Recitation Material: Linear Algebra

10-606

September 2025

Matrix Applications and Properties

Problem 1: System of Equations in Matrix Form (3x3)

Consider the following system of equations:

$$2x + y - z = 8$$

$$3x - 2y + 4z = -2$$

$$x + y + z = 4$$

Write this system in matrix form $A\mathbf{x} = \mathbf{b}$ and discuss how the solution can be found using the inverse of A.

Solution: We can write the system in matrix form as:

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 4 \end{bmatrix}$$

This system can be solved by computing the inverse of A, i.e., $\mathbf{x} = A^{-1}\mathbf{b}$.

Problem 2: Matrix Multiplication (3x3)

Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

Compute AB and BA. What do you observe?

Solution: We have

$$AB = \begin{pmatrix} 33 & 27 & 21 \\ 84 & 69 & 54 \\ 138 & 114 & 90 \end{pmatrix},$$

and

$$BA = \begin{pmatrix} 90 & 123 & 129 \\ 54 & 75 & 78 \\ 18 & 27 & 27 \end{pmatrix}.$$

Notice that $AB \neq BA$, demonstrating that matrix multiplication is not commutative.

Problem 3: Associative Property of Matrix Multiplication (3x3)

Verify the associative property of matrix multiplication by calculating $(AB)C$ and $A(BC)$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}.$$

Problem 4: Distributive Property of Matrix Multiplication (3x3)

Let

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 4 \\ 9 & 8 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Verify the distributive property by calculating $A(B + C)$ and $AB + AC$.

Norms and Their Applications

Problem 1: Manhattan Norm ($\|\mathbf{x}\|_1$)

Consider the vector

$$\mathbf{x} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}.$$

Calculate the $\|\mathbf{x}\|_1$ (Manhattan norm) of the vector.

Solution: The Manhattan norm (or ℓ_1-norm) is defined as the sum of the absolute values of the vector components:

$$\|\mathbf{x}\|_1 = |3| + |-4| + |5| = 3 + 4 + 5 = 12.$$

Problem 2: Euclidean Norm ($\|\mathbf{x}\|_2$)

Given the vector

$$\mathbf{y} = \begin{bmatrix} 6 \\ 8 \\ -2 \end{bmatrix},$$

calculate the Euclidean norm $\|\mathbf{y}\|_2$.

Solution: The Euclidean norm (or ℓ_2-norm) is the standard length of the vector and is computed as:

$$\|\mathbf{y}\|_2 = \sqrt{6^2 + 8^2 + (-2)^2} = \sqrt{36 + 64 + 4} = \sqrt{104} \approx 10.2.$$

Problem 3: Infinity Norm ($\|\mathbf{x}\|_\infty$)

For the vector

$$\mathbf{z} = \begin{bmatrix} -10 \\ 5 \\ 7 \end{bmatrix},$$

calculate the infinity norm $\|\mathbf{z}\|_\infty$.

Solution: The infinity norm (or ℓ_∞-norm) is defined as the maximum absolute value of the vector components:

$$\|\mathbf{z}\|_\infty = \max(|-10|, |5|, |7|) = 10.$$

Problem 4: General p-Norm

Given the vector

$$\mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

and $p = 3$, calculate the $\|\mathbf{w}\|_3$.

Solution: The p-norm is defined as:

$$\|\mathbf{w}\|_p = \left(\sum_{i=1}^{n} |w_i|^p \right)^{1/p}.$$

For $p = 3$, we have:

$$\|\mathbf{w}\|_3 = \left(|1|^3 + |-3|^3 + |4|^3\right)^{1/3} = (1 + 27 + 64)^{1/3} = (92)^{1/3} \approx 4.54.$$

Problem 5: Application in Machine Learning - Frobenius Norm

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & -9 \end{bmatrix}.$$

Compute the Frobenius norm $\|A\|_F$.

Solution: The Frobenius norm is a matrix norm and is defined as the square root of the sum of the squares of all matrix elements:

$$\|A\|_F = \sqrt{2^2 + (-1)^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + (-9)^2}$$

$$= \sqrt{4 + 1 + 9 + 16 + 25 + 36 + 49 + 64 + 81} = \sqrt{285} \approx 16.88.$$

Vector Space Basics

Problem 1: What is / isn't a vector space

1. True or false: the set of all 2×2 invertible matrices with real entries is a vector space under usual addition and scalar multiplication. **Solution: False. Not closed under addition (the zero matrix is not invertible).**

2. Is the set of polynomials of degree exactly 3 (not at most 3) a vector space? Why or why not? **Solution: No. Not closed under addition ($x^3 - x^3 = 0$ is not a polynomial of degree 3).**

3. Is $\{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 7\}$ a vector space under pointwise addition and scalar multiplication? **Solution: No. Not closed under scaling ($2f(0) = 14 \neq 7$). Also zero function not included.**

4. Is the set of continuous functions on $[0, 1]$ with the usual operations a vector space? **Solution: Yes. Closed under addition and scalar multiplication.**

5. Let $A \in \mathbb{R}^{n \times m}$ be a real-valued matrix. Is the set of all vectors $x \in \mathbb{R}^m$ such that $Ax = 0$ a vector space (under usual addition and scalar multiplication)? **Solution: Yes. 0 is in the set; closed under scalar multiplication and addition.**

Problem 2: Basis & Dimension

1. Give a basis for the set of $n \times n$ real-valued matrices. **Solution: Let $E_{i,j}$ be the matrix with 1 in the (i, j)-th entry and 0 elsewhere. The set $\{E_{i,j} : 1 \leq i, j \leq n\}$ is a basis.**

2. What's the dimension of the set of $n \times n$ *symmetric* matrices. **Solution: $n(n + 1)/2$. In a symmetric matrix you can freely choose the n elements on the diagonal, and the $n(n - 1)/2$ elements above the diagonal. The remaining elements are determined by these choices. This gives a total of $n + n(n - 1)/2 = n(n + 1)/2$ degrees of freedom.**

3. What's a basis for the set of $n \times n$ symmetric matrices? **Solution: Using the same notation as above, we can take $\{E_{i,i} : 1 \leq i \leq n\} \cup \{E_{i,j} : 1 \leq i < j \leq n\}$ as a basis.**

4. What is the dimension of the space spanned by $\{1, x, x^2, \sin(x)\}$ as functions on $\mathbb{R}$? **Solution: 4. These functions are linearly independent.**

Problem 3: Linear Dependence & Span

1. Are the vectors $(1, 1, 0), (0, 1, 1), (1, 2, 1)$ linearly dependent or independent? **Solution: Dependent; third = first + second.**

2. Does $\{(1, 2, 3), (2, 4, 6), (0, 1, 0)\}$ span $\mathbb{R}^3$? **Solution: No. First two are multiples; combined with $(0, 1, 0)$ we get a two-dimensional subspace.**