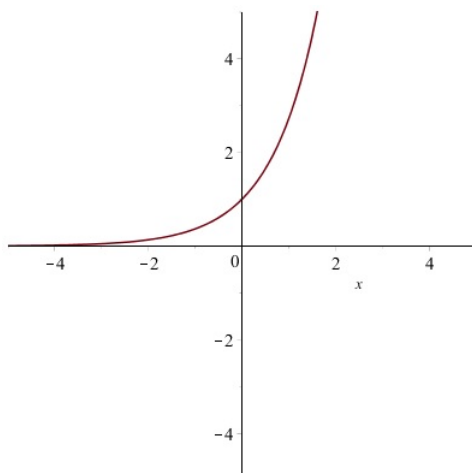


Understanding functions (UF)

Problem 1 The graph of $g(x) = e^x$ is given below.



(a) Find the domain and range of g .

Solution: Domain: $(-\infty, \infty)$, Range: $(0, \infty)$

(b) Find the values of $g(1)$, $g(0)$, $g(-1)$ and plot the points $(1, g(1))$, $(0, g(0))$, and $(-1, g(-1))$ on the graph below.

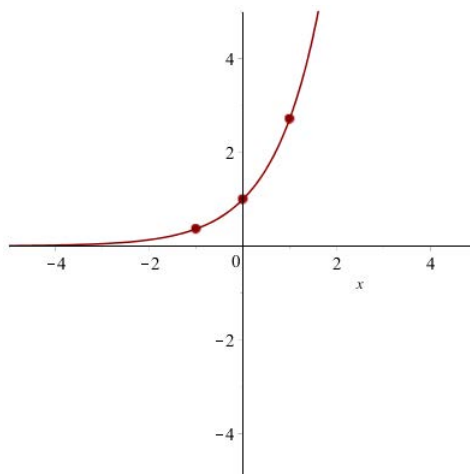
Solution:

$$g(1) = e^1 = e$$

$$g(0) = e^0 = 1$$

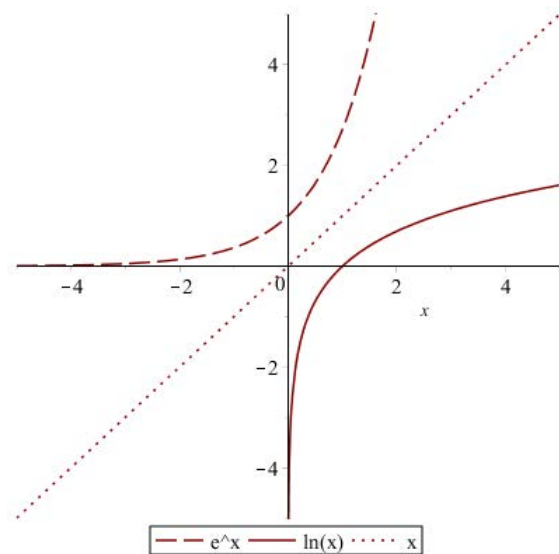
$$g(-1) = e^{-1} = \frac{1}{e}$$

These are values of $g(x) = e^x$ that you should know.



(c) Graph $h(x) = \ln(x)$ on the same axis below.

Solution: Recall: $\ln(x)$ is the inverse of e^x . To find the graph of $\ln(x)$ we reflect the graph of e^x over the line $y = x$. Since the points $\left(-1, \frac{1}{e}\right), (0, 1), (1, e)$ are on $y = e^x = g(x)$, the points $\left(\frac{1}{e}, -1\right), (1, 0), (e, 1)$ are on the graph of $y = \ln(x) = g^{-1}(x) = h(x)$



(d) Find the domain and range of h .

Solution: The domain of $h(x) = \ln(x)$ is $(0, \infty)$. The range is $(-\infty, \infty)$.

(e) Find the values of $h(1), h(0), h(-1), h(e), h\left(\frac{1}{e}\right)$, or say x not in the domain.

Solution:

$$h(1) = \ln(1) = 0$$

$h(0)$ is not defined, 0 is not in the domain

$h(-1)$ is not defined, -1 is not in the domain

$$h(e) = \ln(e) = 1$$

$$h\left(\frac{1}{e}\right) = \ln\left(\frac{1}{e}\right) = \ln(e^{-1}) = -1 \ln(e) = -1$$

These are values of $h(x) = \ln(x)$ that you should know.

Problem 2 Given $y(t) = t - \frac{\pi}{3}$ and $w(t) = \sin(t)$. Find:

(a) $y(w(t))$

Solution: $y(w(t)) = y(\sin(t)) = \sin(t) - \frac{\pi}{3}$

(b) $w(y(t))$

Solution: $w(y(t)) = w\left(t - \frac{\pi}{3}\right) = \sin\left(t - \frac{\pi}{3}\right)$

(c) $w\left(y\left(\frac{4\pi}{3}\right)\right)$

Solution: $w\left(y\left(\frac{4\pi}{3}\right)\right) = \sin\left(\frac{4\pi}{3} - \frac{\pi}{3}\right) = \sin(\pi) = 0$

(d) $y(w(\frac{4\pi}{3}))$

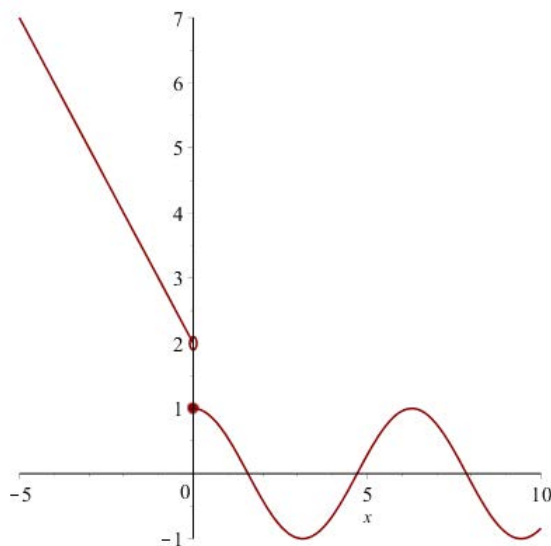
Solution: $y\left(w\left(\frac{4\pi}{3}\right)\right) = \sin\left(\frac{4\pi}{3}\right) - \frac{\pi}{3} = \frac{-\sqrt{3}}{2} - \frac{\pi}{3}$

You should know values of $\sin(x)$ and $\cos(x)$ for all values found on the standard unit circle.

Problem 3 Define $g(x) = \begin{cases} |x - 2| & \text{if } x < 0 \\ \cos(x) & \text{if } x \geq 0 \end{cases}$

(a) Sketch a graph of g

Solution:



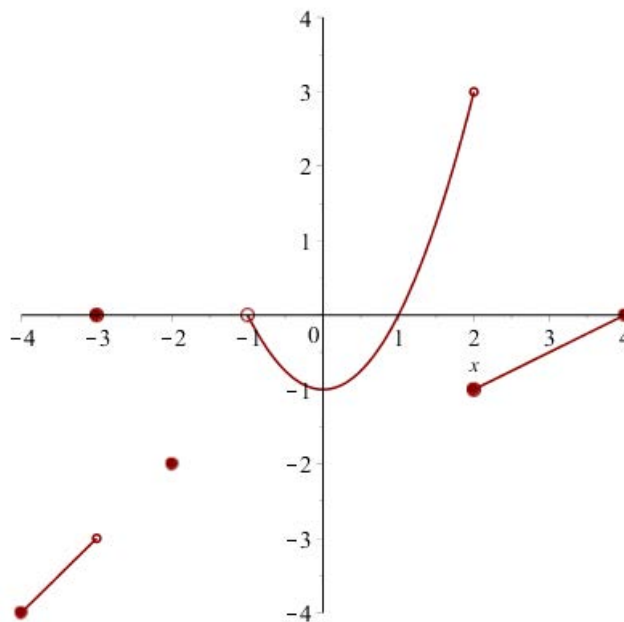
(b) Find the domain and range of g

Solution: Domain: $(-\infty, \infty)$, Range: $[-1, 1] \cup (2, \infty)$

(c) Find the values of $g(\pi)$ and $g(-\pi)$

Solution: $g(\pi) = \cos(\pi) = -1$ and $g(-\pi) = |-\pi - 2| = \pi + 2$

Problem 4 The entire graph of $f(x)$ is given below.



(a) Find the domain and range of f

Solution: Domain: $[-4, -3] \cup \{-2\} \cup (-1, 4]$ Range: $[-4, -3] \cup \{-2\} \cup [-1, 3]$

(b) Find the values of $f(-3)$, $f(-2)$, $f(-1)$, $f(2)$

Solution: $f(-3) = 0$, $f(-2) = -2$, $f(-1)$ does not exist, $f(2) = -1$

(c) Find the intervals on which $f(x)$ is positive. Find the intervals on which $f(x)$ is negative.

Solution: $f(x)$ is positive on $(1, 2)$. $f(x)$ is negative on $[-4, -3)$, $\{-2\}$, $(-1, 1)$, $[2, 4)$

(d) Find the intervals on which f is increasing. Find the intervals on which f is decreasing.

Solution: f is increasing on $(-4, -3)$, $(0, 2)$, $(2, 4)$. f is decreasing on $(-1, 0)$

(e) True or False: $f(1.5) < f(2)$

Solution: False, $f(2) < f(1.5)$

Problem 5 Determine if the function is even, odd, or neither.

(a) $h(x) = x^4 + x^2 - 3$

Solution: A function is even if $f(x) = f(-x)$, for all x in the domain, which means its graph is symmetric about the y-axis. A function is odd if $f(-x) = -f(x)$, for all x in the domain, which means its graph is symmetric about the origin.

$$h(x) = x^4 + x^2 - 3$$

$$h(-x) = (-x)^4 + (-x)^2 - 3 = x^4 + x^2 - 3$$

$h(x) = h(-x)$. Hence h is even. This can be verified by graphing h and seeing that its graph is symmetric about the y-axis.

(b) $s(t) = t^2 - t$

Solution:

$$s(t) = t^2 - t$$

$$s(-t) = (-t)^2 - (-t) = t^2 + t$$

This does not equal $s(t)$ so $s(t)$ is not even.

$$-s(t) = -(t^2 - t) = -t^2 + t$$

This does not equal $s(-t)$ so s is not odd. Hence, $s(t)$ is neither even, nor odd.

(c) We know that $\sin(\theta)$ is odd and $\cos(\theta)$ is even. Is $g(\theta) = \tan(\theta)$ even, odd, or neither?

Solution:

$$g(\theta) = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\begin{aligned} g(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)} \\ &= \frac{-\sin(\theta)}{\cos(\theta)} \\ &= -\frac{\sin(\theta)}{\cos(\theta)} \end{aligned}$$

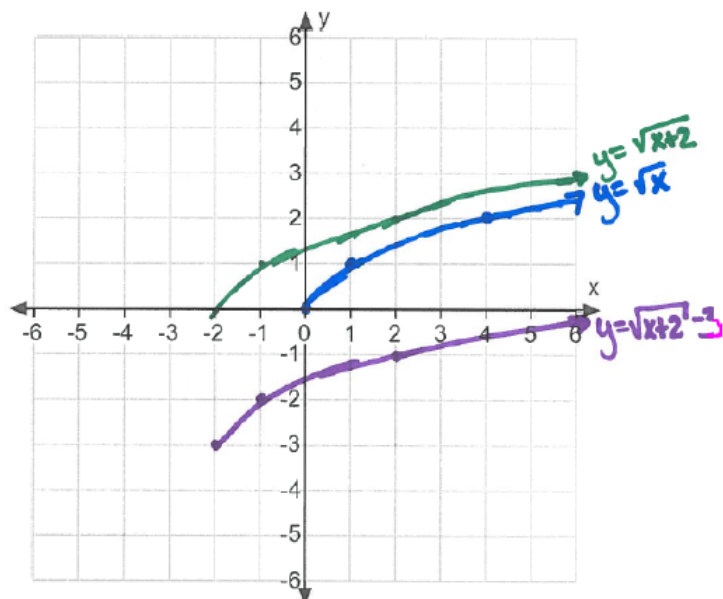
$$-g(\theta) = -\frac{\sin(\theta)}{\cos(\theta)}$$

$$g(-\theta) = -g(\theta) \implies g \text{ odd}$$

Problem 6 Using the known graphs of $y = \sqrt{x}$ and $y = \frac{1}{x}$, sketch the graphs of the following using transformations.

(a) $y = \sqrt{x+2} - 3$

Solution: This is a shift of $y = \sqrt{x}$ moved left 2 units and down three units.



$y = \sqrt{x}$

x	y
0	0
1	1
4	2

$y = \sqrt{x+2}$

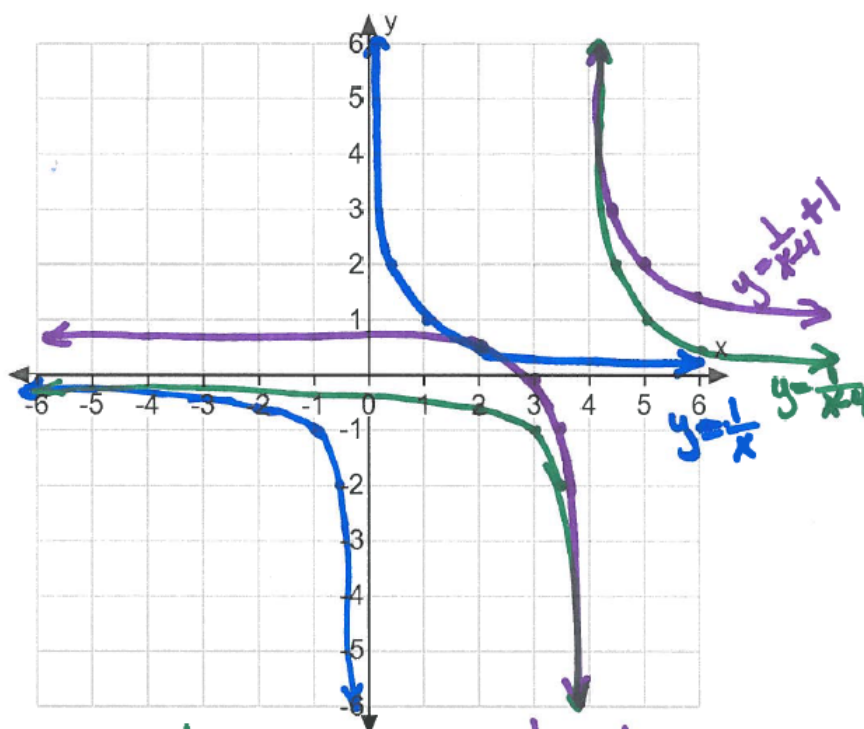
x	y
-2	0
-1	1
2	2

$y = \sqrt{x+2} - 3$

x	y
-2	-3
-1	-2
2	-1

(b) $y = \frac{1}{x-4} + 1$

Solution: This is a shift of $y = \frac{1}{x}$ moved right 4 units and up one unit.



$$y = \frac{1}{x}$$

x	y
-2	-1/2
-1	-1
-1/2	-2
1/2	2
1	1
2	1/2

HA: $y=0$
VA: $x=0$

$$y = \frac{1}{x-4}$$

x	y
2	-1/2
3	-1
3 1/2	-2
4 1/2	2
5	1
6	1/2

HA: $y=0$
VA: $x=4$

$$y = \frac{1}{x-4} + 1$$

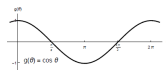
x	y
2	1/2
3	0
3 1/2	-1
4 1/2	3
5	2
6	1 1/2

HA: $y=1$
VA: $x=4$

Problem 7 Find the domain of the function. Determine whether the function is odd, even, or neither.

(a) $f(x) = \frac{x}{\sqrt{x^2 - 9}}$

Solution: To find the domain, recall that a rational expression cannot have 0 in the denominator and a square root expression cannot have a negative number under the square root. Thus, $x^2 - 9 > 0$.
 $\implies (x-3)(x+3) > 0$ The zeros are located at $x = -3, 3$. From this we can draw a sign chart for the expression, $x^2 - 9$, and test values.



We see that $x^2 - 9 > 0$ on the interval $(-\infty, -3) \cup (3, \infty)$. Thus our domain is $(-\infty, -3) \cup (3, \infty)$.

Next we check for even/odd/neither.

$$f(-x) = \frac{-x}{\sqrt{(-x)^2 - 9}} \text{ which does not equal } f(x) \text{ so } f \text{ is not even}$$

$$-f(x) = -\frac{x}{\sqrt{x^2 - 9}} \text{ which is equal to } f(-x) \text{ so } f \text{ is odd.}$$

(b) $g(x) = \frac{\sin(x)}{x}$

Solution: To find the domain, recall that a rational expression cannot have 0 in the denominator. Therefore, our domain is $(-\infty, 0) \cup (0, \infty)$

Next we check for even/odd/neither.

$$f(-x) = \frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x} \text{ which equals } f(x) \text{ so } f \text{ is even}$$

(c) $h(t) = \ln(t^3 - 1)$

Solution: To find the domain, recall that we cannot take the natural logarithm of 0 or a negative number. Therefore, $t^3 - 1 > 0$. $\implies (t - 1)(t^2 + t + 1) > 0$. The zero is located at $t = 1$. From this we can draw a sign chart and test values.



We see that $t^3 - 1 > 0$ on the interval $(1, \infty)$. Thus our domain is $(1, \infty)$.

Next we check for even/odd/neither.

$h(t)$ is neither even nor odd because if t is in the domain, then $-t$ is not in the domain.

Problem 8 Let g be a one-to-one function and let g^{-1} be its inverse. **True or False:** If the point $(2, 1/5)$ lies on the graph of g , then the point $(2, 5)$ lies on the graph of g^{-1} .

Solution: This statement is **false**: we have $g(2) = 1/5 \iff 2 = g^{-1}(1/5)$. The notation g^{-1} never, in this course, means $1/g$.

Problem 9 For any function f defined on $(-\infty, \infty)$, we can define f_e and f_o as follows: $f_e = \frac{f(x) + f(-x)}{2}$ and $f_o = \frac{f(x) - f(-x)}{2}$

(a) Show $f_e = \frac{f(x) + f(-x)}{2}$ is even

Solution:

$$f_e = \frac{f(x) + f(-x)}{2}$$

We want to show this is an even function, so we need to show $f_e(-x) = f_e(x)$ for all x

$$\begin{aligned} f_e(-x) &= \frac{f(-x) + f(-(-x))}{2} \\ &= \frac{f(-x) + f(x)}{2} \\ &= f_e(x) \end{aligned}$$

(b) Show $f_o = \frac{f(x) - f(-x)}{2}$ is odd

Solution:

$$f_o = \frac{f(x) - f(-x)}{2}$$

We want to show this is an odd function, so we need to show $f_o(-x) = -f_o(x)$ for all x First, we'll find $-f_o$

$$-f_o(x) = -\frac{f(x) - f(-x)}{2}$$

Now we'll find $f_o(-x)$

$$\begin{aligned} f_o(-x) &= \frac{f(-x) - f(-(-x))}{2} \\ &= \frac{f(-x) - f(x)}{2} \\ &= -\frac{f(x) - f(-x)}{2} \\ &= -f_o(x) \end{aligned}$$

(c) Show that $f(x) = f_e(x) + f_o(x)$, for all x

Solution: We want to show that $f(x) = f_e(x) + f_o(x)$, for all x . From parts a and b, we know

$f_e = \frac{f(x) + f(-x)}{2}$ and $f_o = \frac{f(x) - f(-x)}{2}$. Therefore,

$$\begin{aligned} f_e(x) + f_o(x) &= \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \\ &= \frac{f(x) + f(-x) + f(x) - f(-x)}{2} \\ &= \frac{2f(x)}{2} \\ &= f(x) \end{aligned}$$

We have just proved that every function defined on $(-\infty, \infty)$ can be written as the sum of an odd function and an even function.

(d) Is there a function that is both even and odd?

Solution: $f(x) = 0$ is both even and odd because $f(x) = f(-x) = 0$ for all x and $-f(x) = f(-x) = 0$ for all x

(e) Something to think about: Write each function as an even function plus an odd function.

(i) $f(x) = \sin(x)$

Solution: $f_e(x) = 0$ and $f_o(x) = \sin(x)$. $f(x) = 0 + \sin(x)$

(ii) $f(x) = x^3 + 2x^2 + x$

Solution: $f_e(x) = 2x^2$ and $f_o(x) = x^3 + x$. $f(x) = 2x^2 + x^3 + x$

(iii) $f(x) = \ln(x)$

Solution: $f(x) = \ln(x)$ is not defined on $(-\infty, \infty)$. It is only defined on $(0, \infty)$. It doesn't make sense to think about the natural logarithm as even or odd.

(iv) $f(x) = e^x$

Solution: $f_e(x) = \frac{f(x) + f(-x)}{2} = \frac{e^x + e^{-x}}{2}$ and $f_o(x) = \frac{f(x) - f(-x)}{2} = \frac{e^x - e^{-x}}{2}$.

$$f(x) = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$$
