Optimization (O)

SUMMARY: How to Solve an Optimization Problem

(a) Identify **variables**; draw and label the picture of the problem

(b) Identify the **objective function** (a quantity to be optimized);
write a **formula** for the objective function in terms of variables of the problem

(c) Identify the **constraint(s)**;
use the constraint(s) to express all the variables in terms of a **single variable**

(d) Write the **objective function in terms of a single variable**;
find the **interval of interest**

(e) Using the methods of calculus, find the **global maximum/minimum**;
justify your answer

REMINDER: The Interval of Interest tells you the method to use to solve an optimization problem.
CASE 1: The interval of interest is a closed interval $[a, b]$

In this case, the Extreme Value Theorem (EVT) guarantees that both global extrema of f exist!
If f has an global minimum and an global maximum which either occur at a critical point or at a boundary point (which means a or b). To find an extreme values of f on $[a, b]$, we:

- Find all critical points and plug them into f.
- Evaluate f at both boundary points.
- Compare the values. The biggest of those values is the maximum value of f on $[a, b]$, and the least one is the minimum value of f on $[a, b]$.

CASE 2: The interval of interest is any interval.

In this case, the global minimum/maximum **may not exist**. Our approach is then:

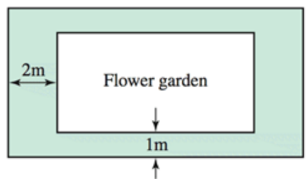
- Find all critical points of f on the interval.
- Use 1st or 2nd Derivative Test to classify those critical points as **local maxima/minima**.
- If there is *exactly one local extremum*, then it is an global extremum of the same type. (That is, if it is a local minimum then it is automatically a global minimum...)

OTHER CASES (e.g., An interval with multiple local extrema, etc) can be approached similar to how we graphed functions: check the boundary points, check the critical points, use the sign chart etc.

Recitation Questions

Problem 1 *Suppose you want to maximize a continuous function on a closed interval, but you find that it only has one local extremum on the interval which happens to be a local minimum. Where else should you check for the solution?* ***EXPLAIN****.*

Problem 2 *A rectangular flower garden with an area of $30\,m^2$ is surrounded by a grass border $1\,m$ wide on two sides and $2\,m$ wide on the other two sides (see figure). What dimensions of the garden minimize the combined area of the garden and borders?*



(a) *Label the picture with variables.*

(b) *What are you trying to maximize or minimize? Write an equation for it in terms of the variables from (a).*

(c) *What is your constraint? Write a constraint equation in terms of the variables from (a).*

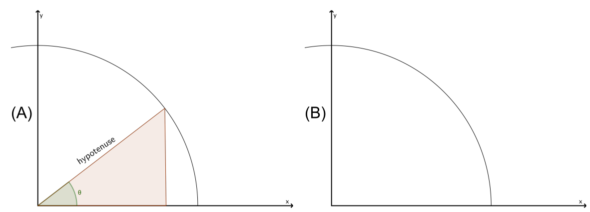
(d) *Reduce your optimization equation to one variable using the constraint equation.*

(e) *What is the interval on which your variable makes sense? Is it open or closed? What does this mean for the method of finding the global max or min?*

(f) *Use the appropriate method to find and justify your global extremum.*

(g) *Be sure to answer the question asked in the original problem.*

Problem 3 *A part of a circle centered at the origin with radius $r = 7$ cm is given in the figure (A) below. A right triangle is formed in the first quadrant (see figure (A)). One of its sides lies on the x-axis. Its hypotenuse runs from the origin to a point on the circle. The hypotenuse makes an angle θ with the x-axis.*



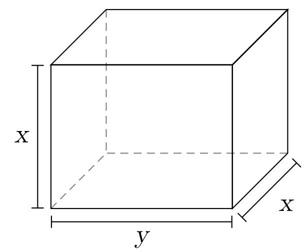
Make sure to label the picture.

(a) *Draw 2 more examples of such a triangle in the figure (B).*

(b) *Express the area of such a triangle as a function of θ and state its domain.*

(c) *Find the value of θ which maximizes the area in part (b). Show your work and justify your answer.*

Problem 4 *A box with two square sides is constructed to have a volume of 27. Denote the dimensions of the length of the base as y, and the width and height as x, as labeled in the diagram below. Find the dimensions needed to minimize the surface area of the box.*



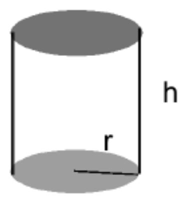
Solve the problem by performing the following steps.

(a) *Find a formula for $S(x)$ the surface area of the box, as a function of only the variable x.*

(b) *Find theinterval of interest for $S(x)$.*

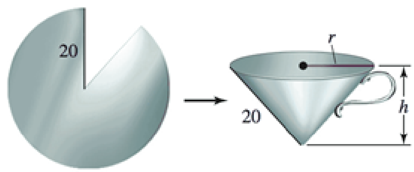
(c) *Find the x-value that gives the minimum surface area over all such boxes.*

Problem 5 *Find the radius of a cylindrical container with a volume of* $2\pi\ m^3$ *that minimizes the surface area.*



(**HINT: surface area, $S = 2\pi rh + 2r^2\pi$; volume, $V = r^2\pi h$.**)

Problem 6 *A cone is constructed by cutting a sector from a circular sheet of metal with radius 20. The cut sheet is then folded and welded. Find the radius and height of the cone with maximum volume that can be formed this way.*



Problem 7 *What point on the parabola $y = 5 - x^2$ is closest to the point $(4, 7)$?*

Problem 8 *A rectangle is constructed with one side on the positive x-axis, one side on the positive y-axis, and the vertex opposite the origin on the line $y = 10 - 2x$. What dimensions maximize the area of the rectangle? What is the maximum area?*

Problem 9 *Suppose you own a tour bus and you book groups of 20 to 80 people for a day tour. The cost per person is \$30 minus \$0.25 for every ticket sold. If gas and other miscellaneous costs are \$300, how many tickets should you sell to maximize your profit? Treat the number of tickets as a nonnegative real number.*