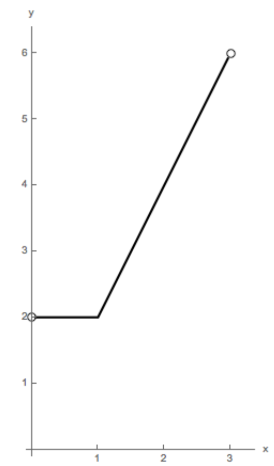
Understanding functions (UF)

SUMMARY: What we need to know about functions

- State the definition of a function.
- Find the domain and range of a function.
- Distinguish between functions by considering their domains.
- Recognize different representations of the same function.
- Determine where a function is positive or negative.
- Determine algebraically whether a function is even, odd, or neither.
- Use symmetry when graphing even or odd functions.
- Plot basic functions.
- Apply appropriate transformations to graphs of basic functions
  (vertical and horizontal shifts, vertical and horizontal stretching and reflecting)
- Perform basic operations and compositions on functions.
- Work with piecewise defined functions.
- Determine if a function is one-to-one.
- Define and work with inverse functions.
- Plot inverses of basic functions.
- Find inverse functions (algebraically and graphically).
- Find the largest interval containing a given point where the function is invertible.
- Determine the intervals on which a function has an inverse.
- Relate the domain and range of f and the domain and range of f^{-1}.

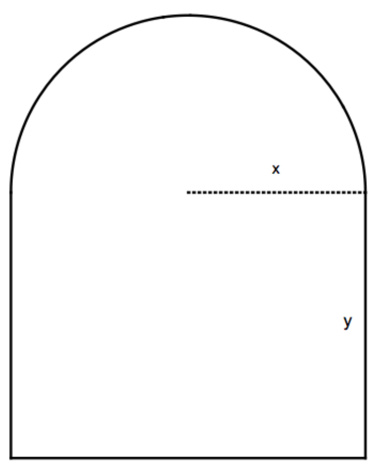
Problem 1 *Find a formula for the function f whose graph is given in the figure below. What is the domain of f? What is the range of f? Is the function f linear?*



Solution: $f(x) = \begin{cases} 2 & \text{if } 0 < x \leq 1 \\ 2 + 2(x-1) & \text{if } 1 < x < 3 \end{cases}$

Domain of $f = (0,3)$, *range of* $f = [2,6)$.
f *is not linear; because its graph is NOT a line; it is a piecewise defined function.*

Problem 2 *A mirror has the shape of a rectangle surmounted by a semicircle (see figure). The area of the mirror is 32* in^2. *Let x be the radius of the semicircle (that lies on top of the rectangle). Express the perimeter of the mirror P (in inches) as a function of x (in inches). Is P a polynomial, a rational function or a transcendental function?*



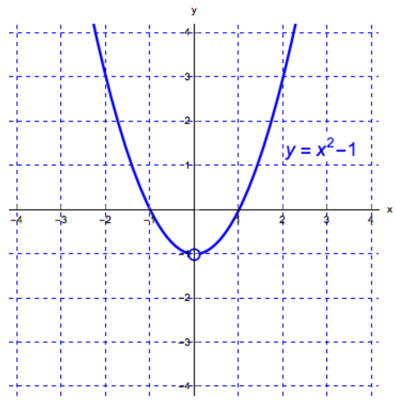
Solution: *Since the area* $A = 32 = 2xy + \frac{x^2\pi}{2}$, *it follows that* $y = \frac{16}{x} - \frac{x\pi}{4}$ *and that* $P(x) = 2x + 2y + x\pi = 2x + \frac{32}{x} - \frac{x\pi}{2} + x\pi = 2x + \frac{x\pi}{2} + \frac{32}{x}$

Problem 3 *Define* $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 0 \\ ? & \text{if } x > 0 \end{cases}$

(a) *Find an expression for "?" such that f will be even.*

Solution: *If f is even then $f(x) = f(-x)$.*

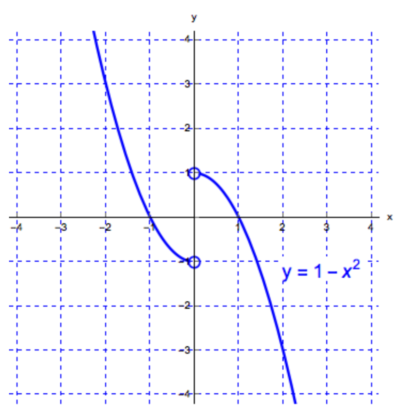
$$\begin{aligned} f(x) &= f(-x) \text{ for } x > 0 \\ &= (-x)^2 - 1 \\ &= x^2 - 1 \end{aligned}$$



(b) *Find an expression for "?" such that f will be odd.*

Solution: *If f is odd then $f(-x) = -f(x)$.*

$$\begin{aligned} -f(x) &= f(-x) \text{ for } x > 0 \\ &= (-x)^2 - 1 \\ &= x^2 - 1 \\ &\implies f(x) = -x^2 + 1 \end{aligned}$$



Problem 4 *Given* $y(t) = t - \dfrac{\pi}{3}$ *and* $w(t) = \sin(t)$*. Find:*

(a) $y(w(t))$

Solution: $y(w(t)) = y(\sin(t)) = \sin(t) - \frac{\pi}{3}$

(b) $w(y(t))$

Solution: $w(y(t)) = w\left(t - \frac{\pi}{3}\right) = \sin\left(t - \frac{\pi}{3}\right)$

(c) $w\left(y\left(\frac{4\pi}{3}\right)\right)$

Solution: $w\left(y\left(\frac{4\pi}{3}\right)\right) = \sin\left(\frac{4\pi}{3} - \frac{\pi}{3}\right) = sin(\pi) = 0$

(d) $y(w(\frac{4\pi}{3}))$

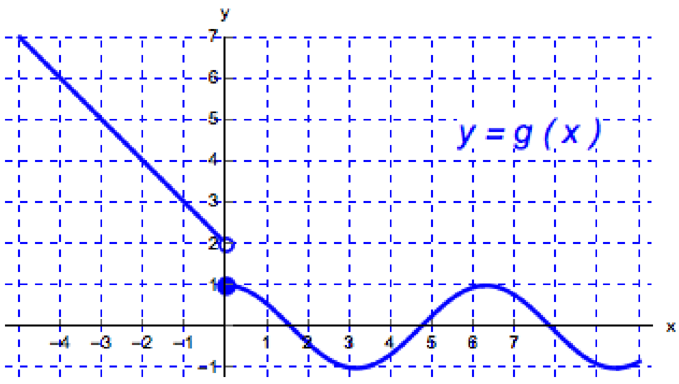
Solution: $y\left(w\left(\frac{4\pi}{3}\right)\right) = \sin\left(\frac{4\pi}{3}\right) - \frac{\pi}{3} = \frac{-\sqrt{3}}{2} - \frac{\pi}{3}$

You should know values of $sin(x)$ and $cos(x)$ for all values found on the standard unit circle.

Problem 5 *Define* $g(x) = \begin{cases} |x-2| & \text{if } x < 0 \\ \cos(x) & \text{if } x \geq 0 \end{cases}$

(a) *Sketch a graph of g*

Solution:



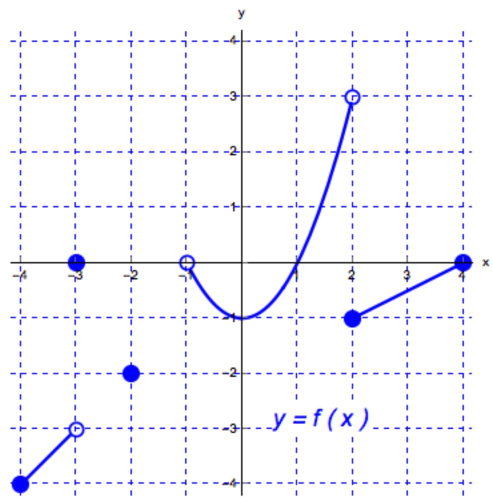
(b) *Find the domain and range of g*

Solution: *Domain:* $(-\infty, \infty)$, *Range:* $[-1, 1] \cup (2, \infty)$

(c) *Find the values of $g(\pi)$ and $g(-\pi)$*

Solution: $g(\pi) = \cos(\pi) = -1$ *and* $g(-\pi) = |-\pi - 2| = \pi + 2$

Problem 6 *The entire graph of $f(x)$ is given below.*



(a) *Find the domain and range of f*

Solution: *Domain:* $[-4,-3] \cup \{-2\} \cup (-1,4]$ *Range:* $[-4,-3) \cup \{-2\} \cup [-1,3)$

(b) *Find the values of $f(-3), f(-2), f(-1), f(2)$*

Solution: $f(-3) = 0, f(-2) = -2, f(-1)$ *does not exist,* $f(2) = -1$

(c) *Find the intervals on which $f(x)$ is positive. Find the intervals on which $f(x)$ is negative.*

Solution: $f(x)$ *is postive on* $(1,2)$. $f(x)$ *is negative on* $[-4,-3), \{-2\}, (-1,1), [2,4)$

(d) *Find the intervals on which f is increasing. Find the intervals on which f is decreasing.*

Solution: f *is increasing on* $(-4,-3), (0,2), (2,4)$. f *is decreasing on* $(-1,0)$

(e) *True or False: $f(1.5) < f(2)$*

Solution: *False,* $f(2) < f(1.5)$

Problem 7 *Determine if the function is even, odd, or neither.*

(a) $h(x) = x^4 + x^2 - 3$

Solution: *A function is even if $f(x) = f(-x)$, for all x in the domain, which means its graph is symmetic about the y-axis. A function is odd if $f(-x) = -f(x)$, for all x in the domain, which means its graph is symmetic about the origin.*

$$h(x) = x^4 + x^2 - 3$$

$$h(-x) = (-x)^4 + (-x)^2 - 3 = x^4 + x^2 - 3$$

$h(x) = h(-x)$. Hence h is even. This can be verified by graphing h and seeing that its graph is symmetric about the y-axis.

(b) $s(t) = t^2 - t$

Solution:

$$s(t) = t^2 - t$$
$$s(-t) = (-t)^2 - (-t) = t^2 + t$$

This does not equal $s(t)$ so $s(t)$ is not even.

$$-s(t) = -(t^2 - t) = -t^2 + t$$

This does not equal $s(-t)$ so s is not odd. Hence, $s(t)$ is neither even, nor odd.

(c) *We know that $\sin(\theta)$ is odd and $\cos(\theta)$ is even. Is $g(\theta) = \tan(\theta)$ even, odd, or neither?*

Solution:

$$g(\theta) = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\begin{aligned} g(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)} \\ &= \frac{-\sin(\theta)}{\cos(\theta)} \\ &= -\frac{\sin(\theta)}{\cos(\theta)} \end{aligned}$$

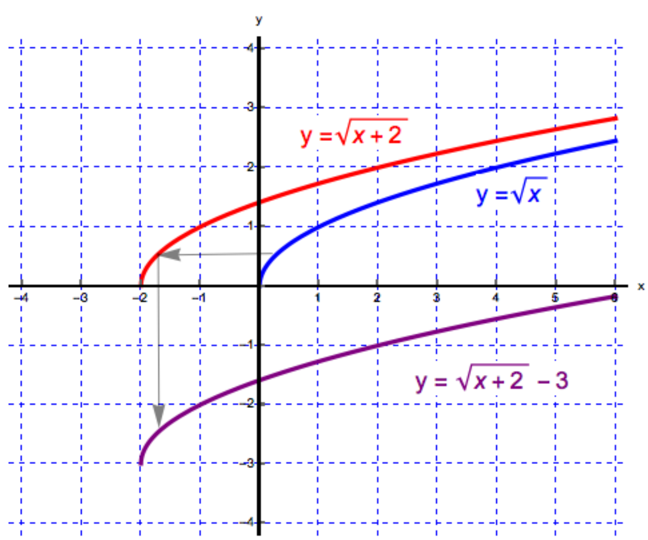
$$-g(\theta) = -\frac{\sin(\theta)}{\cos(\theta)}$$

$$g(-\theta) = -g(\theta) \implies g \text{ odd}$$

Problem 8 *Using the known graphs of $y = \sqrt{x}$ and $y = \frac{1}{x}$, sketch the graphs of the following using transformations.*

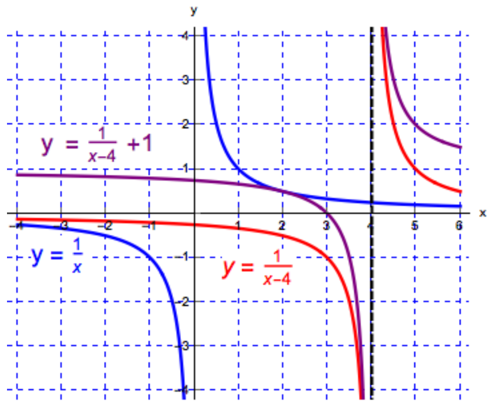
(a) $y = \sqrt{x+2} - 3$

Solution: *This is a shift of $y = \sqrt{x}$ moved left 2 units and down three units.*



(b) $y = \dfrac{1}{x-4} + 1$

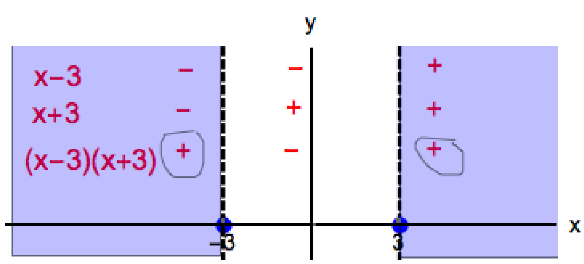
Solution: *This is a shift of* $y = \dfrac{1}{x}$ *moved right 4 units and up one unit.*



Problem 9 *Find the domain of the function. Determine whether the function is odd, even, or neither.*

(a) $f(x) = \dfrac{x}{\sqrt{x^2-9}}$

Solution: *To find the domain, recall that a rational expression cannot have 0 in the denominator and a square root expression cannot have a negative number under the square root. Thus,* $x^2 - 9 > 0$. $\implies (x-3)(x+3) > 0$ *The zeros are located at* $x = -3, 3$. *From this we can draw a sign chart for the expression,* $x^2 - 9$, *and test values.*



We see that $x^2 - 9 > 0$ *on the interval* $(-\infty, -3) \cup (3, \infty)$. *Thus our domain is* $(-\infty, -3) \cup (3, \infty)$.

Next we check for even/odd/neither.

$f(-x) = \dfrac{-x}{\sqrt{(-x)^2 - 9}}$ *which does not equal* $f(x)$ *so* f *is not even*

$-f(x) = -\dfrac{x}{\sqrt{x^2-9}}$ *which is equal to* $f(-x)$ *so* f *is odd.*

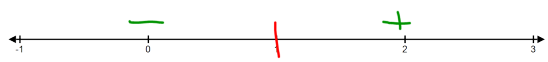
(b) $g(x) = \dfrac{\sin(x)}{x}$

Solution: *To find the domain, recall that a rational expression cannot have 0 in the denominator. Therefore, our domain is* $(-\infty, 0) \cup (0, \infty)$

Next we check for even/odd/neither.
$f(-x) = \dfrac{\sin(-x)}{-x} = \dfrac{-\sin(x)}{-x} = \dfrac{\sin(x)}{x}$ *which equals* $f(x)$ *so* f *is even*

(c) $h(t) = \ln(t^3 - 1)$

Solution: *To find the domain, recall that we cannot take the natural logarithm of 0 or a negative number. Therefore,* $t^3 - 1 > 0. \implies (t-1)(t^2+t+1) > 0$*. The zero is located at* $t = 1$*. From this we can draw a sign chart and test values.*



We see that $t^3 - 1 > 0$ *on the interval* $(1, \infty)$*. Thus our domain is* $(1, \infty)$*.*

Next we check for even/odd/neither.
$h(t)$ *is neither even nor odd because if* t *is in the domain, then* $-t$ *is not in the domain.*

Problem 10 *Let* g *be a one-to-one function and let* g^{-1} *be its inverse.* ***True or False:*** *If the point* $(2, 1/5)$ *lies on the graph of* g*, then the point* $(2, 5)$ *lies on the graph of* g^{-1}*.*

Solution: *This statement is* ***false****: we have* $g(2) = 1/5 \iff 2 = g^{-1}(1/5)$*. The notation* g^{-1} *never, in this course, means* $1/g$*.*

Problem 11 *Each of the following functions are invertible on their given domains. For each one find a formula for its inverse and give the domain and range of the inverse.*

(a) *The function* f *defined by* $f(x) = x^2 - 4x - 5$ *for every* $x \geq 2$*.*

Solution: *To help find the a formula for* f^{-1} *we will first "complete the square":*

$$\begin{aligned} x^2 - 4x - 5 &= x^2 - 4x + 4 - 4 - 5, \\ &= (x-2)^2 - 9. \end{aligned}$$

Setting $y = f(x) = x^2 - 4x - 5 = (x-2)^2 - 9$*, we can follow the procedures outlined for algebraically finding the formula for an inverse function.*

$$\begin{aligned} & y = (x-2)^2 - 9 \\ & \implies y + 9 = (x-2)^2 \\ & \implies \sqrt{y+9} = |x-2| \\ & \implies \sqrt{y+9} = x - 2 \quad \text{(since } x \geq 2\text{)} \\ & \implies \sqrt{y+9} + 2 = x \\ & \implies 2 + \sqrt{x+9} = y \quad \text{(interchange } x \text{ and } y \text{ along with minor rewriting)} \end{aligned}$$

Therefore we have that f^{-1} is defined by $f^{-1}(x) = 2 + \sqrt{x+9}$. The domain of $f^{-1}(x)$ is $[-9, \infty)$ and the range is $[2, \infty)$.

(b) *The function g defined by $g(u) = \sqrt[4]{u+2}$.*

Solution: *Following the procedure to algebraically find the formula for the inverse function we have*

$$\begin{aligned} z &= \sqrt[4]{u+2} \\ &\implies z = (u+2)^{1/4} \\ &\implies z^4 = u+2 \\ &\implies z^4 - 2 = u \\ &\implies u^4 - 2 = z \quad \text{(interchange } u \text{ and } z\text{)} \end{aligned}$$

Therefore we have that g^{-1} is defined by $g^{-1}(u) = u^4 - 2$. The domain of g^{-1} is $[0, \infty)$ and the range is $[-2, \infty)$.

(c) *The function h defined by $h(t) = 1/(t+2)^2$ for every $t > -2$.*

Solution: *Following the procedure to algebraically find the formula for the inverse function we have*

$$\begin{aligned} s &= \frac{1}{(t+2)^2} \\ &\implies (t+2)^2 = \frac{1}{s} \\ &\implies |t+2| = \sqrt{\frac{1}{s}} \\ &\implies t+2 = \sqrt{\frac{1}{s}} \quad \text{(since } t > -2\text{)} \\ &\implies t = \sqrt{\frac{1}{s}} - 2 \\ &\implies s = \frac{1}{\sqrt{t}} - 2 \quad \text{(interchange } s \text{ and } t\text{)} \end{aligned}$$

Therefore we have h^{-1} is defined by $h^{-1} = \dfrac{1}{\sqrt{t}} - 2$. The domain of h^{-1} is $(0, \infty)$ and the range is $(-2, \infty)$.

Problem 12 *Explain what each of the following means:*

(a) $f^{-1}(x)$

Solution: *This denotes the inverse function of f, f^{-1}, evauluated at x.*

(b) $f(x^{-1})$

Solution: *This means $f\left(\dfrac{1}{x}\right)$.*

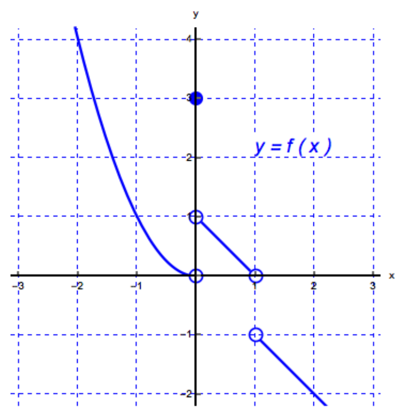
(c) $(f(x))^{-1}$

Solution: *This means $f(x)$ raised to the -1 power, i.e.* $\dfrac{1}{f(x)}$.

Problem 13 *If $f(x)$ represents the number of packages of buns needed for x packages of hotdogs, what does $f^{-1}(x)$ represent?*

Solution: *$f^{-1}(x)$ represents the numbers of packages of hotdogs needed for x packages of buns.*

Problem 14 *We're given the following graph of a function:*



Use this graph to answer the following questions:

(a) *What is the domain of this function?*

Solution: $(-\infty, 1) \cup (1, \infty)$

(b) *What is the range of this function?*

Solution: $(-\infty, -1) \cup (0, \infty)$

(c) *What is the value of $f(0)$, $f(1)$, and $f(2)$?*

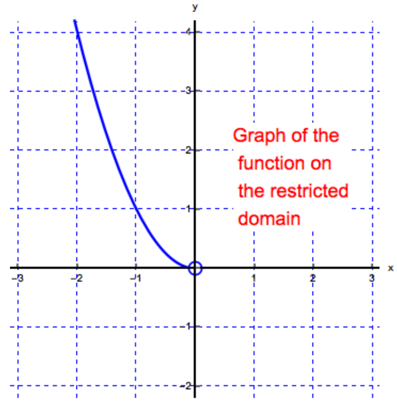
Solution: *$f(0) = 3$, $f(1)$ does not exist, $f(2) = -2$*

(d) *Does this function have an inverse? (Why or why not?)*

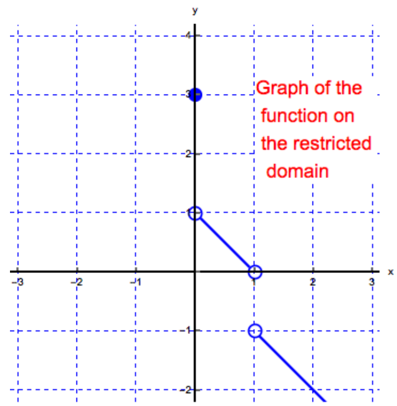
Solution: *No, the function does not have an inverse. It is not one-to-one (that is, it does not pass the horizontal line test).*

(e) *Find at least two intervals on which the function is one-to-one.*

Solution: *The function becomes one-to-one when we restrict its domain to $(-\infty, 0)$:*



The function also becomes one-to-one when we restrict its domain to $[0, \infty)$*:*

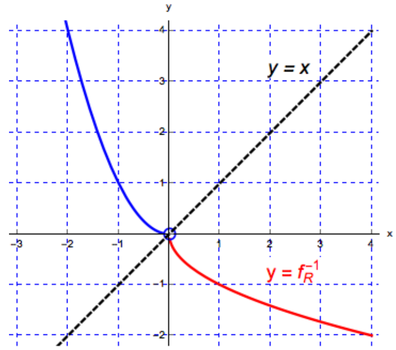


(f) *Find* $f^{-1}(3)$ *on a restricted domain of* f*.*

Solution: *In this case restrict the domain of* f *to* $[0, \infty)$*. By definition we have* $f^{-1}(3) = y \iff 3 = f(y)$*. Looking at the second graph of the restricted function we see that* $f(0) = 3$*, that is,* $f^{-1}(3) = 0$*. (If we restrict the domain of* f *to* $(-\infty, 0)$*, then* $f^{-1}(3) = y \iff 3 = f(y)$ *implies that* $f(-1.7) \approx 3$*. Hence* $-1.7 \approx f^{-1}(3)$*.)*

(g) *Using the restricted domain for* f *of* $(-\infty, 0)$*, sketch a graph of* f^{-1}*.*

Solution: *To graph the inverse, we can think of the graph being reflected over the line* $y = x$*. Another way to obtain the graph is to remember than if* (x, y) *is a point on the graph of* f*, then* (y, x) *is a point on the graph of* f^{-1}*. For example,* $(-1, 1)$ *is on the graph of* f *so* $(1, -1)$ *is on the graph of* f^{-1}*.*



(h) *Using the restricted domain for f of $[0, \infty)$, sketch a graph of f^{-1}.*

Solution: *We can graph the inverse of f on $[0, \infty)$ in pieces from left to right. Since $(0, 3)$ is a point on the graph of f, the point $(3, 0)$ in on the graph of f^{-1}. Then, we see on the graph of f, there is a linear piece going from $(0, 1)$ to $(1, 0)$. When we imagine reflecting this part of the graph of f over the line $y = x$, it reflects onto itself. The last piece of the graph of f is a line from $(1, -1)$ which appears to also contain the point $(-2, 2)$. Reflecting this part of f over the line $y = x$, we obtain a line starting at $(-1, 1)$ and through the point $(-2, 2)$.*