Proof Techniques
Quiz 1 Practice

10-607

Disclaimer: This isn't exactly a "specimen paper." That is, some problems here are going to be more difficult, and some are going to be easier. However, this is a good place to start to find out which concepts you need to work on more. This also isn't meant to reflect the length or format of the actual quiz.

1. A subset $A \subseteq \mathbb{R}$ is defined to be closed under multiplication if and only if $xy \in S$ whenever $x \in S$ and $y \in S$.

   (a) Let $S = \{x \in \mathbb{R} \mid \exists k \in \mathbb{Z}, x = 5^k\}$. Prove that S is closed under multiplication.

   (b) Suppose that $T \subseteq \mathbb{R}$ is closed under multiplication **and** has the property that $\frac{1}{x} \in T$ whenever $x \in T$ and $x \neq 0$. Prove that $\forall x, y, \in \mathbb{R}$, if $x \in T, x \neq 0$ and $y \notin T$ then $xy \notin T$.

   (c) Prove that the set of irrational numbers is not closed under multiplication. Recall irrationals $= \mathbb{R} \setminus \mathbb{Q}$.

Ans: a) Let $x \in S$ such that $x = 5^i$ for some $i \in \mathbb{Z}$. Let $y \in S$ such that $y = 5^j$ for some $j \in \mathbb{Z}$. Then, $xy = 5^i 5^j = 5^{i+j}$. Since $i, j \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under addition, $xy = 5^k$ for some integer $k = i + j$. By definition of S, $xy \in S$.

b) Let $x \in T, x \neq 0, y \notin T$. Assume for the sake of contradiction that $xy \in T$. Since $x \neq 0$ and T is closed under multiplication, and $\frac{1}{x} \in T$, let us multiply xy by $\frac{1}{x}$: $\frac{1}{x} \cdot xy = y$. By closure of T, we then have that $y \in T$, which violates our original assumption that $y \notin T$. Thus, the statement holds.

c) Proof by counterexample: $\sqrt{2}$ is a well known irrational number. $\sqrt{2} \cdot \sqrt{2} = 2$, which is a rational number, obviously. Thus, they are not closed under multiplication.

2. Prove that for all integers a and b, if $ab + 3b - a - 1$ is even then a is odd or b is odd.

Ans: We give a proof by contraposition. Suppose that a is even and b is even. Then, we have that $a = 2j, b = 2k$ for some integers j and k. Substituting these expressions in, we have that $ab+3b-a-1$ is equal to $2j \cdot 2k + 3 \cdot (2k) - 2j - 1 =$

$2(2jk + 3k - j) - 1$. By definition of odd numbers, this number is odd, i.e., the contrapositive of the original statement holds.

3. Prove that if n is an integer and m is an even integer, then $3n^2 + n + m$ is even.

Ans: Since m is even, we can write $m = 2u$ for some $u \in \mathbb{Z}$. We consider two cases depending on the parity of n.

If n is even then $n = 2k$ for some $k \in \mathbb{Z}$. So

$$3n^2 + n + m = 12k^2 + 2k + 2u = 2(6k^2 + k + u),$$

which is even (since it's some integer multiplied by two).

If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. So

$$\begin{aligned} 3n^2 + n + m &= 3(4k^2 + 4k + 1) + (2k + 1) + 2u \\ &= 12k^2 + 14k + 4 + 2u \\ &= 2(6k^2 + 7k + 2 + u), \end{aligned}$$

which is again even for the same reason.

4. Let $p(x), q(x), r(x, y), s(x, y)$ be predicates defined on a universe of discourse U. Write out the logical negation of each of the following statements, fulling propagating negations through any quantifiers and logical expressions so that negations appear only in front of the individual predicates we defined above:

(a) $\exists x \in U, \forall y \in U, (r(x, y) \vee \neg s(x, y))$

(b) $\forall x \in U, (p(x) \leftrightarrow q(x))$

Ans: (a) $\forall x \in U, \exists y \in U, (\neg r(x, y) \wedge s(x, y))$

(b) $\exists x \in U, (p(x) \wedge \neg q(x)) \vee (q(x) \wedge \neg p(x))$

5. Define:

- R: it is raining
- S: the sprinklers are on
- W: the grass is wet

- H: I do my homework
- P: I pass the class

Translate each of the following sentences into propositional logic:

- If it rains, the grass is wet.
- The grass is wet only if it rains or the sprinklers are on.
- Unless it's raining, the sprinklers are on.
- I pass the class if and only if I do my homework.
- I will pass only if I do my homework.
- Either it is raining or the sprinklers are on, but not both.
- The grass is wet if the sprinklers are on.
- The grass is not wet.
- It is not the case that both it is raining and the sprinklers are on.
- If the grass is wet then either the sprinklers are on or it is raining.

Ans:

- $R \to W$.
- $W \to (R \vee S)$.
- $\neg R \to S \quad$ (equiv. $R \vee S$).
- $P \leftrightarrow H$.
- $P \to H$.
- $(R \vee S) \wedge \neg(R \wedge S)$.
- $S \to W$.
- $\neg W$.
- $\neg(R \wedge S)$.
- $W \to (S \vee R)$.

6. Consider the following predicates:

- P(x): x passed the midterm

- H(x): x handed in every homework
- S(x): x studies daily
- F(x,y): x is friends with y
- C(x): x cheated

Translate the following English sentences into predicate logic.

- Every student who studies daily passes the midterm.
- Someone passed the midterm without handing in every homework.
- Not everyone who passed studied daily.
- No cheaters passed.
- Exactly one student didn't pass.
- Everyone has a friend.
- Someone is friends with everyone.
- At most one student cheated.
- If anyone is friends with someone who studies daily, then they also study daily.
- There is a student who is friends with no one.

Ans:

- $\forall x \left(S(x) \to P(x)\right)$.
- $\exists x \left(P(x) \land \neg H(x)\right)$.
- $\neg \forall x \left(P(x) \to S(x)\right)$ (equiv. $\exists x \, (P(x) \land \neg S(x))$).
- $\forall x \left(C(x) \to \neg P(x)\right)$.
- $\exists x \left(\neg P(x) \land \forall y \, (\neg P(y) \to y = x)\right)$.
- $\forall x \, \exists y \, F(x, y)$.
- $\exists x \, \forall y \, F(x, y)$.
- $\forall x \, \forall y \left((C(x) \land C(y)) \to x = y\right)$.
- $\forall x \left((\exists y \, (F(x, y) \land S(y))) \to S(x)\right)$.
- $\exists x \, \forall y \, \neg F(x, y)$.