# Recitation Material: Induction Problems

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## Induction Problems

### Problem 1: Sums of Geometric Progressions

Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term $a$ and common ratio $r$:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}, \quad \text{when } r \neq 1,$$

where $n$ is a nonnegative integer.

### Problem 2: Proving Inequalities

Use mathematical induction to prove the inequality

$$n < 2^n$$

for all positive integers $n$.

### Problem 3: Divisibility of a Sequence

Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer $n$.

### Problem 4: Number of Subsets of a Finite Set

Use mathematical induction to show that if $S$ is a finite set with $n$ elements, then $S$ has $2^n$ subsets.

### Problem 5: Odd Pie Fights

An odd number of people stand in a yard at distinct mutual distances. Each person throws a pie at their nearest neighbor. Use induction to show there is at least one survivor, that is, one person not hit by a pie.

## Problem 6: Find the Error in the Proof

Find the error in this "proof" of the clearly false claim that every set of lines in the plane, no two of which are parallel, meet in a common point.

**"Proof:"** Let $P(n)$ be the statement that every set of $n$ lines in the plane, no two of which are parallel, meet in a common point. We will attempt to prove that $P(n)$ is true for all positive integers $n \geq 2$.

**Basis Step:** The statement $P(2)$ is true because any two lines in the plane that are not parallel meet in a common point (by the definition of parallel lines).

**Inductive Step:** The inductive hypothesis is the statement that $P(k)$ is true for the positive integer $k$, that is, it is the assumption that every set of $k$ lines in the plane, no two of which are parallel, meet in a common point. To complete the inductive step, we must show that if $P(k)$ is true, then $P(k+1)$ must also be true. That is, we must show that if every set of $k$ lines in the plane, no two of which are parallel, meet in a common point, then every set of $k+1$ lines in the plane, no two of which are parallel, meet in a common point. So, consider a set of $k+1$ distinct lines in the plane. By the inductive hypothesis, the first $k$ of these lines meet in a common point $p_1$. Moreover, by the inductive hypothesis, the last $k$ of these lines meet in a common point $p_2$.

We will show that $p_1$ and $p_2$ must be the same point. If $p_1$ and $p_2$ were different points, all lines containing both of them must be the same line because two points determine a line. This contradicts our assumption that all these lines are distinct. Thus, $p_1$ and $p_2$ are the same point. We conclude that the point $p_1 = p_2$ lies on all $k+1$ lines. We have shown that $P(k+1)$ is true assuming that $P(k)$ is true. That is, we have shown that if we assume that every $k$, $k \geq 2$, distinct lines meet in a common point, then every $k+1$ distinct lines meet in a common point. This completes the inductive step.

We have completed the basis step and the inductive step, and supposedly we have a correct proof by mathematical induction.

## Problem 7: Game of Matches

Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

## Problem 8: Postage Problem

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.