Recitation Material: Linear Algebra

10-606

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Matrix Calculus

1. Let $f(x) = x^T Ax$, where $A \in \mathbb{R}^{n\times n}$. Compute the gradient of $f(x)$ with respect to x. What if A is symmetric?
2. Let $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$. Compute the derivative of $f(x) = x^T Ax + b^T x$, where $b \in \mathbb{R}^n$.
3. For linear regression, the loss function is $L(\beta) = \|y - X\beta\|^2$, where $X \in \mathbb{R}^{m\times n}$ is the design matrix, $\beta \in \mathbb{R}^n$ is the parameter vector, and $y \in \mathbb{R}^m$ is the target vector. Compute the gradient of $L(\beta)$ with respect to β.
4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be
$$f(x, y) = \begin{pmatrix} x^2 y \\ \sin(x + y) \end{pmatrix}.$$
Compute the Jacobian matrix $J_f(x, y)$.
5. Let $b \in \mathbb{R}^n$ and $g(x) = (b^\top x)^2$.
   (a) Compute $\nabla g(x)$.
   (b) Compute the Hessian $\nabla^2 g(x)$.
   (c) Is $\nabla^2 g(x)$ positive semidefinite?
6. Let $f : \mathbb{R}^n \to \mathbb{R}$. Let $J_x(y)$ be the Jacobian of a vector y with respect to the vector x (i.e., $J(y)_{ij} = \partial y_i/\partial x_j$). Show that the Hessian of f obeys
$$H(f) = J(\nabla_x f(x)).$$
Recall that the Hessian of a function g satisfies $H(g)_{ij} = \partial^2 g/\partial x_i \partial x_j$.

Convexity and Optimization

1. Let $f(x) = x^2 + 3x + 5$. Is f convex?
2. Compute the Hessian of the function $f(x, y) = x^2 + y^2 + xy$. Can we conclude whether or not f is convex?

3. Show that the Hessian of $f(x) = x^\top Ax$ is $A + A^\top$.
4. Consider the loss function in linear regression $L(\beta) = \|y - X\beta\|^2$. Prove that this loss function is convex.
5. Perform two iterations of gradient descent for $f(x) = x^2 + 4x + 4$, starting at $x_0 = 3$, with a learning rate $\alpha = 0.1$.
6. Consider the function $f(x, y) = x^2 + y^2$. Starting at $(x_0, y_0) = (1, 2)$, perform one iteration of gradient descent with learning rate $\alpha = 0.1$.