HOMEWORK 3
LINEAR ALGEBRA *

10-606 MATHEMATICAL FOUNDATIONS FOR MACHINE LEARNING

START HERE: Instructions

- **Collaboration Policy**: Please read the collaboration policy in the syllabus.
- **Late Submission Policy:** See the late submission policy in the syllabus.
- **Submitting your work:** You will use Gradescope to submit answers to all questions.
  - **Written:** For written problems such as short answer, multiple choice, derivations, proofs, or plots, please use the provided template. Submissions can be handwritten onto the template, but should be labeled and clearly legible. If your writing is not legible, you will not be awarded marks. Alternatively, submissions can be written in LaTeX. Each derivation/proof should be completed in the boxes provided. To receive full credit, you are responsible for ensuring that your submission contains exactly the same number of pages and the same alignment as our PDF template.
  - **Latex Template:** https://www.overleaf.com/read/fxdwkrtthhgc#037764

Question	Points
Linear Systems and Linear Algebra	14
Projection Matrices	10
Matrix Memories	5
Eigenvectors and eigenvalues	8
Total:	37

*Compiled on Monday 8th September, 2025 at 07:10

Instructions for Specific Problem Types

For "Select One" questions, please fill in the appropriate bubble completely:

Select One: Who taught this course?

- ● Matt Gormley
- ○ Marie Curie
- ○ Noam Chomsky

If you need to change your answer, you may cross out the previous answer and bubble in the new answer:

Select One: Who taught this course?

- ● Henry Chai
- ○ Marie Curie
- ~~●~~ Noam Chomsky

For "Select all that apply" questions, please fill in all appropriate squares completely:

Select all that apply: Which are scientists?

- ■ Stephen Hawking
- ■ Albert Einstein
- ■ Isaac Newton
- □ I don't know

Again, if you need to change your answer, you may cross out the previous answer(s) and bubble in the new answer(s):

Select all that apply: Which are scientists?

- ■ Stephen Hawking
- ■ Albert Einstein
- ■ Isaac Newton
- □ I don't know

For questions where you must fill in a blank, please make sure your final answer is fully included in the given space. You may cross out answers or parts of answers, but the final answer must still be within the given space.

Fill in the blank: What is the course number?

10-606	10-60~~6~~7

1 Linear Systems and Linear Algebra (14 points)

1. For each statement, indicate whether it is true or false.

   (a) (1 point) A system of n linear equations with n unknowns has at least one solution.
   ◯ True ◯ False

   (b) (1 point) A system of n linear equations with n unknowns has at most one solution.
   ◯ True ◯ False

   (c) (1 point) If a square matrix is full rank, it cannot be inverted.
   ◯ True ◯ False

   (d) (2 points) Rank of matrix product $\mathbf{AB}$ can be greater than the rank of either $\mathbf{A}$ or $\mathbf{B}$.
   ◯ True ◯ False

2. (3 points) Consider the following matrix:

$$\mathbf{B} = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 4 & 4 & -1 & 3 \\ 0 & 0 & -2 & -1 \\ -2 & -1 & 0 & 0 \end{bmatrix}$$

   Compute the matrix rank of $\mathbf{B}$. You may do it by hand or by using Python and numpy. You must show your work. If you use Python and numpy, Just paste your working code in the Work box.

Rank

Work

3. (6 points) Rewrite the following system in matrix form and solve it by Gaussian Elimination.

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 &= 5 \\ -3x_1 + 5x_2 + x_3 &= 0 \\ 6x_1 - 4x_2 &= 1 \end{aligned}$$

For each row operation you use to put the matrix in upper triangular form, state the row operation and show the resulting matrix and RHS. For example, you can use the following format to indicate that you are replacing row2 with the result of row1 + row2:

$$r_2 \leftarrow r_1 + r_2 : \begin{pmatrix} 3 & -7 & -2 & 5 \\ 0 & -2 & -1 & 5 \\ 6 & -4 & 0 & 1 \end{pmatrix}.$$

Then, as you back-substitute to find the solutions, state the value you assign to each variable in turn, and what equation and previously-assigned variables you are using. For example, $x_2 + \frac{1}{12}x_3 = -\frac{5}{2}$, and $x_3 = -16$, so $x_2 = \frac{11}{2}$. The solution to this equation system is $x_1 = \frac{23}{6}$, $x_2 = \frac{11}{2}$, $x_3 = -16$.

Solution

Work

2 Projection Matrices (10 points)

A projection matrix maps a vector $\mathbf{y}$ onto the column space of a matrix $\mathbf{X}$ by finding the point that lies in the column space of $\mathbf{X}$ which is closest (in ℓ_2 norm) to $\mathbf{y}$. For a matrix $\mathbf{X}$, the projection matrix is given by $\mathbf{P} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$ and the projection of $\mathbf{y}$ onto the column space of $\mathbf{X}$ is given by $\mathbf{P}\mathbf{y}$. We will see next week in class how the projection matrix is closely related to linear regression. For now, we study some of $\mathbf{P}$'s fundamental properties.

1. (5 points) Prove that the projection matrix is necessarily symmetric. A matrix $\mathbf{A}$ is said to be symmetric if $\mathbf{A} = \mathbf{A}^T$.

2. (5 points) Prove that the projection matrix is necessarily idempotent. A matrix $\mathbf{A}$ is said to be idempotent if $\mathbf{A}\mathbf{A} = \mathbf{A}$.

3 Matrix Memories (5 points)

Matrix Memories store a single pattern pair $\mathbf{s} \rightarrow \mathbf{t}$ by encoding the outer product of a target vector $\mathbf{t}$ with the input stimulus $\mathbf{s}$. In this problem, we will examine exactly when it is possible to store multiple pattern pairs. Suppose we wish to store the K pattern pairs below. Note that the kth stimulus $\mathbf{s}^{(k)} = \begin{bmatrix} s_1^{(k)} & s_2^{(k)} & s_3^{(k)} & \dots & s_m^{(k)} \end{bmatrix}^T$ is paired with the kth target $\mathbf{t}^{(k)} = \begin{bmatrix} t_1^{(k)} & t_2^{(k)} & t_3^{(k)} & \dots & t_m^{(k)} \end{bmatrix}^T$—the superscript (k) is simply a label indicating which pair we are referring to.

$$\begin{bmatrix} s_1^{(1)} & s_2^{(1)} & s_3^{(1)} & \dots & s_m^{(1)} \end{bmatrix}^T \rightarrow \begin{bmatrix} t_1^{(1)} & t_2^{(1)} & t_3^{(1)} & \dots & t_m^{(1)} \end{bmatrix}^T$$

$$\begin{bmatrix} s_1^{(2)} & s_2^{(2)} & s_3^{(2)} & \dots & s_m^{(2)} \end{bmatrix}^T \rightarrow \begin{bmatrix} t_1^{(2)} & t_2^{(2)} & t_3^{(2)} & \dots & t_m^{(2)} \end{bmatrix}^T$$

$$\begin{bmatrix} s_1^{(K)} & s_2^{(K)} & s_3^{(K)} & \dots & s_m^{(K)} \end{bmatrix}^T \rightarrow \begin{bmatrix} t_1^{(K)} & t_2^{(K)} & t_3^{(K)} & \dots & t_m^{(Ka)} \end{bmatrix}^T$$

A Matrix Memory for multiple pattern pairs encodes the weight matrix as the sum of the outer products of the target/stimulus pairs: $\mathbf{W} = \sum_{k=1}^K \mathbf{t}^{(k)} \otimes \mathbf{s}^{(k)} = \sum_{k=1}^K \mathbf{t}^{(k)} (\mathbf{s}^{(k)})^T$, where $(\mathbf{s}^{(k)})^T$ is the transpose of $\mathbf{s}^{(k)}$. Under this definition we have that the i, jth entry in $\mathbf{W}$ is given by:

$$W_{ij} = \sum_{k=1}^K t_i^{(k)} (s_j^{(k)})^T$$

The Matrix Memory takes a new stimulus vector $\mathbf{s}$ as input and computes the output response as $\mathbf{r} = \mathbf{W}\mathbf{s}$. If we wish to compute the response vector corresponding to the (l)th original pattern pair, we do the same $\mathbf{r}^{(l)} = \mathbf{W}\mathbf{s}^{(l)}$. Z

1. (5 points) **Prove that** for the lth response vector $\mathbf{r}^{(l)}$ for the lth stimulus $\mathbf{s}^{(l)}$ will equal the target $\mathbf{t}^{(l)}$ if all **pairs** of stimulus vectors $(\mathbf{s}^{(k)}, \mathbf{s}^{(l)}) \forall k \neq l$ are orthonormal to each other.

4 Eigenvectors and eigenvalues (8 points)

1. (2 points) Consider a matrix A, represented as the eigendecomposition $A = Q\Lambda Q^{-1}$ for

$$Q = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

so that

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

Compute the product Av for $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, first by directly performing the matrix multiplication.

2. (2 points) Now, express v in the basis given by the eigenvectors of A. That is, write v as a linear combination of the two eigenvectors.

3. (2 points) Using your answer to the question above, write the product Av as a linear combination of the eigenvectors of A.

4. (1 point) Find all eigenvalues and corresponding eigenvectors for the matrix A if

$$A = \begin{bmatrix} 2 & 7 \\ -1 & 6 \end{bmatrix}$$

5. (1 point) Find all eigenvalues and corresponding eigenvectors for the matrix A if

$$A = \begin{bmatrix} 1 & -1 \\ \frac{4}{9} & \frac{-1}{3} \end{bmatrix}$$