HW 2: PMATH 945

Due Jan. 26 at 11:59 pm

(1) (a) Show that if $R \cong R_1 \times R_2$ for commutative rings R_1, R_2, then $\mathrm{Spec} R \cong \mathrm{Spec} R_1 \coprod \mathrm{Spec} R_2$ (where $\coprod$ is disjoint union, i.e. coproduct, of topological spaces) as ringed spaces (or equivalently, as schemes).

Consider the map $\mathrm{Spec} R_1 \to \mathrm{Spec} R$ *which sends* $[\mathfrak{p}]$ *to* $[\mathfrak{p} \times R_2]$*; the ideal* $\mathfrak{p} \times R_2$ *is prime since* $(R_1 \times R_2)/(\mathfrak{p} \times R_2) \cong R_1/\mathfrak{p}$ *is a domain. Symmetrically, we have a map* $\mathrm{Spec} R_2 \to \mathrm{Spec} R$ *sending* $[\mathfrak{p}] \mapsto [R_1 \times \mathfrak{p}]$*. Together, these give a map* $\mathrm{Spec} R \to \mathrm{Spec} R_1 \coprod \mathrm{Spec} R_2$ *which is clearly injective.*
Consider a prime ideal $\mathfrak{p} \subset R_1 \times R_2$*. Let* e_1, e_2 *be the image of* $(1,0)$ *and* $(0,1)$ *in* $R/\mathfrak{p}$*. These satisfy* $e_1 e_2 = 0$*, so one of these is zero. That is,* $(0,1) \in \mathfrak{p}$ *or* $(1,0) \in \mathfrak{p}$*. WLOG can assume the former. This means* $\mathfrak{p} \supset R_2$*, so* $\mathfrak{p} = (\mathfrak{p} \cap R_1) \times R_2$*. So, the map* $\mathrm{Spec} R \to \mathrm{Spec} R_1 \coprod \mathrm{Spec} R_2$ *is surjective and thus a bijection. If* $D(f,g)$ *is an open subset of* $\mathrm{Spec} R$*, then its preimage in* $\mathrm{Spec} R_1 \coprod \mathrm{Spec} R_2$ *is* $D(f) \cup D(g)$*, since* $(f,g) \in \mathfrak{p} \times R_2$ *if and only if* $f \in \mathfrak{p}$*, and similarly with factors switched. Since these are bases of the respective topologies, this map is a homeomorphism.*
Consider the pushforward of $\mathcal{O}_{\mathrm{Spec} R}$ *to* $\mathrm{Spec} R_1 \coprod \mathrm{Spec} R_2$*. This is a sheaf of rings, and its sections on* $D(f) \cup D(g)$ *are* $R_{(f,g)} \cong (R_1)_f \times (R_2)_g$*, which is the sections on* $D(f) \cup D(g)$*. Since the open sets* $D(f)$ *and* $D(g)$ *are disjoint, the sections of the structure sheaf on the disjoint union are also this product, so we obtain an isomorphism of the sections on all open sets of this form. Since these are a basis of the topology, this shows we have an isomorphism of rings.*

(b) Show that the topological space $\mathrm{Spec}(R)$ is disconnected if and only if $R \cong R_1 \times R_2$ for some commutative rings R_1, R_2 (recall that by assumption, a commutative ring has a unit, and thus is not zero).

If $\mathrm{Spec}(R)$ *is disconnected, then* $\mathrm{Spec}(R) = U \cup V$ *where both* U *and* V *are open and* $U \cap V = \emptyset$*. We have a map of rings* $R \to \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ *by restriction. This map is injective by the identity property of sheaves, and surjective by the gluing property (since* $U \cap V = \emptyset$*, any pair of sections glues). This shows that* R *is a direct product.*

(c) If we choose R_i for $i \in \mathbb{Z}_{\geq 0}$, consider the direct product $R = \prod_{i=0}^{\infty} R_i$. Show that $\coprod_{i=0}^{\infty} \mathrm{Spec}(R_i)$ is a proper subset of $\mathrm{Spec}(R)$ (Vakil suggests looking at the direct sum $\oplus_{i=0}^{\infty} R_i$ for an ideal not contained in any of the obvious ones).

The injection sends $\mathfrak{p} \subset R_i$ *to the direct product of* $\mathfrak{p}$ *with* R_j *for* $j \neq i$*. The arguments as before show that this is an injective map and a homeomorphism to its image. However, we can't apply the argument about surjectivity, since the identity in* $R = \prod_{i=0}^{\infty} R_i$ *is not a sum of identities in the individual factors (since we can't take infinite sums).*
The direct sum $\oplus_{i=0}^{\infty} R_i$ *is an ideal, and since it contains the identity in each copy of* R_i*, it isn't contained in any ideal of the form given above. This ideal is contained in a maximal ideal, which must not be of the form above, so the map to* $\mathrm{Spec}(R)$ *is not surjective.*

(2) Let M be an R-module.

(a) Prove there is a sheaf of $\mathcal{O}_X$-modules $\tilde{M}$ on $X = \mathrm{Spec}(R)$ such that $\tilde{M}(D(f)) = R_f \otimes_R M$.

To construct this sheaf, we use Th. 2.5.1 from Vakil. It suffices to define it on a base, which the formula $\tilde{M}(D(f)) = R_f \otimes_R M$ *does. The restriction map if* $D(g) \subset$

$D(f)$ is induced by the ring homomorphism $R_f \to R_g$ (the element f is invertible in R_g since it is not contained in any maximal ideals). Since $R_f \to R_g \to R_h$ is the same as the induced map $R_f \to R_h$, this defines a presheaf.

Thus, finally, we need to check the sheaf conditions. For identity, consider a section $m \in M_f$ and open cover $D(f) = \cup_{i \in I} D(f_i)$. If $m|_{D(f_i)} = 0$ for all i, then $f_i^{n_i} m = 0$ for some n_i. Since the ideal generated by the $f_i^{n_i}$ in R_f is the whole ring, this implies that $m = 0$.

For gluing, consider the same open cover, and $\frac{m_i}{f_i^{n_i}}$ for $m_i \in M$. As in Vakil, pg. 133, we can assume that $m_i f_j^{n_j} = m_j f_i^{n_i}$, by replacing f_i by a higher power. Since $1 = \sum b_i f_i^{n_i}$, we have $m = \sum m_i b_i \in M_f$ is the desired glued section.

(b) Prove that the stalks $\tilde{M}_{[\mathfrak{p}]}$ is the localization $R_\mathfrak{p} \otimes_R M$ for $R_\mathfrak{p}$ the localization at the multiplicative set $R \setminus \mathfrak{p}$.

Obviously, we have a map $M \to \tilde{M}_{[\mathfrak{p}]}$, and any element $f \in R \setminus \mathfrak{p}$ acts invertibly on this stalk, since $[\mathfrak{p}] \in D(f)$. Thus, by the universal property of localizations, we obtain a map $R_\mathfrak{p} \otimes_R M \to \tilde{M}_{[\mathfrak{p}]}$. This map is injective, since if m/f has image 0 in the stalk, its restriction to some $D(g)$ for $g \notin \mathfrak{p}$ is 0, i.e. we must have $g^n m = 0$ for some n, showing $m/f = 0$ in $R_\mathfrak{p} \otimes_R M$. This map is surjective, since any element of $\tilde{M}_{[\mathfrak{p}]}$ must be represented by a germ, which can be restricted to an open set in a basis, which can be taken to be $D(f)$, so is of the form m/f^n.

(3) Show that for any affine scheme $\mathrm{Spec}(R)$ and $f \in R$, the restriction $\mathcal{O}_X|_{D(f)}$ of the structure sheaf to $D(f)$ and the subspace topology make the ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ into an affine scheme.

If $(D(f), \mathcal{O}_X|_{D(f)})$ is an affine scheme, it must be $\mathrm{Spec}(R_f)$, since R_f is the sections of the structure sheaf. The bijection $D(f) \cong \mathrm{Spec}(R_f)$ sends a prime ideal in R to the ideal $R_f\mathfrak{p}$, which is still prime if $f \notin \mathfrak{p}$, i.e. $[\mathfrak{p}] \notin D(f)$. This is continuous since the preimage of $D(g/f^n)$ for $g \in R$ is $D(gf) = D(g) \cap D(f)$, since this is the set of prime ideals that don't contain f and g.

The inverse map sends prime ideals $\mathfrak{q}$ in R_f to $\mathfrak{q} \cap R$. This is continuous since the preimage of $D(g)$ is $D(g)$, considering $g \in R_f$.

Thus, we have a homeomorphism. We need to produce an isomorphism of the structure sheaves. For $D(g) \subset \mathrm{Spec}(R_f)$, the sections are $\mathcal{O}_{\mathrm{Spec}(R_f)}(D(g)) = (R_f)_g = R_{fg}$. Thus, we can construct an isomorphism of sheaves by identifying this with $\mathcal{O}_{\mathrm{Spec}(R)}(D(gf))$.