Recitation Material: Linear Algebra

10-606

September 2025

Matrix Applications and Properties

Problem 1: System of Equations in Matrix Form (3x3)

Consider the following system of equations:

$$2x + y - z = 8$$

$$3x - 2y + 4z = -2$$

$$x + y + z = 4$$

Write this system in matrix form $A\mathbf{x} = \mathbf{b}$ and discuss how the solution can be found using the inverse of A.

Problem 2: Matrix Multiplication (3x3)

Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

Compute AB and BA. What do you observe?

Problem 3: Associative Property of Matrix Multiplication (3x3)

Verify the associative property of matrix multiplication by calculating $(AB)C$ and $A(BC)$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}.$$

Problem 4: Distributive Property of Matrix Multiplication (3x3)

Let

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Verify the distributive property by calculating $A(B + C)$ and $AB + AC$.

Norms and Their Applications

Problem 1: Manhattan Norm ($\|\mathbf{x}\|_1$)

Consider the vector

$$\mathbf{x} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}.$$

Calculate the $\|\mathbf{x}\|_1$ (Manhattan norm) of the vector.

Problem 2: Euclidean Norm ($\|\mathbf{x}\|_2$)

Given the vector

$$\mathbf{y} = \begin{bmatrix} 6 \\ 8 \\ -2 \end{bmatrix},$$

calculate the Euclidean norm $\|\mathbf{y}\|_2$.

Problem 3: Infinity Norm ($\|\mathbf{x}\|_\infty$)

For the vector

$$\mathbf{z} = \begin{bmatrix} -10 \\ 5 \\ 7 \end{bmatrix},$$

calculate the infinity norm $\|\mathbf{z}\|_\infty$.

Problem 4: General p-Norm

Given the vector

$$\mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

and $p = 3$, calculate the $\|\mathbf{w}\|_3$.

Problem 5: Application in Machine Learning - Frobenius Norm

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & -9 \end{bmatrix}.$$

Compute the Frobenius norm $\|A\|_F$.

Vector Space Basics

Problem 1: What is / isn't a vector space

1. True or false: the set of all 2×2 invertible matrices with real entries is a vector space under usual addition and scalar multiplication.
2. Is the set of polynomials of degree exactly 3 (not at most 3) a vector space? Why or why not?
3. Is $\{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 7\}$ a vector space under pointwise addition and scalar multiplication?
4. Is the set of continuous functions on $[0, 1]$ with the usual operations a vector space?
5. Let $A \in \mathbb{R}^{n \times m}$ be a real-valued matrix. Is the set of all vectors $x \in \mathbb{R}^m$ such that $Ax = 0$ a vector space (under usual addition and scalar multiplication)?

Problem 2: Basis & Dimension

1. Give a basis for the set of $n \times n$ real-valued matrices.
2. What's the dimension of the set of $n \times n$ *symmetric* matrices.
3. What's a basis for the set of $n \times n$ symmetric matrices?
4. What is the dimension of the space spanned by $\{1, x, x^2, \sin(x)\}$ as functions on $\mathbb{R}$?

Problem 3: Linear Dependence & Span

1. Are the vectors $(1, 1, 0), (0, 1, 1), (1, 2, 1)$ linearly dependent or independent?
2. Does $\{(1, 2, 3), (2, 4, 6), (0, 1, 0)\}$ span $\mathbb{R}^3$?