to the set obtained from the union $\mathsf{u}_C \cup \mathsf{u}_{C'} \cup \mathsf{u}_{C''}$. Thus, we need only prove the corresponding transitivity for any decomposition of 1 in a ring as the sum of 3 orthogonal idempotents $e + e' + e''$, in which case it is clear.

(2) Using the union $\mathsf{u}_C \cup \mathsf{u}_{C'}$ again, this is just the fact that for any idempotent, we have $e(Ae \otimes_{eA} M) = M$, giving the required isomorphism of $\gamma_{C'C}\delta_{CC'}$ to the identity.

(3) By assumption, if (C, C', C'') are colinear, then we can assume that the line joining them is generic in the span of these faces. Let $\mathcal{H}_0$ be the set (possibly empty) of hyperplanes that contain all three faces, $\mathcal{H}_1$ the set of hyperplanes separating C and C', and $\mathcal{H}_2$ the set separating C' and C.

Choose a point in $\phi \in \mathsf{u}_C$. We have a functor ${}_CT_\phi \overset{L}{\otimes} -\colon D^b(\mathcal{A}_\phi\text{-mod}) \to \mathcal{E}_C^{(p)}$ given by the tensor products with ${}_{\phi'}T_\phi$ for all $\phi' \in \mathsf{u}_C$. Now consider the derived tensor product with ${}_{C'}T_C$; since the image of $\mathcal{A}_\phi$ is projective, the composition is the functor of tensor product ${}_{\psi'}T_\phi$ for all $\psi' \in \mathsf{u}_{C'}$, i.e. tensor product with ${}_{C'}T_\phi$. For any point $\psi' \in \mathsf{u}_{C'}$, we can find a point in the same chamber such that the straight line to ϕ passes through any hyperplanes in $\mathcal{H}_0$ that separating ψ and ϕ before crossing any hyperplanes in $\mathcal{H}_1$. We can choose $\psi \in \mathsf{u}_{C'}$ on the same side as ϕ of all hyperplanes in $\mathcal{H}_0$, so ${}_{\psi'}T_\psi \overset{L}{\otimes}_{\mathcal{A}_\psi} {}_\psi T_\phi \cong {}_{\psi'}T_\phi$ by Lemma 4.6. That is, we have

$$ {}_{C'}T_C \overset{L}{\otimes}_{\mathcal{A}_C} {}_CT_\phi \cong {}_{C'}T_\psi \overset{L}{\otimes}_{\mathcal{A}_\psi} {}_\psi T_\phi. $$

Applying this result a second time with χ an element of u_C on the same side of all hyperplanes in $\mathcal{H}_0$ as ϕ and ψ, we have

$$ {}_CT_{C'} \overset{L}{\otimes}_{\mathcal{A}_{C'}} {}_{C'}T_{C''} \overset{L}{\otimes}_{\mathcal{A}_{C''}} {}_{C''}T_\phi \cong {}_CT_\phi \cong {}_CT_{C''} \overset{L}{\otimes}_{\mathcal{A}_{C''}} {}_{C''}T_\phi. $$

Since the projective modules ${}_{C''}T_\phi$ for all ϕ are generators for $\mathcal{A}_{C''}$-mod, this establishes that $\phi_{CC'}\phi_{C'C''} = \phi_{CC''}$. Furthermore, since these isomorphisms are induced by the natural tensor product maps, they are appropriately associative.

(4) Now, assume that C and C' are both d-dimensional, and differ across a face of codimension 1. As before, let $\mathcal{H}_0$ be the hyperplanes that contain both these faces. Note that for each $\phi \in \mathsf{u}_C$, there is a unique chamber intersecting u_C separated from ϕ by all hyperplanes in $\mathcal{H}_0$ and no others. Let ϕ' lie in this face. Then, we have that ${}_{\phi''}T_\phi$ can also be written as $\mathrm{RHom}_{\mathcal{A}_{\phi'}}({}_{\phi'}T_{\phi''}, {}_{\phi'}T_\phi)$ for all $\phi'' \in \mathsf{u}_C$, using Lemma 4.6 to show that ${}_{\phi'}T_\phi \cong {}_{\phi'}T_{\phi''} \overset{L}{\otimes}_{\mathcal{A}_{\phi''}} {}_{\phi''}T_{\phi'}$ and the fact that the inverse of a derived equivalence is its adjoint.

Now let $\psi, \psi' \in \mathsf{u}_{C'}$ be elements not separated from ϕ, ϕ' respectively by any hyperplane in $\mathcal{H}_0$. Applying the argument above and Lemma 4.6 again, we see that

$$ {}_{C'}T_C \overset{L}{\otimes}_{\mathcal{A}_C} {}_C T_\phi = \mathrm{RHom}_{\mathcal{A}_{\psi'}}({}_{\psi'}T_{C'}, {}_{\psi'}T_\phi). $$

The adjoint version of Lemma 4.6 then implies that

$$ \mathrm{RHom}_{\mathcal{A}_{C'}}({}_{C'}T_C, {}_{C'}T_C \overset{L}{\otimes}_{\mathcal{A}_C} {}_C T_\phi) = \mathrm{RHom}_{\mathcal{A}_{\psi'}}({}_{\psi'}T_C, {}_{\psi'}T_\phi) = {}_C T_\phi. $$

Again, since the projectives ${}_C T_\phi$ are generate, the functors $\phi_{C'C}$ are thus an equivalence of derived categories. □

Note that Losev shows that when C and C'' are top dimensional faces and (C, C', C'') are colinear with $C' \subset \bar{C} \cap \bar{C}''$, then $\phi_{CC''}$ is not just any equivalence of categories, but a partial Ringel duality functor in an appropriate sense (or rather, the degrading of one) and a perverse equivalence [Los, Thm. 9.10]. It would be interesting to consider whether this is true in the case where C and C'' are lower dimensional faces with the same span.

4.5. **The coherent Schober.** Of course, it is a bit inelegant to consider this Schober over in the case where $\mathbb{k} = \mathbb{F}_p$ for some large p; it would preferable to send $p \to \infty$ and replace the algebra $\mathcal{A}$ quantizing $\mathbb{F}_p[\mathfrak{M}]$ with the non-commutative resolution A.

In order to do this, we must feed every object that appeared in the quantum Schober through the woodchipper of Theorem 2.21, which allowed us to construct A in the first place. Applying this result to the bimodule ${}_{\phi'}T_\phi$, we send the wall-crossing functor to tensor product with a bimodule ${}_{\phi'_{1/p}}\hat{\mathsf{T}}_{\phi_{1/p}}$ over the categories $\hat{\mathsf{B}}_{\phi'_{1/p}}$ and $\hat{\mathsf{B}}_{\phi_{1/p}}$. Applying Theorem 2.21 again, but now to the gauge group Q, we shows that we can describe the resulting bimodule as the completion of ${}_{\phi'_{1/p}}\mathsf{T}_{\phi_{1/p}}$, the bimodule given by the Hom spaces in the quotient $\bar{\mathsf{B}} = \mathsf{B}^Q/(\mathfrak{t}_F)$ of the category for Q with the Lie algebra $\mathfrak{t}_F$ set to 0 in the morphism spaces (which is well-defined since $h = 0$ in the pth root conventions).

We can easily extend the presentation of Theorem 2.7 to $\bar{\mathsf{B}}$; essentially the only change needed is that we expand the set of objects to include all of $\mathfrak{t}_Q$. In particular, as in Section 2.5, we can replace the object set with just the elements of $\bar{\Lambda}_Q$ and form a category B^{Λ_Q} by choosing a representative $\eta_{\mathbf{a}}$ for each chamber. Note that the set $\bar{\Lambda}_Q$ contains the sets $\bar{\Lambda}$ and $\bar{\Lambda}'$ corresponding to the flavors ϕ and ϕ', and the corresponding full subcategories are exactly $\mathsf{B}^{\bar{\Lambda}}$ and $\mathsf{B}^{\bar{\Lambda}'}$ as defined in Section 2.5. Composing the equivalence of Theorem 2.21 with the equivalences

$$\mathsf{B}_{\phi_{1/p}} \cong \mathsf{B}^{\Lambda} \qquad \mathsf{B}_{\phi'_{1/p}} \cong \mathsf{B}^{\Lambda'} \qquad \bar{\mathsf{B}} \cong \mathsf{B}^{\Lambda_Q}, \tag{4.1}$$

we have that:

Lemma 4.12 *The bimodule ${}_{\phi'}T_{\phi}$ matches with the completion of the bimodule ${}_{\Lambda'}\mathsf{T}_{\Lambda}$ that sends $\mathbf{a} \in \Lambda, \mathbf{b} \in \Lambda'$ to*

$$(\mathbf{b}, \mathbf{a}) \mapsto \operatorname{Hom}_{\bar{\mathsf{B}}}(\eta_{\mathbf{a}}, \eta_{\mathbf{b}}).$$

This latter bimodule is independent of p, and thus can be defined over any field, in particular over $\mathbb{Q}$.

This has a very simple consequence for the structure of the category of modules over the algebra $\mathcal{A}_C$. Consider the set $\bar{\Lambda}_C = \cup_{\phi \in \mathfrak{u}_C} \bar{\Lambda}^{\mathbb{R}}_{\phi}$; this set is independent of p, and describes all the chambers that appear in the preimage of the star of C under the map $\mathfrak{t}_Q \to \mathfrak{t}_F$. As with all objects here, we and obtain the result that:

Proposition 4.13 *The category $\mathcal{A}_C$-mod$_{v'}$ for $\Bbbk$ a field of large positive characteristic p is equivalent to the category of modules over the completion of $\mathsf{B}^{\bar{\Lambda}_C}(\Bbbk)$, the subcategory of the completion $\bar{\mathsf{B}}(\Bbbk)$ with object set $\eta_{\mathbf{a}}$ for $\mathbf{a} \in \Lambda_C$.*

As we exploited earlier, the latter category is well-defined over any base ring, in particular over $\mathbb{Q}$. By analogy with the noncommutative resolution A, we let:

$$A_C = \bigoplus_{\bar{\mathbf{a}}, \bar{\mathbf{b}} \in \bar{\Lambda}_C} \operatorname{Hom}_{\mathsf{B}^{\bar{\Lambda}_C}(\mathbb{Q})}(\bar{\mathbf{a}}, \bar{\mathbf{b}}).$$

This ring has a presentation directly analogous to that of A given in section 3.7. We need only adjust Proposition 3.28 by changing the vertex set to be Λ_C.

Definition 4.14 *Let $\mathcal{E}^{\mathbb{Q}}_C = D^-(A_C\text{-mod})$, and $\phi^{\mathbb{Q}}_{CC'}$ be derived tensor product with the bimodule ${}_{\Lambda_C}\mathsf{T}_{\Lambda_{C'}}$.*

Note that if C is maximal dimensional, then A_C is a noncommutative crepant resolution of $\mathfrak{M}$ by Corollary 3.27, so $D^-(A_C\text{-mod}) \cong D^-(\mathrm{Coh}(\mathfrak{M}))$. Unfortunately we know no such convenient geometric interpretation of the other categories that appear for smaller strata.

Theorem 4.15 *The assignment $\mathcal{E}^{\mathbb{Q}}_C$ and $\phi^{\mathbb{Q}}_{CC'}$ above defines a $\widehat{W}_F$-equivariant Schober.*

Proof. The required isomorphisms are all induced by composition of maps, so in order to show that the Schober relations hold, it is enough to check that we have the Schober relations mod infinitely many primes p. This is clear from comparison with the Schober $\mathcal{E}^{(p)}$ of Theorem 4.11 via the functor of Lemma 3.12. □

Note the similarity of this action with that defined using the "magic windows" approach of [HLS]. It would be quite interesting to understand how these approaches compare when the same symplectic singularity can be written as both a Higgs and Coulomb branch.

For a fixed basepoint, we can choose a D-equivalence between the nccr A_C for a maximal dimensional face, and the commutative resolution $\tilde{\mathfrak{M}}$. This shows that:

Corollary 4.16 *The functors $\phi_{CC'}$ define an action of π on $D^b(\mathrm{Coh}(\tilde{\mathfrak{M}}_{\mathbb{Q}}))$.*

A long-standing conjecture of Bezrukavnikov and Okounkov connects these actions to enumerative geometry, as discussed in [Oko18, §3.2]:

Conjecture 4.17 *The action of π on $\mathrm{Coh}(\tilde{\mathfrak{M}}_{\mathbb{Q}})$ categorifies the monodromy of the quantum connection.*

A positive resolution to this conjecture has been announced by Bezrukavnikov and Okounkov, but as of the current moment, the proof has not appeared. Of course, it would be quite interesting to understand whether the Schober discussed above contains deeper information about the quantum D-module.

Glossary

V	The matter representation	1, 2, 6, 11, 14, 43, 44
G	The gauge group	1, 2, 6, 11, 14, 20, 42, 43
$\mathfrak{M}$	The Coulomb branch—the quotient of the convolution algebra of a modified affine Grassmannian as defined in [BFN18]	1, 7, 22, 30, 42–44
$\tilde{\mathfrak{M}}$	The resolution of the Coulomb branch $\mathfrak{M}$ defined by taking symplectic redution with a GIT quotient of $\mathfrak{M}_Q$	1, 4, 5, 7, 22–24, 27, 30, 35, 42, 44
A	The noncommutative resolution of $\mathfrak{M}$ constructed by quantization. Also, the sum of morphisms in $\mathsf{B}^{\bar{\Lambda}}(\mathbb{Q})$.	1, 2, 4, 19, 30, 31, 40, 41
pth root	The conventions for the extended category adopted in Definition 2.12	2, 13, 15–17, 40, 43, 44
H	The normalizer $N^{\circ}_{GL(V)}(G)$	6, 42
F	The flavor group H/G	6, 7, 11, 14, 29, 42
T_*	A maximal torus of the group $*$	6–8, 20, 42, 43
$\mathfrak{t}_{*,\dagger}$	The Lie algebra of the torus T_* over $\dagger = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	6, 40
Q	The preimage in the normalizer H of a fixed torus T_F in the flavor group F	6, 7, 22, 26, 40, 43, 44

ϕ	The flavor: a fixed cocharacter $\phi\colon \mathbb{C}^* \to T_F$	6, 11, 13, 18, 19, 29–31, 34
$\mathcal{A}$	The Iwahori Coulomb branch $\mathcal{A} = H_*^{BM,\widetilde{G((t))}}(\mathfrak{X}_{V[[t]]} \times_{V((t))} \mathfrak{X}_{V[[t]]})$	6, 7, 18, 20, 38, 40, 44
S_*	The symmetric algebra on $\tilde{\mathfrak{t}}^*$, that is, the ring of functions on the affine variety $\tilde{\mathfrak{t}}$, with the parameter h specialized at $h = *$	6, 9–11, 15, 16
$\mathcal{A}^{\mathrm{sph}}$	The quantum Coulomb branch $\mathcal{A}^{\mathrm{sph}} = H_*^{BM,\widetilde{G((t))}}(\mathcal{Y}_{V[[t]]} \times_{V((t))} \mathcal{Y}_{V[[t]]})$	6–8, 20, 21, 44
$\mathfrak{M}_Q$	The Coulomb branch $\mathfrak{M}$ attached to the group Q acting on V with its usual action	7, 22, 23, 27, 42
K	The Langlands dual $T_F^\vee$, or equivalently, the Pontryagin dual $\mathrm{Hom}(\mathfrak{t}_{\mathbb{Z}}^*, \mathbb{C}^*)$.	7, 22, 23, 26
δ	A parameter between the open interval $(0,1) \subset \mathbb{R}$ used in the definition of $\mathscr{B}$	7, 8, 13
$\mathfrak{t}_{1,\mathbb{R}}$	Let $\mathfrak{t}_{1,\mathbb{R}} \subset \tilde{\mathfrak{t}}_{\mathbb{R}} = d\nu^{-1}(1)$ be the real lifts of the cocharacter ϕ.	7, 8, 13, 18, 24, 26, 27, 35, 38
φ_i^{mid}	The average of φ_i^+ and $-\varphi_i^-$	7, 8, 13–15, 17
$\widehat{W}$	The affine Weyl group of G. The semi-direct product of W and the coweight lattice of T	8, 9, 12, 18, 29, 44
${}_\eta\mathfrak{X}_{\eta'}$	The subspace $\{(g, v(t)) \in G((t)) \times U_\eta \mid g \cdot v(t) \in U_{\eta'}\}/\mathrm{Iwa}_\eta \subset \mathfrak{X}_\eta$	8, 11
$\mathscr{B}$	The extended BFN category, defined in Definition 2.4	8, 10–13, 43
τ	The cocharacter $\tau\colon \mathbb{C}^* \to GL(T^*V)$ that acts trivially on V and weight -1 on V^*	8, 20, 25, 26
y_w	The homology classes that give the action of $\widehat{W}$ in $\mathscr{B}$	9, 10
$r(\eta, \eta')$	The homology classes corresponding to fibers over torus fixed points	9, 10, 16, 17
u	The homology classes corresponding crossing a root hyperplane	9, 10
W	The Weyl group of G	11, 12, 21, 24–26, 43
${}_{\phi+\nu}\mathscr{T}_\phi$	The $\mathscr{B}_{\phi+\nu}$ - $\mathscr{B}_\phi$ bimodule formed by the appropriate quotient of $\mathscr{T}(\nu)$, the morphisms of weight ν in $\mathscr{B}^Q$	11, 43
${}_{\phi+\nu}T_\phi$	The twisting bimodule ${}_{\phi+\nu}\mathscr{T}_\phi(o, o)$	11, 23, 34, 44
$\widehat{\mathscr{B}}$	The completion of the extended BFN category with respect to the different maximal ideals of S_1.	12, 26, 43
$\widehat{\mathscr{B}}_\upsilon$	The subcategory of $\widehat{\mathscr{B}}$ where we only allow objects (η, υ) with $\eta \in \mathfrak{t}_\tau, \upsilon \in \upsilon' + \mathfrak{t}_{\mathbb{Z}}$.	12, 15, 16, 18
$\widehat{\mathscr{A}_\upsilon}$	The subcategory of $\widehat{\mathscr{B}}$ where we only allow objects of the form (o, υ) for $\upsilon \in \upsilon' + \mathfrak{t}_{\mathbb{Z}}$.	12, 18, 20
B	The extended BFN category with pth root conventions	13, 16–18, 26, 28, 31, 44

$\mathsf{C}_{\mathbf{a}}$	The chamber $\mathsf{C}_{\mathbf{a}} = \{\xi \in \mathfrak{t}_{1,\mathbb{R}} \mid a_i < \varphi_i^{\mathrm{mid}}(\xi) < a_i + 1 \text{ for } i = 1, \dots, d\}$	15, 29, 35
$\widehat{\mathsf{B}}$	The completion with respect to grading of the extended BFN category B with pth root conventions	16, 18, 26, 40
A_p	The subcategory of B defined in Definition 2.20	18, 19, 29
Λ	The set of vectors such that $\mathsf{C}_{\mathbf{a}}$ contains $\xi_{1/p}$ for $\xi \in \mathfrak{t}_{1,\mathbb{Z}}$.	18, 28, 34, 35, 44
$\bar{\Lambda}$	The quotient set $\Lambda/\widehat{W}$.	18, 19, 28, 29, 31, 44
$\mathsf{B}^{\bar{\Lambda}}(\Bbbk)$	The category over the base ring $\Bbbk$ with object set $\bar{\Lambda} \subset \mathbb{Z}^d$ and morphism spaces $\mathrm{Hom}_{\mathsf{B}}(\eta_{\mathbf{a}}, \eta_{\mathbf{b}})$ for $\eta_{\mathbf{a}}$ an arbitrary element of the chamber $\mathsf{C}_{\mathbf{a}}$.	19, 30, 41, 42
$\mathscr{W}_\phi$	The coherent sheaf of generically Azumaya algebras on $\tilde{\mathfrak{M}}$ or its pushforward to $\mathfrak{M}$, such that $\Gamma(\tilde{\mathfrak{M}}; \mathscr{W}_\phi) = \mathcal{A}_1^{\mathrm{sph}}$ with the quantization parameter ϕ	22–24, 27, 44
$\mathscr{H}_\phi$	The coherent sheaf of generically Azumaya algebras on $\tilde{\mathfrak{M}}$ or its pushforward to $\mathfrak{M}$, such that $\Gamma(\tilde{\mathfrak{M}}; \mathscr{W}_\phi) = \mathcal{A}_1$ with the quantization parameter ϕ	22–24, 44
$\mathcal{Q}_h$	The right module quotient $\mathcal{Q}_h = \mathcal{A}_{h,Q}^{\mathrm{sph}}/\mathfrak{k} \cdot (\mathcal{A}_{h,Q}^{\mathrm{sph}})$. If $h = 0$, this is the algebra $\mathbb{F}_p[\mu^{-1}(0)]$	22–24
$\hat{\mathfrak{M}}$	The formal neighborhood in $\mathfrak{M}$ of the fiber over the origin in $\mathfrak{t}/W$	24, 44
$\hat{\mathscr{W}}_\phi$	The restriction of $\mathscr{W}_\phi$ to the formal neighborhood $\hat{\mathfrak{M}}$	24, 28
$\hat{\mathscr{H}}_\phi$	The restriction of $\mathscr{H}_\phi$ to the formal neighborhood $\hat{\mathfrak{M}}$	24–26
$\hat{\mathcal{Q}}_\mu$	The vector bundle on $\hat{\mathfrak{M}}$ defined by $\hat{\mathcal{Q}}_\mu := e_\mu \hat{\mathscr{H}}_\phi e_{0,\tau}$	25, 26
$\mathscr{B}^Q$	The extended BFN category attached to the pair (Q, V), defined in Definition 2.4	26, 27, 40
$\mathcal{Q}_\mu^{\Bbbk}$	The vector bundle on $\tilde{\mathfrak{M}}$ defined by Hamiltonian reduction of $\mathsf{R}_\mu(\Bbbk) = \mathrm{Hom}_{\mathscr{B}^Q(\Bbbk)}(\tau, -\tilde{\mu}_{1/p} + o)$	27–29, 44
$\Lambda^{\mathbb{R}}$	The set of vectors such that $\mathsf{C}_{\mathbf{a}}$ is non-empty.	29, 30, 35, 44
$\mathcal{Q}_\phi^{\mathbb{Q}}$	The tilting generator obtained by summing the vector bundles $\mathcal{Q}_\mu^{\mathbb{Q}}$ over representatives of the set $\bar{\Lambda}^{\mathbb{R}}$ for generic ϕ.	29, 30
$\Phi_w^{\phi',\phi}$	The functor of derived tensor product with the twisting bimodule ${}_{w\phi'}T_\phi$.	34, 35

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