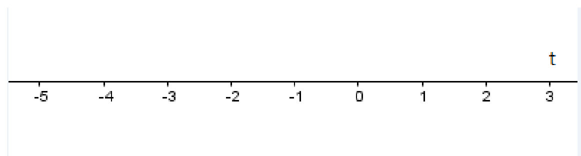
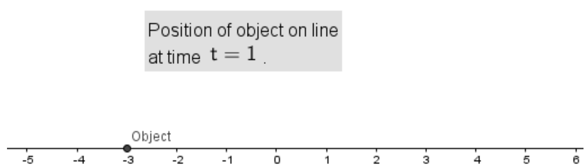
An application of limits (AAOL) - Solutions

Problem 1 *The position, $s(t)$, of an object moving along a horizontal line is given by $s(t) = t^2 - 4$, where s is in meters and t is in seconds, $0 \le t < 5$.*

(a) *Mark the position of the object on the line at time $t = 1$:*



Solution: $s(1) = 1^2 - 4 = -3$



(b) *Find the average velocity, $v_{\rm AV}$, of the object during the time interval $[1, 3]$.*

Solution: *The average velocity over $[1, 3]$ is*

$$\frac{s(3) - s(1)}{3 - 1} = \frac{5 - (-3)}{2} = \frac{8}{2} = 4m/s$$

(c) *Compute the average velocity, $v_{\rm AV}(t)$, of the object during the time interval*

(i) $[1, t]$, *for $1 < t < 5$;*

Solution: *The average velocity,$v_{\rm AV}(t)$ over $[1, t]$ is*

$$\begin{aligned} v_{\rm AV}(t) = \frac{s(t) - s(1)}{t - 1} &= \frac{(t^2 - 4) - (-3)}{t - 1} \\ &= \frac{t^2 - 1}{t - 1} = t + 1, \quad 1 < t < 5 \end{aligned}$$

(ii) $[t, 1]$, *for $0 \le t < 1$.*

Solution: *The average velocity over $[t, 1]$ is*

$$\begin{aligned} v_{\rm AV}(t) = \frac{s(1) - s(t)}{1 - t} &= \frac{(-3) - (t^2 - 4)}{1 - t} \\ &= \frac{1 - t^2}{1 - t} = 1 + t, \quad 0 < t < 1 \end{aligned}$$

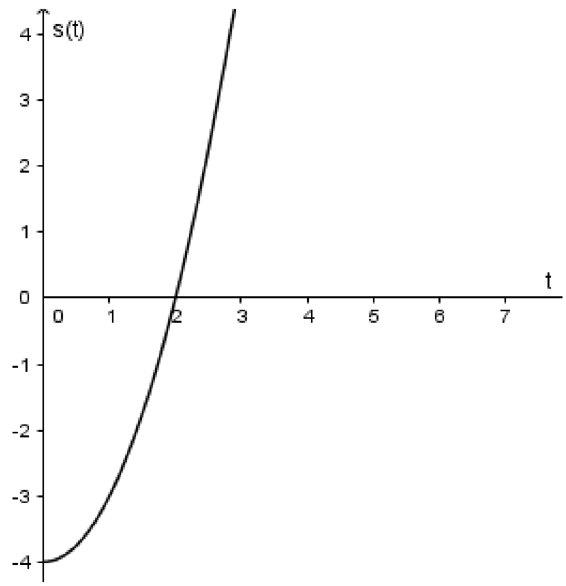
Note: $v_{\rm AV}(t) = \dfrac{s(1) - s(t)}{1 - t} = \dfrac{s(t) - s(1)}{t - 1} = 1 + t, \quad 0 \le t < 5$

(d) *Find the instantaneous velocity, v_{inst}, of the object at $t = 1$. Justify your answer.*

Solution: *The instantaneous velocity of the object at $t = 1$ is given by $\lim\limits_{t\to 1} \dfrac{s(t) - s(1)}{t - 1}$. This limit has form $\dfrac{0}{0}$.*

$$\begin{aligned} v_{\text{inst}} &= \lim_{t\to 1} v_{\text{AV}}(t) = \lim_{t\to 1} \frac{s(t) - s(1)}{t - 1} \\ &= \lim_{t\to 1} (t + 1) = 2. \end{aligned}$$

(e) *The position-time graph of the function s is given in the figure below.*



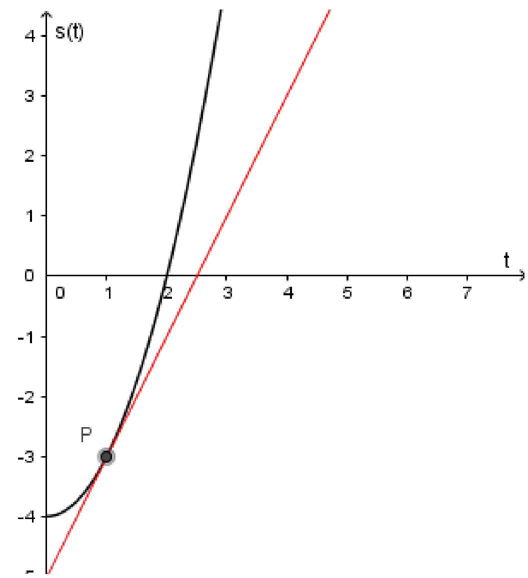
(i) *Assume P is a point on the graph of s. Fill in the blank.*

$$P = (1, \underline{\qquad}).$$

Solution: $P = (1, \underline{-3})$

(ii) *Plot the point P and draw the tangent line at this point in the figure above.*

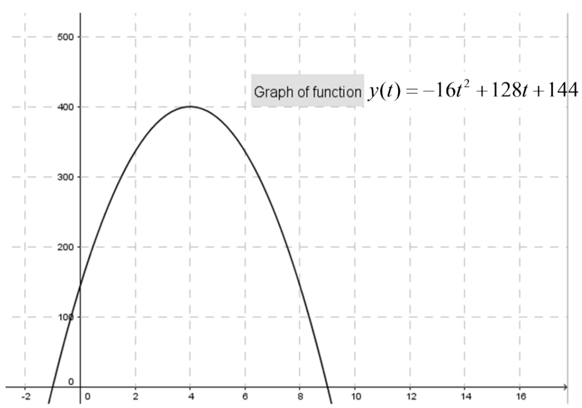
Solution:



(iii) *Find the slope, m_{tan}, of the tangent line in part (ii). Explain.*

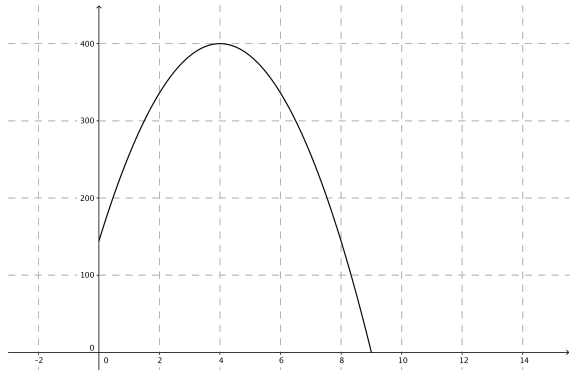
Solution: *The slope of the tangent line at $t = 1$ is the same as the instantaneous velocity at $t = 1$. Therefore $m_{\text{tan}} = v_{\text{inst}} = 2$.*

Problem 2 *Part of the given parabola can be used to model the "position-time" graph of a ball thrown straight up into the air. The graph gives the height of the ball in feet t seconds after being thrown into the air.*
Let the function f be defined by $f(t) = -16t^2 + 128t + 144$.
Use this graph, and the given function, f, to answer the following questions.



(a) *Mark the part of the parabola that can be used to model the position of the ball.*

Solution:



(b) *What are the units on the t axis? What are the units on the y axis?*

Solution: *The units on the t axis are "seconds" (for time), while the units on the y axis are "feet" (for height).*

(c) *If you were watching a movie of the ball being thrown, is the graph a picture of the path that the ball follows? Why or why not?*

Solution: *No, the position-time graph is* not *the path the ball follows. The graph shows the height of the ball at a given* time. *The ball is thrown straight up and has no horizontal movement so its path is on a vertical line.*

(d) *Let $f(t)$ denote the height of the ball at time $t, t \geq 0$. What is the height of the ball at time $t = 0$?*

Solution: *The height can be found by finding $f(0)$*

$$f(0) = -16(0)^2 + 128(0) + 144$$
$$f(0) = 144 \textit{ feet}$$

(e) *When will the ball hit the ground?*

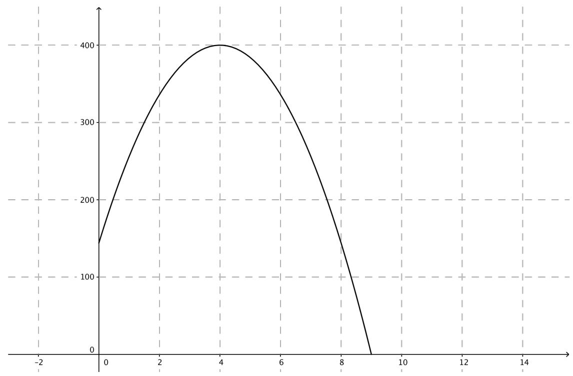
Solution: *The ball will hit the ground when the height $f(t)$ equals zero.*

$$0 = -16t^2 + 128t + 144$$
$$0 = -16(t^2 - 8t - 9)$$
$$0 = -16(t + 1)(t - 9)$$
$$t = -1, 9$$

The ball will hit the ground at $t = 9$ or 9 seconds after the ball is thrown into the air.

(f) *What is the domain of the position function, f, of the ball?*

Solution: *The domain of f is the interval $[0, 9]$. The ball is thrown straight up at $t = 0$ and hits the ground at $t = 9$. With this domain, the position-time graph of the ball is given by*



(g) *Use the table of values to find the average velocity of the ball between $t = 8.9$ and $t = 9$ seconds.*

t	$\approx f(t)$
8.9	15.84
8.99	1.6
8.999	0.159984
8.9999	0.015998
9	0

Solution: *The average velocity of the ball between $t = 8.9$ seconds and $t = 9$ seconds is*

$$\frac{f(9) - f(8.9)}{9 - 8.9} = \frac{0 - 15.84}{0.1} = -158.4 \textit{ feet per second}$$

(h) *Use the table of average velocities to approximate the instantaneous velocity of the ball when it hits the ground.*

Time Interval	*Average Velocity*
$[8.9, 9]$	$\frac{f(9) - f(8.9)}{.1} = \frac{0 - 15.84}{.1} = -158.4$
$[8.99, 9]$	$\frac{f(9) - f(8.99)}{.01} = \frac{0 - 1.5984}{.01} = -159.84$
$[8.999, 9]$	$\frac{f(9) - f(8.999)}{.001} = \frac{0 - .159984}{.001} = -159.984$
$[8.9999, 9]$	$\frac{f(9) - f(8.999)}{.0001} = \frac{0 - .0159998}{.0001} = -159.998$

Solution: *The instantaneous velocity of the ball hitting the ground appears to be -160 ft/sec.*

(i) *Compute $v_{AV(t)}$ the average velocity of the ball on the time interval $[t, 9]$, where $t < 9$.*

Solution:

$$
\begin{aligned}
v_{AV(t)} &= \frac{f(9) - f(t)}{9 - t} \\
&= \frac{0 - (-16t^2 + 128t + 144)}{9 - t} \\
&= \frac{16t^2 - 128t - 144)}{9 - t} \\
&= \frac{16(t^2 - 8t - 9)}{9 - t} \\
&= \frac{16(t - 9)(t + 1)}{9 - t} \\
&= \frac{-16(9 - t)(t + 1)}{9 - t} \\
&= -16(t + 1) \textit{ feet per second}
\end{aligned}
$$

(j) *Compute $v(9)$, the **instantaneous** velocity of the ball at $t = 9$.*

Solution:

$$
\begin{aligned}
v(9) &= \lim_{t \to 9^-} v_{AV(t)} \\
&= \lim_{t \to 9^-} \frac{f(9) - f(t)}{9 - t}
\end{aligned}
$$

Notice that this is an indeterminate form, with form $\frac{0}{0}$.

$$
\begin{aligned}
v(9) &= \lim_{t \to 9^-} v_{AV(t)} \\
&= \lim_{t \to 9^-} \frac{f(9) - f(t)}{9 - t} \\
&= \lim_{t \to 9^-} -16(t + 1) \\
&= -16(9 + 1) \\
&= -160 \textit{ feet per second}
\end{aligned}
$$