Regressions with Matrix Algebra

One Explanatory Variable
(I.e., the easy case)

First, some matrix multiplication…

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

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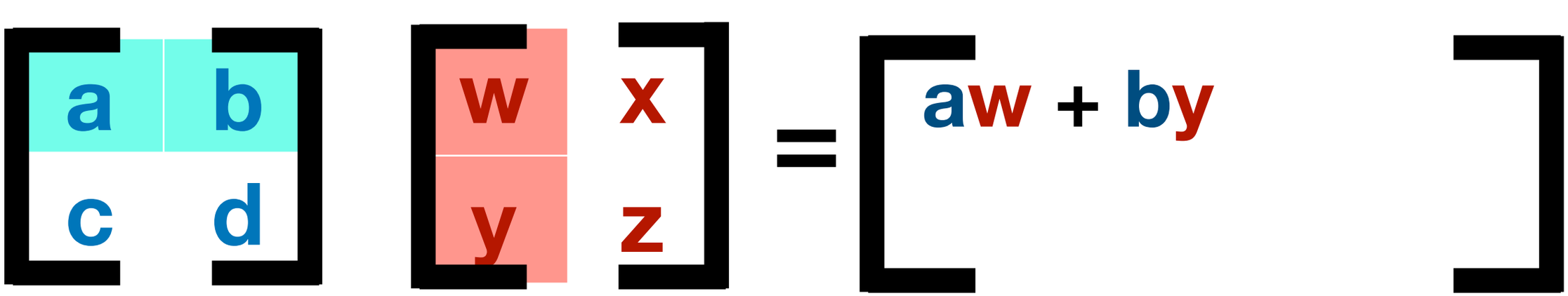

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ & \end{bmatrix}$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & \end{bmatrix}$$

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Our Data

x	y
2	2
2	4
3	6
4	2
4	6

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x	y
2	2
2	4
3	6
4	2
4	6

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

$$Y = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \\ 6 \end{bmatrix}$$

Our Goal

$$B = (X'X)^{-1}X'Y$$

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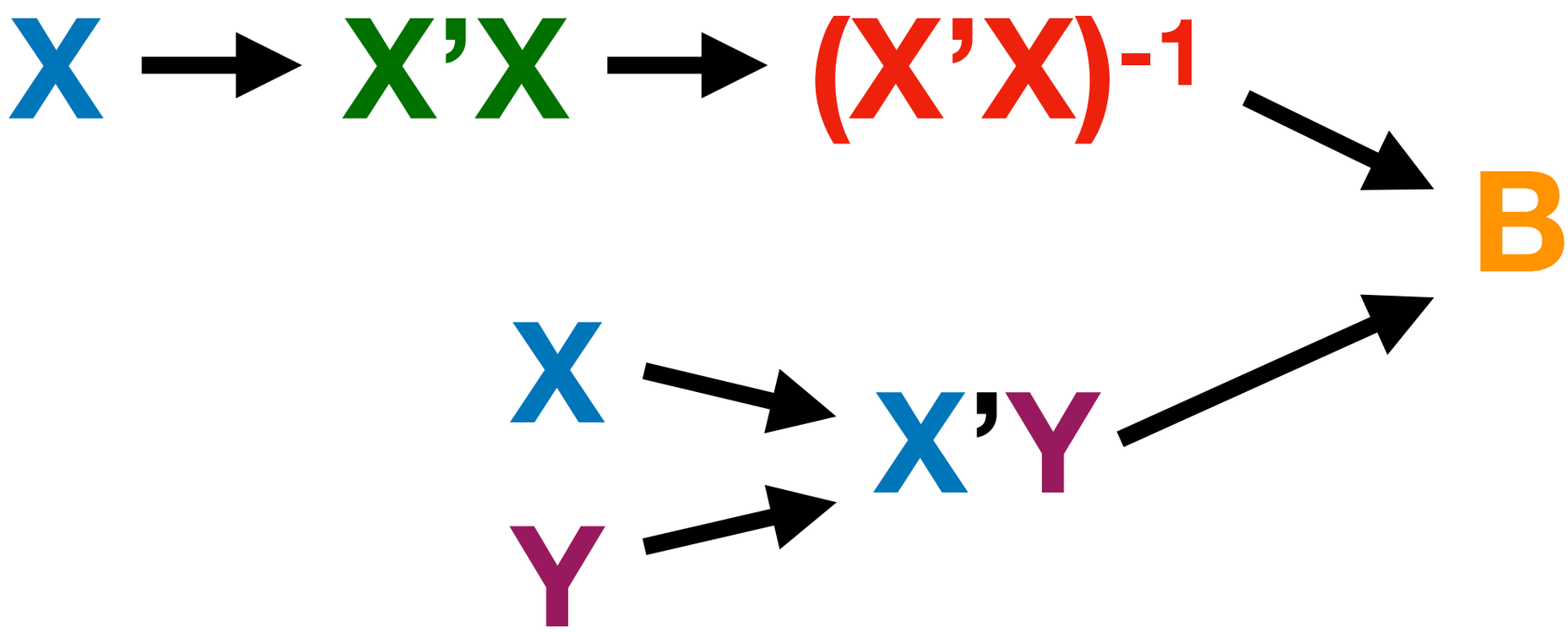
Just break it up into smaller steps:



X and X Transpose

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix} \qquad X' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix}$$

Step 1: Calculate X'X

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

Step 1: Calculate X'X

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(1) + (1)(1) + (1)(1) + (1)(1) + (1)(1) & (1)(2) + (1)(2) + (1)(3) + (1)(4) + (1)(4) \\ (2)(1) + (2)(1) + (3)(1) + (4)(1) + (4)(1) & (2)(2) + (2)(2) + (3)(3) + (4)(4) + (4)(4) \end{bmatrix}$$

Step 1: Calculate X'X

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix}$$

Step 2: Calculate $(X'X)^{-1}$

$$(X'X)^{-1} = \frac{\text{Adj}(X'X)}{|X'X|}$$

Step 2a: Calculate |X'X|

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = ad - bc$$

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Step 2a: Calculate $|X'X|$

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = ad - bc$$

$$X'X = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix}$$

$$|X'X| = (5)(49) - (15)(15) = 20$$

Step 2b: Calculate Adj(X'X)

$$\text{Adj}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Step 2b: Calculate Adj(X'X)

$$\text{Adj}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{Adj}\left(\begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix}\right) = \begin{bmatrix} 49 & -15 \\ -15 & 5 \end{bmatrix}$$

Step 2c: Calculate $(X'X)^{-1}$

$$(X'X)^{-1} = \frac{\text{Adj}(X'X)}{|X'X|}$$

Step 2c: Calculate $(X'X)^{-1}$

$$(X'X)^{-1} = \frac{\text{Adj}(X'X)}{|X'X|}$$

$$= \frac{1}{20}\begin{bmatrix} 49 & -15 \\ -15 & 5 \end{bmatrix}$$

Step 2c: Calculate $(X'X)^{-1}$

$$(X'X)^{-1} = \frac{\text{Adj}(X'X)}{|X'X|}$$

$$= \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

Step 3: Calculate X'Y

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \\ 6 \end{bmatrix}$$

Step 3: Calculate X'Y

$$\mathbf{X'Y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(2) + (1)(4) + (1)(6) + (1)(2) + (1)(6) \\ (2)(2) + (2)(4) + (3)(6) + (4)(2) + (4)(6) \end{bmatrix}$$

Step 3: Calculate X'Y

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ 62 \end{bmatrix}$$

Step 4: Put it all together

$$B = (X'X)^{-1}X'Y$$

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$$= \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix} \begin{bmatrix} 20 \\ 62 \end{bmatrix}$$

Step 4: Put it all together

$$B = (X'X)^{-1}X'Y$$

$$= \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix} \begin{bmatrix} 20 \\ 62 \end{bmatrix}$$

$$= \begin{bmatrix} (2.45)(20) + (-0.75)(62) \\ (-0.75)(20) + (0.25)(62) \end{bmatrix}$$

Step 4: Put it all together

$$B = (X'X)^{-1}X'Y$$

$$= \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix} \begin{bmatrix} 20 \\ 62 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 \\ 0.5 \end{bmatrix}$$

Tricks for Checking Your Work

1. Each number in the X'X matrix has a substantive interpretation:

$$X'X = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix} = \begin{bmatrix} \text{Number of observations} & \text{Sum of } x \\ \text{Sum of } x & \text{Sum of } x^2 \end{bmatrix}$$

Tricks for Checking Your Work

2. X'X and $(X'X)^{-1}$ are both symmetric matrices

$$X'X = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix} \qquad (X'X)^{-1} = \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

Tricks for Checking Your Work

3. X'X $(X'X)^{-1}$ is the identity matrix (I)

$$X'X\,(X'X)^{-1} = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix} \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} (5)(2.45) + (15)(-0.75) & (5)(-0.75) + (15)(0.25) \\ (15)(2.45) + (49)(-0.75) & (15)(-0.75) + (49)(0.25) \end{bmatrix}$$

Tricks for Checking Your Work

3. X'X $(X'X)^{-1}$ is the identity matrix (I)

$$X'X\,(X'X)^{-1} = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix} \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$