Sublinear Approximation of Centrality Indices in Large Graphs

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Abstract

The identification of important vertices or edges is a ubiquitous problem in the analysis of graphs. There are many application-dependent measures of importance, such as centrality indices (e.g. degree centrality, closeness centrality, betweenness centrality, and eigencentrality) and local triangle counts. Traditional computational models assume that the entire input fits into working memory, which is impractical for very large graphs. Exact algorithms on very large graphs in practice hold the graph in distributed memory, where a collection of processors partition the graph. Distributed graph algorithms must optimize communication in addition to execution time. The data stream model is an alternative approach to large data scale that assumes only sequential access to the input, which is handled in small chunks. Data stream algorithms use sublinear memory and a small number of passes and seek to optimize update time, query time, and post processing time.

In this dissertation, we consider the application of distributed data stream algorithms to the sublinear approximation of several centrality indices and local triangle counts. We pay special attention to the recovery of *heavy hitters* - the largest elements relative to the given index.

The first part of this dissertation focuses on serial graph stream algorithms. We present new algorithms providing a streaming approximation of degree centrality, a semi-streaming constant-pass approximation of closeness centrality, and a semi-streaming approximation of HITS on large collections of small graphs. We achieve our results by way of counting sketches, sampling sketches, and subspace embeddings.

The second part of this dissertation considers vertex-centric distributed graph stream algorithms. We develop hybrid pseudo-asynchronous communication protocols tailored to managing communication on distributed graph algorithms with asymmetric computational loads. We use this protocol as a framework to develop distributed streaming algorithms utilizing cardinality sketches. We present new algorithms for estimating vertex- and edge-local triangle counts, with special attention paid to heavy hitter recovery. We also utilize ℓ_p sampling sketches for the adjacency information of high degree vertices to boost the performance of the sampling of random walks and subtrees. We present hybrid exact-approximating distributed algorithms for sublinearly sampling random walks, simple paths, and subtrees from scale-free graphs. We use these algorithms to approximate κ-path centrality as a proxy for recovery the top-k betweenness centrality elements.

Contents

Chapter 1

Introduction

Many modern computing problems focus on complex relationship networks arising from real-world data. Many of these complex systems such as the Internet, communication networks, logistics and transportation systems, biological systems, epidemiological models, and social relationship networks map naturally onto graphs [WF94]. A natural question that arises in the study of such networks is how to go about identifying "important" vertices and edges. How one might interpret importance within a graph is contingent upon its domain. Accordingly, investigators have devised a large number of importance measures that account for different structural properties. These measures implicitly define an ordering on graphs, and typically only the top elements vis-á-vis the ordering are of analytic interest.

However, most traditional RAM algorithms scale poorly to large datasets. This means that very large graphs tend to confound standard algorithms for computing various important orderings. Newer computational models such as the data stream model and the distributed memory model were introduced to address these scalability concerns. The data steam model assumes only sequential access to the data, and permits a sublinear amount of additional working memory. The time to update, query, and post process this data structure, as well as the number of passes and amount of additional memory are the important resources to optimize in the data stream model. The data stream model is a popular computational model for handling scalability in sequential algorithms. The distributed memory model partitions the data input across several processors, which may need to subsequently communicate with each other. The amount of communication is an important optimization resource. In practical terms, minimizing the amount of time processors spend waiting on their communication partners is also important.

Although both models have been applied to very large graphs independently, there is relatively little literature focusing on the union of the two models of computation. In this work we devise distributed data stream algorithms to approximate orderings of vertices and edges of large graphs. We focus in particular on recovering the heavy hitters of these orderings. We consider the sublinear

approximation of classic centrality scores, as well as local and global triangle counts. We also describe space-efficient methods for sampling random walks and subtrees in scale-free vertex-centric distributed graphs, and their application to estimating some centrality indices.

1.1 Data Stream Models

The data stream model: A stream $\sigma = \langle a_1, a_2, \ldots, a_m \rangle$ is a sequence of elements in the universe $\mathcal{U}$, $|\mathcal{U}| = n$. We assume throughout that that the hardware has working memory storage capabilities $o(\min\{m, n\})$. We will use the notation $[p] = \{1, 2, \ldots, p-1, p\}$ for $p \in \mathbb{Z}_{>0}$ throughout for compactness. For $t \in [m]$, we will sometimes refer to the state of σ after reading t updates as $\sigma(t)$. A streaming algorithm $\mathcal{A}$ accumulates a data structure $\mathcal{S}$ while reading over σ. We will sometimes use the notation $\mathcal{D}(\sigma)$ to indicate the data structure state after $\mathcal{A}$ has accumulated σ. Authors generally assume $|\mathcal{D}| = \widetilde{O}(1) = O(\log m + \log n)$, where here the tilde suppresses logarithmic factors. Except where noted otherwise, we will assume the base 2 logarithm in our presentation.
The semi-streaming model: Unfortunately, logarithmic memory constraints are not always possible. In particular, it is known that many fundamental properties of complex structured data such matrices and graphs require memory linear in some dimension of the data [M+11, McG09]. In such cases, the logarithmic requirements of streaming algorithms are sometimes relaxed to $O(n \operatorname{polylog} n)$ memory, where $\operatorname{polylog} n = \Theta(\log^c n)$ for some constant c. In the case of matrices, here n refers to one of the matrix's dimensions, whereas for graphs n refers to the number of vertices. This is usually known as the *semi-streaming model*, although some authors also use the term to refer to $O(n^{1+\gamma})$ for small γ [FKM+05, M+05].
The frequency vector: A stream σ is often thought of as updates to a hypothetical frequency vector $f(\sigma)$, which holds a counter for each element in $\mathcal{U}$. We will drop the parameterization of f where it is clear. We will sometimes parameterize $f(t)$ to refer to f after reading $t \in [m]$ updates from σ. $\mathcal{D}$ can be thought of as a lossy compression of f that only preserves some statistic thereof.
Dynamic streams: If f is subject to change, then we call σ a *turnstile* or *dynamic* stream. Such a σ's elements are of the form (i, c), where i is an index of f (an element of $\mathcal{U}$), $c \in [-L, \ldots, L]$ for some integer L, and (i, c) indicates that $f_i \leftarrow f_i + c$. In the *cash register* model only positive updates are permitted, whereas in the *strict* turnstile model all elements of f are guaranteed to retain nonnegativity.
Data sketching: Let $\circ$ be the concatenation operator on streams. For $\mathcal{A}$ a streaming algorithm, we call its data structure $\mathcal{S}$ a *sketch* if there is an operator $\oplus$ such that, for any streams σ_1 and σ_2,

$$\mathcal{S}(\sigma_1) \oplus \mathcal{S}(\sigma_2) = \mathcal{S}(\sigma_1 \circ \sigma_2). \tag{1.1}$$

Linear sketching: A sketch $\mathcal{S}$ is a *linear sketch* if it is a linear function of $f(\sigma)$ of fixed dimension. For streams σ_1 and σ_2 with frequency vectors f_1 and

f_2, scalars a and b, and linear sketch transform $\mathcal{S}$,

$$a\mathcal{S}(f_1) + b\mathcal{S}(f_2) = \mathcal{S}(af_1 + bf_2). \tag{1.2}$$

The graph stream model: In this model, the stream σ consists of edge insertions on n vertices. The *dynamic graph stream model* also allows edge deletions, If the graph is weighted, the stream updates the weight of the corresponding edge, possibly bringing it into existence or, in the case of dynamic streams, deleting it.

1.2 Serial Graph Stream Algorithms

Streaming Degree Centrality: Classic degree centrality is still one of the most commonly applied centrality measures. Indeed, indegree centrality is known to correlate well with PageRank, and so can be used as a proxy in some scenarios [UCH03]. We demonstrate a streaming $(1+\varepsilon, \delta)$-approximation for degree centrality of a graph that uses $O(\varepsilon^{-1} \log n \log(1/\delta))$ space. This algorithm is a straightforward application of CountMinSketch, where we interpolate a graph stream as updates to a vector of the degrees of the vertices. Accordingly, our algorithm has $O(\log(1/\delta))$ update and query times.

Furthermore, we can simultaneously maintain a list of the approximate top k degree vertices for a space overhead of

$$O\left(k \log \frac{n}{\delta} + \frac{\sum_{q=k+1}^{n} n_q^2}{n_k^2} \varepsilon^{-2} \log \frac{n}{\delta}\right), \tag{1.3}$$

where n_x is the degree of vertex x. This follows a procedure very similar to that given by Charikar et al. in [CCFC02]. For practical scale-free graphs and reasonably small values of k, the space dependence upon n remains logarithmic.

Constant-pass semi-streaming closeness centrality: The closeness centrality of a vertex is the inverse of all of its shortest paths, and is a common centrality index of interest in applications. We describe a semi-streaming $(r^{\log_2 5} - 1, \delta)$-approximation algorithm for the vertex centrality of a graph that uses $O(n^{1+1/r})$ space and $\log r$ passes, where r is an accuracy parameter. The algorithm builds on an algorithm developed by Ahn, Guha, and McGregor that builds a $(r^{\log_2 5} - 1)$-spanning sparsifying subgraph of an input graph [AGM12b]. Ahn, Guha, and McGregor's algorithm depends upon building random subtrees by iteratively querying ℓ_p-sampling sketches in $\log r$ passes. We then compute the closeness centrality of this subgraph, which bounds the shortest paths distance error by design. For an almost-linear runtime, we can instead employ the fast, scalable algorithm due to Cohen et al. at the expense of some additional error [CDPW14].

Semi-streaming HITS HITS is an algorithm that outputs two indices, an authoritativenss index, and a hubbiness index, for each vertex in a graph. If A is the adjacency matrix of an input graph, these scores converge to the left dominant eigenvectors of $A^T A$ and AA^T, respectively [Kle99]. We build

on a sketching algorithm due to Gilbert, Park, and Wakin to estimate the dominant singular vectors of A [GPW12]. This enables us to estimate the authoritativeness and hubbiness scores of the underlying graph using space $O(nr\varepsilon^{-2}(\log(1/\varepsilon) + \log(1/\delta)))$, where r is the rank of A. Unfortunately, this means that the algorithm with only be efficient if the graph is made of many disconnected components. On the other hand, it is efficient for approximating HITS on many small parallel graphs.

1.3 Vertex-Centric Distributed Streaming Graph Algorithms

Hybrid pseudo-asynchronous communication for vertex-centric distributed algorithms: Modern distributed graph algorithms partition the vertices of input graphs to processors, which communicate as required. The iconic Pregel system executes communication as global rounds of communication between all processors [MAB+10]. However, Pregel-like systems have trouble with scale-free graphs, as high-degree vertices create storage, processing, and communication imbalances. Meanwhile, fully asynchronous approaches address this concern by sub-partitioning high-degree vertices across processors [PGA14]. However, this approach introduces development and computational overhead due to the nature of peer-to-peer routing in implementations, such as MPI. We propose a hybrid "pseudo-asynchronous" approach where rounds of point-to-point communication occur over partitions of processors, taking advantage of modern hybrid distributed-shared memory architectures. Once a node's participation in its partitions is complete, it can drop out of the communication exchange and continue computation. We propose three protocols that route messages from source to destination over MPI.

1. **Node Local Routing**. Cores on the same compute node exchange messages destined for the target core offset, followed by cores at the same core offset but different nodes exchanging messages to their final destinations.

2. **Node Remote Routing**. Cores with the same core offset but different nodes exchange messages destined for the target node, followed by cores at the same core offset but different nodes exchanging messages.

3. **Node Local Node Remote Routing**. Cores locally exchange messages, followed by a remote exchange via a lattice, followed by a second local exchange.

We show in rigorous experiments that these routing exchanges exhibit better scaling characteristics than Pregel- or fully-asynchronous-style communication protocols on distributed algorithms over large graphs.

DegreeSketch and local triangle count heavy hitters: Counting global and local triangles is a canonical problem in both the random access and data stream models. In the data stream model it is known that $\Omega(n^2)$ space is

required to even decide whether a graph has any triangles [BYKS02]. Vertex-local triangle counts are similarly fraught in the data stream model, as they require $\Omega(n)$ space to even write them down. Recent advances in the graph stream literature achieve low variance via clever sampling from edge streams [BBCG08, LK15, SERU17], and have been distributed to achieve better variance [SHL+18, SLO+18].

We examine a different line of analysis altogether: utilizing *cardinality sketches* to estimate local triangle counts via a sublinearization of the set intersection method. We discuss trade-offs between different estimators of the intersection of cardinality sketches, as well as different cardinality sketches themselves. We develop DegreeSketch, a distributed sketch data structure composed of cardinality sketches that can answer point queries about unions and intersections of vertex neighborhoods in a manner reminiscent of CountSketch.

We demonstrate DegreeSketch as a tool for estimating the edge- and vertex-local triangle count heavy hitters of large graphs.

1. **Edge-local triangle count heavy hitters**. We recover the edge-local triangle count heavy hitters of a graph using linear time and $O(n(\varepsilon^{-2}\log\log n + \log n))$ worst-case distributed memory

2. **Vertex-local triangle counts**. We recover the verte-local triangle counts (or, alternately, the heavy hitters) of a graph using linear time and $O(n(\varepsilon^{-2}\log\log n + \log n))$ worst-case distributed memory

We analyze the performance of these algorithms on numerous large graphs, and discuss many practical optimizations to the underlying algorithms. We also discuss the limitations of the approach relating to high variance on small intersections.

Distributed sampling of random walks, simple paths, and subtrees via fast ℓ_p-sampling sketches: The sampling of random walks is a core subroutine in many graph algorithms. However, random walks suffer from sampling from high degree vertices in scale-free graphs. Very large vertex neighborhoods stored in RAM can overwhelm the space and computation constraints of a graph partitioning in a Pregel-like system, whereas each sampling incurs nontrivial communication overhead in a delegated subpartitioning. We address this problem by applying Fast ℓ_p sampling sketches to high degree vertices in scale free graphs. While the adjacency neighborhoods of most vertices are small enough to be stored explicitly in a vertex-centric distributed graph, we record substreams of high degree vertices in fast memory, e.g. NVRAM. We make the following contributions:

1. **Hybrid sublinear random walk sampling**. We ingest a graph using some partitioning and store sparse columns of the adjacency matrix. Once a column grows sufficiently large, we write it as a stream to NVRAM and construct a logarithmic number of ℓ_p sampling sketches, where $p \in \{0, 1\}$. All subsequent additions to the row are appended to the NVRAM stream and inserted into the sketches. When sampling a neighbor from

a high degree vertex, we consume one of the sampling sketches to yield the next neighbor. Once all sampling sketches of a particular vertex are consumed, we take another pass over its recorded stream and accumulate new sketches. The algorithm requires

$$O\left(\sum_{x\in\mathcal{V}} \min\left\{d_x, \log^3 n \log(1/\delta)\right\}\right)$$

space for unweighted graphs, and

$$O\left(\sum_{x\in\mathcal{V}} \min\left\{d_x, \varepsilon^{-1}\log(1/\varepsilon)\log^3 n \log(1/\delta)\right\}\right)$$

space for weighted graphs, where d_x is the degree of vertex x.

2. **Hybrid sublinear random simple path sampling**. In a manner similar to random walk sampling, we construct ℓ_p sampling sketches on demand, while forwarding sampled path histories. We use these path histories when accumulating sampling sketches so as to ignore edges that would result in loops.

3. **Hybrid sublinear random subtree sampling**. In a manner similar to random simple path sampling, we construct ℓ_p sampling sketches of sparse column vectors of the vertex-edge incidence matrix. Exact or sketched columns are transmitted along with walk histories so that they can be summed together with the next sampled column, allowing us to sample from tree neighbors without forming loops.

Sublinear distributed κ-path centrality: The κ-path centrality of a vertex is the probability that a random simple path (a non-self-intersecting random walk) of length $\leq \kappa$ will include it. κ-path centrality was introduced as a cheaper, more scalable alternative to betweenness centrality, and is shown to empirically agree on heavy hitters [KAS+13]. κ-path centrality can be approximated via a Monte Carlo simulation of $T = 2\kappa^2 n^{1-2\alpha} \ln n$ random simple paths, where α is an accuracy parameter. We sublinearize this simulation algorithm to estimate the κ-path centrality of an input graph using our hybrid sublinear random simple path sampling scheme. This affords us an algorithm that can empirically recover the betweenness centrality heavy hitters of a graph using space sublinear in the size of the graph.

Chapter 2

Background and Notation

This chapter introduces the basic concepts and notation that we will use throughout this document. Section 2.1 lays out the basic graph definitions and notation conventions that will be referenced later. Section **??** defines the vector and matrix definitions used throughout the rest of the document. Section **??** describes k-universal hash families, an important concept for many sketches. Section **??** describes some of the sketching concepts that we will reference throughout this document.

2.1 Graph Definitions and Notation

Throughout this document we will consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{w})$. We assume that $\mathcal{G}$ has no self loops, and that where $|\mathcal{V}| = n$ and $|\mathcal{E}| = m$. For convenience of reference and indexing, we will often assume that $\mathcal{V} = [n]$ and $E = [m]$. We denote an edge connecting $x, y \in \mathcal{V}$ as $xy \in \mathcal{E}$. In general we will assume that $\mathcal{G}$ is an undirected graph, except where noted otherwise. If $\mathcal{G}$ is a weighted graph, then $\mathbf{w} \in \mathbb{R}^{\binom{n}{2}}$ is the vector of edge weights. For $x, y \in \mathcal{V}$, $\mathbf{w}_{xy} \in \mathbb{R}_{\geq 0}$ is the weight associated with the edge xy if $xy \in \mathcal{E}$, and is zero otherwise. If $\mathcal{G}$ is unweighted, then $\mathbf{w}_e = 1$ for every $e \in \mathcal{E}$.

Let $A \in \mathbb{R}^{n \times n}$ be the *adjacency matrix* of $\mathcal{G}$, where $A_{x,y} = \mathbf{w}_{xy}$. Let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix, where $D_{x,x}$ is the *degree* or *valency* of vertex $x \in \mathcal{V}$, which can be computed as the row sum of the xth row of A. We define $L = D - A$ as the *Laplace Matrix* or *Laplacian* of $\mathcal{G}$.

Consider the signed vertex-edge incidence matrix, $B \in \mathbb{R}^{\binom{n}{2} \times n}$, given by

$$B_{xy,z} = \begin{cases} 1 & \text{if } xy \in \mathcal{E} \text{ and } x = z \\ -1 & \text{if } xy \in \mathcal{E} \text{ and } y = z \\ 0 & \text{else.} \end{cases} \tag{2.1}$$

Here we let x, y, and z range over $\mathcal{V}$. Let $W \in \mathbb{R}^{n \times n}$ be a diagonal matrix such that $W_{x,y} = \sqrt{w_{xy}}$. Then if G is undirected, we can alternatively write the

Laplacian as

$$L = BWW^T B^T. \tag{2.2}$$

If $\mathcal{G}$ is unweighted, then we can simply write $L = BB^T$.

A *path* in $\mathcal{G}$ is a series of edges $(x_1x_2, x_2x_3, \ldots, x_{\ell-1}x_\ell)$ where the tail of each edge is the head of the following edge in the path. The length, alternatively weight, of a path is the sum of the weights of all of its edges. If G is unweighted, this is simply the number of edges in the path. For vertices $x, y \in \mathcal{V}$, the *distance* $d_{\mathcal{G}}(x, y)$ between x and y in $\mathcal{G}$ is the length of the shortest path that begins with an edge with head x and ends with an edge with tail y. There may be more than one such path. If the graph is clear from context, we may omit the subscripts and write $d(x, y)$. If there is no path connecting x to y in $\mathcal{G}$, then $d_{\mathcal{G}}(x, y) = \infty$. We call a path *simple* if it visits every vertex no more than once.

2.2 Centrality Indices

A centrality index is any map $\mathcal{C}$ that assigns to every $x \in \mathcal{V}$ a nonnegative score. The particulars of $\mathcal{C}$ are usually assumed to be conditioned only on the structure of $\mathcal{G}$. Consequently, we can identify the centrality index on $\mathcal{G}$ as a function $\mathcal{C}_{\mathcal{G}} : \mathcal{V} \to \mathbb{R}_{\geq 0}$. For $x \in \mathcal{V}$, we will call $\mathcal{C}_{\mathcal{G}}(x)$ the centrality score of x in $\mathcal{G}$. Typically, for $x, y \in \mathcal{V}$, $\mathcal{C}_{\mathcal{G}}(x) > \mathcal{C}_{\mathcal{G}}(y)$ implies that x is more important than y in $\mathcal{G}$ with respect to the property that $\mathcal{C}$ measures. We will generally drop the subscript from $\mathcal{C}$ when it is clear from context. It is important to note that if $\mathcal{G}$ changes, so may the mapping $\mathcal{C}$. At times, we will write $\mathcal{C}(\mathcal{G})$ or $\mathcal{C}(\mathcal{V})$ to denote the set of all centrality scores of the vertices in $\mathcal{G}$.

Researchers have considered more exotic centrality indices that rely on metadata, such as vertex and edge colorings [KKM+16]. Such notions of centrality are most likely out of scope for the research proposed by this document.

2.3 Vector and Matrix definitions and notation

For a vector $v \in \mathbb{R}^n$, we denote the ℓ_p norm as follows:

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}. \tag{2.3}$$

As $p \to 0$, this quantity converges to the special case of the ℓ_0 norm:

$$\|v\|_0 = \sum_{i=1}^{n} v_i^0 = |\{i \in [n] \mid v_i \neq 0\}|. \tag{2.4}$$

Here we define $0^0 = 0$. Throughout this document we will mostly be concerned with the ℓ_0 and ℓ_1 norms of matrix rows and columns. The p-th frequency moment F_p of a vector v is related to its ℓ_p norm in the following way:

$$F_p(v) = \|v\|_p^p = \sum_{i=1}^{n} v_i^p. \tag{2.5}$$

We will also sometimes be interested in matrix norms. For $i \in [n]$ and $j \in [m]$, we will write the i, jth element of M as $M_{i,j}$. We will also write the ith row and jth column of M as $M_{i,:}$ and $M_{:,j}$, respectively. For a matrix $M \in \mathbb{R}^{n \times m}$, we define the Fröbenius norm as follows:

$$\|M\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} M_{i,j}^2 \right)^{1/2}. \tag{2.6}$$

Given $A \in \mathbb{R}^{n \times d}$, let $A = U \Sigma V^T$ be its singular value decomposition (SVD), where $\Sigma \in \mathbb{R}^{n \times n}$ is a diagonal matrix and U and V are orthonormal. Set $A_k = U_k \Sigma_k V_k^T$, where U_k and V_k are the leading k columns of U and V, respectively, and $\Sigma_k \in \mathbb{R}^{k \times k}$ is a diagonal matrix whose entries are the first k entries of Σ. A_k is known to solve the optimization problem

$$\min_{\widetilde{A} \in \mathbb{R}^{n \times d} : \text{rank}(\widetilde{A}) \leq k} \|A - \widetilde{A}\|_F.$$

That is, A_k is the rank-k matrix which has the smallest Fröbenius residual with A. This is also true of the spectral norm. We will sometimes denote the rank-k truncated SVD of a product of matrices $A_1 \cdots A_n$ as $[A_1 \cdots A_n]_k$. We use the notation $A^+ = V \Sigma^{-1} U^T$ to denote the Moore-Penrose Pseudoinverse of A.

2.4 k-Universal Hash Families

Many critical results in the sketching literature depend on k-universal hash families.

Definition 2.4.1. *A hash family from sets $\mathcal{X}$ to $\mathcal{Y}$ is a set of functions $\mathcal{H}$ such that for all $h \in \mathcal{H}$, $h : \mathcal{X} \to \mathcal{Y}$. Such a family $\mathcal{H}$ is k-universal if for all $x_1, \ldots, x_k \in \mathcal{X}$ and for all $y_1, \ldots, y_k \in \mathcal{Y}$, the following holds:*

$$\Pr_{h \in_R \mathcal{H}} [h(x_1) = y_1 \wedge \cdots \wedge h(x_k) = y_k] = \frac{1}{|\mathcal{Y}|^k}.$$

A k-universal hash family $\mathcal{H}$ has the property that for any collection of $x_1, \ldots, x_k \in \mathcal{X}$, if $h \in_R \mathcal{H}$ then $(h(x_1), \ldots, h(x_k))$ is distributed uniformly in $\mathcal{Y}^k$. This property is of critical importance. If an algorithm requires a k-wise independent uniform projection over elements, then a k-universal hash function suffices. Moreover, so long as $k \ll n$, a sampled hash function is relatively efficient to store.

Such hash functions underly many of the fundamental results enabling sketching algorithms, discussed in detail in Chapter **??**.

2.5 Approximation Paradigms

Throughout this document we will consider several different approximation problems. For the purpose of discussion, we will consider a nonspecific centrality index $\mathcal{C}$ and a graph $\mathcal{G}$. The general approach to approximate C is to

generate a function $\widetilde{\mathcal{C}} : \mathcal{V} \to \mathbb{R}_{\geq 0}$ such that $\widetilde{\mathcal{C}}(x)$ is a "good approximation" of $\mathcal{C}(x)$ for every $x \in \mathcal{V}$. However, in many applications it is not necessary to exhaustively list the centrality for every vertex in the graph. Instead, many applications require only the top k vertices with respect to $\mathcal{C}$, for $k \ll n$. Consequently, we will also be interested in the problem of maintaining approximations of the top k vertices with respect to $\mathcal{C}$.

For $x, y \in \mathbb{R}$, $\varepsilon > 0$, we will use the compact notation $x = (1 \pm \varepsilon)y$ to denote the situation where $(1 - \varepsilon)y \leq x \leq (1 + \varepsilon)y$. We will refer to an algorithm as an (ε, δ)-approximation of quantity Q if it is guaranteed to output $\widetilde{Q}$ such that $\widetilde{Q} = (1 \pm \varepsilon)Q$ with probability at least $1 - \delta$.

2.5.1 Total Centrality Approximation

We will consider the problem ApproxCentral$(\mathcal{C}, \mathcal{G}, \varepsilon)$ as the problem of producing a function $\widetilde{\mathcal{C}_{\mathcal{G}}}$ such that, for all $x \in \mathcal{V}$, $\widetilde{\mathcal{C}_{\mathcal{G}}}(x) = (1 \pm \varepsilon)\mathcal{C}_{\mathcal{G}}(x)$. It is important to note that such strong approximations may not be attainable for some indices in the streaming or even semi-streaming model. In such cases we will be forced to accept unbounded approximations that exhibit good empirical performance. We will call this relaxed problem, with no specified worst case bounds, UBApproxCentral$(\mathcal{C}, \mathcal{G})$.

2.5.2 Top-k Centrality Approximation

It is worth repeating the estimation of the centrality of every vertex in a graph is unnecessary for many applications. Indeed, in many cases a set of the top k central vertices suffices, for a reasonable choice of k. We will consider the problem ApproxTopCentral$(\mathcal{C}, \mathcal{G}, k, \varepsilon)$ as the problem of producing a list $\mathcal{V}_k$ of k vertices such that, if v_{i_k} is the k-th largest index of $\mathcal{V}$ with respect to $\mathcal{C}_{\mathcal{G}}$, then for every $v \in \mathcal{V}_k$, $\mathcal{C}_{\mathcal{G}}(v) \geq (1 - \varepsilon)\mathcal{C}_{\mathcal{G}}(v_{i_k})$. We will similarly relax the requirements to UBApproxTopCentral$(\mathcal{C}, \mathcal{G}, k)$ to allow for unbounded approximations.

Chapter 3

Streaming Degree Centrality

In this chapter we present an $\widetilde{O}(1)$ space and update time algorithm for approximating degree centrality in one pass. The algorithm is a straightforward application of CountMinSketch to the degree counting problem, and so instead of maintaining $O(n)$ counters we maintain a $\widetilde{O}(1)$ sketch that can be queried for degrees. We also discuss streaming heavy hitter recovery.

3.1 Introduction and Related Work

In an undirected and unweighted graph, the degree centrality of a vertex is simply calculated as the number of adjoining edges in the graph. If the graph is weighted, degree centrality is usually generalized to the sum of the weights of the adjoining edges. In either case, the degree centrality of vertex $x \in [n]$ is equal to the sum of the xth row of the adjacency matrix A. In a directed graph, the indegree (outdegree) centrality of vertex x is the number of incoming (outgoing) edges to (from) x, conventionally corresponding to the xth column (row) of A.

$$\mathrm{DC}(x) = |\{(u,v) \in E \mid x \in \{u,v\}\}| = \|A_{x,:}\|_1 = \|A_{:,x}\|_1 \tag{3.1}$$

$$\mathrm{IDC}(x) = |\{(u,v) \in E \mid x = v\}| = \|A_{x,:}\|_1 \tag{3.2}$$

$$\mathrm{ODC}(x) = |\{(u,v) \in E \mid x = u\}| = \|A_{:,x}\|_1 \tag{3.3}$$

Though simple, degree centrality is still widely used as a benchmark in many applications [BV14]. Indeed, it is competitive with more sophisticated notions of centrality in some contexts [UCH03]. Moreover, degree centrality is known to correlate well with PageRank, making it a decent proxy when computing PageRank is not practical [UCH03]. Moreover, even a naïve streaming implementation of degree centrality is efficient to compute compared to other centrality indices.

However, the naïve implementation requires the maintenance of $\Omega(n)$ counters over a graph stream.

We will improve upon this constraint using the famous CountMinSketch, an early and important sketch performing approximate counting [CM05]. CountMinSketch maintains $t = O(\log(1/\delta))$ sets of $r = O(\varepsilon^{-1})$ counters, using 2-universal hash functions $h_1, \ldots, h_t : [n] \to [r]$. These function define a matrices $C^{(1)}, \ldots, C^{(t)} \in \mathbb{R}^{r \times n}$ as follows. Initialize $C^{(1)} = \cdots = C^{(t)} = \{0\}^{tr \times n}$. Then, for each $i \in [t]$ and each $j \in [n]$, set $C^{(i)}_{h_i(j),j} = 1$. $C^{(1)}, \ldots, C^{(t)}$ are sparse matrices with 1 nonzero entry in every column. For $v \in \mathbb{R}^n$ let $\mathcal{S}(v) \in \mathbb{R}^{t \times r}$ be the matrix whose ith column is given by $C^{(i)}v$. $\mathcal{S}(v)$ is the CountMinSketch object, and can be computed in $\widetilde{O}(\text{nnz}(v))$ time.

Theorem 3.1.1 shows that the minimum of $\{\mathcal{S}_{1,h_1(j)}, \mathcal{S}_{2,h_2(j)}, \ldots \mathcal{S}_{t,h_t(j)}\}$ is a biased estimator of v_j with bounded error. Given a vector $v \in \mathbb{R}^n$, we use the notation v_{-i} to denote the vector whose elements are all the same as v aside from the ith element, which is zero.

Theorem 3.1.1 (Theorem 1 of [CM05])**.** *Let $v \in \mathbb{R}^n$ be the frequency vector of a strict turnstile stream and let $\mathcal{S}$ be its accumulated* CountMinSketch *with parameters (ε, δ). For all $j \in [n]$ the following holds with probability at least $1 - \delta$:*

$$0 \leq \widetilde{v}_j - v_j \leq \varepsilon \|v_{-j}\|_1$$

CountMinSketch guarantees error in terms of the ℓ_1 norm, which is not as tight as the ℓ_2 bounds offered by the more general CountSketch, which has the added benefit of operating on turnstile streams [CCFC02]. However, the CountSketch data structure depends upon a second set of 2-universal hash functions and adds a $1/\varepsilon$ multiplicative factor to the definition of r, as well as involving larger constants [CM05]. All of the results in the following are stated in terms of CountMinSketch as they yield more practical implementations, but similar results could be obtained using CountSketch.

3.2 Turnstile Streaming Degree Centrality

Given a stream updating adjacency matrix A, it is a simple to interpolate it as a stream updating a vector d storing the degree of every vertex in the graph. If computing indegree, simply convert an update (x, y, c) to (y, c), where (y, c) is interpreted as "add c to d_y". Outdegree is similar, and if $\mathcal{G}$ is undirected the two measures are equivalent. Here $c \in \{1, -1\}$ in all cases if $\mathcal{G}$ is unweighted, where $c = 1$ implies an edge insertion and $c = -1$ implies an edge deletion. We assume a strict turnstile model where each edge can only be inserted or deleted, possibly more than once. Thus accumulated vector d is such that d_x is exactly the degree centrality of vertex x.

We can generalize this model to account for weighted $\mathcal{G}$. In this case edge weights could change after insertion, and the vector d is such that $d_x = \|A_{x,:}\|_1$ or $d_x = \|A_{x,:}\|_1$. Rather than the number of edges incident upon x, d_x denotes

the amount of weight incident upon x. As the treatment for weighted and unweighted $\mathcal{G}$ is the same, we will not distinguish in the following.

An immediate consequence of this formulation is that one can accumulate CountMinSketch on a strict turnstile graph stream to obtain $\mathcal{S}$ and perform point queries as to the degrees of vertices. However, the error guarantees of Theorem 3.1.1 are tight only for $x \in \mathcal{V}$ where $d_x \geq \phi\|d\|_1$ for a nontrivial fraction ϕ. It is desirable in particular to compute and return these heavy hitters. While this can be achieved trivially with a second pass, single pass algorithms suffice.

Cash Register Model. We describe an algorithm similar to the algorithm of Theorem 6 of [CM05]. We maintain a heap H of candidate heavy hitters and $\|d(t)\|$ for update t. The latter is monotonically increasing because the stream is insert only. Upon receiving (x_t, y_t, c_t), update $\mathcal{S}(t)$ as usual and then query it for $\widetilde{d(t)}_x$ and $\widetilde{d(t)}_y$. Instead perform these updates for only the head or tail in out or in degree centrality, respectively. If $\widetilde{d(t)}_x \geq \phi\|d(t)\|_1$, add $\left(x, \widetilde{d(t)}_x\right)$ to H, and similarly for y. Let $\left(z, \widetilde{d(t^*)}_z\right) = \min(H)$ and remove it from H if $\widetilde{d(t^*)}_z < \phi\|d(t)\|_1$. Once done, we iterate through $z \in H$ and using $\mathcal{S}$ output z such that $\widetilde{d}_z \geq \phi\|d\|_1$.

Theorem 3.2.1. *Let $\mathcal{G}$ with degree vector d be given in a cash register stream. An algorithm can return every vertex with degree $\geq \phi\|d\|_1$, and avoid returning any vertices with degree $< (\phi - \varepsilon)\|d\|_1$ with probability $1 - \delta$. The algorithm requires space $O\left(\varepsilon^{-1} \log \frac{n}{\delta}\right)$ and update time $O\left(\log \frac{n}{\delta}\right)$.*

Proof. Recall that $\|d(t)\|_1$ increases monotonically with t. If $t < t^*$ and a vertex's estimate is smaller than $\phi\|d(t)\|_1$, it cannot be larger than $\phi\|d(t)^*\|_1$ without its estimate due to $\mathcal{S}$ increasing by time t^*. We check the estimate for each vertex when an incident edge is updated and the estimates do not underestimate, so no heavy hitters are omitted.

Call the event where $\widetilde{d}_x > \phi\|d_{-x}\|_1 \wedge d_x < (\phi - \varepsilon)\|d\|_1$ for some $x \in \mathcal{V}$ a *miss*. By Theorem 3.1.1, an accumulated CountMinSketch data structure $\mathcal{S}$ with parameters ε and δ^* guarantees that

$$
\begin{aligned}
n\delta^* &> \sum_{x \in \mathcal{V}} \Pr\left[\widetilde{d}_x - d_x > \varepsilon\|d_{-x}\|_1\right] \\
&\geq \sum_{x \in \mathcal{V}} \Pr\left[\widetilde{d}_x - (\phi - \varepsilon)\|d\|_1 > \varepsilon\|d\|_1 \wedge d_x < (\phi - \varepsilon)\|d\|_1\right] \\
&= \sum_{x \in \mathcal{V}} \Pr\left[\widetilde{d}_x > \phi\|d\|_1 \wedge d_x < (\phi - \varepsilon)\|d\|_1\right] && \text{i.e. sum of misses} \\
&\geq \Pr\left[\bigvee_{x \in \mathcal{V}} \widetilde{d}_x > \phi\|d\|_1 \wedge d_x < (\phi - \varepsilon)\|d\|_1\right]. && \text{Union bound}
\end{aligned}
$$

We have shown that the probability that *any* miss occurs is less than $n\delta^*$. By setting $\delta^* = \frac{\delta}{n}$ we guarantee that a miss occurs with probability at most δ, obtaining our desired result. This also fixes the space and update time complexities. □

Strict Turnstile Model. The presence of negative updates necessitates a more complex algorithm, as $\|d(t)\|_1$ is no longer monotonic. We describe an algorithm similar to Theorem 7 of [CM05]. Assume that n is a power of two for convenience of notation. For $j \in \{0, 1, \ldots, \log n\}$ a CountMinSketch sketch $\mathcal{S}^{(\ell)}$. For each degree update (x, c), apply the update $\left(\left\lfloor \frac{x}{2^j} \right\rfloor, c\right)$ to $\mathcal{S}^{(j)}$ for each j. Note that $\mathcal{S}^{(j)}$ receives at most $2^{\log n - j}$ distinct elements, which correspond to the dyadic ranges $\{1, \ldots, 2^j\}$, $\{2^j + 1, \ldots, 2 \cdot 2^j\}$, $\ldots$, $\{(2^{\log n - j} - 1)2^j, \ldots, 2^{\log n - j} \cdot 2^j\}$. Moreover the sketches form a hierarchy, where the range elements of $\mathcal{S}^{(j)}$ subdivide those of $\mathcal{S}^{(j+1)}$, forming a natural binary search structure, allowing us to query for heavy hitters.

Algorithm 1 Strict Turnstile Degree Centrality Heavy Hitters

Input: σ - stream of edge updates
ϕ - threshold
ε, δ - approximation parameters

Accumulation:

for $j \in \{0, 1, \ldots, \log n\}$ **do**
  $\mathcal{S}^{(j)} \leftarrow$ empty independent CountMinSketch(ε, δ^*)
$D \leftarrow 0$
for $(x, y, c) \in \sigma$ **do**
  $D \leftarrow D + 2c$
  for $j \in \{0, 1, \ldots, \log n\}$ **do**
    Insert $\left(\left\lfloor \frac{x}{2^j} \right\rfloor, c\right)$ and $\left(\left\lfloor \frac{y}{2^j} \right\rfloor, c\right)$ into $\mathcal{S}^{(j)}$

Query:

return AdaptiveSearch$(\log n, 0, (\phi + \varepsilon)D)$

Functions:

function AdaptiveSearch$(j, r, thresh)$
  $\widetilde{d}_r \leftarrow$ query $\mathcal{S}^{(j)}$ for r
  if $\widetilde{d}_r \geq thresh$ **then**
    if $j = 0$ **then**
      return $(r, \widetilde{d}_r)$
    else
      AdaptiveSearch$(j - 1, 2r, thresh)$
      AdaptiveSearch$(j - 1, 2r + 1, thresh)$

Algorithm 1 summarizes this approach. We perform a recursive binary search, starting with $\mathcal{S}^{\log n}$. For the dyadic range sums at layer j that are greater than $(\phi + \varepsilon)\|d\|_1$, we query the partitioned halves of the dyadic ranges

at layer $j-1$, until we obtain the heavy hitters at layer 0. If a queried range sum is below the threshold, it is discarded. Elements are only returned when a recursive chain of queries makes it all the way to layer 0, which is a normal CountMinSketch over the stream.

Theorem 3.2.2. *Let $\mathcal{G}$ with degree vector d be given in a strict turnstile stream. Algorithm 1 can return every vertex with degree $\geq (\phi+\varepsilon)\|d\|_1$, and avoid returning any vertices with degree $< \phi\|d\|_1$ with probability $1-\delta$. The algorithm requires space $O\left(\varepsilon^{-1}\log n \log\frac{2\log n}{\phi\delta}\right)$ and update time $O\left(\log n\log\frac{2\log n}{\phi\delta}\right)$.*

Proof. First note that for each $x\in\mathcal{V}$ where $d_x \geq (\phi+\varepsilon)\|d\|_1$, each dyadic range of which it is a part must have a sum no less than $(\phi+\varepsilon)\|d\|_1$. Consequently, Algorithm 1 must return this vertex as CountSketches cannot underestimate any of these sums.

Call the event where $\widetilde{d}_x > (\phi+\varepsilon)\|d_{-x}\|_1 \wedge d_x < \phi\|d\|_1$ for some $x\in\mathcal{V}$ a *miss*. Clearly, for each $j\in\{0,1,\ldots,\log n\}$ there are at most $1/\phi$ range sums greater than $\phi\|d\|_1$. Consequently the algorithm makes at most twice this number of queries at any particular layer, assuming there are no misses. This makes for a maximum of $\frac{2\log n}{\phi}$ queries in total. The sketches are independent, so these queries all have the same probability of failure δ^*. Assume that $\widetilde{d}_x^{(j)}$ is the queried output of some range x from $\mathcal{S}^{(j)}$ in a run of Algorithm 1, and let $Q=\{(x,j)\mid \mathcal{S}^{(j)} \text{ is queried for } x\}$. By Theorem 3.1.1, the CountMinSketch data structures guarantee that

$$\begin{aligned}
\frac{\phi\delta^*}{2\log n} &> \sum_{(x,j)\in Q}\Pr\left[\widetilde{d}_x^{(j)} - d_x > \varepsilon\|d_{-x}\|_1\right] \\
&\geq \sum_{(x,j)\in Q}\Pr\left[\widetilde{d}_x^{(j)} - \phi\|d\|_1 > \varepsilon\|d\|_1 \wedge d_x < \phi\|d\|_1\right] \\
&= \sum_{(x,j)\in Q}\Pr\left[\widetilde{d}_x^{(j)} > (\phi+\varepsilon)\|d\|_1 \wedge d_x < \phi\|d\|_1\right] \qquad \text{i.e. sum of misses} \\
&\geq \Pr\left[\bigvee_{(x,j)\in Q}\widetilde{d}_x^{(j)} > (\phi+\varepsilon)\|d\|_1 \wedge d_x < \phi\|d\|_1\right]. \qquad \text{Union bound}
\end{aligned}$$

We have shown that the probability that *any* miss occurs is less than $\frac{\phi\delta^*}{2\log n}$. By setting $\delta^* = \frac{2\log n}{\phi}$ we guarantee that a miss occurs with probability at most δ, obtaining our desired result. This also fixes the space and update time complexities. □

Chapter 4

Semi-Streaming Closeness Centrality

4.1 Introduction and Related Work

4.1.1 ℓ_p Sampling Sketches

First, we define a ℓ_p-sampling sketch on vector $v \in \mathbb{R}^n$ as follows.

Definition 4.1.1. *Let Π be a distribution on real $r \times n$ matrices, where r depends on ε and δ. Suppose $v \in \mathbb{R}^n$ and sample $S \sim \Pi$. Suppose further that there is a procedure that, given Sv, can output (i, P) where $i \in [n]$ is an index of v sampled with probability $P = (1 \pm \varepsilon)\frac{|v_i|^p}{\|v\|_p^p}$, failing with probability at most δ. Then Π is an ℓ_p sampling sketch.*

Monemizadeh and Woodruff proved the existence of such sampling sketches and provided upper bounds for $p \in [0, 2]$ [MW10]. Jowhari et al. [JST11] improved upon these space bounds. In particular, they proved that ℓ_0 and ℓ_1 sampling sketches require $O(\log^2 n \log \delta^{-1})$ and $O(\varepsilon^{-1} \log \varepsilon^{-1} \log^2 n \log \delta^{-1})$ space, respectively. Note that both have only a polylogarithmic dependence on n, and ℓ_0-sampling sketches can achieve arbitrary sampling precision at no additional cost. Interestingly, the main ingredients in these sketches are applications of Tug-of-War and CountSketch, defined in Sections **??** and **??**.

The full details are too verbose for the purposes of this document. The core idea of Jowhari et al.'s approach is as follows:

1. Sample a uniform random scaling factor $t_i \in_R [0, 1]$ for every $i \in [n]$.
2. Given a streaming vector $v \in \mathbb{R}^n$, rescale each update to v_i by $1/t_i^{1/p}$, obtaining $z \in \mathbb{R}^n$ such that $z_i = v_i/t_i^{1/p}$.
3. In one pass, accumulate CountSketch and ℓ_2 norm approximation data structures of z (C and L') and an ℓ_p norm approximation of v (L).

4. Compute the output z^* of C, and a sparse approximation $\widetilde{z}$.

5. Using z^*, $\widetilde{z}$, L, and L', compute real r and s satisfying a specific relationship with the norms of v and $z - \widetilde{z}$.

6. Compute $j = \text{argmax}_i |z_i^*|$

7. If s, r, and z_j^* satisfy certain inequalities, output FAIL.

8. Otherwise, output $(j, z_j^* t_j^{1/p})$.

See [JST11] for a full treatment of the algorithm and its analysis.

As these sampling sketches are linear, they can be added together. That is, if we have two vectors of the same length $v, v' \in \mathbb{R}^n$ and S is a sketch matrix drawn from such a sampling distribution, then $Sv + Sv' = S(v + v')$. This observation will prove important in Section 4.1.2.

4.1.2 ℓ_p-Sampling Graph Sparsification

In a recent series of papers, Ahn, Guha, and McGregor used ℓ_0- and ℓ_1-sampling sketches on the columns of the vertex-edge incidence matrix to solve or approximating a number of other semi-streaming graph problems [AGM12a, AGM12b]. Their approach depends on ℓ_p sampliing sketches as discussed in Section 4.1.1. Consider the columns of the vertex-edge incidence matrix B (see Equation (2.1)), and a series of ℓ_0 sampling sketches $S_1, S_2, \ldots, S_t$, for some $t = O(\log n)$. The graph sparsification algorithms all proceed along the following basic premise. Algorithms depending on ℓ_1 sampling are similar. First, read the stream defining B (which can be adapted from a stream defining A) and sketch each column of B using each of $S_1, S_2, \ldots, S_t$. If $x \in [n]$ is a vertex of $\mathcal{G}$, then $B_{:,x}$ is a vector whose nonzero entries correspond to edges of which it is an endpoint. Thus, $S_1(B_{:,x})$ can recover a uniformly sampled neighbor of x, say y, with high probability. Then, $S_2(B_{:,x}) + S_2(B_{:,y}) = S_2(B_{:,x} + B_{:,y})$ can sample a neighbor of the supervertex $(x + y)$, since the row values indexed by the edge (x, y) are cancelled out when adding the two row vectors by the definition of B. New sketches are required for each contraction, as otherwise the samples will not be independent and the guarantees of the sampling fails. The sparisification of the graph so obtained may then be used, possibly over several passes, to learn nontrivial information about G using a semi-streaming algorithm. Algorithms using ℓ_1-sampling sketches are similar.

In particular, we are interested in their construction of sparse graph spanners.

Definition 4.1.2. *An α-spanner of a graph $\mathcal{G}$ is a sparse subgraph $\mathcal{H}$ of $\mathcal{G}$ such that for all $x, y \in \mathcal{V}$, the following holds:*

$$d_{\mathcal{H}}(x, y) \leq \alpha d_{\mathcal{G}}(x, y). \tag{4.1}$$

If one is able to construct such a spanner $\mathcal{H}$ using only $O(n \operatorname{polylog} n)$ memory in a small number of passes over $\mathcal{G}$, then one can estimate vertex-vertex distances on $\mathcal{G}$ in the semi-streaming model. Unfortunately, there is presently no known algorithm that can do so in a single pass. The following result corresponds to Theorem 5.1 of [AGM12b].

Theorem 4.1.1 ([AGM12b])**.** *Given an unweighted graph $\mathcal{G}$, there is a randomized algorithm that constructs a $(k^{\log_2 5} - 1)$-spanner in $\log k$ passes using $\widetilde{O}(n^{1+1/k})$ space.*

The rough idea of the algorithm is as follows. Over the course of $\log k$ passes, iteratively contract $\mathcal{G}$ by cleverly choosing vertices from whom to sample ℓ_0 neighbors. Let $\widetilde{\mathcal{G}}_0 = \mathcal{G}$. In pass i, contract graph $\widetilde{\mathcal{G}}_{i-1}$ by partitioning its vertex set into $\widetilde{O}(n^{2^i/k})$ subsets, using an ℓ_0 sampling sketch for each partition. These sampled edges give us a graph $\mathcal{H}_i$. Finally, perform a clever clusting procedure to collapse sampled vertices in the neighborhood of a high degree vertex into a single supervertex. This compression defines $\widetilde{\mathcal{G}}_i$. In every pass, we ensure that the graph is compressed by a certain amount, ensuring favorable memory use. For full details, see [AGM12b].

4.2 Semi-Streaming Constant-Pass Closeness Centrality

More complex notions of centrality require more complex data. Closeness centrality, itself an early measure of graph centrality, defines centrality of a vertex in terms of the reciprocal of the vertex's distance to the graph's other vertices [BV14]. Thus, a vertex with high closeness can be thought of as one that can communicate with the other vertices in the graph relatively quickly. Specifically, one calculates the closeness centrality of a vertex x as

$$\mathrm{CC}(x) = \frac{1}{\sum_{y \in \mathcal{V}} d(x, y)}. \tag{4.2}$$

Consequently, an implementation of exact closeness centrality requires one to solve the all pairs shortest paths problem, making it extremely expensive to compute on a large dense graph. Consequently, on a large evolving graph maintaining up-to-date measures of closeness centrality requires a more sophisticated approach. It is worth pointing out that $CC(x)$ is not defined if the underlying graph is not strongly connected, and indeed it was probably not intended for such graphs. While there are generalizations to the definition that account for disconnected graphs, their details are out of the scope of this document [BV14].

Computing the closeness centrality of an evolving graph is an interesting problem, but one that is eclipsed by the attention paid to its cousin betweenness centrality, covered in Section 9.1.1. However, online solutions have been suggested in the literature [WC14]. In particular, given the information required

to solve betweenness centrality for all vertices one can also solve the closeness centrality. Thus, it might be expected that a solution for betweenness centrality can be adapted into a solution for closeness centrality. However, computing closeness centrality is strictly easier than betweenness centrality, so some solutions that work for closeness centrality in pass- or space-constrained settings may not generalize to betweenness centrality.

A natural avenue of inquiry is whether the ℓ_p-norm sampling approach developed by Ahn, Guha, and McGregor discussed in Section 4.1.2 can be applied to approximating centrality. One of their results of particular interest to our analysis is the construction of graph spanners,. Theorem 4.1.1 provides an algorithm for computing a $(k^{\log_2 5} - 1)$-spanner using $\widetilde{O}(n^{1+1/k})$ space taking $\log k$ passes over a stream defining B [AGM12b].

An α-spanner $\mathcal{H}$ of $\mathcal{G}$ can then be used to approximate the closeness centrality of the vertices in $\mathcal{G}$. Discounting the time spent constructing the sketched spanner, the time needed to compute $CC_H(x)$ (Equation (4.2)) will be substantially less than the time required to compute $CC_{\mathcal{G}}(x)$, particularly if $\mathcal{G}$ is dense. Moreover, an appropriate choice of k places this formulation in the semi-streaming model, making it much more space efficient than any existing algorithm.

Though this approximation of closeness centrality is a fairly obvious application of Ahn, Guha, and McGregor's work, to our knowledge it has not be formulated in prior literature. First note that if $\mathcal{H}$ is an α-spanner of $\mathcal{G}$, then for any $x \in \mathcal{V}$, the following holds:

$$\begin{aligned} CC_{\mathcal{H}}(x) &= \frac{1}{\sum_{y \in \mathcal{V}} d_{\mathcal{H}}(x, y)} && \text{Equation (4.2)} \\ &\geq \frac{1}{\sum_{y \in \mathcal{V}} \alpha d_{\mathcal{G}}(x, y)} && \text{Equation (4.1)} \\ &\geq \frac{1}{\alpha} CC_{\mathcal{G}}(x). \end{aligned}$$

Since $\mathcal{H}$ is a spanning subgraph of $\mathcal{G}$, this implies that for all $x \in \mathcal{V}$, $\mathrm{CC}_{\mathcal{H}}(x) \in [\frac{1}{\alpha}, 1]\mathrm{CC}_{\mathcal{G}}(x)$. Hence, $\mathrm{CC}_{\mathcal{H}}(x)$ is an α-approximation of $\mathrm{CC}_{\mathcal{G}}(x)$ for all $x \in \mathcal{V}$. This implies that the algorithm corresponding to Theorem 4.1.1 can be used to solve ApproxCentral$(\mathrm{CC}, \mathcal{G}, 1 - (k^{\log_2 5} - 1)^{-1})$. As k increases, the guaranteed bound becomes less tight, but the space complexity of $\mathcal{H}$ improves. We summarize the result in the following theorem,

Corollary 4.2.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an unweighted graph with $|\mathcal{V}| = n$ and $|\mathcal{E}| = m$. Let $\mathcal{H}$ be the subgraph spanner of $\mathcal{G}$ output by the algorithm corresponding to Theorem 4.1.1, where we choose k sufficiently large. Then $CC_{\mathcal{H}}(\mathcal{G})$ solves* ApproxCentral$(CC, \mathcal{G}, 1 - (k^{\log_2 5} - 1)^{-1})$ *in the semi-streaming model using* $\log k$ *passes over the stream defining $\mathcal{G}$.*

This technique has two major drawbacks. First, while vertices with high closeness centrality in $\mathcal{G}$ might have similarly high closeness centrality in the

sketched spanner, unlike the results in Section **??** there are no theoretical guarantees thereof. Indeed, this spanner construction does not imply a procedure for maintaining a list of the top closeness central vertices in $\mathcal{H}$ or $\mathcal{G}$. Second, this approach requires $\log k$ passes over the stream updating B. This means that this algorithm is not appropriate for real-time analysis of evolving graphs, and $\mathcal{H}$ must be treated as a static graph for any subsequent analysis. However, the underlying algorithm due to Ahn, Guha, and McGregor is a very recent discovery. It may be possible to construct such a graph in a single pass using related techniques.

Chapter 5

Semi-Streaming HITS

In this section we will briefly discuss some preliminary results in this research area. Much of this section is a straightforward application of results from Chapters 2 and **??**. Unsurprisingly, a cursory analysis yields the most promising results for approximating degree centrality in the streaming model. Degree centrality has the advantage of being the most simple of the many centrality indices, and one which can be succinctly captured using $O(n)$ bits of memory. We also apply existing algorithms to yield semi-streaming approximations of closeness centrality and HITS, a variant of eigencentrality.

This chapter is intended as a brief overview of the relevant concepts from the sketching and streaming literature. Section **??** describes how CountSketch, as described in Section **??**, can be used to maintain a streaming approximation of the top k degree central vertices in a graph. Section 4.2 shows how prior results in graph sparsification yields a multi-pass semi-streaming algorithm for approximating closeness centrality. Section **??** explains a procedure for approximating HITS, a variant of eigencentrality, using the direct singular vector approximation techniques discussed in Section **??**. These results are meant to motivate the claim that other, more widely used, centrality indices can be approximated in the semi-streaming model.

5.1 Introduction and Related Work

5.2 ℓ_2-Subspace Embeddings

The defintion of a turnstile data stream can be easily generalized to instead update a matrix $M \in \mathbb{R}^{n \times d}$. In such a stream σ the elements are of the form (i, j, c), and are interpreted as "add c to $M_{i,j}$". When streaming a graph $\mathcal{G}$, the data structure being aggregated is commonly the adjacency matrix $A \in \mathbb{R}^{n \times n}$ or sometimes the vertex-edge incidence matrix B.

A transformation S drawn from a linear sketch distribution on $k \times n$ matrices can be used to accumulate a sketch of a matrix M given a turnstile stream as

follows. Initialize $SM = \{0\}^{k \times d}$ and apply updates (i, j, c) in order: $SM \leftarrow SM + S(c(e_i * e_j^T))$. An example turnstile algorithm might accumulate SM and then compute some function thereon as an approximation of the function applied to M.

Sketching has had much success in the numerical linear algebra literature by way of *subspace embeddings*.

Definition 5.2.1. *A $(1 \pm \varepsilon)$ ℓ_2-subspace embedding for the column space of an $n \times d$ matrix A is a matrix S for which for all $x \in \mathbb{R}^d$*

$$\|SAx\|_2^2 = (1 \pm \varepsilon)\|Ax\|_2^2.$$

If S is a $(1 \pm \varepsilon)$ ℓ_2-subspace embedding for the columnspace of A, then by the definition it is good at preserving the action of A's columns. This will prove to be a very useful property. Much emphasis in the literature is placed upon finding sketch distributions Π whose elements are $(1 \pm \varepsilon)$ ℓ_2-subspace embeddings for arbitrary matrices with high probability.

Definition 5.2.2. *Suppose Π is a distribution on $r \times n$ matrices, where r is a function of n, d, ε, and δ. Suppose that for any fixed $A \in \mathbb{R}^{n \times d}$, a matrix $S \sim \Pi$ is a $(1 \pm \varepsilon)$ ℓ_2-subspace embedding for A with probability at least $1 - \delta$. Then we call Π an (ε, δ) oblivious ℓ_2-subspace embedding.*

If the dimension $r \ll n$, then SA will be efficient to store. If S is sparse, then SA will also be efficient to compute. Computing the product SA is known to be the computational bottleneck in many applications, and so such a distribution Π is very desirable.

The earliest known oblivious ℓ_2-subspace embeddings satisfy a stronger property: they are Johnson-Lindenstrauss Transforms.

Definition 5.2.3. *Suppose that a matrix $S \in \mathbb{R}^{r \times n}$ has the property that, for any f-element subset $V \subset \mathbb{R}^n$, for all $v, v' \in V$*

$$|\langle Sv, Sv' \rangle - \langle v, v' \rangle| \leq \varepsilon \|v\|_2 \|v'\|_2.$$

Then S forms a (ε, δ, f) Johnson-Lindenstrauss Transform, alternately JLT(ε, δ, f).

Johnson-Lindenstrauss Transforms, by their definition, are good at preserving the inner product. Such transforms satisfy the conditions of the Johnson-Lindenstrauss Lemma, hence their name, and require $r = \Omega(\log(f)/\varepsilon^2)$. Tamas Sarlos famously used Johnson-Lindenstrauss transforms to give some of the first semi-streaming algorithms for approximating regression, matrix multiplication, and low-rank approximation [Sar06]. Others have since refined this approach [CW09, CW17].

There are numerous ways to construct Johnson-Lindenstrauss Transforms. A matrix consisting entirely of scaled gaussian random variables suffices, for example. Achlioptas showed that a scaled matrix of variables drawn uniformly from $\{-1, 1\}$ also suffices, while having obviously more favorable computational features [Ach01]. However, computing SA for such a matrix S will still prove

very expensive in practice. Consequently, researchers placed much focus on sparsifying these matrices. Nelson and Nguyễn showed that such matrices must have $\Omega(\log(f/d)/\varepsilon \log(1/\varepsilon))$ nonzero entries [NN14]. The tightest known construction is $O(\log(f/\delta)/\varepsilon)$, given by Kane and Nelson [KN14]. If $nnz(A)$ is the number of nonzero entries of a matrix A, then a JLT sketch of A can be performed using this construction in $O(nnz(A) \cdot \log(f/\delta)/\varepsilon)$ time. Ailon and Chazelle devised a different JLT construction by performing coordinate sampling over the input vector x, after applying a random rotation and the Hadamard transform [AC06]. Such a transform can be applied to a matrix in $O(nd \log n)$ time, which can be more efficient depending on d and δ.

Clarkson and Woodruff given an entirely different oblivious $(1 \pm \varepsilon)$ ℓ_2 subspace embedding using none other than CountSketch [CW17]. Their forumulation does not satisfy the requirements of a JLT, but allows for the construction of a sketch that can be applied in $O(nnz(A))$ time. However, this construction requires that S have $r = O(d^2/(\delta\varepsilon^2))$ rows, which is large if d is not much smaller than n.

Going forward, we will state results that depend on $(1 \pm \varepsilon)$ ℓ_2-subspace embeddings as such, without delving into the details of which formulation is used. We invite the interested reader to examine Woodruff's excellent book on the subject for more details [W+14].

5.2.1 Direct Singular Vector Approximation

It is desirable for some situations to have bounded error guarantees on the singular vectors themselves. Fortunately, Gilbert, Park, and Wakin approach the problem of approximating right singular vectors directly by sketching using JLTs [GPW12] [AH14]. Their proposed algorithm is very simple: Sketch the left-hand side of a matrix, compute the truncated SVD of the sketch, and return the singular values and right singular vectors. The following theorem, corresponding to Theorem 1 of [GPW12] summarizes their results:

Theorem 5.2.1 ([GPW12])**.** *Let $A \in \mathbb{R}^{n\times d}$ with $n \geq m$ and $\operatorname{rank}(A) = r \leq d$. Let $A = U\Sigma V^T$ be the r-truncated SVD of A. Let $\varepsilon, \delta \in (0,1)$ be accuracy and precision parameters. Suppose that $S \in \mathbb{R}^{m\times n}$ is randomly drawn from a JLT distribution satisfying*

$$m \geq \frac{r\log(42/\varepsilon) + \log(2/\delta)}{f(\varepsilon/\sqrt{2})}$$

Let $SA = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^T$ be the r-truncated SVD of SA. Then with probability $1-\delta$, $\operatorname{rank}(SA) = r$ and for $j \in [r]$ each of the following hold:

$$\widetilde{\Sigma}_{j,j} = (1 \pm \varepsilon)^{\frac{1}{2}}\Sigma_{j,j},$$

and

$$\|V_{:,j} - \widetilde{V}_{:,j}\|_2 \leq \min\left\{\sqrt{2}, \frac{\varepsilon\sqrt{1+\varepsilon}}{\sqrt{1-\varepsilon}} \max_{i\neq j} \frac{\sqrt{2}\Sigma_{i,i}\Sigma_{j,j}}{\min_{c\in[-1,1]}\{|\Sigma_{i,i}^2 - \Sigma_{j,j}^2(1+c\varepsilon)|\}}\right\}. \quad (5.1)$$

The error bounds on the singular vectors given by Equation (5.1) are not tight, particularly when the singular values of the matrix are close together. By Corollary 1 of [GPW12], the algorithm works when $m = O(r\varepsilon^{-2}(\log(1/\varepsilon) + \log(1/\delta)))$. The approach therefore requires only $O(dr\varepsilon^{-2}(\log(1/\varepsilon) + \log(1/\delta)))$ space to store SA, assuming a constant bound on the size of its entries. An immediate consequence of this is that the approach is only semi-streaming for a square matrix if $r = O(\operatorname{polylog} n)$.

It is important to note that Li et al. proved the negative result that any general sketching algorithm guaranteed to approximate a top singular vector of a square matrix with probability $\leq \frac{1}{2}$ requires $\Omega(n^{3/2})$ space [LNW14]. This implies that directly approximating the top singular vector of a graph's adjacency matrix is intractable if high precision is demanded. This negative result will remain important throughout the rest of this document.

5.3 Semi-Streaming HITS

Given a graph $\mathcal{G}$, the HITS algorithm iteratively computes two symbiotic scores - *authoritativeness* and *hubbiness* [Kle99]. A vertex with high authoritativeness has many incoming edges to vertices with high hubbiness, while a vertex with high hubbiness has many outgoing edges to vertices with high authoritativeness. The iterative process computing authoritativeness and hubbiness converges to the left dominant eigenvector of the matrices A^TA and AA^T, respectively [BV14]. These two vectors are the dominant right and left singular vectors of A. We will discuss HITS in greater detail in Section **??**.

There is a large body of work in the streaming and sketching literature concerned with providing low-rank approximations to matrices [Sar06, W$^+$14, CW09, CW17]. Sections **??** and **??** summarize some of these approximations. One can use these procedures to accumulate a rank 1 approximation of a general matrix of the form $\widetilde{\sigma}_1\widetilde{u}_1\widetilde{v}_1^T$ [A$^+$13]. However, the majority of these approach seek to minimize the error between the residual of the rank-k approximation sketch and the residual of the rank-k SVD (i.e. Equation (**??**)) rather than approximating the singular values and vectors directly. Thus, the singular value and corresponding vectors so found do not have guarantees as to their error with respect to the corresponding singular value and vectors of the original matrix. However, so long as relatively large values in the dominant eigenvector of the original matrix are also large in the approximated dominant eigenvector, then they may still be useful in applications.

The results in Section **??** due to Gilbert, Park, and Wakin provide a way forward for bounding the error on approximating *singular vectors*. Though not immediately applicable to approximating eigenvectors, it is known that the right and left singular vectors of a matrix A inform the eigenvalues of A^TA and AA^T, respectively. It is widely known that if the singular value decomposition (SVD) of $A = U\Sigma V^T$, then $A^TA = V\Sigma^2V^T$ and $AA^T = U\Sigma^2U^T$. Thus, the eigenvalues of A^TA and AA^T are squares of the singular values of A, and the eigenvectors of A^TA are the right singular vectors of A, while the eigenvectors

of AA^T are the left singular vectors of A. Consequently, an approximation of the top right and left singular vectors of A constitute approximations of the dominant eigenvectors of A^TA and AA^T, respectively.

Recall that for a graph $\mathcal{G}$, the HITS algorithm outputs authoritativeness and hubbiness scores for each vertex. These scores converge to the left dominant eigenvectors of the matrices A^TA and AA^T, respectively [Kle99]. In what follows, we Thus, a straightforward application of Gilbert, Park and Wakin's approximation to A and A^T's right singular vectors suffices, where the only element of interest in each approximation is the dominant right singular vector. We summarize this result in the corollary to Theorem 5.2.1 below.

Corollary 5.3.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with $|\mathcal{V}| = n$ and $|\mathcal{E}| = m$. Let $A \in \mathbb{R}^{n\times n}$ with be $\mathcal{G}$'s adjacency matrix, and assume $\operatorname{rank}(A) = r \leq n$. Let $C_{auth}(\mathcal{G})$ and $C_{hub}(\mathcal{G})$ be the output of the HITS algorithm, and let $\widetilde{C}_{auth}(\mathcal{G})$ be the dominant right singular vector of the output of the algorithm associated with Theorem 5.2.1 applied to A (and $\widetilde{C}_{hub}(\mathcal{G})$ defined equivalently in terms of A^T). For $j \in [r]$, assume that σ_j is the jth singular value of A. Then with probability $1-\delta$, the following holds:*

$$\|C_{auth}(\mathcal{G}) - \widetilde{C}_{auth}(\mathcal{G})\|_2 \leq \min\left\{\sqrt{2}, \frac{\varepsilon\sqrt{1+\varepsilon}}{\sqrt{1-\varepsilon}} \max_{i\neq j} \frac{\sqrt{2}\sigma_i\sigma_j}{\min\limits_{c\in[-1,1]}\{|\sigma_i^2 - \sigma_j^2(1+c\varepsilon)|\}}\right\}. \tag{5.2}$$

Let ξ be the right side of Equation (5.2). In particular, for every $x \in V$, the following holds with probability $1-\delta$:

$$\widetilde{C}_{auth}(x) = C_{auth}(x) \pm \xi. \tag{5.3}$$

The corresponding statements are also true for $\widetilde{C}_{hub}(G)$ and $C_{hub}(x)$.

As discussed in Section **??**, the space of the algorithm is dependent on r, the rank of A, with complexity $O(nr\varepsilon^{-2}(\log(1/\varepsilon)+\log(1/\delta)))$. Hence this algorithm will be efficient only when applied to graphs with $\operatorname{rank}(A) = r = O(\operatorname{polylog} n)$. That is, the algorithm will only be in the semi-streaming model for graphs with many disconnected components. It is therefore not appropriate in general.

Moreover, as noted in Section **??** the error bound on the dominating eigenvector (Equations (5.1) and (5.2)) may be too loose for some applications. The worst-case bounds on the individual HITS scores for vertices are also additive rather than multiplicative as desired, and may be quite poor for some vertices. However, the error can only be severe for a relatively small number of vertices due to Equation (5.2). To make matters worse, the negative result cited in that section indicate that any general sketching algorithm guaranteed to approximate a top singular vector of a square matrix with probability $\leq \frac{1}{2}$ requires $\Omega(n^{3/2})$ space [LNW14]. Thus, it appears that the best possible general approximation with bounded error for HITS sublinear in m is not much better than the above result.

The other spectral notions of centrality discussed in Section **??** depend on the spectrum of matrices whose structure does not benefit from the trick applied

here to HITS. In particular, it would be very desirable to obtain a sketching procedure by which to tightly approximate the top eigenvector of a general matrix in the semi-streaming model. However, there are at present no known methods for doing so. That said, the rank-1 approximation method detailed at the beginning of the section may be empirically useful in some applications. This claim requires empirical evaluation, which we discuss below in Section **??**.

Chapter 6

Pseudo-Asynchronous Communication for Vertex-Centric Distributed Algorithms

Chapter 7

DegreeSketch and local triangle count heavy hitters

Chapter 8

Distributed sampling of random walks, simple paths, and subtrees via fast ℓ_p-sampling sketches

Chapter 9

Sublinear distributed κ-path centrality

9.1 Introduction and Related Work

9.1.1 Betweenness Centrality

Betweenness is a more recent, though still relatively old, measure of centrality that is related to closeness centrality [Fre77]. The betweenness of a vertex is defined in terms of the proportion of shortest paths that pass through it. Thus, a vertex with high betweenness is one that connects many other vertices to each other - such as a boundary vertex connecting tightly-clustered subgraphs. For $x, y, z \in \mathcal{V}$, suppose that $\lambda_{y,z}$ is the number of shortest paths from y to z and $\lambda_{y,z}(x)$ is the number of such paths that include x. Then the betweenness centrality of x is calculated as

$$\mathrm{BC}(x) = \sum_{\substack{x \notin \{y,z\} \\ \lambda_{y,z} \neq 0}} \frac{\lambda_{y,z}(x)}{\lambda_{y,z}}. \tag{9.1}$$

An implementation of betweenness centrality must solve the all pairs *all* shortest paths problem, which is strictly more difficult than the all pairs shortest paths solution required by closeness centrality. Moreover, there is no known algorithm for computing the betweenness centrality of a single vertex using less space or time than the best algorithm for computing the betweenness centrality for all of the vertices. The celebrated Brandes algorithm, the best known algorithm for solving betweeness centrality, requires $\Theta(nm)$ time ($\Theta(nm + n^2 \log n)$ for weighted graphs) and space no better than that required to store the graph [Bra01].

A significant number of algorithms that attempt to alleviate this time cost by approximating the betweenness centrality of some or all vertices have been proposed. Some of these approaches depend on adaptively sampling and com-

puting all of the single-source shortest paths of a small number of vertices [BKMM07, BP07], while others sample shortest paths between random pairs of vertices [RK16]. A recent advancement incrementalizes the latter approach to handle evolving graphs [BMS14].

Fortunately, researchers have directed much effort in recent years toward maintaining the betweenness centrality of the vertices of evolving graphs [GMB12, WC14, KMB15]. The most recent of these approaches keep an eye toward parallelization across computing clusters to maintain scalability.

If the evolving graph in question is sufficiently small that storing it in working memory is feasible, then existing solutions suffice to solve the problem in a reasonably efficient fashion. However, none of the existing solutions adapt well to the semi-streaming model, as they each require $\Omega(m)$ memory. Indeed, directly approximating betweenness centrality seems likely to be infeasible using sublinear memory.

9.1.2 κ-Path Centrality

Kourtellis et al. attempt to approximate the top-k betweenness central vertices by way of relaxing its computation with the related κ-path centrality [KAS+13]. The authors define the κ-centrality of a vertex x as the probability that a random simple path of length at most k over all other source vertices y passes through x. Kourtellis et al. provide a randomized algorithm for approximating the κ-path centrality of the vertices of a graph, and demonstrate that on real-world networks vertices with high approximate κ-path centrality empirically correlate well with high betweenness-centrality vertices. Moreover, this algorithm is much more efficient than other attempts to approximate betweenness centrality, as it sidesteps the need to approximate the all pairs all shortest paths problem. Finally, κ-Path centrality is desirable in that it depends only on κ-local features of the graph when considering any particular vertex.

9.1.3 Semi-Streaming Approximation of Betweenness Centrality Heavy Hitters via κ-Path Centrality

It is unclear how to sublinearize approximating the betweenness centrality of a vertex. Indeed, each solution to the SSSP problem requires $\Omega(m)$ memory. As this is the basis of the on- and off-line betweenness centrality approximation algorithms discussed in Section 9.1.1, variations on their approach is unlikely to yield a semi-streaming solution. Moreover, the spanner-based approach deployed in Section 4.2 to approximate closeness centrality does not apply. This is because betweenness centrality is defined in terms of the number of all shortest paths between two vertices, information which is lost when building the sparse spanner.

In order to find a way forward, it is helpful to step back and assess what betweenness centrality purports to measure. Vertices with relatively high betweenness centrality scores are those that connect communities - the same vertices should also have high κ-path centrality scores, as discussed in Section 9.1.2.

Moreover, the existing κ-path centrality approximation algorithm operates by way of sampling simple paths from $\mathcal{G}$. This algorithm looks suspiciously similar to the vertex-edge adjacency matrix sampling algorithms developed by Ahn, Guha, and McGregor discussed in Section 4.1.2.

While these results are promising, they do not solve the problem of approximating betweenness centrality in the semi-streaming turnstile model. First, the κ-path centrality approximation algorithm requires a static graph. Second, there is at present no known algorithm that can approximate κ-path centrality using $o(m)$ space. Third, as discussed by Kourtellis et al. in [KAS+13], the sampling algorithm requires $2\kappa n^{1-2\alpha} \ln n$ independent samples, where α is an accuracy parameter. Finally, while empirical correlation is promising, there are no theoretical guarantees that κ-path centrality will accurately capture the top-k betweenness central vertices, which is a highly desirable result. If an algorithm were to arise that simultaneously solves the first three problems, it might be used to approximate the betweenness centrality of the vertices of an arbitrary evolving graph in the semi-streaming model.

It is the authors' belief that ℓ_p-sparsification sketches of the type discussed in Section **??** might afford such an algorithm. Such methods certainly allow for randomly sampling trees (or, with some minor modifications, simple paths) in G in the semi-streaming model. The κ-path centrality approximation algorithm given by Kourtellis et al. is a simple Monte Carlo simulation, which affords a loose additive error. It may well be that a modified algorithm, or indeed a slightly different measure of centrality, suffices to easily sublinearize the approximation problem.

Even in this best case, however, it is important to note that the best we can hope is a semi-streaming algorithm that solves UBApproxCentral$(\text{BC}, \mathcal{G})$ well in practice. A solution to UBApproxTopCentral$(C, \mathcal{G}, k)$ is also desirable, but would require a single-pass algorithm that simultaneously maintains a heap - similar to the CountSketch heap algorithm discussed in Section **??**. Hence, even once such an algorithm is derived it will require extensive empirical evaluation on both real and synthetic data. Similar to the evaluation proposed in Section **??**, we propose an extensive performance comparison between such an algorithm once derived and existing techniques. Even if the empirical accuracy is substandard compared to state-of-the-art techniques, significant time and space improvements represent a contribution toward improving the state of the art.

9.2 Sublinear Distributed κ-Path Centrality

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