HOMEWORK 4
MATRIX CALCULUS *

10-606 MATHEMATICAL FOUNDATIONS FOR MACHINE LEARNING

START HERE: Instructions

- **Collaboration Policy**: Please read the collaboration policy in the syllabus.
- **Late Submission Policy:** See the late submission policy in the syllabus.
- **Submitting your work:** You will use Gradescope to submit answers to all questions.
  - **Written:** For written problems such as short answer, multiple choice, derivations, proofs, or plots, please use the provided template. Submissions can be handwritten onto the template, but should be labeled and clearly legible. If your writing is not legible, you will not be awarded marks. Alternatively, submissions can be written in LaTeX. Each derivation/proof should be completed in the boxes provided. To receive full credit, you are responsible for ensuring that your submission contains exactly the same number of pages and the same alignment as our PDF template.
  - **Latex Template:** https://www.overleaf.com/read/nbjzsbnrppnt#a66b94

Question	Points
Projection Matrices	10
Matrix rank and inverses	14
Singular Value Decomposition (SVD)	4
Total:	28

*Compiled on Monday 15th September, 2025 at 23:34

Instructions for Specific Problem Types

For "Select One" questions, please fill in the appropriate bubble completely:

Select One: Who taught this course?

- ● Matt Gormley
- ○ Marie Curie
- ○ Noam Chomsky

If you need to change your answer, you may cross out the previous answer and bubble in the new answer:

Select One: Who taught this course?

- ● Henry Chai
- ○ Marie Curie
- ⊗ Noam Chomsky

For "Select all that apply" questions, please fill in all appropriate squares completely:

Select all that apply: Which are scientists?

- ■ Stephen Hawking
- ■ Albert Einstein
- ■ Isaac Newton
- □ I don't know

Again, if you need to change your answer, you may cross out the previous answer(s) and bubble in the new answer(s):

Select all that apply: Which are scientists?

- ■ Stephen Hawking
- ■ Albert Einstein
- ■ Isaac Newton
- □ I don't know

For questions where you must fill in a blank, please make sure your final answer is fully included in the given space. You may cross out answers or parts of answers, but the final answer must still be within the given space.

Fill in the blank: What is the course number?

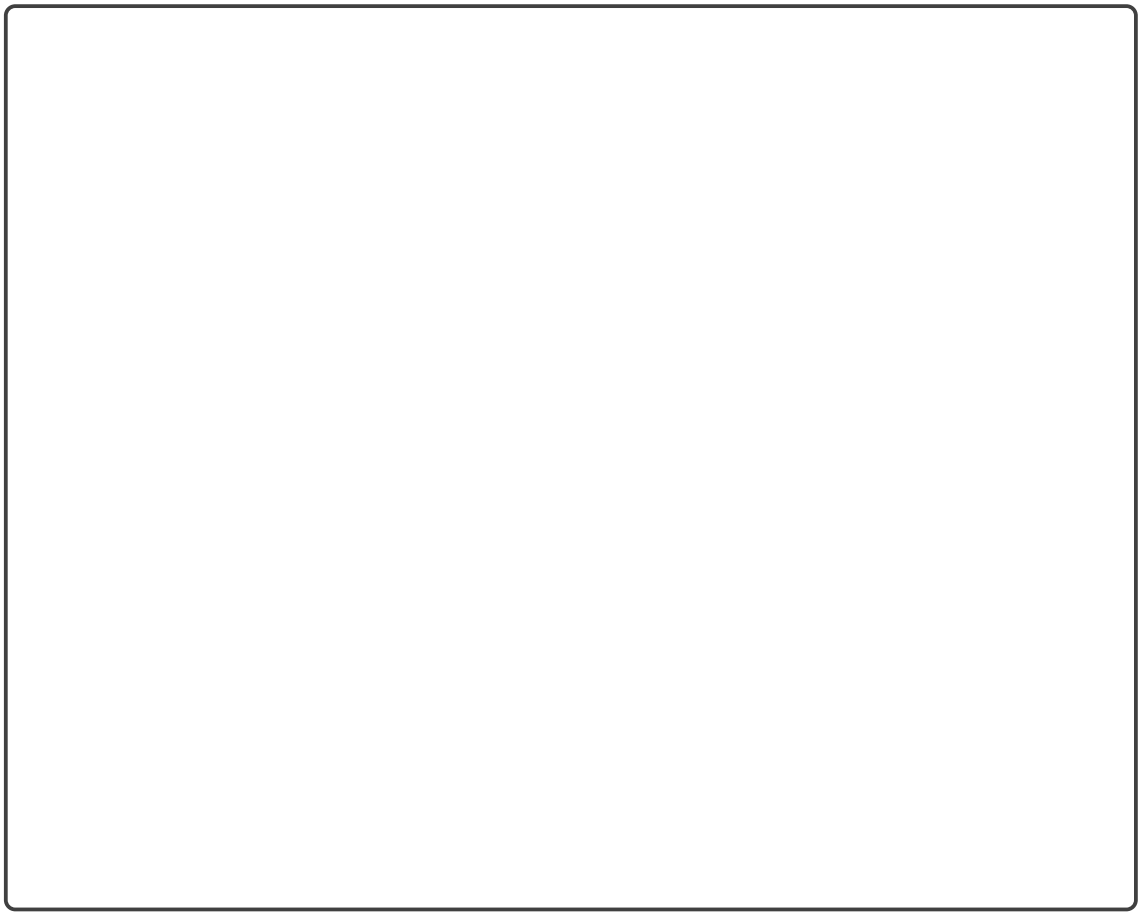
10-606	10-6067

1 Projection Matrices (10 points)

A projection matrix maps a vector $\mathbf{y}$ onto the column space of a matrix $\mathbf{X}$ by finding the point that lies in the column space of $\mathbf{X}$ which is closest (in ℓ_2 norm) to $\mathbf{y}$. For a matrix $\mathbf{X}$, the projection matrix is given by $\mathbf{P} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$ and the projection of $\mathbf{y}$ onto the column space of $\mathbf{X}$ is given by $\mathbf{P}\mathbf{y}$.

1. (5 points) Prove that the projection matrix is necessarily symmetric. A matrix $\mathbf{A}$ is said to be symmetric if $\mathbf{A} = \mathbf{A}^T$.

2. (5 points) Prove that the projection matrix is necessarily idempotent. A matrix $\mathbf{A}$ is said to be idempotent if $\mathbf{AA} = \mathbf{A}$.



2 Matrix rank and inverses (14 points)

1. (2 points) Construct a 4×3 matrix, A_1, that has a rank of 1. Explain why your matrix is rank-1.

2. (2 points) Construct a 4×3 matrix, A_2, that has a rank of 2. Explain why your matrix is rank-2. Note that you need to explain why the matrix is exactly rank 2, i.e., not of either higher *or* lower rank.

3. (2 points) Construct a 4×3 matrix, A_3, that has a rank of 3. Explain why your matrix is rank-3.

4. (1 point) Is it possible to construct a 4×3 matrix of rank 4?

   ◯ Yes ◯ No

5. (1 point) If a square matrix is full rank, it cannot be inverted.

   ◯ True ◯ False

6. (1 point) If a square matrix A is invertible, then all of its rows are linearly independent.

◯ True ◯ False

7. (1 point) Consider a matrix

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

. Is A invertible?

◯ Yes ◯ No

8. (2 points) Consider the system $ABC - D = A$ where A, B, C, D are all square matrices. Solve the system for C. Explicitly note whenever you need to assume that a specific matrix is invertible.

9. (2 points) Consider the system $AB(C - D) = -C$ where A, B, C, D are all square matrices. Solve the system for C. Explicitly note whenever you need to assume that a specific matrix is invertible.

3 Singular Value Decomposition (SVD) (4 points)

Singular Value Decomposition (SVD) is a powerful matrix factorization technique with wide applications, from image compression to machine learning. One of its most well-known uses is in building recommender systems. By decomposing a user-movie ratings matrix, SVD can uncover latent (hidden) factors that describe user preferences.

The SVD of a matrix A is given by $A = U\Sigma V^T$, where:

- U and V are orthogonal matrices.
- Σ is a diagonal matrix whose diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ are the singular values of A.

1. (1 point) What transformation does applying a matrix A to a vector x (i.e., calculating Ax) correspond to, according to its SVD, $A = U\Sigma V^T$?

   ◯ A single scaling operation.

   ◯ A rotation, followed by a scaling, followed by another rotation.

   ◯ A translation followed by a rotation.

   ◯ A projection followed by a scaling.

2. (1 point) What relationship does the matrix U have to AA^T?

   ◯ U is a diagonal matrix containing the eigenvalues of AA^T.

   ◯ U's rows are the eigenvectors of AA^T

   ◯ U's columns are the eigenvectors of AA^T.

   ◯ U is equal to $AA^T(A^TA)$

3. (1 point) *Any* matrix has a singular value decomposition.

   ◯ True

   ◯ False

4. (1 point) Suppose that a matrix $A \in \mathbb{R}^{5\times 4}$ has singular values $\{3, 5, -1, 0, 0\}$. What is the rank of A?

   ◯ 2

   ◯ 3

   ◯ 5

   ◯ 1

4 Collaboration Questions

After you have completed all other components of this assignment, report your answers to these questions regarding the collaboration policy. Details of the policy can be found in the syllabus.

1. Did you receive any help whatsoever from anyone in solving this assignment? If so, include full details.
2. Did you give any help whatsoever to anyone in solving this assignment? If so, include full details.
3. Did you find or come across code that implements any part of this assignment? If so, include full details.

Your Answer