## Overall plan:

Hyperbolic PDEs, 1: linear advection eqtn. + first order solutions (stability!)

Hyperbolic PDEs, 2: Finite volume solution; second-order solution; flux limiters

Hyperbolic PDEs, 3: Burger's eqtn. (nonlinear!) and the Riemann problem

Comp. fluid dynamics (over 2-3 days):

- the Euler equations
- the Riemann problem as applied to Euler eqtns.
- pieces of a hydro code (differences w/ linear advection + Burger's)
- solving the Sod shock tube w/ 2nd order scheme + Riemann solver
- maybe: 2D solver.

## Pre-class discussion questions:

1) Qualitatively, what happens to the Gaussian pulse and top hat for FTCS vs. the upwind method for $C \sim 1$?

2) Qualitatively, what happens to the Gaussian pulse and top hat after one period (i.e., one loop through the box) as you change $C$ from $\simeq 1$ to $\ll 1$? (Think in terms of the shape of the pulse. Is one more forgiving than the other?)

**Stability** What is stability region for FTCS?

Test one Fourier mode: $a_i^n = A^n e^{Ii\theta}$ (using Zingale I for $\sqrt{-1}$, $\theta$ = phase)

Stable if $\left| A^{n+1} / A^n \right| \leq 1$ (modes do not grow!)

FTCS: $$A^{n+1} e^{Ii\theta} = \underbrace{A^n e^{Ii\theta}}_{\text{cell } i} - \frac{c}{2}\left( \underbrace{A^n e^{I(i+1)\theta}}_{\text{cell } i+1} - \underbrace{A^n e^{I(i-1)\theta}}_{\text{cell } i-1} \right)$$

factor: $$A^{n+1} e^{Ii\theta} = A^n e^{Ii\theta} - \frac{c}{2} A^n e^{Ii\theta} \underbrace{\left( e^{I\theta} - e^{-I\theta} \right)}$$

So we get $$A^{n+1} e^{Ii\theta} = A^n e^{Ii\theta} - C A^n e^{Ii\theta} I \sin(\theta)$$

recall $\sin x = \frac{1}{2I}\left( e^{Ix} - e^{-Ix} \right)$

Divide each side by $A^n e^{I\theta}$:

$$\frac{A^{n+1}}{A^n} = 1 - IC\sin\theta$$

stability is $\left|\frac{A^{n+1}}{A^n}\right|^2 \leq 1$:

$$\left|\frac{A^{n+1}}{A^n}\right| = \left(1 + C^2\sin^2\theta\right)^{\frac{1}{2}}$$

so $$\left|\frac{A^{n+1}}{A^n}\right|^2 = 1 + C^2\sin^2\theta$$ , which is $>1$ for any $C>0$!

FTCS is always unstable to all Fourier modes!

What about upwinding?

recall: $$\frac{a_i^{n+1} - a^n}{\Delta t} = -u \frac{a_i^n - a_{i-1}^n}{\Delta x}$$

move terms: $$a^{n+1} = a^n - \underset{=c}{\boxed{\frac{u\Delta t}{\Delta x}}}\left(a_i^n - a_{i-1}^n\right)$$

Put in $a_i^n = A^n e^{Ii\theta}$:

$$A^{n+1} e^{Ii\theta} = A^n e^{Ii\theta} - c\left(A^n e^{Ii\theta} - A^n e^{I(i-1)\theta}\right)$$

Divide both sides by $a_i^n$:

$$\frac{A^{n+1}}{A^n} = 1 - c\left(1 - e^{-I\theta}\right)$$

Recall, $e^{I\theta} = \cos\theta + I\sin\theta$, so

$$e^{-I\theta} = \cos(-\theta) + I\sin(-\theta) = \cos\theta - I\sin\theta$$

So, we get:

$$\frac{A^{n+1}}{A^n} = 1 - C\left(1 - (\cos\theta - I\sin\theta)\right)$$

$$= \underbrace{1 - C + C\cos\theta}_{\mathbb{R}} \quad - \underbrace{CI\sin\theta}_{\mathbb{I}}$$

$$\left|\frac{A^{n+1}}{A^n}\right| = \left[(1 - C + C\cos\theta)^2 + C^2\sin^2\theta\right]^{\frac{1}{2}}$$

$$= \Big[1 - C + C\cos\theta - C + C^2 - C^2\cos\theta + C\cos\theta - C^2\cos\theta$$
$$+ \underbrace{C^2\cos^2\theta + C^2\sin^2\theta}_{= C^2}\Big]^{\frac{1}{2}}$$

term in [ ... ] simplifies:

$$1 - 2C + 2C\cos\theta - 2C^2\cos\theta + 2C^2$$

$$= 1 - 2C\left[1 - \cos\theta + C\cos\theta - C\right]$$

$$= 1 - 2C(1-C)(1-\cos\theta)$$

so,

$$\left|\frac{A^{n+1}}{A^n}\right|^2 = 1 - 2C(1-C)\underbrace{(1-\cos\theta)}_{\geq 0 \text{ for any } \theta}$$

minus sign means

$\left|\frac{A^{n+1}}{A^n}\right| \leq 1$ for $2C(1-C) \geq 0$!

by inspection, see this means $C \geq 0$ and $C \leq 1$

$\boxed{\text{so, upwinding stable for } 0 \leq C \leq 1}$ (but $C > 0$ or nothing happens!)

## Today's in-class assignment

1) Code review: pair up, share your code. Fdbk. from partner!
   - what's confusing?
   - what doesn't work?
   - fix it!

2) Make sure your code can handle both positive and negative velocities! ($u<0$, $u>0$)
   → if you don't get the same answer for $u=-1$ and $u=+1$, something is wrong!

3) What happens when we evolve upwind for $T=10$ periods for $u=1$, $u=-1$, $C=1$? What about $C=0.1, 0.01$?