(In)determinate Forms (IF) - Solutions

Problem 1 *Which of the following limits are of the form* $\dfrac{0}{0}$*?*

(a) $\lim\limits_{x\to 7} \dfrac{\sin(x-7)}{|x-7|}$

(b) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin(x)}{x}$

(c) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{|\cos(x)|}{x}$

(d) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\pi - 2x}{\cos(x)}$

(e) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-4}$

(f) $\lim\limits_{x\to 4} \dfrac{x^2-4}{x-4}$

(g) $\lim\limits_{x\to 4} \dfrac{x^2-16}{x-4}$

(h) $\lim\limits_{x\to 3} \dfrac{x-4}{\sqrt{25-x^2}-4}$

(i) $\lim\limits_{x\to 3} \dfrac{x-3}{\sqrt{25-x^2}-4}$

Solution: *(a), (d), (g) and (i)*

Problem 2 *State the form of the limit and then evaluate the limit.*

(a) $\lim\limits_{x\to 6} \dfrac{4x^2-144}{x-6}$

Solution: $\lim\limits_{x\to 6} \dfrac{4x^2-144}{x-6} = \lim\limits_{x\to 6} \dfrac{4(x^2-36)}{x-6}$.

Form: $\dfrac{0}{0}$

$$\begin{aligned}\lim_{x\to 6} \frac{4x^2-144}{x-6} &= \lim_{x\to 6} \frac{4(x-6)(x+6)}{x-6}\\ &= \lim_{x\to 6} 4(x+6)\\ &= 4(12) = 48\end{aligned}$$

(b) $\lim_{x \to 6} \dfrac{x-6}{\sqrt{2x-8}-2}$

Solution:

$$\textit{Form:}\frac{0}{0}$$

$$\begin{aligned}
\lim_{x \to 6} \frac{x-6}{\sqrt{2x-8}-2} &= \lim_{x \to 6} \frac{x-6}{\sqrt{2x-8}-2} \cdot \frac{\sqrt{2x-8}+2}{\sqrt{2x-8}+2} \\
&= \lim_{x \to 6} \frac{(x-6)(\sqrt{2x-8}+2)}{2x-8-4} \\
&= \lim_{x \to 6} \frac{(x-6)(\sqrt{2x-8}+2)}{2(x-6)} \\
&= \lim_{x \to 6} \frac{\sqrt{2x-8}+2}{2} \\
&= \frac{\sqrt{12-8}+2}{2} \\
&= \frac{4}{2} = 2
\end{aligned}$$

(c) $\lim_{x \to 2} \dfrac{(3x-2)^2-16}{x-2}$

Solution:

$$\textit{Form:}\frac{0}{0}$$

$$\begin{aligned}
\lim_{x \to 2} \frac{(3x-2)^2-16}{x-2} &= \lim_{x \to 2} \frac{((3x-2)-4)((3x-2)+4)}{x-2} \\
&= \lim_{x \to 2} \frac{(3x-6)(3x+2)}{x-2} \\
&= \lim_{x \to 2} \frac{3(x-2)(3x+2)}{x-2} \\
&= \lim_{x \to 2} 3(3x+2) \\
&= 3(6+2) = 24
\end{aligned}$$

(d) $\lim_{x \to 6} \dfrac{\frac{x}{x-2} - \frac{3}{2}}{x-6}$

Solution: $\quad \textit{Form: } \dfrac{0}{0}$

$$
\begin{aligned}
\lim_{x\to 6} \frac{\frac{x}{x-2} - \frac{3}{2}}{x-6} &= \lim_{x\to 6} \frac{\frac{2x}{2(x-2)} - \frac{3(x-2)}{2(x-2}}{x-6} \\
&= \lim_{x\to 6} \frac{\frac{2x-3x+6}{2(x-2)}}{x-6} \\
&= \lim_{x\to 6} \frac{\frac{-x+6}{2(x-2)}}{x-6} \\
&= \lim_{x\to 6} \frac{-(x-6)}{2(x-2)(x-6)} \\
&= \lim_{x\to 6} \frac{-1}{2(x-2)} \\
&= \frac{-1}{2(6-2)} = \frac{-1}{2(4)} = \frac{-1}{8}
\end{aligned}
$$

(e) $\lim\limits_{x\to 1} \dfrac{\sqrt{5x-2} - \sqrt{3}}{x-1}$

Solution:

$$
\begin{aligned}
&\textit{Form:}\frac{0}{0} \\
\lim_{x\to 1} \frac{\sqrt{5x-2} - \sqrt{3}}{x-1} &= \lim_{x\to 1} \frac{\sqrt{5x-2} - \sqrt{3}}{x-1} \cdot \frac{\sqrt{5x-2} + \sqrt{3}}{\sqrt{5x-2} + \sqrt{3}} \\
&= \lim_{x\to 1} \frac{(5x-2) - 3}{(x-1)(\sqrt{5x-2} + \sqrt{3})} \\
&= \lim_{x\to 1} \frac{5(x-1)}{(x-1)(\sqrt{5x-2} + \sqrt{3})} \\
&= \lim_{x\to 1} \frac{5}{\sqrt{5x-2} + \sqrt{3}} \\
&= \frac{5}{\sqrt{5(1)-2} + \sqrt{3}} \\
&= \frac{5}{2\sqrt{3}}
\end{aligned}
$$

Problem 3 *Evaluate the following limits.*

(a) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\cos(x)}{x}$

Solution: *Apply the Quotient Law and use continuity of the numerator and denominator.*

$$\lim_{x\to \frac{\pi}{2}} \frac{\cos(x)}{x} = \frac{0}{\frac{\pi}{2}} = 0$$

(b) $\lim\limits_{x\to 3} \dfrac{\sqrt{25-x^2} - 4}{x-4}$

Solution: *Apply the Quotient Law and use continuity.*

$$\lim_{x\to 3}\frac{\sqrt{25-x^2}-4}{x-4}=\frac{\sqrt{25-(3)^2}-4}{3-4}=\frac{4-4}{-1}=0$$

(c) $\displaystyle\lim_{x\to 3}\frac{\sqrt{25-x^2}-4}{x-3}$

Solution: *Since* $\lim_{x\to 3}(\sqrt{25-x^2}-4)=0$ *and* $\lim_{x\to 3} x-3=0$, *this limit has form* $\frac{0}{0}$.

$$\lim_{x\to 3}\frac{\sqrt{25-x^2}-4}{x-3}=\lim_{x\to 3}\frac{\sqrt{25-x^2}-4}{x-3}\frac{\sqrt{25-x^2}+4}{\sqrt{25-x^2}+4}=$$
$$\lim_{x\to 3}\frac{25-x^2-16}{(x-3)(\sqrt{25-x^2}+4)}=\lim_{x\to 3}\frac{9-x^2}{(x-3)(\sqrt{25-x^2}+4)}=\lim_{x\to 3}\frac{(3+x)(3-x)}{(x-3)(\sqrt{25-x^2}+4)}=$$
$$\lim_{x\to 3}\frac{-(3+x)}{\sqrt{25-x^2}+4}=\frac{-6}{\sqrt{25-9}+4}=\frac{-6}{\sqrt{16}+4}=\frac{-6}{8}=\frac{-3}{4}$$

Problem 4 *Two polynomials, h and g, are given*

$$h(x)=\frac{x^2-4}{4}$$
$$g(x)=x-2$$

State the form of the limit, evaluate the limit, or state that it does not exist. Justify your answer.

(a) $\displaystyle\lim_{x\to 2}\frac{h(x)}{g(x)}$

Solution: $\displaystyle\lim_{x\to 2}\frac{h(x)}{g(x)}=\lim_{x\to 2}\frac{\frac{x^2-4}{4}}{x-2}$

Form: $\frac{0}{0}$

$$\begin{aligned}\lim_{x\to 2}\frac{\frac{x^2-4}{4}}{x-2}&=\lim_{x\to 2}\frac{(x-2)(x+2)}{4(x-2)}\\&=\lim_{x\to 2}\frac{x+2}{4}\\&=\frac{4}{4}\\&=1\end{aligned}$$

(b) $\displaystyle\lim_{x\to 3}\frac{g(x)-g(3)}{x-3}$

Solution: $\displaystyle\lim_{x\to 3}\frac{g(x)-g(3)}{x-3}=\lim_{x\to 3}\frac{x-2-1}{x-3}$

Form: $\frac{0}{0}$

$$\begin{aligned}\lim_{x\to 3}\frac{g(x)-g(3)}{x-3} &= \lim_{x\to 3}\frac{x-2-1}{x-3}\\ &= \lim_{x\to 3} 1\\ &= 1\end{aligned}$$

(c) $\displaystyle\lim_{x\to 4}\frac{h(x)-h(4)}{x-4}$

Solution: $\displaystyle\lim_{x\to 4}\frac{h(x)-h(4)}{x-4} = \lim_{x\to 4}\frac{\frac{x^2-4}{4}-3}{x-4}$

Form: $\dfrac{0}{0}$

$$\begin{aligned}\lim_{x\to 4}\frac{h(x)-h(4)}{x-4} &= \lim_{x\to 4}\frac{\frac{x^2-4}{4}-3}{x-4}\\ &= \lim_{x\to 4}\frac{x^2-4-12}{4(x-4)}\\ &= \lim_{x\to 4}\frac{x^2-16}{4(x-4)}\\ &= \lim_{x\to 4}\frac{(x-4)(x+4)}{4(x-4)}\\ &= \lim_{x\to 4}\frac{x+4}{4}\\ &= 2\end{aligned}$$

Problem 5 *Evaluate each of the following limits. Possible answers include a number, $+\infty$, $-\infty$ and "Does Not Exist" (DNE). Make sure to state the form of the limit. Justify your answer.*

(a) $\displaystyle\lim_{x\to 3^-}\frac{x^2-3}{x^2-x-6}$

Solution: $\displaystyle\lim_{x\to 3^-}\frac{x^2-3}{x^2-x-6} = \lim_{x\to 3^-}\frac{x^2-3}{(x-3)(x+2)}$ *is of the form* $\dfrac{\#}{0}$.

Since $\displaystyle\lim_{x\to 3^-}(x^2-3)=6$, *and* $\displaystyle\lim_{x\to 3^-}(x^2-x-6)=\lim_{x\to 3^-}(x-3)(x+2)=0$, *and since*

$x^2-x-6=(x-3)(x+2)<0$, *if* $x<3$ *and* x *close to 3.*

$$\lim_{x\to 3^-}\frac{x^2-3}{x^2-x-6}=-\infty$$

(b) $\displaystyle\lim_{x\to 5^+}\frac{x^2+6}{x^2-3x-10}$

Solution: *The limit is of the form* $\frac{\#}{0}$.

$\lim_{x\to 5^+} \frac{x^2+6}{x^2-3x-10} = \lim_{x\to 5^+} \frac{x^2+6}{(x-5)(x+2)} = \infty$, *because*

$\lim_{x\to 5^+} (x^2+6) = 31$ *and* $x^2-3x-10 = (x-5)(x+2) > 0$, *for* $x > 5$ *and* x *close to 5.*

(c) $\lim_{x\to 1} \frac{4-x}{x^2-2x+1}$

Solution: *The limit is of the form* $\frac{\#}{0}$.

Checking left and right sided limits we see:

$$\lim_{x\to 1^+} \frac{4-x}{x^2-2x+1} = \lim_{x\to 1^+} \frac{4-x}{(x-1)^2} = \infty$$

and

$$\lim_{x\to 1^-} \frac{4-x}{x^2-2x+1} = \lim_{x\to 1^-} \frac{4-x}{(x-1)^2} = \infty$$

Since $\lim_{x\to 1^-} \frac{4-x}{x^2-2x+1} = \lim_{x\to 1^+} \frac{4-x}{x^2-2x+1} \implies \lim_{x\to 1} \frac{4-x}{x^2-2x+1} = \infty$

(d) $\lim_{x\to 2} \frac{-e^x}{(2-x)^3}$

Solution: *The limit is of the form* $\frac{\#}{0}$.

Checking right limit we have:

$$\lim_{x\to 2^+} \frac{-e^x}{(2-x)^3} = \infty$$

and checking the left limit we have:

$$\lim_{x\to 2^-} \frac{-e^x}{(2-x)^3} = -\infty$$

Since $\lim_{x\to 2^+} \frac{-e^x}{(2-x)^3} \neq \lim_{x\to 2^-} \frac{-e^x}{(2-x)^3} \implies \lim_{x\to 2} \frac{-e^x}{(2-x)^3}$ *Does not exist*

(e) $\lim_{x\to 1} \frac{\sin x}{\sqrt{2-x^2}-1}$

Solution: *The limit is of the form* $\dfrac{\#}{0}$.

Checking right limit we have:

$\lim\limits_{x\to 1^+} \dfrac{\sin x}{\sqrt{2-x^2}-1} = -\infty,$

$\lim\limits_{x\to 1^+} \sin x = \sin 1 > 0$, *since* $1 < \dfrac{\pi}{2}$. *Explanation: Note that for x near 1 and such that* $x > 1$, *we have that*

$x^2 > 1$, *and by multiplying by* (-1), *we get*

$-x^2 < -1$, *and by adding 2 on both sides, we get*

$2 - x^2 < 2 - 1$, *and by taking the square root, we get*

$\sqrt{2-x^2} < \sqrt{1}$, *or*

$\sqrt{2-x^2} < 1$, *and by subtracting 1 from both sides, we get*

$\sqrt{2-x^2} - 1 < 0$.

Therefore the numerator is positive, and denominator is ***negative*** *and goes to 0.*

Checking the left limit we have: $\lim\limits_{x\to 1^-} \dfrac{\sin x}{\sqrt{2-x^2}-1} = +\infty$

Explanation: Note that and for x near 1 such that $x < 1$, *we have that*

$x^2 < 1$, *and by multiplying by* (-1), *we get*

$-x^2 > -1$, *and by adding 2 on both sides, we get*

$2 - x^2 > 2 - 1$, *and by taking the square root, we get*

$\sqrt{2-x^2} > \sqrt{1}$, *or*

$\sqrt{2-x^2} > 1$, *and by subtracting 1 from both sides, we get*

$\sqrt{2-x^2} - 1 > 0$.

Therefore the numerator is positive, and denominator is ***positive*** *and goes to 0.*

Since $\lim\limits_{x\to 1^+} \dfrac{\sin x}{\sqrt{2-x^2}-1} \neq \lim\limits_{x\to 1^-} \dfrac{\sin x}{\sqrt{2-x^2}-1} \implies \lim\limits_{x\to 1} \dfrac{\sin x}{\sqrt{2-x^2}-1}$ *does not exist.*

Problem 6 *A piecewise defined function* f *is given by*

$$f(x) = \begin{cases} \dfrac{2x-3}{x-2} & \textit{if} \quad x < 2 \\[2ex] \dfrac{x^2-5x+6}{x^2-4} & \textit{if} \quad x > 2 \end{cases}$$

Determine the form of the limit, then find the limit.

(a) $\lim\limits_{x\to 2^+} f(x)$

Solution: *Since* $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} \dfrac{x^2-5x+6}{x^2-4}$, *the form of the limit is* $\dfrac{0}{0}$.

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^2-5x+6}{x^2-4} = \lim_{x\to 2^+} \frac{(x-3)(x-2)}{(x-2)(x+2)} = \lim_{x\to 2^+} \frac{x-3}{x+2} = -\frac{1}{4}$$

(b) $\lim\limits_{x\to 2^-} f(x)$

Solution: *Since* $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \dfrac{2x-3}{x-2}$, *the form of the limit is* $\dfrac{\#}{0}$.

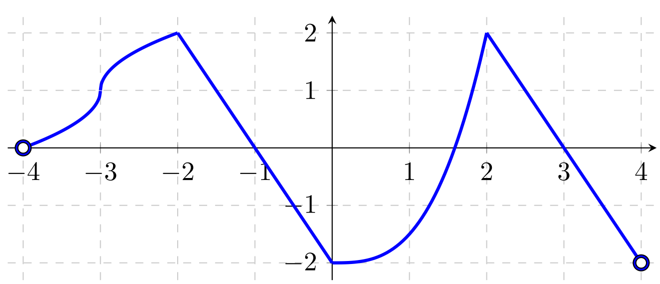
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{2x-3}{x-2} = -\infty,$$

since the numerator positive, the denominator negative and goes to 0.

(c) $\lim\limits_{x\to 2} f(x)$

Solution: *The limit does not exist, because* $\lim\limits_{x\to 2^+} f(x) \neq \lim\limits_{x\to 2^-} f(x)$.

Problem 7 *The graph of a function $g(x)$ with domain $(-4, 4)$ is given in the figure. This portions of this graph in the intervals $(-2, 0)$ and $(2, 4)$ are straight lines. The portion on the interval $(0, 2)$ is not parabolic.*



For each of the limits below, give the form of the limit, then evaluate the limit.

(a) $\lim\limits_{x\to -3} \dfrac{g(x) + \sin\left(\frac{\pi}{4}x\right)}{x\sqrt{x+4}}$.

Solution: *The numerator and denominator are each continuous at $x = -3$, and at $x = -3$ the denominator is nonzero. The entire fraction is continuous at $x = -3$. To find the limit, we can plug in the value. (We would call this form either "$\frac{\#}{\#}$" or "Continuous").*

$$\begin{aligned}\lim_{x\to -3} \frac{g(x) + \sin\left(\frac{\pi}{4}x\right)}{x\sqrt{x+4}} &= \frac{g(-3) + \sin\left(\frac{-3\pi}{4}\right)}{-3\sqrt{-3+4}} \\ &= \frac{1 - \frac{1}{\sqrt{2}}}{-3}.\end{aligned}$$

(b) $\displaystyle\lim_{x\to 0^+} \frac{e^x}{|g(x)+2|}$.

Solution: $\displaystyle\lim_{x\to 0^+} e^x = e^0 = 1$ *and* $\displaystyle\lim_{x\to 0^+} |g(x)+2| = 0$, *so this limit has form $\frac{\#}{0}$. One-sided limits with form $\frac{\#}{0}$ are either $+\infty$ or $-\infty$, so we need to check the sign of the fraction.*

We know the range of the famous function e^x is $(0, \infty)$, so the numerator here is always positive. The denominator is an absolute value, so it is not negative either. As long as x is near, but not equal, to 0, $g(x) + 2$ will be close, but not equal, to 0, so $|g(x) + 2|$ will be positive.

$\displaystyle\lim_{x\to 0^+} \frac{e^x}{|g(x)+2|} = \infty.$

(c) $\displaystyle\lim_{x\to 3} \frac{x^2+2x-15}{x+g(x)-3}$

Solution: $\displaystyle\lim_{x\to 3}(x^2 + 2x - 15) = 0$ *and* $\displaystyle\lim_{x\to 3}(x + g(x) - 3) = 0$, *so this limit has form $\frac{0}{0}$. In order to use algebra to simplify this fraction, we need a formula for $g(x)$ for x near 3. From the graph we see that in the interval $(2, 4)$, the graph of g is a straight line with slope -2. The equation of that line is $y = -2x + 6$, so for x close to 3, we know $g(x) = -2x + 6$.*

$$\begin{aligned}\lim_{x\to 3} \frac{x^2+2x-15}{x+g(x)-3} &= \lim_{x\to 3} \frac{x^2+2x-15}{x+(-2x+6)-3} \\ &= \lim_{x\to 3} \frac{x^2+2x-15}{-x+3} \\ &= \lim_{x\to 3} \frac{(x-3)(x+5)}{-(x-3)} \\ &= \lim_{x\to 3} -(x+5) \\ &= -8.\end{aligned}$$