Limit Laws (LL)

The Limit Laws:

Continuity If a function f is **continuous at** a, then

$$\lim_{x \to a} f(x) = f(a)$$

Composition Limit Law If a function f is **continuous at** $b = \lim_{x \to a} g(x)$, then

$$\lim_{x \to a} f(g(x)) = f\Big(\lim_{x \to a} g(x)\Big)$$

For the limit laws below, assume $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist. The following properties hold.

Sum Limit Law

$$\lim_{x \to a} \big(f(x) + g(x)\big) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

Difference Limit Law

$$\lim_{x \to a} \big(f(x) - g(x)\big) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

Product Limit Law

$$\lim_{x \to a} \big(f(x) \cdot g(x)\big) = \big(\lim_{x \to a} f(x)\big) \cdot \big(\lim_{x \to a} g(x)\big)$$

Quotient Limit Law

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ provided } \lim_{x \to a} g(x) \neq 0$$

Limit laws also hold for one-sided limits.

Recitation Questions

Problem 1 *The following argument shows*

$$\lim_{x\to 3} \frac{5x^3 - 4\sqrt{x}}{\sqrt{x^5 - 87}} = \frac{135 - 4\sqrt{3}}{\sqrt{156}}.$$

State which limit law is used to justify each step. (A particular step may have more than one limit law as a justification.)

$$\begin{aligned}
\lim_{x\to 3} \frac{5x^3 - 4\sqrt{x}}{\sqrt{x^5 - 87}} &= \frac{\lim_{x\to 3}\left(5x^3 - 4\sqrt{x}\right)}{\lim_{x\to 3}\sqrt{x^5 - 87}} \\
&= \frac{\left(\lim_{x\to 3} 5\right)\left(\lim_{x\to 3}(x^3)\right) - \left(\lim_{x\to 3} 4\right)\left(\lim_{x\to 3}\sqrt{x}\right)}{\sqrt{\lim_{x\to 3}(x^5 - 87)}} \\
&= \frac{5\lim_{x\to 3}(x^3) - 4\lim_{x\to 3}\sqrt{x}}{\sqrt{\lim_{x\to 3}(x^5 - 87)}} \\
&= \frac{5(\lim_{x\to 3} x)^3 - 4\sqrt{3}}{\sqrt{\lim_{x\to 3}(x^5) - \lim_{x\to 3}(87)}} \\
&= \frac{5(3)^3 - 4\sqrt{3}}{\sqrt{3^5 - 87}} \\
&= \frac{135 - 4\sqrt{3}}{\sqrt{156}}
\end{aligned}$$

Problem 2 *Find the limit and justify your answer.*

(a) $\lim_{x \to 0} |x|$

(b) $\lim_{x \to 2} \ln \left(\sin (x-2) + e^x \sin \left(\frac{\pi x}{4} \right) \right)$

(c) $\lim_{x \to 1} \frac{3 + 2\cos\left(\frac{\pi x}{3}\right)}{4x - 2x^3}$

Problem 3 *Suppose* $f(x) = \begin{cases} x^2 - ax & \text{if } x < 3 \\ \\ a2^x + 7 + a & \text{if } x > 3 \end{cases}$

Find a *so that* $\lim\limits_{x \to 3} f(x)$ *exists.*

Problem 4 *Determine the value of* $\lim_{x \to 0} \left(x^2 \cos\left(\frac{1}{x}\right)\right)$. ***EXPLAIN***.

Problem 5 *For all* x *near 0, the inequalities* $1 - \frac{x^2}{6} \leq \frac{\sin(x)}{x} \leq 1$ *are true. Use these inequalities to find* $\lim_{x \to 0} \frac{\sin(x)}{x}$. ***EXPLAIN***.

Problem 6 *Two functions, h and g, are given*

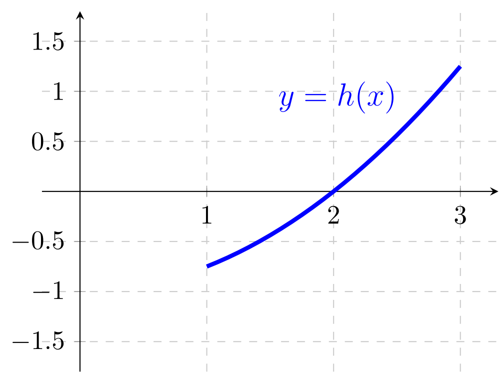
$$h(x) = \frac{x^2 - 4}{4},\ 1 < x < 3$$

$$g(x) = x - 2,\ 1 < x < 3$$

The graph of the function h is given in the figure below. Let f be a function defined on the interval $(1, 3)$ that satisfies the following inequalities

$$g(x) \le f(x) \le h(x),\ 1 < x < 3$$

(a) *In the figure below, sketch and label the graph of g and a possible graph of f. (All three functions have a common domain $(1, 3)$.)*



(b) *Evaluate the limit, or state that it does not exist. Justify your answer.*

(i) $\lim_{x \to 2} f(x)$

(ii) $\lim_{x \to 2} \frac{f(x) + 2}{x - 1}$

(iii) $\lim_{x \to 2} g(1 + e^{f(x)})$