Octave software note

Depending on your Octave installation, you may need to install the signal processing package. That package defines the `dct` fast discrete cosine transform function required by this project. This is not required by MATLAB, which automatically loads the signal processing functions by default.

Here is how you can set up Octave:

1. Start up Octave
2. Run the commands to install required packages (only required once):

   ```
   pkg install -forge control
   pkg install -forge signal
   ```

3. Then before running this project (every time), run:

   ```
   pkg load signal
   ```

Project 2: Digital Audio Compression

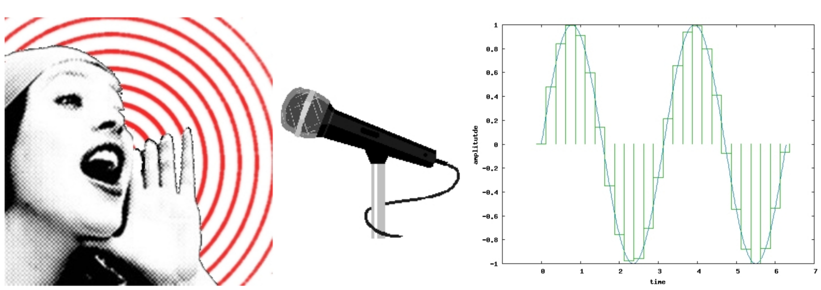
The compression of digital audio data is an important topic. Compressing (reducing) the data storage requirements of digital audio allows us to fit more songs into our phones and download them faster. We will apply ideas from interpolation, least-squares and other topics to reduce the storage requirements of digital audio files. All of our approaches will replace the original audio signal with approximations that are made from a linear combination of cosine functions.

You will learn the basic ideas behind data compression methods used in mp3 and other algorithms. You'll also encounter several interesting MATLAB/Octave commands in the course of this project, including commands for manipulating sound.

Computers and sound

Sound is a complicated phenomenon. It's normally caused by a moving object in air (or other medium), for example a loudspeaker cone moving back and forth. The motion in turn causes air pressure variations that travel through the air like waves in a pond. Our eardrums convert the pressure variations into the phenomenon that our brain processes as hearing.

Computers "hear" sounds using a microphone instead of an eardrum. The microphone converts pressure variations into an electric potential with amplitude corresponding to the intensity of the pressure. The computer then processes the electrical signal using a technique called sampling. The computer samples the signal by measuring its amplitude at regular intervals, often 44100 times per second. Each measurement is stored as a number with fixed precision, often 16 bits. The following diagram illustrates the sampling process showing a simple wave sampled at regular intervals[1]:



Sample values
0.000000000000000
0.479425538604203
0.841470984807897
0.997494986604054
0.909297426825682
0.598472144103956
0.141120008059867
-0.35078322768962
-0.756802495307928
-0.977530117665097
-0.958924274663138
...

[1] Image adapted from Franz Ferdinand, ©2005

Computers emit sound by more or less reversing the above process. Samples are fed to a device that generates an electric potential proportional to the sample values. A speaker or other similar device then converts the electric signal into air pressure waves.

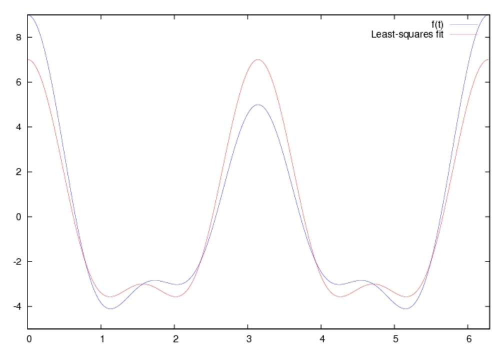
The rate at which the measurements are made is called the *sampling rate*. A common sampling rate is 44,100 times per second (used by compact disc, or CD, audio). The numbers used to store the sampled audio signal are usually not double-precision floating-point numbers (64 bits per number) but instead lower-precision numbers. For example, compact discs use 16 bit numbers to store their samples.

The *bit rate* of a set of digital audio data is the storage in bits required for each second of sound. If the data has fixed sampling rate and precision (like CD audio), the bit rate is simply their product. For example, the bit rate of one channel of CD audio is 44100 samples/second $\times$ 16 bits/sample = 705600 bits/second. The bit rate is a general measure of storage, and is not always simply the product of sampling rate and precision. For example, we will presently encounter a way of encoding data with variable precision.

Large storage requirements limit the amount of audio that can be stored on compact discs, flash memory and other media. Large file sizes also work out to long download times for retrieving songs from the internet. For these reasons (and others), there is a lot of interest in shrinking the storage requirements of sampled sound.

Least-squares data compression

Least-squares data fitting can be thought of as a method for replacing a (large) set of data with a model and a (smaller) set of model coefficients that approximate the data in an optimal way–namely, by minimizing the Euclidean norm of the residual difference between the data and the model.



Consider the following simple example. Let the function $f(t) = \cos(t) + 5\cos(2t) + \cos(3t) + 2\cos(4t)$. A plot of $f(t)$ for $0 \le t \le 2\pi$ appears as the blue curve in the figure. Let's say we are given a data set of 1000 discrete function values of $f(t)$ regularly spaced over the interval $0 \le t \le 2\pi$. We can fully interpolate the data by setting up the model matrix A (using MATLAB/Octave notation):

```
t = linspace (0,2*pi,1000)';
b = cos(t) + 5*cos(2*t) + cos(3*t) + 2*cos(4*t);
A = [ones(size(t)), cos(t), cos(2*t), cos(3*t), cos(4*t)];
```

and then solving the linear system $Ax = b$ with the command `x=A\b`. Try it! Note that the solution vector components match the function coefficients.

Some of the coefficients are not as large as others in this simple example. We can approximate the function f with a least-squares approximation that omits parts of the model corresponding to smaller coefficients. For example, set up the least-squares model

```
A = [cos(2*t), cos(4*t)];
```

and solve the corresponding least-squares system `x=A\b`. This model uses only two coefficients to describe the data set of 1000 data points. The resulting fit is reasonable, and is displayed by the red curve in the figure. The plot was made with the command `plot (t,b,'-b',t,A*x,'-r')`.

The cosine function oscillates with a regular frequency. The multiples of t in the above example correspond to different frequencies (the larger the multiple of t is, the higher the frequency of oscillation). The least-squares fit computed the best approximation to the data using only two frequencies.

Exercise

Experiment with different least-squares models for the above example by omitting different frequencies (that is–omitting different columns if `A`). Plot your experiments and briefly describe the results.

Manipulating sound in MATLAB and Octave

MATLAB and Octave provide lots of commands that make it relatively easy to read in, manipulate, and listen to digital audio signals. Accompanying this project, you will find a short sound file: `https://github.com/bwlewis/enr230-audio-compression/raw/main/shostakovich.wav`. The file is sampled at 22050 samples per second and 16 bits per sample (exactly 1/2 the bit rate of CD audio), and is about 40 seconds long. If you don't care for the tune, you are free to experiment with any audio samples that you wish.

The MATLAB/Octave command to load an audio file is:

```
%% download an audio file to 'tune.wav'...
urlwrite('https://github.com/bwlewis/enr230-audio-compression/raw/main/shostakovich.wav',...
         'shostakovich.wav');

%% read in the downloaded file...
[b,R] = audioread ('shostakovich.wav');
N = length(b);
```

The returned vector b contains the sound samples (it's very long!), R is the sampling rate, and N is the number of samples. Note that, even though the precision of the data is 16 bits, MATLAB and Octave represent the samples as double-precision internally. You can listen to the sample you just loaded with the command:

```
sound (b,R);
```

Some versions of MATLAB and Octave may have slightly different syntax–use the help command for more detailed information.

Sampled audio data is generally much more complicated looking than the simple example in the last section, confirmed by viewing the data with the command `plot(b)` (try it!). However, it too can be interpolated and/or least-squares fit with a cosine model:

$$y = c_0 + c_1 \cos \omega_1 t + c_2 \cos \omega_2 t + \cdots + c_{n-1} \omega_{n-1} t,$$

for some positive integer $n-1$ and frequencies ω_j. A famous and important result from information theory called the Shannon-Nyquist theorem requires that the highest frequency in our model, ω_{n-1}, be less than half the sampling rate. That is, our cosine model assumes that the audio data is filtered to cut-off all frequencies above half the sampling rate.

The cosine model requires additional technical assumptions on the data. Recall that the cosine function is an even function, and the sum of even functions is an even function. Therefore, the model also assumes that the data is even. The usual approach taken to satisfy this requirement of the model is to simply assume that the data is extended outside of the interval of interest to make it even.

The above-mentioned conditions (cut-off frequency, extension beyond the interval boundaries) are in general important to consider, but we won't get in to the details in this project. Instead, we focus on the basic ideas behind compression methods like mp3.

Computing the model interpolation coefficients with the DCT

Let the vector b contain one second of sampled audio, and assume that the sampling rate is N samples per second (b is of length N). It's tempting to proceed just as in the simple example above by setting up an interpolation model (don't try this!):

```
t = linspace (0,2*pi,N)';
A = [ones(size(t)), cos(t), cos(2*t), cos(3*t), ..., cos((N/2-1)*t)];
x = A\b;
```

Aside from a few technical details, this method could be used to interpolate an audio signal. However, consider the size of the quantities involved. At the CD-quality sampling rate, $N = 44100$, and the matrix A is gigantic (44100×22050)! This problem is unreasonably large.

Fortunately, there exists a remarkable algorithm called the Fast Discrete Cosine Transform (DCT) that can compute the solution with extreme efficiency. The DCT is a variation on the famous fast Fourier transform (FFT)–one of the most important algorithms ever devised. The DCT produces scaled versions of the model coefficients for us with the command:

```
c = dct(b);
```

The returned coefficient vector c is of the same length as b.

To investigate the plausibility of the DCT, we can try it out on our simple example:

```
% Simple example revisited
t = linspace (0,2*pi,1000)';
b = cos(t) + 5*cos(2*t) + cos(3*t) + 2*cos(4*t);
x = dct(b);
N = length(b);
w = sqrt(2/N);
f = linspace(0, N/2, N)';
plot (f(1:8),w*x(1:8),'x');
```

The variable w is a scaling factor produced by the DCT algorithm and the vector f is the frequency scale for the model coefficients computed by the DCT and stored in x. The frequency range from 0 to $N/2-1$ corresponds to half the sampling rate (assumed here to be N). We can think of the `dct(b)` command as essentially computing `A\b` for the full interpolation model using the frequencies in the vector f. Your plot should show that we closely compute the model coefficients (i.e., a value of 1 at frequency 1, 5 at frequency 2, etc.)

We can reconstruct the original signal from the model coefficients with the command:

```
y = idct(x);     % The reconstructed data is in y.
plot (t, b, '-r', t, y,'-b');
```

The plots should overlay each other. The `idct` command is the inverse of the `dct` command. We can think of `idct(x)` as computing the product Ax for an appropriate model matrix A and coefficient vector x.

Digital filtering

The DCT algorithm can be used to not only interpolate data, but to compute a least-squares fit to the data by omitting frequencies. The process of computing a least-squares fit to digitized signals by omitting frequencies is called digital filtering. Digital filtering can reduce the storage requirements of digital audio by simply lopping off parts of the data that correspond to specific frequencies. Of course, cutting out frequencies affects the sound quality of data. However, the human ear is not equally sensitive to all frequencies. In particular, we generally don't perceive very high and very low frequencies nearly as well as mid-range frequencies. In some cases, we can filter out these frequencies without significantly affecting quality. An easy way to filter specific frequencies in MATLAB and Octave is to generate a mask. Consider this example:

```
[b,R] = audioread('shostakovich.wav');
N = length(b);
c = dct(b);                % Compute the interpolation model coefficients
w = sqrt(2/N);
f = linspace(0, R/2, N)';
plot (f,w*c);     % Shows a plot of the frequencies coefficients for the sample
% Generate a mask of zeros and ones. m is 0 for every frequency above 2000, 1 otherwise.
% This mask will cut-off all frequencies above 2000 cycles/second.
m = (f < 2000);
plot (f, w*m .* c);   % Display the filtered frequency coefficients.
y = idct(m .* c);     % Generate a filtered sound sample data set
sound(y, R);          % Listen to the result!
```

Exercise

Experiment with several frequency cut-off values in the above example. Listen to your results.

Exercise

Exhibit how to construct a single mask that will cut off frequencies below 200 *and* above 5000 cycles/second.

Exercise

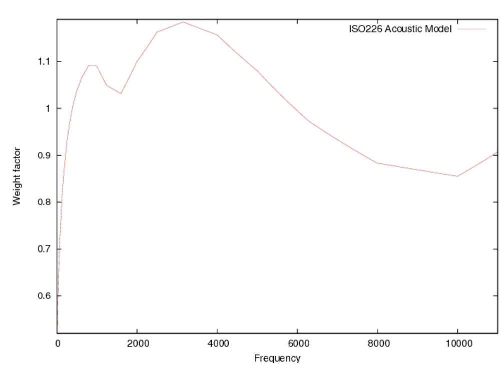
How much does the above code reduce the storage requirement of the sample (in bit rate)?

The ideas behind mp3

Digital fifiltering is an effective technique for compressing audio data in many situations, especially telephony. Cutting out entire frequency ranges is rather a brute-force method, however. There are more effective ways to reduce the storage required of digital audio data, while also maintaining a high-quality sound.

One idea is this: rather than cutting out "less-important" frequencies altogether, we could store the corresponding model coefficients with lower precision–that is, with fewer bits. This technique is called quantization.

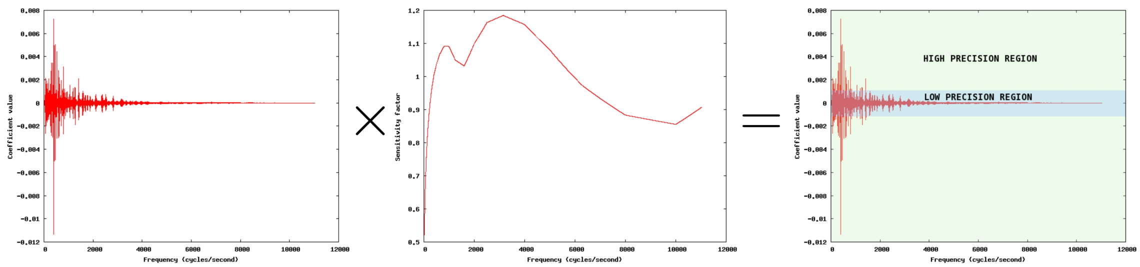
We've already encountered that we don't hear all frequencies equally well. This phenomenon has been studied extensively, and as a result we have a pretty good model of "average" human frequency perception. The fifigure shows an example perceptual frequency response curve from zero to 11kHz



adapted from an international standard ISO226, generated with the psyweight and iso226 functions included with this project. We can use such models to help determine which frequencies in a data set are important, and which ones are less important to the perceived sound. We will then store the less-important frequencies with less precision. The idea is to focus the compression on parts of the signal that are perceptually less important.

Here is an illustration of an audio compression method similar to (but much simpler than) mp3 compression: Note that with the illustrated choice of quantization bands, the bulk of the model

Sample model coefficients | Perceptual model | Weighted coefficients (shown with quantization bands)



coefficients lie in the low precision storage realm in the above example. Our compression method will encode all of the corresponding unweighted model coefficients that fall in that band with low precision. Note that the perceptual model is only used to help decide which coefficients are important to keep at full precision. The precise cutoff between low- and high-precision storage will govern the overall compression obtained by this method.

For example, let's say that 90% of the coefficients lie in the low-precision part of the illustration. Suppose that we store those coefficients with only 8-bit numbers, and the remaining ones with 16-bit numbers. The resulting data will only require about 55% of the storage space used by the original sample data set of entirely 16-bit numbers.

Exercise

What is the bit rate of the compressed audio sample discussed in the last paragraph, assuming 22,050 samples per second?

We can achieve higher compression by either widening the low-precision region, or by lowering the precision used to store the coefficients, or both. The algorithm used in mp3 compression uses similar techniques to achieve up to a 10:1 compression of CD audio and still maintain a high perceived quality of sound.

Quantization in MATLAB and Octave

MATLAB and Octave do not easily represent quantized numbers internally. We can, however, simulate the result of quantization in double-precision with the following function, `quantize.m`:

```
function y = quantize (x, bits)

m = max(abs(x));
y = x/m;
y = floor((2^bits - 1)*y/2);
```

```
y = 2*y/(2^bits -1);
y = m*y;
```

MP3-like compression with MATLAB and Octave

Before proceeding you will need to download the functions `psyweight.m` and `iso226.m`:

```
baseurl = 'https://raw.githubusercontent.com/bwlewis/enr230-audio-compression/main/';
urlwrite([baseurl, 'quantize.m'], 'quantize.m');
urlwrite([baseurl, 'iso226.m'], 'iso226.m');
urlwrite([baseurl, 'psyweight.m'], 'psyweight.m');
```

The following code example illustrates our discussion of audio data compression with actual audio samples. You will need the above `quantize.m` function, as well as the standard perceptual model of human hearing provided in the functions `psyweight.m` and `iso226.m`.

```
[b,R] = audioread ('shostakovich.wav');    % Load an audio sample
N  = length(b);
c = dct(b);        % Compute the interpolation model coefficients
w = sqrt(2/N);
f = linspace(0,R/2,N)';
% Generate a perceptual model (requires psyweight.m, iso226.m)
p = psyweight(f);
% Let's look at the weighted coefficients and pick a cutoff value
plot (f,w*p.*c)

% Pick a cutoff value and split the coefficients into low- and high-precision sets:
cutoff = 0.00015
mask = (abs(w*c)<cutoff);
low = mask .* c;
high = (1-mask) .* c;
% This plot illustrates the cutoff region:
plot(f,w*high,'-r',f,w*low,'-b')

% Now pick a precision (in bits) for the low precision data set:
lowbits = 8
% We won't quantize the high-precision set of coefficients (high), only the
% low precision part (requires quantize.m):
low = quantize(low, lowbits);

% Finally, let's reconstruct our compressed audio sample and listen to it!
y = idct(low+high);
sound (y,R);
```

Exercise

Experiment with the above code, trying out different cutoff values (`cutoff`) and precision values (`lowbits`). Listen to your results. What is the lowest bit rate that you can find that still sounds good to you?