Homework 2

1. (1 point each part) For each of the following constant-coefficient systems $y' = Ay$, determine if the system is stable, asymptotically stable, or unstable:

   (a)
$$A = \begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}$$

   (b)
$$A = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

   (c)
$$A = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}$$

   (d)
$$A = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}$$

2. (5 points) Consider the method of lines applied to the diffusion equation in one space dimension, $u_t = au_{xx}$, with $a > 0$ a constant, $u = 0$ at $x = 0$, $x = 1$ for $t \geq 0$, and with given initial values. Formulate the method of lines using the central difference approximation to the derivative u_{xx}, as in Example 1.3 of the text (as we did in Lecture 1), to arrive at a linear constant coefficient ODE system $y' = Ay$ with A symmetric. Find the eigenvalues of A to determine whether the problem is unstable, stable or asymptotically stable. (Hint: Try eigenvector $\mathbf{v}^k$ in the form $v_i^k = \sin(ik\pi\Delta x)$, $i = 1, \ldots, m$, for $1 \leq k \leq m$, and use the sum of angles formula for sin).

3. (5 points) Consider the method of lines applied to the advection equation in one space dimension, $u_t + au_x = 0$, with $a > 0$ a constant, on the spatial domain $0 \leq x \leq 1$, with boundary condition $u = 0$ at $x = 0$,

and with given initial values. Formulate the method of lines using the first order backward difference approximation to the derivative u_x, (i.e. $u_x(x_j) \approx \frac{u(x_j)-u(x_{j-1})}{\Delta x}$), to arrive at a linear constant coefficient ODE system $y' = Ay$. Find the eigenvalues of A to determine whether the problem is unstable, stable or asymptotically stable. Next, consider approximating the derivative u_x by the first order forward difference approximation (see the Scholarpedia MOL article for the definition of the forward difference approximation). For this, use the boundary condition $u = 0$ at $x = 1$. Find the eigenvalues of A to determine whether the problem is unstable, stable or asymptotically stable.

4. (6 points) Write a well-structured MATLAB code implementing the explicit Euler method for systems of standard-form ODEs. Apply the code to the following problem. (Note that the system is uncoupled, but you are to treat it in your code as if it were coupled.) Turn in your code, plots and explanation on Canvas.

$$\begin{aligned} y_1' &= -y_1 \\ y_2' &= -100(y_2 - \sin(t)) + \cos(t) \end{aligned}$$

for $0 \le t \le 1$, with initial value $y_1 = 1$, $y_2 = 2$. Try this for 2 sequences of stepsizes:

- $h = .01$
- $h = .05$

Plot the solutions obtained. Explain your results.