Quiz 2 Practice Problems

10-606

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1 Matrix Inverses

1. True/False: If $AB = I$, then $BA = I$. **Solution:** True. If $AB = I$, then A and B are inverses of each other, so $BA = I$ as well.

2. Prove or disprove: Every square diagonal matrix is invertible. **Solution:** False. A diagonal matrix is invertible iff all diagonal entries are nonzero.

3. Compute the inverse of $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$. **Solution:** det $= 2 \cdot 3 - 5 \cdot 1 = 1$. Inverse $= \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

4. Suppose A is invertible. Show that $(A^T)^{-1} = (A^{-1})^T$. **Solution:** $(A^T)(A^{-1})^T = (A^{-1}A)^T = I^T = I$. Thus $(A^{-1})^T$ is the inverse of A^T.

5. Explain, without resorting to calculation, why a matrix with two identical rows cannot be invertible. **Solution:** Identical rows $\implies$ linearly dependent $\implies$ determinant $= 0$, so not invertible.

6. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$. Is A invertible? **Solution:** No. Last row is zero $\implies$ rank $< 3 \implies$ singular.

7. If A is invertible, does $Ax = b$ has a unique solution for all b? **Solution:** Yes. Solution is $x = A^{-1}b$. Existence and uniqueness follow from invertibility.

8. If $Ax = b$ has no solution, show that $ABx = b$ has no solution for any matrix B. **Solution:** Suppose it did, then $z = Bx$ satisfies $Az = b$. But such a z cannot exist by assumption.

9. A set of vectors $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^n$ are *orthonormal* if $v_i^\intercal v_j = 1$ if $i = j$ and 0 otherwise. A matrix $Q \in \mathbb{R}^{n \times n}$ is orthonormal if its column and row vectors are orthonormal. If Q is orthonormal, show that $Q^\intercal = Q^{-1}$. **Solution:** Let $Q = [v_1, \ldots, v_n]$ where $\{v_1, \ldots, v_n\}$ are orthonormal. Then $(Q^\intercal Q)_{ij} = v_i^\intercal v_j = \delta_{ij}$. That is, $Q^\intercal Q = I$, implying that $Q^\intercal$ is the inverse of Q.

Linear Systems

1. Can the following system be solved? $\{x + y = 2,\ 2x + 2y = 5\}$. **Solution:** Inconsistent: first eq $\implies 2x+2y = 4$, but second requires 5. No solution.

2. How many solutions are there to $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} x = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$. **Solution:** Row 2 is multiple of row 1. Solutions: $x_1 + 2x_2 = 3 \implies x_1 = 3 - 2x_2$. Infinitely many.

3. Row-reduce $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$. **Solution:** Final echelon form: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

4. For which a is $\begin{bmatrix} 1 & a \\ 2 & 4 \end{bmatrix}$ invertible? **Solution:** $\det = 4 - 2a$. Invertible iff $a \neq 2$.

5. Solve the system: $\{x+y+z = 6,\ x-y+z = 2,\ 2x+z = 5\}$. **Solution:** $x = 1, y = 2, z = 3$.

6. True/False: If the equation $Ax = 0$ has only the trivial solution $x = 0$, then the columns of A form a basis for $\mathbb{R}^n$. **Solution:** True. Trivial solution means full column rank $= n$. Columns are independent and span $\mathbb{R}^n$.

7. Solve $Ax = b$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$. **Solution:** You could row-reduce, or notice that $\det A = -2$. Hence $A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$. Multiplying by b gives $x = (1, 2)^\intercal$.

8. Construct 3 equations in 3 unknowns with no solutions. **Solution:** Example: $x + y + z = 1$, $2x + 2y + 2z = 2$, $x + y + z = 3$. First and third inconsistent.

Eigenvalues and Eigenvectors

1. Find the characteristic polynomial of $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ for any $a \in \Re$ (i.e., compute $\det(A - \lambda I) = 0$). **Solution:** $(1-\lambda)^2$. Only eigenvalue=1, algebraic multiplicity 2.

2. What sort of geometric action does the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ correspond to? **Solution:** 90 degree rotation CCW.

3. What sort of geometric action does the matrix $\begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$ correspond to? **Solution:** Flips the vector along the x-axis; scales it by 3.

4. Prove that an orthogonal matrix (see Section 1) always has eigenvalues of magnitude 1. **Solution:** We showed above that $Q^\intercal Q = I$. Let x be eigenvector of Q with eigenvalue λ, so $Qx = \lambda x$. Multipying both sides by, $(Qx)^\intercal$:

$$x^\intercal Q^\intercal Qx = (x^\intercal Q^\intercal)\lambda x = \lambda x^\intercal Qx = \lambda^2 x^\intercal x.$$

   The left hand side is equal to $x^\intercal x$. Hence $x^\intercal x = \lambda^2 x^\intercal x$, implying that $\lambda^2 = 1$.

5. If 0 is an eigenvalue of A, what does this tell you about the rank of A? **Solution:** A is not full rank (its kernel has dimension at least 1).

6. Let A have eigenvectors v_1, v_2 with eigenvalues λ_1, λ_2. Suppose $x = 3v_1 - 2v_2$. What is Ax in terms of v_1, v_2? Explain geometrically what happened to x.

   Solution: Since $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, $Ax = 3\lambda_1 v_1 - 2\lambda_2 v_2$. Geometrically, x is decomposed into eigen-directions. Each component is scaled by its eigenvalue: stretched/compressed (if $|\lambda_i| \neq 1$), flipped (if $\lambda_i < 0$), or left unchanged (if $\lambda_i = 1$).