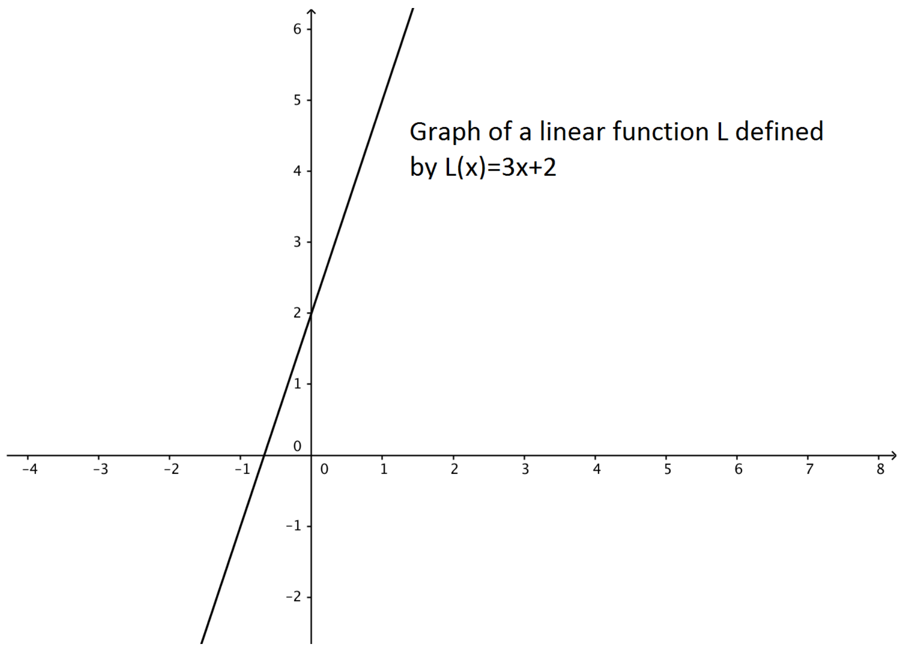
2.1: The Idea of Limits

Problem 1

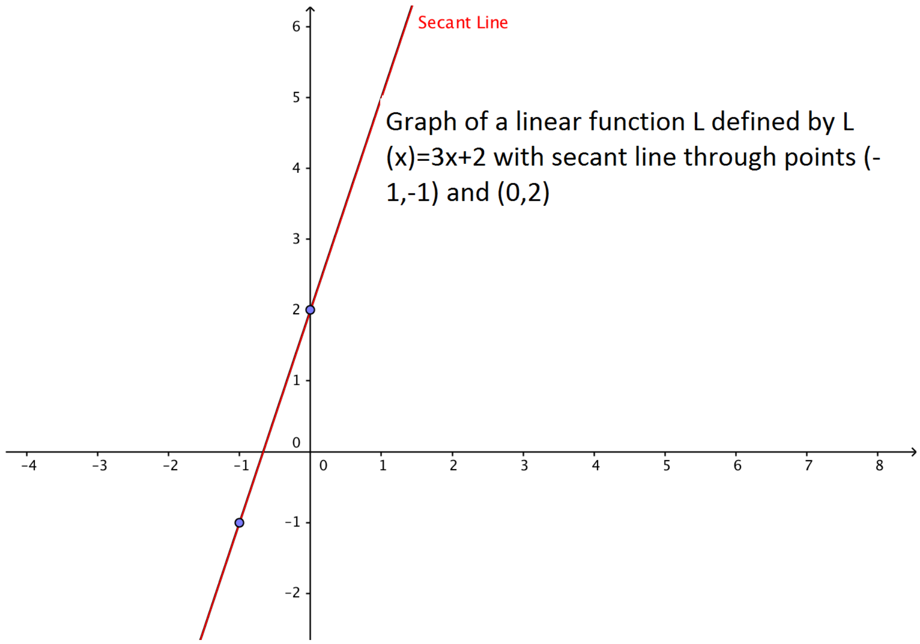
(a) *What does a secant line to the graph of a linear function look like? What does a tangent line to the graph of a linear function look like?*

Solution: *A graph of a linear function L defined by $L(x) = mx + b$, is a line, where m is the slope of the line and b is the y-intercept.*

For instance, here is the graph of a particular linear function

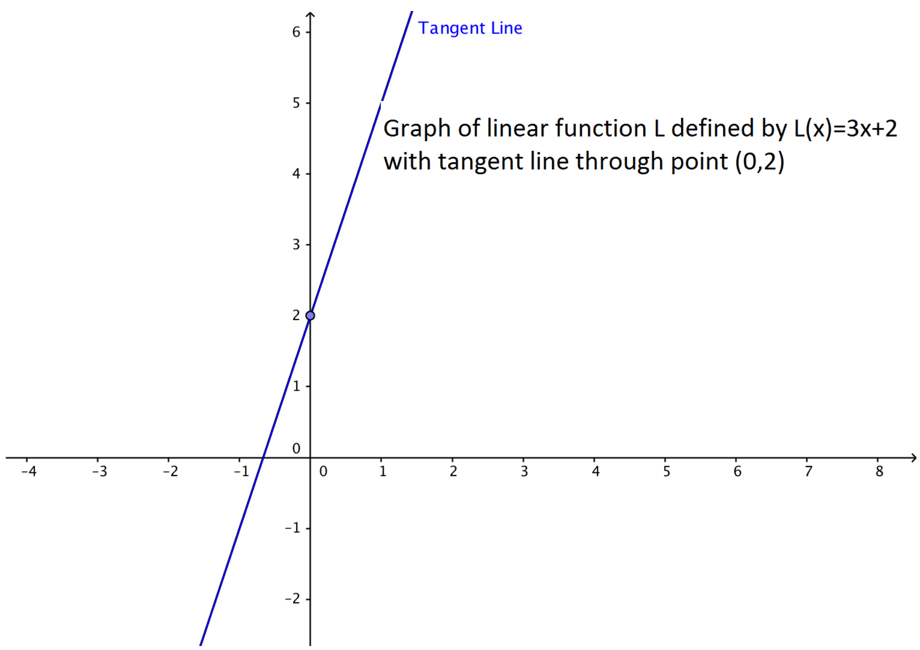


If we draw a secant line between two points of this graph we have



So the secant line is identical to the line itself.

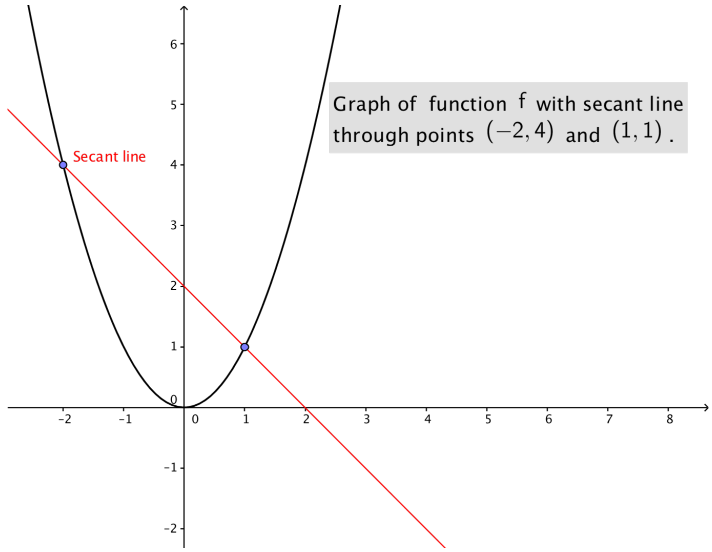
If we draw a tangent line at one point on this graph we have

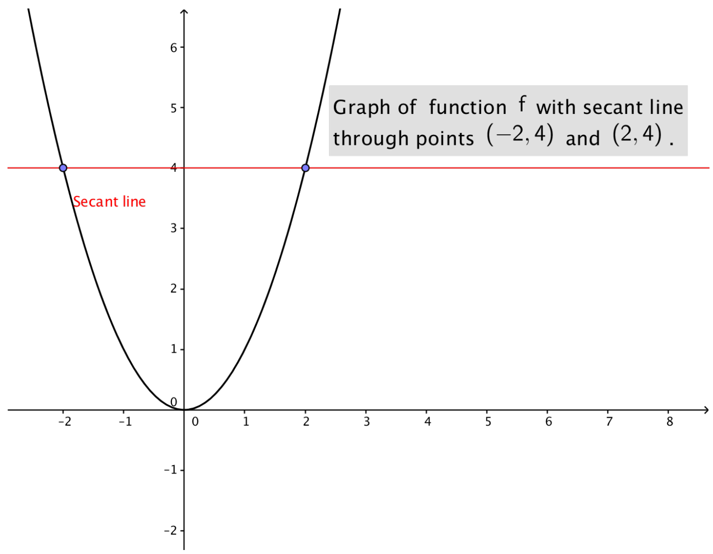


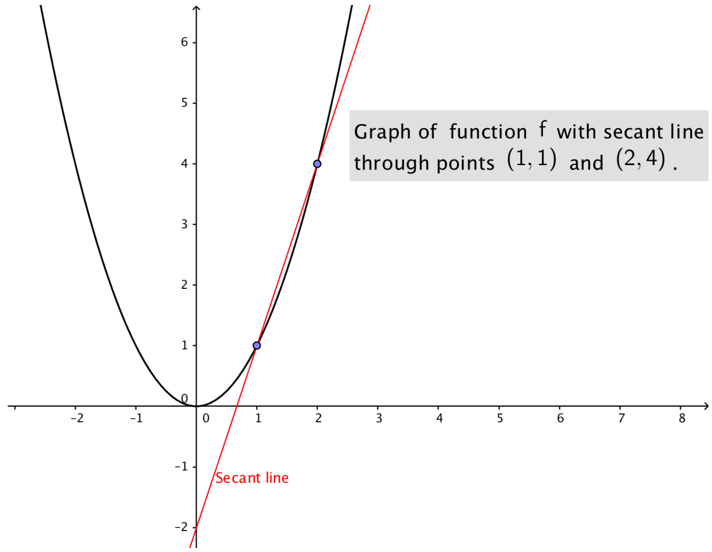
So the tangent line is identical to the line itself.

(b) *What might a secant line and tangent line of the function f, defined by $f(x) = x^2$, look like?*

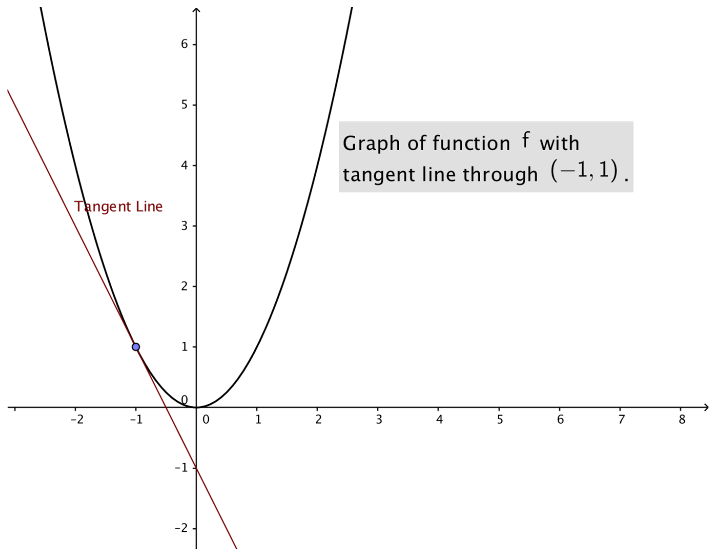
Solution:


Graph of function f with secant line through points $(-2, 4)$ and $(1, 1)$.
Secant line

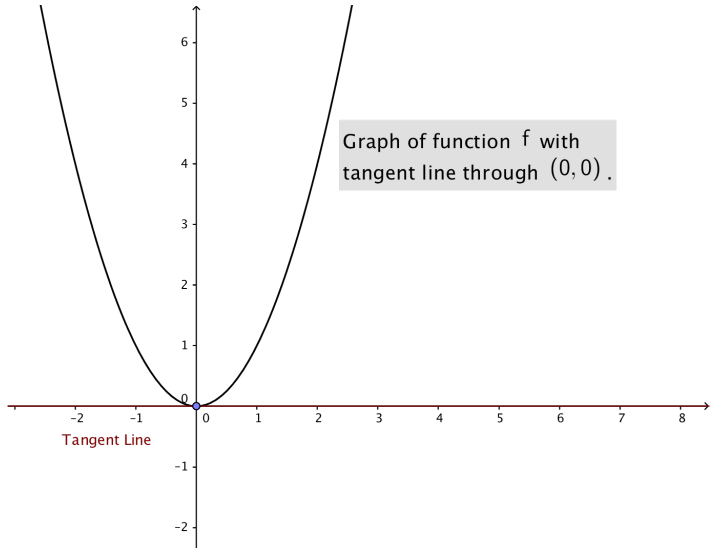

Graph of function f with secant line through points $(-2, 4)$ and $(2, 4)$.
Secant line

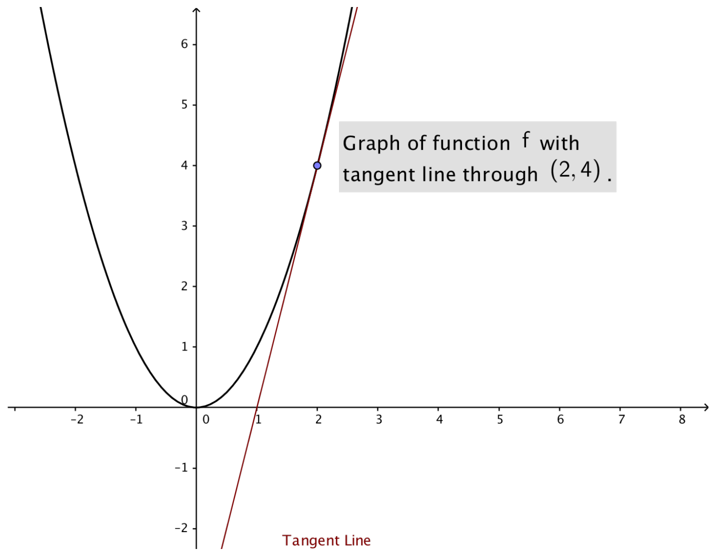


Graph of function f with secant line through points $(1,1)$ and $(2,4)$.
Secant line

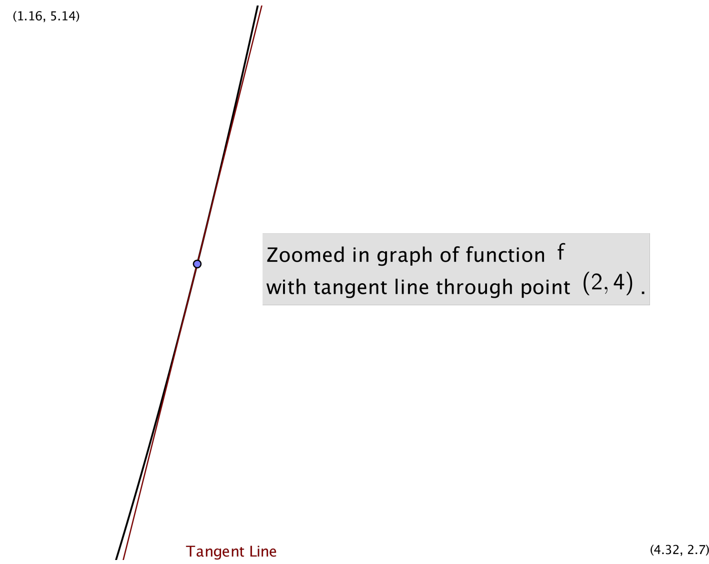


Graph of function f with tangent line through $(-1,1)$.
Tangent Line



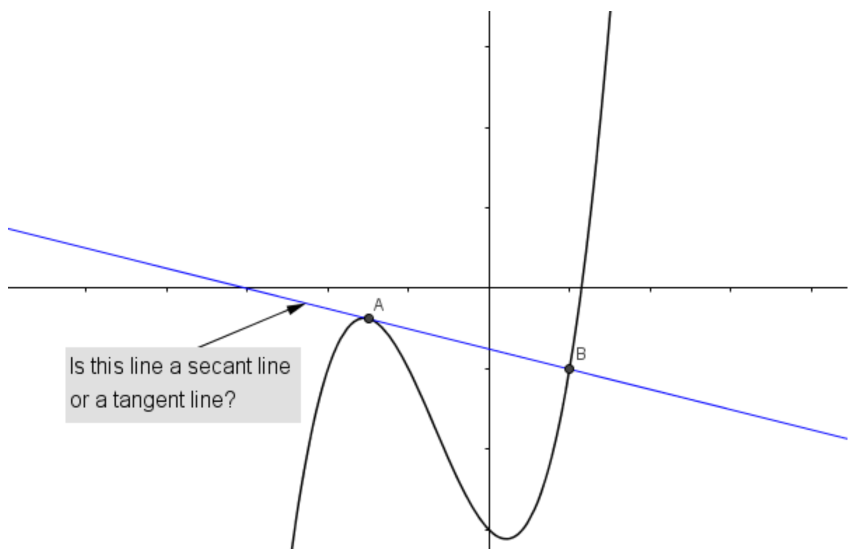


There is an important difference between secant lines and tangent lines! When we zoom in enough, at an appropriate point, *the tangent line looks* nearly *indistinguishable from the graph itself:*



Secant lines usually don't have this property.

(c) *In the graph below, is the given line a secant line or a tangent line at the point $A(a, f(a))$?*

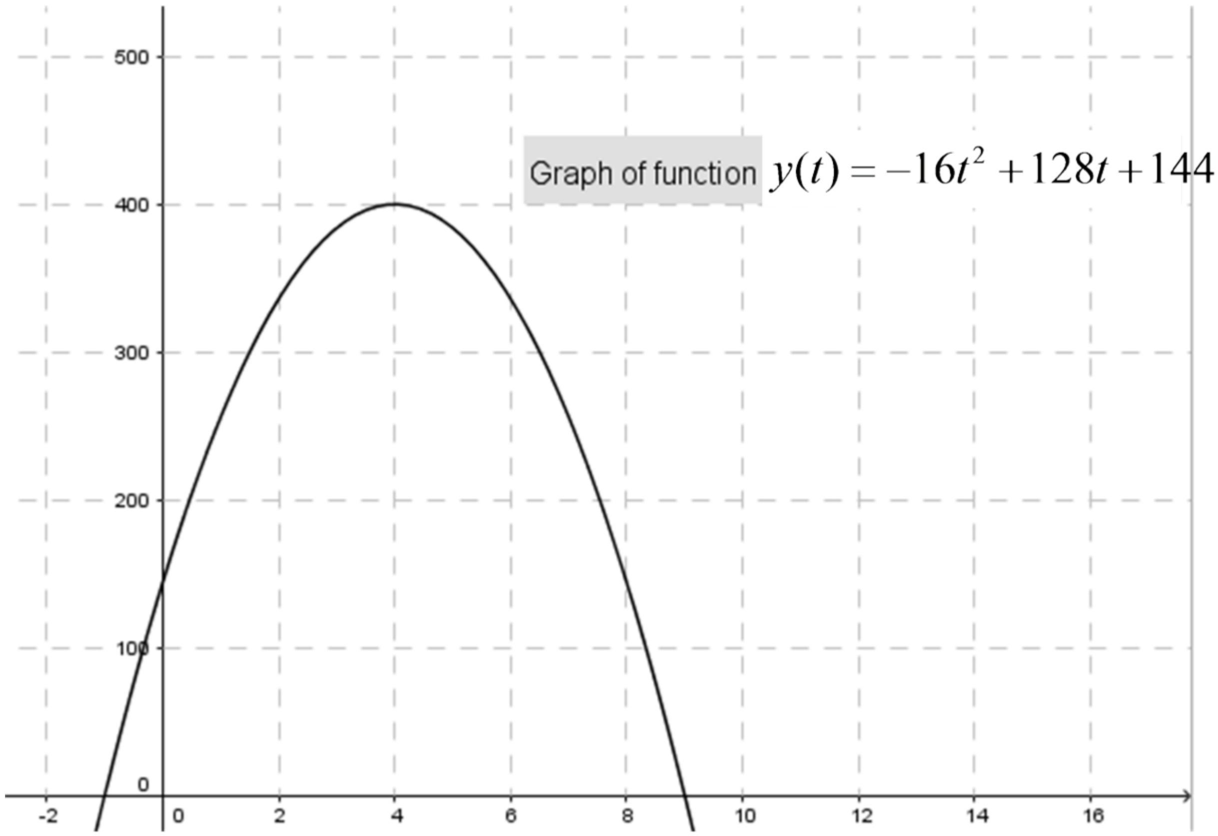


Solution: *This is a trick question!*

The given line is a tangent line at $A(a, f(a))$—when we zoom in enough the graph is nearly indistinguishable from its tangent line at that point. But, it can also be considered a secant line through A and B.

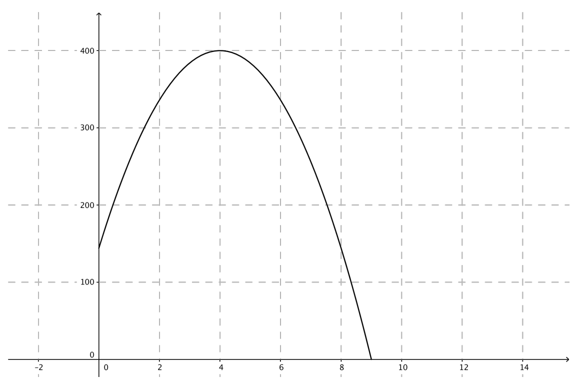
By convention however, since we have drawn the graph by emphasizing only one point of intersection we usually interpert such a line as a tangent line.

Problem 2 *Part of the given parabola can be used to model to "position-time"graph of a ball thrown straight up into the air. The graph gives the height of the ball in feet t seconds after being thrown into the air. Use this graph, and the given function, to answer the following questions.*



(a) *Mark the part of the parabola that can be used to model the position of the ball.*

Solution:



(b) *What are the units on the t axis? What are the units on the y axis?*

Solution: *The units on the t axis are "seconds" (for time), while the units on the y axis are "feet" (for height).*

(c) *If you were watching a movie of the ball being thrown, is the graph a picture of the path that the ball follows? Why or why not?*

Solution: *No, the position-time graph is* not *the path the ball follows. The graph shows the height of the ball at a given* time. *The ball is thrown straight up and has no horizontal movement so its path is on a vertical line.*

(d) *Let $f(t)$ denote the height of the ball at time $t, t \geq 0$. What is the height of the ball at time $t = 0$?*

Solution: *The height can be found by finding $f(0)$*

$$f(0) = -16(0)^2 + 128(0) + 144$$
$$f(0) = 144 \textit{ feet}$$

(e) *When will the ball hit the ground?*

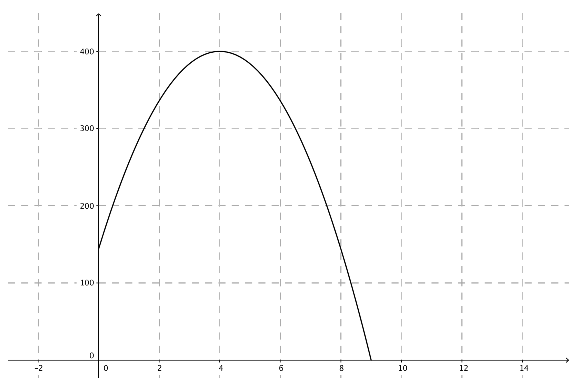
Solution: *The ball will hit the ground when the height $f(t)$ equals zero.*

$$0 = -16t^2 + 128t + 144$$
$$0 = -16(t^2 - 8t - 9)$$
$$0 = -16(t + 1)(t - 9)$$
$$t = -1, 9$$

The ball will hit the ground at $t = 9$ or 9 seconds after the ball is thrown into the air.

(f) *What is the domain of the position function, f, of the ball?*

Solution: *The domain of f is the interval $[0, 9]$. The ball is released at $t = 0$ and hits the ground at $t = 9$. With this domain, the position-time graph of the ball is given by*



(g) *Use the table of values to find the average velocity of the ball between $t = 8.9$ and $t = 9$ seconds.*

t	$\approx f(t)$
8.9	15.84
8.99	1.6
8.999	0.159984
8.9999	0.015998
9	0

Solution: *The average velocity of the ball between $t = 8.9$ seconds and $t = 9$ seconds is*

$$\frac{f(9) - f(8.9)}{9 - 8.9} = \frac{0 - 15.84}{0.1} = -158.4 \textit{ feet per second}$$

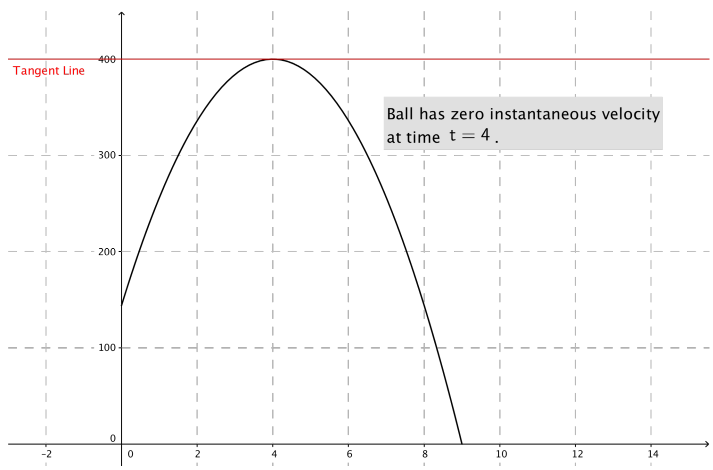
(h) *Use the table of average velocities to approximate the instantaneous velocity of the ball when it hits the ground.*

Time Interval	*Average Velocity*
$[8.9, 9]$	$\dfrac{f(9) - f(8.9)}{.1} = \dfrac{0 - 15.84}{.1} = -158.4$
$[8.99, 9]$	$\dfrac{f(9) - f(8.99)}{.01} = \dfrac{0 - 1.5984}{.01} = -159.84$
$[8.999, 9]$	$\dfrac{f(9) - f(8.999)}{.001} = \dfrac{0 - .159984}{.001} = -159.984$
$[8.9999, 9]$	$\dfrac{f(9) - f(8.999)}{.0001} = \dfrac{0 - .0159998}{.0001} = -159.998$

Solution: *The instantaneous velocity of the ball hitting the ground appears to be -160 ft/sec.*

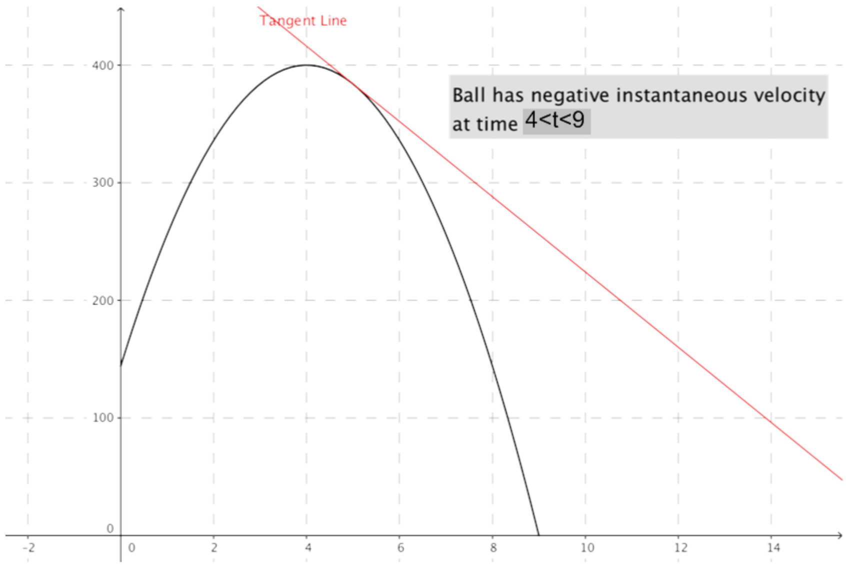
(i) *Use the graph to determine if, at any moment in time, the ball has instantaneous velocity equal to 0. Why or why not?*

Solution: *The ball has zero instantaneous velocity when the graph $f = f(t)$ has a tangent line with zero slope:*



(j) *For which times is the instantaneous velocity of the ball negative? What happens to the height of the ball when its velocity is negative?*

Solution: *The instantaneous velocity of the ball is negative for $4 < t < 9$:*



The height of the ball is decreasing at those times.