Linear systems

Many mathematical problems reduce to (possibly with a bit of manipulation).

$$\overleftrightarrow{A}\vec{x} = \vec{b}$$

matrix unknowns solutions

For example:

$$\begin{aligned} 17x + 3y - 9z + 2 &= 0 \\ -9x + 2y + 4z - 9 &= 0 \\ \tfrac{1}{2}x - 7y + z + 1 &= 0 \end{aligned}$$

becomes

$$\underbrace{\begin{pmatrix} 17 & 3 & 9 \\ -9 & 2 & 4 \\ \frac{1}{2} & -7 & 1 \end{pmatrix}}_{\overleftrightarrow{A}} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} -2 \\ 9 \\ 1 \end{pmatrix}}_{\vec{b}}$$

$N = 3$ eqtns, 3 unk. straightforward, but the algebra gets tedious as N grows

How to make this easier to solve?

* Convert this system to be an upper-triangular matrix by Gaussian elimination, which can then be solved by substitution *

→ this is what you do when you solve sets of equations by hand by substitution!

$$\underbrace{A^{(n-1)}}_{\text{upper triangular}} x = \underbrace{\underline{b}^{(n-1)}}_{\text{transformed RHS}}$$

$$\begin{pmatrix} A_{11}^{(n-1)} & \cdots & A_{1n}^{(n-1)} \\ & \ddots & \vdots \\ \emptyset & & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} b_1^{(n-1)} \\ \vdots \\ b_N^{(n-1)} \end{pmatrix}$$

Solve the last row (Nth row): $x_n = b_N / A_{NN}$

[note: dropped $(n-1)$ superscripts for clarity in this slide]

then the N-1st row:

$$A_{N-1,N-1} x_{N-1} + A_{N,N-1} x_N = b_{N-1}$$

so,

$$x_{N-1} = \frac{b_{N-1} - A_{N,N-1} x_n}{A_{N-1,N-1}}$$

etc.

with the general form

$$x_i = \frac{1}{A_{i,i}^{(n-1)}} \left[b_i - \sum_{j=i+1}^{N} A_{ij} x_j \right]$$

for $i = N-1, N-2, \dots, 1$