Sets, Vectors, and Matrices
Quiz 1 Practice

10-606

Disclaimer: This isn't exactly a "specimen paper." That is, some problems here are going to be more difficult, and some are going to be easier. However, this is a good place to start to find out which concepts you need to work on more. This also isn't meant to reflect the length or format of the actual quiz. :)

1 Set Operations

Key Points: Review addition, subtraction, products, unions, intersections, types for sets.

1. What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

> **Ans:** The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$.
>
> Hence, $A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1),$
> $(0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$.

2. Let $A = \{x \in \mathbb{N} : 3 \leq x < 16\}$ and $B = \{x \in \mathbb{N} : x \text{ is even}\}$.
   - Find $A \cap B$.
   - Find $A \setminus B$.

> **Ans:**
>
> - $A \cap B$ will be the set of natural numbers that are both at least 3 and less than 16, and even. That is, $A \cap B = \{x \in \mathbb{N} : 3 \leq x < 16 \land x \text{ is even}\} = \{4, 6, 8, 10, 12, 14\}$.
> - $A \setminus B$ is the set of all elements that are in A but not B. So this is $\{x \in \mathbb{N} : 3 \leq x < 16 \land x \text{ is odd}\} = \{3, 5, 7, 9, 11, 13, 15\}$.
>
>   Note this is the same set as $A \cap \overline{B}$.

3. Recall $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ (the integers). Let $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ be the positive integers. Let $2\mathbb{Z}$ be the even integers $2\mathbb{Z} = \{\dots, -4, -2, 0, 2, 4, \dots\}$, $3\mathbb{Z}$ be the multiples of 3, $3\mathbb{Z} = \{\dots, -6, -3, 0, 3, 6, \dots\}$

   - Is $\mathbb{Z}^+ \subseteq 2\mathbb{Z}$? Explain.
   - Is $2\mathbb{Z} \subseteq \mathbb{Z}^+$? Explain.
   - Find $2\mathbb{Z} \cap 3\mathbb{Z}$. Describe the set in words, and using set notation.
   - Express $\{x \in \mathbb{Z} : \exists y \in \mathbb{Z}(x = 2y \vee x = 3y)\}$ as a union or intersection of two sets already described in this problem.

Ans:

- No.
- No.
- $2\mathbb{Z} \cap 3\mathbb{Z}$ is the set of all integers which are multiples of both 2 and 3 (so multiples of 6). Therefore $2\mathbb{Z} \cap 3\mathbb{Z} = \{x \in \mathbb{Z} : \exists y \in \mathbb{Z}(x = 6y)\}$.
- $2\mathbb{Z} \cup 3\mathbb{Z}$.

2 Set Builder Notation

Key Points: Review how to convert between natural language expression to set notation (and vice versa).

1. Describe the set of all odd numbers between 100 and 200 using set builder notation.

Ans:

$$\{p \mid p = 2n + 1, n \in \mathbb{Z}, 100 < p < 200\}$$

Alternatively, we can specify the range for n directly:

$$\{2n + 1 \mid n \in \mathbb{Z}, 50 \leq n \leq 99\}$$

2. Describe each of the following sets both in words and by listing out enough elements to see the pattern.

- $\{x : x + 3 \in \mathbb{N}\}$
- $\{x \in \mathbb{N} : x + 3 \in \mathbb{N}\}$
- $\{x : x \in \mathbb{N} \vee -x \in \mathbb{N}\}$
- $\{x : x \in \mathbb{N} \wedge -x \in \mathbb{N}\}$
- $\{x \in \mathbb{Z} : x^2 \in \mathbb{N}\}$
- $\{x^2 : x \in \mathbb{N}\}$

Ans:

- This is the set of all numbers which are 3 less than a natural number (i.e., that if you add 3 to them, you get a natural number). The set could also be written as $\{-3, -2, -1, 0, 1, 2, \ldots\}$ (note that 0 is a natural number, so -3 is in this set because $-3 + 3 = 0$).
- This is the set of all natural numbers which are 3 less than a natural number. So here we just have $\{0, 1, 2, 3, \ldots\}$.
- This is the set of all integers (positive and negative whole numbers, written $\mathbb{Z}$). In other words, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- Here we want all numbers x such that x and $-x$ are natural numbers. There is only one: 0. So we have the set $\{0\}$.
- The set of integers that pass the condition that their square is a natural number. Well, every integer, when you square it, gives you a non-negative integer, so a natural number. Thus $A = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.
- Here we are looking for the set of all x^2's where x is a natural number. So this set is simply the set of perfect squares. $B = \{0, 1, 4, 9, 16, \ldots\}$.

3 Set Comprehension

Key Points: Python syntax (chained for loops using one line, set comprehension, conditionals, range function, etc.)

Write a Python set comprehension that does the following:

1. $\{(a, b, c) \in \mathbb{Z}_+^3 \mid a \text{ is even or } b \text{ is divisible by } 5\}$

Ans:

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{(a, b, c) for a in range(1, 21) for b in range(1, 21)
    for c in range(1, 21) if a % 2 == 0 or b % 5 == 0}
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2. $\{(a, b) \in \mathbb{Z}^2 \mid a^2 + b^2 < 4\}$

Ans:

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{(a, b) for a in range(-2, 3) for b in range(-2, 3)
    if a**2 + b**2 < 4}
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3. $\{x \in \mathbb{Z} \mid x \text{ is odd}\}$

Ans:

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{x for x in range(-50, 51) if x % 2 != 0}
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4 Vectors

Key Points: Vector addition, subtraction, scalar multiplication, dot product, outer and inner product, transpose, magnitude, L1 norm, types

1. Given $\vec{a} = (8, 5)$ and $\vec{b} = (-3, 6)$ compute each of the following.

   - $6\vec{a}$
   - $7\vec{b} - 2\vec{a}$
   - $\|10\vec{a} + 3\vec{b}\|$

Ans:

- $6\vec{a} = 6(8, 5) = (48, 30)$
- $7\vec{b} - 2\vec{a} = 7(-3, 6) - 2(8, 5) = (-21, 42) - (16, 10) = (-37, 32)$
- $10\vec{a} + 3\vec{b} = 10(8, 5) + 3(-3, 6) = (71, 68)$, so $\|10\vec{a} + 3\vec{b}\| = \sqrt{71^2 + 68^2} = \sqrt{9665}$

2. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$. Calculate The dot product $\mathbf{u} \cdot \mathbf{v}$.

Ans:
$\mathbf{u} \cdot \mathbf{v} = (1)(4) + (2)(-1) + (3)(0) = 4 - 2 + 0 = 2.$

3. Given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, find its transpose A^T.

Ans:

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

4. Let $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Calculate the inner product $\langle \mathbf{a}, \mathbf{b} \rangle$ and the outer product $\mathbf{a}\mathbf{b}^T$.

Ans: The inner product is the same as the dot product for real vectors:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a} \cdot \mathbf{b} = (2)(3) + (1)(-4) = 6 - 4 = 2$$

The outer product is:

$$\mathbf{a}\mathbf{b}^T = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -4 \end{pmatrix} = \begin{pmatrix} (2)(3) & (2)(-4) \\ (1)(3) & (1)(-4) \end{pmatrix} = \begin{pmatrix} 6 & -8 \\ 3 & -4 \end{pmatrix}$$

5. Given a vector $\mathbf{v} = \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix}$, calculate the L_1 norm, denoted as $||\mathbf{v}||_1$.

Ans: The L_1 norm (or Manhattan norm) is the sum of the absolute values of the vector's components.

$$||\mathbf{v}||_1 = |-3| + |5| + |-2| = 3 + 5 + 2 = 10$$

5 Matrices

Key Points: Matrix operations and properties (addition, subtraction, scalar multiplication), matrix multiplication, trace, types

1. Given two matrices $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ -2 & 6 \end{pmatrix}$, calculate the sum $A + B$ and the difference $A - B$.

   Ans:
   $A + B = \begin{pmatrix} 2+5 & 1+0 \\ 3-2 & 4+6 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 1 & 10 \end{pmatrix}$
   $A - B = \begin{pmatrix} 2-5 & 1-0 \\ 3-(-2) & 4-6 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 5 & -2 \end{pmatrix}$

2. If $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, find the scalar multiple $3A$.

   Ans: $3A = 3\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3\times 1 & 3\times 2 \\ 3\times 0 & 3\times 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 0 & 9 \end{pmatrix}$

3. Let $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$. Compute the matrix product CD.

   Ans:
   $CD = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} (1)(5)+(2)(7) & (1)(6)+(2)(8) \\ (3)(5)+(4)(7) & (3)(6)+(4)(8) \end{pmatrix}$
   $= \begin{pmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$

4. Find the trace of the matrix $E = \begin{pmatrix} 4 & 7 & 1 \\ 2 & 9 & 5 \\ 3 & 8 & 6 \end{pmatrix}$.

   Ans: The trace is the sum of the elements on the main diagonal. $Tr(E) = 4 + 9 + 6 = 19$

5. Classify the following matrices as either a row vector, a column vector, a square matrix, or a diagonal matrix:

   (a) $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

   (b) $G = \begin{pmatrix} 4 & -1 \end{pmatrix}$

   (c) $H = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

   (d) $I = \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$

Ans:

(a) Diagonal matrix

(b) Row vector

(c) Column vector

(d) Square matrix