HW 4: PMATH 945

Due Feb. 21 at 11:59 pm

(1) An element $m \in M$ of a module over an integral domain A is called *torsion* if $am = 0$ for some $a \in A - \{0\}$. A module is called *torsion* if every element of it is torsion, and *torsion-free* if no non-zero element of it is torsion.

Assume X is an integral locally Noetherian scheme. Call a quasi-coherent sheaf $\mathcal{F}$ on X *torsion (torsion-free)* if its stalk $\mathcal{F}_p$ at every point $p \in X$ is a torsion (torsion-free) $\mathcal{O}_{X,p}$-module.

(a) Prove that every quasi-coherent sheaf $\mathcal{F}$ has a unique subsheaf $\mathcal{F}_{tors}$ such that $\mathcal{F}_{tors}$ is torsion and $\mathcal{F}/\mathcal{F}_{tors}$ is torsion-free.

(b) Prove that the following are equivalent:

(i) $\mathcal{F}$ is torsion-free

(ii) $\mathrm{Ass}_X\mathcal{F}$, the set of associated points to $\mathcal{F}$, only contains the generic point η of X

(iii) for all non-empty open sets U, the map $\mathcal{F}(U) \to \mathcal{F}_\eta$ is injective

(c) Prove that the following are equivalent:

(i) $\mathcal{F}$ is torsion

(ii) $\mathrm{Ass}_X\mathcal{F}$, the set of associated points to $\mathcal{F}$, does not contain the generic point η of X

(iii) $\mathcal{F}_\eta = 0$

(2) Let X be a Noetherian scheme.

(a) Show that $\mathfrak{N}(U) = \{f \in \mathcal{O}_X(U) \mid f^n = 0 \text{ for some } n\}$ is a coherent sheaf.

(b) Show that $X - \{\mathrm{Supp}(\mathfrak{N})\}$ is a reduced scheme.

(c) Show that any embedded point in $\mathrm{Ass}_X\mathcal{O}_X$ lies in $\mathrm{Supp}(\mathfrak{N})$ (Vakil gives a long hint on pg. 195. I think a better hint is to think about how the vanishing locus $V(f)$ and support $\mathrm{Supp}f$ of $f \in \mathcal{O}_X(U)$ compare).