Recitation Material: Linear Algebra

10-606

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PSD Matrices and Inverses

1. Show that if A is invertible, then its eigenvalues are all nonzero. Conversely, if an eigenvalue is zero, why can't A^{-1} exist? **Solution:** If $Av = \lambda v$ and A^{-1} exists, then $v = A^{-1}Av = \lambda A^{-1}v$, so $\lambda \neq 0$. Conversely, if $\lambda = 0$, then $Av = 0$ for $v \neq 0$. If A^{-1} existed, then $v = A^{-1}Av = A^{-1}0 = 0$, contradicting that v is nonzero.

2. If A is invertible, verify that

$$(A + uv^\top)^{-1} = A^{-1} - \frac{A^{-1}u\,v^\top A^{-1}}{1 + v^\top A^{-1}u},$$

as long as the denominator is nonzero. **Solution:** Let $s = 1 + v^\intercal A^{-1}u$ be the denominator (which is a scalar!). Consider multiplying the right hand side by $(A + uv^\intercal)$ on the right, which gives

$$\begin{aligned}
&A^{-1}(A + uv^\intercal) - \frac{1}{s}(A^{-1}uv^\intercal A^{-1})(A + uv^\intercal) \\
&= A^{-1}A + A^{-1}uv^\intercal - \frac{1}{s}(A^{-1}uv^\intercal + A^{-1}uv^\intercal A^{-1}uv^\intercal) \\
&= I + A^{-1}uv^\intercal - \frac{A^{-1}u}{s}(I + v^\intercal A^{-1}u)v^\intercal \\
&= I + A^{-1}uv^\intercal - \frac{A^{-1}u}{s}sv^\intercal \\
&= I.
\end{aligned}$$

Since the left hand side multiplied by $A + uv^\intercal$ is also I, this gives the result.

3. A *norm* is a function ρ that satisfies (i) $\rho(x) \geq 0$ for all x, (ii) $\rho(x) = 0$ if and only if $x = 0$, (iii) $\rho(cx) = |c|\rho(x)$ for all x and scalars c, and (iv) $\rho(x + y) \leq \rho(x) + \rho(y)$ for all x, y. If A is a positive-definite matrix A, show that $\rho(x) = \|x\|_A$ where $\|x\|_A\sqrt{x^\intercal Ax}$ defines a norm. Recall that A is positive-definite if $x^\intercal Ax > 0$ for all nonzero x. Hint: You may use the Cauchy-Schwarz inequality $x^\intercal Ay \leq \|x\|_A\|y\|_A$ without proof. (i) and (ii) are immediately implied by the fact that A is positive definite. For (iii),

compute $\|cx\|_A = \sqrt{(cx)^\intercal A(cx)} = \sqrt{c^2 x^\intercal Ax} = |c|\sqrt{x^\intercal Ax} = |c|\|x\|_A$ as desired. For (iv), we will show that $\|x+y\|_A^2 \leq (\|x\|_A + \|y\|_A)^2$. We have

$$\begin{aligned}\|x+y\|_A^2 = (x+y)^\intercal A(x+y) &= x^\intercal Ax + 2x^\intercal Ay + y^\intercal Ay \\ &= \|x\|_A^2 + 2x^\intercal Ay + \|y\|_A^2.\end{aligned}$$

(Note we've used that $x^\intercal Ay = y^\intercal Ax$.) Meanwhile,

$$(\|x\|_A + \|y\|_A)^2 = \|x\|_A^2 + \|y\|_A^2 + 2\|x\|_A\|y\|_A.$$

From here the desired inequality follows after applying Cauchy-Schwarz.

SVD and Rank

1. What's the singular value decomposition of a matrix A? **Solution:** For any $A \in \mathbb{R}^{m\times n}$, there exists some $U \in \mathbb{R}^{m\times m}, V \in \mathbb{R}^{n\times n}$ orthogonal and $\Sigma \in \mathbb{R}^{m\times n}$ diagonal (and nonnegative) such that $A = U\Sigma V^T$.

2. Show that if $A = uv^\top$ with $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$ and $u, v \neq 0$, then $\text{rank}(A) = 1$. **Solution.** $\text{col}(A) = \text{span}\{u\}$ since each column j equals $v_j u$. Hence $\text{rank} = 1$.

3. What is the relation between singular values of A and eigenvalues of $A^\top A$? **Solution.** If $A = U\Sigma V^T$ is the SVD of A, then $A^T = V\Sigma^2 U^T$ is the SVD of A^2, implying that the eigenvalues of A^T are $\sigma_1^2, \ldots, \sigma_n^2$, the squares of the singular values of A.

4. Compute the nonzero singular value of $A = uv^\top$. **Solution.** Use the logic implied by the previous question. We have $A^\top A = vu^\top uv^\top = \|u\|^2\, vv^\top$. Eigenvalues and eigenvectors of $A^\top A$ thus satisfy $\|u\|_2 vv^\top x = \lambda x$. Now, since $vv^\top$ has rank one, we know that the eigenvector x has to be in the span of v. In fact, we can take $x = v$ to see that the nonzero eigenvalue of $A^\top A$ is $\|u\|^2\|v\|^2$. Thus $\sigma_1(A) = \|u\|\,\|v\|$.

5. If $A = U\Sigma V^\top$, what is the SVD of $A^\top$? **Solution.** $A^\top = V\Sigma^\top U^\top$ with the same nonzero singular values and swapped singular vectors.