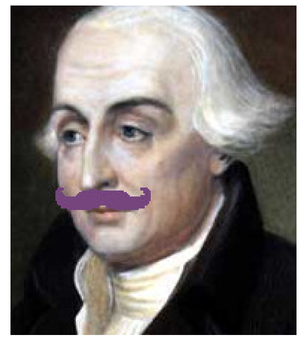
# Derivative Notation is Awful

This idea of differentiation we're using to compute derivatives of functions comes from the late 1600's. And it shows! There are hundreds of years of crufty and weird notation we need to at least recognize. This note organizes notation by approximate date and inventor.

In every example below, assume that $f$ is a real-valued, differentiable function of a real variable (usually called $x$).

## Lagrange's notation (1749)



You're familiar with this one! The derivative of $f(x)$ is written $f'(x)$. There is also an interpretation of the apostrophe (aka "prime") as an *operator* on a function. What I mean is:

$$(f(x))' = f'(x).$$

Note the subtle difference here. On the left of that last equals sign the prime symbol is *doing something* to the function $f$ to return the thing on the right side of the equals sign, which represents the derivative itself (another function). In some very specific sense, the $'$ symbol used on the left is a kind of function that takes one function as input and returns another as output.

The derivative of $f'(x)$—if it exists—is written $f''(x)$, often called the 2nd derivative. The derivative of $f''(x)$—if it exists—is written, wait for it, $f'''(x)$.

Now, guess how the derivative of $f'''(x)$ is written! Give up? It's $f^{(4)}(x)$. Obviously. Note the parenthesis as a bad attempt to avoid confusion with $f^4(x)$ which in some books might indicate "the function $f$ evaluated at $x$ then raised to the fourth power". Madness!

If you're reading some weirdly Roman-influenced text you might see the author using Roman numerals (of all things!). So, like, $f^{\mathrm{iv}}(x)$, $f^{\mathrm{v}}(x)$, etc. (without parentheses in the superscripts).

## Leibniz's notation (1684) [4]

*(They were really into wigs back then...)*

This notation is used just as much as the Lagrange notation above. If the domain variable is called $x$, then the derivative *operator* is written $\frac{d}{dx}$. Think of this as a function that takes a differentiable function as its argument and produces a function as its output. Thus,

$$\frac{d}{dx}\big(f(x)\big) = \frac{df}{dx}(x).$$

On the left, we have a function $f$ whose derivative with respect to $x$ is being taken. On the right, a new function called $\frac{df}{dx}$, the first derivative of $f$. NOTE(!) that the symbols $\frac{d}{dx}$ and $\frac{df}{dx}$ **are not** fractions—they are simply symbols for functions. Except, of course, when they aren't (see below).

Although this seems janky, it's actually a pretty good notation for many reasons. In particular, it explicitly indicates the dependent variable which is nice. Also, let's say we just have the graph of some function with $x$ and $y$ variables. We want to differentiate this *curve* (that is, find its slope at each point). The derivative of the curve using Leibniz's notation is $\frac{dy}{dx}$. This has a very natural interpretation in terms of slope:

$$\frac{dy}{dx} \qquad \text{reminds you of} \qquad \frac{\text{change in } y}{\text{change in } x}.$$

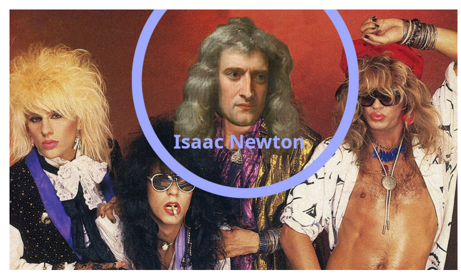
*Conceptually*, think of $dy$ as a tiny change in $y$ and $dx$ as a tiny change in $x$. And, again conceptually, the symbol $\frac{dy}{dx}$ reminds us of the ratio that defines slope. In fact that is literally almost how Leibniz thought about things, but it took until 1966 until Abraham Robinson formalized those ideas mathematically [3].

The Leibniz notation represents 2nd and higher order derivatives as:

$$\begin{aligned} \tfrac{d}{dx}\Big(f(x)\Big) &= \tfrac{df}{dx}(x) \\ \tfrac{d}{dx}\Big(\tfrac{df}{dx}(x)\Big) &= \tfrac{d^2f}{dx^2}(x) \\ \tfrac{d}{dx}\Big(\tfrac{d^2f}{dx^2}(x)\Big) &= \tfrac{d^3f}{dx^3}(x) \\ \vdots \quad & \quad \vdots \end{aligned}$$

Note $\frac{d}{dx}\Big(\frac{df}{dx}(x)\Big) = \frac{d^2f}{dx^2}(x)$ and **not** $\frac{d^2f}{d^2x^2}$, what you might expect if this were algebra class. Welcome to calculus! Remember $dx$ is a symbol that conceptually means a tiny change in $x$—not $d$ times $x$.

## Newton's notation (1684) [2]




This is the notation Isaac Newton used to write derivatives, which he called "fluxions." It's still used in some books (often when differentiating with respect to time), but is less common than the Lagrange and Leibniz notation. The table below shows Lagrange and Leibniz notation on the left, and the equivalent Newton notation on the right:

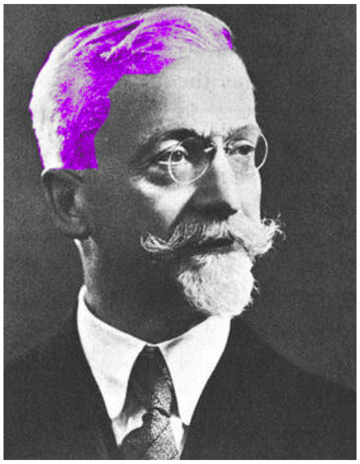
$$\begin{aligned} y' &= \tfrac{dy}{dx} = \dot{y} \\ y'' &= \tfrac{d^2y}{dx^2} = \ddot{y} \\ y''' &= \tfrac{d^3y}{dx^3} = \dddot{y} \end{aligned}$$

Beyond three, the dots get a little silly but Newton persevered with outrageous symbols like: $\overset{\vdots\vdots}{y}$. We won't be using this notation in class!

## Differentials (ca 1900)

Above we said that the symbol $\frac{dy}{dx}$ is a *function*—that is $\frac{dy}{dx} = \frac{dy}{dx}(x)$—except when it isn't. You can thank the famous French mathematician Élie Cartan for that.

Cartan helped to formalize the intuitive idea that $dx$ and $dy$ represent tiny changes in $x$ and $y$. He called them *differentials*—a term that even today has several possible interpretations. One way uses local linear approximation similar to the definition of derivative.



Let $y = f(x)$ be a differentiable function with derivative $f'(x)$ (using Lagrange's notation). Let $dx$ be a variable representing a nonzero real number. Define $dy$ as:

$$dy = f'(x)dx.$$

Note that here, $dy$ is a function of $x$ **and** of $dx$ since it depends on both of them. Defined in this way, $dy$ and $dx$ are called "differentials."

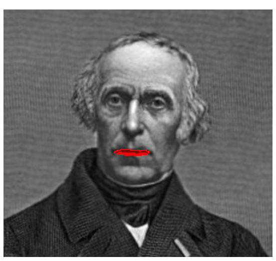
If you divide each side of the above equation by $dx$ you get:

$$\frac{dy}{dx} = f'(x),$$

which is just the usual way we write the derivative! Thus, we can think of $dy/dx$ the usual way—as a function of $x$—or, in differential form we can think separately of $dy$ (as a function of both $x$ and $dx$) and $dx$ (as a number).

Differentials are useful in at least two important applications: integration using substitution, and in the solution of some differential equations.

## Arbogast's D-notation (1800) and partial derivatives [1]



This is a differential operator notation that we *will* use a lot, but not until Calc 3 and differential equations courses. It is widely used in the kinds of problems encountered there. It is often mis-attributed to Euler. Here is the gist:

$$\begin{array}{rcrcl} & & Df & = & \frac{d}{dx}\Big(f\Big) \\ & & (Df)(x) & = & \frac{df}{dx}(x) \\ & & D^2 f & = & \frac{d^2}{dx^2}\Big(f\Big) \\ f_x & = & \partial_x f & = & \frac{\partial}{\partial x}\Big(f\Big) \\ f_{xx} & = & \partial_{xx} f & = & \frac{\partial^2}{\partial x^2}\Big(f\Big) \end{array}$$

Those last few are called "partial" derivatives that we will spend a lot of time on in the third semester (Calc 3).

## Appendix: Non-standard numbers and infinitesimals

Mathematician and Dilbert boss look-alike contest winner, Abe Robinson

Abraham Robinson devised a "non-standard analysis" that directly implements the intuitive ideas of Leibniz representing $dx$ and $dy$ as infinitely tiny quantities [3]. All he had to do was invent a new kind of number that is larger than any real number. Why not?

We normally assume that the real numbers are unbounded. Robinson throws this out and invents numbers bigger than any real number. So if $x$ is any real number and $N$ is a non-standard number then $N > x$. And, weirdly, for *any* positive real number $p$:

$$0 < \frac{1}{N} < p$$

These things: $1/N$ for a non-standard number $N$, are called *infinitesimals*. Robinson developed a version of calculus based on this.

Remarkably, he showed the non-standard approach to be essentially identical to the usual calculus, demonstrating that the assumption that real numbers are unbounded is not necessary (for calculus, anyway). Some tricky problems are actually easier to solve with the non-standard approach. Perhaps Leibniz was ahead of his time!

# References

[1] Arbogast, Louis F. A.. Du calcul des derivations, 1800.

[2] Newton, Isaac. De analysi per aequationes numero terminorum infinitas, 1669.

[3] Robinson, Abraham. Non-standard analysis. North-Holland Publishing Co., Amsterdam, 1966.

[4] Roero, Clara Silvia. "Gottfried Wilhelm Leibniz, first three papers on the calculus (1684, 1686, 1693)." Landmark Writings in Western Mathematics 1640-1940. Elsevier Science, 2005. 46-58.

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Lagrange: Wikipedia `https://commons.wikimedia.org/wiki/File:Lagrange_crop.jpg`

Newton: Godfrey Kneller, Portrait of Sir Isaac Newton, 1689. From
`https://exhibitions.lib.cam.ac.uk/linesofthought/artifacts`

Cartan: Wikipedia `https://en.wikipedia.org/wiki/File:Elie_Cartan.jpg`

Robinson: London Mathematical Society
`https://mathshistory.st-andrews.ac.uk/LMS/robinson_lms_obit.pdf`