

Derivatives as functions (DAF)

Problem 1 (a) If $f'(2)$ exists, then

- (i) $\lim_{x \rightarrow 2} f(x)$ must exist, but $\lim_{x \rightarrow 2} f(x) \neq f(2)$
- (ii) $\lim_{x \rightarrow 2} f(x) = f(2)$.
- (iii) $\lim_{x \rightarrow 2} f(x) = f'(2)$
- (iv) $\lim_{x \rightarrow 2} f(x)$ need not exist.

Solution: The correct answer is (ii):

$$\begin{aligned} f'(2) \text{ exists} &\iff f \text{ differentiable at } x = 2 \\ &\implies f \text{ continuous at } x = 2 \\ &\iff \lim_{x \rightarrow 2} f(x) = f(2) \end{aligned}$$

(b) Assuming that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, we can conclude

- (i) $\frac{0}{0} = 1$
- (ii) the tangent line to $y = \sin x$ at $(0, 0)$ has slope 1.
- (iii) you can cancel the x 's.
- (iv) for all x near 0, $\sin x = x$.
- (v) for all x near 0, $\sin x \approx x$.

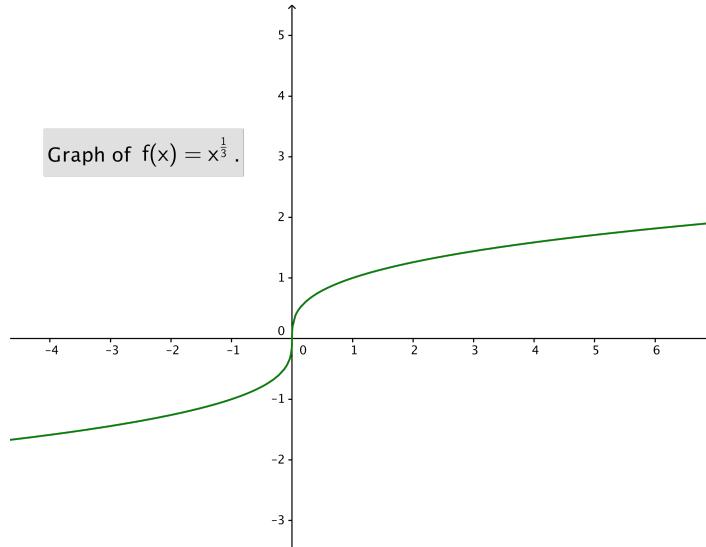
Solution: This problem has two correct answers: (ii) and (v).

Statement (ii) is true:

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{\sin(h+0) - \sin(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\ &= 1 \\ &\implies \text{slope of tangent line at } (0, 0) \text{ is 1} \end{aligned}$$

Statement (v) is true: When x is near 0 the tangent line $y = x$ is a good approximation to f .

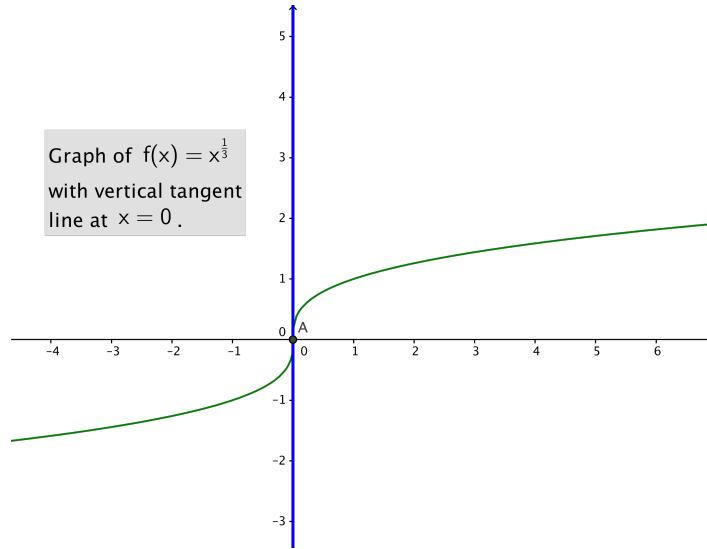
Problem 2 Define the function f by $f(x) = x^{1/3}$ and consider the graph of this function:



Which of the following two statements are true:

- (a) The graph of f has a tangent line at $x = 0$.

Solution: This statement is **true!** The function f has a vertical tangent at $x = 0$:

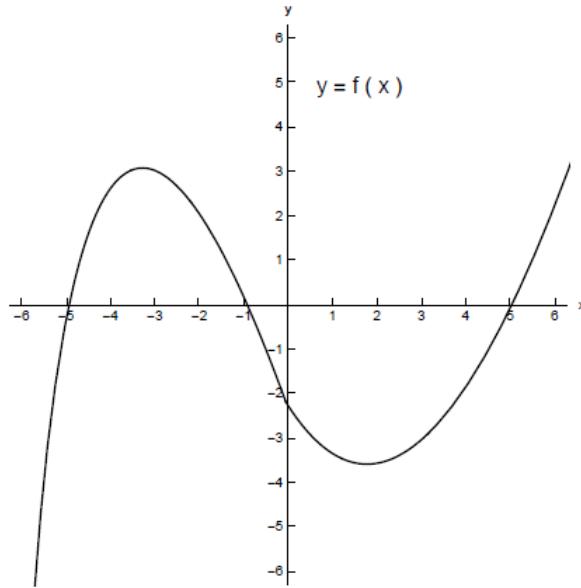


- (b) The derivative $f'(0)$ is defined.

Solution: This statement is **false!**

$$\begin{aligned}
 \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{h^{1/3}}{h} \\
 &= \lim_{h \rightarrow 0^+} \frac{1}{h^{2/3}} \\
 &= \infty \text{ because the limit is of the form: } \frac{\text{pos}}{0^+} \\
 \implies f'(0) &\text{ is undefined}
 \end{aligned}$$

Problem 3 Suppose we are given the graph of a function f :



(a) Use this graph to find the following: (Assume all values will be integers or $+\infty$ or $-\infty$)

- (i) all x where $f(x) = 0$,

Solution: $f(x)$ is zero when the function crosses the x -axis. Therefore $f(x) = 0$ when $x = -5, x = -1$, and $x = 5$.

- (ii) all x where $f(x) > 0$,

Solution: $f(x)$ is positive when the graph of the function is above the x -axis. Therefore $f(x) > 0$ on $(-5, -1) \cup (5, \infty)$.

- (iii) all x where $f(x) < 0$, and

Solution: $f(x)$ is negative when the graph of the function is below the x -axis. Therefore $f(x) < 0$ on $(-\infty, -5) \cup (-1, 5)$.

- (iv) all x where $f(x)$ attains a local maximum and all x where f attains a local minimum.

Solution: $f(x)$ has a local maximum at $x = -3$. $f(x)$ has a local minimum at $x = 2$.

Without sketching the graph of f' find

- (b) (i) all x where $f'(x) = 0$,

Solution: $f'(x)$ is zero when the tangent line has a slope of zero, which is approximately at $x = -3$ and $x = 2$. Note, for this question, these are the same answers as the (local) highest and lowest point for the graph of f .

- (ii) all x where $f'(x) > 0$,

Solution: $f'(x)$ is positive when the slope of the tangent line is positive. Observe that f is increasing on $(-\infty, -3), (2, \infty)$ and this same set of intervals is where the tangent lines have positive slope. Therefore $f'(x) > 0$ on $(-\infty, -3), (2, \infty)$.

(iii) all x where $f'(x) < 0$, and

Solution: $f'(x)$ is negative when the slope of the tangent line is negative. Observe that f is decreasing on $(-3, 2)$ and this interval is where the tangent lines have negative slope. Therefore $f'(x) < 0$ on $(-3, 2)$.

(iv) On the following intervals, does $f'(x)$ seem to be increasing or decreasing?

- i. $(-\infty, 0)$

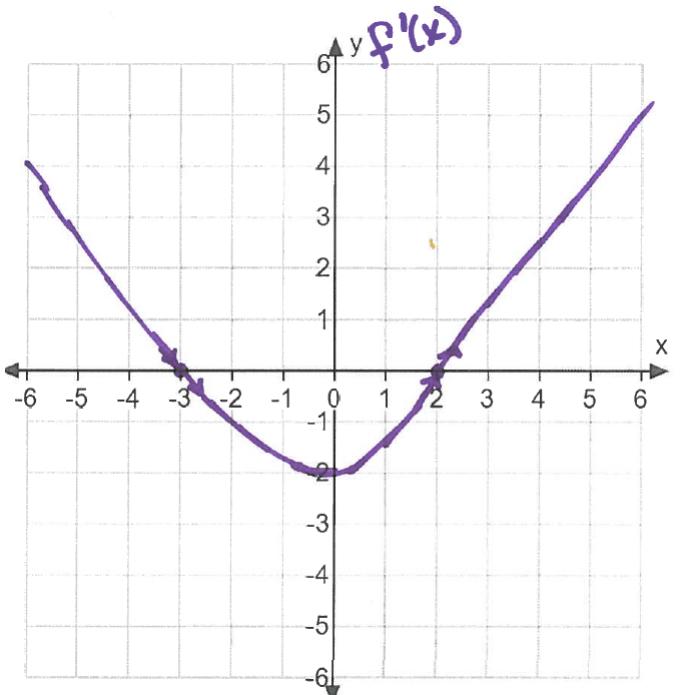
Solution: decreasing

- ii. $(0, \infty)$

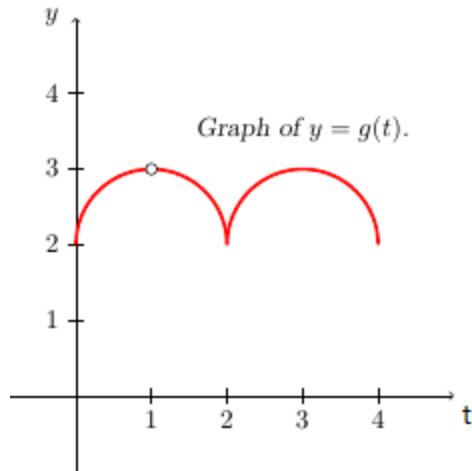
Solution: increasing

(c) Sketch a graph of f' .

Solution: The graph of f' is approximately



Problem 4 Use the graph of g



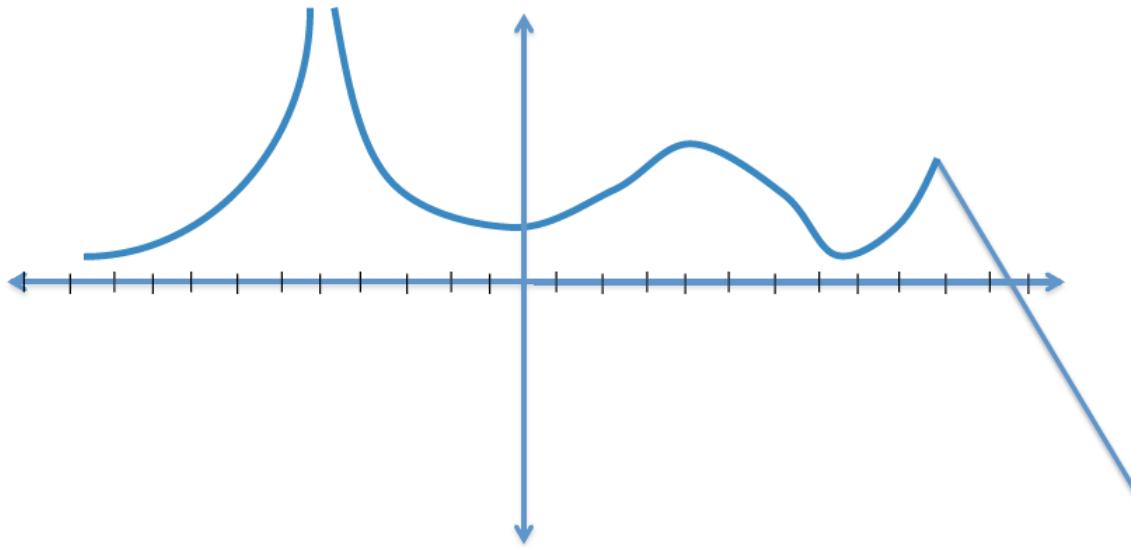
- (a) Find the values of t in $(0, 4)$ at which g is not continuous.

Solution: g is not continuous at $t = 1$.

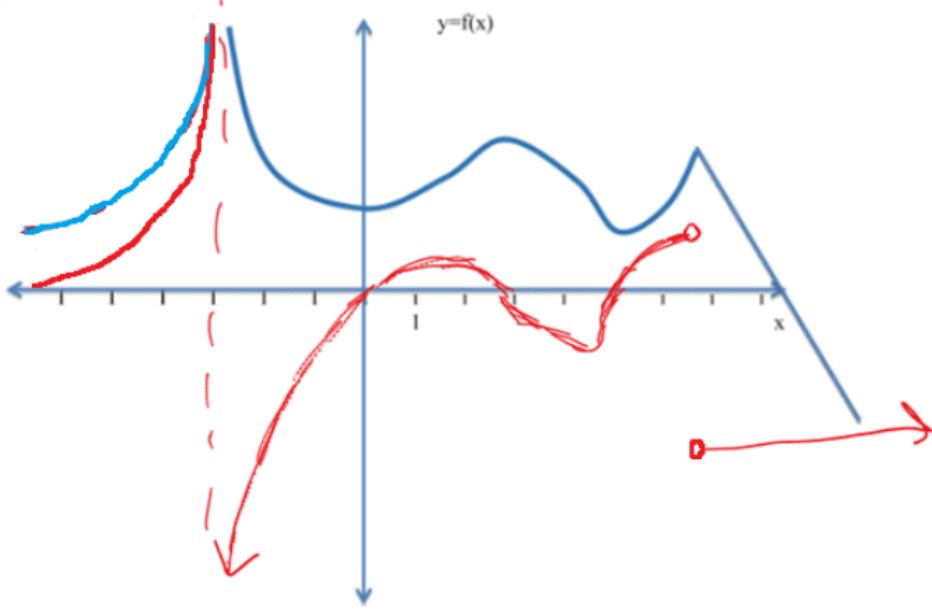
- (b) Find the values of t in $(0, 4)$ at which g is not differentiable.

Solution: g is not differentiable at $t = 1$ and $t = 2$.

Problem 5 Given the following graph of a function h sketch a graph of the derivative h' .



Solution: The graph of the derivative is in red. **Important Note:** Despite being drawn on the same graph, the “units” for f and f' are not the same!



Problem 6

(a) Fill in the blanks

$$f'(x) = \lim_{\text{???}} \frac{\text{????}}{h}$$

if the limit exists.

Solution:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists.

(b) Let

$$f(x) = \frac{1}{x+4}.$$

Use the (limit) definition of derivative in (a) to find $f'(x)$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h+4} - \frac{1}{x+4}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{x+4-(x+h+4)}{(x+h+4)(x+4)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{(x+h+4)(x+4)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x+h+4)(x+4)} = \frac{-1}{(x+4)^2} \end{aligned}$$

Problem 7 Let $f(x) = |5 - x|$.

- (a) For $a < 5$, find $f'(a)$.

Solution: Recall that

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}.$$

When $a < 5$, $0 < 5 - a$ and so $f(a) = |5 - a| = 5 - a$. Since we are considering the limit as x approaches a , (and therefore interested in values of x really close to a), we may also assume that $x < 5$ and therefore $f(x) = 5 - x$. Thus

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(5 - x) - (5 - a)}{x - a} \\ &= \lim_{x \rightarrow a} \frac{-(x - a)}{x - a} \\ &= \lim_{x \rightarrow a} -1 = -1. \end{aligned}$$

- (b) For $a > 5$, find $f'(a)$.

Solution: When $a > 5$, $0 > 5 - a$ and so $f(a) = |5 - a| = -(5 - a) = a - 5$. Since we are considering the limit as x approaches a , we may also assume that $x > 5$ and therefore $f(x) = x - 5$. Thus

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \rightarrow a} \frac{(x - 5) - (a - 5)}{x - a} \\ &= \lim_{x \rightarrow a} \frac{x - a}{x - a} \\ &= \lim_{x \rightarrow a} 1 = 1. \end{aligned}$$

- (c) Determine whether $f'(5)$ exists.

Solution: If $f'(5)$ exists then $\lim_{x \rightarrow 5} \frac{f(x) - f(5)}{x - 5}$ exists. But when x is close to 5 it may be that $x > 5$ or $x < 5$. So in order to find this limit, we have to check whether the two one-sided limits are equal.

This means, $\lim_{x \rightarrow 5^-} \frac{f(x) - f(5)}{x - 5} = \lim_{x \rightarrow 5^+} \frac{f(x) - f(5)}{x - 5}$

$$\begin{aligned} \lim_{x \rightarrow 5^-} \frac{f(x) - f(5)}{x - 5} &= \lim_{x \rightarrow 5^-} \frac{-(x - 5) - (0)}{x - 5} = -1 \\ \lim_{x \rightarrow 5^+} \frac{f(x) - f(5)}{x - 5} &= \lim_{x \rightarrow 5^+} \frac{(x - 5) - (0)}{x - 5} = 1 \\ \lim_{x \rightarrow 5^-} \frac{f(x) - f(5)}{x - 5} &\neq \lim_{x \rightarrow 5^+} \frac{f(x) - f(5)}{x - 5} \end{aligned}$$

Therefore, $f'(5)$ does not exist.

$$f'(a) = \begin{cases} 1 & \text{if } a > 5 \\ \text{undefined} & \text{if } a = 5 \\ -1 & \text{if } a < 5 \end{cases}$$

- (d) Sketch a graph of the function $f(x)$ and its derivative $f'(x)$

Solution:

