# Math 104C Homework #7

**Due on May 15 by 11:59 PM**

**Topics**: (05/08 – 05/15) DFT, FFT

Video: How to submit homework on Gradescope or copy and paste (`https://youtu.be/quBWbQ5opT0`)

## I. For presentation

1. (Exploration; Analysis) Let $F_n$ be the $n$-by-$n$ Fourier matrix, where $\omega = e^{-2\pi i/n}$. Prove (a) $F_n^H F_n = I$, (b) $F_n^T = F_n$, and (c) $F_n^{-1} = \overline{F_n}$.

$$F_n = \frac{1}{\sqrt{n}} \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \cdots & \omega^{n-1} \\ \omega^0 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \omega^0 & \omega^3 & \omega^6 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}.$$

2. (Exploration; Core & computation) (a) Write a code that computes DFT of the vectors $u = [3/4, 1/4, -1/4, 1/4]$ and $v = [1, 0, -1/2, 0, 1, 0, -1/2, 0]$. Do not use package for this problem. You will practice core knowledge via this problem. When you construct Fourier matrix, do it as efficient as possible using **vectorized coding**. In Python, `1j` is the keyword for $\sqrt{-1}$ in Python. (b) From the result of (a), verify the theorem that the DFT $y$ of a real $n$-vector $x$ satisfies (i) $y_0$ is real and (ii) $y_{n-k} = \bar{y}_k$ for $k = 1, \ldots, n-1$. (c) Write a code (function) that computes the inverse DFT of the result of (a); this should consist of just several lines using special property of Fourier matrix. Also compare the result of IDFT of DFT of $u$ or $v$ with the original vector. For this HW, you will need to use `numpy.real(y)`, if `y` is the DFT of $u$ or $v$; because of rounding errors, the result is likely to be complex even though it must be real in theory.

3. Devise a version of the following theorem that applies to odd number $n$.

   ($n$-even version)

   For an even integer $n$, let $t_j = c + j(d-c)/n$ for $j = 0, \ldots, n-1$, and let $x = (x_0, \ldots, x_{n-1})$ denote a vector of $n$ real numbers. Define $\vec{a} + \vec{b}i = F_n x$, where $F_n$ is the Discrete Fourier Transform. Then the function

$$P_n(t) = \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{n/2-1} \left( a_k \cos\frac{2k\pi(t-c)}{d-c} - b_k \sin\frac{2k\pi(t-c)}{d-c} \right) + \frac{a_{n/2}}{\sqrt{n}} \cos\frac{n\pi(t-c)}{d-c}$$

   satisfies $P_n\left(t_j\right) = x_j$ for $j = 0, \ldots, n-1$.

4. Give complete details of establishing the theorem on trigonometric interpolation for real data. In the notes, I provided only some verbal instructions.

   (a) Start with trigonometric interpolation of complex data without proof: Given an interval $[c, d]$ and positive integer $n$, let $t_j = c + j(d-c)/n$ for $j = 0, \ldots, n-1$, and let $x = (x_0, \ldots, x_{n-1})$ denote a vector of $n$ numbers. Define $\vec{a} + \vec{b}i = F_n x$, where $F_n$ is the Discrete Fourier Transform matrix. Then the complex function

$$Q(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (a_k + ib_k)\, e^{i2\pi k(t-c)/(d-c)}$$

satisfies $Q\left(t_j\right) = x_j$ for $j = 0, \dots, n-1$.

Prove the second part about real data: Furthermore, if the $x_j$ are real, the real function

$$P(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left( a_k \cos \frac{2\pi k(t-c)}{d-c} - b_k \sin \frac{2\pi k(t-c)}{d-c} \right)$$

satisfies $P\left(t_j\right) = x_j$ for $j = 0, \dots, n-1$.

(b) Prove the trigonometric identity: For $t = j/n$ and $j \in Z$, we have

$$\cos 2(n-k)\pi t = \cos 2k\pi t \quad \text{and} \quad \sin 2(n-k)\pi t = -\sin 2k\pi t.$$

(c) Prove the trigonometric interpolation for real data: For an even integer $n$, let $t_j = c + j(d-c)/n$ for $j = 0, \dots, n-1$, and let $x = (x_0, \dots, x_{n-1})$ denote a vector of $n$ real numbers. Define $\vec{a} + \vec{b}i = F_n x$, where $F_n$ is the Discrete Fourier Transform. Then the function

$$\begin{aligned} P_n(t) = & \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{n/2-1} \left( a_k \cos \frac{2k\pi(t-c)}{d-c} - b_k \sin \frac{2k\pi(t-c)}{d-c} \right) \\ & + \frac{a_{n/2}}{\sqrt{n}} \cos \frac{n\pi(t-c)}{d-c} \end{aligned}$$

satisfies $P_n\left(t_j\right) = x_j$ for $j = 0, \dots, n-1$.

## II. Not for presentation

The following problems are for your own study, but not for presentation. They have been already discussed during lectures.

5. Prove the following. Let $\{y_k\}$ be the DFT of $\left\{x_j\right\}$, where the $x_j$ are real numbers. Then,

   (a) $y_0$ is real, and

   (b) $y_{n-k} = \bar{y}_k$ for $k = 1, \dots, n-1$.

   (c) If $n$ is even, $y_{n/2}$ is also real.

6. (Formation) Answer the following. DFT($n$) and FFT($n$) mean the Discrete Fourier Transform and the Fast Fourier Transform of dimension $n$ and $n = 2^m$.

   (a) (True/False) The main trick of FFT is to reduce DFT($n$) to DFT($n/2$).

   (b) Give the complexity of DFT($n$) and FFT($n$) in big 'O' notation.

(c) Given a real data $x = [1, 2, 3, \cdots, 10]$, consider its DFT $y$. Give all the entries of $y$ that are definitely real.

(d) (True/False) DFT of a vector is a complex vector.

(e) (True/False) DFT of a real vector is a complex vector.

(f) (True/False) IDFT(DFT($x$)) is a real vector if $x$ is a real vector.

(g) (True/False) In machine computation of IDFT(DFT($x$)) is a real vector if $x$ is a real vector.

(h) (True/False) When you want to exploit the power of FFT, it is practical to choose evenly space $p$ nodes, where $p$ is a prime number.

End of homework