# Computations for graphing functions (CFGF) - Solutions

**Problem 1** *Let $f$ be a function given by': $f(x) = \left(\tan^{-1}(x)\right)^2$.*

(a) *State the domain of $f$. Write your answer in interval notation.*

***Solution:*** *Domain of $f = (-\infty, +\infty)$*

(b) *Find the critical points of $f$. Show your work!*

***Solution:*** *Since*

$$f'(x) = 2\tan^{-1}(x) \cdot \frac{1}{x^2+1} = \frac{2\tan^{-1}(x)}{x^2+1},$$

*the only critical points of $f$ are the points where $f'(x) = 0$.*

$$f'(x) = 0 \quad \equiv \quad \tan^{-1}(x) = 0 \quad \equiv \quad x = 0.$$

*Therefore, the function $f$ has the only critical point at $x = 0$.*

(c) *Show that* $$f''(x) = \frac{2 - 4x\tan^{-1}(x)}{(x^2+1)^2}.$$

***Solution:*** $$f''(x) = \left(\frac{2\tan^{-1}(x)}{x^2+1}\right)' = \frac{2\frac{x^2+1}{x^2+1} - 2\tan^{-1}(x)(2x)}{(x^2+1)^2} = \frac{2 - 4x\tan^{-1}(x)}{(x^2+1)^2}$$

(d) *Use the **second derivative test** to determine whether critical point(s) of $f$ correspond to local minimums, local maximums, or whether the test is inconclusive.*

***Solution:*** *Since*

$$f''(0) = \frac{2 - 4(0)\tan^{-1}(0)}{(0^2+1)^2} = 2 > 0,$$

*it follows by the Second derivative test that the function $f$ has a* ***local minimum*** *at $x = 0$.*

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**Problem 2** *Determine the following information about the given function and then graph the function:*

$$f(x) = \frac{x^2+x+1}{x^2}$$

*Domain*
*$x, y$-intercepts*
*Symmetry*
*Asympotes*
*Intervals of increasing/decreasing*
*Maxima/Minima*
*Intervals of concavity*
*Inflection points*

***Solution:***

(a) ***Domain***

*The function is a rational function, and so the domain of the function is all real numbers except where the denominator equals zero.*

$$x^2 = 0 \quad \Longrightarrow \quad x = 0$$

*So the domain of $f$ is $(-\infty, 0) \cup (0, \infty)$.*

(b) ***$x$, $y$-intercepts***
*To find any $x$-intercept(s), set $y = 0$ and solve:*

$$\begin{aligned} \frac{x^2+x+1}{x^2} = 0 &\Longrightarrow x^2+x+1 = 0 \\ &\Longrightarrow x = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)} \end{aligned}$$

*which has no real solutions. Thus, $f$ has no $x$-intercepts.*

*Since $x = 0$ is not in the domain of $f$, $f$ has no $y$-intercepts as well.*

(c) ***Symmetry***

*Note that $f(1) = 3$ and $f(-1) = 1$. So it cannot be for all values of $x$ that either $f(-x) = f(x)$ or $f(-x) = -f(x)$. So $f$ is neither even nor odd, and therefore $f$ has no symmetry.*

(d) ***Asymptotes***

***Vertical Asymptotes:*** *Our only candidate is $x = 0$, and so we compute the two one-sided limits:*

$$\lim_{x\to 0^-} \frac{x^2+x+1}{x^2} = \infty$$

$$\lim_{x\to 0^+} \frac{x^2+x+1}{x^2} = \infty$$

*Therefore, $x = 0$ is the only vertical asymptote of $f$.*

***Horizontal Asymptotes:*** *We compute the following limits:*

$$\lim_{x\to \infty} \frac{x^2+x+1}{x^2} = 1$$

$$\lim_{x\to -\infty} \frac{x^2+x+1}{x^2} = 1$$

*we checked both ends and so the only horizontal asymptote of $f$ is $y = 1$.*

(e) ***Increasing/Decreasing***

$$\begin{aligned} f'(x) &= \frac{x^2(2x+1)-(x^2+x+1)(2x)}{x^4} \\ &= \frac{2x^3+x^2-2x^3-2x^2-2x}{x^4} \\ &= \frac{-x^2-2x}{x^4} \\ &= \frac{-x-2}{x^3} \end{aligned}$$

*To find where $f'$ is positive and where $f'$ is negative, we need to find where $f'(x) = 0$ and where $f'(x)$ does not exist. Clearly, $f'(x)$ does not exist when $x = 0$. To find when $f'(x) = 0$, we solve:*

$$\begin{aligned} \frac{-x-2}{x^3} = 0 &\implies -x-2 = 0 \\ &\implies -x = 2 \\ &\implies x = -2 \end{aligned}$$

*Since $x = 0$ is not in the domain of $f$, $x = -2$ is the only critical point of $f$. To see where $f$ is increasing and decreasing, consider the following sign chart for $f'$:*

| | $(-\infty,-2)$ | $(-2,0)$ | $(0,\infty)$ |
|---|---|---|---|
| $f'$ | (-) | (+) | (-) |
| | $f'(-3) = \frac{-1}{27}$ | $f'(-1) = 1$ | $f'(1) = -3$ |

*So we see that $f$ is increasing on $(-2, 0)$, and $f$ is decreasing on $(-\infty, -2)$ and $(0, \infty)$.*

(f) ***Local Extrema***

*$f'$ changes from negative to positive at $x = -2$, so this is the location of a local minimum. $f'$ also changes from positive to negative at $x = 0$, but $f$ is not defined at $x = 0$ and so this is not a local extreme value. $f$ has a local minimum at $\left(-2, \frac{3}{4}\right)$.*

(g) ***Concavity***

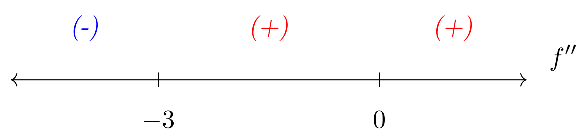
$$\begin{aligned} f''(x) &= \frac{x^3(-1) - (-x-2)(3x^2)}{x^6} \\ &= \frac{-x^3 + 3x^3 + 6x^2}{x^6} \\ &= \frac{2x^3 + 6x^2}{x^6} \\ &= \frac{2x+6}{x^4} \\ &= \frac{2(x+3)}{x^4} \end{aligned}$$

*To find where $f''$ is positive and where $f''$ is negative, we need to find where $f''(x) = 0$ and where $f''(x)$ does not exist. Clearly, $f''(x)$ does not exist when $x = 0$. To find when $f''(x) = 0$, we solve:*

$$\begin{aligned} \frac{2(x+3)}{x^4} = 0 &\implies 2(x+3) = 0 \\ &\implies x = -3 \end{aligned}$$

*The denominator of $f''$ is always positive so the sign of $f''$ depends on the numerator. When $x < -3$, $f'' < 0$ and when $x > -3$, $f'' > 0$.*

*To see where $f$ is concave up and concave down, consider the following sign chart for $f''$:*

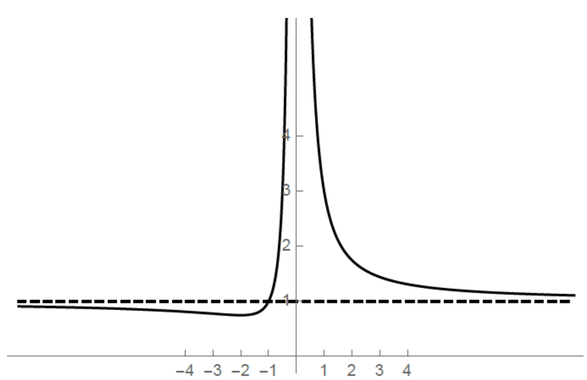



*So we see that $f$ is concave up on $(-3, 0)$ and $(0, \infty)$, and $f$ is concave down on $(-\infty, -3)$.*

(h) ***Inflection Points***

*$f''(x)$ changes sign from negative to positive at $x = -3$ so $f$ has an inflection point at $\left(-3, \frac{7}{9}\right)$*

(i) ***The graph of** $f$*



**Problem 3** *Given that:*

$$\begin{array}{cccc}
\lim_{x\to-\infty} f(x) = 0 & \lim_{x\to-3^-} f(x) = \infty & \lim_{x\to-3^+} f(x) = -\infty & \lim_{x\to4^-} f(x) = \infty \\
\lim_{x\to4^+} f(x) = -\infty & f(1) = 1 & f(5) = -2 & \lim_{x\to9} f(x) = 3 \\
f'(1) \neq 0 & f'(7) = 0 & f'(x) = 2 \text{ for } x > 9 & f''(1) = 0
\end{array}$$

*the domain of $f$ is: $(-\infty,-3)\cup(-3,4)\cup(4,9)\cup(9,\infty)$ and $f$ is continuous on its domain. The following sign chart for the first and second derivatives of $f$:*


*find the following:*

(a) *Critical points.*

(b) *Intervals where $f$ is increasing/decreasing.*

(c) *Local extrema.*

(d) *Inflection points.*

(e) *Intervals of concavity.*

(f) *Sketch the graph of $f$.*

***Solution:*** (a) *Critical points.*
*The critical points of $f$ occur at points in the domain of $f$ where either $f'(x) = 0$ or where $f'(x)$ does not exist. Even though $f'(-3), f'(4)$, and $f'(9)$ do not exist, all three of those points are not in the domain of $f$ and therefore are not critical points. We are given that $f'(7) = 0$, and so $x = 7$ is a critical point of $f$. Therefore, $x = 7$ is the only critical point of $f$.*

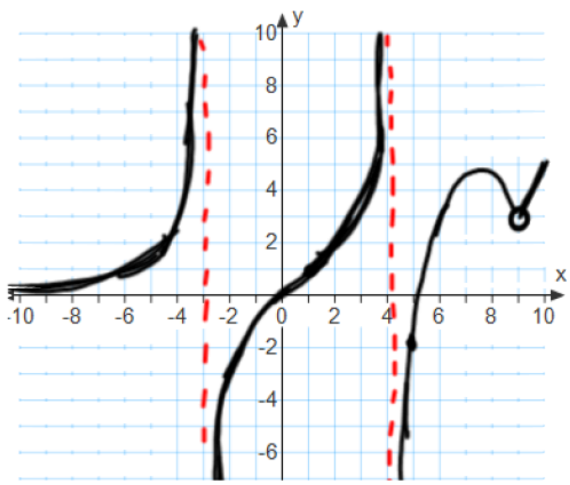
(b) *Intervals where $f$ is increasing/decreasing.*
*$f$ is increasing when $f'(x) > 0$. From the sign chart and our critical points, these are the intervals $(-\infty,-3)$, $(-3,4)$, $(4,7)$, and $(9,\infty)$. $f(x)$ is decreasing when $f'(x) < 0$. From the sign chart, this is on the interval $(7,9)$.*

(c) *Local extrema.*
*Using the first derivative test, $f(7)$ is a local maximum because the derivative changes sign from positive to negative. So, $x = 7$ is the only local extremum of $f$, and it is a local maximum.*

(d) *Inflection points.*
*Possible inflection points occur where $f''(x) = 0$ or where $f''(x)$ does not exist. We are given that $f''(1) = 0$. In addition, $f''(x)$ does not exist at $x = -3, 4, 9$. However, these are not inflection points because $f$ is not defined at these points. Since $f''$ changes sign at $x = 1$, this is in fact an inflection point of $f$. We are given that $f(1) = 1$, and so the only inflection point of $f$ is the point $(1,1)$.*

(e) *Intervals of concavity.*
$f(x)$ *is concave up when* $f''(x) > 0$. *From the sign chart, this is on the intervals* $(-\infty, -3)$ *and* $(1, 4)$.
$f(x)$ *is concave down when* $f''(x) < 0$. *From the sign chart, this is on the interval* $(-3, 1)$ *and* $(4, 9)$.

(f) *Sketch the graph of* $f$.

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**Problem 4** *Determine the following information about the given function and then graph the function:*

$$f(x) = 3\sin\left(\frac{x}{2}\right), [-2\pi, 2\pi]$$

*Domain*
$x, y$*-intercepts*
*Symmetry*
*Asympotes*
*Intervals of increasing/decreasing*
*Maxima/Minima*
*Inflection points*
*Intervals of concavity*
*Sketch a graph of* $f$

***Solution:*** (a) *Domain: This was given as* $[-2\pi, 2\pi]$

(b) $x, y$*-intercepts:*
*x-interpects occur when* $f(x) = 3\sin\left(\frac{x}{2}\right) = 0 \implies \frac{x}{2} = 0 + k\pi$ *where* $k$ *is an integer,* $-1 \leq k \leq 1$.
*On the domain this is* $x = -2\pi, 0, 2\pi$. *y-intercepts:* $f(0) = 3\sin\left(\frac{0}{2}\right) = 0$ *The y-intercept is the point* $(0, 0)$

(c) *Symmetry:*
$f(-x) = 3\sin\left(\frac{-x}{2}\right) = -3\sin\left(\frac{x}{2}\right) \neq f(x)$ *so* $f$ *is not even* $-f(x) = -3\sin\left(\frac{x}{2}\right) = f(-x)$ *so* $f$ *is odd*

(d) *Intervals of increasing/decreasing:*

$$f'(x) = 3\cos\left(\frac{x}{2}\right) \cdot \frac{1}{2} = \frac{3}{2}\cos\left(\frac{x}{2}\right)$$

*Since $f'$ is defined everywhere on $(-2\pi, 2\pi)$ to find the critical points we just solve the equation $f'(x) = 0$ where $-2\pi < x < 2\pi$:*

$$\begin{aligned} f'(x) = 0 &\iff \frac{3}{2}\cos\left(\frac{x}{2}\right) = 0 \\ &\iff \cos\left(\frac{x}{2}\right) = 0 \\ &\iff -2\pi < x < 2\pi \text{ and } x/2 = \pi/2 + n\pi \text{ with } n \text{ an integer} \\ &\iff -2\pi < x < 2\pi \text{ and } x = \pi + n2\pi \text{ with } n \text{ an integer} \\ &\iff x = -\pi \text{ and } x = \pi. \end{aligned}$$

*So the only critical points are $x = -\pi$ and $x = \pi$.*

*If $-2\pi < x < -\pi$, it follows that $-\pi < x/2 < -\pi/2$ so $x/2$ lies in the third quadrant and $\cos(x/2) < 0$.*
*If $-\pi < x < \pi$, it follows that $-\pi/2 < x/2 < \pi/2$ so $x/2$ lies in the fourth and first quadrants and $\cos(x/2) > 0$.*
*If $\pi < x < 2\pi$, it follows that $\pi/2 < x/2 < \pi$ so $x/2$ lies in the second quadrant and $\cos(x/2) < 0$*
$\implies$ *$f$ is increasing on the interval $(-\pi, \pi)$, and $f$ is decreasing on the intervals $(-2\pi, -\pi)$ and $(\pi, 2\pi)$.*

(e) *Maxima/minima:*

*The first derivative test states we have a local maximum if the sign of $f'$ changes from positive to negative, and a local minimum if the sign of $f'$ changes from negative to positive.*

*Therefore there is a local maximum at $x = \pi$ and local minimum at $x = -\pi$ by the results in part (d).*

(f) *Intervals of concavity: Concavity can change at interior points in $[-2\pi, 2\pi]$ where $f''$ is undefined or equal to 0 are only* candidates *for the inflection points.*

$$f''(x) = \frac{-3}{2}\sin\left(\frac{x}{2}\right) \cdot \frac{1}{2} = \frac{-3}{4}\sin\left(\frac{x}{2}\right)$$

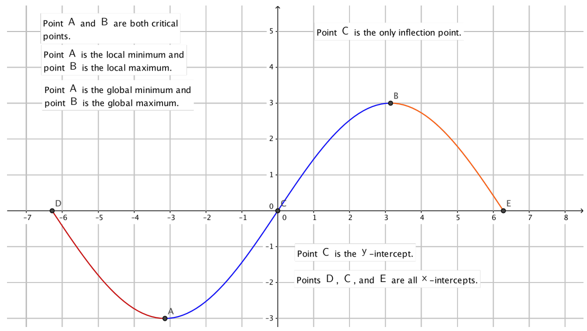
$\frac{-3}{4}\sin\left(\frac{x}{2}\right) = 0$ *when* $\sin\left(\frac{x}{2}\right) = 0$ *which in the domain of $-2\pi < x < 2\pi$ occurs at $x/2 = n\pi$ with $n$ an integer. $-2\pi < x < 2\pi$ and $x = n2\pi$ with $n$ an integer $\implies x = 0$*
*We have a candidate for an inflection point at $x = 0$.*
*If $-2\pi < x < 0$, it follows that $-\pi < x/2 < 0$ so $x/2$ lies in the third and fourth quadrant and $\sin(x/2) < 0$.*
*If $0 < x < 2\pi$, it follows that $0 < x/2 < \pi$ so $x/2$ lies in the first and second quadrant and $\sin(x/2) > 0$.*

*So $f$ is concave up on the interval $(-2\pi, 0)$, and $f$ is concave down on the interval $(0, 2\pi)$.*

(g) *Inflection points: Inflection points occur where the concavity of a function changes. Those interior points in $[-2\pi, 2\pi]$ where $f''$ is undefined or equal to 0 are only* candidates *for the inflection points. We found in part f that we had a candidate for an inflection point at $x = 0$. We verified in part f that concavity changes at $x = 0$.*

*Since $f(0) = 3\sin\left(\frac{0}{2}\right) = 0$, the inflection point is $(0, 0)$*

(h) *Sketch the graph:*




**Problem 5** *Determine the following information about the given function and then graph the function:*

$$f(x) = x\ln(x)$$

*Domain*
*$x, y$-intercepts*
*Symmetry*
*Intervals of increasing/decreasing*
*Maxima/Minima (including absolute)*
*Intervals of concavity*
*Inflection points*
*Given $\lim_{x\to 0^+} f(x) = 0$, sketch a graph $f$.*

***Solution:*** (a) *Domain: $(0, \infty)$*

(b) *$x, y$-intercepts:*
*x-interpects occur when $f(x) = x\ln(x) = 0$. We would think this occurs when either $x = 0$ or $ln(x) = 0$. However, $x = 0$ is not in the domain. We only have $ln(x) = 0$ which is at $x = 1$. y-intercepts: None given the domain.*

(c) *Symmetry: Given the domain of the function, it does not make sense to check for even/odd.*

(d) *Intervals of increasing/decreasing:*

$$f'(x) = \ln(x) + 1$$

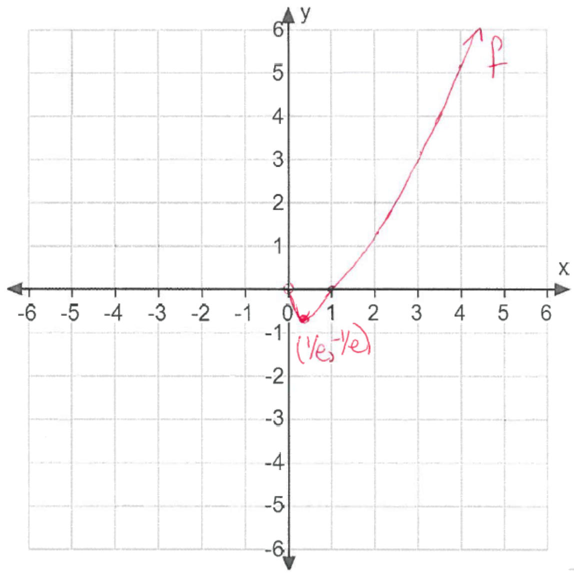
*Find: $f'(x) = \ln(x) + 1 = 0 \implies \ln(x) = -1 \implies x = e^{-1}$*

*Since $f'$ possibly changes its sign at critical points only, the sign of $f'$ is constant on intervals $(0, e^{-1})$ and $(e^{-1}, \infty)$. Since $f'(1) = 1 > 0$, and $f'(e^{-3}) = -3 + 1 = -2 < 0$, it follows that $f'$ is negative on $(0, e^{-1})$ and positive on $(e^{-1}, \infty)$.*
*Therefore, $f$ is decreasing on $(0, e^{-1})$ and $f$ is increasing on $(e^{-1}, \infty)$.*

(e) *Maxima/Minima*
*Since the function is decreasing on $(0, e^{-1})$ and $f$ is increasing on $(e^{-1}, \infty)$., the absolute minimum value of $f$ is attained at $x = e^{-1}$. The absolute minimum value of $f(x) = f(e^{-1}) = e^{-1}\ln e^{-1} = -e^{-1} = -\frac{1}{e}$*
*The absolute maximum value of $f$ does not exist, since the only critical point is a minimum and $\lim_{x\to\infty} f(x) = +\infty$*

(f) *Intervals of concavity and inflection points:* $f'(x) = \ln(x) + 1 \implies f''(x) = \frac{1}{x} > 0$ *on the domain of* $f$*. Concave up on its domain and there are no inflection points.*

(g) *Given* $\lim_{x \to 0^+} f(x) = 0$*, sketch a graph* $f$*:*

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**Problem 6** *Sketch a graph of the function* $f$ *defined by*

$$f(x) = \begin{cases} 2x - 2\tan^{-1}(x) - x\ln(x^2+1) & \text{for } x \leq 1 \\ \sqrt{x-2} & \text{for } x \geq 2 \end{cases}$$

(a) *Find the domain of* $f$*.*

***Solution:*** *The domain of* $f$ *is* $(-\infty, 1] \cup [2, \infty)$.

(b) *Show that* $f'(x) = -\ln(x^2+1)$ *on the interval* $(-\infty, 1)$.

***Solution:*** *For* $x < 1$ *we have*

$$\begin{aligned} f'(x) &= 2 - \frac{2}{1+x^2} - \ln(x^2+1) - x \cdot \frac{1}{x^2+1} \cdot 2x \\ &= 2 - \frac{2}{1+x^2} - \ln(x^2+1) - \frac{2x^2}{x^2+1} \\ &= 2 - \frac{2+2x^2}{x^2+1} - \ln(x^2+1) \\ &= 2 - 2 \cdot \frac{1+x^2}{x^2+1} - \ln(x^2+1) \\ &= 2 - 2 - \ln(x^2+1) = -\ln(x^2+1) \end{aligned}$$

(c) *Compute a formula for* $f'$ *on the interval* $(2, \infty)$.

***Solution:*** *For $x > 2$ we have*

$$f'(x) = \frac{1}{2\sqrt{x-2}}$$

(d) *Compute a formula for $f''$ on the interval $(-\infty, 1)$.*

***Solution:*** *For $x < 1$ we have*

$$f''(x) = \frac{-1}{x^2+1} \cdot 2x = \frac{-2x}{x^2+1}$$

(e) *Compute a formula for $f''$ on the interval $(2, \infty)$.*

***Solution:*** *For $x > 2$ we have*

$$f''(x) = \frac{-1}{4(x-2)^{3/2}}$$

(f) *List all the asymptotes (with limit justifications, if appropriate). (If $f$ does not have a particular asymptote write "NONE".)*

(i) *Vertical asymptotes:*

***Solution:*** *NONE: For $x \geq 2$, $f(x) = \sqrt{x-2}$ so no vertical asymptotes there. For $x \leq 1$, $f(x) = 2x - 2\tan^{-1}(x) - x\ln(x^2+1)$ and we can look at each term. The term $2x$ is a polynomial. The term $-2\tan^{-1}(x)$ is bounded. Finally, the term $-x\ln(x^2+1)$. We would be concerned about where $x^2 + 1 = 0$ but that never happens because of the squared term.*

(ii) *Horizontal asymptotes:*

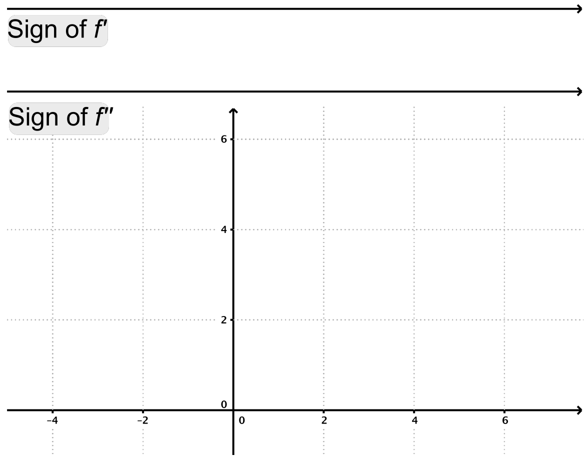
***Solution:*** *No H.A. as $x \to \infty$:*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \underbrace{\sqrt{x-2}}_{\text{form } \infty} = \infty$$

*No H.A. as $x \to -\infty$:*

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} (2x - 2\tan^{-1}(x) - x\ln(x^2+1)) \\
&= \lim_{x\to-\infty} \left[x \cdot (2 - \ln(x^2+1)) - 2\tan^{-1}(x)\right] \\
&= \lim_{x\to-\infty} \left[x \cdot (2 + \ln(x^2+1)^{-1} - 2\tan^{-1}(x)\right] \\
&= \lim_{x\to-\infty} \left[\underbrace{x}_{\to -\infty} \cdot \left(2 + \ln\left(\underbrace{\frac{1}{x^2+1}}_{\to 0}\right)\right) - 2\underbrace{\tan^{-1}(x)}_{\to -\pi/2}\right] \\
&= \lim_{x\to-\infty} \left[\underbrace{x}_{\to -\infty} \cdot \left(2 + \underbrace{\ln\left(\frac{1}{x^2+1}\right)}_{\to -\infty}\right) - \underbrace{2\tan^{-1}(x)}_{\to -\pi}\right] \\
&= \infty
\end{aligned}$$

(g) *In the following sign chart for $f'$, list the points (in the domain of $f$) where $f'(x) = 0$ or $f'(x)$ is undefined, indicate the sign of $f'$ on each subinterval, and identify any local extrema (with "local max" or "local min".) In the following sign chart for $f''$, list the points (in the domain of $f$) where $f''(x) = 0$ or $f''(x)$ is undefined, indicate the sign of $f''$ on each subinterval, and identify any inflection points. Use this information to sketch a graph of $f$ in the blank grid. (Be sure to indicate any $x$- or $y$-intercepts.)*




***Solution:*** ***Finding critical points of $f$:***

*For $x < 1$ we have*

$$\begin{aligned} f'(x) = 0 &\iff -\ln(x^2+1) = 0 \\ &\iff \ln(x^2+1) = 0 \\ &\iff x^2 + 1 = e^0 = 1 \\ &\iff x = 0. \end{aligned}$$

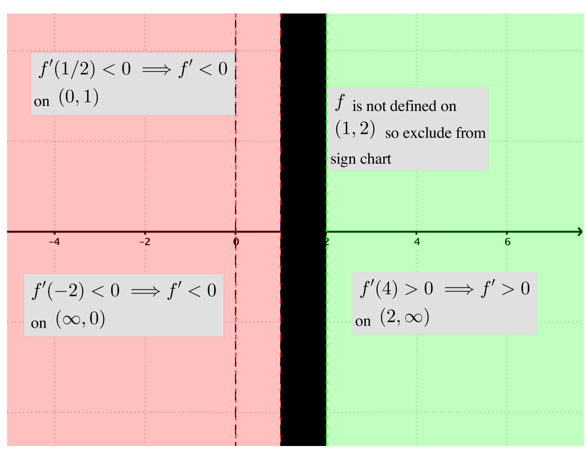
*Critical point at $0$ when $x < 1$.*

*For $x > 2$ we have*

$$f'(x) = 0 \iff \frac{1}{2\sqrt{x-2}} = 0$$

*This last equation is never true. Hence no critical points when $x > 2$.*

*Sign chart of $f'$:*



*Using the sign chart above, we see that the sign of $f'$ does not change at $x = 0$ and is therefore neither a maximum or minimum. So $f$ does not have a local maximum or local minimum.*

*Using the sign chart above, we see $f'$ is positive on $(2, \infty)$ and thus intervals where $f$ is increasing: $(2, \infty)$*

*Using the sign chart above, we see $f'$ is negative on $(-\infty, 0)$, $(0, 1)$ and thus intervals where $f$ is decreasing: $(-\infty, 0)$, $(0, 1)$.*

***Find candidates for inflection points:***

*For $x < 1$ we have*

$$\begin{aligned} f''(x) = 0 &\iff \frac{-2x}{x^2+1} = 0 \\ &\iff x = 0. \end{aligned}$$

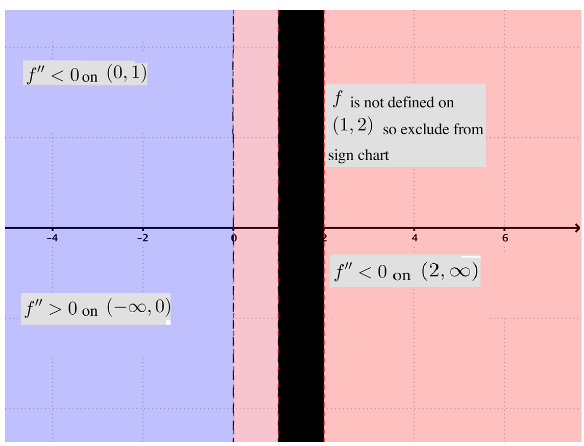
*So, $x = 0$ is a candidate for an inflection point.*

*For $x > 2$ we have*

$$f''(x) = 0 \iff \frac{-1}{4(x-2)^{3/2}} = 0$$

*This last equation has no solution. Hence no candidates for inflection points when $x > 2$.*

*We need to determine the sign of $f''$ on the intervals $(-\infty, 0), (0, 1), (2, \infty)$.*

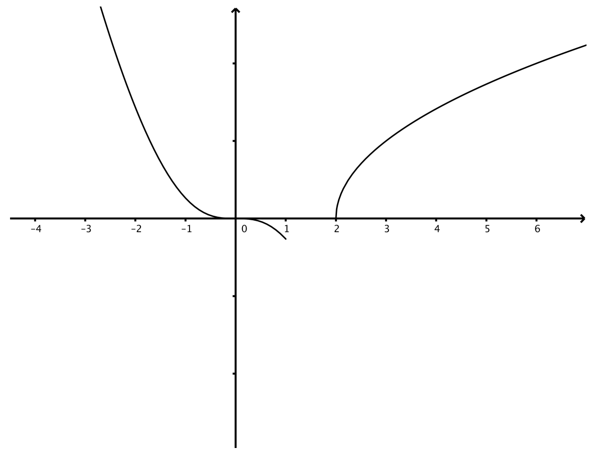



*$f''$ changes sign at $x = 0$ so $(0, 0)$ is an inflection point.*

*intervals where $f$ is concave up: $(-\infty, 0)$*

*intervals where $f$ is concave down: $(0, 1)$, $(2, \infty)$*

*Final sketch of graph:*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−3
−2
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