Recitation 1: Logic & Basic Proofs

1 Truth tables and equivalences

Decide if the following statements are equivalent. You can use truth tables or logical basic identities.

a) $p \to q$ vs. $\neg p \lor q$.

b) $p \land (q \lor r)$ vs. $(p \land q) \lor (p \land r)$.

c) $p \leftrightarrow q$ vs. $(p \land q) \lor (\neg p \land \neg q)$.

d) $\neg(p \land q)$ vs. $\neg p \lor \neg q$.

e) $(p \to r) \land (q \to r)$ vs. $(p \lor q) \to r$.

Solution. All pairs are equivalent. Sketches: (a) Truth table; (b) Truth table or rule of distribution; (c)

$$\begin{aligned}
p \leftrightarrow q &\equiv (p \to q) \land (q \to p) \\
&\equiv (\neg p \lor q) \land (\neg q \lor p) \\
&\equiv (\neg p \land (\neg q \lor p)) \lor (q \land (\neg q \lor p)) \\
&\equiv ((\neg p \land \neg q) \lor (\neg p \land p)) \lor ((q \land \neg q) \lor (q \land p)) \\
&\equiv ((\neg p \land \neg q) \lor F) \lor (F \lor (q \land p)) \\
&\equiv (\neg p \land \neg q) \lor (q \land p).
\end{aligned}$$

(d) truth table or De Morgan; (e) $(\neg p \lor r) \land (\neg q \lor r) \equiv (\neg p \land \neg q) \lor r \equiv \neg(p \lor q) \lor r \equiv (p \lor q) \to r$.

Decide whether each is a tautology / contradiction / contingent:

f) $(p \to q) \lor (q \to p)$

g) $(p \land (p \to q)) \to q$

h) $(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)$

Solution. (a) tautology; (b) tautology (captures modus ponens); (c) tautology (negating both sides preserves biconditional).

Prove using equivalence laws (no truth tables):

i) $p \to (q \to r) \equiv (p \land q) \to r$.

j) $(p \to q) \land (p \to r) \equiv p \to (q \land r)$.

Solution. (a) $p \to (q \to r) \equiv \neg p \lor (\neg q \lor r) \equiv (\neg p \lor \neg q) \lor r \equiv \neg(p \land q) \lor r \equiv (p \land q) \to r$.
(b) LHS $\equiv (\neg p \lor q) \land (\neg p \lor r) \equiv \neg p \lor (q \land r) \equiv p \to (q \land r)$.

2 From English to Logic

Convert the following sentences to propositional logic.

a) "Alice will meet Bob unless Bob is seeing Claire."

b) "You cannot access the dataset if you lack credentials unless the PI has approved you."

c) "The server restarts iff the watchdog fails or the temperature exceeds the threshold, but not both."

d) "If the model overfits, then either the dataset is too small or the regularizer is disabled."

Solution. (a) Let A = "Alice will meet Bob" and B = Bob is seeing Claire. Then a solution is: $\neg B \to A$.
(b) Let C = "have credentials", A = "PI approved", D = "can access dataset". Then a solution is: $\neg A \to (\neg C \to \neg D)$.
(c) Let R = "server restarts", W = "watchdog fails", T = "temperature exceeds threshold". Then: $R \leftrightarrow \big((W \vee T) \wedge \neg(W \wedge T)\big)$.
(d) Let O = "model overfits", S = "dataset too small", Reg = "regularizer enabled". Then: $O \to \big(S \vee \neg\text{Reg}\big)$.

3 Inference rules & short derivations

a) From $A \vee B$, $\neg A \to C$, $\neg B \to C$, is C entailed?

b) Show: from $p \to (q \vee r)$ and $\neg q$ infer $p \to r$.

Solution. (a) Not entailed. Countermodel: take $A = T$, $B = T$, $C = F$. Then $A \vee B$ true; both $\neg A$ and $\neg B$ are false so $\neg A \to C$ and $\neg B \to C$ are vacuously true; yet C is false.
(b) Assume p. From $p \to (q \vee r)$ get $q \vee r$. Since $\neg q$ we obtain r.

4 Basic Proofs

Here $a \bmod b = c$ is the same as $a \equiv c \bmod b$.

1. Show: If $a \mid b$ and $a \mid c$ then $a \mid (mb + nc)$ for any $m, n \in \mathbb{Z}$.

   Solution. $b = au$, $c = av \Rightarrow mb + nc = a(mu + nv)$.

2. Show that $a^2 \bmod 4 = 0$ or $a^2 \bmod 4 = 1$ for all $a \in \mathbb{Z}$.

   Solution. If a even: $a = 2k \Rightarrow a^2 = 4k^2 \equiv 0 \pmod 4$. If a odd: $a = 2k + 1 \Rightarrow a^2 = 4k^2 + 4k + 1 \equiv 1 \pmod 4$.

3. Show: For all $x, y \in \mathbb{Z}$, if $x \equiv y \pmod m$ then $x^2 \equiv y^2 \bmod m$.

   Solution. Suppose $x - y = km$ for some $k \in \mathbb{Z}$. Then $x + y = km + 2y$, so

   $$x^2 - y^2 = (x - y)(x + y) = (km)(km + 2y) = k^2m^2 + 2kmy = m(mk^2 + 2ky),$$

   which is a multiple of m.

4. Prove the triangle inequality. That is, show that for all $x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$.

   Solution. Since both $|x| + |y|$ and $|x + y|$ are nonnegative by definition, it suffices to prove that $(|x+y|)^2 \leq (|x| + |y|)^2$. To see this, write

$$(|x| + |y|)^2 = x^2 + 2|x||y| + y^2 \geq x^2 + 2xy + y^2 = |x+y|^2.$$