HW 5: PMATH 945

Due Mar. 7 at 11:59 pm

(1) (a) Let X be a reduced locally Noetherian scheme. Describe how rational maps $X \dashrightarrow \mathbb{A}^1_{\mathbb{Z}}$ are the same thing as rational functions.
   (b) Show that a rational map $\pi\colon X \dashrightarrow Y$ of integral schemes is dominant if and only if π sends the generic point of X to the generic point of Y.
   (c) Let k be a field. Call an extension field $E \supset k$ *finitely-generated* if there is a finite subset of E contained in no proper subfield of E. Describe an equivalence between the categories:
      (i) objects: finite type integral affine k-schemes; morphisms: dominant rational maps of k-schemes.
      (ii) objects: finitely generated field extensions of k; morphisms: k-algebra homomorphisms in the opposite direction. (That is: take the opposite category to the obvious one.)

(2) Let X be a $\mathbb{F}_p$-scheme. Explain how to define an endomorphism $F_X\colon X \to X$ such that:
   (a) If $X = \operatorname{Spec} A$, then F_X is induced by the Frobenius map on A which sends $a \mapsto a^p$.
   (b) For any morphism of $\mathbb{F}_p$-schemes $\pi\colon X \to Y$, we have $\pi \circ F_X = F_Y \circ \pi$.

   Prove that if X is locally of finite type over $\mathbb{F}_p$, then F is finite.

(3) (a) Let Y_1 and Y_2 be closed subschemes of X. Prove that there is a unique smallest closed subscheme $Y_1 \cup Y_2$ containing Y_1 and Y_2 and that its underlying set is the union of the underlying sets of Y_1 and Y_2.
   (b) Fix a finitely generated A-module M and an element $m \in M$. Consider $V(\operatorname{Ann} m)$ as a closed subscheme of $\operatorname{Spec} A$; we call this the **scheme-theoretic support** of m. Show that $\operatorname{Supp} M = V(\operatorname{Ann} M)$ is the union of the closed subschemes $\operatorname{Supp} m$ and that the underlying set of this subscheme is the support as defined before (the closed subset where $\tilde{M}$ has non-vanishing stalks).
   (c) Show that for any coherent sheaf $\mathcal{F}$ on X, this defines a closed subscheme structure on $\operatorname{Supp} \mathcal{F}$ (the closed subset where $\mathcal{F}$ has non-vanishing stalks).

(4) Consider $Y = \mathbb{P}^3_k = \operatorname{Proj}(k[x, y, z, w])$ and the subscheme $X = V(wz - xy, x^2 - wy, y^2 - xz)$. Show that $X \cong \mathbb{P}^1_k$.