Mathematical Foundations for Machine Learning

Linear Systems

TA: Jocelyn

Slide credits: Pat Virtue

Today

Homework

- Review HW1
- Answer any HW2 questions

Linear Systems

- Systems of equations
- Fitting linear models to data
- Python implementation

Exercise

Given the following system of equations, *how* would you solve for acceptable values for $\theta_1, \theta_2, \theta_3$?

Alien coins! Your friend E.T. is helping you to learn alien currency. There are three different types of coins that have values $\theta_1, \theta_2, \theta_3$. There are three different piles of coins. E.T. is kind enough to tell us the total value of each of the three piles.

$$\begin{aligned}
3\theta_1 + 0\theta_2 + 5\theta_3 &= 36 \\
4\theta_1 + 3\theta_2 + 2\theta_3 &= 46 \\
2\theta_1 + 2\theta_2 + 1\theta_3 &= 25
\end{aligned}$$

Exercise

Given the following system of equations, how would you solve for acceptable values for $\theta_1, \theta_2, \theta_3$?

$$3\theta_1 + 0\theta_2 + 5\theta_3 = 36$$
$$4\theta_1 + 3\theta_2 + 2\theta_3 = 46$$
$$2\theta_1 + 2\theta_2 + 1\theta_3 = 25$$

Notation alert!

Linear Algebra

Linear algebra allows us to represent and operate upon sets of linear equations.

$$3\theta_1 + 0\theta_2 + 5\theta_3 = 36$$
$$4\theta_1 + 3\theta_2 + 2\theta_3 = 46$$
$$2\theta_1 + 2\theta_2 + 1\theta_3 = 25$$

$$V\boldsymbol{\theta} = \boldsymbol{u} \qquad V = \begin{bmatrix} 3 & 0 & 5 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \boldsymbol{u} = \begin{bmatrix} 36 \\ 46 \\ 25 \end{bmatrix} \quad \boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

Possible way to solve: $\boldsymbol{\theta} = V^{-1}\boldsymbol{u}$

Notation alert!

Exercise

Write a system of equations to help us fit a line to the following data:

$\mathcal{D} = \left\{\left(x^{(i)}, y^{(i)}\right)\right\}_{i=1}^{N}$, where N is the number of points in the dataset

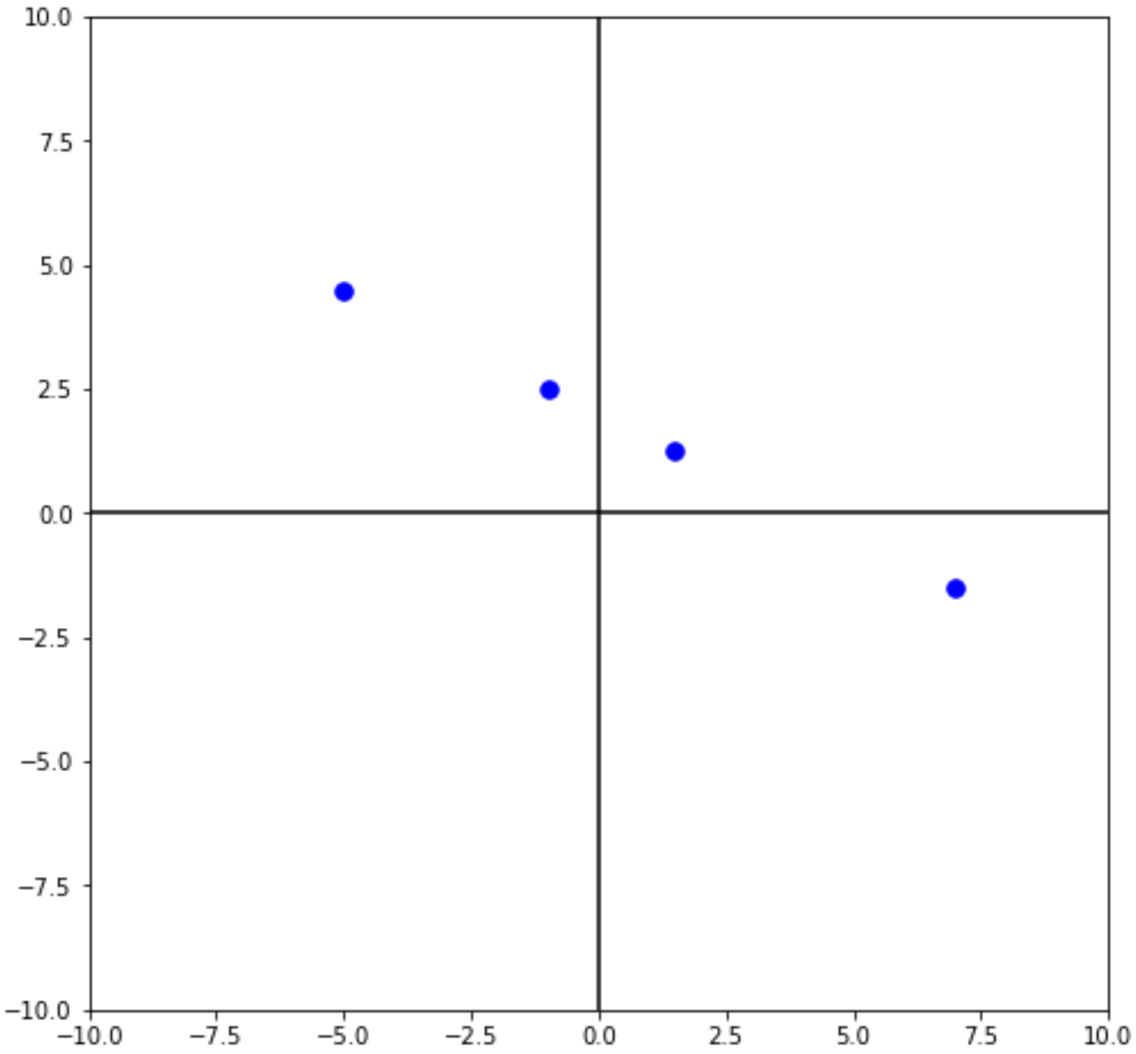
Specifically:

$$
\begin{aligned}
\mathcal{D} = \{&(-1,\ \ 2.5),\\
&(7,\ -1.5),\\
&(-5,\ 4.5),\\
&(1.5,\ 1.25)\}
\end{aligned}
$$

Poll 1

Can we fit a linear model to this data?

$$\mathcal{D} = \left\{\left(x^{(i)}, y^{(i)}\right)\right\}_{i=1}^{4}$$
$$= \{(-1,\ \ 2.5),$$
$$(7,\ -1.5),$$
$$(-5,\ 4.5),$$
$$(1.5,\ 1.25)\}$$



Linear and Affine

Notation alert!

$\forall$

$\mathbb{R}^M$

Linear combination (of a set of terms)

Multiplying each term by a scalar and adding the results

e.g. Given a set of terms $\mathcal{S} = \{x_1, x_2, x_3\}$, where $x_i \in \mathbb{R} \; \forall \; i \in \{1 \dots 3\}$

$w_1 x_1 + w_2 x_2 + w_3 x_3$ is a linear combination of $\mathcal{S}$ if $w_i \in \mathbb{R} \; \forall \; i \in \{1 \dots 3\}$

e.g. Given a set of terms $\mathcal{S} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, where $\boldsymbol{v}_i \in \mathbb{R}^M \; \forall \; i \in \{1 \dots 3\}$

$w_1 \boldsymbol{v}_1 + w_2 \boldsymbol{v}_2 + w_3 \boldsymbol{v}_3$ is a linear combination of $\mathcal{S}$ if $w_i \in \mathbb{R} \; \forall \; i \in \{1 \dots 3\}$

Linear and Affine

Affine combination (of a set of terms)

Affine allows for an additional scalar term to be added to a linear combination. Often called an offset or bias term

$w_1 x_1 + w_2 x_2 + w_3 x_3 + b$, where $b \in \mathbb{R}$

$w_1 \boldsymbol{v}_1 + w_2 \boldsymbol{v}_2 + w_3 \boldsymbol{v}_3 + b$, where $b \in \mathbb{R}$

Linear vs Affine Models

What linear usually means depends on the domain:

Linear algebra:

Linear usually means strictly linear

Geometry, algebra:

Linear usually means affine

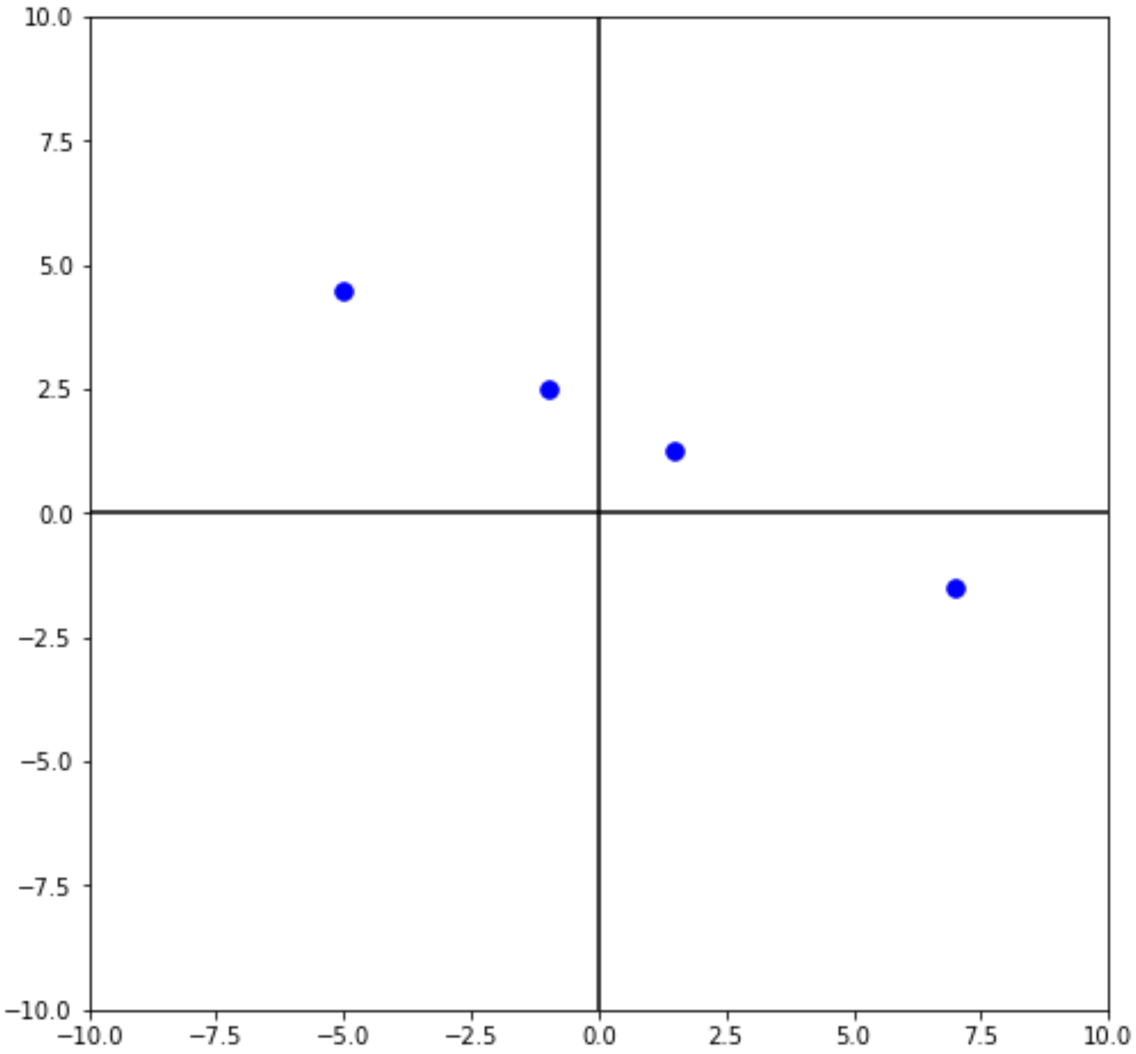
Machine learning:

Linear usually means affine but we often transform affine to strictly linear to make the linear algebra and notation easier

Poll 1

Can we fit a linear model to this data?

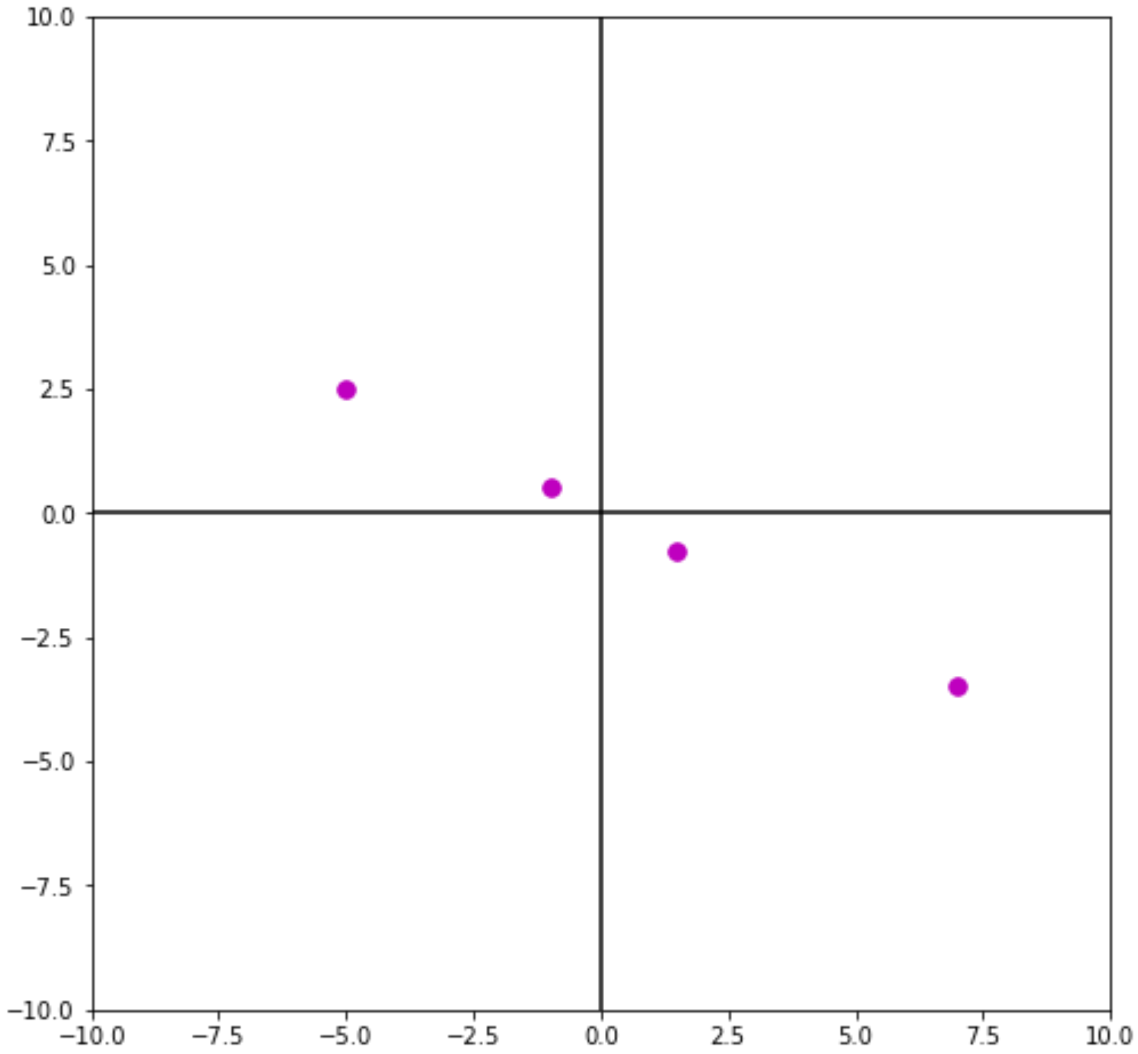
$$\mathcal{D} = \left\{\left(x^{(i)}, y^{(i)}\right)\right\}_{i=1}^{4}$$
$$= \{(-1,\ \ 2.5),$$
$$(7,\ -1.5),$$
$$(-5,\ 4.5),$$
$$(1.5,\ 1.25)\}$$



Linear (but not affine) model will include the origin

We don't need an offset (or bias) term to fit this data

$$
\begin{aligned}
\mathcal{D} &= \left\{\left(x^{(i)}, y^{(i)}\right)\right\}_{i=1}^{4} \\
&= \{(-1,\ \ 0.5), \\
&\quad\ (7,\ -3.5), \\
&\quad\ (-5,\ 2.5), \\
&\quad\ (1.5,\ -0.75)\}
\end{aligned}
$$



Linear (Affine) in Higher Dimensions

What are these linear shapes called for 1-D, 2-D, 3-D, M-D input?

$\boldsymbol{x} \in \mathbb{R}$ $\boldsymbol{x} \in \mathbb{R}^2$ $\boldsymbol{x} \in \mathbb{R}^3$ $\boldsymbol{x} \in \mathbb{R}^M$

$$y = \boldsymbol{w}^T\boldsymbol{x} + b$$

Python implementation

Now we will go over how to implement some of these concepts in python.

We will walk through the tutorial at this link:

https://shorturl.at/bhTX1

Python implementation

Implement the following matrix operations discussed in class using numpy functions

$$\boldsymbol{X} = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

- Matrix transpose: what is $\boldsymbol{X}^T$
- Matrix multiplication / matrix-vector products: what is $\boldsymbol{X}\boldsymbol{u}$
- Compute eigenvalues and eigenvectors of $\boldsymbol{X}$