

Recitation Material: Probability and Distributions

TA

October 2024

Basic Probability and Random Variables

Problem 1: Coin Toss

A fair coin is flipped 5 times. What is the probability of getting exactly 2 heads?

Problem 2: Rolling a Die

A six-sided die is rolled twice. What is the probability that the sum of the outcomes is greater than or equal to 10?

Probability Tables

Problem 3: Expected Value from Probability Table

A random variable Y has the following probability distribution:

y	$P(Y = y)$
0	0.1
1	0.3
2	0.4
3	0.2

Find the expected value of Y .

Uniform Distribution and Continuous Distributions

Problem 4: Uniform Distribution

A random variable X is uniformly distributed between -5 and 5. What is the probability that X is between -3 and 3?

Problem 5: Cumulative Distribution Function (CDF)

Given a continuous random variable X with probability density function $f(x) = 3x^2$ for $0 \leq x \leq 1$, find the cumulative distribution function (CDF).

Discrete Distributions: Bernoulli, Categorical, Poisson

Problem 6: Categorical Distribution

A random variable Z takes values in the set $\{A, B, C\}$ with probabilities $P(Z = A) = 0.2$, $P(Z = B) = 0.5$, and $P(Z = C) = 0.3$. What is the probability that Z is either A or C ?

Problem 7: Poisson Distribution

If the average number of cars passing through a toll booth in an hour is 7, what is the probability that exactly 4 cars pass through in a given hour?

Continuous Distributions: Gaussian, Multivariate Gaussian

Problem 8: Gaussian Distribution

A random variable X is normally distributed with mean $\mu = 15$ and variance $\sigma^2 = 9$. Find the probability that X is between 12 and 18.

Problem 9: Multivariate Gaussian Covariance

Consider the bivariate normal distribution $p(x_1, x_2)$ with mean vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix}$. What is the covariance between x_1 and x_2 ?

Combinatorics

Problem 10: Committee Selection

A committee of 3 people is to be formed from a group of 6 men and 4 women. What is the probability that the committee consists of exactly 2 men and 1 woman?

Problem 11: Dice Rolling and Union

Consider two events: A : "The result of rolling a fair six-sided die is greater than 4" and B : "The result of rolling a fair six-sided die is even." Find $P(A \cup B)$ and $P(A \cap B)$.

Problem 12: Probability Table (Modified)

Consider the following joint probability table for random variables X and Y :

X/Y	1	2
0	0.05	0.15
1	0.10	0.20
2	0.10	0.05
3	0.20	0.15

Find the following:

- The marginal $P(X = 1)$
- The marginal $P(Y = 2)$
- The joint $P(X = 3, Y = 1)$
- The conditional $P(X = 2 \mid Y = 2)$
- The conditional $P(Y = 1 \mid X = 3)$

Combinatorics

Problem 13: Arranging Books

In how many ways can 4 different math books and 3 different science books be arranged on a shelf if books of the same subject must be kept together?

Continuous Distributions

Problem 14: Uniform Distribution

A random variable X is uniformly distributed between 2 and 8. Find the probability that X is between 4 and 6.

System Reliability

Problem 15: Power System Failure

An energy system relies on two power plants: plant A and plant B. If and only if both plants fail, then the energy system fails. Consider two events: A :

{plant A fails}, B : {plant B fails}. Assume that $P(A) = 4\%$, $P(B) = 6\%$, and $P(A \cup B) = 8\%$.

- a) What is the failure probability, p_F , of the energy system?
- b) Are A and B independent events?
- c) What is p_F given A ? What is p_F given B ? What is p_F given $A \cup B$?

Bernoulli and Poisson Distributions

Problem 16: Bernoulli Distribution

A light bulb has a 90% probability of working. Let X be a Bernoulli random variable representing whether the light bulb works ($X = 1$ if it works, $X = 0$ if it fails). Find $E[X]$ and $Var(X)$.

Problem 17: Poisson Distribution

A website gets an average of 10 visits per minute. What is the probability that the site will get exactly 7 visits in a given minute?

Conditional Probability

Problem 18: Conditional Probability and Independence

Consider two events C and D such that $P(C) = 0.6$, $P(D) = 0.4$, and $P(C \cap D) = 0.24$. Are C and D independent? Find $P(D | C)$.

Multivariate Gaussian

Problem 19: Multivariate Gaussian Mean and Variance

Consider the bivariate normal distribution $p(x_1, x_2)$ with mean vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and covariance matrix $\begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$. Find the conditional mean and variance of X_1 given $X_2 = x_2$.

Combinatorics

Problem 20: Committee Selection

A committee of 4 people is to be selected from 5 men and 6 women. What is the probability that the committee consists of exactly 2 men and 2 women?