Extreme and Mean Value Theorems (MVT)

SUMMARY of the Extreme Value Theorem

A function f has a **global maximum** at $x = a$, if $f(a) \geq f(x)$, for all x in the domain of f.

A function f has a **global minimum** at $x = a$, if $f(a) \leq f(x)$, for all x in the domain of f.

Theorem 1 (Extreme Value Theorem (EVT)). If a function f is **continuous on the closed interval** $[a, b]$, then f attains both a global maximum and a global minimum on the closed interval $[a, b]$.

Note: If a function f has a global extremum (minimum or maximum) at $x = c$, then c is either a boundary point (which means that $c = a$ or $c = b$), or f has a critical point at $x = c$.

So, if we have to find an extreme value of f on $[a, b]$, we should check all the boundary and all the critical points of f and compare the values of f at those points. The biggest of those values is the maximum value of f on $[a, b]$, and the smallest one is the minimum value of f on $[a, b]$.

SUMMARY of The Mean Value Theorem

Theorem 2 (The Mean Value Theorem (MVT)). If a function f is **continuous on the closed interval** $[a, b]$, and **differentiable on** (a, b), then there exists a point c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Note 1a: The value $\dfrac{f(b) - f(a)}{b - a}$ is the slope of the (secant) line through the points $(a, f(a))$ and $(b, f(b))$.
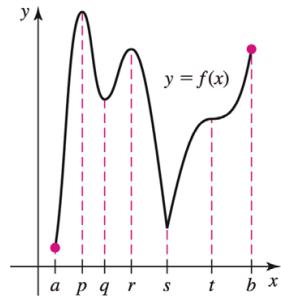
Note 1b: The line tangent to the graph of f at the point where $x = c$ is **parallel** to the line through the points $(a, f(a))$ and $(b, f(b))$.

Note 2a: The value $\dfrac{f(b) - f(a)}{b - a}$ is the average rate of change of f on the interval $[a, b]$.

Note 2b: The instantaneous rate of change f at the point where $x = c$ is equal to the average rate of change of f on $[a, b]$.

Recitation Questions

Problem   **1**   *Find the x-coordinates of the points where the function f has a global max or min.*

Problem 2 *In each problem, sketch a graph of a function meeting the given critera.*

(a) *Sketch a possible graph of a function which is continuous on an open interval $(-1, 5)$ but does not have an global maximum or minimum.*

(b) *Sketch a possible graph of a function with an global minimum, a local maximum and a local minimum, but no global maximum on the interval $[-4, 5]$.*

(c) *Sketch a possible graph of a function f, continuous on $[1, 4]$ with the following properties: $f'(x) = 0$ for $x = 2$ and $x = 3$; f has a global minimum at $x = 4$; f has a global maxmimum at $x = 3$, and f has a local minimum at $x = 2$.*

Problem **3** *Find the x-coordinates of global extrema and global extreme values of f on the given interval.*
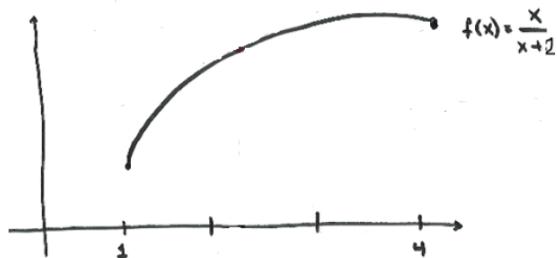
(a) $f(x) = x\sqrt{2 - x^2}$ on $[-\sqrt{2}, \sqrt{2}]$.

(b) $f(x) = x^3 e^{-x}$ on $[-1, 5]$.
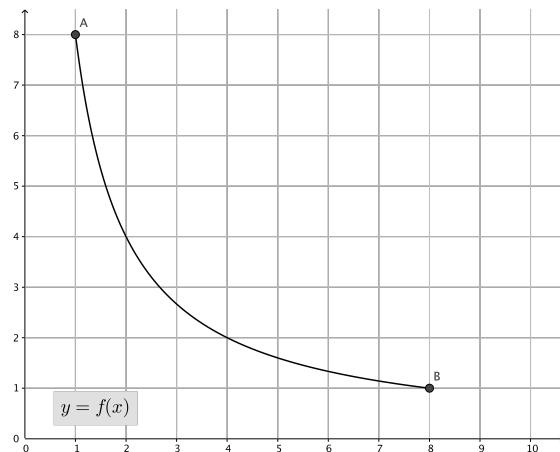
(c) $f(x) = x \ln \left(\dfrac{x}{5} \right)$ on $[0.1, 5]$.

Problem 4 *Verify that the given function satisfies the hypotheses of the Mean Value Theorem in the given interval. Then algebraically find all numbers c that satisfy the conclusion of the Mean Value Theorem. Using the graph provided, label the point(s) c and sketch the secant line through the points $(1, f(1))$ and $(4, f(4))$ and the tangent line at c.*

$$f(x) = \frac{x}{x+2} \qquad on \; [1,4]$$

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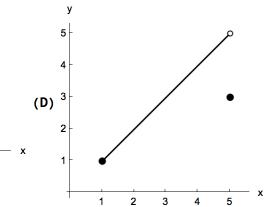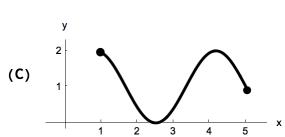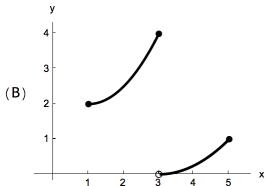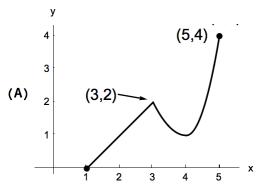
**Problem 5** *A curve is given in the figure below, where $f(x) = 8/x$.*

(a) *In the figure above, draw a secant line joining the points $A = (1, f(1))$ and $B = (8, f(8))$.*

(b) *Find the slope, m_{sec}, of this secant line.*

(c) *Show that the function f satisfies the conditions of the Mean Value Theorem on the interval $[1, 8]$ and find a point (or points) guaranteed to exist by the Mean Value Theorem.*

Problem 6 *Given the four functions on the interval* $[1, 5]$, *answer the questions below.*
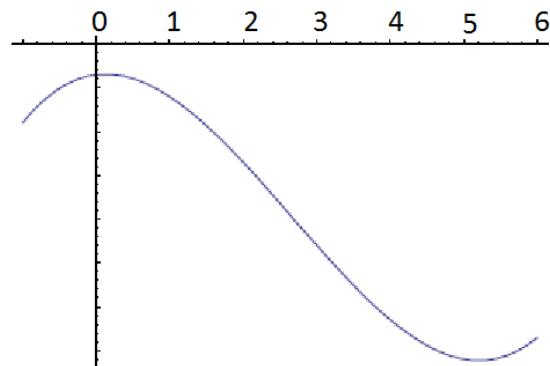
(a) List the functions that satisfy the hypothesis of the Extreme Value Theorem on $[1, 5]$.

(b) For each of the functions, determine if they have a global maximum, a global minimum, both, or neither.

(c) List the functions that satisfy the hypothesis of the Mean Value Theorem on $[1, 5]$.

(d) List the function (or functions) for which there exists a point c in $(1, 5)$ such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1}$$

Problem 7 *Heidi drives from her house in Columbus, OH to Indianapolis, IN for vacation. Google maps says her driving distance is 170 miles. The drive takes her 2.5 hours. The police send her a speeding ticket in the mail, saying she must have sped to arrive so quickly. She is fighting the ticket, saying she just never stopped through the whole drive. Can you prove she broke the 65 mph speed limit at some point during her drive?* **EXPLAIN**.

Problem 8 *Does the function given in the graph below satisfy the hypotheses of the Mean Value Theorem in the interval $[-1, 6]$? If so, estimate the values of all numbers c that satisfy the conclusion of the Mean Value Theorem.*

Problem 9 Let $f(x) = (x-3)^{-2}$. Show that there is no value c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4-1)$. Why does this not contradict the Mean Value Theorem?

Problem   10   *Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. (Hint: Consider the function $h(t) = f(t) - g(t)$, where $f(t)$ and $g(t)$ are the position of the first and second runner at time t, respectively.)*

Problem 11 *Verify that the given function satisfies the hypotheses of the Mean Value Theorem in the given interval. Then algebraically find all numbers c that satisfy the conclusion of the Mean Value Theorem.*

$$f(x) = x + \sin\left(\frac{\pi}{4} \cdot x\right) \qquad on \ [0, 2]$$

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