HW 1: PMATH 945

Due Jan. 17 at 11:59 pm

(1) (a) Suppose E is a topological space equipped with a continuous map $E \to X$. Show that continuous sections of this map form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps of $\sigma\colon U \to E$ such that $\pi \circ \sigma = \mathrm{id}_U$. Show that this forms a sheaf $\mathcal{F}_E$.

(b) For a presheaf $\mathcal{F}$ of sets on X, let $E_{\mathcal{F}}$ be the disjoint union of the stalks $\mathcal{F}_x$ equipped with the obvious map to X (we'll write elements of this as (x, s) for $x \in X, s \in \mathcal{F}_x$), given the topology with basis $U_{V,f} = \{(v, f_v) \text{ for all } v \in V\}$ for V open, $f \in \mathcal{F}(V)$. Show that if $\mathcal{F}$ is a sheaf, the continuous sections of the map $\pi\colon E_{\mathcal{F}} \to X$ are isomorphic to the sheaf $\mathcal{F}$.

(c) For a presheaf $\mathcal{F}$ on X, let $\mathsf{S}(\mathcal{F}) = \mathcal{F}_{E_{\mathcal{F}}}$. Show that S is a functor $\mathsf{Set}_X^{\mathrm{pre}} \to \mathsf{Set}_X$ that sends every presheaf to its sheafification.

(2) Suppose $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is an exact sequence of sheaves of abelian groups on a topological space X . If $\pi : X \to Y$ is a continuous map, show that

$$0 \longrightarrow \pi_*\mathscr{F} \longrightarrow \pi_*\mathscr{G} \longrightarrow \pi_*\mathscr{H}$$

is exact.

(3) (a) State a universal property which can be used to define the tensor product of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.

(b) Give an explicit construction of a sheaf (be sure it is not just a presheaf) satisfying this property for given $\mathcal{F}$ and $\mathcal{G}$.

(c) Show that the stalk of a tensor product of sheaves over $\mathcal{O}_X$ is the tensor product of the stalks over the local ring $\mathcal{O}_{X,p}$.