Recitation Material: Probability and Distributions

TA

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Basic Probability and Random Variables

Problem 1: Coin Toss

A fair coin is flipped 5 times. What is the probability of getting exactly 2 heads?

Solution: The problem follows a binomial distribution with $n = 5$ and $p = 0.5$. The probability of getting exactly 2 heads is:

$$0.3125$$

Problem 2: Rolling a Die

A six-sided die is rolled twice. What is the probability that the sum of the outcomes is greater than or equal to 10?

Solution: the probability is:

$$P(\text{sum} \geq 10) = \frac{6}{36} = \frac{1}{6} \approx 0.1667$$

Probability Tables

Problem 3: Expected Value from Probability Table

A random variable Y has the following probability distribution:

y	$P(Y = y)$
0	0.1
1	0.3
2	0.4
3	0.2

Find the expected value of Y.

Solution: The expected value is:

$$E[Y] = \sum_y yP(Y = y) = 0\times0.1+1\times0.3+2\times0.4+3\times0.2 = 0+0.3+0.8+0.6 = 1.7$$

Uniform Distribution and Continuous Distributions

Problem 4: Uniform Distribution

A random variable X is uniformly distributed between -5 and 5. What is the probability that X is between -3 and 3?

Solution:

$$P(-3 \leq X \leq 3) = 0.6$$

Problem 5: Cumulative Distribution Function (CDF)

Given a continuous random variable X with probability density function $f(x) = 3x^2$ for $0 \leq x \leq 1$, find the cumulative distribution function (CDF).

Solution: The CDF $F(x)$ is given by:

$$F(x) = \int_0^x 3t^2\,dt = \left[t^3\right]_0^x = x^3 \quad \text{for } 0 \leq x \leq 1$$

Discrete Distributions: Bernoulli, Categorical, Poisson

Problem 6: Categorical Distribution

A random variable Z takes values in the set $\{A, B, C\}$ with probabilities $P(Z = A) = 0.2$, $P(Z = B) = 0.5$, and $P(Z = C) = 0.3$. What is the probability that Z is either A or C?

Solution: The probability that Z is either A or C is:

$$P(Z = A \text{ or } Z = C) = P(Z = A) + P(Z = C) = 0.2 + 0.3 = 0.5$$

Problem 7: Poisson Distribution

If the average number of cars passing through a toll booth in an hour is 7, what is the probability that exactly 4 cars pass through in a given hour?

Solution: This is a Poisson distribution with $\lambda = 7$. The probability of seeing exactly 4 cars is:

$$P(X = 4) = \frac{e^{-7}7^4}{4!}$$

Continuous Distributions: Gaussian, Multivariate Gaussian

Problem 8: Gaussian Distribution

A random variable X is normally distributed with mean $\mu = 15$ and variance $\sigma^2 = 9$. Find the probability that X is between 12 and 18.

Solution: We standardize X using $Z = \frac{X-\mu}{\sigma}$. For $X = 12$, $Z = \frac{12-15}{3} = -1$. For $X = 18$, $Z = \frac{18-15}{3} = 1$. And use the standard normal table.

Problem 9: Multivariate Gaussian Covariance

Consider the bivariate normal distribution $p(x_1, x_2)$ with mean vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix}$. What is the covariance between x_1 and x_2?

Solution: The covariance between x_1 and x_2 is the off-diagonal element of the covariance matrix:

$$\mathrm{Cov}(x_1, x_2) = 0.5$$

Combinatorics

Problem 10: Committee Selection

A committee of 3 people is to be formed from a group of 6 men and 4 women. What is the probability that the committee consists of exactly 2 men and 1 woman?

Solution:

$$P(2 \text{ men}, 1 \text{ woman}) = 0.5$$

Problem 11: Dice Rolling and Union

Consider two events: A: "The result of rolling a fair six-sided die is greater than 4" and B: "The result of rolling a fair six-sided die is even." Find $P(A \cup B)$ and $P(A \cap B)$.

Solution: $P(A \cap B) = \frac{1}{6}$. $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$.

Problem 12: Probability Table (Modified)

Consider the following joint probability table for random variables X and Y:

X/Y	1	2
0	0.05	0.15
1	0.10	0.20
2	0.10	0.05
3	0.20	0.15

Find the following:

- The marginal $P(X = 1)$
- The marginal $P(Y = 2)$
- The joint $P(X = 3, Y = 1)$
- The conditional $P(X = 2 \mid Y = 2)$
- The conditional $P(Y = 1 \mid X = 3)$

Solution: The marginal $P(X = 1) = 0.10 + 0.20 = 0.30$. The marginal $P(Y = 2) = 0.15 + 0.20 + 0.05 + 0.15 = 0.55$. The joint $P(X = 3, Y = 1) = 0.20$. The conditional $P(X = 2 \mid Y = 2) = \frac{P(X=2,Y=2)}{P(Y=2)} = \frac{0.05}{0.55} \approx 0.0909$. The conditional $P(Y = 1 \mid X = 3) = \frac{P(X=3,Y=1)}{P(X=3)} = \frac{0.20}{0.20+0.15} = \frac{4}{7} \approx 0.5714$.

Combinatorics

Problem 13: Arranging Books

In how many ways can 4 different math books and 3 different science books be arranged on a shelf if books of the same subject must be kept together?

Solution: 288

Continuous Distributions

Problem 14: Uniform Distribution

A random variable X is uniformly distributed between 2 and 8. Find the probability that X is between 4 and 6.

Solution:

$$P(4 \leq X \leq 6) = \frac{1}{3}$$

System Reliability

Problem 15: Power System Failure

An energy system relies on two power plants: plant A and plant B. If and only if both plants fail, then the energy system fails. Consider two events: A: {plant A fails}, B: {plant B fails}. Assume that $P(A) = 4\%$, $P(B) = 6\%$, and $P(A \cup B) = 8\%$.

a) What is the failure probability, p_F, of the energy system?

b) Are A and B independent events?

c) What is p_F given A? What is p_F given B? What is p_F given $A \cup B$?

Solution: a)

$$P(A \cap B) = 2\%$$

b) For A and B to be independent, $P(A \cap B) = P(A) \times P(B) = 0.04 \times 0.06 = 0.0024 = 0.24\%$. Since $P(A \cap B) = 2\%$, A and B are not independent.

c) p_F given $Ais = 0.5$
p_F given $Bis = \approx 0.333$
p_F given $A \cup B = 0.25$

Bernoulli and Poisson Distributions

Problem 16: Bernoulli Distribution

A light bulb has a 90% probability of working. Let X be a Bernoulli random variable representing whether the light bulb works ($X = 1$ if it works, $X = 0$ if it fails). Find $E[X]$ and $Var(X)$.

Solution: $E[X] = p = 0.9$ $Var(X) = p(1-p) = 0.9 \times 0.1 = 0.09$

Problem 17: Poisson Distribution

A website gets an average of 10 visits per minute. What is the probability that the site will get exactly 7 visits in a given minute?

Solution: This is a Poisson distribution with $\lambda = 10$. The probability of getting exactly 7 visits is:

$$P(X = 7) = \frac{e^{-10} 10^7}{7!}$$

Conditional Probability

Problem 18: Conditional Probability and Independence

Consider two events C and D such that $P(C) = 0.6$, $P(D) = 0.4$, and $P(C \cap D) = 0.24$. Are C and D independent? Find $P(D \mid C)$.

Solution: C and D are independent. The conditional probability $P(D \mid C)$ is:

$$P(D \mid C) = 0.4$$

Multivariate Gaussian

Problem 19: Multivariate Gaussian Mean and Variance

Consider the bivariate normal distribution $p(x_1, x_2)$ with mean vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$. Find the conditional mean and variance of X_1 given

$X_2 = x_2$.

Solution: The conditional mean and variance are given by:

$$\mu_{X_1|X_2=x_2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2) = 1 + \frac{2}{5}(x_2 - 3)$$

$$\sigma^2_{X_1|X_2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = 4 - \frac{2^2}{5} = 4 - \frac{4}{5} = 3.2$$

Combinatorics

Problem 20: Committee Selection

A committee of 4 people is to be selected from 5 men and 6 women. What is the probability that the committee consists of exactly 2 men and 2 women?

Solution:

$$P(2 \text{ men, } 2 \text{ women}) = \frac{5}{11} \approx 0.4545$$