Math 104C Homework #6

Due on May 8 by 11:59 PM

Topics: (05/01 – 05/08) GMRES, preliminary of trigonometric interpolation

Video: How to submit homework on Gradescope or copy and paste (`https://youtu.be/quBWbQ5opT0`)

I. For presentation

1. (Exploration; Analysis) Let $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. Define the *modulus* of a complex number $z = a + b\mathrm{i}$ by $|z| := \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$, where $\bar{z} := a - b\mathrm{i}$ represents complex conjugate. And the Euler formula is given by $e^{i\theta} := \cos(\theta) + \mathrm{i}\sin(\theta)$, where $\theta \in \mathbb{R}$.

   (a) Prove that $|zw| = |z||w|$ for any $z, w \in \mathbb{C}$.

   (b) Prove that $|z| = |\bar{z}|$ for any $z \in \mathbb{C}$.

   (c) Show that if $|z| = 1$, then $z^{-1} = \bar{z}$. In particular, $e^{-\theta\mathrm{i}} = \overline{e^{\theta\mathrm{i}}}$.

   (d) For any $\alpha, \beta \in \mathbb{R}$, prove that $e^{\mathrm{i}(\alpha+\beta)} = e^{\mathrm{i}\alpha}e^{\mathrm{i}\beta}$.

2. (Exploration; Analysis) Prove that the complex dot-product $(v, u) = \sum_{j=1}^{n} u_j\overline{v_j}$, where $u, v \in C^n$ is an inner product. That is, for $u, v, w \in C^n$ and $a, b \in C$, (a) $(v, u) = \overline{(u, v)}$, (b) $(au+bv, w) = a(u, w)+b(v, w)$, and (c) $(u, u) \geq 0$ and $(u, u) = 0$ if and only if $u = \vec{0}$. Also, show that (d) $(u, av + bw) = \overline{a}(u, v) + \overline{b}(u, w)$ and (e) $(u + v, u + v) = (u, u) + 2\mathrm{Re}[(u, v)] + (v, v)$ as a consequence of the previous properties, where $\mathrm{Re}[z]$ is the real part of z.

3. (Exploration; Analysis) Prove the following.

   Let n be a positive integer. A complex number ω satisfying $\omega^n = 1$ is called a *root of unity*. If, in addition, $\omega^k \neq 1$ for $1 \leq k < n$, then it is called *primitive root of unity*.

   Let ω be a primitive n-th root of unity, and k be an integer. Then

$$\sum_{j=0}^{n-1} \omega^{jk} = \begin{cases} n & \text{if } k/n \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}.$$

   In particular, $\omega = e^{\pm 2\pi\mathrm{i}/n}$ satisfies this.

II. Not for presentation

The following problems are for your own study, but not for presentation. They have been already discussed during lectures.

4. (Formation) Answer the following.

   (a) What problem does the GMRES answer?

   (b) Give the definition of the Krylov space, including the settings.

   (c) Give two main reasons why restarting GMRES is desired.

   (d) What can we say about 2-norm of residual as GMRES proceeds?

(e) Summarize conditioning of methods for least square problems: via normal equation, QR - Gram-Schmidt, and QR - Householder.

(f) Explain why we take the opposite sign of the first entry of the input vector when we construct Householder reflector.

(g) (True/False) QR factorization via Householder reflector does not encode orthogonalization because it has nothing to do with Gram-Schmidt orthogonalization, but only some reflections.

(h) Suppose that you are conducting QR factorization of 10-by-7 matrix A. In the fourth iteration, where you construct a Householder reflector $\hat{H}_4$ in a smaller dimension and embed it to H_4 in the full dimension, what is the length of the input vector of $\hat{H}_4$? Also, give the expression of that vector using A and the full dimension Householder reflectors H_1, H_2, and H_3.

End of homework