

# Regressions with Matrix Algebra

One Explanatory Variable  
(i.e., the easy case)

First, some matrix  
multiplication...

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

# First, some matrix multiplication...

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & \end{bmatrix}$$

# First, some matrix multiplication...

The diagram illustrates the multiplication of two 2x2 matrices. The first matrix has elements  $a$  and  $b$  in the top row (highlighted in cyan) and  $c$  and  $d$  in the bottom row. The second matrix has elements  $w$  and  $x$  in the top row (highlighted in red) and  $y$  and  $z$  in the bottom row (also highlighted in red). The result is shown as a single row in brackets:  $aw + by$  followed by  $ax + bz$ . In this result,  $aw$  and  $ax$  are red,  $by$  and  $bz$  are blue, and the plus signs and the variable  $a$  in  $ax$  are black.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \end{bmatrix}$$

# First, some matrix multiplication...

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & \end{bmatrix}$$

# First, some matrix multiplication...

The diagram illustrates the multiplication of two 2x2 matrices. The first matrix has elements  $a$  and  $b$  in the top row, and  $c$  and  $d$  in the bottom row. The second matrix has elements  $w$  and  $x$  in the top row, and  $y$  and  $z$  in the bottom row. The resulting matrix is shown as a 2x2 grid of expressions: the top row contains  $aw + by$  and  $ax + bz$ , and the bottom row contains  $cw + dy$  and  $cx + dz$ . In the original image, the terms  $w$ ,  $y$ ,  $x$ , and  $z$  are red, while  $a$ ,  $b$ ,  $c$ , and  $d$  are blue. The background of the first matrix is light blue, and the background of the second matrix is light red.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

# First, some matrix multiplication...

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

# Our Data

x	y
2	2
2	4
3	6
4	2
4	6



# Our Data

x	y
2	2
2	4
3	6
4	2
4	6

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

$$Y = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \\ 6 \end{bmatrix}$$

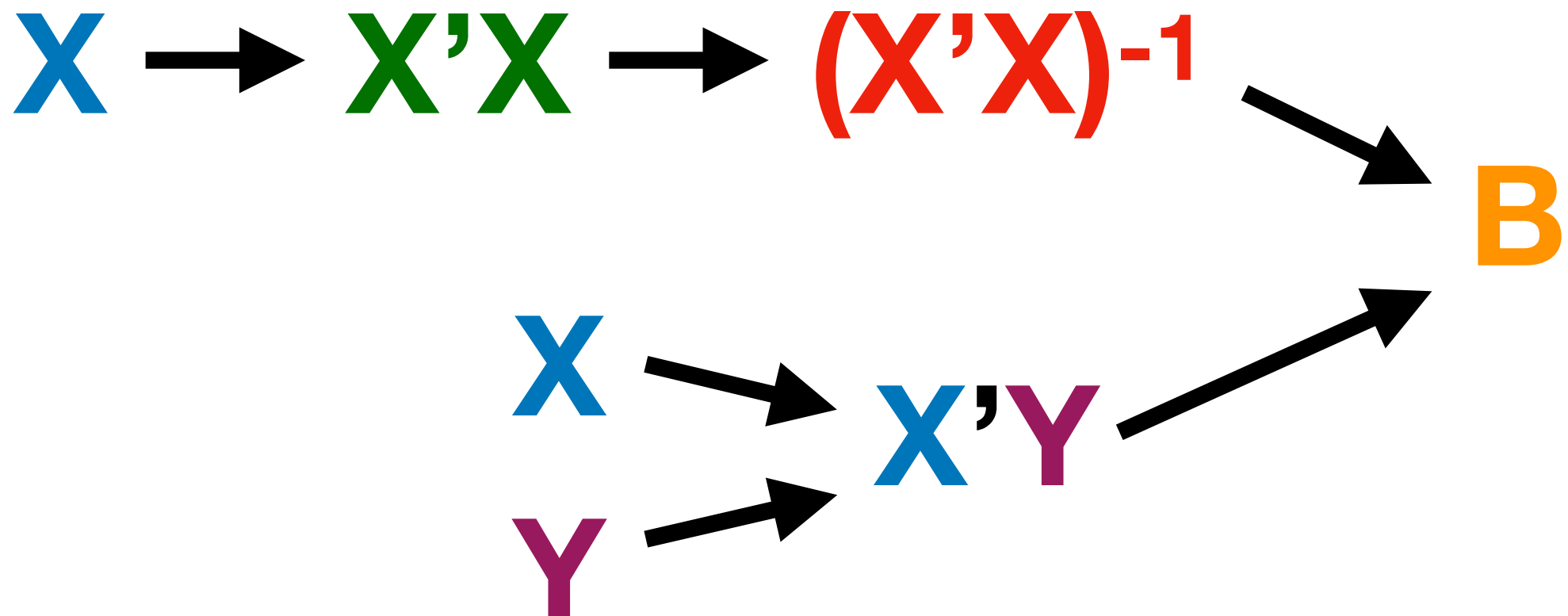
# Our Goal

$$\mathbf{B} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

# Our Goal

$$B = (X'X)^{-1}X'Y$$

Just break it up into smaller steps:



# X and X Transpose

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix} \quad X' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix}$$

# Step 1: Calculate $X'X$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

# Step 1: Calculate $X'X$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(1) + (1)(1) + (1)(1) + (1)(1) + (1)(1) & (1)(2) + (1)(2) + (1)(3) + (1)(4) + (1)(4) \\ (2)(1) + (2)(1) + (3)(1) + (4)(1) + (4)(1) & (2)(2) + (2)(2) + (3)(3) + (4)(4) + (4)(4) \end{bmatrix}$$

# Step 1: Calculate $X'X$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix}$$

Step 2: Calculate  $(X'X)^{-1}$

$$(X'X)^{-1} = \frac{\text{Adj}(X'X)}{|X'X|}$$



Step 2a: Calculate  $|X'X|$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

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$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

$$X'X = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix}$$

Step 2a: Calculate  $|X'X|$

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

$$X'X = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix}$$

$$|X'X| = (5)(49) - (15)(15) = 20$$

Step 2b: Calculate  $\text{Adj}(\mathbf{X}'\mathbf{X})$

$$\text{Adj}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Step 2b: Calculate Adj( $X'X$ )

$$\text{Adj}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{Adj}\left(\begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix}\right) = \begin{bmatrix} 49 & -15 \\ -15 & 5 \end{bmatrix}$$

Step 2c: Calculate  $(X'X)^{-1}$

$$(X'X)^{-1} = \frac{\text{Adj}(X'X)}{|X'X|}$$

Step 2c: Calculate  $(X'X)^{-1}$

$$(X'X)^{-1} = \frac{\text{Adj}(X'X)}{|X'X|}$$

$$= \frac{1}{20} \begin{bmatrix} 49 & -15 \\ -15 & 5 \end{bmatrix}$$

Step 2c: Calculate  $(X'X)^{-1}$

$$(X'X)^{-1} = \frac{\text{Adj}(X'X)}{|X'X|}$$

$$= \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix}$$



Step 3: Calculate  $X'Y$

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \\ 6 \end{bmatrix}$$

# Step 3: Calculate $X'Y$

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(2) + (1)(4) + (1)(6) + (1)(2) + (1)(6) \\ (2)(2) + (2)(4) + (3)(6) + (4)(2) + (4)(6) \end{bmatrix}$$

# Step 3: Calculate $X'Y$

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 20 \\ 62 \end{bmatrix}$$

# Step 4: Put it all together

$$\mathbf{B} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

# Step 4: Put it all together

$$B = (X'X)^{-1}X'Y$$

$$= \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix} \begin{bmatrix} 20 \\ 62 \end{bmatrix}$$

# Step 4: Put it all together

$$B = (X'X)^{-1}X'Y$$

$$= \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix} \begin{bmatrix} 20 \\ 62 \end{bmatrix}$$

$$= \begin{bmatrix} (2.45)(20) + (-0.75)(62) \\ (-0.75)(20) + (0.25)(62) \end{bmatrix}$$

# Step 4: Put it all together

$$B = (X'X)^{-1}X'Y$$

$$= \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix} \begin{bmatrix} 20 \\ 62 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 \\ 0.5 \end{bmatrix}$$

# Tricks for Checking Your Work

1. Each number in the  $X'X$  matrix has a substantive interpretation:

$$X'X = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix} = \begin{bmatrix} \text{Number of observations} & \text{Sum of } x \\ \text{Sum of } x & \text{Sum of } x^2 \end{bmatrix}$$



# Tricks for Checking Your Work

2.  $X'X$  and  $(X'X)^{-1}$  are both symmetric matrices

$$X'X = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix} \quad (X'X)^{-1} = \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

# Tricks for Checking Your Work

3.  $X'X (X'X)^{-1}$  is the identity matrix (**I**)

$$X'X (X'X)^{-1} = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix} \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} (5)(2.45) + (15)(-0.75) & (5)(-0.75) + (15)(0.25) \\ (15)(2.45) + (49)(-0.75) & (15)(-0.75) + (49)(0.25) \end{bmatrix}$$

# Tricks for Checking Your Work

3.  $X'X (X'X)^{-1}$  is the identity matrix ( $I$ )

$$X'X (X'X)^{-1} = \begin{bmatrix} 5 & 15 \\ 15 & 49 \end{bmatrix} \begin{bmatrix} 2.45 & -0.75 \\ -0.75 & 0.25 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$