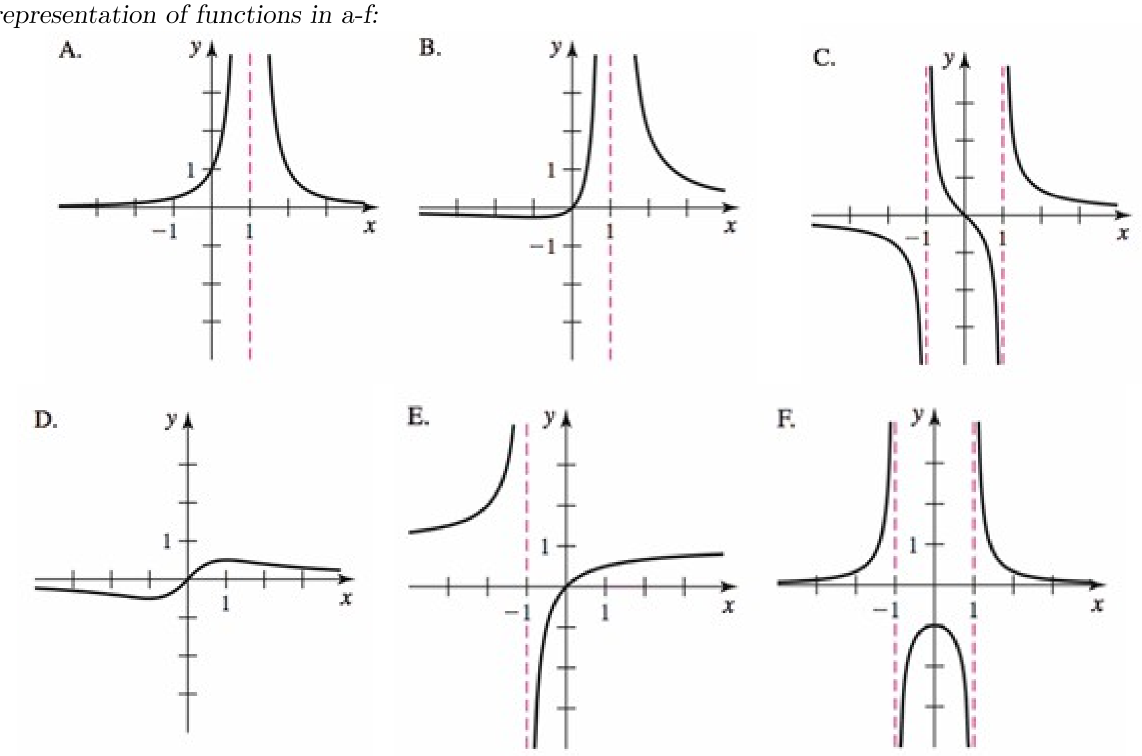
Using limits to detect asymptotes (ULTDA) - Solutions

Problem 1 *Without using a graphing utility, match each graph of functions in A-F with the algebraic representation of functions in a-f:*



(a) *The function f defined by* $f(x) = \dfrac{x}{x^2+1}$.

Solution: *Since there is no real number x such that $x^2 + 1 = 0$, we have no candidates for vertical asymptotes. Hence the graph of f should not contain any vertical asymptotes. The only listed graph with no vertical asymptote is graph D.*

(b) *The function g defined by* $g(x) = \dfrac{x}{x^2-1}$.

Solution: Candidates *for vertical asymptotes:*

$$x^2 - 1 = 0 \implies x = 1 \text{ or } x = -1.$$

Test of candidate $x = 1$:

$$\lim_{x\to 1^+} \frac{x}{x^2-1} = \lim_{x\to 1^+} \frac{x}{(x-1)(x+1)} = \infty$$

Note: the limit is of the form $\dfrac{\#}{0}$, the numerator positive, the denominator also positive. Therefore, vertical asymptote at $x = 1$.

Test of candidate $x = -1$:

$$\lim_{x \to -1^-} \frac{x}{x^2 - 1} = \lim_{x \to -1^-} \frac{x}{(x-1)(x+1)} = -\infty$$

Note: the limit is of the form $\frac{\#}{0}$, the numerator negative, the denominator positive. Therefore, vertical asymptote at $x = -1$.

Since $g(0) = 0$, the only listed graph that can match is graph C.

(c) *The function h defined by $h(x) = \frac{1}{x^2 - 1}$.*

Solution: Candidates *for vertical asymptotes: $x = 1$ and $x = -1$.*

Test of candidate $x = 1$:

$$\lim_{x \to 1^+} \frac{1}{x^2 - 1} = \lim_{x \to 1^+} \frac{1}{(x-1)(x+1)} = \infty$$

Note: the limit is of the form $\frac{\#}{0}$, the numerator positive, the denominator positive. Therefore, vertical asymptote at $x = 1$.

Test of candidate $x = -1$:

$$\lim_{x \to -1^-} \frac{1}{x^2 - 1} = \lim_{x \to -1^-} \frac{1}{(x-1)(x+1)} = \infty$$

Note: the limit is of the form $\frac{\#}{0}$, the numerator positive, the denominator positive. Therefore, vertical asymptote at $x = -1$.

Since $h(0) = -1$, the only listed graph that can match is graph F.

(d) *The function a defined by $a(x) = \frac{x}{(x-1)^2}$.*

Solution: *Candidate for the vertical asymptote: $x = 1$.*

Test of candidate $x = 1$:

$$\lim_{x \to 1^+} \frac{x}{(x-1)^2} = \infty$$

Note: the limit is of the form $\frac{\#}{0}$, the numerator positive, the denominator positive. Therefore, vertical asymptote at $x = 1$.

Since $a(0) = 0$ the graph is graph B.

(e) *The function s defined by $s(x) = \frac{1}{(x-1)^2}$.*

Solution: *Candidate for the vertical asymptote:* $x = 1$.

Test of candidate $x = 1$*:*

$$\lim_{x\to 1^+} \frac{1}{(x-1)^2} = \infty$$

Note: the limit is of the form $\frac{\#}{0}$, *the numerator positive, the denominator positive.*
Therefore, vertical asymptote at $x = 1$.

Since $s(0) = 1$ *the graph is graph A.*

(f) *The function* r *defined by* $r(x) = \dfrac{x}{x+1}$.

Solution: *Candidate for the vertical asymptote:* $x = -1$.

Test of candidate $x = -1$*:*

$$\lim_{x\to -1^+} \frac{x}{x+1} = -\infty$$

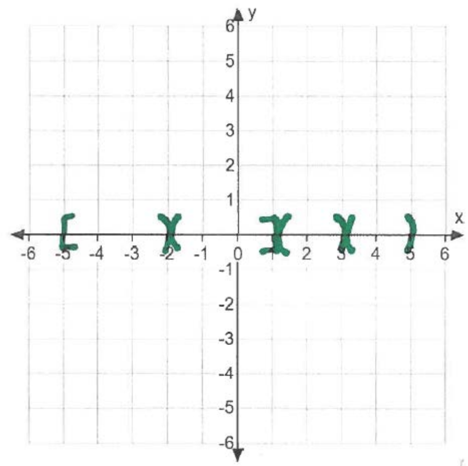
Note: the limit is of the form $\frac{\#}{0}$, *the numerator negative, the denominator positive.*
Therefore, vertical asymptote at $x = -1$.

Since $r(0) = 0$ *the graph is graph E.*

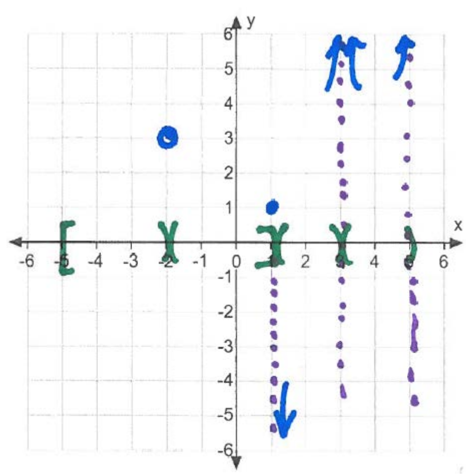
Problem 2 *Sketch a possible graph of a function* g *that satisfies the following conditions:*

Domain:$[-5,-2)\cup(-2,3)\cup(3,5)$	$\lim_{x\to -2} g(x) = 3$
$g(1) = 1$	$\lim_{x\to 1^+} g(x) = -\infty$
$\lim_{x\to 3} g(x) = \infty$	$\lim_{x\to 5^-} g(x) = \infty$
$\lim_{x\to 1^-} g(x) = 1$	$g(-5) = -1.8$

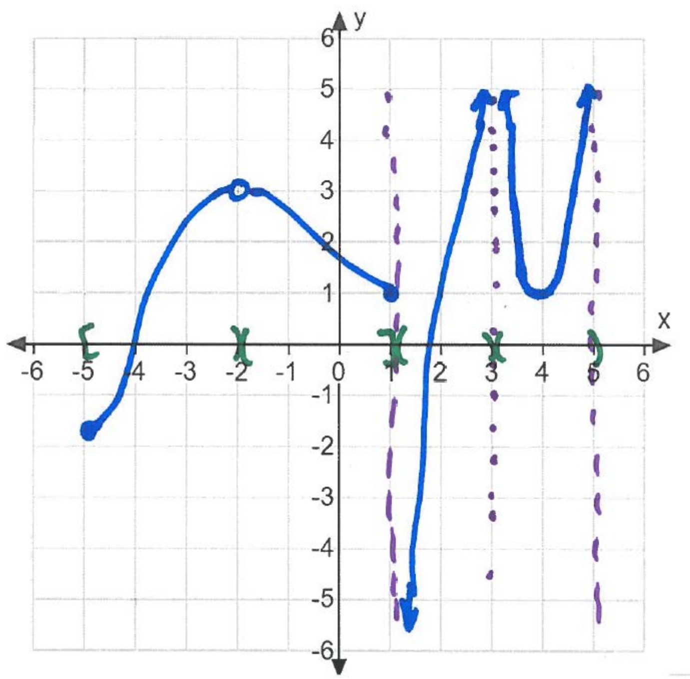
Solution: *First we will draw our domain on the x-axis using green. We will use a bracket when a point is in the domain and curved parenthesis when it is not.*



Next, we draw and open circle at $(-2,3)$ *because we know* $\lim_{x\to -2} g(x) = 3$*. We also draw a closed circle at (1,1) and (-5,-1.8) because we know* $g(1) = 1$ *and* $g(-5) = -1.8$*. Finally we draw in our vertical asymptotes in purple as indicated by the infinite limits at* $x = 1, x = 3, x = 5$*. We draw the tails of the asmyptotes in blue to indicate whether the function* g *approaches* $-\infty$ *or* ∞*.*



Finally, we connect our graph together.



Problem 3 *Let f be a function given by $f(x) = \ln(1+x)$.*

(a) *Find the domain of f. Write your answer in interval notation.*

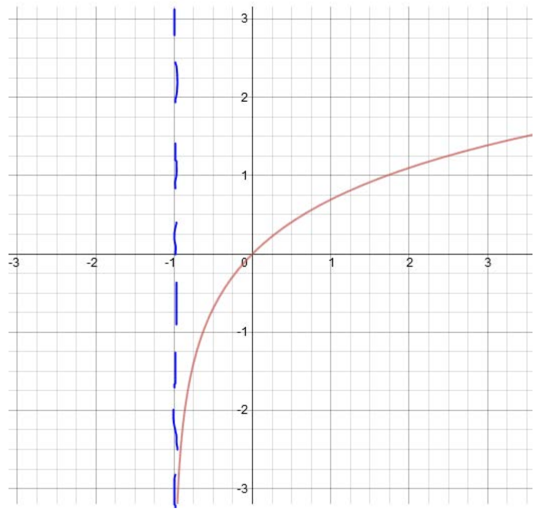
Solution: $1 + x > 0 \implies x > -1 \implies$ *domain:*$(-1, +\infty)$

(b) *Find the vertical asymptotes of f and* ***EXPLAIN*** *and justify your answer.*

Solution: *The function $f(x) = \ln(1+x)$ has a vertical asymptote at $x = -1$ since $\lim_{x \to -1^+} \ln(1+x) = -\infty$. (There is no Theorem/Test to reference here, since it is checking the definition of a vertical asymptote.)*
Alternatively, $g(x) = \ln(x)$ has a vertical asymptote at $x = 0$. f is g shifted one unit left so f will have a vertical asymptote at $x = -1$.

(c) *Sketch a graph of f*

Solution: *The graph of $f(x) = \ln(1+x)$ is show below.*



Problem 4 *Let* $f(x) = \dfrac{\ln(x)}{x-2}$.

(a) *Evaluate the limit.*

$$\lim_{x \to 2^-} \frac{\ln(x)}{x-2}$$

Solution: *This limit is of the form* $\dfrac{\#}{0}$, *the numerator positive, and the denominator is negative. Therefore,* $\displaystyle\lim_{x \to 2^-} \frac{\ln x}{x-2} = -\infty$.

(b) *Find the vertical asymptotes of* f. ***EXPLAIN*** *and justify your answer.*

Solution: $x = 2$ *is a vertical asymptote of* f *since* $\displaystyle\lim_{x \to 2^-} \frac{\ln x}{x-2} = -\infty$ *calculated in part (a).*
On the other hand, we know that
$\displaystyle\lim_{x \to 0^+} \ln(x) = -\infty$.
So, it follows that
$\displaystyle\lim_{x \to 0^+} \frac{\ln(x)}{x-2} = \infty$,
since the limit is of the form $\dfrac{\infty}{\#}$, *the numerator negative, and the denominator negative. That means* $x = 0$ *is a vertical asymptote of* f *since* $\displaystyle\lim_{x \to 0^+} \frac{\ln x}{x-2} = \infty$

Problem 5 *Select the meaning of* $\displaystyle\lim_{x \to \infty} f(x) = 6$. *Support your explanation graphically.*

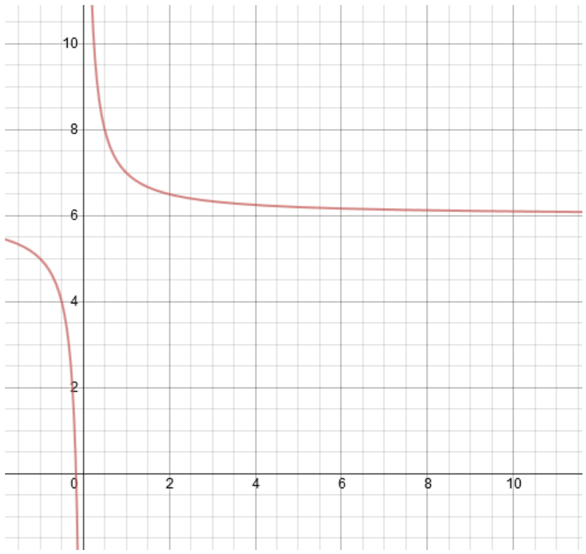
(a) *As x becomes arbitrarily negatively large, $f(x)$ approaches 6.*

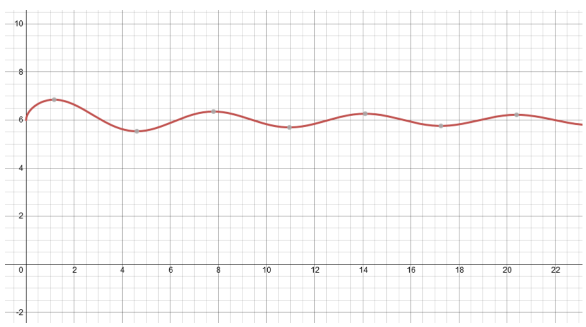
(b) *As x becomes arbitrarily positively large, $f(x)$ approaches 6.*

(c) *As x approaches 6, $f(x)$ becomes arbitrarily negatively large.*

(d) *As x approaches 6, $f(x)$ becomes arbitrarily positively large.*

Solution: *As x becomes arbitrarily positively large, $f(x)$ approaches 6. Here are two examples of what this function might look like:*





Problem 6 *Evaluate the following limits.*

(a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt[3]{x^9+5}}{3x^3+\sqrt{4x^6+1}}$

Solution: *This limit is of the form:* $\dfrac{\infty}{\infty}$.

This will be a very detailed solution. Make sure you can justify every step below.

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt[3]{x^9+5}}{3x^3+\sqrt{4x^6+1}} &= \lim_{x\to\infty} \frac{\sqrt[3]{x^9+5}}{3x^3+\sqrt{4x^6+1}} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\frac{\sqrt[3]{x^9+5}}{x^3}}{\frac{3x^3+\sqrt{4x^6+1}}{x^3}} \\
&= \lim_{x\to\infty} \frac{\sqrt[3]{\frac{x^9+5}{x^9}}}{\frac{3x^3}{x^3}+\sqrt{\frac{4x^6+1}{x^6}}} \\
&= \lim_{x\to\infty} \frac{\sqrt[3]{1+\frac{5}{x^9}}}{3+\sqrt{4+\frac{1}{x^6}}} \\
&= \frac{\lim\limits_{x\to\infty} \sqrt[3]{1+\dfrac{5}{x^9}}}{\lim\limits_{x\to\infty}\left(3+\sqrt{4+\dfrac{1}{x^6}}\right)} \\
&= \frac{\sqrt[3]{\lim\limits_{x\to\infty}\left(1+\dfrac{5}{x^9}\right)}}{3+\lim\limits_{x\to\infty}\sqrt{4+\dfrac{1}{x^6}}} \\
&= \frac{\sqrt[3]{1+\lim\limits_{x\to\infty}\dfrac{5}{x^9}}}{3+\sqrt{\lim\limits_{x\to\infty}\left(4+\dfrac{1}{x^6}\right)}} \\
&= \frac{\sqrt[3]{1+5\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)^9}}{3+\sqrt{4+\lim\limits_{x\to\infty}\dfrac{1}{x^6}}} \\
&= \frac{\sqrt[3]{1+5\left(\lim\limits_{x\to\infty}\dfrac{1}{x}\right)^9}}{3+\sqrt{4+\left(\lim\limits_{x\to\infty}\dfrac{1}{x}\right)^6}} \\
&= \frac{\sqrt[3]{1+5\left(0\right)^9}}{3+\sqrt{4+\left(0\right)^6}} \\
&= \frac{1}{3+\sqrt{4}} \\
&= \frac{1}{5}
\end{aligned}$$

(b) $\displaystyle\lim_{x\to\infty} \frac{\sin(9x)}{5x}$

Solution: *Since* $-1 \le \sin(9x) \le 1$ *for all* x *, and we can divide by* $5x > 0$, $\dfrac{-1}{5x} \le \dfrac{\sin(9x)}{5x} \le \dfrac{1}{5x}$.
We have $\lim\limits_{x\to\infty}\left(-\dfrac{1}{5x}\right) = -\dfrac{1}{5}\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right) = 0$ *and* $\lim\limits_{x\to\infty}\dfrac{1}{5x} = 0$. *The Squeeze theorem implies that*

$$\lim_{x\to\infty}\frac{\sin(9x)}{5x} = 0$$

Problem 7 *The function* f *is defined by* $f(x) = \dfrac{6e^x + 1}{3e^x + 5}$.

(a) *Find all vertical asymptotes of* f. ***EXPLAIN*** *and justify your answer by using appropriate limits.*

Solution: *Since the exponential function,* e^x, *is always positive, it follows that the denominator of the function* f *is also always positive. Since the exponential function,* e^x, *is continuous on* $(-\infty, \infty)$, *it follows that the functions* $6e^x + 1$ *and* $3e^x + 5$ *are also continuous on* $(-\infty, \infty)$. *Therefore, the function* f *is also continuous there, since it is a quotient of two continuous functions, and the function in the denominator is never* 0.

Therefore, f *has no vertical asymptotes, since it is continuous on* $(-\infty, \infty)$.

(b) *Find all horizontal asymptotes of* f. ***EXPLAIN*** *and justify your answer by using appropriate limits.*

Solution: *Check End Behavior as* $x \to \infty$:

$\lim\limits_{x\to\infty}\dfrac{6e^x + 1}{3e^x + 5}$ *is of the form:* $\dfrac{\infty}{\infty}$

$$\lim_{x\to\infty}\frac{6e^x + 1}{3e^x + 5} = \lim_{x\to\infty}\frac{6e^x + 1}{3e^x + 5}\cdot\frac{\frac{1}{e^x}}{\frac{1}{e^x}} =$$

$$= \lim_{x\to\infty}\frac{6 + \frac{1}{e^x}}{3 + \frac{5}{e^x}} = \frac{6+0}{3+0} = 2$$

The line $y = 2$ *is a horizontal asymptote of* f *because* $\lim\limits_{x\to\infty} f(x) = 2$.

Check End Behavior as $x \to -\infty$:

$$\lim_{x\to-\infty}\frac{6e^x + 1}{3e^x + 5} = \frac{0+1}{0+5} = \frac{1}{5}$$

The line $y = \dfrac{1}{5}$ *is a horizontal asymptote of* f *because* $\lim\limits_{x\to-\infty} f(x) = \dfrac{1}{5}$.

Problem 8 *The function* f *is defined by* $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$.

(a) *Find all vertical asymptotes of* f. ***EXPLAIN*** *and ustify your answer by using appropriate limits.*

Solution: *Candidate for vertical asymptotes:* $3x - 5 = 0 \implies x = 5/3$.
Test of candidate $x = 5/3$*:*

$$\lim_{x \to \frac{5}{3}^+} \frac{\sqrt{2x^2+1}}{3x-5} = \infty$$

Note: the limit is of the form $\dfrac{\#}{0}$, *the numerator positive, and the denominator* $3x - 5 = 3(x - 5/3)$ *is positive, since* $x > 5/3$.

The line $x = \dfrac{5}{3}$ *is a vertical asymptote of* f *since* $\displaystyle\lim_{x \to \frac{5}{3}^+} f(x) = \infty$.

(b) *Find all horizontal asymptotes of* f*.* ***EXPLAIN*** *and justify your answer by using appropriate limits..*

Solution: *Check End Behavior as* $x \to \infty$*:*

$\displaystyle\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}$ *is of the form:* $\dfrac{\infty}{\infty}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{3 - \frac{5}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}} \\
&= \frac{\sqrt{2}}{3}
\end{aligned}$$

The line $y = \dfrac{\sqrt{2}}{3}$ *is a horizontal asymptote of* f *since* $\displaystyle\lim_{x\to\infty} f(x) = \frac{\sqrt{2}}{3}$.
Check End Behavior as $x \to -\infty$*:*

$\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$ *is of the form:* $\dfrac{\infty}{\infty}$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \cdot \frac{\frac{1}{-x}}{\frac{1}{-x}} \\
&= \lim_{x\to-\infty} \frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{-3 + \frac{5}{x}} \\
&= \lim_{x\to-\infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{-3 + \frac{5}{x}} \\
&= \frac{-\sqrt{2}}{3}
\end{aligned}$$

The line $y = -\dfrac{\sqrt{2}}{3}$ *is a horizontal asymptote of* f *since* $\displaystyle\lim_{x\to-\infty} f(x) = -\frac{\sqrt{2}}{3}$.

Problem 9 *A piecewise defined function f is given by*

$$f(x)=\begin{cases}\dfrac{2x-3}{x-2} & \text{if} \quad x<2\\[2ex] \dfrac{x^2-5x+6}{x^2-4} & \text{if} \quad x>2\end{cases}$$

(a) *Find all vertical asymptotes.* ***EXPLAIN*** *and justify your answer by using appropriate limits.*

Solution: *Since* $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{2x-3}{x-2}$, *the form of the limit is* $\dfrac{\#}{0}$.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{2x-3}{x-2} = -\infty,$$

since the numerator positive, the denominator negative and goes to 0.
Therefore, the line $x=2$ is a vertical asymptote of f since $\lim_{x\to 2^-} f(x) = -\infty$.

Note: The line $x=-2$ is not a vertical asymptote, since for values of x near -2, $f(x)=\dfrac{2x-3}{x-2}$. *So, f is continuous at -2.*

(b) *Find all horizontal asymptotes.* ***EXPLAIN*** *and justify your answer by using appropriate limits.*

Solution: *Check End Behavior as $x\to\infty$:*

$\lim_{x\to\infty} \dfrac{x^2-5x+6}{x^2-4}$ *is of the form:* $\dfrac{\infty}{\infty}$

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2-5x+6}{x^2-4} &= \lim_{x\to\infty} \frac{x^2-5x+6}{x^2-4}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\\ &= \lim_{x\to\infty} \frac{1-\frac{5}{x}+\frac{6}{x^2}}{1-\frac{4}{x^2}}\\ &= 1\end{aligned}$$

Therefore, the line $y=1$ is a horizontal asymptote of f since $\lim_{x\to\infty} f(x) = 1$.

Check End Behavior as $x\to-\infty$:

$\lim_{x\to-\infty} \dfrac{2x-3}{x-2}$ *is of the form:* $\dfrac{\infty}{\infty}$

$$\lim_{x\to-\infty}\frac{2x-3}{x-2} = \lim_{x\to-\infty}\frac{2x-3}{x-2}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$$
$$= \lim_{x\to-\infty}\frac{2-\frac{3}{x}}{1-\frac{2}{x}}$$
$$= 2$$

Therefore, the line $y = 2$ is a horizontal asymptote of f since $\lim_{x\to-\infty} f(x) = 2$.

Problem 10 *For the piecewise function f defined by*

$$f(x) = \begin{cases} \sin(x) & \text{if } \quad x \le 0 \\ \dfrac{x^2}{x^2-4} & \text{if } \quad 0 < x < 2 \\ \dfrac{x^2}{x^2+4} & \text{if } \quad 2 \le x \end{cases}$$

(a) *Find all vertical asymptotes.* ***EXPLAIN*** *and justify your answer by using appropriate limits.*

Solution: *Candidates for vertical asymptote: $x^2 - 4 = 0 \implies x = -2,\ x = 2$ (Each piece of f is continuous at all other points).*

Note: $x = -2$ is not a vertical asymptote, since $f(x) = \sin(x)$ near -2, so $\lim_{x\to-2} f(x) = \lim_{x\to-2}\sin(x) =$ $\sin(-2)$.

$$\lim_{x\to2^-}\frac{x^2}{x^2-4} = \lim_{x\to2^-}\frac{x^2}{(x-2)(x+2)} = -\infty$$

Note: the limit is of the form $\frac{\#}{0}$, the numerator positive, and the denominator negative.

The line $x = 2$ is a vertical asymptote of f because $\lim_{x\to2^-} f(x) = -\infty$. This is the only vertical asymptote of f.

(b) *Find all horizontal asymptotes.* ***EXPLAIN*** *and justify your answer by using appropriate limits.*

Solution: *Check End Behavior as $x \to \infty$:*

$\lim_{x\to\infty} \frac{x^2}{x^2+4}$ *is of the form:* $\frac{\infty}{\infty}$

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2}{x^2+4} &= \lim_{x\to\infty} \frac{x^2}{x^2+4} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{1}{1+\frac{4}{x^2}} \\ &= 1\end{aligned}$$

The line $y = 1$ is a horizontal asymptote of f since $\lim_{x\to\infty} f(x) = 1$.

Check End Behavior as $x \to -\infty$:

$\lim_{x\to-\infty} \sin(x)$ *does not exist, since the function oscillates.*

For example, $f(-n\pi) = \sin(-n\pi) = 0$, while $f(-(2n+1)\frac{\pi}{2}) = \sin\left(-(2n+1)\frac{\pi}{2}\right) = 1$, for all positive integers n.
There is no horizontal asymptote as $x \to -\infty$ since $\lim_{x\to-\infty} f(x)$ does not exist.

Problem 11 *For the function g defined by*

$$g(t) = \frac{t^2+7t+11}{t-3}$$

(a) *Find all vertical asymptotes.* ***EXPLAIN*** *and justify your answer by using appropriate limits.*

Solution: *The function g is a rational function which is continuous on the intervals $(-\infty, 3)$ and $(3, \infty)$. The only candidates for a vertical asymptote is $t = 3$, where g is not defined.*
Test of candidate $t = 3$:

$$\lim_{t\to 3^+} \frac{t^2+7t+11}{t-3} = \infty$$

Note: the limit is of the form $\frac{\#}{0}$, the numerator positive, and the denominator also positive.
The line $t = 3$ is a vertical asymptote of g since $\lim_{t\to 3^+} g(t) = \infty$.

(b) *Find all horizontal asymptotes.* ***EXPLAIN*** *and justify your answer by using appropriate limits.*

Solution: *To find the limits, we need to divide the numerator and denominator by the highest power of t in the denominator.*

$$\lim_{t\to\infty} \frac{t^2+7t+11}{t-3} = \lim_{t\to\infty} \frac{t+7+\frac{11}{t}}{1-\frac{3}{t}} = \infty$$

Note: the last limit is of the form $\dfrac{\infty}{\#}$*, the numerator positive, the denominator positive.*

Therefore, g *has no horizontal asymptotes as* $t \to \infty$*.*

$$\lim_{t\to-\infty} \frac{t^2+7t+11}{t-3} = \lim_{t\to-\infty} \frac{t+7+\frac{11}{t}}{1-\frac{3}{t}} = -\infty$$

Note: the last limit is of the form $\dfrac{\infty}{\#}$*, the numerator negative, the denominator positive.*

Therefore, g *has no horizontal asymptotes as* $t \to -\infty$ *either.*

Problem 12 *Sketch a possible graph of a function that satisfies all of the given properties. (You do not need to find a formula for the function.)*

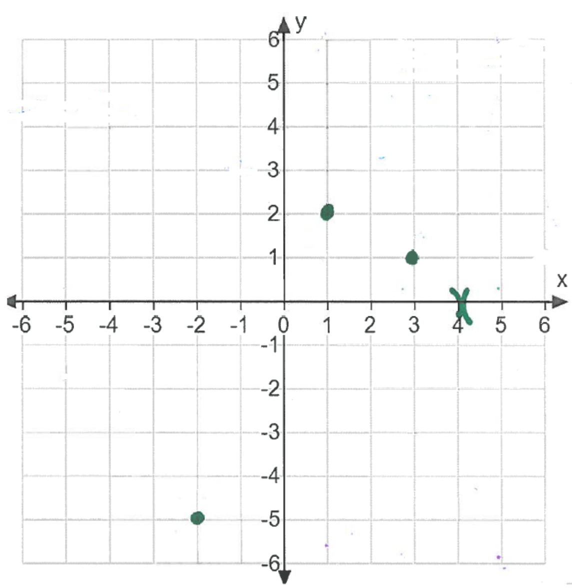
$$\lim_{x\to-2^-} f(x) = \infty \qquad f(-2) = -5 \qquad f(1) = 2$$

$$\lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = 4 \qquad \lim_{x\to5} f(x) = \infty$$

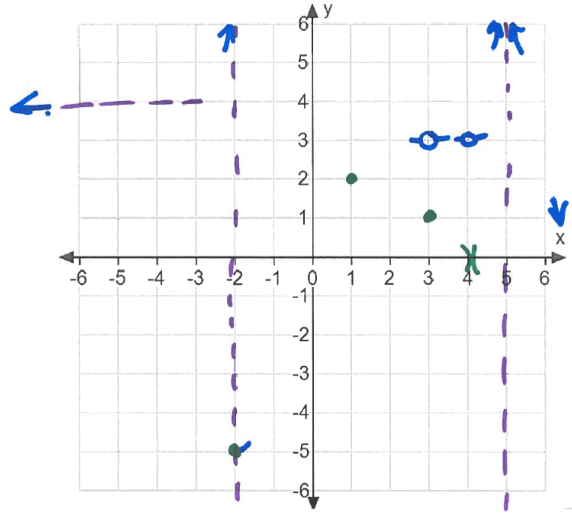
$$\lim_{x\to3} f(x) = 3 \qquad f(3) = 1$$

$$\lim_{x\to-2^+} f(x) = -5 \qquad f(4) \text{ is undefined} \qquad \lim_{x\to4} f(x) = 3$$

Solution: *There are many correct solutions to this problem. One possible solution can be constructed as follows: First we draw the given points. We'll also draw open brackets on the x-axis at* $x = 4$ *since* $f(x)$ *is not defined at* $x = 4$ *(This is seen in green.)*



Then we draw the vertical and horizontal asymptotes (in purple) and arrows indicating how the graph approaches the asymptote (in blue). We also draw end-behavior as x approaches ∞ and $-\infty$ (in blue). We can also draw tails where we know limits at particular x-values. Such as $\lim_{x \to -2^+} f(x) = -5$.



Finally we connect our graph together (seen in blue)