Constructing the Wigner function in phase space for simple arbitrary quantum systems using the trace of a displaced parity operator

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Abstract

On the quantum scale there are two meaningful physical observables of any state known as position and momentum. These two properties of all quantum states do not commute and hence measurements cannot be made of both. This has lead for the need of a map of these two variables in a expanse known as the phase space. Since the Creation of the Wigner function in phase space, many physicists have taken to this as their choice of representation of a quantum state. In order to compute the Wigner function, the traditional integral method is the usual approach. Here we look to compute the functions by use of a displaced parity operator, and provide examples of how to exercise such a method. Our examples include the computation of the coherent state, A cat state created by the superposition of two coherent states, and a method of creating Wigner functions for simple harmonic oscillator states.

Introduction

The Wigner function has been under investigation by Physicists since it was first proposed by Eugene Wigner in 1932, for which his derivation can be seen in [1]. This function displays the quantum state in phase space, and is extremely useful due to its visual intuitiveness. As an example of how powerful it is, it can be seen as the choice of representation in one of the most ground breaking pieces of work in recent decades [2]. From [3], we know the function is a quasi-probability density due to the fact that it can be negative in places, which is actually representing the quantum interference from measurement of the state. The phase space is a powerful system that can be used to represent any quantum state. In [4],it is stated that the phase space is a collection of position and momentum points, for which each particle in three dimensions has three position and as many momentum co-ordinates. However, it is well known that beyond the classical limit, position and momentum do not commute, so individual points cannot be measured. This alone is an adequate reason to warrant the application of the Wigner function, as it displays a map of all possible positions and momentum values of the state.

An example of a Wigner function can be seen in Fig.1 and was obtained from [5].We learn from [6] that the state shown is classified as a cat state due to it being a superposition of two coherent states, and simply because Fig.1 is comparable to the Schrdinger cat state (peak at dead and alive). This will be solved later on in this paper.

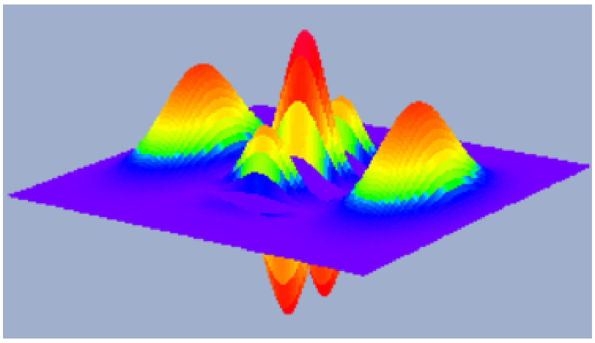

Figure 1: *Picture of the Wigner function in phase space taken from [5].This is known as a cat state due to the two distinct states, and the values separating them is known as quantum interference.*

Traditionally, the Wigner function (which is dependant on position and momentum), is given as an integral of a transformed density matrix $\hat{\rho}$, combined with a Fourier transform. An example of this form can be seen in [7], where we write it in an equivalent form,

$$W(q,p) = \left(\frac{1}{2\pi\hbar}\right)^n \int_{-\infty}^{+\infty} dz \left\langle q - \frac{z}{2} \middle| \hat{\rho} \middle| \frac{z}{2} + q \right\rangle e^{\frac{ip\cdot z}{\hbar}} \quad (1)$$

A good description of this comes from [3], which tells us that the bra-ket segment denotes the quantum jump from $|q - \frac{z}{2}\rangle$ to the point $|\frac{z}{2} + q\rangle$ in space. In order to incorporate momentum dependence, a Fourier transform on $\hat{\rho}$ over distance z is applied.

for which the definition of the symbols used is best given by [8], in which it states that "where $q = [q_1, q_2, ..., q_n]$ and $p = [p_1, p_2, ..., p_n]$ are n-dimensional vectors representing the classical phase-space position and momentum values, $z = [z_1, z_2, ..., z_n]$,$\hbar$ is Plancks constant".

As the Wigner function is a quasi-probability distribution, we have normalisation in place so that our functions, when formed, can be interpreted as probability distributions, therefore

$$\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dp W(q,p) = Tr[\hat{\rho}] = 1 \quad (2)$$

In order to use this form of the Wigner function, we simply have to substitute in the appropriate density matrix and directly evaluate the integral, which can become quite complex. Instead we look to a different method given by [9], in which we construct a Wigner distribution using the trace of a displaced parity operator. [8] shows that this method can be generalised and used to construct the Wigner function for any quantum state.

The Method

To construct Wigner functions, we look to utilise the fact that the function can be represented by the expectation value of a displaced parity operator. This is a well known method and used by many scientists [10], and is more useful than the integral method as it can be solved for any quantum state. This is most comprehensibly shown in [11] where the function is written as,

$$\begin{aligned} W(q,p) &= \langle \hat{D}^{\dagger}(q,p)\Pi\hat{D}(q,p)\rangle_{\rho} \\ &= Tr[\rho\{\hat{D}^{\dagger}(q,p)\Pi\hat{D}(q,p)\}] \quad (3) \end{aligned}$$

where the displacement operator $\hat{D}$, is well defined in [12]. This source tells us that the operators role is to shift a state by given displacement, i.e $\hat{D}(x)|\psi\rangle = |\psi + x\rangle$.It can also be seen from [13] that when dependant on position and momentum, it can be written in the exponential form $\hat{D}(q,p) = e^{i(p\hat{q}-q\hat{p})}$. The source also shows that the operator can be written in terms of coherent states, so that $\hat{D}(\alpha) = e^{i(\alpha\hat{a^{\dagger}}-\alpha^{\star}\hat{a})} = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a^{\dagger}}}e^{-\alpha^{\star}\hat{a^{\dagger}}}$ where $\hat{a^{\dagger}}$ and $\hat{a}$ creation and annihilation operators respectively. The parity operator Π, is best explained by J. Mimih in [14] where he shows that the operator makes all co-ordinates undergo a change in sign. So, mathematically for a state $|\psi\rangle$, the operation is simply $\Pi|\psi\rangle = |-\psi\rangle$. In this paper we will only concern ourselves with solving systems which are known as pure states, Except for the cat states which is a mixture of two coherent states and has a straight forward density matrix which will be seen in the cat state section. The pure states have a density matrix which is given below.

$$\hat{\rho} = |\psi\rangle\langle\psi| \quad (4)$$

Here we can implement $\hat{\rho}$ into the expectation value form of the Wigner function shown in (3), and with further computation we will be able to create the functions for the state. Alternatively, we can use the trace of a displaced parity operator also shown in (3), in order to get exponential functions for our arbitrary systems. It is well known that the trace of an outer product of two vectors is just the inner product of both vectors, which is mathematically equivalent to the exponential shown,

$$\begin{aligned} Tr[|A\rangle\langle B|] &= \langle A|B\rangle \\ &= e^{-\frac{1}{2}(|B|^2+|A|^2-2B^{\star}A)} \quad (5) \end{aligned}$$

This exponential is analogous to the final form of a Wigner function, allowing us to use it to directly compute distributions in phase space with pure states. Using rules of superposition, we can also find the Wigner function for mixed states, as seen in [15].

It is important to now mention that Trace is invariant under cyclic permutations. This implies to us that all operators inside the trace commute with each other, allowing us to rearrange them, making the function easy to calculate.

The Coherent State

The coherent state was first discovered in 1926 by Schrdinger where his work can be seen in [16]. Only in 1963 was it given a full theoretical description by R. Glauber, which is why a common alias for the coherent state is the Glauber state. From his work in [17], we learn that the coherent state is the most classical harmonic oscillator state as the uncertainty is minimised. In this article he also gives three mathematical definitions, however the one which is the most relevant for finding Wigner functions using a displaced parity operator, is

$$|\beta\rangle = D(\hat{\beta})|0\rangle \tag{6}$$

where the coherent state $|\beta\rangle$ is defined as

$$|\beta\rangle = e^{-\frac{|-\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle \tag{7}$$

Where β is any complex number.

In words, equation (6) essentially means that the coherent state is just a vacuum state displaced by an eigenstate β. The methods used in this paper are based on lectures given by M. Everitt at Loughborough university. As the state $|\phi\rangle$ is given by

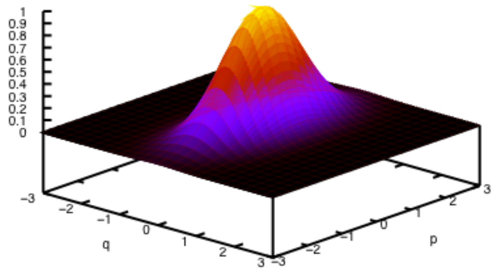

Figure 2: *Wigner function of the coherent state $|\beta\rangle$ in two dimensional phase space, taken from [18].This state has the least uncertainty making it the most classical of all quantum states.*

the complex number $|\beta\rangle$, we can now write the density matrix as

$$\hat{\rho} = |\beta\rangle\langle\beta| \tag{8}$$

Which will then allow us to give an trace of the displaced parity operator with respect to the coherent state

$$W(\alpha) = Tr[\Pi \hat{D}^{\dagger}(\alpha)|\beta\rangle\langle\beta|\hat{D}(\alpha)] \tag{9}$$

If we then act on the state $|\beta\rangle$ and its respective ket with the displacement operators, we see that the above equation becomes

$$W(\alpha) = Tr[|-(\beta-\alpha)\rangle\langle\beta-\alpha|] \tag{10}$$

Here we have also acted on $|\beta-\alpha\rangle$ with the operator Π. By using the relation given in (5) we can see that the above Wigner function becomes

$$W(\alpha) = e^{-2|\beta-\alpha|^2} \tag{11}$$

This above expression can then be used to plot the Wigner function for a coherent state in phase space, for which visualisations can be seen in [2] [15]. A Wigner function for a coherent state can also be seen in Fig.2 which was taken from [18]. We can clearly see that this state is a Gaussian distribution with an elegant symmetry to it. The coherent state is an extremely powerful model for physicists of all fields. Not only is the coherent state a displaced vacuum state $|0\rangle$, it can also show the behaviour of a cooper pair of electrons in superconductivity, for which details were found from [19].

The Cat State

To reiterate from earlier, we know from [6] that the cat state is a superposition of two coherent states, making it a mixed state. To create this, we use the coherent state $|\beta\rangle$ and add it to the state $|-\beta\rangle$. The overall mixed state can be given by

$$|\psi\rangle = |\beta\rangle + |-\beta\rangle \tag{12}$$

This then means that the density matrix for the cat state is denoted by $\hat{\rho} = (|\beta\rangle + |-\beta\rangle)(\langle\beta| + \langle-\beta|)$ and upon expanding this expression, we see that our matrix is

$$\begin{aligned}\hat{\rho} = |\beta\rangle\langle\beta| + |\beta\rangle\langle-\beta| \\ + |-\beta\rangle\langle\beta| + |-\beta\rangle\langle-\beta|\end{aligned} \tag{13}$$

Using the principle of superposition, we calculate the Wigner function for each term, and then simply add them together to get the distribution for the cat state. We already know the Wigner function for the term $|\beta\rangle\langle\beta|$ from our calculation of the coherent state, so our first step will be to calculate the function for $|-\beta\rangle\langle-\beta|$, where we

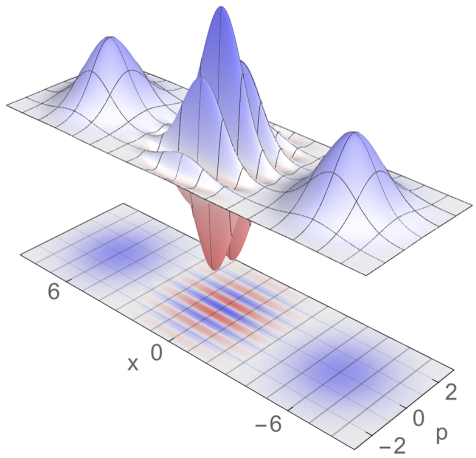

Figure 3: *The Wigner function of two coherent states $|\beta\rangle$ and $|-\beta\rangle$, super imposed on each other in two dimensional phase space, taken from [20]. Their are some oscillations about the origin which visualise the decoherence between the two states.*

see that

$$\begin{aligned}W_{|-\beta\rangle\langle-\beta|}(\alpha) &= Tr[|(\beta+\alpha)\rangle\langle-(\beta+\alpha)|] \\ &= e^{-2|\beta+\alpha|^2}\end{aligned} \tag{14}$$

For our asymmetric terms, our Wigner functions are not quite as graceful when compared to previous terms. For a value of $|\beta\rangle\langle-\beta|$ for ρ, the Wigner function is given by

$$\begin{aligned}&W_{|\beta\rangle\langle-\beta|}(\alpha) \\ &\quad = Tr[|-(\beta-\alpha)\rangle\langle-(\beta+\alpha)|] \\ &\quad = e^{-\frac{1}{2}(|\beta+\alpha|^2+|\beta-\alpha|^2-2(\beta-\alpha)^\star(\beta+\alpha))}\end{aligned}$$

and for the term with opposite signs $|-\beta\rangle\langle\beta|$, we see that the function is

$$\begin{aligned}&W_{|-\beta\rangle\langle\beta|}(\alpha) \\ &\quad = e^{-\frac{1}{2}(|\beta+\alpha|^2+|\beta-\alpha|^2-2(\beta-\alpha)(\beta+\alpha)^\star)}\end{aligned}$$

Having found the Wigner function for each term in the definition of ρ, we can now use our superposition rule

$$W_{cat}(\alpha) = W_{|\beta\rangle\langle\beta|}(\alpha) + W_{|-\beta\rangle\langle-\beta|}(\alpha) + W_{|\beta\rangle\langle-\beta|}(\alpha) + W_{|-\beta\rangle\langle\beta|} \quad (15)$$

to give us our complete function for the cat state, W_{cat}. So, concatenating all our terms into one expression, we find,

$$W_{cat}(\alpha) = e^{-2|\beta-\alpha|^2} + e^{-2|\beta+\alpha|^2} + e^{-\frac{1}{2}(|\beta-\alpha|^2+|\beta+\alpha|^2)} (e^{-(\beta+\alpha)(\beta-\alpha)^*} + e^{-(\beta+\alpha)^*(\beta-\alpha)})$$

Using what we have just calculated, we can create plots similar to Fig.3 which was taken from [20]. Being able to picture the cat state in this way allows us to visualise significant quantum phenomena, such as the decoherence which can be seen between both Gaussian curves. The true significance of the cat state is outlined in [21] where it mentions several applications of cat states, this includes phenomena such as fault tolerant quantum computing and open destination teleportation.

To calculate the Wigner function for this case and the coherent state, we have only used linear algebra, and therefore no integration has been carried out. This method is quick and easy for these simple cases especially the coherent state, and also allows one to calculate the Wigner functions without knowing the wave function of the state, which is not the case with the traditional integral method.

Simple Harmonic Oscillator

The simple harmonic oscillator (SHO), is one of the most important models in quantum mechanics. To approve this we read from [22] which states almost all oscillations in any system can be approximated to the simple harmonic oscillator if the amplitude of oscillations is small.

To calculate the Wigner function for the SHO, we adopt a slightly different approach where we instead view the Wigner function as the expectation value of a displaced parity operator, which is shown in (5). It is also convenient to now use a shorthand term known to physicists as the kernel which is defined as $\Delta(\alpha) = \hat{D}^\dagger(\alpha)\Pi\hat{D}(\alpha)$, also our calculations are made in terms of the number state $|n\rangle$ which is an orthogonal basis defined as $|n\rangle = |0, 1, 2...\rangle$. For the rest of the paper, all sums are for zero to infinity unless stated otherwise. To commence, we define our state $|\psi\rangle$, as

$$|\psi\rangle = \sum_n c_n|n\rangle \quad (16)$$

noticing that if we pre-multiply by another number $\langle m|$, we find

$$\langle m|\psi\rangle = \sum_n c_n\langle m|n\rangle = c_m$$
$$\delta_{mn} = \langle m|n\rangle \quad (17)$$

Next, We introduce the identity matrix, $I_n = \sum_n |n\rangle\langle n|$, and introduce two of them into the Wigner function such that

$$\begin{aligned} W(\alpha) &= \langle\psi|I_n\Delta(\alpha)I_m|\psi\rangle \\ &= \langle\psi|\sum_n|n\rangle\langle n|\Delta(\alpha)\sum_m|m\rangle\langle m||\psi\rangle \\ &= \sum_{nm}\langle\psi|n\rangle[\langle n|\Delta(\alpha)|m\rangle]\langle m|\psi\rangle \\ &= \sum_{nm}c_n^\star c_m\Delta(\alpha)_{mn} \quad (18) \end{aligned}$$

where $\langle n|\Delta(\alpha)|m\rangle = \Delta_{mn}(\alpha)$. Here it is now appropriate to evaluate the kernel. Using our definition for the displacement operator with respect to coherent states, $D\hat{(\alpha)}$ and the series form of the parity operator which can be found in, [14] to be $\Pi = \sum_k(-1)^k|k\rangle\langle k|$, we find that the kernel is

$$\begin{aligned} \Delta_{mn}(\alpha) &= \langle n|\hat{D}^\dagger(\alpha)\Pi\hat{D}(\alpha)|m\rangle \\ &= \langle n|e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a^\dagger}}e^{-\alpha^\star\hat{a^\dagger}} \\ \sum_k(-1)^k|k\rangle\langle e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a^\dagger}}e^{-\alpha^\star\hat{a^\dagger}}|m\rangle \quad (19) \end{aligned}$$

Which can be rearranged and simplified to find

$$\begin{aligned} \Delta_{mn}(\alpha) = e^{-\frac{1}{2}|\alpha|^2}\sum_k(-1)^k\langle n|e^{\alpha\hat{a^\dagger}}e^{-\alpha^\star\hat{a}}|k\rangle \\ \langle k|e^{-\alpha\hat{a^\dagger}}e^{\alpha^\star\hat{a}}|m\rangle \quad (20) \end{aligned}$$

Noticing that the displaced number states $e^{\alpha^\star\hat{a}}|m\rangle$ and $e^{-\alpha^\star\hat{a}}|k\rangle$ can be denoted as new states, say Φ_m and Ψ_k respectively, we see that (20) becomes

$$\begin{aligned} \Delta_{mn}(\alpha) &= e^{-\frac{1}{2}|\alpha|^2} \\ &\sum_k(-1)^k\langle\Phi_n|\Psi_k\rangle\langle\Psi_k|\Phi_m\rangle \quad (21) \end{aligned}$$

In order to calculate this, we just need to compute the results for one $|\Phi\rangle$ state and one $|\Psi\rangle$ state as one inner product is just the complex conjugate of the other. So to begin we will evaluate $|\Phi_m\rangle$, where we use the power series form of the exponential given in [23], as $e^x = \sum_i \frac{x^i}{i!}$ allowing us to define the state by

$$\begin{aligned} |\Phi_m\rangle = e^{\alpha^\star} &= \sum_i\frac{(\alpha^\star\hat{a})^i}{i!}|m\rangle \\ &= \sum_i\frac{\alpha^{\star i}}{i!}\hat{a}^i|m\rangle \quad (22) \end{aligned}$$

We can see from [24] that the operation $\hat{a}$ has on our state is $\hat{a}|m\rangle = \sqrt{m}|m-1\rangle$, so if we evaluate $\hat{a^i}|m\rangle$ by inspection,

$$\begin{aligned} \hat{a}|m\rangle &= \sqrt{m}|m-1\rangle \\ \hat{a^2}|m\rangle &= \sqrt{m}\sqrt{m-1}|m-2\rangle \\ \hat{a^3}|m\rangle &= \sqrt{m}\sqrt{m-1}\sqrt{m-2}|m-3\rangle \end{aligned}$$

we find that

$$\begin{aligned} \hat{a^i}|m\rangle &= \sqrt{\frac{i!}{(m-i)!}}|m-i\rangle \\ \therefore |\Phi_m\rangle &= \sum_{i=0}^m\frac{\alpha^{\star,i}}{i!}\sqrt{\frac{i!}{(m-i)!}}|m-i\rangle \quad (23) \end{aligned}$$

Here we introduce a new term j, where $j = m-i$, and pre-multiply by the term $\langle j|$ to give

$$\langle |j|\Phi_m\rangle = \sum_{j=0}^{m} \frac{\alpha^{\star,m-j}}{m-j!}\sqrt{\frac{m-j!}{(j)!}}\langle j|m-i\rangle$$
$$= \frac{\alpha^{\star,m-j}}{m-j!}\sqrt{\frac{m!}{(j)!}}$$

$$\langle j|m-i\rangle = \delta_{j,m-i} \quad (24)$$

In order to finish computing the Wigner function for the SHO states, all that is left to do is replicate the following method for $\langle\Psi|$, and then input (24) and the solution to $\langle\Psi|$ into an array and then find the inner product in the form

$$\langle\Psi_k|\Phi_m\rangle = [\Psi_0^\star, \Psi_1^\star, ...][\Phi_0^\star, \Phi_1^\star, ...]^T \quad (25)$$

As we know our other inner product is just the complex conjugate, we can now compute the kernel,

$$\Delta_{nm}(\alpha) = e^{-|\alpha|^2}\sum_k(\Phi_n * \Psi_k)(\Psi_k * \Phi_m) \quad (26)$$

and hence the Wigner function

$$W(\alpha) = \sum_{nm} c_n^\star c_m e^{-|\alpha|^2} \sum_k(\Phi_n * \Psi_k)(\Psi_k * \Phi_m) \quad (27)$$

This method may seem prolonged, nevertheless it is one with simple maths to carry out as there are no untidy integrals to evaluate. Furthermore it can also be done without knowing the wave function of the state. However, as there is need for construction of an array on a computer or similar type of device, many may have the integral method as their preference as it can be carried out using just pen and paper.

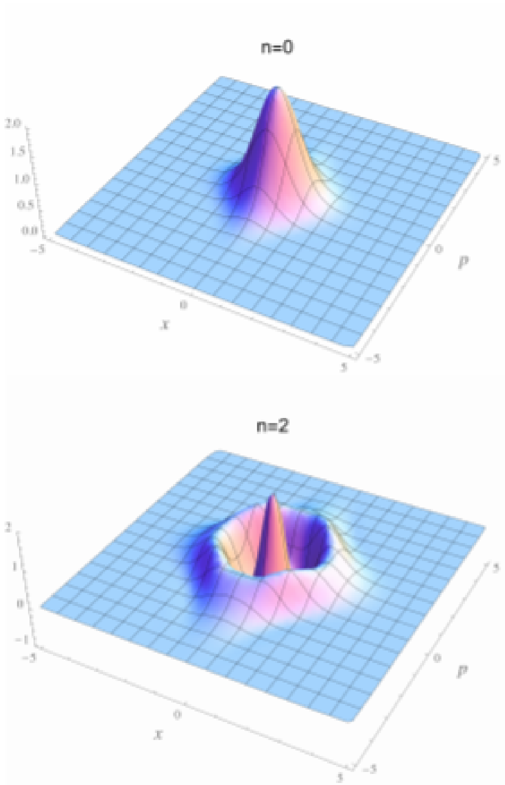

Figure 4: *Wigner functions for the vacuum state $|0\rangle$ and the second excited state $|2\rangle$ in two dimensional phase space as seen in [25]. The vacuum state is identical in shape to the coherent state however it has no displacement. The second excited state is an even waveform so the parity has no effect on the state.*

Wigner functions for the vacuum state $|0\rangle$ and the second excited state $|2\rangle$ of the SHO can be seen in 4 which were taken from [25].

Conclusion

We have shown that our method for producing Wigner functions using the expectation value of a displaced parity operator can produce distributions in phase space for arbitrary quantum systems. This approach can be extended to all quantum systems by adopting methods shown in this review. The Wigner function has also been shown to display quantum phenomena like decoherence and superposition of states in a phase space. Furthermore, there are benefits to using this method over the integral method if the physicist wishing to create Wigner functions is adept with any coding software. It can be a more proficient and sometimes a potentially less stressful approach to creating these functions, but if the physicist excels at computing integrals, the traditional method can be a faster way to compute the Wigner function.

References

[1] E. Wigner, "On the quantum correction for thermodynamic equilibrium," *Phys. Rev.*, vol. 40, pp. 749–759, Jun 1932.

[2] S. Deleglise, I. Dotsenko, C. Sayrin, J. Bernu, M. Brune, J.-M. Raimond, and S. Haroche, "Reconstruction of non-classical cavity field states with snapshots of their decoherence," *nat*, vol. 455, pp. 510–514, Sept. 2008.

[3] W. Schleich, *Quantum Optics in Phase Space*. Wiley, 2015.

[4] T. Tao, "Phase space," 2016.

[5] Q. Coherence and I. T. Group, "http://www.physics.umd.edu/qcoh/," 24/03/2017.

[6] H. Jeong and T. C. Ralph, "Schrodinger cat states for quantum information processing," *eprint arXiv:quant-ph/0509137*, sep 2005.

[7] J. Brogaard, "Wigner function formalism in quantum mechanics."

[8] T. Tilma, M. J. Everitt, J. H. Samson, W. J. Munro, and K. Nemoto, "Wigner functions for arbitrary quantum systems," *Phys. Rev. Lett.*, vol. 117, p. 180401, Oct 2016.

[9] A. Royer, "Wigner function as the expectation value of a parity operator," *Phys. Rev. A*, vol. 5, pp. 449–450, Feb 1977.

[10] K. Banaszek, C. Radzewicz, K. Wodkiewicz, and J. S. Krasinski, "Direct measurement of the wigner function by photon counting," *Phys. Rev. A*.

[11] R. P. Rundle, T. Tilma, J. H. Samson, and M. J. Everitt, "Quantum state reconstruction made easy: a direct method for tomography," *ArXiv e-prints*, may 2016.

[12] C. Gerry and P. I. Q. O. C. E. C. U. Knight

[13] V. Potoek and S. M. Barnett, "On the exponential form of the displacement operator for different systems," *Physica Scripta*, vol. 90, no. 6, p. 065208, 2015.

[14] C. C. Gerry and J. Mimih, "The parity operator in quantum optical metrology," *Contemporary Physics*, vol. 51, pp. 497–511, nov 2010.

[15] A. M. Ozorio de Almeida and O. Brodier, "Letter to the editor: Parity measurements, decoherence and spiky wigner functions," *Journal of Physics A Mathematical General*, vol. 37, pp. L249–L255, jun 2004.

[16] E. Schrodinger, "Der stetige ubergang von der mikro- zur makromechanik," *Naturwissenschaften*, vol. 14, pp. 664–666, jul 1926.

[17] R. J. Glauber, "Coherent and incoherent states of the radiation field," *Phys. Rev.*, vol. 131, pp. 2766–2788, Sep 1963.

[18] U. o. L. Stuart K Reed, "http://www.personal.leeds.ac.uk/ chm-skr/ccs.html," 06/04/2017.

[19] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, "Theory of superconductivity," *Phys. Rev.*, vol. 108, pp. 1175–1204, Dec 1957.

[20] Reidel, "http://blog.jessriedel.com/2015/03/24/how-to-think-about-quantum-mechanics-part-4-quantum-indeterminism-as-an-anomaly/," 07/04/2017.

[21] S. S. J. B. R. B. J. C. D. H. W. I. J. J. C. L. R. O. R. R. D. Leibfried, E. Knill and D. Wineland., "Creation of a six atom 'schrodinger cat state,"

[22] D. Griffiths, *Introduction to Quantum Mechanics*. Pearson international edition, Pearson Prentice Hall, 2005.

[23] D. http://mathworld.wolfram.com/SeriesExpansion.h Uznanski, "From mathworld–a wolfram web resource, created by eric w. weisstein.," 8/04/2017.

[24] R. Feynman, *Statistical Mechanics: A Set Of Lectures*. Advanced Books Classics, Avalon Publishing, 1998.

[25] S. Das, M. P. G. Robbins, and M. A. Walton, "Generalized uncertainty principle corrections to the simple harmonic oscillator in phase space," *Canadian Journal of Physics*, vol. 94, pp. 139–146, jan 2016.