Maximums and minimums (MAM) - Solutions

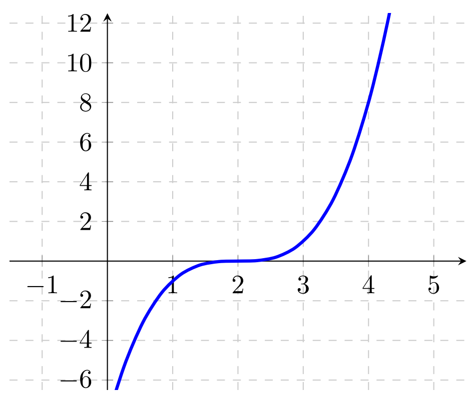
Problem 1 *Determine whether the following statements are true or false and give either an explanation or a counterexample.*

(a) *The function $f(x) = \sqrt{x}$ has a local maximum on the interval $[0, 1]$.*

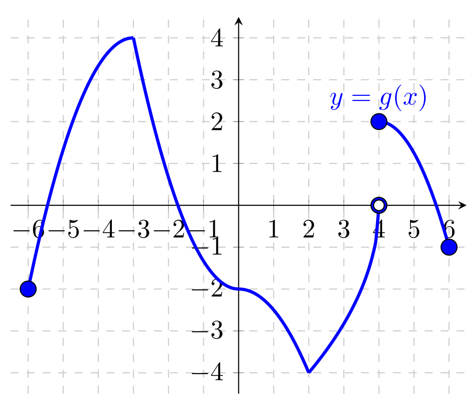
Solution: *False. Since $\sqrt{x}$ is increasing over the entire region $[0, 1]$, the only candidate for a local maximum would be $x = 1$. But by definition, endpoints are never local extrema. So $f(x) = \sqrt{x}$ has no local maximum on the interval $[0, 1]$.*

(b) *If $f'(2) = 0$, then $x = 2$ is either a local maximum or local minimum of f.*

Solution: *False. Consider $f(x) = (x - 2)^3$. This has derivative $f'(x) = 3(x - 2)^2$, so that $f'(2) = 0$. We can see from the graph of f below, that $x = 2$ is not a local extremum for f.*



Problem 2 *The entire graph of a function g is given below.*



Based on the graph of g, answer the questions below.

(a) *List the x-coordinates of all critical points of g.*

Solution: $x = -3$, $x = 0$, $x = 2$, and $x = 4$.

(b) *List the x-coordinates of all critical points of g where $g'(x) = 0$.*

Solution: $x = 0$

(c) *List the x-coordinates of all critical points of g where $g'(x)$ is **undefined**.*

Solution: $x = -3$, $x = 2$, and $x = 4$.

(d) *List the x-coordinates of all local maximums of g.*

Solution: $x = -3$ and $x = 4$.

(e) *List the x-coordinates of all local minimums of g.*

Solution: $x = 2$

(f) *List all intervals where g is both decreasing AND concave down.*

Solution: $(0, 2)$ and $(4, 6)$.

(g) *List all intervals where g is both decreasing AND concave up.*

Solution: $(-3, 0)$

(h) *List all intervals where g is both increasing AND concave down.*

Solution: $(-6, -3)$

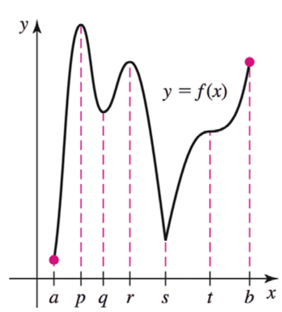
(i) *List all intervals where g is both increasing AND concave up.*

Solution: $(2, 4)$

(j) *List the x-coordinates of all inflection points of g.*

Solution: $x = -3$, $x = 0$, and $x = 2$.

Problem 3 *For each point in the interval (a,b) and identified on the graph below, determine if the function f has a critical point, a local max or min at that point.*



Solution: *(p) The function f has a critical point and a local maximum at $x = p$.*

(q) The function f has a critical point and a local minimum at $x = q$.

(r) The function f has a critical point and a local maximum at $x = r$

(s) The derivative of f does not exist at $x = s$ because the function f has a corner at $x = s$. The function has a critical point and a local minimum at $x = s$.

(t) The function f has a critical point but not a local maximum or minimum at $x = t$.

Problem 4 *Locate the critical points and use the second derivative test to determine whether they correspond to local maxima or local minima.* ***EXPLAIN****.*

$f(x) = (x + c)^4$ *where c is a positive constant*

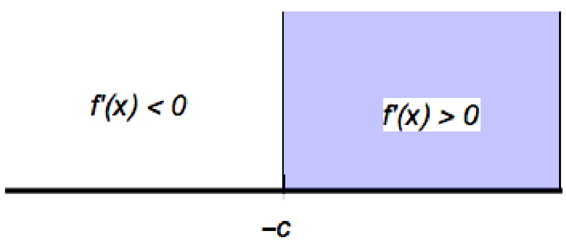
Solution:

$$f'(x) = 4(x + c)^3, f''(x) = 12(x + c)^2$$

To find the critical points: $f'(x) = 0 = 4(x + c)^3$. This occurs when $x = -c$

Using the second derivative test: $f''(-c) = 12(-c + c)^2 = 0$. The second derivative test was inconclusive so we need to use the first derivative test.

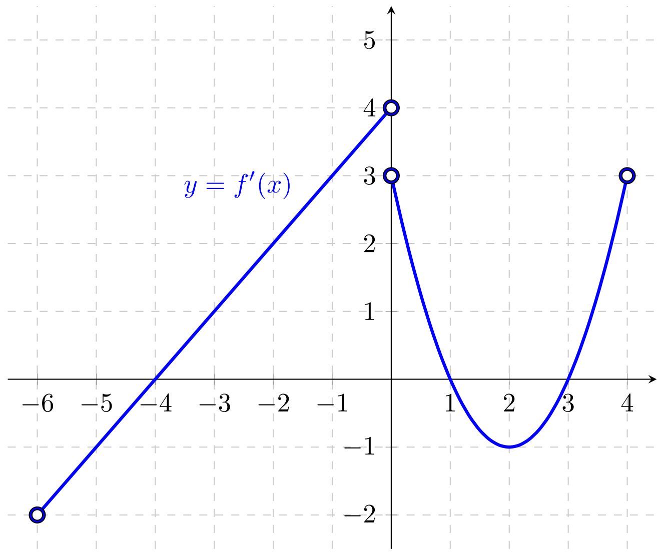
Making a sign chart we see that if we plug something a little less than $-c$ into f', we'll get a negative number. If we plug something a little greater than $-c$ into f', we'll get a positive number.



The derivative of f, f', does not change sign on the intervals $(-\infty, -c)$ and on $(-c, \infty)$, therefore, we have the above chart. From the chart, we can conclude that f has a local minimum at $x = -c$, and that f also has a global minimum there. We could also have used our understanding of the family of functions $g(x) = x^4$ to reason $x = -c$ would be a minimum.

At $x = -c$, f has a local minimum by the first derivative test since $x = -c$ is a critical point of f and the sign of f' changes from negative to positive around $x = -c$.

Problem 5 *Let f be a function that is continuous on its domain $(-6, 4)$. The graph of f', the derivative of f, is given below.*



Based on the graph of f', answer the question below.

(a) *List x-coordinates of all critical points of f.*

Solution: *Notice, $f'(-4) = f'(1) = f'(3) = 0$, and $f'(0)$ is not defined. Therefore, the function f has critical points at $x = -4$, $x = 0$, $x = 1$ and $x = 3$.*

(b) *List x-coordinates of all critical points of f where $f'(x) = 0$.*

Solution: $x = -4$, $x = 1$ *and* $x = 3$.

(c) *List x-coordinates of all critical points of f where $f'(x)$ is* ***undefined****.*

Solution: $x = 0$

(d) *List x-coordinates of all local minimums of f.*

Solution: $x = -4$ *and* $x = 3$*. At these points the sign of $f'(x)$ changes from negative to positive.*

(e) *List x-coordinates of all local maximums of f.*

Solution: $x = 1$ *At this point the sign of $f'(x)$ changes from positive to negative.*

(f) *List all intervals where f is decreasing and concave down.*

Solution: $(1, 2)$ *Notice, $f'(x)$ is negative and decreasing there.*

(g) *List all intervals where f is decreasing and concave up.*

Solution: $(-6, -4)$ *and* $(2, 3)$*. Notice, $f'(x)$ is negative and increasing there.*

(h) *List all intervals where f is increasing and concave down.*

Solution: $(0, 1)$ *Notice, $f'(x)$ is positive and decreasing there.*

(i) *List all intervals where f is increasing and concave up.*

Solution: $(-4, 0)$ *and* $(3, 4)$. *Notice, $f'(x)$ is positive and increasing there.*

(j) *List x-coordinates of all inflection points of f.*

Solution: $x = 0$, *and* $x = 2$ *$f'(x)$ is increasing on $(-6, 0)$ and decreasing on $(0, 2)$, then increasing on $(2, 4)$.*

Problem 6 *Find the critical points of f on the given interval. Determine whether the function f has a local maximum, local minimum or no local extremum at each critical point.* ***EXPLAIN***.

(a) $f(x) = x\sqrt{2 - x^2}$ on $(-\sqrt{2}, \sqrt{2})$.

Solution:

$$\begin{aligned} f'(x) &= \sqrt{2 - x^2} + x\left(\frac{1}{2\sqrt{2 - x^2}}(-2x)\right) \\ &= \sqrt{2 - x^2} - \frac{x^2}{\sqrt{2 - x^2}} \\ &= \frac{2 - x^2 - x^2}{\sqrt{2 - x^2}} \\ &= \frac{2(1 - x^2)}{\sqrt{2 - x^2}} \end{aligned}$$

Critical points of f occur where $f'(x) = 0$ or where $f'(x)$ does not exist. Solving $f'(x) = 0$ yields that $2(1 - x^2) = 0$, or $x = \pm 1$. $f'(x)$ does not exist when $2 - x^2 \leq 0$, but the points $x = \pm\sqrt{2}$ are not in the domain.

Let us use the Second Derivative Test to determine whether the function f has a local maximum or local minimum at the points where $x = \pm 1$.

$$f''(x) = \frac{d}{dx}\frac{2(1 - x^2)}{\sqrt{2 - x^2}} = \frac{2x(x^2 - 3)}{(2 - x^2)^{\frac{3}{2}}}$$

Since $f''(-1) = 4$, the point $(-1, -1)$ is a local minimum by the Second Derivative Test. Since $f''(1) = -4$, the point $(1, 1)$ is a local maximum by the Second Derivative Test.

(b) $f(x) = x^3 e^{-x}$ on $(-1, 5)$.

Solution:

$$\begin{aligned} f'(x) &= 3x^2 e^{-x} + x^3(-e^{-x}) \\ &= x^2 e^{-x}(3 - x) \end{aligned}$$

Notice that $f'(x)$ always exists, and so all of the critical points of f occur when $f'(x) = 0$. Solving this equation:

$$x^2 e^{-x}(3 - x) = 0$$
$$x^2(3 - x) = 0$$
$$x = 0 \qquad \textit{or} \qquad x = 3$$

It is easier to use the First derivative test in order to determine whether the function f has a local maximum, local minimum or no local extremum at the point where $x = 0$. Since $-1 < 0 < 1$, $f'(-1) = 4e > 0$, and $f'(1) = 2e^{-1} > 0$, it follows that the sign of f' does not change at $x = 0$. The function f has **no local extremum** at $x = 0$.

Similarly, we will use the First derivative test in order to determine whether the function f has a local maximum, local minimum or no local extremum at the points where $x = 3$. Since $1 < 3 < 4$, $f'(1) = 2e^{-1} > 0$, and $f'(4) = -16e^{-4} < 0$ it follows that the sign of f' changes from positive to negative at $x = 3$. The function f has **a local maximum** at $x = 3$ *by the First Derivative Test.*

(c) $f(x) = x \ln\left(\frac{x}{5}\right)$ *on* $(0, 5)$.

Solution:

$$\begin{aligned} f'(x) &= \ln\left(\frac{x}{5}\right) + x \cdot \frac{5}{x} \cdot \frac{1}{5} \\ &= \ln\left(\frac{x}{5}\right) + 1 \end{aligned}$$

Notice that $f'(x)$ exists for all values in $(0, 5)$, and so all of the critical points of f occur when $f'(x) = 0$. Solving this equation:

$$\ln\left(\frac{x}{5}\right) + 1 = 0$$
$$\ln\left(\frac{x}{5}\right) = -1$$
$$\frac{x}{5} = e^{-1}$$
$$x = 5e^{-1} = \frac{5}{e}$$

Let's use the Second Derivative Test to determine whether the function f has a local maximum, local minimum or no local extremum at the points where $x = \frac{5}{e}$.

$$\begin{aligned} f''(x) &= \frac{d}{dx}(\ln\left(\frac{x}{5}\right) + 1) \\ &= \frac{d}{dx}(\ln(x) - \ln(5) + 1) = \frac{1}{x} \end{aligned}$$

Since, $f''\left(\frac{5}{e}\right) > 0$, the point $\left(\frac{5}{e}, -\frac{5}{e}\right)$ is a local minimum by the Second Derivative Test.

Problem 7 *Let $f(x) = \dfrac{1}{1 + x^2}$. Find the following for f:*

(a) f' and f''

Solution:

$$\begin{aligned} f'(x) &= \frac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} \\ &= \frac{-2x}{(1+x^2)^2} \end{aligned}$$

$$\begin{aligned} f''(x) &= \frac{(1+x^2)^2(-2) - (-2x)(2)(1+x^2)(2x)}{(1+x^2)^4} \\ &= \frac{-2(1+x^2) + 8x^2}{(1+x^2)^3} \\ &= \frac{6x^2 - 2}{(1+x^2)^3} \end{aligned}$$

(b) *Critical points*

Solution: *Since $1+x^2 > 0$ for all x, f is differentiable over all real numbers. Thus all critical points of f occur when $f'(x) = 0$. But a fraction equals 0 if and only if its numerator equals 0. So*

$$f'(x) = 0 \qquad \Longrightarrow \qquad -2x = 0 \qquad \Longrightarrow \qquad x = 0$$

Hence, the only critical point is $x = 0$.

(c) *Local extrema (and check your answers with both the first and second derivative tests)*

Solution: *At $x = 0$, f' changes sign from positive to negative. Thus f goes from increasing to decreasing, and therefore by the first derivative test $x = 0$ is a local maximum of f.*

For the second derivative test, we have that

$$f''(0) = \frac{6(0)^2 - 2}{(1+0^2)^3} = \frac{-2}{1} = -2 < 0$$

and thus we again conclude that $x = 0$ is a local maximum of f.

(d) *Inflection points.* ***EXPLAIN****.*

Solution: *By the results in part*

$$\begin{aligned} f''(x) &= \frac{6x^2 - 2}{(1+x^2)^3} \\ &= \frac{6(x^2 - \frac{1}{3})}{(1+x^2)^3} \\ &= \frac{6(x - \frac{1}{\sqrt{3}})(x + \frac{1}{\sqrt{3}})}{(1+x^2)^3} \end{aligned}$$

which implies that the only candidates for inflection points are points where $x = \pm\frac{1}{\sqrt{3}}$. *On the other hand, since* $-1 < -\frac{1}{\sqrt{3}} < 0$, $f''(-1) = \frac{1}{2}$, *and* $f''(0) = -2$, *it follows that* f'' *changes the sign at* $x = -\frac{1}{\sqrt{3}}$. *Similarly, since* $0 < x = \frac{1}{\sqrt{3}} < 1$, $f''(0) = -2$, $f''(1) = \frac{1}{2}$, *it follows that* f'' *changes the sign at* $x = \frac{1}{\sqrt{3}}$, *too. Since,*

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{3}{4}$$
$$f\left(-\frac{1}{\sqrt{3}}\right) = \frac{3}{4}$$

The function f *has inflection points at* $\left(-\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$, $\left(\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$ *because* f *is continuous at those points and* f *changes concavity around those points.*

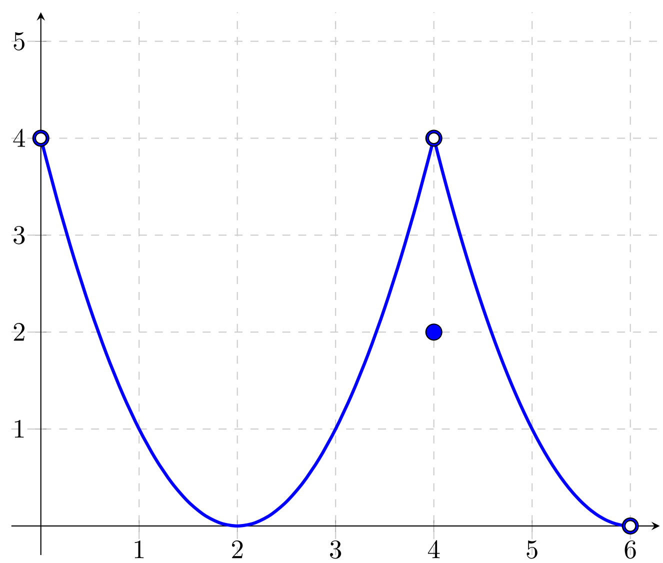
Problem 8 *Sketch a possible graph of a function* f *that has the following properties:*

(a) f *is defined on the interval* $(0, 6)$.

(b) f *has no local maximums.*

(c) f *has exactly two local minimums.*

Solution: *This could be the graph of* f. *Notice two local minimums: at* $x = 2$ *and at* $x = 4$.



Problem 9 *Consider the parabola $f(x) = ax^2 + bx + c$ where a, b, c are constants. For what values of a, b, c is f concave up? For what values of a, b, c is f concave down?*

Solution:

$$f'(x) = 2ax + b$$
$$f''(x) = 2a$$

This means the sign of a will determine whether f is concave up or down. When $a < 0$, f is concave down, which makes sense because then the graph of f is a downward opening parabola. When $a > 0$, f is concave up, which makes sense because then the graph of f is an upward openng parabola. If $a = 0$, f has no concavity because it is a linear function.