HW 4: PMATH 945

Due Feb. 21 at 11:59 pm

(1) An element $m \in M$ of a module over an integral domain A is called *torsion* if $am = 0$ for some $a \in A - \{0\}$. A module is called *torsion* if every element of it is torsion, and *torsion-free* if no non-zero element of it is torsion.

Assume X is an integral locally Noetherian scheme. Call a quasi-coherent sheaf $\mathcal{F}$ on X *torsion (torsion-free)* if its stalk $\mathcal{F}_p$ at every point $p \in X$ is a torsion (torsion-free) $\mathcal{O}_{X,p}$-module.

(a) Prove that every quasi-coherent sheaf $\mathcal{F}$ has a unique subsheaf $\mathcal{F}_{tors}$ such that $\mathcal{F}_{tors}$ is torsion and $\mathcal{F}/\mathcal{F}_{tors}$ is torsion-free.

Let $\mathcal{F}'_{tors}(U) = \{m \in \mathcal{F}(U) | fm = 0$ for some nonzero $f \in \mathcal{O}_X(U)\}$. This is a presheaf, since the restriction of m and f to an open subset V of U will still satisfy $f|_V \neq 0$ (since X is integral) and $fm|_V = 0$, so $m|_V \in \mathcal{F}'_{tors}(V)$; let $\mathcal{F}_{tors}$ be the sheafification of $\mathcal{F}_{tors}$. If $m_p \in (\mathcal{F}_{tors})_p$, then $fm|_V = 0$ for some $f \neq 0$, and $f_p \neq 0$ since X is integral, so m_p is torsion for all $m_p \in (\mathcal{F}_{tors})_p$.
To show that this is coherent, it suffices to show that for any affine U, then map $\mathcal{F}_{tors}(U)_f \to \mathcal{F}_{tors}(U_f)$ for U affine is an isomorphism. This map is injective since $\mathcal{F}$ is coherent, and localization is flat. if $m \in \mathcal{F}_{tors}(U_f)$, then $m = m'/f^n$ for $m' \in \mathcal{F}(U)$, and m being torsion implies $(g/f^k)m = 0$ so $f^\ell gm = 0 \in \mathcal{F}_{tors}(U)$. Thus m is torsion in $\mathcal{F}(U)$, and so the map $\mathcal{F}_{tors}(U)_f \to \mathcal{F}_{tors}(U_f)$ is surjective and thus an isomorphism.
On the other hand, if we consider the stalks of $\mathcal{F}/\mathcal{F}_{tors}$, these are the same as the stalks of $\mathcal{G} = \mathcal{F}/\mathcal{F}'_{tors}$, since sheafification preserves stalks. In particular, if $m_p + (\mathcal{F}'_{tors})_p \in \mathcal{G}_p$ is torsion, then $f_p(m_p + (\mathcal{F}'_{tors})_p) = 0$ for some $f_p \neq 0$. Then $f_p m_p \in (\mathcal{F}'_{tors})_p$, that is, for some neighborhood U of p, we have $fm \in \mathcal{F}'_{tors}(U)$. Thus, $gfm = 0$ for some $g \neq 0$. Since $\mathcal{O}_X(U)$ is a domain, $gf \neq 0$ and $m \in \mathcal{F}'_{tors}(U)$. Thus, m is trivial in $\mathcal{G}_p$, and $\mathcal{G}_p$ is torsion-free. This shows that $\mathcal{F}/\mathcal{F}_{tors}$ is torsion-free as a sheaf.
Now, assume that $m \in \mathcal{F}(U)$ has m_p has torsion for all p. By the quasi-compactness of U, this means that U has a finite cover by U_i such that $f_i m|_{U_i} = 0$ for $f_i \neq 0 \in \mathcal{F}(U_i)$. This shows that $m \in \mathcal{F}_{tors}(U)$; thus any subsheaf $\mathcal{H}$ of $\mathcal{F}$ which is torsion lies in $\mathcal{F}_{tors}$. On the other hand, if $\mathcal{H} \to \mathcal{F}_{tors}$ is not surjective, then $\mathcal{F}/\mathcal{H}$ contains the subsheaf $\mathcal{F}_{tors}/\mathcal{H}$, which is torsion and non-zero. Thus, $\mathcal{F}/\mathcal{H}$ is not torsion-free. Thus $\mathcal{F}_{tors}$ is unique.

(b) Prove that the following are equivalent:

(i) $\mathcal{F}$ is torsion-free

(ii) $\mathrm{Ass}_X\mathcal{F}$, the set of associated points to $\mathcal{F}$, can only contain the generic point η of X

(iii) for all non-empty open sets U, the map $\mathcal{F}(U) \to \mathcal{F}_\eta$ is injective

(i) $\Rightarrow$ (ii): Assume that $p \neq \eta$, and that p is an associated point of $\mathcal{F}$. Let U be an affine neighborhood of p. Then $\mathcal{F}(U)$ has a section m whose support is the closure of $\{p\}$ in U. Since p is not generic, there is some $f \neq 0$ such that $p \in V(f)$. We have $m|_{D(f)} = 0$ since the stalk of m at all these points is 0. This implies that $m/1 \in \mathcal{F}(U) = 0$, that is, $f^k m = 0$ for some k. Thus, m is torsion. The statement we want is the contrapositive. (ii) $\Rightarrow$ (iii): The map $\mathcal{F}(U) \to \prod_{x \in \mathrm{Ass}_X \mathcal{F}} \mathcal{F}_x$ is injective; since any non-empty open set contains η, the map $\mathcal{F}(U) \to \mathcal{F}_\eta$ is injective. (iii) $\Rightarrow$ (i): If the map $\mathcal{F}(U) \to \mathcal{F}_\eta$ is injective, that means that every non-zero section has non-zero stalk at η. Since any non-zero element $f \in \mathcal{O}_X(U)$ has an inverse $1/f \in \mathcal{O}_{X,\eta}$, we can only have $fm = 0$ if $m_\eta = 0$, which implies $m = 0$, so $\mathcal{F}(U)$ is torsion-free as a $\mathcal{O}_X(U)$-module for all U, and so $\mathcal{F} = 0$.

(c) Prove that the following are equivalent:

(i) $\mathcal{F}$ is torsion

(ii) $\mathrm{Ass}_X\mathcal{F}$, the set of associated points to $\mathcal{F}$, does not contain the generic point η of X

(iii) $\mathcal{F}_\eta = 0$

(i) $\Rightarrow$ (ii): If $\mathcal{F}$ is torsion and p is an associated point, there is an affine open U containing p, and $m \in \mathcal{F}(U)$ such that $\mathrm{Supp}\, m = U \cap \overline{\{p\}}$. Since m_p is torsion, we can shrink U so that there is a function $f \in \mathcal{O}_X(U)$ such that $fm = 0$. Thus $\mathrm{Supp}\, m \subset V(f)$, and so p is not generic. (ii) $\Rightarrow$ (iii): Since the support of $\mathcal{F}$ is the closure of the set of associated points, the generic point of η is not in $\mathrm{Supp}\,\mathcal{F}$, and by definition $\mathcal{F}_\eta = 0$. (iii) $\Rightarrow$ (i): Consider $m_p \in \mathcal{F}_p$. This is represented by a section $m \in \mathcal{F}(U)$ for a neighborhood U. Since $\mathcal{F}_\eta = 0$, we have $\mathrm{Supp}(m) \subset V(f)$ for some f. As argued above, this shows that $f^k m = 0$ for some k, and so m is torsion. Since $f_p^k m_p = 0$, m_p is torsion, and so $\mathcal{F}$ is a torsion sheaf.

(2) Let X be a Noetherian scheme.

(a) Show that $\mathfrak{N}(U) = \{f \in \mathcal{O}_X(U) \mid f^n = 0 \text{ for some } n\}$ is a coherent sheaf.

First, we need to show that $\mathfrak{N}$ is a sheaf. The existence of restriction maps is inherited from $\mathcal{O}_X$, as is the identity axiom (any sub-presheaf of a sheaf is separated). Note that since X is Noetherian, any open subset of it is quasi-compact. Thus, for any open cover of an open U, it has a finite subcover U_i. If $f_i \in \mathfrak{N}(U_i)$ and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then

these glue to a section $f \in \mathcal{O}_X(U)$, and because the cover is finite, $f^n = 0$ for some n.
Now, we need to show that for U affine, $\mathfrak{N}(U)_f = \mathfrak{N}(U_f)$. The map exists by the universal property of localizations. Since $\mathfrak{N}(U) \subset \mathfrak{O}_X(U)$, this map remains injective after localization, showing $\mathfrak{N}(U)_f \subset \mathfrak{O}_X(U)_f \cong \mathfrak{O}_X(U_f)$. Furthermore, any element of $\mathfrak{N}(U_f)$ can be written as a/f^k. If this element is nilpotent, then $a^n/f^{kn} = 0$ in $\mathcal{O}_X(U_f) \cong \mathcal{O}_X(U)_f$. Thus, $a^n f^p = 0$ for some p, and so af is nilpotent.

(b) Show that $X - \{\mathrm{Supp}(\mathfrak{N})\}$ is a reduced scheme. *Assume not. Then,*

there is a point p such that $\mathcal{O}_{X,p}$ has non-trivial radical, i.e. $f \in \mathcal{O}_{X,p}$ satisfies $f \neq 0$ but $f^n = 0$. By the definition of stalk, this means that f has a representative $f' \in \mathcal{O}_X(U)$ such that $f'_p = f$ and $(f')^n = 0$, perhaps after shrinking the open subset U. Thus, $f' \in \mathfrak{N}(U)$, and since $f'_p \neq 0$, this means $p \in \mathrm{Supp}(\mathfrak{N})$

(c) Show that any embedded point in $\mathrm{Ass}_X\mathcal{O}_X$ lies in $\mathrm{Supp}(\mathfrak{N})$ (Vakil gives a long hint on pg. 195. I think a better hint is to think about how the vanishing locus $V(f)$ and support $\mathrm{Supp} f$ of $f \in \mathcal{O}_X(U)$ compare). *Let x be an embedded point in $mathrmAss_X\mathcal{O}_X$. By as-*

sumption, for any affine neighborhood of x, there is a $f \in \mathcal{O}_X(U)$ such that $\mathrm{Supp}(f) = \overline{\{x\}} \cap U$. Now, consider the vanishing locus $V(f) \subset U$. This is closed, and obviously contains the complement of $\overline{\{x\}} \cap U$. Since x is embedded, this complement contains the generic point of every component of U, so $U = V(f)$, since $V(f)$ is closed. This implies that $f \in \mathfrak{N}(U)$, and $f^n = 0$ for some n. Thus $x \subset \mathrm{Supp}(f) \subset \mathrm{Supp}(\mathfrak{N})$.