Definite integrals (DI) - Solutions

Problem 1 *Consider the following limit of Riemann sums of a function g on $[a, b]$:*

$$\lim_{n\to\infty} \sum_{k=1}^{n} (x_k^* + \cos(x_k^*))\Delta x \ , \ [0, \pi].$$

Express the limit as a definite integral. Use geometry to evaluate the resulting definite integral.

Solution: *We have*

$$\int_0^{\pi} (x + \cos(x))\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} (x_k^* + \cos(x_k^*))\Delta x.$$

To evaluate this definite integral, we'll start by writing $\int_0^{\pi} (x + \cos(x))\, dx = \int_0^{\pi} x\, dx + \int_0^{\pi} \cos(x)\, dx.$

The first term is the definite integral $\int_0^{\pi} x\, dx$ *which is the net area under the graph of* $y = x$ *on the interval* $[0, \pi]$. *This is a triangle of base* π *and height* π, *so it has area* $\frac{1}{2}\pi^2$.

The second term $\int_0^{\pi} \cos(x)\, dx$ *is the net area between the graph of* $y = \cos(x)$ *and the* x*-axis on the interval* $[0, \pi]$. *In the interval* $[0, \frac{\pi}{2}]$ *the function is positive, and in the interval* $[\frac{\pi}{2}, \pi]$ *the function is negative. By the symmetry of the cosine function, these two regions have the same geometric area, so the net area is zero.*

$$\int_0^{\pi} (x + \cos(x))\, dx = \int_0^{\pi} x\, dx + \int_0^{\pi} \cos(x)\, dx = \frac{\pi^2}{2}.$$

Problem 2 *Let $f(x)$ and $g(x)$ be functions for which we only know the following:*

$$\int_1^4 f(x)\, dx = 7 \qquad \int_2^4 f(x)\, dx = 5 \qquad \int_1^4 g(x)\, dx = 2$$

Compute the following integrals, if possible. If it is not possible, give examples explaining why not.

(a) $\int_1^4 (8f(x) - 7g(x))\, dx$

Solution:

$$\begin{aligned} \int_1^4 (8f(x) - 7g(x))\, dx &= 8\int_1^4 f(x)\, dx - 7\int_1^4 g(x)\, dx \\ &= 8(7) - 7(2) \\ &= 56 - 14 = 42 \end{aligned}$$

(b) $\int_1^2 (-f(x))\, dx$

Solution: First notice that

$$\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx$$

Therefore,

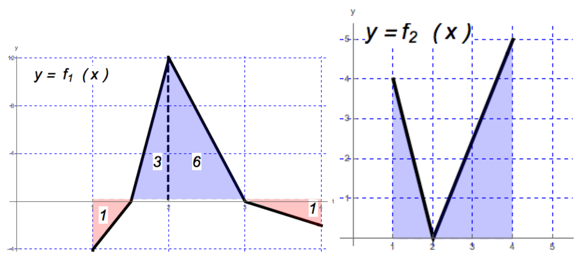
$$\int_1^4 f(x)\,dx - \int_2^4 f(x)\,dx = \int_1^2 f(x)\,dx$$

So

$$\begin{aligned}\int_1^2 (-f(x))\,dx &= -\int_1^2 f(x)\,dx \\ &= -\left(\int_1^4 f(x)\,dx - \int_2^4 f(x)\,dx\right) \\ &= -(7-5) = -2.\end{aligned}$$

(c) $\displaystyle\int_1^4 |f(x)|\,dx$

Solution: We are not given enough information to solve this integral because we do not know the regions where f is positive or negative. Consider the following two functions $f_1(x)$ and $f_2(x)$:



Just using geometry, one can check that

$$\int_1^4 f_1(x)\,dx = 7 \qquad \int_2^4 f_1(x)\,dx = 5 \qquad \int_1^4 f_2(x)\,dx = 7 \qquad \int_2^4 f_2(x)\,dx = 5$$

and so both f_1 and f_2 satisfy the assumptions of f. But notice that

$$\int_1^4 |f_1(x)|\,dx = 11 \qquad \text{and} \qquad \int_1^4 |f_2(x)|\,dx = 7$$

These two examples demonstrate that we were not given enough information to solve this problem.

(d) $\displaystyle\int_1^4 (2 - x + f(x))\,dx$

Solution: First notice that since the integral is linear over addition:

$$\int_1^4 (2 - x + f(x))\,dx = \int_1^4 2\,dx - \int_1^4 x\,dx + \int_1^4 f(x)\,dx = \int_1^4 2\,dx - \int_1^4 x\,dx + 7. \qquad (1)$$

By using geometry, we can see that

$$\int_1^4 2\,dx = 2(4-1) = 6$$

$$\int_1^4 x\,dx = 1(4-1) + \frac{1}{2}(4-1)(4-1) = 3 + \frac{9}{2} = \frac{15}{2}.$$

Then substituting into equation (1) gives:

$$\int_1^4 (2 - x + f(x))\,dx = 6 - \frac{15}{2} + 7 = \frac{11}{2}.$$

Problem 3 *Evaluate the following sums:*

(a) $\displaystyle\sum_{k=1}^{4} k^5$

Solution: $\displaystyle\sum_{k=1}^{4} k^5 = 1^5 + 2^5 + 3^5 + 4^5 = 1 + 32 + 243 + 1024 = 1300.$

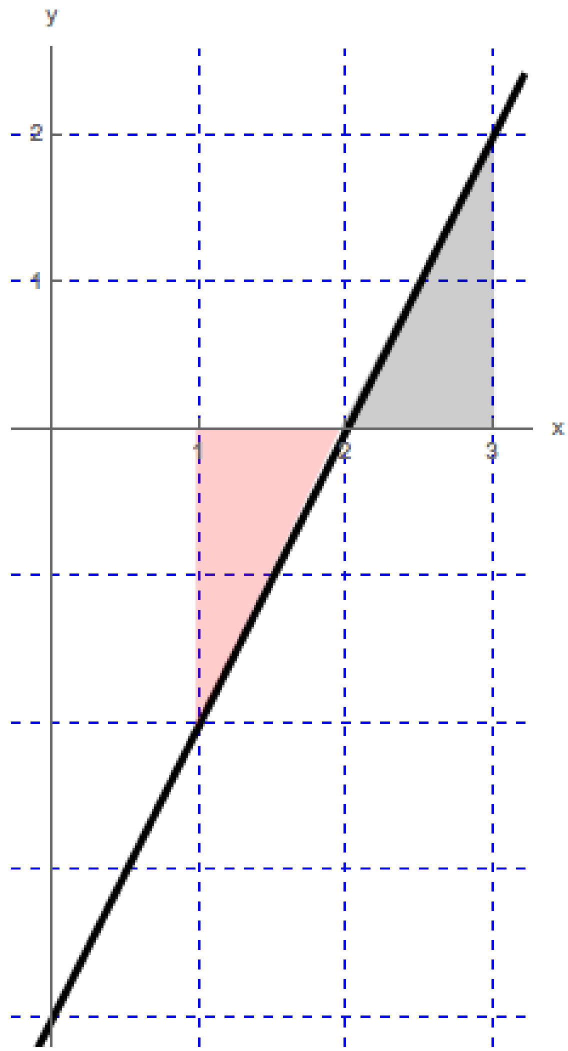
(b) $\displaystyle\sum_{k=1}^{400} (5(k+1)^2 + 3)$

Solution:

$$\begin{aligned}
\sum_{k=1}^{400} (5(k+1)^2 + 3) &= \sum_{k=1}^{400} (5(k^2 + 2k + 1) + 3) \\
&= \sum_{k=1}^{400} (5k^2 + 10k + 8) \\
&= \sum_{k=1}^{400} 5k^2 + \sum_{k=1}^{400} 10k + \sum_{k=1}^{400} 8 \\
&= 5\sum_{k=1}^{400} k^2 + 10\sum_{k=1}^{400} k + 8(400) \\
&= 5\left(\frac{400(400+1)(2(400)+1)}{6}\right) + 10\left(\frac{400(400+1)}{2}\right) + 3,200 \\
&= 5(200)(401)(267) + 10(200)(401) + 3,200 \\
&= 107,067,000 + 802,000 + 3,200 = 107,872,200
\end{aligned}$$

Problem 4 *Use geometry to evaluate the definite integral. Sketch the graph of the function and shade the relevant regions.*

(a) $\displaystyle\int_1^3 (2x - 4)\,dx$



Solution:

We want to find the net area of the shaded region. We have two identical triangles, each with base 1 and height 2. Since the triangles will have identical area and one lies above and the other below the x-axis, the total area will be 0.

Therefore, $\int_1^3 (2x-4)\,dx = 0$

(b) $\int_1^3 |2x-4|\,dx$

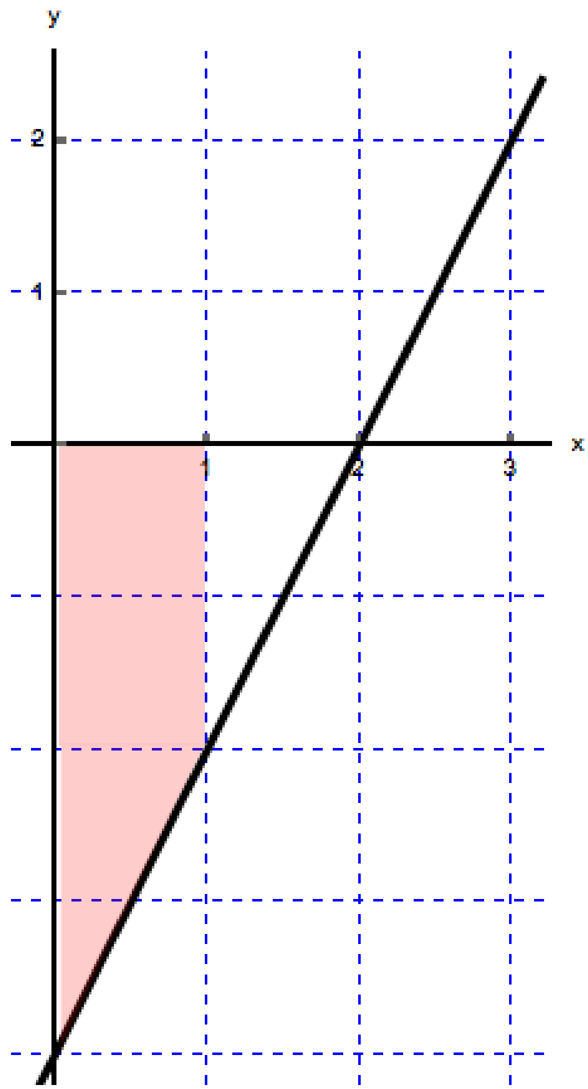
Solution: *The functions changes the sign at $x = 2$. Therefore, we will split the integral into two integrals:*

$$\int_1^3 |2x-4|\,dx = \int_1^2 |2x-4|\,dx + \int_2^3 |2x-4|\,dx.$$

Since the function is negative for $x < 2$ and positive for $x > 2$, we write

$$\int_1^3 |2x-4|\,dx = -\int_1^2 (2x-4)\,dx + \int_2^3 (2x-4)\,dx = 1 + 1 = 2.$$

(c) $\int_0^1 (2x-4)\,dx$

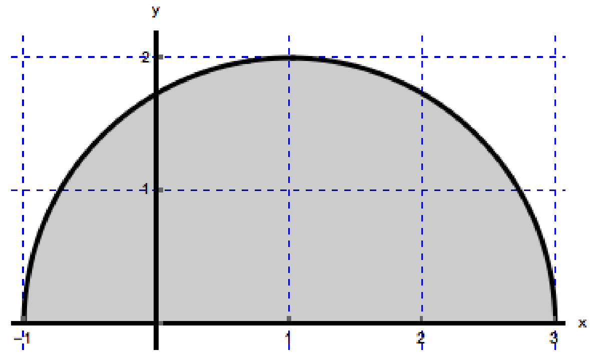


Solution:

Here we need to find the area of a trapezoid. The trapezoid is made up of a rectangular top with area $(2)(1) = 2$ and triangle bottom with area $(1/2)(1)(2) = 1$. This gives us an area of 3, but, since f is negative on the interval $(0,1)$, the value of the integral is the negative of this area.

Therefore, $\int_0^1 (2x-4)\,dx = -3$

(d) $\int_{-1}^3 \sqrt{4-(x-1)^2}\,dx$



Solution:

We are looking for the area of the entire semi-circle above the x-axis. This is a circle with radius 2 so we have $(1/2)\pi 2^2 = 2\pi$

$$\int_{-1}^{3} \sqrt{4-(x-1)^2}\,dx = 2\pi$$

Problem 5 (a) *If f is an odd function, why is it true that $\int_{-a}^{a} f(x)\,dx = 0$? Support your reasoning with a picture.*

Solution: *If f is odd, then the regions between the graph of f and the x-axis from $[-a, 0]$ and $[0, a]$ are reflections of each other through the origin. Thus, these two regions will have the same area but with opposite signs since they are on opposite sides of the x-axis. They will therefore cancel each other out.*

(b) *If f is an even function, why is it true that $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$? Support your reasoning with a picture.*

Solution: *If f is even, then the regions between the graph of f and the x-axis from $[-a, 0]$ and $[0, a]$ are reflections of each other through the y-axis. Thus, these two regions will have the same area with the same sign since they are on the same sides of the x-axis. So you can only find one of these areas and then double it.*

Problem 6 (a) *Find the following definite integral:*

$$\int_{-4}^{4} \frac{x^2 \sin^3(x)}{\sqrt{x^4+1}}\,dx$$

Solution: *Let $f(x) = \dfrac{x^2\sin^3(x)}{\sqrt{x^4+1}}$. Then notice that*

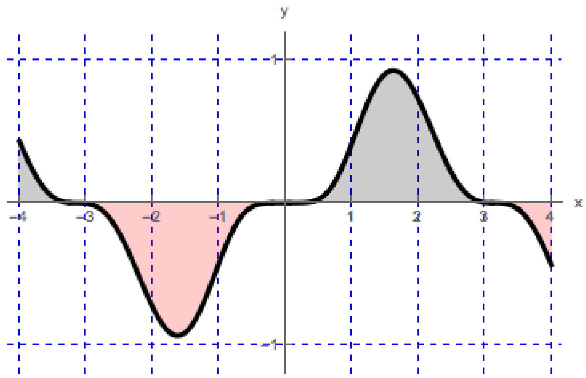
$$\begin{aligned} f(-x) &= \frac{(-x)^2 \sin^3(-x)}{\sqrt{(-x)^4+1}} \\ &= \frac{x^2(-\sin(x))^3}{\sqrt{x^4+1}} \quad \text{(since } \sin(x) \text{ is an odd function)} \\ &= \frac{-x^2\sin^3(x)}{\sqrt{x^4+1}} \\ &= -f(x). \end{aligned}$$

Thus, f is an odd function and therefore

$$\int_{-4}^{4} \frac{x^2 \sin^3(x)}{\sqrt{x^4+1}}\, dx = 0.$$

We can illustrate our computation with the graph of the function f, where

$$f(x) = \frac{x^2 \sin^3(x)}{\sqrt{x^4+1}}, \text{ for } -4 \le x \le 4.$$



(b) *Suppose that f is an even function. Given that $\int_0^6 f(x)\, dx = 13$, find $\int_{-6}^{6} (5f(x)+14)\, dx$.*

Solution: *First notice that*

$$\int_{-6}^{6} (5f(x)+14)\, dx = 5\int_{-6}^{6} f(x)\, dx + \int_{-6}^{6} 14\, dx. \tag{2}$$

Since f is even, we know that

$$\int_{-6}^{6} f(x)\, dx = 2\int_{0}^{6} f(x)\, dx = 2(13) = 26. \tag{3}$$

We also know that $g(x) = 14$ is also an even function. We have: $\int_{-6}^{6} 14\, dx = 2\int_0^6 14\, dx$

$$2\int_0^6 14\, dx = 2(14)(6) = 168. \tag{4}$$

Then substituting equations (3) and (4) into equation (2) gives:

$$\int_{-6}^{6} (5f(x)+14)\, dx = 5(26) + 168 = 130 + 168 = 298.$$

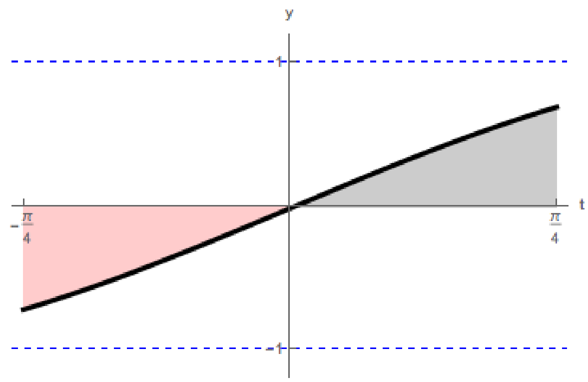
Problem 7 *Evaluate the following integrals using symmetry arguments.*

(a) $\int_{-\pi/4}^{\pi/4} \sin(t)\, dt$

Solution: *Since the function* $f(t) = \sin(t)$ *is odd,* $\displaystyle\int_{-\pi/4}^{\pi/4} \sin(t)\,dt = 0$

We can illustrate our computation with the graph of the function f*, where*

$f(t) = \sin t$*, for* $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$.



(b) $\displaystyle\int_{-2}^{2} (1 + x + 3x^7 - x^9)\,dx$

Solution: *In order to take advantage of symmetry, we will split the given polynomial into a sum of its even and odd parts. We will then integrate each part separately:*

$$\int_{-2}^{2} (1 + x + 3x^7 - x^9)\,dx = \int_{-2}^{2} 1\,dx + \int_{-2}^{2} (x + 3x^7 - x^9)\,dx$$

We will first integrate an even function:
$\displaystyle\int_{-2}^{2} 1\,dx = 2\int_{0}^{2} 1\,dx = 2(2) = 4$
Then an odd function:
$\displaystyle\int_{-2}^{2} (x + 3x^7 - x^9)\,dx = 0$

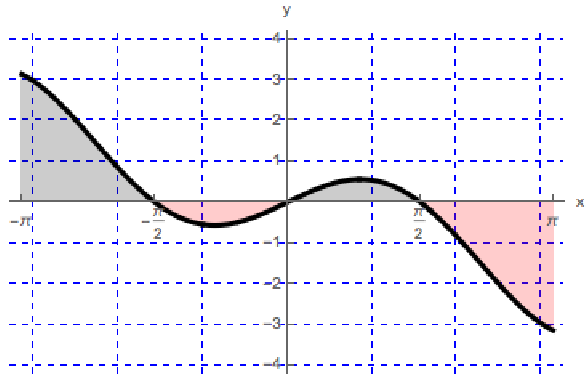
Therefore: $\displaystyle\int_{-2}^{2} (1 + x + 3x^2 - x^9)\,dx = 4$

(c) $\displaystyle\int_{-\pi}^{\pi} x\cos(x)\,dx$

Solution: $\displaystyle\int_{-\pi}^{\pi} x\cos(x)\,dx = 0$ *, since the product of an odd and even function is odd.*

We can illustrate our computation with the graph of the function f*, where*

$f(x) = x\cos x$*, for* $-\pi \le x \le \pi$.


y
4
3
2
1
x
$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
π
-1
-2
-3
-4