Linear Approximation (LA)

SUMMARY of Linear Approximation:

Definition
If a function f is **differentiable** at $x = a$, then a **linear approximation to** f **at** a is a function given by

$$L_a(x) = f(a) + f'(a)(x - a)$$

Note 1: When the value of a is understood, the subscript is sometimes dropped from the notation. In this case, it is written as just $L(x)$ instead of $L_a(x)$.
Note 2: The graph of L is the line tangent to the graph of f at the point where $x = a$.
Note 3: If x is near a, the value $f(x)$ can be approximated by the value of $L(x)$.
Note 4: If the graph of f is concave down *on the interval* between a and x, then the approximation $f(x) \approx L(x)$ is an overestimate. If the graph of f is concave up *on the interval* between a and x, then the approximation $f(x) \approx L(x)$ is an underestimate. (The concavity has to be consistent across the interval, not just at a point.)

A **differential**, df of f at x is given by

$$df = f'(x)dx$$

Note 3: If we consider a point x and if L is the linear approximation of f at x, then for any point $x + dx$ the following holds

$$f(x + dx) \approx L(x + dx) = f(x) + f'(x)dx = f(x) + df$$

Note 4: The **increment** of f, Δf, is given by

$$\Delta f = f(x + dx) - f(x) \approx L(x + dx) - f(x) =$$
$$f(x) + f'(x)dx - f(x) = f'(x)dx = df$$

Therefore, $\Delta f \approx df$.

Recitation Questions

Problem 1 (a) *Find the linearization, $L(x)$, of the function $f(x) = e^{2x}$ at $a = 0$.*

(b) *Using the linearization, $L(x)$, from the part (a), approximate e.*

(c) *Is the estimation found in part (b) an overestimate or an underestimate?* ***EXPLAIN***.

Problem 2 *Complete steps (i)-(vii) below in order to estimate the following values using linear approximation:*

(a) $\cos\left(\frac{31\pi}{180}\right)$

(b) $\sqrt[3]{8.13}$

(i) *Identify the function,* $f(x)$.

(ii) *Find the nearby value where the function can be easily calculated,* $x = a$.

(iii) *Find* $\Delta x = dx$.

(iv) *Find the linear approximation,* $L(x)$.

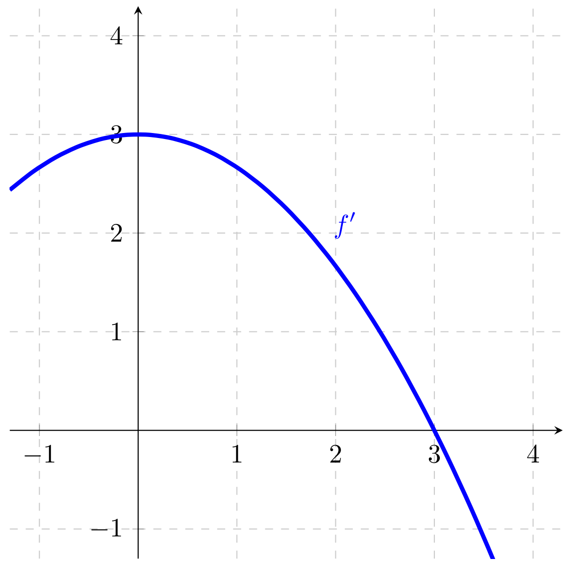
(v) *Compute the approximate value of the expression using the linear approximation.*

(vi) *Compare the approximated value to the value given by your calculator.*

(vii) *Compare* dy *and* Δy *using the value given by your calculator.*

Problem 3 *Estimate the value of* $\sin\left(\frac{178\pi}{180}\right)$. *Indicate whether your value is an overestimate or an underestimate.*

Problem 4 *Consider the graph of $f'(x)$ given below. Suppose you know that $f(3) = 7$. Can you approximate $f(2.98)$ and $f(3.02)$? Explain your answer. Are these overestimates or underestimates?*

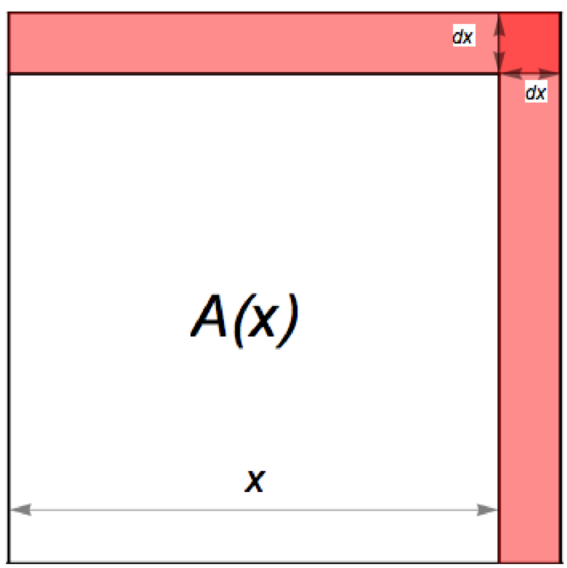


Problem 5 *Consider a square with a side x. Let A be the area of the square.*

(a) *Compute ΔA, the change in area if the side increases by $\Delta x = dx$.*

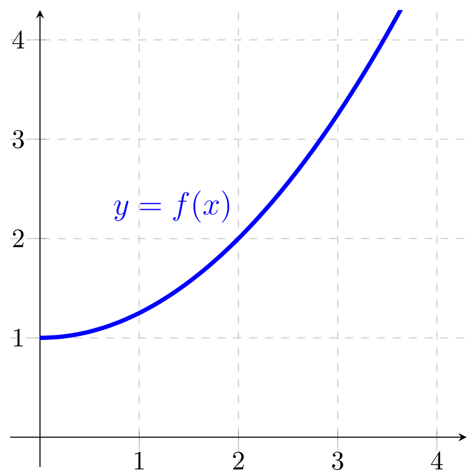
(b) *Compute dA, the differential of A at x, and compare it to ΔA.*

(c) *In the figure below the shaded part represents the change ΔA. Shade the part that represents dA.*



Problem 6 *Estimate the amount of paint needed to apply a coat of paint .05 cm thick to a hemispherical dome with diameter 50m. Is this value an underestimate or an overestimate?*

Problem 7 *The graph of a function f is given below.*



(a) *Given that $f'(2) = 1$, find the linear approximation L to the function f at $a = 2$.*

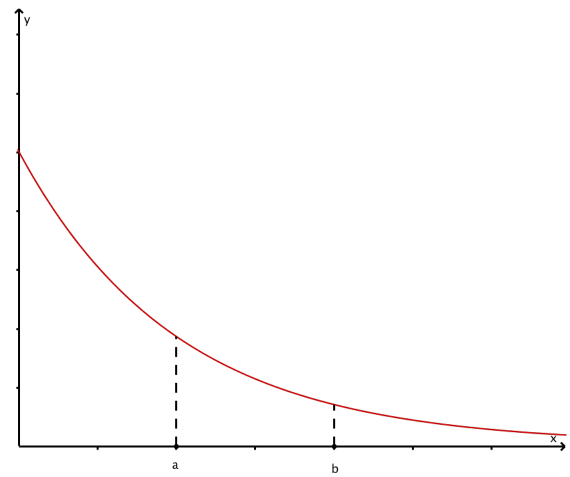
(b) *Sketch the graph of L in the figure above.*

(c) *Use the linear approximation L to estimate the value of $f(3)$. Is this an underestimate or overestimate?* ***EXPLAIN****.*

(d) *When x changes from $a = 2$ to $a + \Delta x = 3$, the change in the* **function** *$y = f(x)$, Δy, is given by $\Delta y = f(a + \Delta x) - f(a)$. Draw and label Δy and Δx in the figure above.*

(e) *When x changes from $a = 2$ to $a + \Delta x = 3$, the change in the* **linear approximaton***, dy, is given by $dy = L(a+\Delta x) - L(a) = f'(a)\Delta x$. Draw and label $L(x)$, dx and dy (differential) in the figure above.*

Problem 8 *The figure shows the graph of a function f. Let $L_a(x)$ be the linear approximation of f at a.*



Circle ALL the correct statements below.

(a) $L_a(b) < f(b)$

(b) $L_a(b) > f(b)$

(c) $L_a(a) < f(a)$

(d) $L_a(a) > f(a)$

(e) *No statement (a) – (d) is correct.*

Problem 9 *By using linear approximation, determine which of the following is the best estimate of* $e^{0.002}$.

(a) *1.0010005001667983416*6

(b) *1.0020020013340002667*5

(c) *1.0030045045033770260*1

(d) *1.0202013400267558101*6

Problem 10 *Find a formula for the differential of the following functions.*

(a) $y = 3x^6e^x$.

(b) $z = \ln(1 + t^2)$.

(c) $\theta = \tan^{-1}(r^3)$.

Problem 11 *In your own words, explain why $L_a(x)$ is a good approximation of the function f for x values $x \approx a$.*