Parabolic PDEs

for a PDE $Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0$,

Parabolic PDEs satisfy $B^2 - AC = 0$

In astrophysics, conduction/diffusion are the standard examples:

$$\vec{F} = -k\vec{\nabla}\phi$$

flux of temp. or concentration

coefficient (may not be constant)

temp. or concentration

Which can be rewritten as a conservation law:

$$\phi_t + \vec{\nabla}\cdot\vec{F} = 0$$

, which turns into a diffusion eqn:

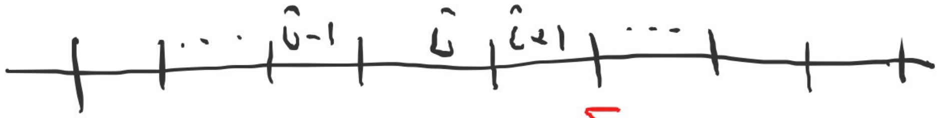
$$\phi_t = -\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot(k\vec{\nabla}\phi)$$ (general form)

In 1D w/constant k,

$$\frac{\partial \phi}{\partial t} = k \frac{\partial^2 \phi}{\partial x^2}$$

[both diffusion and conduction in astrophysics use non-constant k, but it adds to the algebra but conceptually stays the same].

Explicit expression:



$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = k \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{\Delta x^2}$$

[reminder: $n, n+1$ = time
$i, i \pm 1$ = space]

shuffling around terms:

$$\phi_i^{n+1} = \phi_i^n + \boxed{\frac{k\Delta t}{\Delta x^2}}\left(\phi_{i+1}^n - 2\phi_i^n + \phi_{i+1}^n\right)$$

→ gives a clue to stability criterion

stability analysis: to not be numerically unstable,

$$\Delta t < \frac{1}{2} \frac{\Delta x^2}{\kappa}$$

or

$$\Delta t = \frac{C}{2} \frac{\Delta x^2}{\kappa}, \quad C < 1$$

<u>very</u> restrictive due to Δx^2! even worse in plasmas for conduction, since $\kappa \propto T^{5/2}$, so

$$\Delta t \propto \frac{\Delta x^2}{T^{5/2}} \leftarrow \text{ouch!}$$

$\longrightarrow$ lends itself to an implicit solution!

Implicit discretization: (stable for all $c > 0$, but not necessarily accurate)

(the "Implicit Euler method)

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = k \, \frac{\phi_{i+1}^{\boxed{n+1}} - 2\phi_i^{\boxed{n+1}} + \phi_{i-1}^{\boxed{n+1}}}{\Delta x^2}$$

Define $\alpha \equiv \dfrac{k \Delta t}{\Delta x^2}$ and reshuffle:

$$\phi_i^{n+1} + 2\alpha \phi_i^{n+1} - \alpha \phi_{i+1}^{n+1} - \alpha \phi_{i-1}^{n+1} = \phi_i^n$$

or, rearranging:

$$-\alpha \phi_{i-1}^{n+1} + (1 + 2\alpha)\phi_i^{n+1} - \alpha \phi_{i+1}^{n+1} = \phi_i^n$$

which starts to look a lot like $Ax = b$

(or $A\phi^{n+1} = \phi^n$)

Which ends up looking like:

$$\begin{pmatrix} \boxed{(1+2\alpha) \quad -\alpha} & 0 & 0 & \cdots & & \\ -\alpha & (1+2\alpha) & -\alpha & 0 & \cdots & \\ 0 & -\alpha & (1+2\alpha) & -\alpha & 0 & \cdots \\ \cdots & & & \ddots & & \\ \cdots & & & & \boxed{-\alpha \quad (1+2\alpha)} & \end{pmatrix} \begin{pmatrix} \vdots \\ \phi_{i-1}^{n+1} \\ \phi_i^{n+1} \\ \phi_{i+1}^{n+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \phi_{i-1}^{n} \\ \phi_i^{n} \\ \phi_{i+1}^{n} \end{pmatrix}$$

This is a <u>tridiagonal matrix</u> and is easily solvable w/ libraries!

but, end points need to be modified for boundary conditions! (see Zingale 10.19 and following)

Implicit Euler is only 1st order in time; how can we do better?

$\longrightarrow$ use time-centered $\nabla^2\phi$ to solve! ("Crank-Nicholson method")

so, $\phi_t = k\,\phi_{xx}$ becomes:

$$\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = k\,\nabla^2\phi_i^{n+\frac{1}{2}} = k\cdot\underbrace{\tfrac{1}{2}\left(\nabla^2\phi_i^{n+1}+\nabla^2\phi_i^n\right)}_{\text{averaging in time to get 2nd order!}}$$

turns into:

$$\phi_i^{n+1}-\phi_i^n = \frac{k}{2}\left[\underbrace{\frac{\phi_{i+1}^{n+1}-2\phi_i^{n+1}+\phi_{i-1}^{n+1}}{\Delta x^2}}_{\text{time } n+1} + \underbrace{\frac{\phi_{i+1}^{n}-2\phi_i^{n}+\phi_{i-1}^{n}}{\Delta x^2}}_{\text{time } n}\right]$$

Define $\alpha \equiv \frac{k\Delta t}{\Delta x^2}$ as before, group n+1 and n terms:

$$\underbrace{\phi_i^{n+1} - \frac{\alpha}{2}\left(\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}\right)}_{\text{time } n+1} = \underbrace{\phi_i^n + \frac{\alpha}{2}\left(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n\right)}_{\text{time } n}$$

doing some shuffling:

$$\underbrace{-\frac{\alpha}{2}\phi_{i-1}^{n+1} + (1+\alpha)\phi_i^{n+1} - \frac{\alpha}{2}\phi_{i+1}^{n+1}}_{\text{tri-diagonal matrix elements for } Ax=b} = \underbrace{\phi_i^n + \underbrace{\frac{\alpha}{2}\left(\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n\right)}_{= k\frac{\Delta t}{2}\nabla^2\phi_i^n}}_{\text{looks complicated, but we know all of this info already!}}$$