## Differentials (ca 1900)

The derivative symbol $\frac{dy}{dx}$ is a *function*—that is $\frac{dy}{dx} = \frac{dy}{dx}(x)$—except when it isn't. You can thank the French mathematician Élie Cartan for that.

Cartan helped to formalize the intuitive idea that $dx$ and $dy$ represent tiny changes in $x$ and $y$. He called them *differentials*—a term that even today has several possible interpretations. One uses local linear approximation inspired by the definition of derivative.

Let $y = f(x)$ be a differentiable function with derivative $f'(x)$ (using Lagrange's notation). Let $dx$ be a variable representing a nonzero real number. Define $dy$ as:

$$dy = f'(x)dx.$$

Note that here, $dy$ is a function of $x$ **and** of $dx$ since it depends on both of them. Defined in this way, $dy$ and $dx$ are called "differentials."

If you divide each side of the above equation by $dx$ you get:

$$\frac{dy}{dx} = f'(x),$$

which is just the usual way we write the derivative! Thus, we can think of $dy/dx$ the usual way—as a function of $x$—or, in differential form we can think separately of $dy$ (as a function of both $x$ and $dx$) and $dx$ (as a number).

Differentials are useful in at least two important applications: integration using substitution, and in the solution of some differential equations.