HW 2: PMATH 945

Due Jan. 26 at 11:59 pm

(1) (a) Show that if $R \cong R_1 \times R_2$ for commutative rings R_1, R_2, then $\mathrm{Spec} R \cong \mathrm{Spec} R_1 \coprod \mathrm{Spec} R_2$ (where $\coprod$ is disjoint union, i.e. coproduct, of topological spaces) as ringed spaces (or equivalently, as schemes).

(b) Show that the topological space $\mathrm{Spec}(R)$ is disconnected if and only if $R \cong R_1 \times R_2$ for some commutative rings R_1, R_2 (recall that by assumption, a commutative ring has a unit, and thus is not zero).

(c) If we choose R_i for $i \in \mathbb{Z}_{\geq 0}$, consider the direct product $R = \prod_{i=0}^{\infty} R_i$. Show that $\coprod_{i=0}^{\infty} \mathrm{Spec}(R_i)$ is a proper subset of $\mathrm{Spec}(R)$ (Vakil suggests looking at the direct sum $\oplus_{i=0}^{\infty} R_i$ for an ideal not contained in any of the obvious ones).

(2) Let M be an R-module.

(a) Prove there is a sheaf of $\mathcal{O}_X$-modules $\tilde{M}$ on $X = \mathrm{Spec}(R)$ such that $\tilde{M}(D(f)) = R_f \otimes_R M$.

(b) Prove that the stalks $\tilde{M}_{[\mathfrak{p}]}$ is the localization $R_{\mathfrak{p}} \otimes_R M$ for $R_{\mathfrak{p}}$ the localization at the multiplicative set $R \setminus \mathfrak{p}$.

(3) Show that for any affine scheme $\mathrm{Spec}(R)$ and $f \in R$, the restriction $\mathcal{O}_X|_{D(f)}$ of the structure sheaf to $D(f)$ and the subspace topology make the ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ into an affine scheme.