HW 5: PMATH 945

Due Mar. 7 at 11:59 pm

(1) (a) Let X be a reduced locally Noetherian scheme. Describe how rational maps $X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$ are the same thing as rational functions.

A rational map $X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$ is a morphism from a dense open subset $U \subset X$ to $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[t])$ (note that an open subset of a reduced locally Noetherian scheme is dense if and only if it contains all associated points). Such a morphism must be induced by a map of rings $\mathbb{Z}[t] \to \mathcal{O}_X(U)$, which is, in turn induced by choosing any element of $\mathcal{O}_X(U)$ which is the image of t. This image is the corresponding rational function.

(b) Show that a rational map $\pi\colon X \dashrightarrow Y$ of integral schemes is dominant if and only if π sends the generic point of X to the generic point of Y.

By replacing X and Y by open affine subsets, we can assume that they are affine and that the rational map is an honest morphism. Thus, we have a map $B = \mathcal{O}_Y(Y) \to A = \mathcal{O}_X(X)$. The map π is dominant if and only if the map $B \to A$ is injective. This is equivalent to saying that the preimage of the ideal $(0) \subset A$ is $(0) \subset B$ which is to say, the generic point is sent to the generic point.

(c) Let k be a field. Call an extension field $E \supset k$ *finitely-generated* if there is a finite subset of E contained in no proper subfield of E. Describe an equivalence between the categories:

(i) objects: finite type integral affine k-schemes; morphisms: dominant rational maps of k-schemes.

(ii) objects: finitely generated field extensions of k; morphisms: k-algebra homomorphisms in the opposite direction. (That is: take the opposite category to the obvious one.)

We can define a functor by sending X a finite type integral affine k-scheme to its function field $K(X)$. This is finitely generated as an extension of k, since the functions of X are finitely generated as a k-algebra. A dominant map $X \to Y$ maps generic point to generic point, and thus induces a k-algebra homomorphism $K(Y) \to K(X)$, and compatibility with composition shows that this is a functor. This functor hits every object since if $\{y_1, dots, y_n\} \subset K(Y)$ is a finite subset not contained in any proper subfield, then the image A of $k[y_1, \dots, y_n]$ in $K(Y)$ is a finitely generated k-algebra whose fraction field is $K(Y)$. Any k-algebra homomorphism $K(Y) \to K(X)$ is induced by a map of A into $K(X)$; the subring generated by A and any finite subset of $K(X)$ not contained in any subfield gives a finite generated k-algebra B. The map $A \to B$ defines a rational map

$X \dashrightarrow Y$, which is dominant, since the map $K(Y) \to K(X)$ and thus $A \to B$ is injective.

(2) Let X be a $\mathbb{F}_p$-scheme. Explain how to define an endomorphism $F_X \colon X \to X$ such that:
  (a) If $X = \operatorname{Spec} A$, then F_X is induced by the Frobenius map on A which sends $a \mapsto a^p$.
  (b) For any morphism of $\mathbb{F}_p$-schemes $\pi \colon X \to Y$, we have $\pi \circ F_X = F_Y \circ \pi$.

Prove that if X is locally of finite type over $\mathbb{F}_p$, then F_X is finite.

If $X = \operatorname{Spec} A$, we already have the definition of F_X. Note that for any prime ideal P, we have $a^p \in P$ if and only if $a \in P$, so F_X acts on the points of X by the identity map.

Thus, for X arbitrary, let F_X be the morphism of schemes whose underlying map is the identity, and where the pullback map $\mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is the Frobenius map. This has desired commutation with any map of schemes, since we can reduce to the affine case, and then use the fact that $\phi(a)^p = \phi(a^p)$ for all $\mathbb{F}_p$-algebra homomorphisms.

In order to show that F_X is finite, we can reduce to the case of X affine since finiteness is a "reasonable" property, and can be checked affine-locally on the target. For $X = \operatorname{Spec} A$, we need to show that A is finitely generated as a module over A^p. Since A is finitely generated as an $\mathbb{F}_p$-algebra, it's enough to check this in the case $A = \mathbb{F}_p[x_1, \dots, x_n]$. This ring is generated as an A^p-module by $x_1^{a_1} \cdots x_n^{a_n}$ for all $a_i < p$.

(3) (a) Let Y_1 and Y_2 be closed subschemes of X. Prove that there is a unique smallest closed subscheme $Y_1 \cup Y_2$ containing Y_1 and Y_2 and that its underlying set is the union of the underlying sets of Y_1 and Y_2.

Consider the union of the underlying sets of Y_1 and Y_2. We endow this with a scheme structure by defining $\mathcal{O}^{\mathrm{pre}}_{Y_1 \cup Y_2}(U) = \operatorname{image}(\mathcal{O}_{Y_1 \cup Y_2}(U) \to \mathcal{O}_{Y_1}(U \cap Y_1) \oplus \mathcal{O}_{Y_2}(U \cap Y_2))$ and $\mathcal{O}_{Y_1 \cup Y_2}$ its sheafification. If $U' = \operatorname{Spec} A \subset X$, then we have $U \cap Y_i = \operatorname{Spec} A/I_i$, and $\mathcal{O}_{Y_1 \cup Y_2}(U' \cap (Y_1 \cup Y_2)) = A/(I_1 \cap I_2)$ since this is the image of the map to $A/I_1 \oplus A/I_2$. This shows that with this scheme structure, the inclusion of $Y_1 \cup Y_2$ is a closed subscheme. Now, assume that Z is a subscheme containing both Y_1 and Y_2. Intersecting again with U', we find that $Z \cap U' = V(J)$ for some idea in J. Since $J \subset I_1, I_2$, so $J \subset I_1 \cap I_2$, so Z contains $Y_1 \cup Y_2$.

  (b) Fix a finitely generated A-module M and an element $m \in M$. Consider $V(\operatorname{Ann} m)$ as a closed subscheme of $\operatorname{Spec} A$; we call this the **scheme-theoretic support** of m. Show that $\operatorname{Supp} M = V(\operatorname{Ann} M)$ is the union of the closed subschemes $\operatorname{Supp} m$ and that the underlying set of this subscheme is the support as defined before (the closed subset where $\tilde{M}$ has non-vanishing stalks).

We can translate this into a statement about ideals: $\operatorname{Ann} M$ *is the intersection of the ideal* $\operatorname{Ann} m$ *for* $m \in M$*. In fact, it's enough to let* m *range over a finite generating set of* M*. That shows that* $\operatorname{Supp} M = \bigcup_{m \in M} \operatorname{Supp} m$*.*

For a given prime P*, the point* $[P]$ *is in the support if and only* $P \supset \operatorname{Ann} m$ *for some* $m \in M$*; this is the case if and only if* $ms \neq 0$ *for all* $s \notin P$*, that is, if* m *has non-zero localization in the image in the localization* M_P*. Thus, this happens if and only if* $M_P \neq 0$*.*

(c) Show that for any coherent sheaf $\mathcal{F}$ on X, this defines a closed subscheme structure on $\operatorname{Supp} \mathcal{F}$ (the closed subset where $\mathcal{F}$ has non-vanishing stalks).

Given a coherent sheaf $\mathcal{F}$*, its restriction to an affine open set* $U \cong \operatorname{Spec} A$ *is* $\tilde{M}$ *for a finitely generated module* M*. Thus, we can define a subsheaf* $\mathcal{I}$ *of* $\mathcal{O}_X$ *given by the elements* $\mathcal{I}(V)$ *whose restriction to each* $U \cap V$ *with* U *affine vanishes on the subscheme* $\operatorname{Supp} M$*. The vanishing subshceme of this sheaf of ideals defines a closed subscheme whose intersection with* U *is* $\operatorname{Supp} M$*. In particular, its underlying closed subset is exactly the points where the stalk is non-zero, since we've already made this calculation for affine schemes.*

(4) Consider $Y = \mathbb{P}^3_k = \operatorname{Proj}(k[x, y, z, w])$ and the subscheme $X = V(wz - xy, x^2 - wy, y^2 - xz)$. Show that $X \cong \mathbb{P}^1_k$. *The intersection with the open set defined by* $D^+(x)$ *is*

$$\frac{w}{x}\frac{z}{x} - \frac{y}{x} \qquad 1 - \frac{w}{x}\frac{y}{x} \qquad (\frac{y}{x})^2 - \frac{z}{x}$$

so

$$\frac{z}{x} = (\frac{y}{x})^2 \qquad \frac{w}{x} = \frac{x}{y}$$

and the intersection with the subscheme X *is* $\operatorname{Spec}(k[\frac{y}{x}, \frac{x}{y}]) = \mathbb{A}^1 - pt$*.*

The intersection with the open set defined by $D^+(y)$ *is the same, with the pairs* (x, y) *and* (z, w) *switched.*

The intersection with the open set defined by $D^+(w)$ *is*

$$\frac{z}{w} - \frac{x}{w}\frac{y}{w} \qquad (\frac{x}{w})^2 - \frac{y}{w} \qquad (\frac{y}{w})^2 - \frac{z}{w}\frac{x}{w}$$

so

$$\frac{y}{w} = (\frac{x}{w})^2 \qquad \frac{z}{w} = (\frac{x}{w})^3$$

so the intersection with the subscheme X *is* $\operatorname{Spec}(k[\frac{x}{w}]) = \mathbb{A}^1$*.*

Similarly, $D^+(z)$ *is* $\operatorname{Spec}(k[\frac{y}{z}]) = \mathbb{A}^1$*. To see how these glue together, consider the open set* $D^+(xyzw)$*. On this open set* $u = \frac{x}{w} = \frac{y}{x} = \frac{z}{y}$*, so gluing together* $D^+(w)$ *and* $D^+(z)$ *creates a copy of* $\mathbb{P}^1_k$ *with coordinate* u*. The open sets* $D^+(x)$ *and* $D^+(y)$ *are identified with the intersection of these open sets and thus don't contribute any other points.*