Math 104C Homework #4

Due on May 1st by 11:59 PM

Topics: (04/25 – 04/30) QR - Gram-Schmidt , QR - Householder,

Video: How to submit homework on Gradescope or copy and paste (`https://youtu.be/quBWbQ5opT0`)

Special instruction 1: To alleviate workload during Group project preparations, only #2 will be looked at and graded this time. However, other problems can appear in the quiz when QR - Householder is covered.

Special instruction 2: There is no HW4 presentation due to CP1. If you have signed up for presentation, present your answer to Dongyang or Jea-Hyun during their office hours. This is to ensure a good enough quality of shared answers since the answer will be directly uploaded to Canvas Discussions board.

I. For presentation

1. (Exploration; Analysis) Show that the Gram-Schmidt orthogonalization of an $m \times m$ matrix requires approximately m^3 multiplications and m^3 additions. (*Hint: Look at the algorithm and carefully count the operations.*)

2. (Exploration; Analysis) Let A be a full rank m-by-n matrix $m \geq n$ and let $P = A(A^TA)^{-1}A^T$. (a) Show that P is an orthogonal projection onto column space of A. That is, (i) $P^2 = P$, (ii) P is symmetric, and $Pv \in \text{Col}(A)$ for any $v \in \mathbb{R}^m$. Also, (b) show that $P = \frac{vv^T}{v^Tv}$, where $v \in \mathbb{R}^m$ is an orthogonal projection onto span$\{v\}$.

3. (Exploration; Analysis) Prove the following

   (a) Prove that if P is an orthogonal projection defined on $\mathbb{R}^m$, that is, P satisfies (i) $P^2 = P$, (ii) P is symmetric, then for $v \in \mathbb{R}^m$, Pv and $v - Pv$ are perpendicular.

   (b) Prove that, for a vector space V, if $P : V \to V$ is a projection, i.e., $P^2 = P$, then for any $v \in \text{R}(P)$, we have $Pv = v$, where $\text{R}(P)$ is the range of P.

   (c) Prove that Householder reflectors are symmetric, orthogonal matrices.

4. (Exploration; Analysis) Prove that, for m-by-n matrix Q with $m \geq n$ whose columns are mutually orthonormal, (a) $||u||_2 = ||Qu||_2$ for any vector $u \in \mathbb{R}^n$, (b) $||v||_2 = ||Q^Tv||_2$ if v belongs to the column space of Q, and (c) give an example where $||v||_2 \neq ||Q^Tv||_2$ for v not belonging to the column space of Q.

End of homework