Notation overview

10-606

1 Summation and Product Notation

Sums

Sums are denoted with the Greek letter $\sum$. General form:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n.$$

Examples.

$$\sum_{i=1}^{5} i = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{j=0}^{3} (2^j) = 2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15$$

$$\sum_{x \in \{2,4,6\}} x^2 = 2^2 + 4^2 + 6^2 = 4 + 16 + 36 = 56$$

Products

Products are denoted with the Greek letter $\prod$. General form:

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot a_3 \cdots a_n$$

Examples.

$$\prod_{i=1}^{4} i = 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

$$\prod_{j=1}^{3} (j+1) = (1+1)(2+1)(3+1) = 2 \cdot 3 \cdot 4 = 24$$

$$\prod_{x \in \{1,2,3\}} 2^x = 2^1 \cdot 2^2 \cdot 2^3 = 2 \cdot 4 \cdot 8 = 64$$

Properties

- Linearity of sums:

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i \ + \ \sum_{i=1}^{n} b_i$$

- Constants can be pulled out:

$$\sum_{i=1}^{n} c \cdot a_i = c \cdot \sum_{i=1}^{n} a_i$$

- Products are multiplicative:

$$\prod_{i=1}^{n} (a_i b_i) = \left(\prod_{i=1}^{n} a_i \right) \left(\prod_{i=1}^{n} b_i \right)$$

Multiple Summations and Products

Summations and products can be combined or nested. The order and scope of the indices matters.

Double Summation.

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

means "add up all the a_{ij} as i runs from 1 to m and j runs from 1 to n." This is equivalent to summing over all pairs (i, j).

Example:

$$\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j) = (1+1)+(1+2)+(1+3)+(2+1)+(2+2)+(2+3) = 18.$$

Double Product.

$$\prod_{i=1}^{m}\prod_{j=1}^{n} a_{ij}$$

means "multiply all the a_{ij} over all pairs (i, j)."

Example:

$$\prod_{i=1}^{2}\prod_{j=1}^{2}(i+j) = (1+1)(1+2)(2+1)(2+2) = 2\cdot 3\cdot 3\cdot 4 = 72.$$

Sums over Products. You can also have products inside a sum:

$$\sum_{i=1}^{n}\Big(\prod_{j=1}^{m} a_{ij}\Big).$$

Example:

$$\begin{aligned}\sum_{i=1}^{2}\left(\prod_{j=1}^{3}(i+j)\right) &= \big[(1+1)(1+2)(1+3)\big] + \big[(2+1)(2+2)(2+3)\big] \\ &= (2\cdot 3\cdot 4) + (3\cdot 4\cdot 5) = 24+60 = 84.\end{aligned}$$

Products over Sums. Or sums inside a product:

$$\prod_{i=1}^{n}\Big(\sum_{j=1}^{m} a_{ij}\Big).$$

Example:

$$\begin{aligned}\prod_{i=1}^{2}\left(\sum_{j=1}^{3}(i+j)\right) &= \big[(1+1)+(1+2)+(1+3)\big]\cdot\big[(2+1)+(2+2)+(2+3)\big] \\ &= (2+3+4)(3+4+5) = 9\cdot 12 = 108.\end{aligned}$$

2 Quantifiers: $\exists$ and $\forall$

When we talk about sets and mathematical statements, we often want to say things like:

- "*Some* number has property X."
- "*All* numbers have property Y."

Mathematics has special symbols for this:

- Existential quantifier: $\exists$ ("there exists")
- Universal quantifier: $\forall$ ("for all")

The Existential Quantifier ($\exists$)

- Meaning: "There exists at least one element with this property."
- Symbol: $\exists x \in S : P(x)$, where $P(x)$ is some proposition
- Read: "There exists an x in set S such that property $P(x)$ holds."

Example:

$$\exists x \in \mathbb{R} : x^2 = 4$$

There exists a real number whose square is 4. This is true, since $x = 2$ or $x = -2$.

The Universal Quantifier ($\forall$)

- Meaning: "Every element has this property."
- Symbol: $\forall x \in S : P(x)$, where $P(x)$ is some proposition
- Read: "For all x in set S, property $P(x)$ holds."

Example:

$$\forall x \in \mathbb{R} : x^2 \geq 0.$$

For every real number, its square is non-negative. This is true.

Combinations

We can combine universal and existential quantifiers. The order matters a lot: $\forall$ then $\exists$ means something very different from $\exists$ then $\forall$. Always read quantifiers carefully, left to right.

Example 1.

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : y = x + 1$$

For every real number x, there exists a real number y such that $y = x + 1$. This is true, because given any x, we can always find y one unit bigger.

Example 2.

$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R} : y = x + 1$$

There exists a single real number y such that $y = x+1$ for *every* real x. This is false, because no single y can equal $x + 1$ for all x at once.

Example 3.

$$\forall x \in \mathbb{N}, \exists \text{ a prime } p : p > x$$

For every natural number x, there exists a prime number p larger than x. This is true, and is another way of stating that there are infinitely many primes.

Example 4.

$$\exists f : \mathbb{R} \to \mathbb{R}, \forall x \in \mathbb{R}, f(x) \geq 0$$

There exists a function from $\mathbb{R}$ to $\mathbb{R}$ such that for all real numbers x, $f(x) \geq 0$. This is true — for example, the function $f(x) = 1$ works.

Negation Rules

A useful fact (important in proofs and algorithms) is that negation flips quantifiers:

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

"Not everything works" means "something fails."

$$\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

"Nothing works" means "everything fails."

Why Does This Matter for ML?

Quantifiers let us write conditions precisely:

- **Existential:** "There exists a parameter θ such that our loss is minimized."
$$\exists \theta \in \mathbb{R}^d : L(\theta) = \min_{\theta^*} L(\theta^*).$$
- **Universal:** "For all data points, our classifier predicts correctly."
$$\forall (x, y) \in \text{dataset} : f(x) = y.$$

In practice, we often cannot satisfy the universal condition exactly (too strict), so optimization in ML is usually about balancing "for all" vs. "there exists" statements.

Quick Exercises

1. Write $\forall x \in \mathbb{R} : |x| \geq 0$ in words. Is it true?
2. Express in symbols: "There exists a natural number greater than 100."
3. Are these true or false?
   - $\forall x \in \mathbb{R} : x^2 > 1$
   - $\exists x \in \mathbb{R} : x^2 > 1$
4. Translate into words and decide if true:
$$\exists x \in \mathbb{N} : x + 5 = 2$$
5. Translate into words and decide if true:
$$\forall x \in \mathbb{R} : x^2 + 1 > 0$$
6. Which of these statements is true? Which is false?
$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : y > x$$
$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R} : y > x$$

7. Express in symbols: “Every student in the class has solved at least one homework problem.”

8. Express in symbols: “There is one student in the class who solved all the homework problems.”

9. Decide whether the following is true or false:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : xy = 1$$

10. Fill in the correct quantifiers ($\forall$ or $\exists$) to make the following statement true:

$$?\ x \in \mathbb{R},\ ?\ y \in \mathbb{R} : x + y = 0$$

11. Compare the strength of the following two statements:

$$\forall x \in \mathbb{N} : x^2 \geq x$$

$$\exists x \in \mathbb{N} : x^2 \geq x$$

Which one is stronger? Which is weaker?

Answers

1. True

2. $\exists x \in \mathbb{N} : x \geq 100$.

3. False, True.

4. There exists a natural number x such that $x + 5 = 2$. True.

5. For all real numbers x, $x^2 + 1 > 0$. True.

6. True, False

7. Let S be the set of students and H the set of homework problems. Let $A(x, h)$ be the statement, “x has solved problem h”. $\forall x \in S\ \exists h \in H\ A(x, h)$.

8. $\exists x \in S\ \forall h \in H\ A(x, h)$.

9. False (consider $x = 0$).
10. $\forall, \exists$
11. The universal statement is stronger.