Homework 2 due Friday May 17 11:59pm PST

PSTAT 176/276, Spring 2024

1. Trinomial Interest Rate Tree.

   Consider a trinomial interest rate model: every period interest rates can increase by 0.25% (for example, from 3% to 3.25%), stay constant, or fall by 0.25%. We shall assume that the respective $\widetilde{\mathbb{P}}$-probabilities are 0.3, 0.3 and 0.4. Initial interest rate is $R_0 = 3\%$.

   (1) Write a Monte Carlo simulator which generates paths of R_n for a given number of periods $n = 1, 2, \dots, N$ with $N = 12$. Generate and save a database of $M = 1000$ scenarios of the trinomial interest rates (R_n).

   (2) Using the above fixed database, estimate the zero-coupon bond prices $B(0,T)$ (as a Monte Carlo estimate for the risk neutral expectation of receiving \$1 at T) for $T = 1, 2, \dots, 12$. Note that you will be re-using the same paths from (1) for different maturity dates T.

   (3) Report the resulting bond prices, and the corresponding yields $y_{0,T}$. Include a plot of the resulting yield curve $T \mapsto y_{0,T}$.

2. The Vasicek model says that interest rates evolve according to

$$r_{t+h} := e^{-\kappa h} \cdot r_t + (1 - e^{-\kappa h})\bar{r} + \sqrt{(1 - e^{-2\kappa h})}\sqrt{\frac{\sigma^2}{2\kappa}}\, Z \tag{1}$$

   where $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable, independent of everything else and $\kappa, \bar{r}, \sigma$ are model parameters. The period length h (aka Δt in other notations) can be arbitrary. You apply (1) recursively starting from a given r_0 to obtain $r_h, r_{2h}, r_{3h}, \dots$.

   Recall that bond prices in a continuous-time model are given according to

$$B_{0,T} = \widetilde{\mathbb{E}}\Big[\exp\Big(-\int_0^T r_s ds\Big)\Big].$$

   For the purposes of this question we will approximate above via the sum

$$B_{0,T} = \widetilde{\mathbb{E}}\Big[\exp\big(-\sum_{k=1}^{K} r_{kh} \cdot h\big)\Big],$$

   where r_{kh} is the interest rate after applying eqn (1) k time-steps.

   Take $T = 3$, $h = 1/24$, and model parameters $\kappa = 2$, $\bar{r} = 0.05$, $\sigma = 0.04$ and initial interest rate $r_0 = 0.04$. This means that $K = 72$ above.

   Using $M = 1000$ simulations estimate the prices of 1-year, 2-year and 3-year Floors with floor rate $R = 0.03$. Note: you can reuse the same simulated paths for the different floor maturities again.

3. (Uniform Stratified Sampling) The point of this problem is to illustrate stratification with uniform strata. We will price an Out-of-the-money Call in a Black-Scholes model with the lognormal random variable S_T. The parameters are set as $S_0 = 60$, $r = 0.04$, $T = 1$, $\sigma = 0.25$. The Call strike is $K = 65$.

   (1) Simulate independent, identically distributed uniform random variables $\{U_i, i = 1, \ldots, n\}$, $n = 400$ on the unit interval $[0, 1]$, and define

$$V_i = \frac{i - 1 + U_i}{400}; \quad i = 1, \ldots, 400.$$

   (2) Using V_i and the cumulative normal distribution function $\Phi(x) := \int_{-\infty}^{x} e^{-u^2/2}/\sqrt{2\pi}\mathrm{d}u$, $x \in \mathbb{R}$, define the normal random variable

$$W_i := \Phi^{-1}(V_i); \quad i = 1, \ldots, 400.$$

   (3) Using the normal random variables, evaluate the time-0 Call option price by the stratified Monte Carlo simulation. Also, compute the standard error of the stratified Monte Carlo estimator.

4. **PSTAT 276** students ONLY. Non-uniform Stratified Sampling. The stratification will be on the underlying U's, and the strata are $A_1 = [0, 0.3]$, $A_2 = [0.3, 0.6]$, $A_3 = [0.6, 0.8]$, $A_4 = [0.8, 1]$. Thus, the corresponding p_i's are: $p_1 = p_2 = 0.3$, $p_3 = p_4 = 0.2$.

   We wish to use a total of $n = 400$ samples, split across the strata as $N_1 = 40$, $N_2 = N_3 = N_4 = 120$. Note that the split is not proportional to the p_i, $i = 1, 2, 3, 4$, but this is still fine.

   Evaluate the Call option price using the non-uniform stratified sampling as in the previous problem.