Recitation Material: Linear Algebra

10-606

September 2025

Linear Operators, Rank, Eigenvalues

Definitions/Basic Properties

1. Define a *functional* and give an example.
2. What does it mean for a function $f : V \to V$ to be *linear*?
3. Prove that a linear operator maps the origin to the origin. That is, if f is linear, then $f(0) = 0$.
4. Prove that if A is positive semi-definite then its eigenvalues are nonnegative.

Conceptual Understanding

4. What's the geometric meaning of having multiple linearly independent eigenvectors corresponding to the same eigenvalue?
5. What happens if an eigenvalue of a matrix is equal to zero? Interpret this geometrically.

Applied Problems

7. Let $L = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.
   (a) Compute its eigenvalues and eigenvectors.
   (b) Is it positive semi-definite?
8. Suppose a linear transformation ℓ has rank 3. What can you say about the dimension of its null space if $\ell : \mathbb{R}^5 \to \mathbb{R}^5$?

Matrices & Systems of Equations

Definitions/Basic Properties

1. Prove that the system $Ax = b$ has a unique solution if A is full rank.
2. What are the three possible outcomes for a system of linear equations depending on the rank of its coefficient matrix relative to the number of unknowns?

Conceptual Understanding

3. Why do elementary row operations preserve the solution set of a linear system?
4. Explain why the number of nonzero rows in REF equals the rank of the matrix.

Applied Problems

5. Solve the system using Gaussian elimination:

$$\begin{cases} x + y + z = 3 \\ 2x + 3y + z = 7 \\ x + 2y + 2z = 5 \end{cases}$$

6. Solve the system using Gaussian elimination:

$$\begin{cases} 2x + y - z = 1 \\ 4x - 6y = -2 \\ -2x + 7y + 2z = 9 \end{cases}$$