HW 3: PMATH 945

Due Feb. 7 at 11:59 pm

(1) Consider the graded ring $R = \mathbb{Z}[x,y,z]/(zy^2 = x(x^2 - z^2))$.

(a) Consider the set $V(pR) \subset \operatorname{Proj}(R)$ of homogeneous ideals containing pR for $p \in \mathbb{Z}$ prime. Describe the points of this set.

These are in bijection with the homogeneous prime ideals of $\mathbb{F}_p[x,y,z]/(zy^2 - x(x^2 - z^2))$ that don't contain the irrelevant ideal. The ring $\mathbb{F}_p[x,y,z]/(zy^2 - x(x^2 - z^2))$ is a domain since $zy^2 - x(x^2 - z^2)$ is an irreducible polynomial (if it factors, then it factors after substituting $z = 1$, and $y^2 - x(x^2 - 1)$ doesn't factor because $x(x^2 - 1)$ isn't a square). Thus, we have the ideal (0) (whose preimage in R is (p)). If (a,b,c) is in $\overline{\mathbb{F}}_p^3$ and satisfy $cb^2 = a(a^2 - c^2)$, then we can consider the map $\mathbb{Z}[x,y,z]/(zy^2 = x(x^2 - z^2)) \to \overline{\mathbb{F}}_p[u])$ defined by $x \mapsto au, y \mapsto bu, z \mapsto cu$. This is a homogeneous map to a domain, so its kernel is prime and homogeneous, and contains pR. Two points give the same prime if, up to multiplication by a scalar, they are in the same orbit of the Galois group of $\overline{\mathbb{F}}_p$ over $\mathbb{F}_p$. In $\operatorname{Proj} \mathbb{F}_p[x,y,z]/(zy^2 = x(x^2 - z^2))$, the open set $D^+(x)$ is isomorphic to $\operatorname{Spec}(\mathbb{F}_p[y/x, z/x]/((z/x)(y/x)^2 = (1 - (z/x)^2)))$, so aside from a generic point, each point of this open set is given by a maximal ideal, whose residue field is $\mathbb{F}_{p^n}$, so z/x and y/x are sent to elements $a, b \in \overline{\mathbb{F}}_p$. This is one of the ideals before for the point $(1, a, b)$; the same argument applies to y and z and shows that the point of $V(pR)$ are:

- *The generic point (p).*
- *The Galois orbits of $(a,b,c) \in \mathbb{P}^2(\overline{\mathbb{F}}_p)$ that satisfy $cb^2 = a(a^2 - c^2)$. Explicitly, if $f(z) \in \mathbb{F}_p[z]$ is the minimal polynomial of b/a and $g(z,w) \in \mathbb{F}_p[z,w]/(f(z))$ is the minimal polynomial of c/a over $\mathbb{F}_p[z]/(f(z))$, then the ideal is generated by $(p, x^{\deg f} f(y/x), x^{\deg g} g(y/x, z/x)$.*

(b) Consider the set $V(zR) \subset \operatorname{Proj}(R)$ of homogeneous ideals containing zR. Describe the points of this set.

These are in bijection with the homogeneous prime ideals of $\mathbb{Z}[x,y]/(x^3)$ that don't contain the irrelevant ideal. Since any prime ideal will contain x, these are the same as homogeneous prime ideals of $\mathbb{Z}[y])$ that don't contain the irrelevant ideal. There's the ideal (0) which is the generic point. The other primes are generated by (p) for $p \in \mathbb{Z}$ a prime.

(2) Fix a graded ring $S_\bullet$ and a homogeneous ideal I. Show that the following are equivalent:

(a) $V(I) = \emptyset$.

(b) If $\{f_i\}_{i \in I}$ are homogeneous generators of I, then $\bigcup_{i \in I} D^+(f_i) = \operatorname{Proj}(S)$.

(c) $\sqrt{I} \supset S_+$.

If $\{f_i\}_{i \in I}$ are homogeneous generators of I are homogeneous generators of I, then $\bigcup_{i \in I} D^+(f_i)$ is the set of homogeneous prime ideals not containing S_+ that don't contain one of the f_i. Of course, that's the complement of $V(I)$, so (1) and (2) are equivalent. (In contrast to Vakil, I am using $D^+(f_i)$ to mean the ideals in Proj not containing f_i even if $\deg f_i = 0$; note that in this case $D^+(f_i)$ might not be affine).

If $\sqrt{I} \supset S_+$, then any homogeneous prime ideal containing I (and thus $\sqrt{I}$ by primality) contains S_+ and thus doesn't represent a point in $\operatorname{Proj}(S_\bullet)$. Thus $V(\emptyset) = \emptyset$, so (3) implies (1).

On the other hand, if we take a homogeneous element g of positive degree, then we can take $D^+(g) \cong \operatorname{Spec}(S_g)_0$. The intersection of $D^+(f_i)$ with $D^+(g)$ is the open set

$D(f_i^{\deg g}/g^{\deg f_i})$ *(note that this is true even though f might have degree 0 since g has positive degree). Since these cover $D^+(g)$, we have that in S_g,*

$$1 = \sum_{i \in I} a_i f_i^{\deg g}/g^{\deg f_i}$$

where a_i is only non-zero for finitely many i. Multiplying by g^m for m sufficiently large, we then find that $g^m \in (f_i)_{i \in I}$ for some m. Thus, $g \in \sqrt{I}$, so (2) implies (3).

(3) (a) Show that for a scheme X, $p \mapsto \overline{\{p\}}$ is a bijection between the points of the scheme and its irreducible closed subsets (the inverse of this map sends a closed subset to its generic point).

The closure $\overline{\{p\}}$ is obviously closed, and irreducible since if $\overline{\{p\}} = C_1 \cup C_2$, p must lie in one of C_1 or C_2, in which case C_1 or C_2 must contain all of $\overline{\{p\}}$.
Now, assume that C is closed and irreducible. Consider an affine subscheme $U = \mathrm{Spec}(A)$ with $U \cap C$ non-empty. The intersection $U \cap C$ is irreducible, since if $U \cap C$ is the union of two closed sets in U, then C is the union of their closures. Thus, $U \cap C = V(\mathfrak{p})$ for a unique prime ideal $\mathfrak{p} \subset R$. Let $p = [\mathfrak{p}]$; The closure $\overline{\{p\}}$ is a subset of C, and $C = \overline{\{p\}} \cup (C \cap U^c)$. Since the latter set is a proper subset, by irreducibility, we must have $C = \overline{\{p\}}$.
This shows that $p \mapsto \overline{\{p\}}$ is a surjective map from points to irreducible closed sets. If $C = \overline{\{p\}} = \overline{\{q\}}$, then for any affine open $U = \mathrm{Spec}(A)$ with $U \cap C$ non-empty, we must have $p, q \in U$, but that would mean we have inclusions both directions of the corresponding prime ideals. Thus $p = q$. This shows we have a bijection.

(b) Show that a scheme X is integral if and only if it is reduced and irreducible.

If X is not reduced, $\mathcal{O}_{X,p}$ is not a domain for some p, and so there is a neighborhood of p where $\mathcal{O}_X(U)$ is not a domain, so X is not integral. If X is not irreducible, then $X = C_1 \cup C_2$. The open $U = X \setminus (C_1 \cap C_2)$ is thus disconnected, and so $\mathcal{O}_X(U)$ is not a domain (the function which is 1 on $X \setminus C_1$ and 0 on $X \setminus C_2$ is exists by gluing and is a zero-divisor). This shows that X is not integral.
Now, assume X is irreducible and reduced. If there is an open set U with $\mathcal{O}_X(U)$ not a domain, then there are two different elements $x, y \in \mathcal{O}_X(U)$ such that $xy = 0$. Consider the open subsets $D(x) = \{p \in U \mid x \notin \mathfrak{m}_{X,p}\}, D(y) \subset U$. These are both non-empty, since X is reduced, so no section of $\mathcal{O}_X(U)$ vanishes at all points. On the other hand, since $xy = 0$, these sets are disjoint. Thus, their complements show that X is not irreducible, contradicting the assumption that $\mathcal{O}_X(U)$ is not a domain.

(c) Show that if X is integral and η is its generic point, then $\mathcal{O}_{X,\eta} = \mathrm{Frac}(R)$ for any affine open subset $\mathrm{Spec}(R) \subset X$.

Consider any affine open $U = \mathrm{Spec}\, R$ of X. This contains η, and it is identified with the point of the ideal (0). Thus, the stalk at this point is the fraction field of R.

(4) Let A be a Noetherian ring. Show that any projective A-scheme is Noetherian and of finite type over A.

Any projective A-scheme is $\mathrm{Proj}(S)$ for S a graded A-algebra generated by finitely many homogeneous elements of positive degree $s_1, \ldots, s_k$ which necessarily generate the irrelevant ideal. The scheme $\mathrm{Proj}(S)$ has a cover by the affine open sets $D^+(s_i) = \mathrm{Spec}((S_{s_i})_0)$. The ring $R_i = (S_{s_i})_0$ is finitely generated as an A-algebra: if $d_i = \deg f_i$, *it's spanned by the monomials* $s_1^{a_1} \cdots s_k^{a_k}/s_i^b$ *for* $d_i b = \sum d_i a_i$*. For all but finitely many choices of* a_i*, we have* $a_j \geq d_i$ *for some* j*, and thus*

$$\frac{s_1^{a_1} \cdots s_k^{a_k}}{s_i^b} = \frac{s_1^{a_1} \cdots s_j^{a_j - d_i} \cdots s_k^{a_k}}{s_i^{b-d_j}} \cdot \frac{s_j^{d_i}}{s_i^{d_j}}$$

This shows that the collection of such monomials with all $a_j \leq d_i$ *is a finite set of generators.*

Since R_i *is a finitely generated* A*-algebra and* A *is Noetherian,* R_i *is Noetherian by the Hilbert basis theorem. Since* $\operatorname{Proj}(S)$ *has a finite cover by affine Noetherian schemes of finite type over* A*, the scheme* $\operatorname{Proj}(S)$ *itself is Noetherian and finite type over* A*.*

Extra exercises (if the above wasn't enough): 4.3.F, 4.4.A, 4.5.F, 5.2.F, 5.3.E, 5.4.A, 5.4.I