Math 104C Homework #8

Due on May 29 by 11:59 PM

Topics: (05/15 – 05/22) Trig interpolation, signal processing

Video: How to submit homework on Gradescope or copy and paste (`https://youtu.be/quBWbQ5opT0`)

Update: (2024-05-24 01:00 PM) Problems on eigenvalue problems are added. To even out the workload coming from more problems to explore than usual, some problems are not graded when it comes to HW credit or will be graded more generously if selected (see comments that follow those problems). But they are all important problems and may appear in the quiz.

I. For presentation

1. (Exploration; Analysis) Prove, for any $\theta \in \mathbb{R}$,

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

2. (Exploration; Analysis) Let $n \geq 1$ and k, l be integers. Then

$$\sum_{j=0}^{n-1} \cos\frac{2\pi jk}{n} \cos\frac{2\pi jl}{n} = \begin{cases} n & \text{if both } (k-l)/n \text{ and } (k+l)/n \text{ are integers} \\ \frac{n}{2} & \text{if exactly one of } (k-l)/n \text{ and } (k+l)/n \text{ is an integer} \\ 0 & \text{if neither is an integer} \end{cases}$$

$$\sum_{j=0}^{n-1} \cos\frac{2\pi jk}{n} \sin\frac{2\pi jl}{n} = 0$$

$$\sum_{j=0}^{n-1} \sin\frac{2\pi jk}{n} \sin\frac{2\pi jl}{n} = \begin{cases} 0 & \text{if both } (k-l)/n \text{ and } (k+l)/n \text{ are integers} \\ \frac{n}{2} & \text{if } (k-l)/n \text{ is an integer and } (k+l)/n \text{ is not} \\ -\frac{n}{2} & \text{if } (k+l)/n \text{ is an integer and } (k-l)/n \text{ is not} \\ 0 & \text{if neither is an integer} \end{cases}$$

Hint: Use the result of Problem 1 and orthogonality theorem on sum of powers of primitive n-th root of unity.

(Comment: This problem will not be graded. But I encourage you to study at least the model answer that will be posted on Canvas.)

3. (Exploration; Computation) Write a code that implements trigonometric least square. You may use fft package (`numpy.fft.fft`, `numpy.fft.ifft`, `numpy.real`, `numpy.imag`) offered by NumPy appropriately.

   (a) Data: $x = [2.5433, 1.8744, 1.695, 1.6759, 2.7873, 1.6847, 1.7283, 1.8506, 2.5267, 2.1502, 1.2686, 2.2924, 2.1646, 2.2637, 1.2958, 2.076]$ at $t = [jh]_{j=0}^{15}$, where $h = L/16$ and $L = 3$.

   (b) Plot the data.

(c) Find a function belonging to $V = \text{span}\{f_k(t)\}_{k=0}^{m} \cup \{g_\ell\}_{\ell=1}^{m-1} \subset C_{\text{per}}[0, L]$, set of all continuous, periodic functions defined on $[0, L]$, where $m = 5$, $f_k(t) = \cos\frac{2\pi k t}{L}$ $(j = 0, 1, 2, \cdots, m)$ and $g_\ell(t) = \sin\frac{2\pi \ell t}{L}$ $(\ell = 1, 2, \cdots, m-1)$

(d) Plot, on the same axis as the given data, the least square fitting function on a fine grid, say, at 100 grid points that evenly divides $[0, L]$.

4. (Exploration; Core) Answer the following. Also, give **brief explanations** about your answers. (Also, you are encouraged to give reference if your answer depends on facts that are not easy to recall from memory.)

   (a) Explain the relation between DFT and FFT.

   (b) (True/False) Given a function $P(t) = f_1(t) - 2f_2(t) + 3f_3(t) + 1/2f_4(t)$ that interpolates (t_i, x_i) for $i = 1, 2, 3, 4$, $P(t) = f_1(t) - 2f_2(t)$ is the best function that is a linear combination of $\{f_1, f_2\}$ in the sense of least square.

   (c) (True/False) Given a function $P(t) = f_1(t) - 2f_2(t) + 3f_3(t) + 1/2f_4(t)$ that interpolates (t_i, x_i) for $i = 1, 2, 3, 4$, $P(t) = f_1(t) - 2f_2(t)$ is the best function that is a linear combination of $\{f_1, f_2\}$ in the sense of least square if $A = \left[f_i(t_j)\right]_{i,j=1}^{4}$ is orthogonal.

   (d) (True/False) Consider a data x of length 8 and its DFT y. If x is a real vector, the least square fitting of x using wave number 0, 1, 2 by can be computed by taking $y_0, y_1, \cdots, y_5$.

   (e) (True/False) Consider a data x of length 8 and its DFT y. If x is a real vector, the least square fitting of x using wave number 0, 1, 2 by can be computed by taking y_0, y_1, y_2.

   (f) (True/False) In trigonometric least square fitting using m basis functions, taking the m functions from highest frequency results in the best result.

   (g) (True/False) Assume the settings in Problem 7. Then, $F_m(t) = \sum_{k=0}^{m-1} y_k f_k(t)$ is the least square solution that fits the data $x = [x_j] = [F(t_j)]_{j=0}^{n-1}$.

5. (Exploration; Analysis) Prove the following properties of the Rayleigh quotient. Suppose that $\mathrm{A} \in \mathbb{C}^{n\times n}$ is Hermitian with spectrum $\sigma(\mathrm{A}) = \{\lambda_i\}_{i=1}^{n} \subset \mathbb{R}$, where the following ordering is imposed:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n.$$

   (a) $R(x) \in \mathbb{R}$, for all $x \in \mathbb{C}_\star^n$.

   (b) For all $x \in \mathbb{C}_\star^n$,

$$\min_{j=1}^{n} \lambda_j \leq R(x) \leq \max_{j=1}^{n} \lambda_j.$$

   (c) $R(x) = \lambda_k$ iff x is an eigenvector associated to λ_k. (*Hint: You can use the fact that a Hermitian matrix has Cholesky factorization $A = LL^H$ without proof, where L is a lower triangular matrix. In fact, we covered real version of this: If A is symmetric, there is a (real) lower triangular matrix L such that $A = LL^T$.*)

6. (Exploration; Analysis) Prove the following properties of the Rayleigh quotient. Under the same assumptions made in Problem 5, for a fixed $x \in \mathbb{C}_\star^n$, the function

$$Q(\alpha) = \|\mathrm{A}x - \alpha x\|_2^2, \quad \forall \alpha \in \mathbb{C},$$

has a unique global minimum, and, in fact, the Rayleigh quotient is the unique minimizer

$$R(x) = \underset{\alpha\in\mathbb{C}}{\operatorname{argmin}} \|\mathrm{A}x - \alpha x\|_2^2$$

(Comment: If you feel a bit overwhelmed by more problems than usual, at least try something meaningful for this problem.)

II. Not for presentation

The following problems are for your own study, but not for presentation. They have been already discussed during lectures.

7. (Formation; Analysis) Prove the orthogonal function interpolation theorem: Let $f_0(t), \ldots, f_{n-1}(t)$ be functions of t and $t_0, \ldots, t_{n-1}$ be real numbers. Assume that the $n \times n$ matrix

$$A = \begin{bmatrix} f_0(t_0) & f_0(t_1) & \cdots & f_0(t_{n-1}) \\ f_1(t_0) & f_1(t_1) & \cdots & f_1(t_{n-1}) \\ \vdots & \vdots & & \vdots \\ f_{n-1}(t_0) & f_{n-1}(t_1) & \cdots & f_{n-1}(t_{n-1}) \end{bmatrix}$$

is a real $n \times n$ orthogonal matrix. If $y = Ax$, the function

$$F(t) = \sum_{k=0}^{n-1} y_k f_k(t)$$

interpolates $(t_0, x_0), \ldots, (t_{n-1}, x_{n-1})$, that is $F\left(t_j\right) = x_j$ for $j = 0, \ldots, n-1$.

(Comment: This problem will not be graded. But I encourage at least reviewing this theorem.)

8. Let A be an $m \times m$ symmetric matrix and its eigenvalues $\lambda_1, \ldots, \lambda_m$ satisfying $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_m|$. For almost every initial vector, Power Iteration converges linearly to an eigenvector associated to λ_1 with convergence rate constant $S = |\lambda_2/\lambda_1|$.

9. (Formation) Answer the following. Also give **brief explanations** about your answer. (Also, you are encouraged to give reference if your answer depends on facts that are not easy to recall from memory.)

   (a) (True/False) In theory, if A is n-by-n matrix, then Power iteration yields the eigenvector associated with the largest eigenvalue in modulus in n iterations.

   (b) (True/False) Power iteration, as it stands, can find only one eigenvector associated with the largest eigenvalue in modulus.

   (c) (True/False) Inverse Power iteration without shifting, as it stands, can find only one eigenvector associated with the smallest eigenvalue in modulus.

   (d) (True/False) Shifted Inverse Power iteration can find all eigenvalues if there is no technical difficulty and if shifts are chosen appropriately.

   (e) (True/False) Rayleigh Quotient $R(x; A) = (x^H Ax)/x^H x$ always yields an eigenvalue.

   (f) (True/False) Rayleigh Quotient $R(x; A) = (x^H Ax)/x^H x$ always yields an eigenvalue if A is Hermitian.

(g) (True/False) Rayleigh Quotient $R(x; A) = (x^H Ax)/x^H x$ always yields a real number if A is Hermitian.

(h) Write the pseudo algorithm of Power iteration.

(i) Write the pseudo algorithm of Shifted Inverse Power iteration.

(j) Write the spectral theorem on Hermitian matrix. (See the notes `na12eigenvalues.ipynb`.)

(k) State the relation between eigenvalues and eigenvectors of $A \in R^{m\times m}$ and $aA + bI$.

(l) State the relation between eigenvalues and eigenvectors of an invertible matrix $A \in R^{m\times m}$ and A^{-1}.

End of homework