Sets, Vectors, and Matrices
Quiz 1 Practice

10-606

Latex Template: `https://www.overleaf.com/read/pztqstyswzzg#39555d`

Disclaimer: This isn't exactly a "specimen paper." That is, some problems here are going to be more difficult, and some are going to be easier. However, this is a good place to start to find out which concepts you need to work on more. This also isn't meant to reflect the length or format of the actual quiz. :)

1   Set Operations

Key Points: Review addition, subtraction, products, unions, intersections, types for sets.

1. What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

2. Let $A = \{x \in \mathbb{N} : 3 \leq x < 16\}$ and $B = \{x \in \mathbb{N} : x \text{ is even}\}$.

   - Find $A \cap B$.
   - Find $A \setminus B$.

3. Recall $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the integers). Let $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ be the positive integers. Let $2\mathbb{Z}$ be the even integers $2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$, $3\mathbb{Z}$ be the multiples of 3, $3\mathbb{Z} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$

   - Is $\mathbb{Z}^+ \subseteq 2\mathbb{Z}$? Explain.
   - Is $2\mathbb{Z} \subseteq \mathbb{Z}^+$? Explain.

- Find $2\mathbb{Z} \cap 3\mathbb{Z}$. Describe the set in words, and using set notation.
- Express $\{x \in \mathbb{Z} : \exists y \in \mathbb{Z}(x = 2y \lor x = 3y)\}$ as a union or intersection of two sets already described in this problem.

2   Set Builder Notation

Key Points: Review how to convert between natural language expression to set notation (and vice versa).

1. Describe the set of all odd numbers between 100 and 200 using set builder notation.

2. Describe each of the following sets both in words and by listing out enough elements to see the pattern.

   - $\{x : x + 3 \in \mathbb{N}\}$
   - $\{x \in \mathbb{N} : x + 3 \in \mathbb{N}\}$
   - $\{x : x \in \mathbb{N} \lor -x \in \mathbb{N}\}$
   - $\{x : x \in \mathbb{N} \land -x \in \mathbb{N}\}$
   - $\{x \in \mathbb{Z} : x^2 \in \mathbb{N}\}$
   - $\{x^2 : x \in \mathbb{N}\}$

3 Set Comprehension

Key Points: Python syntax (chained for loops using one line, set comprehension, conditionals, range function, etc.)

Write a Python set comprehension that does the following:

1. $\{(a, b, c) \in \mathbb{Z}_+^3 \mid a \text{ is even or } b \text{ is divisible by } 5\}$

2. $\{(a, b) \in \mathbb{Z}^2 \mid a^2 + b^2 < 4\}$

3. $\{x \in \mathbb{Z} \mid x \text{ is odd}\}$

4 Vectors

Key Points: Vector addition, subtraction, scalar multiplication, dot product, outer and inner product, transpose, magnitude, L1 norm, types

1. Given $\vec{a} = (8, 5)$ and $\vec{b} = (-3, 6)$ compute each of the following.

   - $6\vec{a}$
   - $7\vec{b} - 2\vec{a}$
   - $\|10\vec{a} + 3\vec{b}\|$

2. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$. Calculate The dot product $\mathbf{u} \cdot \mathbf{v}$.

3. Given the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, find its transpose A^T.

4. Let $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$. Calculate the inner product $\langle \mathbf{a}, \mathbf{b} \rangle$ and the outer product $\mathbf{ab}^T$.

5. Given a vector $\mathbf{v} = \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix}$, calculate the L_1 norm, denoted as $||\mathbf{v}||_1$.

5 Matrices

Key Points: Matrix operations and properties (addition, subtraction, scalar multiplication), matrix multiplication, trace, types

1. Given two matrices $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ -2 & 6 \end{pmatrix}$, calculate the sum $A + B$ and the difference $A - B$.

2. If $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, find the scalar multiple $3A$.

3. Let $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$. Compute the matrix product CD.

4. Find the trace of the matrix $E = \begin{pmatrix} 4 & 7 & 1 \\ 2 & 9 & 5 \\ 3 & 8 & 6 \end{pmatrix}$.

5. Classify the following matrices as either a row vector, a column vector, a square matrix, or a diagonal matrix:

   (a) $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

   (b) $G = \begin{pmatrix} 4 & -1 \end{pmatrix}$

   (c) $H = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

   (d) $I = \begin{pmatrix} 7 & 8 \\ 9 & 10 \end{pmatrix}$