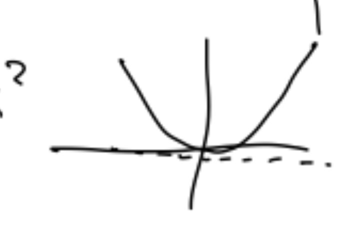
Critical point: $\nabla_x f(x) = 0$

Example: $f(x) = x^2$ 

$f'(x) = 2x$

$f'(0) = 0$

Example: $f(x,y) = x^2 + (y-1)^2$

$\nabla_{x,y} f = [2x,\ 2(y-1)]$

$\nabla_{x,y} f(0,1) = [0,\ 0]$

Critical point: (1) Local min

(2) Local max

(3) Saddle point

$f(x) = x^3$, $\frac{d}{dx} f(x) = 3x^2$

$\Rightarrow$ critical point at $x=0$

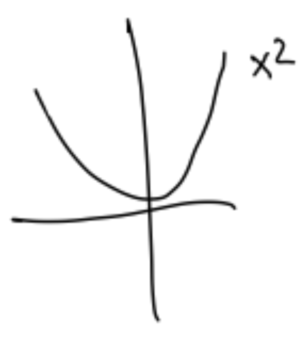
How to tell? Second derivative

Scalar functions:

$f'' > 0 \Rightarrow$ min

$f'' < 0 \Rightarrow$ max

$f'' = 0 \Rightarrow$ could be min/max or saddle point



$f'' = 2$

$f''(0) > 0 \Rightarrow x=0$ is local min

$f'' = -2$

$f''(0) < 0$, $f'(0) = 0$, $x=0$ is local max

$f(x) = x^3$ $f''(x) = 6x$

$f''(0) = 0 \Rightarrow$ saddle point

Matrix equivalent

$\nabla f(x) = \vec{0}$

$H(x) \succ 0$ (Positive definite)

$\Rightarrow$ local min

$H(x) \prec 0$ (Negative definite)

$\Rightarrow$ local max

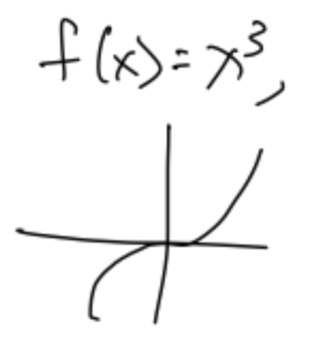
$H(x)$ indefinite

$\Rightarrow$ saddle point

What about $H(x) \succeq 0$?

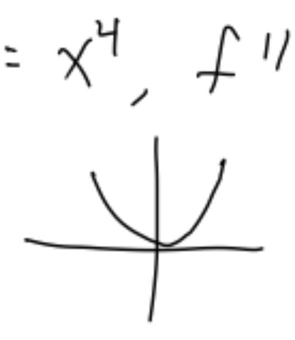
$H(x) \succeq 0$ is nesessary for local min, but not sufficient

Recall $f(x) = x^3$, $f''(0) = 0$

 saddle point

vs

$f(x) = x^4$, $f''(0) = 0$

 local min

Finding critical points

Strategy: (1) set $\nabla f(x) = 0$, solve for x

(2) check $H(x)$

Example: least squares solution to linear system

$$\min_x \|Ax - b\|_2^2$$

Defined even when A not square

Let $f(x) = \|Ax - b\|_2^2$

$= (Ax - b)^T (Ax - b)$

$= x^T A^T A x - 2x^T A^T b + b^T b$

$(x^T A^T - b^T)(Ax - b)$

$= x^T A^T A x - b^T A x - x^T A^T b + b^T b$

$= (x^T A^T b)^T$

$= x^T A^T b$ (scalar)

$\nabla f(x) = 2A^T A x - 2A^T b$

$A^T A$ is symmetric

Set $\nabla f(x) = 0$:

$2A^T A x - 2A^T b = 0$

$x = (A^T A)^{-1} A^T b$

Exists when columns of A are linearly independent

$H(x) = 2A^T A$

$H(x) \succ 0$ because $x^T A^T A x = \|Ax\|^2 > 0$ for any $x \neq 0$

$\Rightarrow x = (A^T A)^{-1} A^T b$ is local min