# Continuity and the Intermediate Value Theorem (CATIVT) - Solutions

**Problem 1** (a) *Let* $f(x) = \dfrac{x-1}{x^2-5x}$. *Then* $f(2) = -\dfrac{1}{6}$ *and* $f(6) = \dfrac{5}{6}$, *but there is no value of* $c$ *between 2 and 6 for which* $f(c) = 0$. *Does this fact violate the Intermediate Value Theorem?*

***Solution:*** *It does not violate the Intermediate Value Theorem.* $f$ *is not continuous at 5 so the conditions of the IVT do not hold and therefore the IVT does not apply.*

(b) *True or False: At some time since you were born your weight in pounds exactly equaled your height in inches.*

***Solution:*** *True: if* $w(t)$ *represents your weight in pounds at time at time* $t$ *and* $h(t)$ *represents your height in inches at time* $t$, *then* $w$ *and* $h$ *are both continuous functions. This implies* $w - h$ *is also continuous. If* $t = 0$ *is the moment you where born and* $t = T_0$ *is the present time, then* $w(0) - h(0) < 0$ *and* $w(T_0) - h(T_0) > 0$. *Hence by the Intermediate Value Theorem there is a point in the past,* $t$, *when* $w(t) - h(t) = 0$ *and therefore your weight in pounds equaled your height in inches.*

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**Problem 2** *For the following function* $g$ *defined by*

$$g(t) = \begin{cases} 5t+7 & \text{if } t < -3 \\ \dfrac{(t-1)(t+2)}{t+2} & \text{if } -3 \leq t < 1 \text{ and } t \neq -2 \\ 4\ln t & \text{if } t \geq 1 \end{cases}$$

*find the* ***Intervals of Continuity****.*
*(**Important Note:** Write your answer as a list of intervals, with each interval separated by a comma.)*

***Solution:*** *Recall that* Intervals of Continuity *means the largest such intervals on which the function is continuous.*

*The function* $g$ *is continuous on* $(\infty, -3)$ *since, in this interval,* $g(t) = 5t + 7$ *is a polynomial and therefore continuous on its domain.*

$g$ *is not continuous at* $t = -3$:

$$\lim_{t \to -3^-} g(t) = \lim_{t \to -3^-} (5t+7) = 5(-3) + 7 = -8$$

*and*

$$\lim_{t \to -3^+} g(t) = g(-3) = \frac{(-3-1)(-3+2)}{-3+2} = \frac{4}{-1} = -4.$$

*Although we have found* $g$ *is not continuous at* $t = -3$, *since* $\lim\limits_{t \to -3^+} g(t) = g(-3)$, $g$ *is continuous from the right at* $t = -3$.

*The function $g$ is continuous on $[-3,-2),(-2,1)$ since, in this interval, $g(t)=\dfrac{(t-1)(t+2)}{t+2}$ is a rational function whose denominator is not zero. $g$ is not continuous at $t=-2$ because it is not defined there. $g$ is continuous on $(1,\infty)$, since, on this interval, $g(t)=4\ln(t)$ is a constant multiple of a logarithmic function.*

*$g$ is continuous at $t=1$:*

$$\lim_{t\to 1^-} g(x)=\lim_{t\to 1^-}\frac{(x-1)(x+2)}{x+2}=\frac{0}{3}=0,$$

*and*

$$\lim_{t\to 1^+} g(x)=g(1)=4\ln(1)=0$$

*So, the Intervals of Continuity are*

$$(-\infty,-3),[-3,-2),(-2,\infty)$$

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**Problem 3** *Determine the value of a constant $b$ for which $f$ is continuous at 0.* ***EXPLAIN***.

$$f(x)=\begin{cases}\dfrac{2x+b}{x-5} & \text{if } x<0\\[2ex] \dfrac{x+16}{x^2-16} & \text{if } x\geq 0\end{cases}$$

***Solution:*** *In order for $f$ to be continuous at $x=0$, the three conditions of continuity have to be satisfied. These criteria are:*

$$\begin{gathered} f \text{ is defined at } x=0\\ \lim_{x\to 0} f(x) \text{ exists}\\ \lim_{x\to 0} f(x)=f(0)\end{gathered}$$

*The first condition is satisfied. In order to satisfy the second condition, $\lim_{x\to 0} f(x)$ exists, we must verify that $\lim_{x\to 0} f(x)=\lim_{x\to 0^+} f(x)=\lim_{x\to 0^-} f(x)$. That is,*

$$\begin{aligned}\lim_{x\to 0^+} f(x)&=\lim_{x\to 0^+}\left(\frac{x+16}{x^2-16}\right)=\frac{16}{-16}=-1\\ \lim_{x\to 0^-} f(x)&=\lim_{x\to 0^-}\frac{2x+b}{x-5}=\frac{b}{-5}\\ \implies \lim_{x\to 0} f(x)&=-1=\frac{-b}{5}\implies b=5\end{aligned}$$

*and $f(0)=-1$. $f$ is continuous at $x=0$ for $b=5$ since $\lim_{x\to 0} f(x)=f(0)=1$.*

*Notice here, it's not enough to just find a value for $b$. Our work must also justify the continuity at $x=0$ for that value of $b$.*

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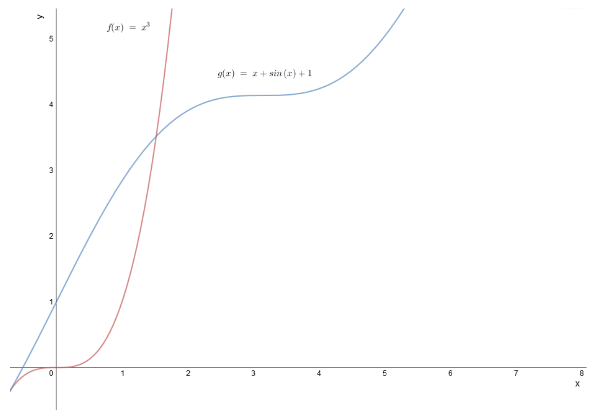
**Problem 4** *Use the Intermediate value theorem to find an interval in which you can guarantee that there is a solution to the equation $x^3=x+\sin(x)+1$.* ***EXPLAIN***. *(Do not use a graphing device or calculator to solve this problem!)*

***Solution:*** *We have*

$$x^3 = x + \sin(x) + 1 \iff x^3 - x - \sin(x) - 1 = 0.$$

*Define* $f(x) = x^3 - x - \sin(x) - 1$.

*Since:*
$f(0) = (0)^3 - (0) - \sin(0) - 1 = -1$ *and*
$f(\pi) = \pi^3 - \pi - \sin(\pi) - 1 = \pi(\pi^2 - 1) - 1 > 3 \cdot (3^2 - 1) - 1 = 23$
*We have:* $-1 < 0 < 23$
*and* $f$ *is continuous on* $[0, \pi]$, *the Intermediate Value Theorem implies there is some* $c$ *with* $0 < c < \pi$ *such that* $f(c) = L = 0$, *that is,* $c^3 = c + \sin(c) + 1$. *That* $c$ *is a solution on the interval* $[0, \pi]$. *(Note: There are many possible intervals, e.g.* $[1, 4]$, *etc. )*




**Problem 5** (a) *True or False: If* $f$ *and* $g$ *are two functions defined on* $(-1, 1)$, *and if* $\lim_{x \to 0} g(x) = 0$, *then it must be true that* $\lim_{x \to 0} (f(x) \cdot g(x)) = 0$.

***Solution:*** *False: Suppose*

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

*and* $g(x) = x$. *Then* $\lim_{x \to 0} g(x) = 0$ *but*

$$\lim_{x \to 0} (f(x) \cdot g(x)) = \lim_{x \to 0} \left( \frac{1}{x} \cdot x \right) = \lim_{x \to 0} 1 = 1.$$

(b) *True or False: If* $f$ *is continuous on* $(-1, 1)$, *and if* $f(0) = 10$ *and* $\lim_{x \to 0} g(x) = 2$, *then*

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 5.$$

***Solution:*** *True: application of quotient rule. Because* $f$ *is continuous,* $lim_{x \to 0} f(x) = f(0) = 10$

(c) *True or False: If* $f$ *is continuous on* $[1, 3]$, *and if* $f(1) = 0$ *and* $f(3) = 4$, *then the equation* $f(x) = \pi$ *has a solution in* $(1, 3)$.

**Solution:** *True: $f$ is continuous on $[1,3]$, $f(1) = 0 < \pi < 4 = f(3)$, and the Intermediate Value Theorem implies there is some $x$ in $(1,3)$ with $f(x) = \pi$.*

(d) *True or False: Let $f$ be a positive function with vertical asymptote $x = 5$. Then*

$$\lim_{x\to 5} f(x) = \infty.$$

**Solution:** *False: Suppose $f$ is defined by*

$$f(x) = \begin{cases} \dfrac{1}{x-5} & \text{if } x > 5 \\ 2 & \text{if } x \le 5. \end{cases}$$

*Then $f$ has a vertical asymptote at $x = 5$:*

$$\lim_{x\to 5^+} \frac{1}{\underbrace{x-5}} = \infty,$$

*because the limit is of the form $\dfrac{\#}{0}$, the numerator positive, and the denominator positive and goes to* 0.
*But, $\lim\limits_{x\to 5^-} f(x) = 2$.*
*Therefore, $\lim\limits_{x\to 5} f(x)$ does not exist*

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**Problem 6** *Let*

$$h(u) = \begin{cases} \dfrac{u^2-5u+4}{u-4} & \text{if } u < 4 \\[2ex] \dfrac{-\sqrt{u+4}}{u-6} & \text{if } u \ge 4, u \ne 6. \end{cases}$$

(a) *What is the domain of $h$?*

**Solution:** *For values of $u < 4$ we check $\dfrac{u^2-5u+4}{u-4}$. This piece of the function is defined everywhere on $(-\infty, 4)$*
*For values of $u \ge 4$ we check $\dfrac{-\sqrt{u+4}}{u-6}$. This piece of the function is defined everywhere except $u = 6$.*
*So $h$ is defined on $[4,6) \cup (6,\infty)$ The domain is $(-\infty,6) \cup (6,\infty)$*

(b) *Find all vertical asymptotes of $h$.* ***EXPLAIN***.

**Solution:** *The only candidates are $u = 6$ and $u = 4$.*

$\lim\limits_{u\to 6^+} \dfrac{-\sqrt{u+4}}{u-6} = -\infty$ *because it is of the form $\dfrac{\#}{0}$, the numerator negative, the denominator positive and goes to* 0.

$$\lim_{u\to 4^-} \frac{u^2-5u+4}{u-4} = \lim_{u\to 4^-} \frac{(u-1)(u-4)}{u-4} = \lim_{u\to 4^-} (u-1) = 3$$

$$\lim_{u\to 4^+} \frac{-\sqrt{u+4}}{u-6} = \frac{-\sqrt{8}}{-2} = \sqrt{2}$$

*The only vertical asympote is $u = 6$, because $\lim\limits_{u\to 6^+} h(u) = -\infty$.*

(c) *Find all horizontal asymptotes of $h$.* ***EXPLAIN.***

***Solution:***

$$\lim_{u\to\infty}\frac{-\sqrt{u+4}}{u-6} = \lim_{u\to\infty}\frac{-\frac{\sqrt{u+4}}{\sqrt{u}}}{\frac{u-6}{\sqrt{u}}}$$

$$= \lim_{u\to\infty}\frac{-\sqrt{\frac{u+4}{u}}}{\frac{u}{\sqrt{u}}-\frac{6}{\sqrt{u}}}$$

$$= \lim_{u\to\infty}\frac{\sqrt{1+\frac{4}{u}}}{\sqrt{u}-\frac{6}{\sqrt{u}}}$$

$$= 0$$

*because it is of the form:* $\dfrac{\#}{\infty}$

*This indicates that $y=0$ is a horizontal asymptote.*
*Checking as $u$ approaches $-\infty$ we have:*

$$\lim_{u\to-\infty}\frac{u^2-5u+4}{u-4} = \lim_{u\to-\infty}\frac{(u-1)(u-4)}{u-4} = \lim_{u\to-\infty}(u-1) = -\infty$$

*Therefore, there is no horizontal asymptote as $u$ approaches $-\infty$*
*There is one horizontal asymptote at $y=0$, because $\lim_{u\to\infty} h(u) = 0$.*

(d) *List the* ***Intervals of Continuity*** *for the function $h$.*

***Solution:*** *Remember that finding Intervals of Continuity means to find the largest such intervals. For values of $u < 4$ we check $\dfrac{u^2-5u+4}{u-4}$. This is a piece of a rational function and is continuous everywhere on $(-\infty, 4)$.*
*For values of $u > 4$ we check $\dfrac{-\sqrt{u+4}}{u-6}$. This piece of the function is continuous everwhere where defined: $(4,6),(6,\infty)$*

*We need to check $u=4$ for continuity. A function is continuous at a point $a$ if:*

$f$ *is defined at* $x=a$
$\lim_{x\to a} f(x)$ *exists*
$\lim_{x\to a} f(x) = f(a)$

*$h$ is defined at $u=4$, $h(4) = \dfrac{-\sqrt{4+4}}{4-6} = \sqrt{2}$*
*Next we check if $\lim_{u\to 4} h(u)$ exists. That is, does $\lim_{u\to 4^-} h(u) = \lim_{u\to 4^+} h(u)$ ?*

$$\lim_{u \to 4^-} h(u) = \lim_{u \to 4^-} \frac{u^2 - 5u + 4}{u - 4}$$

*This is of the form:* $\frac{0}{0^-}$ *so we need to do more algebra*

$$= \lim_{u \to 4^-} \frac{(u-4)(u-1)}{(u-4)}$$

$$= \lim_{u \to 4^-} (u - 1)$$

$$= 4 - 1$$

$$= 3$$

$$\lim_{u \to 4^+} h(u) = \lim_{u \to 4^+} \frac{-\sqrt{u+4}}{u-6}$$

$$= \frac{-\sqrt{4+4}}{4-6}$$

$$= \frac{-\sqrt{8}}{-2}$$

$$= \sqrt{2}$$

$$\lim_{u \to 4^-} h(u) \neq \lim_{u \to 4^+} h(u) \implies \lim_{u \to 4} h(u) \text{ does not exist}$$

*Thus* $h(u)$ *is not continuous at* $u = 4$. *Remark: Since* $\lim_{u \to 4^+} h(u) = h(4)$, $h$ *is right continuous at* $u = 4$. *Its intervals of continuity are* $(-\infty, 4), [4, 6), (6, \infty)$

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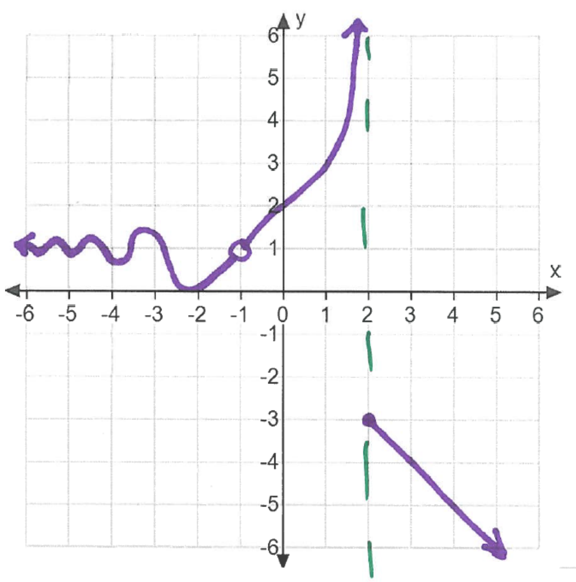
**Problem 7** *Use the graph of* $f$ *to answer the questions below.*



(a) *State the domain of $f$.*

***Solution:*** $(-\infty,-1)\cup(-1,\infty)$

(b) *Find the following values or state "does not exist":*

(i) $\lim_{x\to 2^-} f(x) =$

(ii) $\lim_{x\to 2^+} f(x) =$

(iii) $\lim_{x\to 2} f(x) =$

(iv) $\lim_{x\to -1} f(x) =$

(v) $\lim_{x\to 3} f(x) =$

(vi) $\lim_{x\to -\infty} f(x) =$

(vii) $f(-1) =$

***Solution:*** (i) $\lim_{x\to 2^-} f(x) = \infty$

(ii) $\lim_{x\to 2^+} f(x) = -3$

(iii) $\lim_{x\to 2} f(x)$ *Does not exist*

(iv) $\lim_{x\to -1} f(x) = 1$

(v) $\lim_{x\to 3} f(x) = -4$

(vi) $\lim_{x\to -\infty} f(x) = 1$

(vii) $f(-1)$ *Does not exist*

(c) *State the equation of any vertical asymptotes.*

***Solution:*** $x = 2$

(d) *State the equation of any horizontal asymptotes.*

***Solution:*** $y = 1$

(e) *Find the Intervals of Continuity.*

***Solution:*** $(-\infty,-1),(-1,2),[2,\infty)$

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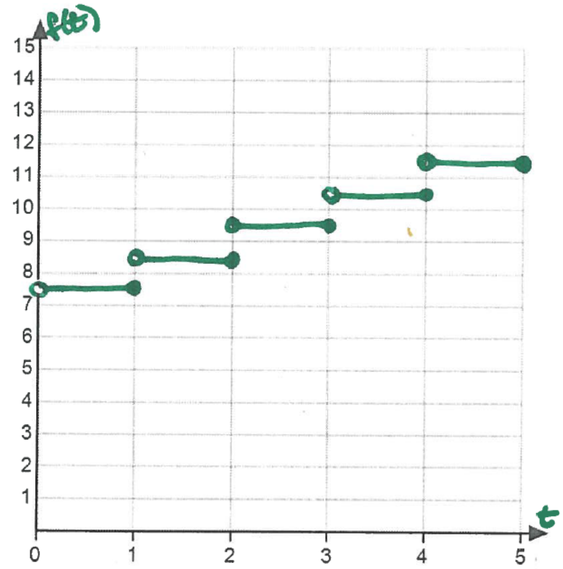
**Problem 8** *Suppose a taxi ride costs \$7.50 for the first mile (or any part of the first mile), plus an additional \$1.00 for each additional mile (or any part of a mile).*

(a) *Graph the function $c = f(t)$ that gives the cost of a taxi ride for $t$ miles, for $0 \le t \le 5$.*

***Solution:***




(b) *Evaluate* $\lim_{t \to 2.9} f(t)$

***Solution:*** $\lim_{t \to 2.9} f(t) = 9.5$

(c) *Evaluate* $\lim_{t \to 3^-} f(t)$ *and* $\lim_{t \to 3^+} f(t)$

***Solution:*** $\lim_{t \to 3^-} f(t) = 9.5$ *and* $\lim_{t \to 3^+} f(t) = 10.5$

(d) *Interpret the meaning of the limits in part (c).*

***Solution:*** *As the number of miles the taxi drives approaches 3, the cost of the taxi ride is $9.50. If one drives for a bit more than 3 miles, the cost is $10.50.*

(e) *On what intervals is the function c continuous? Explain.*

***Solution:*** $(0, 1], (1, 2], (2, 3], (3, 4], (4, 5]$

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**Problem 9** (a) *Given function* $f$ *on an interval* $[a, b]$*:*

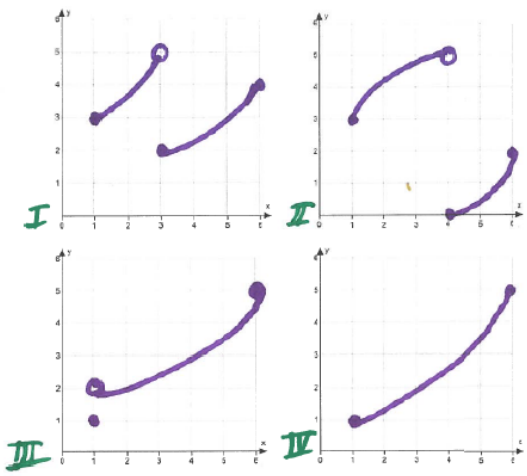
(i) *What are the conditions of the Intermediate Value Theorem?*

***Solution:*** $f$ *is continuous on the interval* $[a, b]$

(ii) *What is the conclusion of the Intermediate Value Theorem?*

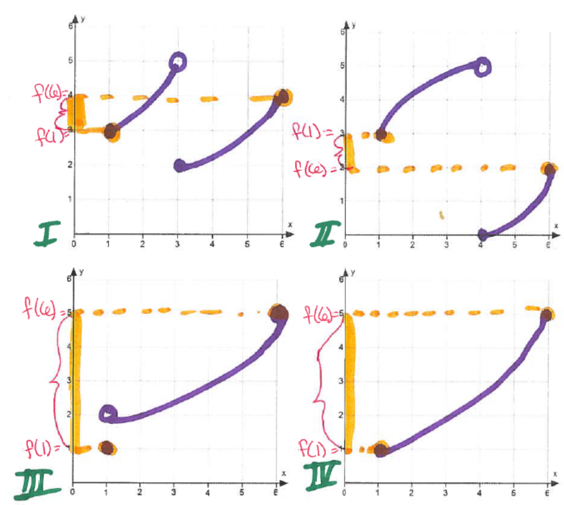
***Solution:*** *For any number* $L$ *strictly between* $f(a)$ *and* $f(b)$*, there is at least one number* $c$ *in* $(a, b)$ *satisfying* $f(c) = L$*. This means that for any* $L$ *strictly between* $f(a)$ *and* $f(b)$*, the horizontal line* $y = L$ *intersects the graph of* $f$*.*

(b) *Given the four functions on the interval* $[1, 6]$*, answer the questions below.*




(i) *For each of the functions I through IV, indicate* $f(1)$ *and* $f(6)$*. Then mark the interval of all numbers strictly between* $f(1)$ *and* $f(6)$*, on the y-axis.*

***Solution:***




(ii) *For each of the functions I through IV, write an interval of all numbers strictly between* $f(1)$ *and* $f(6)$

***Solution:*** $f = I$: $(3, 4)$
$f = II$: $(2, 3)$
$f = III$: $(1, 5)$
$f = IV$: $(1, 5)$

(iii) *List the functions that satisfy the conditions of the Intermediate Value Theorem on* $[1, 6]$

***Solution:*** *Only function IV is continuous on* $[1, 6]$

(iv) *For which of the functions is the following statement true: For any number* $L$ *strictly between* $f(1)$ *and* $f(6)$*, there exists a number* $c$ *in* $(1, 6)$ *satisfying* $f(c) = L$*.*

***Solution:*** *The statement is true for functions I and IV*

(v) *Does the function III satisfy the conclusion of the Intermediate Value Theorem? Why or why not?*

***Solution:*** *No, function $III$ does not satisfy the conclusion of the IVT. For example, take $L = 1.5$. $f(c) = 1.5$ has no solution in $(1, 6)$. Draw the horizontal line $y = 1.5$. What do you notice? This line does not intersect the graph of function III. What does this mean? The function does not attain that value, 1.5. So there is no $c$ in $(1, 6)$ such that $f(c) = 1.5$.*