Recitation Material: Linear Algebra

10-606

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Linear Operators, Rank, Eigenvalues

Definitions/Basic Properties

1. Define a *functional* and give an example. **Solution:** A functional is a linear map $f : V \to \mathbb{R}$. Example: $f(x) = \langle v, x \rangle$ for some fixed v.

2. What does it mean for a function $f : V \to V$ to be *linear*? **Solution:** $f(ax + by) = af(x) + bf(y)$ for all scalars a, b and vectors $x, y \in V$.

3. Prove that a linear operator maps the origin to the origin. That is, if f is linear, then $f(0) = 0$. **Solution:** Since $0 + 0 = 0$, $f(0 + 0) = f(0)$. Since f is linear, the left side is equal to $2f(0)$. Subtract $f(0)$ from both sides.

4. Prove that if A is positive semi-definite then its eigenvalues are nonnegative. **Solution:** Suppose A is PSD with eigenvalue λ and corresponding eigenvector v. Then $Av = \lambda v$. Multiplying each side by $v^\intercal$ we have $0 \le v^\intercal Av = v^\intercal \lambda v = \lambda v^\intercal v = \lambda \|v\|_2^2$. Since norms are nonnegative, λ must be nonnegative as well.

Conceptual Understanding

4. What's the geometric meaning of having multiple linearly independent eigenvectors corresponding to the same eigenvalue? **Solution:** Stretches vectors by the same amount in those directions.

5. What happens if an eigenvalue of a matrix is equal to zero? Interpret this geometrically. **Solution:** The matrix is singular (not invertible). Geometrically, it squashes vectors along the eigenvector direction to 0, reducing dimension.

Applied Problems

7. Let $L = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

    (a) Compute its eigenvalues and eigenvectors.

(b) Is it positive semi-definite?

Solution: Eigenvalues: $\lambda = 2$ (algebraic multiplicity 2). Eigenvectors: any vector of the form $\alpha[1, 0]^T$ for $\alpha \in \mathbb{R}$. Not symmetric, so PSD criterion fails.

8. Suppose a linear transformation ℓ has rank 3. What can you say about the dimension of its null space if $\ell : \mathbb{R}^5 \to \mathbb{R}^5$? **Solution:** By rank-nullity, $\dim(\ker \ell) = 5 - 3 = 2$.

Matrices & Systems of Equations

Definitions/Basic Properties

1. Prove that the system $Ax = b$ has a unique solution if A is full rank. **Solution:** Suppose x_1 and x_2 satisfy the system, i.e., $Ax_1 = b$ and $Ax_2 = b$. Then $A(x_1 - x_2) = 0$, so $x_1 - x_2$ is in the null space. This implies that any two solutions differ only by an element of the null space. But if A is full rank, the null space contains 0 only, hence $x_1 = x_2$.

2. What are the three possible outcomes for a system of linear equations depending on the rank of its coefficient matrix relative to the number of unknowns? **Solution:**

   - Rank $>$ unknowns: this is impossible (Rank is at most the minimum number of rows or columns).
   - Rank $=$ unknowns: unique solution.
   - Rank $<$ unknowns: infinite or no solutions.

Conceptual Understanding

3. Why do elementary row operations preserve the solution set of a linear system? **Solution:** They correspond to algebraic manipulations of equations (swapping, scaling, adding multiples), which don't change the set of solutions.

4. Explain why the number of nonzero rows in REF equals the rank of the matrix. **Solution:** REF operations preserve linear dependence. Each nonzero row is linearly independent of the others; hence the number of nonzero rows it is the dimension of the row space, which is the rank of the matrix.

Applied Problems

5. Solve the system using Gaussian elimination:

$$\begin{cases} x + y + z = 3 \\ 2x + 3y + z = 7 \\ x + 2y + 2z = 5 \end{cases}$$

Solution: From elimination: $x = 1, y = 1, z = 1$.

6. Solve the system using Gaussian elimination:

$$\begin{cases} 2x + y - z = 1 \\ 4x - 6y = -2 \\ -2x + 7y + 2z = 9 \end{cases}$$

Solution: Reducing the augmented matrix gives the unique solution $x = 1,\ y = 1,\ z = 2$.