Recitation Material: Linear Algebra

10-606

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PSD Matrices and Inverses

1. Show that if A is invertible, then its eigenvalues are all nonzero. Conversely, if an eigenvalue is zero, why can't A^{-1} exist?

2. If A is invertible, verify that

$$(A + uv^\top)^{-1} = A^{-1} - \frac{A^{-1}u\,v^\top A^{-1}}{1 + v^\top A^{-1}u},$$

   as long as the denominator is nonzero.

3. A *norm* is a function ρ that satisfies (i) $\rho(x) \geq 0$ for all x, (ii) $\rho(x) = 0$ if and only if $x = 0$, (iii) $\rho(cx) = |c|\rho(x)$ for all x and scalars c, and (iv) $\rho(x + y) \leq \rho(x) + \rho(y)$ for all x, y. If A is a positive-definite matrix A, show that $\rho(x) = \|x\|_A$ where $\|x\|_A\sqrt{x^\intercal Ax}$ defines a norm. Recall that A is positive-definite if $x^\intercal Ax > 0$ for all nonzero x. Hint: You may use the Cauchy-Schwarz inequality $x^\intercal Ay \leq \|x\|_A\|y\|_A$ without proof.

SVD and Rank

1. What's the singular value decomposition of a matrix A?

2. Show that if $A = uv^\top$ with $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$ and $u, v \neq 0$, then $\text{rank}(A) = 1$.

3. What is the relation between singular values of A and eigenvalues of $A^\top A$?

4. Compute the nonzero singular value of $A = uv^\top$.

5. If $A = U\Sigma V^\top$, what is the SVD of $A^\top$?