Math 104C Homework #2

Due on Apr 17 by 11:59 PM

Topics: (04/11 – 04/16) Cubic splines part 2, Bézier curves, Least square

Video: How to submit homework on Gradescope or copy and paste (`https://youtu.be/quBWbQ5opT0`)

Update: Some problems have been changed or moved to the next HW because it is not likely that the topic will be covered before the deadline.

I. For presentation

1. (Exploration) Find k_1, k_2, k_3 in the following cubic spline. Which of the three end conditions-natural, parabolically terminated, or not-a-knot - if any, are satisfied?

$$S(x) = \begin{cases} 4 + k_1 x + 2x^2 - \frac{1}{6}x^3 & \text{on } [0,1] \\ 1 - \frac{4}{3}(x-1) + k_2(x-1)^2 - \frac{1}{6}(x-1)^3 & \text{on } [1,2]. \\ 1 + k_3(x-2) + (x-2)^2 - \frac{1}{6}(x-2)^3 & \text{on } [2,3] \end{cases}$$

2. (Exploration) Show that the Bézier curve defined by the algorithm satisfies its characterization. That is, Given endpoints $(x_1, y_1), (x_4, y_4)$ and control points $(x_2, y_2), (x_3, y_3)$, set

$$\begin{aligned} b_x &= 3(x_2 - x_1) \\ c_x &= 3(x_3 - x_2) - b_x \\ d_x &= x_4 - x_1 - b_x - c_x \\ b_y &= 3(y_2 - y_1) \\ c_y &= 3(y_3 - y_2) - b_y \\ d_y &= y_4 - y_1 - b_y - c_y. \end{aligned}$$

The Bézier curve $B(t) = (x(t), y(t))$ is defined for $0 \le t \le 1$ by

$$\begin{aligned} x(t) &= x_1 + b_x t + c_x t^2 + d_x t^3 \\ y(t) &= y_1 + b_y t + c_y t^2 + d_y t^3. \end{aligned}$$

Show that the curve leaves (x_1, y_1) along the tangent direction $(x_2 - x_1, y_2 - y_1)$ and ends at (x_4, y_4) along the tangent direction $(x_4 - x_3, y_4 - y_3)$.

3. (Exploration) Prove that the 2-norm is a vector norm. You will need to use the Cauchy-Schwarz inequality $|u \cdot v| \le \|u\|_2 \|v\|_2$.

4. (Exploration) Prove the following.

   (a) Let A be an $n \times n$ nonsingular matrix. (i) Prove that $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$. (ii) Let b be an n-vector; then $Ax = b$ has exactly one solution. Prove that this solution satisfies the normal equations. (*Therefore, least square solution is a generalization of usual system of linear equations involving a nonsingular square matrix.*)

(b) Prove that the least square solution minimize the Euclidean norm of the residual. That is, if $\bar{x} \in \mathbb{R}^n$ solves $A^T Ax = A^T b$, where $A \in \mathbb{R}^{(m\times n)}$ and $b \in \mathbb{R}^m$, then $||b - A\bar{x}||_2 \leq ||b - Ax||_2$ for any $\bar{x} \in \mathbb{R}^n$.

5. (Exploration - Computation) Fit the data to the periodic models $F_3(t) = c_1 + c_2 \cos 2\pi t + c_3 \sin 2\pi t$ and $F_4(t) = c_1 + c_2 \cos 2\pi t + c_3 \sin 2\pi t + c_4 \cos 4\pi t$. Find the 2-norm errors $||e||_2$ and compare the fits of F_3 and F_4.

t	y
0	0
1/6	2
1/3	0
1/2	-1
2/3	1
5/6	1

II. Not for presentation

The following problems are for your own study, but not for presentation. They have been already discussed during lectures.

6. (Formation) Answer the following.

(a) Given data points $(x_1, y_1), ..., (x_n, y_n)$, give the definition of the natural cubic spline.

(b) Given data points $(x_1, y_1), ..., (x_n, y_n)$, give the definition of the not-a-knot cubic spline.

(c) The system of equations resulting from the natural cubic spline involves an SDD matrix, and hence always solvable. (True/False)

(d) The system of equations resulting from the parabolically terminated cubic spline involves an SDD matrix of a smaller size than natural cubic spline. (True/False)

(e) Not-a-knot cubic spline with four or more knots is not solvable since the resulting matrix is not SDD. (True/False)

7. (Formation) Find the first endpoint, two control points, and last endpoint for the following one-piece Bézier curves.

$$\begin{cases} x(t) = 3 + 4t - t^2 + 2t^3 \\ y(t) = 2 - t + t^2 + 3t^3 \end{cases}$$

8. (Formation) Answer the following.

(a) Write the problem setting where least square method gives answers to. To be more specific, give the dimension information of A, x, and b for $Ax = b$. Also, for the rest of the subproblems, assume this setting.

(b) (True/False) Least square solution of is the projection of b onto column space A. (Be careful of dimensions and rigorous wording.)

(c) (True/False) Normal equation has a unique solution if and only if A is full rank.

End of homework