HW 1: PMATH 945

Due Jan. 17 at 11:59 pm

(1) (a) Suppose E is a topological space equipped with a continuous map $E \to X$. Show that continuous sections of this map form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps of $\sigma \colon U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$. Show that this forms a sheaf $\mathcal{F}_E$.

Let $\mathcal{F}_E(U) = \{\sigma \colon U \to E \mid \pi \circ \sigma = \mathrm{id}_U\}$. The restriction of a function to a smaller open subset still satisfies the section property; this satisfies the presheaf property, since the sections are actual functions. If $U = \bigcup_{i \in I} U_i$, and $f_1|_{U_i} = f_2|_{U_i}$, then $f_1(x) = f_2(x)$ for all $x \in U_i$ for all i, which implies that $f_1(x) = f_2(x)$ for all $x \in U$. On the other hand, if we have $f_i \in \mathcal{F}_E(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, we can define f by $f(x) = f_i(x)$ for $x \in U_i$; this is consistent because of the overlap property, and continuous since each f_i is continuous. This shows all the sheaf properties.

(b) For a presheaf $\mathcal{F}$ of sets on X, let $E_{\mathcal{F}}$ be the disjoint union of the stalks $\mathcal{F}_x$ equipped with the obvious map to X (we'll write elements of this as (x, s) for $x \in X, s \in \mathcal{F}_x$), given the topology with basis $U_{V,f} = \{(v, f_v) \text{ for all } v \in V\}$ for V open, $f \in \mathcal{F}(V)$. Show that if $\mathcal{F}$ is a sheaf, the continuous sections of the map $\pi \colon E_{\mathcal{F}} \to X$ are isomorphic to the sheaf $\mathcal{F}$.

For $g \in \mathcal{F}(U)$, we can define a section $\sigma_g(x) = (x, g_x)$ for $x \in U$. This is continuous since if $(x, g_x) \in U_{V,f}$, and so for some $U' \subset U \cap V$, we have $g|_{U'} = f|_{U'}$, and so $U' \subset \sigma_g^{-1}(U_{V,f})$. We claim that $g \mapsto \sigma_g$ is the desired isomorphism of sheaves.

Assume that $\sigma(x) = (x, \Sigma(x))$ is a section over U. For each x, the element of the stalk $\Sigma(x)$ is the germ of some section $f_x \in \mathcal{F}(V'_x)$. By continuity, $\sigma^{-1}(U_{V'_x, f^{(x)}})$ contains a neighborhood V_x of x and on this open subset $\Sigma(y) = f^{(x)}_y$ for all $y \in V_x$. In particular, this means $f^{(x)}$ and $f^{(x')}$ agree on $V_x \cap V_{x'}$ since they agree on stalks at every point, so by the gluing property, they glue into a section $f \in \mathcal{F}(U)$.

(c) For a presheaf $\mathcal{F}$ on X, let $\mathsf{S}(\mathcal{F}) = \mathcal{F}_{E_{\mathcal{F}}}$. Show that S is a functor $\mathsf{Set}_X^{\mathrm{pre}} \to \mathsf{Set}_X$ that sends every presheaf to its sheafification.

If $\mathcal{F}$ and $\mathcal{G}$ are presheaves, then a map $\phi \colon \mathcal{F} \to \mathcal{G}$ induces a map $\phi_p \colon \mathcal{F}_p \to \mathcal{G}_p$, and thus a map $E_\phi \colon E_{\mathcal{F}} \to E_{\mathcal{G}}$.

Given $V \subset X$ and $g \in \mathcal{G}(V)$, we have $E_\phi^{-1}(U_{V,g}) = \cup_{\phi(f) = g|_W} U_{W,f}$, so this map is continuous, and thus induces a map $\mathsf{S}(\mathcal{F}) \to \mathsf{S}(\mathcal{G})$.

Since this is clearly compatible with composition, it defines a functor. Furthermore, the map defined earlier, sending $f \in \mathcal{F}(U)$ to the section $x \mapsto f_x \in \mathsf{S}(\mathcal{F})$ is a map of presheaves. Furthermore, if $\mathcal{F}$ is a presheaf and $\mathcal{G}$ is a sheaf, then any morphism $\phi \colon \mathcal{F} \to \mathcal{G}$ induces a map $\mathsf{S}(\mathcal{F}) \to \mathsf{S}(\mathcal{G})$, such that the composition $\mathcal{F} \to \mathsf{S}(\mathcal{F}) \to \mathsf{S}(\mathcal{G}) \cong \mathcal{G}$ is equal to the original ϕ. This shows that $\mathsf{S}(\mathcal{F})$ satisfies the universal property of a sheafification, since ϕ factors through it.

(2) Suppose $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is an exact sequence of sheaves of abelian groups on a topological space X . If $\pi : X \to Y$ is a continuous map, show that

$$0 \longrightarrow \pi_* \mathscr{F} \longrightarrow \pi_* \mathscr{G} \longrightarrow \pi_* \mathscr{H}$$

is exact.

First, consider the kernel of the map $\pi_*\mathscr{F} \longrightarrow \pi_*\mathscr{G}$. If $f \in \pi_*\mathscr{F}(U) = \mathscr{F}(\phi^{-1}(U))$ is in the kernel of the map to $f \in \pi_*\mathscr{G}(U) = \mathscr{G}(\phi^{-1}(U))$, then it is 0 by the injectivity of $\mathscr{F}(\phi^{-1}(U)) \to \mathscr{G}(\phi^{-1}(U))$.

Now, consider $\ker(\mathscr{G} \longrightarrow \mathscr{H})$. *Since this is the kernel of a map between sheaves, it is naturally a sheaf (without any sheafifying, isomorphic to $\mathscr{F}$. That is, the sequence $0 \longrightarrow \mathscr{F}(\phi^{-1}(U)) \longrightarrow \mathscr{G}(\phi^{-1}(U)) \longrightarrow \mathscr{H}(\phi^{-1}(U))$ is exact. Thus, the same is true for $0 \longrightarrow \pi_*\mathscr{F}(U) \longrightarrow \pi_*\mathscr{G}(U) \longrightarrow \pi_*\mathscr{H}(U)$ for all U. This shows that* $\ker(\pi_*\mathscr{G} \longrightarrow \pi_*\mathscr{H})$ *is isomorphic to $\pi_*\mathscr{F}$ via the obvious map.*

(3) (a) State a universal property which can be used to define the tensor product of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.

The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is initial amongst all $\mathcal{O}_X$-module sheaves $\mathcal{M}$ equipped with $\mathcal{O}_X(U)$-bilinear maps $\mathcal{F}(U) \times \mathcal{G}(U) \to \mathcal{M}(U)$ compatible with restriction maps (that is, $\mathcal{M}$ receives a $\mathcal{O}_X$-bilinear map of sheaves from the Cartesian product sheaf $\mathcal{F} \times_{\mathcal{O}_X} \mathcal{G}$).

(b) Give an explicit construction of a sheaf (be sure it is not just a presheaf) satisfying this property for given $\mathcal{F}$ and $\mathcal{G}$.

Consider the presheaf $\mathcal{F} \otimes^{\mathsf{pre}}_{\mathcal{O}_X} \mathcal{G}(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. If $\mathcal{M}$ receives a bilinear map as above, then we have a map $\mathcal{F} \otimes^{\mathsf{pre}}_{\mathcal{O}_X} \mathcal{G}(U) \to \mathcal{M}(U)$ such that the obvious diagram commutes, and we commute with restriction maps. That is, we have a morphism of presheaves $\mathcal{F} \otimes^{\mathsf{pre}}_{\mathcal{O}_X} \mathcal{G} \to \mathcal{M}$ such that the bilinear map $\mathcal{F} \times \mathcal{G} \to \mathcal{M}$ factors through $\mathcal{F} \otimes^{\mathsf{pre}}_{\mathcal{O}_X} \mathcal{G}$. Thus, the sheafification $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F} \otimes^{\mathsf{pre}}_{\mathcal{O}_X} \mathcal{G})^{\mathrm{sh}}$ is a sheaf satisfying the universal property.

(c) Show that the stalk of a tensor product of sheaves over $\mathcal{O}_X$ is the tensor product of the stalks over the local ring $\mathcal{O}_{X,p}$.

We have a natural map $\mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \mathcal{G}_p \to (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_p$ sending $(U, f) \otimes (V, g) \mapsto (U \cap V, f \otimes g)$. This is surjective, since any germ at p of $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ can be represented by an element of $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ for some neighborhood U of p, and this is clearly in the image. Furthermore, any two such representatives must agree after restriction to a smaller neighborhood W. Thus, they have the same image in $\mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \mathcal{G}_p$; this gives a well-defined inverse map.