In [ ]:
```
import numpy as np
```

Question 1

1. We use the conditions of continuity and smoothness at nodes: $S_i(x_{i+1}) = S_{i+1}(x_{i+1})$, $S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$

$S_1(1) = S_2(1)$

$\Rightarrow 4 + k_1 + 2 - \frac{1}{6} = 1$

$\Rightarrow k_1 = 1 - \frac{35}{6} = -\frac{29}{6}$

$S_2(2) = S_3(2)$

$\Rightarrow 1 - \frac{4}{3} + k_2 - \frac{1}{6} = 1$

$k_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$S_2'(2) - S_3'(2) = 0$

$-\frac{4}{3} + 2k_2 - \frac{1}{2} = k_3$

$-\frac{4}{3} + 3 - \frac{1}{2} = k_3 = -\frac{8}{6} + \frac{18}{6} - \frac{3}{6} = \frac{7}{6}$

Natural spline? $(S_1''(x_1) = 0,\ S_{n-1}''(x_n) = 0)$

$S_1''(x) = 4 - x$, $S_1''(0) \neq 0$

<u>Not</u> a natural spline

Parabolically terminated? ($d_1 = d_{n-1} = 0$)

$d_1 = -\frac{1}{6}$

<u>NOT</u> parabolically terminated

Not-a-knot? ($d_1 = d_2$, $d_2 = d_3$)

$d_1 = -\frac{1}{6} = d_2 = -\frac{1}{6} = d_3 = -\frac{1}{6}$ ✓

Yes, not-a-knot holds

Question 2

2. $x'(t) = b_x + 2c_x \cdot t + 3d_x \cdot t^2$

plug in the definition of b_x, c_x, d_x:

$$x'(t) = 3(x_2 - x_1) + 2 \cdot t\big(3(x_3 - x_2) - 3(x_2 - x_1)\big)$$
$$+ 3 \cdot t^2\Big(x_4 - x_1 - 3(x_2 - x_1) - \big(3(x_3 - x_2) - 3(x_2 - x_1)\big)\Big)$$

$$x'(t) = 3(x_2 - x_1) + 6 \cdot t(x_3 - 2x_2 + x_1)$$
$$+ 3t^2\big(x_4 - x_1 - 3(x_3 - x_2)\big)$$

similarly, y is defined the same, we just replace x w/ y:

$$y'(t) = 3(y_2 - y_1) + 6 \cdot t(y_3 - 2y_2 + y_1)$$
$$+ 3t^2\big(y_4 - y_1 - 3(y_3 - y_2)\big)$$

Clearly,

$$x'(0) = 3(x_2 - x_1), \quad y'(0) = 3(y_2 - y_1)$$

Hence, at $t=0$ (at (x_1, y_1)), the curve has <u>direction</u> $(x_2 - x_1, y_2 - y_1)$

Now we examine $t=1$

$$x'(1) = 3(x_2 - x_1) + 6(x_3 - 2x_2 + x_1) + 3(x_4 - x_1 - 3(x_3 - x_2))$$

$$= \underbrace{-3x_1 + 6x_1 - 3x_1} + \underbrace{3x_2 - 12x_2 + 9x_2}$$

$$\underbrace{+6x_3 - 9x_3} + 3x_4 = 3(x_4 - x_3)$$

$$y'(1) = 3(y_2 - y_1) + 6(y_3 - 2y_2 + y_1) + 3(y_4 - y_1 - 3(y_3 - y_2))$$

$$= 3(y_4 - y_3)$$

thus, $(x'(1), y'(1)) = (3(x_4 - x_3), 3(y_4 - y_3))$

which is in the direction of $(x_4 - x_3, y_4 - y_3)$

where $t=1$ corresponds to the terminal point (x_4, y_4)

Question 3

3. Define $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$

a) $\|x\|_2 > 0$ when $x \neq 0$ and $\|x\|_2 = 0$ iff $x = 0$

i) $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$

Assume $\vec{x} \neq \vec{0}$, then there must be an element of x, x_j, s.t. $x_j \neq 0$

Hence, $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} > 0$

ii) $\underline{\|x\|_2 = 0 \Rightarrow x = \vec{0}}$

since $a^2 \geq 0 \quad \forall a$

$$\sqrt{\sum_{i=1}^{n} x_i^2} = 0 \Rightarrow x_i = 0 \quad \forall i \Rightarrow x = \vec{0}$$

$\underline{x = \vec{0} \Rightarrow \|x\|_2 = 0}$

$$x = \vec{0} \Rightarrow x_i = 0 \quad \forall i \Rightarrow \sqrt{\sum_{i=1}^{n} x_i^2} = 0 = \|x\|_2$$

b) $\|kx\|_2 = |k| \cdot \|x\|_2$

$$\|kx\|_2 = \sqrt{\sum_{i=1}^{n} (kx_i)^2} = \sqrt{k^2 \sum_{i=1}^{n} x_i^2}$$

$$= \sqrt{k^2}\sqrt{\sum_{i=1}^{n} x_i^2} = |k| \cdot \|x\|_2$$

() $\|X+Y\|_2 \leq \|X\|_2 + \|Y\|_2$

$$\|X+Y\|_2^2 = (X+Y)\cdot(X+Y) = \|X\|_2^2 + 2|X\cdot Y| + \|Y\|_2^2$$

By Cauchy-Schwartz

$$\leq \|X\|_2^2 + 2\|X\|_2\|Y\|_2 + \|Y\|_2^2 = (\|X\|_2 + \|Y\|_2)^2$$

If we take square root of both sides:

$$\|X+Y\|_2 \leq \|X\|_2 + \|Y\|_2$$

Question 4

4) i) $(A^T)^{-1} = (A^{-1})^T$

We have that $B^{-1}B = BB^{-1} = I$

and $(BC)^T = C^T B^T$

Also $BC = CB = I \Rightarrow C, B$ are inverses ⊛

$$(A^{-1}A)^T = (AA^{-1})^T = I^T = I$$

$$A^T(A^{-1})^T = (A^{-1})^T A^T = I$$

By ⊛, we have shown that $(A^{-1})^T$ is the inverse of A^T

In other words, that $(A^{-1})^T = (A^T)^{-1}$

ii) $A\bar{x} = b \Rightarrow A^T A \bar{x} = A^T b$

$$A\bar{x} = b \Rightarrow \bar{x} = A^{-1} b$$

$$A^T A \bar{x} = A^T A (A^{-1} b) = A^T b$$

This is the normal equation which we have proved using the unique solution $\bar{X}$

b) Let $\bar{x} \in \mathbb{R}^n$ s.t. $A^T A \bar{x} = A^T b$ (1)

and $x \in \mathbb{R}^n$, we show $\|A\bar{x} - b\| \le \|Ax - b\|$

$$\|Ax-b\|_2^2 = \|(A\bar{x} - b) + (Ax - A\bar{x})\|_2^2$$

$$= \|A\bar{x}-b\|_2^2 + \|Ax - A\bar{x}\|_2^2 + \underbrace{\langle A\bar{x}-b, Ax - A\bar{x}\rangle}$$

$$= \|A\bar{x}-b\|_2^2 + \|Ax - A\bar{x}\|_2^2$$

$= \langle A^T(A\bar{x}-b), x - \bar{x}\rangle$

$= 0$ b/c $A^T(A\bar{x}-b) = 0$ by (1)

Since the 2-norm is ≥ 0,

$$\|Ax-b\|_2^2 \ge \|A\bar{x} - b\|_2^2$$

$$\Rightarrow \|Ax-b\|_2 \ge \|A\bar{x}-b\|_2$$

Question 5

In [ ]:

```
from numpy import cos, sin, pi
from numpy.linalg import norm

# Define data
t = np.arange(0,1,1/6, dtype=np.float64)
y = np.array([0.,2.,0.,-1.,1.,1.])

### F3 Model

# Define relevant Matricies
A = np.column_stack((np.ones(6, dtype=np.float64), cos(2.*pi*t), sin(2.*pi*t)))

# Solve the normal equations
sol_1 = np.linalg.solve(A.T @ A, A.T @ y).round(3)

# Compute the 2-norm errors
pred_y = A @ sol_1
SSE_1 = norm(y - pred_y,2).round(3)

print("F3 Model")
print(f"c1: {sol_1[0]}, c2: {sol_1[1]}, c3: {sol_1[2]}")
print(f"SSE: {SSE_1}")
print("")

### F4 Model

# Define relevant Matricies
A = np.column_stack((np.ones(6, dtype=np.float64), cos(2.*pi*t), sin(2.*pi*t), cos(4.*pi

# Solve the normal equations
sol_2 = np.linalg.solve(A.T @ A, A.T @ y).round(3)
```

```python
# Compute the 2-norm errors
pred_y = A @ sol_2
SSE_2 = norm(y - pred_y,2).round(3)

print("F4 Model")
print(f"c1: {sol_2[0]}, c2: {sol_2[1]}, c3: {sol_2[2]}, c4: {sol_2[3]}")
print(f"SSE: {SSE_2}")
```

```
F3 Model
c1: 0.5, c2: 0.667, c3: -0.0
SSE: 2.041

F4 Model
c1: 0.5, c2: 0.667, c3: -0.0, c4: -1.0
SSE: 1.08
```

The more robust model with a fourth parameter has a significantly better fit to the data. Both models have 0 for c3 (the sin term).

Question 6

6) a) $S_i = y_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$

for $i = 1, \dots, n-1$

with the following properties

1) S passes through all points:

$$S_i(x_i) = y_i \quad \forall i = 1, \dots, n-1$$

and

$$S_{n-1}(x_n) = y_n$$

2) continuity

$$S_i(x_i) = y_i, \quad S_i(x_{i+1}) = y_{i+1} \quad \forall i = 1, \dots, n-1$$

3) First derivative continuous

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}) \quad \forall i = 1, \dots, n-2$$

4) Second derivative continuous

$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}) \quad \forall i = 1, \dots, n-2$$

For a natural spline, we also have

$$S_1''(x_1) = S_{n-1}''(x_n) = 0$$

b) For a not-a-knot spline, we also have

$d_1 = d_2$, $d_{n-2} = d_{n-1}$

By the 0°, 1°, and second order continuity conditions, this leads to

$s_1 = s_2$, $s_{n-2} = s_{n-1}$

c) True

d) True

e) False

Question 7

Solution

7. $X(t) = X_1 + bx \cdot t + cx \cdot t^2 + dx \cdot t^3$

$Y(t) = Y_1 + by \cdot t + cy \cdot t^2 + dy \cdot t^3$

we clearly see $X_1 = 3$, $Y_1 = 2$

Next, $bx = 3(X_2 - X_1)$

$4 = 3(X_2 - 3)$

$X_2 = \frac{4}{3} + 3 = \frac{13}{3}$

$by = 3(Y_2 - Y_1)$

$-1 = 3(Y_2 - 2)$

$Y_2 = -\frac{1}{3} + 2 = \frac{5}{3}$

Also, $cx = 3(X_3 - X_2) - bx$

$-1 = 3X_3 - 3 \cdot \frac{13}{3} - 4$

$X_3 = \frac{1}{3}(4 + 13 - 1) = \frac{16}{3}$

$cy = 3(Y_3 - Y_2) - by$

$1 = 3Y_3 - 3 \cdot \frac{5}{3} - (-1)$

$Y_3 = \frac{1}{3}(1 + 5 - 1) = \frac{5}{3}$

Finally $dx = X_4 - X_1 - bx - cx$

$X_4 = 2 + 3 + 4 + (-1)$

$= 8$

$dy = Y_4 - Y_1 - by - cy$

$Y_4 = 3 + 2 + (-1) + (1)$

$Y_4 = 5$

1st end point: $(3, 2)$

two control points: $\left(\frac{13}{3}, \frac{5}{3}\right), \left(\frac{16}{3}, \frac{5}{3}\right)$

Last end point: $(8, 5)$

Check Work

In [ ]:

```
from sympy import symbols, Eq, solve

# Define the symbols for the control points
P0x, P1x, P2x, P3x, P0y, P1y, P2y, P3y = symbols('P0x P1x P2x P3x P0y P1y P2y P3y')
```

```python
# Given endpoint P0 and P3
P0x_value = 3
P0y_value = 2

# Equations for x(t) based on the Bézier curve
eq1 = Eq(-3*P0x_value + 3*P1x, 4)   # Coefficient of t
eq2 = Eq(3*P0x_value - 6*P1x + 3*P2x, -1)  # Coefficient of t^2
eq3 = Eq(-P0x_value + 3*P1x - 3*P2x + P3x, 2)  # Coefficient of t^3

# Equations for y(t) based on the Bézier curve
eq4 = Eq(-3*P0y_value + 3*P1y, -1)   # Coefficient of t
eq5 = Eq(3*P0y_value - 6*P1y + 3*P2y, 1)  # Coefficient of t^2
eq6 = Eq(-P0y_value + 3*P1y - 3*P2y + P3y, 3)  # Coefficient of t^3

# Solve the equations to find the control points
solution = solve((eq1, eq2, eq3, eq4, eq5, eq6), (P1x, P2x, P3x, P1y, P2y, P3y))
print({'P0x': P0x_value, 'P0y': P0y_value}, solution)
```

```
{'P0x': 3, 'P0y': 2} {P1x: 13/3, P1y: 5/3, P2x: 16/3, P2y: 5/3, P3x: 8, P3y: 5}
```

Question 8

8. a) The least squares method gives the value of x that minimizes $\|Ax-b\|_2$:

$$\bar{x} = \min_x \|Ax-b\|_2$$

where $A \in \mathbb{R}^{(m \times n)}$, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$

In statistics, this means finding the x that best fits $Ax=b$.

If A is non-singular, we have the exact solution. Otherwise, x satisfies the normal equation:

$$A^T A x = A^T b$$

b) true

c) true