L'Hôpital's Rule (LHR)

SUMMARY of L'Hôpital's Rule:

Indeterminate Forms - Part I

(1) $\frac{0}{0}$ refers to a limit of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$
(2) $\frac{\infty}{\infty}$ refers to a limit of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$, where $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. (The signs do not need to match.)

How do we determine whether the limit of this form exists and, if it does, how do we evaluate the limit? We can apply L'Hôpital's Rule to limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hôpital's Rule (LHR)

Let $f(x)$ and $g(x)$ be functions that are differentiable near a.
If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $g'(x) \neq 0$ for all x near a, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided that $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists.

L'Hôpital's rule applies also when $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$.

Indeterminate Forms - Part II

(3) $0\cdot\infty$ refers to a limit of the form $\lim_{x\to a} f(x)\cdot g(x)$, where $\lim_{x\to a} f(x) = 0$, and $\lim_{x\to a} g(x) = \pm\infty$
(4)$\infty - \infty$ refers to a limit of the form $\lim_{x\to a}(f(x) - g(x))$, where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$

How do we determine whether the limit of this form exists and, if it does, how do we evaluate the limit? By performing convenient algebraic operations, we show that the limit is equivalent to a limit of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so that we can apply the LHR.

Indeterminate Forms - Part III

(5) 1^∞ refers to a limit of the form $\lim_{x\to a} f(x)^{g(x)}$, where $\lim_{x\to a} f(x) = 1$, and $\lim_{x\to a} g(x) = \infty$
(6) 0^0 refers to a limit of the form $\lim_{x\to a} f(x)^{g(x)}$, where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
(7) ∞^0 refers to a limit of the form $\lim_{x\to a} f(x)^{g(x)}$, where $\lim_{x\to a} f(x) = \infty$, and $\lim_{x\to a} g(x) = 0$

How do we determine whether the limit of this form exists and, if it does, how do we evaluate the limit? By performing the following steps

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} [e^{\ln(f(x))}]^{g(x)} = \lim_{x\to a} e^{g(x)\ln(f(x))} = e^{\lim\limits_{x\to a} g(x)\ln(f(x))}$$

we end up with the limit of simpler form (in the exponent).

Recitation Questions

Problem 1 *State the form of the limit. Determine whether the form is determinate or indeterminate. Evaluate each limit.*

(a) $\displaystyle\lim_{x\to 0}\frac{\sin(x)-\cos(x)+1}{x^2-x}$

(b) $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{x^2}$

(c) $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^4}$

Problem 2 *State the form of the limit. Determine whether the form is determinate or indeterminate. Evaluate each limit.*

(a) $\lim_{x \to \infty} \left(\ln(1 + e^{-x})\right)^x$

(b) $\lim_{x \to \infty} \left(\frac{1}{x} + 1\right)^{\frac{1}{x}}$

(c) $\lim_{x \to \infty} \left(\frac{\arctan(x)}{x}\right)$

(d) $\lim_{x \to \infty} (x - \ln(x))$

(e) $\lim_{x \to \infty} \left(x \ln \left(\frac{1}{x} \right) \right)$

(f) $\lim_{x \to 0^+} (\sin(x) \cot(x))$

Problem 3 *Circle the correct answer in each part:*

(a) *Consider the limit* $\lim_{x\to 0}(\cos(x))^{\sin(x)}$.

(i) *Evaluate the limit.*

i. *the limit DNE*
ii. e
iii. 1
iv. ∞
v. $-\infty$
vi. 0
vii. *none of the previous answers is correct*

(ii) *What Limit Law, rule or technique did you use to find this limit?*

i. *The Squeeze Theorem;*
ii. *L'Hôpital's Rule;*
iii. *The Product Law;*
iv. *evaluated the function at* $x = 0$*, since the function is continuous at* $x = 0$*;*
v. *none of the previous answers is correct*

(b) *Evaluate the limit* $\displaystyle\lim_{x\to 4^-} \frac{\ln(x)}{x-4}$.

(i) *the limit DNE*
(ii) e
(iii) 1
(iv) ∞
(v) $-\infty$
(vi) 0
(vii) *none of the previous answers is correct*

(c) *Evaluate the limit* $\displaystyle\lim_{x\to \infty} \frac{\ln(x)}{x-4}$.

(i) *the limit DNE*
(ii) e
(iii) 1
(iv) ∞
(v) $-\infty$
(vi) 0
(vii) *none of the previous answers is correct*

(d) *Consider the limit*

$$\lim_{h\to 0} \frac{(2+h)^3 - 8}{h} = f'(2).$$

Determine the function f.

(i) *such a function DNE;*

(ii) $f(x) = x^3$;

(iii) $f(x) = (2+x)^3$;

(iv) $f(x) = \dfrac{(2+x)^3}{x}$;

(v) *none of the previous answers is correct*

(e) *Consider the limit:* $\displaystyle\lim_{x\to 0^+} \left(\frac{\sin(x)}{x}\right)^{|\ln(x)|}$.

Determine the form of this limit.

(i) $\dfrac{0}{0}$;

(ii) $\dfrac{\infty}{\infty}$;

(iii) 1^0;

(iv) 0^0;

(v) 1^∞;

(vi) ∞^∞;

(vii) *none of the previous answers is correct*

Problem 4 *Determine the following limits. Use L'Hôpital's Rule if applicable.*

(a) $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

(b) $\lim_{x\to-\infty} x^2 e^x$

(c) $\lim_{x\to\infty} x^{\frac{1}{x}}$

(d) $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$

(e) $\lim_{\theta\to 0^+} (\sin(\theta))^{\tan(\theta)}$

Problem 5 *Let f and g be two functions such that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.*

*We say that f **grows faster** than g as x goes to ∞ if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$, or, equivalently, if*

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

If the limit exists, namely, if $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = M$, for some positive number M,

*then we say that the functions f and g have **comparable growth rates**.*

In other words, we compare the growth rates of functions f and g by computing the limit of the form $\dfrac{\infty}{\infty}$: $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$.

For each of the following pairs of functions, determine which of the pair grows faster or state that they have comparable growth rates. Justify your answer by computing the limit $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$.

(a) $b^x; x^x; b > 1$

(b) $x^x; \left(\dfrac{x}{e}\right)^x$

(c) $x^3; x^3 \cdot \ln(x)$

(d) $a^x; b^x; 0 < a < b$

(e) $\log_a(x); \log_b(x); 1 < a < b$

(f) $\ln^3(x); x^{1/2}$

(g) $x; \ln(x)\sqrt{x}$

(h) *Challenge:* $x^{40}; 1.004^x$ *(Hint: Use the substitution* $x = \ln(t)$.*)*

(i) *Put the following functions in order of growth rate.*

$$x^3 \cdot \ln(x), \ln^3(x), x^x, 1.004^x, x^{40}, x^3$$