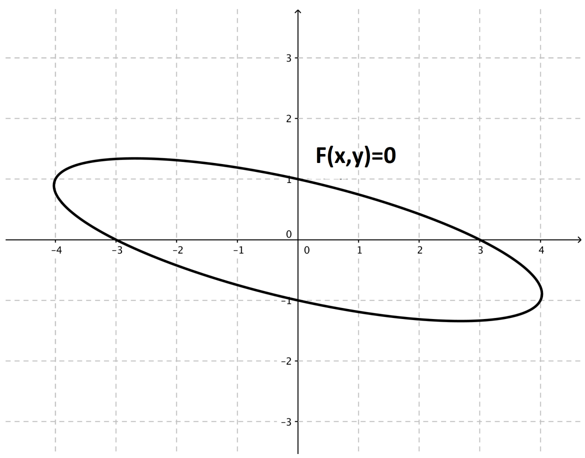
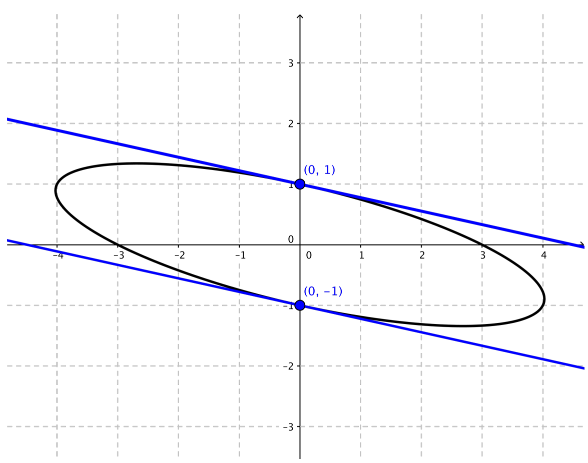
Implicit differentiation (ID) - Solutions

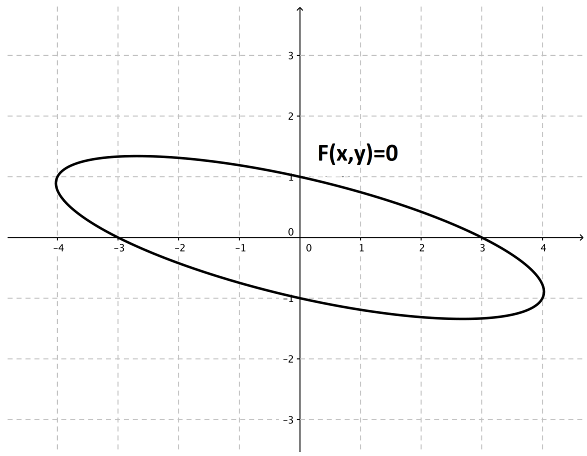
Problem 1 *On the graph below, sketch the tangent lines at $x = 0$. Then, explain why both the x-coordinate and the y-coordinate are generally needed to find the slope of the tangent line at a point on the graph of an equation of the form $F(x, y) = 0$*



Solution: *Given the equation $F(x, y) = 0$, its graph may not pass the "vertical line test". As illustrated in the figure below, fixing a value for x (in this case $x = 0$) will not typically specify a unique point on the graph because there may be more than one corresponding y-value. That is why you need to specify both the x and the y coordinate, to make sure that you are giving the coordinates for a unique point on the graph. In this case, there are two tangent lines to the curve at $x = 0$. One passes through the point $(0, 1)$ and the other through $(0, -1)$.*



Problem 2 *Consider the equation $x^2 + 4xy + 9y^2 = 9$. Note: This equation is equivalent to $x^2 + 4xy + 9y^2 - 9 = 0$. Therefore it has a form $F(x, y) = 0$*



(a) *Find* $\frac{dy}{dx}$.

Solution:

$$\frac{d}{dx}(x^2+4xy+9y^2) = \frac{d}{dx}(9)$$
$$2x + \left(4y + 4x\frac{dy}{dx}\right) + 18y\frac{dy}{dx} = 0$$
$$4x\frac{dy}{dx} + 18y\frac{dy}{dx} = -2x-4y$$
$$(4x+18y)\frac{dy}{dx} = -2x-4y$$
$$\frac{dy}{dx} = \frac{-2x-4y}{4x+18y}$$

provided that $4x+18y \neq 0$.

(b) *Find the equation(s) of the tangent line(s) when* $x=0$. *Draw the tangent line(s) on the above picture.*

Solution: *Plugging into the original equation, we see that when* $x=0$ *we have that*

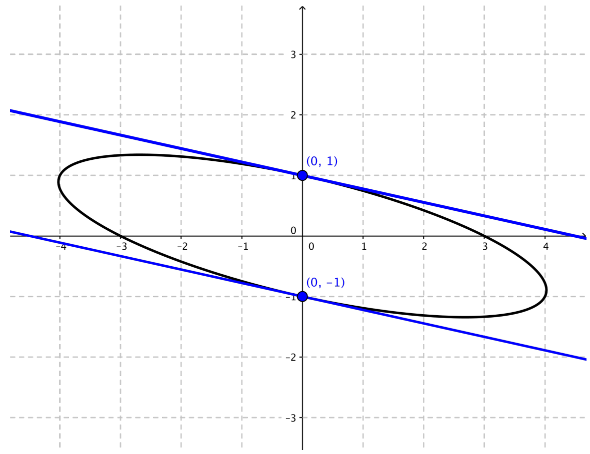
$$0^2+4(0)y+9y^2=9 \qquad \Longrightarrow \qquad y^2=1 \qquad \Longrightarrow \qquad y=\pm 1$$

Then,

$$\left[\frac{dy}{dx}\right]_{(0,1)} = \frac{-4}{18} = -\frac{2}{9}$$
$$\left[\frac{dy}{dx}\right]_{(0,-1)} = \frac{4}{-18} = -\frac{2}{9}$$

So there are two tangent lines to the graph of the given equation when $x=0$. *The two points are* $(0,1)$ *and* $(0,-1)$, *and the tangent lines at both points have slope* $-\frac{2}{9}$. *Thus, the equations of the two tangent lines are*

$$y-1 = -\frac{2}{9}(x-0) \qquad \Longrightarrow \qquad y = -\frac{2}{9}x+1$$
$$y+1 = -\frac{2}{9}(x-0) \qquad \Longrightarrow \qquad y = -\frac{2}{9}x-1$$



(c) *Find the point(s) where the tangent line is horizontal. Draw the point(s) and line(s) on the above picture.*

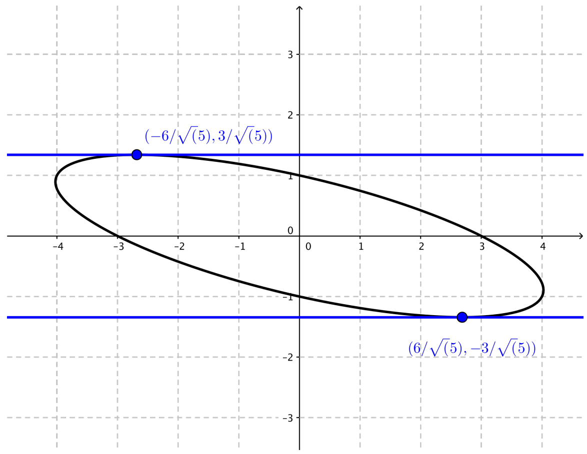
Solution: *A line is horizontal if and only if its slope is 0. So we are looking for the points* (x_0, y_0) *such that* $\left[\frac{dy}{dx}\right]_{(x_0,y_0)} = 0$*. So we solve:*

$$\frac{-2x-4y}{4x+18y} = 0 \quad \Longrightarrow \quad -2x-4y=0 \quad \Longrightarrow \quad x = -2y$$

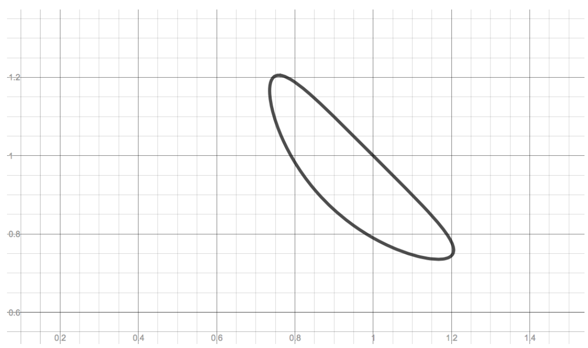
But we also need the point to lie on the the given curve. So, letting $x = -2y$*, we solve:*

$$\begin{aligned} x^2 + 4xy + 9y^2 &= 9 \\ (-2y)^2 + 4(-2y)y + 9y^2 &= 9 \\ 4y^2 - 8y^2 + 9y^2 &= 9 \\ 5y^2 &= 9 \\ y^2 &= \frac{9}{5} \\ y &= \pm\frac{3}{\sqrt{5}} \end{aligned}$$

So there exists two solutions to the given equation where the tangent line to the graph has 0 slope. Those two points are $\left(-\frac{6}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right)$ *and* $\left(\frac{6}{\sqrt{5}}, -\frac{3}{\sqrt{5}}\right)$.



Problem 3 *A part of the curve with equation $\cos(\pi xy) + x + y = 1$ is sketched below.*



(a) *Use the implicit differentiation to find the derivative dy/dx.*

Solution: *Implicitly differentiate equation:*

$$\frac{d}{dx}(\cos(\pi xy) + x + y) = \frac{d}{dx}(1) \implies -\sin(\pi xy) \cdot (\pi y + \pi x \frac{dy}{dx}) + 1 + \frac{dy}{dx} = 0$$

Solve for $\frac{dy}{dx}$*:*

$$\begin{aligned} -\sin(\pi xy) \cdot (\pi y + \pi x \frac{dy}{dx}) + 1 + \frac{dy}{dx} = 0 &\implies -\pi y \sin(\pi xy) - \pi x \frac{dy}{dx} \sin(\pi xy) + 1 + \frac{dy}{dx} = 0 \\ &\implies \big(1 - \pi x \sin(\pi xy)\big) \frac{dy}{dx} = \pi y \sin(\pi xy) - 1 \\ &\implies \frac{dy}{dx} = \frac{\pi y \sin(\pi xy) - 1}{1 - \pi x \sin(\pi xy)} \qquad \textit{if } 1 - \pi x \sin(\pi xy) \neq 0 \end{aligned}$$

(b) *Consider the point $(1, 1)$. Show (algebraically) that this point lies on the curve.*

Solution: $(1,1)$ *lies on this curve:*

$$\begin{aligned}\cos(\pi\cdot 1\cdot 1)+1+1 &= \cos(\pi)+2\\ &= -1+2=1\end{aligned}$$

(c) *Find the equation of the line tangent to the curve at* $(1,1)$*. Draw this line in the figure above.*

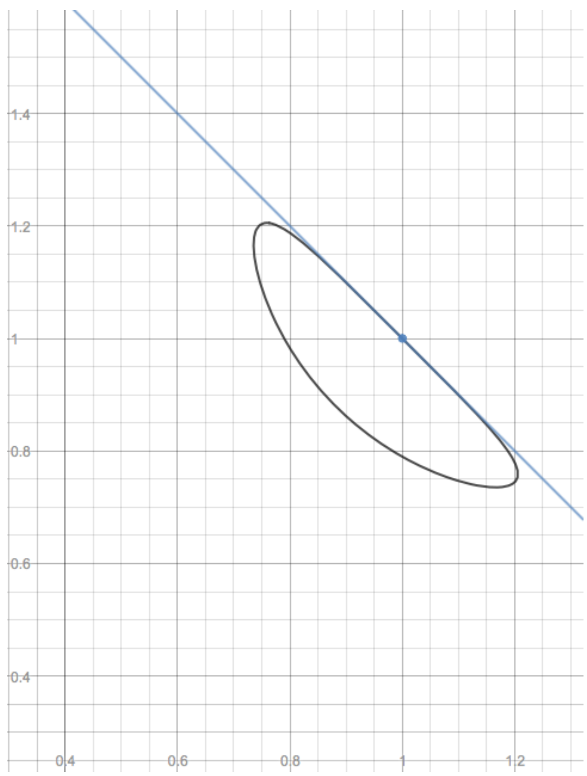
Solution: *Slope of tangent line:*

$$\begin{aligned}\left[\frac{dy}{dx}\right]_{(1,1)} &= \frac{\pi\cdot(1)\cdot\sin(\pi(1)\cdot 1)-1}{1-\pi\cdot(1)\cdot\sin(\pi(1)\cdot(1))}\\ &= \frac{\pi\sin(\pi)-1}{1-\pi\sin(\pi)}\\ &= \frac{-1}{1}=-1\end{aligned}$$

Equation of tangent line:

$$y-1=-1(x-1)\implies y=-x+2$$

Graph of tangent line:



Problem 4 *For each of the curves given by the following equations, find a formula for the slope of the tangent line at a point* (x,y)*.*

(a) $e^{x^2y^3} - 5x + 7y = 36$

Solution:

$$\begin{aligned}
e^{x^2y^3}\left(2xy^3 + x^2(3y^2)\frac{dy}{dx}\right) - 5 + 7\frac{dy}{dx} &= 0 \\
2xy^3e^{x^2y^3} + 3x^2y^2e^{x^2y^3}\frac{dy}{dx} - 5 + 7\frac{dy}{dx} &= 0 \\
3x^2y^2e^{x^2y^3}\frac{dy}{dx} + 7\frac{dy}{dx} &= -2xy^3e^{x^2y^3} + 5 \\
\left(3x^2y^2e^{x^2y^3} + 7\right)\frac{dy}{dx} &= -2xy^3e^{x^2y^3} + 5 \\
\frac{dy}{dx} &= \frac{-2xy^3e^{x^2y^3} + 5}{3x^2y^2e^{x^2y^3} + 7}
\end{aligned}$$

(b) $7 = 22\tan(y) + \dfrac{4}{x} - \dfrac{7}{y}$

Solution:

$$\begin{aligned}
0 &= 22\sec^2(y)\frac{dy}{dx} - \frac{4}{x^2} + \frac{7}{y^2}\frac{dy}{dx} \\
22\sec^2(y)\frac{dy}{dx} + \frac{7}{y^2}\frac{dy}{dx} &= \frac{4}{x^2} \\
\frac{dy}{dx} &= \frac{\frac{4}{x^2}}{22\sec^2(y) + \frac{7}{y^2}}
\end{aligned}$$

(c) $\cos(xy) - x^3 = 5y^3$

Solution:

$$\begin{aligned}
-\sin(xy)\cdot\left(y + x\frac{dy}{dx}\right) - 3x^2 &= 15y^2\frac{dy}{dx} \\
-y\sin(xy) - x\sin(xy)\frac{dy}{dx} - 3x^2 &= 15y^2\frac{dy}{dx} \\
x\sin(xy)\frac{dy}{dx} + 15y^2\frac{dy}{dx} &= -y\sin(xy) - 3x^2 \\
\frac{dy}{dx} &= \frac{-y\sin(xy) - 3x^2}{x\sin(xy) + 15y^2}
\end{aligned}$$

Provided that $x\sin(xy) + 15y^2 \neq 0$. It is worth pointing out that equivalent condition was not necessary for parts (a) and (b) because the denominator of those solutions cannot be 0.

Problem 5 *The volume of a doughnut with an inner radius of a and an outer radius of b is*

$$V = \pi^2\frac{(b+a)(b-a)^2}{4}.$$

Find db/da if the volume of a doughnut is $64\pi^2$ and does not change.

Solution: *We have to (implicitly) differentiate*

$$64\pi^2 = \pi^2 \frac{(b+a)(b-a)^2}{4}.$$

Before doing this we'll perform a bit of algebra to simplify our calculuations:

$$64\pi^2 = \pi^2 \frac{(b+a)(b-a)^2}{4} \implies 256 = (b+a)(b-a)^2.$$

Now we'll differentiate with respect to a:

$$\begin{aligned} 256 = (b+a)(b-a)^2 &\implies 0 = \left(\frac{db}{da}+1\right)(b-a)^2 + (b+a)2\cdot(b-a)\cdot\left(\frac{db}{da}-1\right) \\ &\implies 0 = \frac{db}{da}(b-a)^2 + (b-a)^2 + 2(b^2-a^2)\frac{db}{da} - 2(b^2-a^2). \end{aligned}$$

To finish we solve for $\frac{db}{da}$:

$$\begin{aligned} &0 = \frac{db}{da}(b-a)^2 + (b-a)^2 + 2(b^2-a^2)\frac{db}{da} - 2(b^2-a^2) \\ &\implies -\left(\frac{db}{da}(b-a)^2 + 2(b^2-a^2)\frac{db}{da}\right) = (b-a)^2 - 2(b^2-a^2) \\ &\implies \frac{db}{da} = \frac{(b-a)^2 - 2(b^2-a^2)}{-((b-a)^2 + 2(b^2-a^2))}. \end{aligned}$$

Problem 6 *The curve is given by the equation $x^{1/3} + y^{2/3} = 2$. Find $\frac{d^2y}{dx^2}$.*

Solution: *Take the first derivative:*

$$\frac{1}{3}x^{-2/3} + \frac{2}{3}y^{-1/3}\cdot\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{-1}{3x^{2/3}}}{\frac{2}{3y^{1/3}}} = \frac{-y^{1/3}}{2x^{2/3}}$$

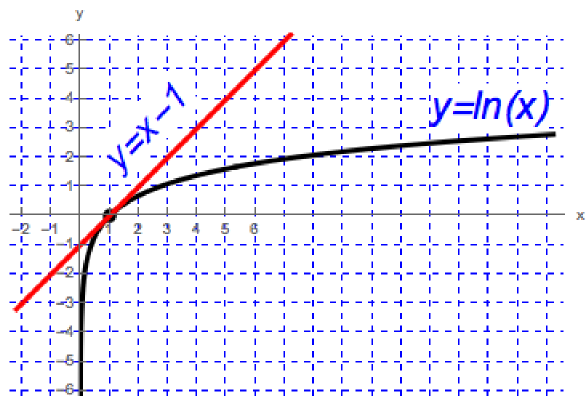
Take the second derivative:

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{3}y^{-2/3}\cdot\frac{dy}{dx}\cdot 2x^{2/3} + y^{1/3}\cdot\frac{4}{3}x^{-1/3}}{4x^{4/3}}$$

We need to substitute in for $\frac{dy}{dx}$ we have:

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{3}y^{-2/3}\cdot\frac{-y^{1/3}}{2x^{2/3}}\cdot 2x^{2/3} + y^{1/3}\cdot\frac{4}{3}x^{-1/3}}{4x^{4/3}}$$

Problem 7 *Sketch both the curve $y = \ln(x)$ and the tangent line to the curve at the point where $x = 1$. Then, write an equation of the tangent line to the curve $y = \ln(x)$ at the point where $x = 1$.*

Solution: 

In order to write an equation of the tangent line we need to find coordinates of the point $P(1, \ln(x))$, and the slope of the line, $y'(1)$.
$P(1, 0)$, and, since $y'(x) = \dfrac{1}{x}$, we get that $y'(1) = 1$. Therefore, the equation is given by $y = x - 1$.

Problem 8 (a) *Let f be a positive differentiable function, defined on an open interval I. Find the formula for the derivative of the function $\ln(f(x))$.*

Solution: $$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

(b) *Using the formula obtained in part (a), compute the derivatives of the following functions.*

(i) $f(x) = \ln(x^2 + x + 1)$

Solution: $$f'(x) = \frac{2x+1}{x^2+x+1}$$

(ii) $f(x) = \ln(\sec(x) + \tan(x))$

Solution: $$f'(x) = \frac{\sec(x)\tan(x) + \sec^2(x)}{\sec(x) + \tan(x)} = \frac{\sec(x)(\tan(x) + \sec(x))}{\sec(x) + \tan(x)} = \sec(x)$$

(iii) $f(x) = \ln(\ln(x))$

Solution: $$f'(x) = \frac{\frac{1}{x}}{\ln(x)} = \frac{1}{x\ln(x)}$$

Problem 9 *Compute $f'(x)$.*

(a) $f(x) = x\ln(x)$

Solution: $$f'(x) = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x)$$

(b) $f(x) = \sin(x)\Big(\ln\left(\sec(x)+1\right)\Big)$

Solution: $$f'(x) = \sin(x)\frac{\sec(x)\tan(x)}{\sec(x)+1} + \cos(x)\Big(\ln\left(\sec(x)+1\right)\Big)$$

(c) $f(x) = 2^x\sqrt{\ln\left(5x+7\right)}$

Solution: $$f'(x) = \frac{2^x}{2\sqrt{\ln\left(5x+7\right)}} \cdot \frac{5}{5x+7} + 2^x\ln(2)\sqrt{\ln\left(5x+7\right)}$$