Recitation Material

TA

October 2024

Convexity and Optimization

Problem 1: Convexity of a Quadratic Function

Let $f(x) = x^2 + 3x + 5$. Prove whether $f(x)$ is convex by calculating its second derivative.

Problem 2: Convexity of a Function in Two Variables

Determine whether the function $f(x, y) = x^2 + y^2 + xy$ is convex by calculating the Hessian matrix.

Problem 3: Gradient Descent on a Simple Function

Perform two iterations of gradient descent for $f(x) = x^2 + 4x + 4$, starting at $x_0 = 3$, with a learning rate $\alpha = 0.1$.

Problem 4: Gradient Descent on a Two-Variable Function

Consider the function $f(x, y) = x^2 + y^2$. Starting at $(x_0, y_0) = (1, 2)$, perform one iteration of gradient descent with learning rate $\alpha = 0.1$.

Linear Regression and Gradient Descent

Problem 5: Linear Regression Gradient Descent Iteration

Consider the linear regression model $y = X\beta$, where $X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $y = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. Starting with $\beta_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, perform one iteration of gradient descent with learning rate $\alpha = 0.01$.

Problem 6: Two Iterations of Gradient Descent for Linear Regression

For the same linear regression model in Problem 5, perform a second iteration of gradient descent starting from $\beta_1 = \begin{pmatrix} 0.12 \\ 0.28 \end{pmatrix}$ with the same learning rate $\alpha = 0.01$.

Problem 7: Convexity in Linear Regression

Consider the loss function in linear regression $L(\beta) = \|y - X\beta\|^2$. Prove that this loss function is convex.

Matrix Calculus for Linear Regression

Problem 8: Derivative of a Quadratic Form

Let $f(x) = x^T Ax$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix. Compute the gradient of $f(x)$ with respect to x.

Problem 9: Gradient of Least Squares Loss

For linear regression, the loss function is $L(\beta) = \|y - X\beta\|^2$, where $X \in \mathbb{R}^{m\times n}$ is the design matrix, $\beta \in \mathbb{R}^n$ is the parameter vector, and $y \in \mathbb{R}^m$ is the target vector. Compute the gradient of $L(\beta)$ with respect to β.

Problem 10: Partial Derivatives in Matrix Calculus

Let $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$. Compute the derivative of $f(x) = x^T Ax + b^T x$, where $b \in \mathbb{R}^n$.

Convexity and Gradient Descent (Continued)

Problem 11: Convexity of L2 Regularization

Show that the L2 regularization term $f(\beta) = \|\beta\|_2^2$ is convex. Use the second derivative test for this function.

Problem 12: Gradient Descent with L2 Regularization

For the linear regression model $y = X\beta$ with an L2 regularization term $\|\beta\|_2^2$, the objective function becomes $L(\beta) = \|y - X\beta\|^2 + \lambda\|\beta\|_2^2$, where $\lambda > 0$ is a regularization parameter. Compute the gradient of $L(\beta)$ with respect to β.

Problem 13: Convergence of Gradient Descent for Convex Functions

Let $f(x) = x^2$. Show that gradient descent converges to the global minimum for this convex function. Assume a learning rate $\alpha = 0.1$ and starting point $x_0 = 3$.

Problem 14: Gradient of Logistic Regression Loss

The loss function for logistic regression is given by:

$$L(\beta) = -\sum_{i=1}^{m} \left[y_i \log(h_\beta(x_i)) + (1 - y_i) \log(1 - h_\beta(x_i)) \right],$$

where $h_\beta(x_i) = \frac{1}{1+e^{-x_i^T \beta}}$ is the sigmoid function. Compute the gradient of $L(\beta)$ with respect to β.

Problem 15: Gradient Descent for Logistic Regression

Given the logistic regression model and the gradient computed in Problem 14, perform one iteration of gradient descent for the dataset:

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad y = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Assume the starting point $\beta_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and learning rate $\alpha = 0.1$.