Recitation 2: Proof Techniques

10-607

1. Prove that there is no smallest positive rational number.

   Solution. We proceed by contradiction. Suppose $r > 0$ is the smallest positive rational. Then $r/2$ is rational and satisfies $0 < r/2 < r$, contradicting minimality.

2. Let $n \in \mathbb{Z}$. Prove that if $3n + 2$ is even, then n is even.

   Solution. We prove the contrapositive. Suppose that n is odd, so $n = 2k+1$ for some $k \in \mathbb{Z}$. Then $3n + 2 = 3(2k + 1) + 2 = 6k + 5$, which is odd, completing the proof.

3. Prove that there is no integer x such that $x^2 \equiv 2 \pmod 3$.

   Solution. Any integer x is either $0, 1$, or 2 mod 3. If $x \equiv 0 \pmod 3$ then $x = 3k$ for some $k \in \mathbb{Z}$ so $x^2 = 0 \pmod 3$. Likewise, if $x \equiv 1 \pmod 3$ then $x = 3k + 1$ for some k, so $x^2 = 9k^2 + 6k + 1 \equiv 1 \pmod 3$. And if $x \equiv 3 \pmod 3$ then $x = 3k + 2$ for some k so $x^2 = 9k^2 + 12k + 4 \equiv 1 \pmod 3$. In no case do we have $x^2 \equiv 2 \pmod 3$.

4. Prove that if ab is even (with $a, b \in \mathbb{Z}$), then a is even or b is even.

   Solution. We prove the contrapositive: if a and b are both odd, then ab is odd. With $a = 2k + 1$, $b = 2\ell + 1$, we have $(2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1$, which is odd.

5. Prove that for all $n \geq 1$, $n! \geq 2^{n-1}$.

   Solution. We proceed via induction. Base case $n = 1$: $1! = 1 \geq 1 = 2^0$. Now assume that the claim holds for some $k \geq 1$. That is, $k! \geq 2^{k-1}$. We will show that $(k + 1)! \geq 2^k$. We have $(k + 1)! = (k + 1)k! \geq (k + 1)2^{k-1} \geq 2 \cdot 2^{k-1} = 2^k$, as desired. This completes the proof.

6. Prove that if $r \in \mathbb{Q}$ and $s \notin \mathbb{Q}$, then $r + s \notin \mathbb{Q}$.

   Solution. Suppose $r \in \mathbb{Q}$ and $s \notin \mathbb{Q}$. Suppose for contradiction that $r + s \in \mathbb{Q}$, meaning that there exist $a, b \in \mathbb{Z}$ such that $r + s = a/b$. By assumption, there exist $c, d \in \mathbb{Z}$ such that $r = c/d$. Therefore,
   $$s = \frac{a}{b} - r = \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd} \in \mathbb{Q},$$
   since $ad - cb, bd \in \mathbb{Z}$. Thus $s \in \mathbb{Q}$, a contradiction.

7. Prove that for all $n \geq 1$, $10^n - 1$ is divisible by 9, i.e., that $9|(10^n - 1)$.

   Solution We proceed by induction. For the base case of $n = 1$ we have $10^n - 1 = 9$, which is clearly divisible by 9. Suppose the claim holds for some $n = kk$; we will show it holds for $k + 1$. Since it holds for k, we have that $10^k - 1 = 9\ell$ for some $\ell \in \mathbb{Z}$. Therefore, $10^{k+1} - k = 10 \cdot 10^k - 1 = 10(9\ell + 1) - 1 = 9(10\ell + 1)$, which is divisible by 9. The claim thus holds for $k + 1$, which completes the proof.

8. For all $r \in \mathbb{R}$, $r \neq 1$, prove that, for all $n \geq 0$,

$$\sum_{j=0}^{n} r^j = \frac{r^{n+1} - 1}{r - 1}. \tag{1}$$

Solution. Fix $r \neq 1$ and let us proceed by induction on n. For $n = 0$, the left hand side of (1) is 1, and the right hand side is $(r-1)/(r-1) = 1$, so the claim holds. Now suppose it holds for some $n = k$ and consider $n = k + 1$. Using the induction hypothesis, we have

$$\sum_{j=0}^{k+1} r^j = \sum_{j=0}^{k} + r^{k+1} = \frac{r^{k+1} - 1}{r - 1} + r^{k+1} = \frac{r^{k+1} - 1 + r^{k+1}(r-1)}{r - 1} = \frac{r^{k+2} - 1}{r - 1},$$

which is as desired. This completes the proof.