Math 104C Homework #3

Due on Apr 24 by 11:59 PM

Topics: (04/18 – 04/23) Least square - various models, QR - Gram-Schmidt

Video: How to submit homework on Gradescope or copy and paste (`https://youtu.be/quBWbQ5opT0`)

I. For presentation

1. (Exploration; Computation) The bloodstream concentration of a drug, measured hourly after administration, is given in the accompanying table. (a) Fit the model $y = c_1 t e^{c_2 t}$. (b) Find the estimated maximum concentration using the found model. (c) Use a root-finding method to estimate the half-life (See below for definition). (d) Suppose that the therapeutic range for the drug is $4 - 15$ng/ml. Use the equation solver of your choice to estimate the time the drug concentration stays within therapeutic levels.

hour	concentration (ng/ml)
1	6.2
2	9.5
3	12.3
4	13.9
5	14.6
6	13.5
7	13.3
8	12.7
9	12.4
10	11.9

Note: "Half-life in the context of medical science typically refers to the elimination half-life. The definition of elimination half-life is the length of time required for the concentration of a particular substance (typically a drug) to decrease to half of its starting dose in the body." (Reference: National Library of Medicine). Here, the starting dose means the maximum concentration.

2. (Exploration; Core) (a) Apply classical Gram-Schmidt process to find a orthogonal basis $\{q_i\}_{i=1}^3$ such that $\text{span}\{w_i\}_{i=1}^j = \text{span}\{q_i\}_{i=1}^j$ for $j = 1, 2, 3$, where $w_1 = (1, -1, 1, -1), w_2 = (1, 1, 3, -1), w_3 = (-3, 7, 1, 3)$. For this problem, you can skip normalization step. (This is meant for hand calculation.) (b) Check your answer (coming up with a way to check the answer is the main problem).

3. (Exploration; Computation) Apply classical Gram-Schmidt and modified Gram-Schmidt method to orthogonalize the 4×3 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \delta & \delta/2 & \delta/3 \\ \delta/2 & \delta/3 & \delta/4 \\ \delta/3 & \delta/4 & \delta/5 \end{bmatrix}$$

where $\delta = 10^{-10}$. Compare accuracy of the results by computing $Q^T Q$ for each method.

4. (Exploration; Computation) Compute the QR factorization and use it to solve the least squares problem.

$$\begin{bmatrix} 1 & 4 \\ -1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \end{bmatrix}$$

5. (Exploration; Analysis) Answer the following.
   (a) Prove that a square matrix is orthogonal if and only if its columns are pairwise orthogonal unit vectors.
   (b) Prove that the product of two orthogonal $m \times m$ matrices is again orthogonal.
   (c) Prove that every nonempty subspace of $\mathbb{R}^n$ has an orthonormal basis.

II. Not for presentation

The following problems are for your own study, but not for presentation. They have been already discussed during lectures.

Throughout the following problems, assume an over determined system $Ax = b$ with $A \in \mathbb{R}^{(m\times n)}$ $(m \geq n)$ full rank, $b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $\bar{x} \in \mathbb{R}^n$ is the least square solution, and $\hat{Q}\hat{R}$ and QR are a reduced and full QR factorization of A respectively unless otherwise mentioned.

6. (Formation) Answer the following. And give a **brief explanation** for each subproblem.
   (a) (True/False) Data linearization for nonlinear data fitting always introduces unfavorable bias since the distance to minimize is distorted.
   (b) (True/False) QR factorization encodes orthogonalization of columns of A.
   (c) (True/False) $\hat{R}$ has the same size as A.
   (d) (True/False) R has the same size as A.

7. (Formation) Answer the following.
   (a) List two ways we learned so far to treat nonlinear data fitting. Also, give one- or two-sentence outline for each method. Remember that we didn't learn nonlinear least square method.
   (b) Write out the Gram-Schmidt orthogonalization with three linearly independent vectors $w_i \in \mathbb{R}^n$ $(i = 1, 2, 3)$. That is, give the expressions for $q_i \in \mathbb{R}^n$ $(i = 1, 2, 3)$, where q_i's are pair-wisely orthogonal unit vectors and $\text{span}\{w_i\}_{i=1}^{j} = \text{span}\{q_i\}_{i=1}^{j}$ for $j = 1, 2, 3$.
   (c) Summarize how to obtain least square solution to $Ax = b$ given the factorization $A = QR$ computed.

End of homework