Derivatives of inverse functions (DOIF) - Solutions

Problem 1 *Explain what each of the following means:*

(a) $\sin^{-1}(x)$

Solution: *This denotes the inverse function to* $\sin(x)$*, sometimes denoted by* $\arcsin(x)$.

(b) $(\sin(x))^{-1}$

Solution: *This means* $\sin(x)$ *raised to the* -1 *power, i.e.* $\dfrac{1}{\sin(x)}$.

(c) $\sin\left(x^{-1}\right)$

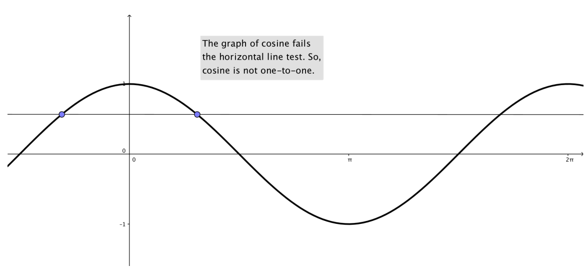
Solution: *This means* $\sin\left(\dfrac{1}{x}\right)$.

Problem 2 *Without using a calculator, determine if the statement below is true or false.*

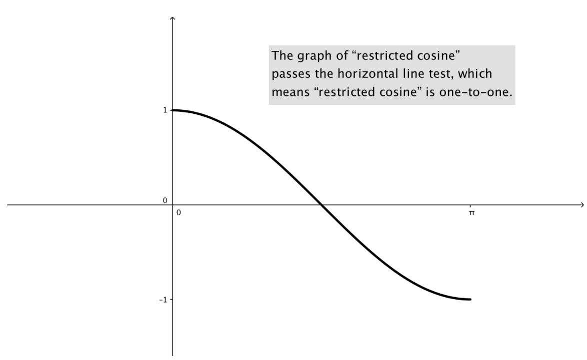
$$\cos^{-1}\big(\cos(7\pi/6)\big) = 7\pi/6$$

Solution: *This statement is* **false**: *the correct statement is* $\cos^{-1}\big(\cos(7\pi/6)\big) = 5\pi/6$. *(Why?)*

Spoiler Alert*: the cosine function is* not *invertible since its graph fails the horizontal line test.*



To produce the inverse cosine we must first restrict the domain of cosine, to the interval $[0, \pi]$*, to produce an invertible function:*

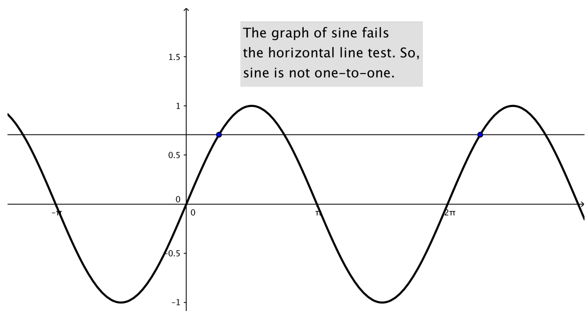


So, the range of $\cos^{-1}$ *is* $[0, \pi]$*. Since* $7\pi/6$ *is not in this range,* $7\pi/6$ *is* never *a possible output of* $\cos^{-1}$.

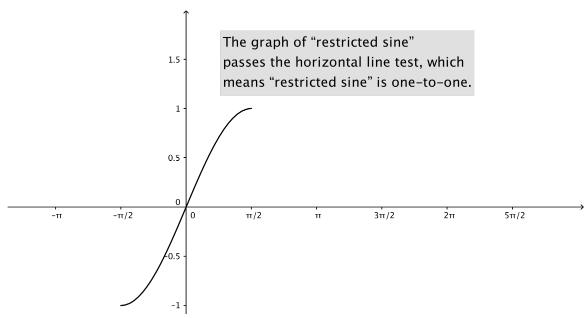
Problem 3 ***True or False:*** $\sin^{-1}(0) = \pi$.

Solution: *This statement is* **false***: the correct statement is* $\sin^{-1}(0) = 0$*. (Why?)*

Spoiler Alert*: the sine function is* not *invertible since its graph fails the horizontal line test.*



To produce the inverse sine we first restrict the domain of sine, to the interval $[-\pi/2, \pi/2]$*, to produce an invertible function:*



So, the range of $\sin^{-1}$ *is* $[-\pi/2, \pi/2]$*. Since* π *is not in this range,* π*, is* never *a possible output of* $\sin^{-1}$.

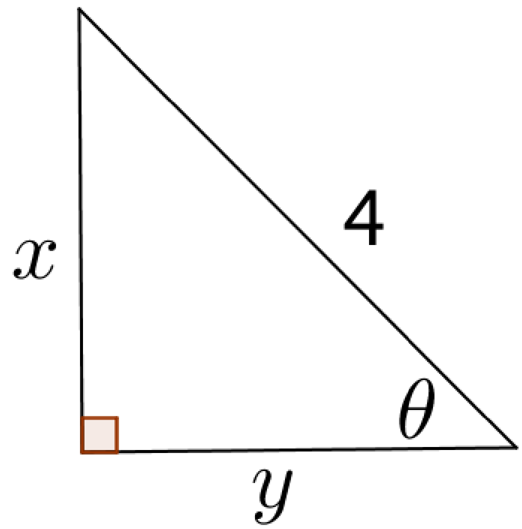
Problem 4 *Simplify each of the following expressions.*

(a) $\cos^{-1}\big(\sin(\pi/2)\big)$

Solution: *By the unit circle,* $\sin(\pi/2) = 1$*, and so we are looking for* $\cos^{-1}(1)$*. The range of* $\cos^{-1}$ *is* $[0, \pi]$*, and so, by properties of inverse functions,* $\cos^{-1}(1) = 0$.

(b) $\tan\big(\sin^{-1}(x/4)\big)$

Solution: *Let* $\theta = \sin^{-1}(x/4)$*, then* $\sin(\theta) = x/4$*. We can then draw the corresponding right triangle:*



Calling the adjacent side y, by the Pythagorean Theorem we obtain

$$4^2 = x^2 + y^2 \implies y = \sqrt{16 - x^2}$$

Remark: Since θ is in the range of $\sin^{-1}$, it follows that $-\pi/2 < \theta < \pi/2$. Therefore, $\cos(\theta) = y/4 > 0$. Therefore, $y > 0$.
Then

$$\begin{aligned}\tan\left(\sin^{-1}(4/x)\right) &= \tan(\theta),\\ &= \frac{x}{y},\\ &= \frac{x}{\sqrt{16-x^2}}.\end{aligned}$$

Note: $\tan\theta$ has the same sign as x, since $y > 0$.

Problem 5 *A table of values for f and f' is shown below. Suppose that f is a one-to-one function and f^{-1} is its inverse.*

x	$f(x)$	$f'(x)$
1	3	4
3	4	5
4	6	3

(a) *Evaluate $f^{-1}(f(x))$ at $x = 3$.*

Solution: $f^{-1}(f(3)) = f^{-1}(4) = 3$

(b) *Evaluate $\frac{d}{dx}f(f(x))$ at $x = 3$.*

Solution:

$$\frac{d}{dx}f(f(x)) = f'(f(x)) \cdot f'(x) \implies \left[\frac{d}{dx}f(f(x))\right]_{x=3} = f'(f(3)) \cdot f'(3)$$
$$= f'(4) \cdot 5 = 3 \cdot 5 = 15$$

(c) *Evaluate* $\frac{d}{dx}\ln((f(x))$ *at* $x = 3$.

Solution:

$$\frac{d}{dx}\ln((f(x)) = \frac{f'(x)}{f(x)}$$
$$\implies \left[\frac{d}{dx}\ln(f(x))\right]_{x=3} = \frac{f'(3)}{f(3)} = \frac{5}{4}$$

(d) *Evaluate* $f^{-1}(x)$ *at* $x = 3$.

Solution: $f^{-1}(3) = 1 \iff 3 = f(1)$

(e) *Evaluate* $\frac{d}{dx}f^{-1}(x)$ *at* $x = 3$.

Solution:

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$
$$\implies \left[\frac{d}{dx}f^{-1}(x)\right]_{x=3} = \frac{1}{f'(f^{-1}(3))}$$
$$= \frac{1}{f'(1)} = \frac{1}{4}$$

(f) *Evaluate* $\lim_{x\to 4}\frac{f(x) - f(4)}{x - 4}$

Solution: $\lim_{x\to 4}\frac{f(x) - f(4)}{x - 4} = f'(4) = 3$

Problem 6 *An object is moving along a horizontal line. Its position (in meters) at the time t (in seconds) is given by $s(t)$.*
What does the value $s^{-1}(5)$ represent?

Solution: $s^{-1}(5)$ *represents the the time when the position of the object is 5m.*

Problem 7 *Given the expression for $f(x)$, find the derivative of f^{-1} at the given point on the graph of f^{-1}, without solving for f^{-1}.*

(a) $f(x) = x^2 + 1$ *(for $x \geq 0$); the point on the graph of f^{-1}: $(5, 2)$.*
Verify your answer by evaluating the derivative of f^{-1} at the given point.

Solution: $(f^{-1})'(5) = \dfrac{1}{f'(2)}$. *Since* $f'(x) = 2x$, $f'(2) = 4$. *Thus,* $(f^{-1})'(5) = \dfrac{1}{4}$.

Verifying: $f^{-1}(x) = \sqrt{x-1} \implies \dfrac{d}{dx}f^{-1}(x) = \dfrac{1}{2\sqrt{x-1}} \implies \left[\dfrac{d}{dx}f^{-1}\right]_{x=5} = \dfrac{1}{2\sqrt{5-1}} = \dfrac{1}{4}$

(b) $f(x) = x^2 - 2x - 3$ *(for* $x \leq 1$*); the point on the graph of* f^{-1}*:* $(12, -3)$.

Solution: $(f^{-1})'(12) = \dfrac{1}{f'(-3)}$. *Since* $f'(x) = 2x - 2$, $f'(-3) = -6 - 2 = -8$. *Thus,* $(f^{-1})'(12) = -\dfrac{1}{8}$.

Notice that for this problem, finding the inverse of $f(x)$ *involves complicated algebra. In this case, it's much easier to find the derivative of* f^{-1} *without solving for* f^{-1}

Problem 8 *Find the slope of the tangent line to the curve* $y = f^{-1}(x)$ *at* $(4, 7)$ *if the slope of the tangent line to the curve* $y = f(x)$ *at* $(7, 4)$ *is* $\dfrac{2}{3}$.

Solution: *Note that the statement "the slope of the tangent line to the curve* $y = f(x)$ *at* $(7,4)$ *is* $\dfrac{2}{3}$*" specifically means that* $f'(7) = \dfrac{2}{3}$. *The slope of the tangent line to the curve* $y = f^{-1}(x)$ *at* $(4, 7)$ *is* $(f^{-1})'(4)$*, and so we use the formula for the derivative of the inverse function to compute:* $(f^{-1})'(4) = \dfrac{1}{f'(7)} = \dfrac{1}{\frac{2}{3}} = \dfrac{3}{2}$.

Problem 9 *Find the derivatives of the following functions:*

(a) $f(x) = \sec^{-1}(\sqrt{x})$.

Solution: $f'(x) = \dfrac{1}{\sqrt{x}\sqrt{x-1}} \cdot \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2x\sqrt{x-1}}$

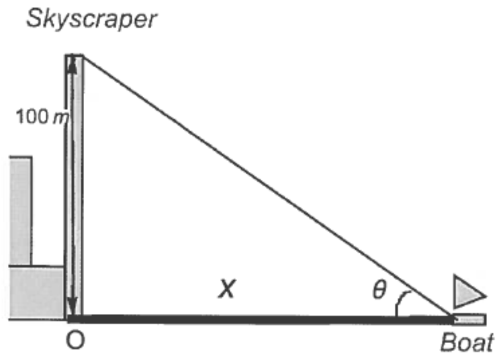
(b) $g(x) = \ln(\sin^{-1}(x))$.

Solution: $g'(x) = \dfrac{1}{\sin^{-1}(x)} \cdot \dfrac{1}{\sqrt{1-x^2}}$.

(c) $h(x) = \dfrac{1}{\tan^{-1}(x^2+4)}$.

Solution: $h'(x) = -\left(\tan^{-1}(x^2+4)\right)^{-2} \cdot \dfrac{1}{1+(x^2+4)^2} \cdot (2x)$.

Problem 10 *A boat sails directly toward a 100-meter skyscraper that stands on the edge of a harbor. The angular size* θ *of the building is the angle formed by lines from the top and bottom of the building to the observer on the boat (see figure below).*



(a) *Express the angle θ as the function of x, the distance of the boat from the building.*

Solution: $\tan\theta = \dfrac{100}{x} \implies \theta = \tan^{-1}\left(\dfrac{100}{x}\right)$

(b) *The boat is sailing directly toward the skyscraper at 3 m/s. Find $\dfrac{d\theta}{dt}$ when the boat is $x = 300m$ from the building.*

Solution:

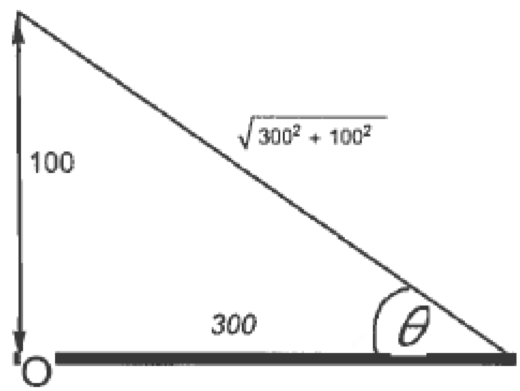
$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{d}{dt}\tan^{-1}\left(\frac{100}{x}\right) \\
&= \frac{d}{dx}\tan^{-1}\left(\frac{100}{x}\right)\frac{dx}{dt} \\
&= \frac{1}{1+\left(\frac{100}{x}\right)^2}\cdot\frac{-100}{x^2}\frac{dx}{dt} \\
&= \frac{1}{1+\left(\frac{100}{x}\right)^2}\cdot\frac{-100}{x^2}\cdot(-3) \\
&= \frac{300}{x^2+100^2}
\end{aligned}$$

$$\begin{aligned}
\left[\frac{d\theta}{dt}\right]_{x=300} &= \frac{300}{300^2+100^2} \\
&= \frac{3}{900+100} \\
&= \frac{3}{1000} \\
&= 0.003\ \textit{rad/sec}
\end{aligned}$$

Alternatively, this problem could be done without inverse functions.

$$\tan\theta = \frac{100}{x}$$
$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{100}{x}\right)$$
$$\sec^2\theta\frac{d\theta}{dt} = \frac{-100}{x^2}\frac{dx}{dt}$$

We have to evaluate $\sec^2\theta$ *at the moment when* $x = 300m$



Using the triangle above, we have:

$\sec\theta = \dfrac{\sqrt{300^2+100^2}}{300} = 100\dfrac{\sqrt{3^2+1^2}}{300} = \dfrac{\sqrt{10}}{3}$, *we obtain:*

$$\left(\frac{\sqrt{10}}{3}\right)^2\left[\frac{d\theta}{dt}\right]_{x=300} = -\frac{100}{300^2}(-3)$$
$$\left[\frac{d\theta}{dt}\right]_{x=300} = \frac{1}{300}\cdot\frac{9}{10} = \frac{3}{1000} \text{ rad/sec}$$