STAT 360 : STOCHASTIC PROCESS II

GROUP 4

MODEL OVERVIEW AND APPLICATIONS:

The Cox-Ingersoll-Ross model (CIR) is a mathematical formula used to model interest rate movements. The CIR model is a öne-factor", equilibrium interest rate model. One factor in the sense that, it models the short â term interest rate. It is used as a method to forecast interest rates and is based on a stochastic differential equation.The CIR model was developed in 1985 by John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross as **an improved version** of the Vasicek Interest Rate model and can be utilized, among other things, to calculate prices for bonds and value interest rate derivatives.The CIR model satisfies the following stochastic differential equation form:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t \quad \kappa, \theta, \sigma > 0$$

where

r(t) - the short term rate

k - the speed of mean reversion

θ - long term rate constant

σ - regulates the volatility

dW(t) - a Wiener process

$\sigma\sqrt{r(t)}dW(t)$ - stochastic volatility

CIR model's key characteristics and assumptions

Positive interest rate: The CIR model assumes that the interest rates are always positive, meaning the short rate $r(t) \geq 0$.

Mean reversion : The model assumes that, interests reverts to a long term mean level.

Volatility:The volatility of interest rates is proportion to the square root of the current rate. This implies that as the rate increase, its standard deviation increase and as it falls and approaches zero, the stochastic term also approaches zero.

Continuous Time: It assumes continuous time, thus the interest rates can change at any instant.

real-world applications

The Cox-Ingersoll-Ross (CIR) model has several real-world applications in finance and economics, including:

1. **Interest Rate Modeling** : The CIR model is widely used to describe the behavior of short-term interest rates, such as the federal funds rate.
2. **Bond Pricing** : The CIR model is used to price bonds, especially those with embedded options, such as callable or putable bonds.
3. **Risk Management** : The CIR model is used to manage interest rate risk, by simulating different interest rate scenarios and estimating potential losses.

4. **Economic Research** : The CIR model is used in macroeconomic research to study the behavior of interest rates and their impact on the economy.

5. **Derivatives Pricing** : The CIR model is used to price exotic derivatives, such as interest rate futures and options on interest rate futures.

6. **Asset Liability Management** : The CIR model is used by banks and financial institutions to manage their asset liability mismatch and estimate potential losses.

THEORETICAL EXPLORATORY ANALYSIS

Derivation of the model in an integral form

$$f(t, r_t) = r_t e^{\kappa t}$$

$$\frac{\partial f}{\partial t} = r\kappa e^{\kappa t}, \quad \frac{\partial f}{\partial r} = e^{\kappa t}, \quad \frac{\partial^2 f}{\partial r^2} = 0.$$

Using the Ito formula,

$$df = rke^{\kappa t}dt + e^{\kappa t}\kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

$$df = \kappa\theta e^{\kappa t}dt + \sigma e^{\kappa t}\sqrt{r_t}dW_t$$

Integrating both sides, we get

$$r_t e^{\kappa t} = r_0 + \int_0^t \kappa\theta e^{\kappa s}ds + \int_0^t \sigma e^{\kappa s}\sqrt{r_s}dW_s$$

$$r_t e^{\kappa t} = r_0 + \kappa\theta\big(\tfrac{1}{\kappa}e^{\kappa t}\big) + \int_0^t \sigma e^{\kappa s}\sqrt{r_s}dW_s$$

$$r_t = \theta + (r_0 - \theta)e^{-\kappa t} + \sigma e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{r_s}dW_s$$

Expectation of CIR model

$$\mathbb{E}[r_t] = \mathbb{E}\big[e^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\big]$$

$$\mathbb{E}[r_t] = e^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \mathbb{E}\big[\int_0^t e^{\kappa s} \sqrt{r_s} dW_s\big]$$

$$\mathbb{E}[r_t] = e^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} \mathbb{E}\big[dW_s\big]$$

Since dW_s is a white noise, $\mathrm{E}\big[\mathrm{dW}_s\big] = 0$

therefore: $\mathrm{E}[\mathrm{r}_t] = e^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t})$

from our model: $\mathbb{E}[r_t] = 21.6875$

Variance of the CIR model

$$Var[r_t] = \mathbb{E}[(r_t)^2] - (\mathbb{E}[r_t])^2$$

$$\mathrm{Var}[\mathrm{r}_t] = \mathbb{E}\Big[\big(e^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t}) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \sqrt{r_s} dW_s\big)^2\Big]$$

$$\text{- } \mathrm{E}\big[\mathrm{e}^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t})\big]^2$$

$$\mathrm{Var}[\mathrm{r}_t] = \mathbb{E}\big[e^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t})\big]^2 + 2\mathbb{E}\Big[\big(e^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t})\big]\sigma e^{-\kappa t}$$

$$\times \mathbb{E}\big[\int_0^t e^{\kappa s} \sqrt{r_s} dW_s)\Big] + \sigma^2 e^{-2\kappa t} \mathbb{E}\Big[\big(\int_0^t e^{\kappa s} \sqrt{r_s} dW_s\big)\Big]$$

$$\text{- } \mathrm{E}\big[\mathrm{e}^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t})\big]^2$$

$$\mathrm{Var}[\mathrm{r}_t] = 2\big[e^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t})\big]\sigma e^{-\kappa t} \mathbb{E}\big[\int_0^t e^{\kappa s} \sqrt{r_s} dW_s\big] + \sigma^2 e^{-2\kappa t} \times$$

$$\mathbb{E}\Big[\big(\int_0^t e^{\kappa s} \sqrt{r_s} dW_s\big)\Big]$$

$$Var[r_t] = 2\big[e^{-\kappa t} r_0 + \theta(1 - e^{-\kappa t})\big]\sigma \cdot e^{-\kappa t}\Bigg[\Big(\int_0^t e^{\kappa s}\sqrt{r_s}\,\mathbb{E}\big[dW_s\big]\Bigg]$$

$$+\sigma^2 e^{-2\kappa t}\mathbb{E}\big[\textstyle\int_0^t e^{2\kappa s} r_s ds\big]$$

$$\mathrm{Var}[\mathrm{r}_t] = \sigma^2 e^{-2\kappa t}\textstyle\int_0^t e^{2\kappa s}\mathbb{E}\big[r_s\big] ds$$

$$\mathrm{Var}[\mathrm{r}_t] = \sigma^2 e^{-2\kappa t}\textstyle\int_0^t e^{2\kappa s}\big(e^{-\kappa s} r_0 + \theta(1 - e^{-\kappa s})\big) ds$$

$$\mathrm{Var}[\mathrm{r}_t] = \tfrac{\sigma^2}{\kappa} r_0(e^{-\kappa t} - e^{-2\kappa t}) + \tfrac{\theta\sigma^2}{2\kappa}(1 - 2e^{-\kappa t} + e^{-2\kappa t})$$

therefore, variance of the CIR model is:

$$\frac{\sigma^2}{\kappa} r_0(e^{-\kappa t} - e^{-2\kappa t}) + \frac{\theta\sigma^2}{2\kappa}(1 - e^{-\kappa t})^2$$

from our model: $Var[r_t] = 0.3791178$

DISCRETIZATION OF THE MODEL:

CHOSEN DISCRETIZATION METHOD:

Euler scheme (Euler - Maruyama approximation)

The Euler- Maruyama method (Maruyama, 1955) is the analogue of the Euler method for ordinary differential equations. We compute the approximate solution either using **(1)** or the **modified form(2):**

$$r_0 = r(0) \ \text{(initial rate)}$$

$$r_{t_{i+1}} = r_{t_i} + \kappa(\theta - r_{t_i})\Delta t + \sigma\sqrt{|r_{t_i}|}\sqrt{\Delta t}Z_i \qquad \textbf{(1)}$$

$$r_{t_{i+1}} = max(r_{t_i} + \kappa(\theta - r_{t_i})\Delta t + \sigma\sqrt{|r_{t_i}|}\sqrt{\Delta t}Z_i, 0) \qquad \textbf{(2)}$$

where

$Z_1, Z_2, ..., Z_{n-1}$ are standard independent $\mathcal{N}(0,1)$ Gaussian variables.

Here r_{t+1} denotes a time-discretized approximation defined on a time partition $t_1, t_2, ..., t_i$ and $i = 1, 2, 3.........., n-1$.

The $\sqrt{\Delta t}Z_i$ factor can be explained by that it has the same distribution as $W_i = Wt_{i+1} - Wt_i$(Wiener process).

Rationale behind choosing Euler - Maruyama

- **Simplicity and Ease of Implementation:** The Euler-Maruyama method is straightforward to implement. It extends the standard Euler method used for ordinary differential equations to handle the stochastic components present in SDEs.
- **Handling Stochastic Components:** The Euler-Maruyama method provides a way to incorporate the randomness introduced by the Wiener process (Brownian motion) in the SDE. It approximates the stochastic integral, which is crucial for correctly simulating the behavior of the SDE over time.

- **Computational Efficiency:** The Euler-Maruyama method is computationally efficient, especially when a high degree of precision is not required. It can provide reasonable approximations with relatively large time steps, making it suitable for simulating long-term dynamics
- **Positive interest rate:** The use of absolute sign or the **max** in the formula helps deals with the negative values, thereby giving us positive rates.

MODEL FITTING WITH DATA

Exploratory analysis for our original data

Mean $= 26.43227$

Median $= 23.59$

Mode $= 62.2$

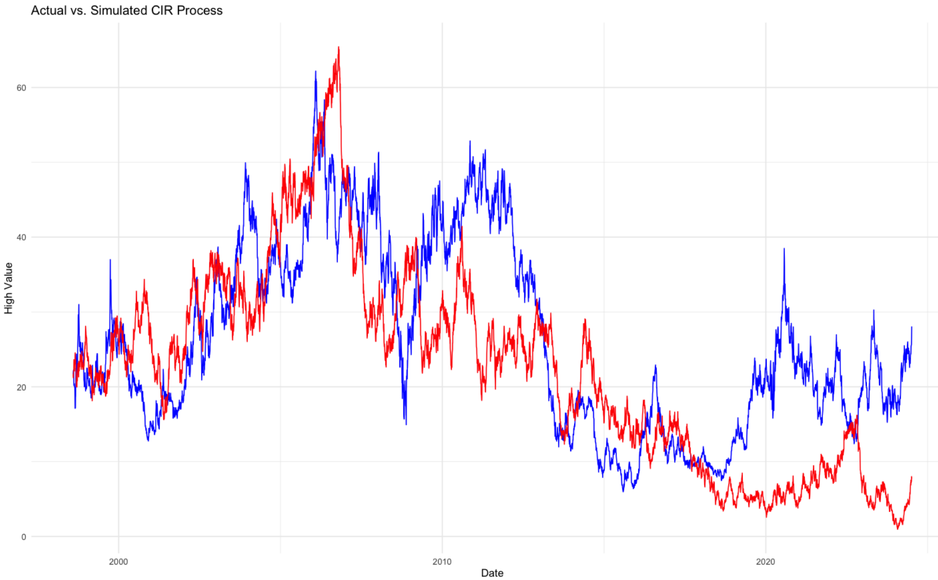
Variance $= 155.1643$

Standard deviation $= 12.4565$

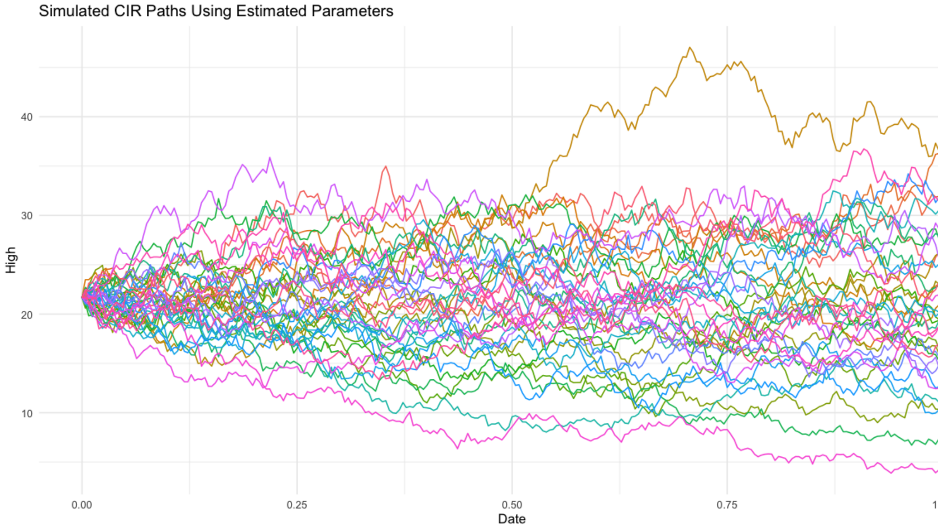
Range $= 56.24$

STEPS/PROCEDURES:

1. import given dataset
2. Converted the given dataset to time series data
3. Plotted time series data to view the actual graph
4. Estimation of parameters : Maximum likelihood Estimation method
5. Descretize data(used Euler - Maruyama approximation) and generate predicted simulation based on your estimated parameters
6. Plot actual vs the simulated data



Simulation plot of a single path



Monte Carlo Simulation

CHECKING MODEL QUALITY

KEY INSIGHTS

CRI Model: $dr_t = 0.00001(0.03880373 - r_t)dt + 2.098856\sqrt{r_t}dW_t$

Parameter Estimates

$\kappa = 0.00001$ (speed of mean reversion)

$\theta = 0.03880373$ (long term mean rate)

$\sigma = 2.098856$ (volatility)

Statistical Summary of the Residuals

ROOT MEAN SQUARE ERROR of the path: 11.00723

Conclusion

Considering a data points of 6521, a RMSE of 11.00723 is not very high, hence;

The CIR model, characterized by the estimated parameters and the distribution of the simulated paths provides a reasonable fit to the actual data. It effectively captures the overall mean-reverting behavior and the general level of volatility in the process. However, the discrepancies observed during periods of high volatility suggest that the model may benefit from further refinement or the inclusion of additional factors to improve its accuracy.