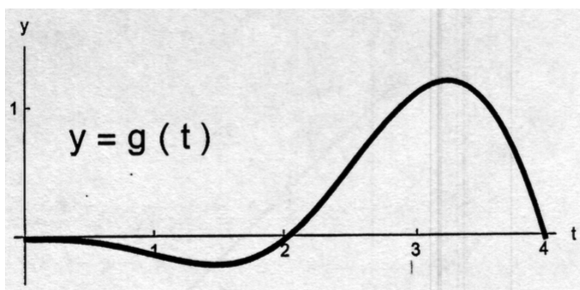
First and Second Fundamental Theorem of Calculus (FFTOC, SFTOC) - Solutions

Problem 1 *True of False: If f is continuous on the closed interval $[a,b]$, then*

$$\frac{d}{dx}\left(\int_a^b f(t)\,dt\right) = f(x)$$

Solution: *False.* $\frac{d}{dx}\left(\int_a^b f(t)\,dt\right) = 0$ *since* $\int_a^b f(t)\,dt$ *is a constant and* $\frac{d}{dx}(\text{constant}) = 0$.

Problem 2 *The graph of g, a continuous function on $[0,4]$, is shown in the figure. Let $A(x) = \int_0^x g(t)\,dt$, for $0 \le x \le 4$.*



(a) *Circle the correct statement about $A(2)$.*

(i) $A(2) = 0$;
(ii) $A(2) > 0$;
(iii) $A(2) < 0$;
(iv) *none of the previous answers.*

Solution: *Correct choice: (iii) $A(2) < 0$. The curve $g(t)$ on the interval $(0,2)$ is below the x-axis and therefore the net area is negative.*

(b) *Circle the correct statement about $A(3.8)$.*

(i) $A(3.8) = 0$;
(ii) $A(3.8) > 0$;
(iii) $A(3.8) < 0$;
(iv) *none of the previous answers.*

Solution: *Correct choice: (ii) $A(3.8) > 0$. The net area under the curve $g(t)$ on the interval $(2,3.8)$ is postive and larger than the negative net area on the interval $(0,2)$. Therefore, the net area on the interval $(0,3.8)$ is positive.*

(c) *Circle the correct statement about* $A'(3.8)$.

(i) $A'(3.8) = 0$;
(ii) $A'(3.8) > 0$;
(iii) $A'(3.8) < 0$;
(iv) *none of the previous answers.*

Solution: *Correct choice: (ii)* $A'(3.8) > 0$ *because* $A'(3.8) = g(3.8)$

(d) *Find the solution of the following initiual value problem:* $y'(x) = g(x)$ *and* $y(0) = 2$.

(i) $y(x) = g(x)$;
(ii) $y(x) = g(x) + 2$;
(iii) $y(x) = A(x)$;
(iv) $y(x) = A(x) + 2$;
(v) $y(x) = g'(x)$;
(vi) $y(x) = g'(x) + 2$;
(vii) *none of the previous answers.*

Solution: *Correct choice: (iv)* $y(x) = A(x) + 2$. $y'(x) = A'(x) = g(x)$ *and* $y(0) = A(0) + 2 = 2$

(e) *Find the expression for* $\displaystyle\int_0^4 |g(t)|\,dt$.

(i) $A(4)$;
(ii) $A(2) - A(4)$;
(iii) $A(4) - A(2)$;
(iv) $A(4) - 2A(2)$;
(v) *none of the previous answers.*

Solution: *Correct choice: (iv)* $A(4) - 2A(2)$.

$$\int_0^4 |g(t)|\,dt = \int_2^4 g(t)\,dt - \int_0^2 g(t)\,dt$$

and

$$\int_2^4 g(t)\,dt = \int_0^4 g(t)\,dt - \int_0^2 g(t)\,dt$$

Therefore,

$$\int_0^4 |g(t)|\,dt = \int_0^4 g(t)\,dt - 2\int_0^2 g(t)\,dt = A(4) - 2A(2)$$

(f) *Find the midpoint Riemann sum for the function* g *on the interval* $[0, 4]$ *with* $n = 2$ *(the number of subintervals).*

(i) $g(1) + g(3)$;
(ii) $g(0) + g(2)$;
(iii) $g(2) + g(4)$;

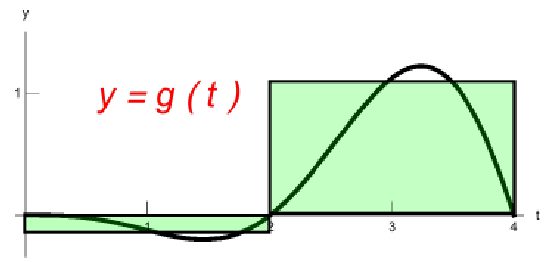
(iv) $2g(1) + 2g(3)$;

(v) $2g(0) + 2g(2)$;

(vi) $2g(2) + 2g(4)$;

(vii) *none of the previous answers.*

Solution: *Correct choice: (iv)* $2g(1) + 2g(3)$



Problem 3 *The limit of Riemann sums for a function f on the interval $[1, 5]$ is given by*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(2x_k^* + \frac{1}{x_k^*}\right) \Delta x \text{ on } [1, 5].$$

(a) *Identify f and express the limit as a definite integral.*

Solution:

$$f(x) = 2x + \frac{1}{x}$$

$$\int_1^5 \left(2x + \frac{1}{x}\right) dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(2x_k^* + \frac{1}{x_k^*}\right) \Delta x$$

(b) *Evaluate the limit of Riemann sums.*

Solution:

$$\int_1^5 \left(2x + \frac{1}{x}\right) dx = \left(x^2 + \ln|x|\right) \Big|_1^5 = (25 + \ln(5)) - (1 + \ln(1)) = 24 + \ln(5)$$

Problem 4 *Compute the following integrals:*

(a) $\displaystyle\int_0^1 e^{5x}\, dx$

Solution: $\displaystyle\int_0^1 e^{5x}\, dx = \left[\frac{1}{5}e^{5x}\right]_0^1 = \frac{1}{5}\left(e^5 - e^0\right) = \frac{1}{5}(e^5 - 1)$

(b) $\displaystyle\int_{-2}^{-1} \frac{1}{x^3}\, dx$

Solution:

$$\begin{aligned}
\int_{-2}^{-1} \frac{1}{x^3}\, dx &= \int_{-2}^{-1} x^{-3}\, dx \\
&= \left[\frac{x^{-2}}{-2}\right]_{-2}^{-1} \\
&= \left[\frac{-1}{2x^2}\right]_{-2}^{-1} \\
&= -\frac{1}{2} - \left(-\frac{1}{8}\right) \\
&= -\frac{1}{2} + \frac{1}{8} = -\frac{3}{8}
\end{aligned}$$

(c) $\displaystyle\int_0^4 \left(3x - 5 + 7\sqrt{16 - x^2}\right)\, dx$

Solution: *First notice that:*

$$\int_0^4 \left(3x - 5 + 7\sqrt{16 - x^2}\right)\, dx = \int_0^4 (3x - 5)\, dx + 7\int_0^4 \sqrt{16 - x^2}\, dx. \tag{1}$$

To get the easy part out of the way first:

$$\int_0^4 (3x - 5)\, dx = \left[\frac{3}{2}x^2 - 5x\right]_0^4 = (24 - 20) - (0 - 0) = 4. \tag{2}$$

So the real issue is in computing $\displaystyle\int_0^4 \sqrt{16 - x^2}\, dx$.

We do not know how to integrate $\sqrt{16 - x^2}$ directly, but recall that the solution set to the equation $x^2 + y^2 = 16$ is a circle with radius 4 centered at the origin. Solving for y we get $y = \pm\sqrt{16 - x^2}$. So if we restrict ourselves to $y = \sqrt{16 - x^2}$ and $0 \leq x \leq 4$, this gives us the upper-right quarter of the circle (draw a picture and convince yourself of this). Since the total area of the circle of $\pi(4)^2 = 16\pi$, the area under this portion of the curve is $\frac{1}{4}(16\pi) = 4\pi$. Thus we have computed (geometrically) that

$$\int_0^4 \sqrt{16 - x^2}\, dx = 4\pi. \tag{3}$$

So, $\displaystyle\int_0^4 \left(3x - 5 + 7\sqrt{16 - x^2}\right)\, dx = 4 + 7(4\pi) = 4 + 28\pi$ *by (1),(2), and (3)*

Problem 5 *Find the derivative of the following functions:*

(a) $\displaystyle F(x) = \int_{\sqrt{x}}^1 \frac{t^2}{2 + 3t^4}\, dt$

Solution: *First notice that*

$$F(x) = \int_{\sqrt{x}}^{1} \frac{t^2}{2+3t^4}\,dt = -\int_{1}^{\sqrt{x}} \frac{t^2}{2+3t^4}\,dt.$$

So we can apply the First Fundamental Theorem of Calculus, along with the chain rule, to compute:

$$\begin{aligned}
F'(x) &= \frac{d}{dx}\left(-\int_{1}^{\sqrt{x}} \frac{t^2}{2+3t^4}\,dt\right) \\
\text{we introduce the variable: } u = \sqrt{x} \implies F'(x) &= \frac{dF}{du}\frac{du}{dx} \\
&= -\frac{u^2}{2+3(u)^4} \cdot \frac{d}{dx}(\sqrt{x}) \\
&= -\frac{u^2}{2+3u^4} \cdot \frac{1}{2\sqrt{x}} \\
\text{substituting } u = \sqrt{x} \implies F'(x) &= -\frac{x}{2+3x^2} \cdot \frac{1}{2\sqrt{x}} \\
&= -\frac{\sqrt{x}}{2(2+3x^2)}
\end{aligned}$$

(b) $G(x) = \displaystyle\int_{x}^{x^3} \sin(7t^2)\,dt$

Solution: *First notice that*

$$\begin{aligned}
G(x) &= \int_{x}^{x^3} \sin(7t^2)\,dt \\
&= \int_{x}^{0} \sin(7t^2)\,dt + \int_{0}^{x^3} \sin(7t^2)\,dt \\
&= -\int_{0}^{x} \sin(7t^2)\,dt + \int_{0}^{x^3} \sin(7t^2)\,dt
\end{aligned}$$

It is worth pointing out that breaking up the integral above at 0 *was arbitrary. Since* $\sin(7t^2)$ *is continuous over* $(-\infty, \infty)$*, we could have chosen any real number in place of* 0*.*

We can now apply the First Fundamental Theorem of Calculus, along with the chain rule, to compute:

$$\begin{aligned}
G'(x) &= \frac{d}{dx}\left(-\int_{0}^{x} \sin(7t^2)\,dt + \int_{0}^{x^3} \sin(7t^2)\,dt\right) \\
&= \frac{d}{dx}\left(-\int_{0}^{x} \sin(7t^2)\,dt\right) + \frac{d}{dx}\left(\int_{0}^{x^3} \sin(7t^2)\,dt\right)
\end{aligned}$$

The left integral requires no chain rule on the FFTOC. $\displaystyle\frac{d}{dx}\left(-\int_{0}^{x} \sin(7t^2)\,dt\right) = -\sin(7x^2)$

The right integral requires chain rule. We'll subsitute $u = x^3$.

$$\begin{aligned}
\frac{d}{dx}\left(\int_0^{x^3} \sin(7t^2)\,dt\right) &= \frac{d}{du}\left(\int_0^{u} \sin(7t^2)\,dt\right)\frac{du}{dx} \\
&= \sin(7u^2)\cdot\frac{d}{dx}(x^3) \\
&= \sin(7u^2)\cdot 3x^2
\end{aligned}$$

$$\textit{substituting } u = x^3 \implies \frac{d}{dx}\left(\int_0^{x^3} \sin(7t^2)\,dt\right) = 3x^2\sin(7x^6)$$

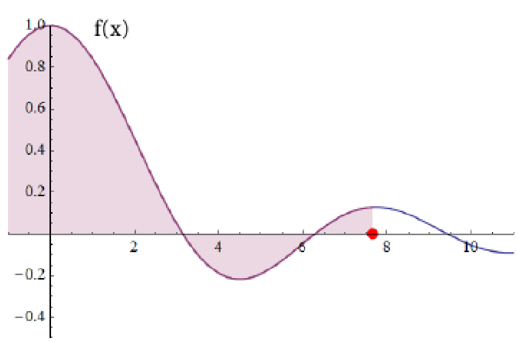
Combining we have:

$$G'(x) = 3x^2\sin(7x^6) - \sin(7x^2).$$

An alternative approach *using SFTOC: Call $h(t) = \sin(7t^2)$ and let H be an antiderivative of h. By SFTOC, $\int_x^{x^3} \sin(7t^2)\,dt = \Big[H(t)\Big]_{t=x}^{x^3}$. Then:*

$$\begin{aligned}
\frac{d}{dx}\left(\int_0^{x^3} \sin(7t^2)\,dt\right) &= \frac{d}{dx}\left(\Big[H(t)\Big]_{t=x}^{x^3}\right) \\
&= \frac{d}{dx}\left(H(x^3) - H(x)\right) \\
&= H'(x^3)\cdot 3x^2 - H'(x) \\
&= h(x^3)\cdot 3x^2 - h(x) \\
&= \sin(7x^6)\cdot 3x^2 - \sin(7x^2)
\end{aligned}$$

Problem 6 *Given the following graph of $y = f(x)$, let $g(x) = \int_{-1}^{x} f(t)\,dt$.*



(a) *Is g continuous? Why or why not?*

Solution: *$g(x)$ represents the area function (or the signed area between the curve $y = f(x)$ and the x-axis from -1 to x). By the FTC(I), $g(x)$ is continuous.*

(b) *Is g differentiable? Why or why not?*

Solution: *Similarly by the FTC(I) the function g is differentiable, and moreover:*

$$\frac{d}{dx}(g(x)) = \frac{d}{dx}\left(\int_{-1}^{x} f(t)\,dt\right) = f(x)$$

(c) *Where does g achieve its absolute maximum and minimum values? Where does g achieve any local extreme values? Assume the domain of g is $[-1, 7.6]$.*

Solution: *Since g is continuous on the closed interval $[-1, 7.6]$, g attains its absolute extreme values at either critical points or endpoints. But we just saw that $g'(x) = f(x)$, and so the critical points of g are just the points where $f(x) = 0$. Thus, the critical points of g are approximately 3.1 and 6.25.*

Since $f = g'$, g is increasing when f is positive and g is decreasing when f is negative. So we have that $x = 3.1$ is a local maximum while $x = 6.25$ is a local minimum of g. This takes care of the local extreme values.

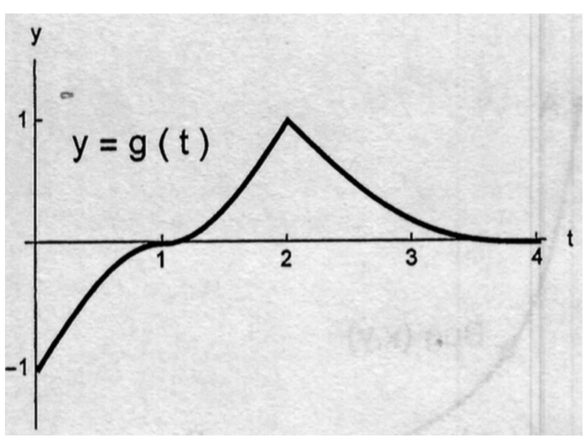
To find the absolute extreme values of g, we need to determine what are the largest and smallest values among $g(-1), g(3.1), g(6.25)$, and $g(7.6)$. And do not forget the $g(x)$ denotes the (signed) area of the region bounded by the curve and the $x - axis$.

Clearly $g(-1) = 0$ since a line segment does not have any area. At any time after $x = -1$ we have a positive (net) area. Thus, our absolute minimum occurs at $x = -1$. To find the absolute maximum of g, we need to compare $g(3.1)$ and $g(7.6)$. We see that we have the greatest area under the curve at $x = 3.1$ because by the time we get to $x = 7.6$, the area we have subtracted off between $x = 3.1$ and $x = 6.25$ is greater than what we have added back on from $x = 6.25$ to $x = 7.6$. Therefore, our absolute maximum occurs at $x = 3.1$.

(d) *Where is the graph of g concave up? Concave down?*

Solution: *g is concave up when when f is increasing, which is on $(-1, 0) \cup (4.25, 7.6)$. Similarly, g is concave down when f is decreasing, which is on $(0, 4.25)$. Remember, $g' = f$.*

Problem 7 *The graph of g, a continuous function $[0, 4]$, is shown in the figure. Let $A(x) = \int_0^x g(t)\,dt$ for* $0 \le x \le 4$.



(a) *Circle the correct statement about* $A(1)$.

(i) $A(1) = 0$;

(ii) $A(1) < 0$;

(iii) $A(1) > 0$;

(iv) *none of the previous answers.*

Solution: *Correct choice: (ii)* $A(1) < 0$. $A(1)$ *is the net area of the region bounded by the curve and the* x*-axis between* $x = 0$ *and* $x = 1$. *On this interval,* $g(t)$ *is negative and therefore,* $A(1)$ *is negative.*

(b) *Circle the correct statement about* $A(1.5)$.

(i) $A(1.5) = 0$;

(ii) $A(1.5) < 0$;

(iii) $A(1.5) > 0$;

(iv) *none of the previous answers.*

Solution: *Correct choice: (ii)* $A(1.5) < 0$ *The net area on the interval* $(1, 1.5)$ *is postive and smaller than the negative net area on the interval* $(0, 1)$. *Therefore, the net area on the interval* $(0, 1.5)$ *is negative.*

(c) *Circle the correct statement about* $A'(1.5)$.

(i) $A'(1.5) = 0$;

(ii) $A'(1.5) < 0$;

(iii) $A'(1.5) > 0$;

(iv) *none of the previous answers.*

Solution: *Correct choice: (iii)* $A'(1.5) > 0$ *because* $A'(1.5) = g(1.5)$

(d) *Circle the correct expression for* $\displaystyle\int_1^4 (g(t) + 1)\,dt$.

(i) $A(4) + 1$;

(ii) $A(4) - A(1)$;

(iii) $A(4) - A(1) + 1$;

(iv) $A(4) + 3$;

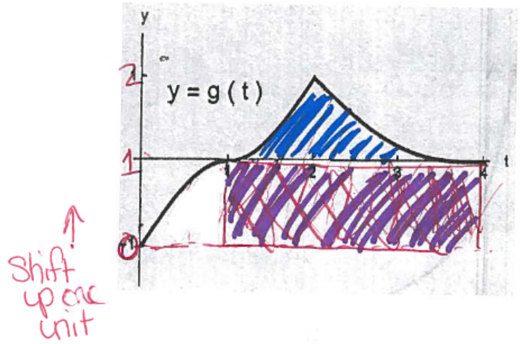
(v) $A(4) - A(1) + 3$;

(vi) *none of the previous answers.*

Solution: *Correct choice: (v)* $A(4) - A(1) + 3$

We can split the integral: $\displaystyle\int_1^4 (g(t) + 1)\,dt = \int_1^4 g(t)\,dt + \int_1^4 1\,dt = A(4) - A(1) + 3$

We can visualize this integal by imagining shifting the graph of g *up one unit. See figure below. Our area we are looking for consists of the area under the curve* g *on the interval* $(1, 4)$ *(the blue region) plus the rectangular region formed below (the purple region).*



The area of the blue region is $\int_1^4 g(t)\,dt = \int_0^4 g(t)\,dt - \int_0^1 g(t)\,dt = A(4) - A(1)$.

The area of the purple region is $\int_1^4 1\,dt$. *This integral is the area under the line* $y = 1$ *on the interval* $(1, 4)$ *which is a rectangular region of area* 3.

Combining these, the total area $\int_1^4 (g(t) + 1)\,dt = A(4) - A(1) + 3$

(e) *Circle the correct statement about* $A(0)$.

(i) $A(0) = 0$;
(ii) $A(0) = 1$;
(iii) $0 < A(0) < 1$;
(iv) $A(0) > 1$;
(v) $A(0) < 0$;
(vi) *none of the previous answers.*

Solution: *Correct choice: (i)* $A(0) = 0$. *By the definition of* A, $A(x) = \int_0^x g(t)\,dt$ *for* $0 \leq x \leq 4$. *So,* $A(0) = \int_0^0 g(t)\,dt = 0$.

(f) *Circle the interval (or intervals) where the function* A *is DECREASING.*

(i) $(0, 1)$;
(ii) $(1, 2)$;
(iii) $(2, 4)$;
(iv) *none of the previous answers.*

Solution: *Correct choice: (i)* $(0, 1)$. A *is decreasing means that the area is decreasing or we are accumulating negative area. This happens when* A*'s derivative,* g *is negative.*

(g) *Circle the value (or values) where the function* A *attains its MAXIMUM.*

(i) $x = 0$;
(ii) $x = 1$;

(iii) $x = 2$;

(iv) $x = 3$;

(v) $x = 4$;

(vi) *none of the previous answers.*

Solution: *Correct choice: (v) $x = 4$. It appears that $A(x) < 0$ on $(0, 2)$ and $A(x) > 0$ on $(2, 4]$. So, the maximum occurs on $[2, 4]$. Since $A' = g$ is postive on $(2, 4)$, and A is increasing there, the maximum is attained at the end point, $x = 4$. In other words, A achieves a maximum when we have the largest potive area. Because we start with negative area and then continue to accumulate positive area until $x = 4$, the maximum occurs at the end point.*

(h) *Circle the value (or values) where the function A attains its MINIMUM.*

(i) $x = 0$;

(ii) $x = 1$;

(iii) $x = 2$;

(iv) $x = 3$;

(v) $x = 4$;

(vi) *none of the previous answers.*

Solution: *Correct choice: (ii) $x = 1$. $A' = g$ is negative on $(0, 1)$ and positive on $(1, 4)$. So A is decreasing on $(0, 1)$ and increasing on $(1, 4)$. So the local and absolute maximum occurs at $x = 1$. In other words, A achieves a minimum when we have the largest negative area. We start by accumulating negative area and then switch to accumulating positive area at $x = 1$.*

(i) *Circle the interval (or intervals) where the function A is CONCAVE DOWN.*

(i) $(0, 1)$;

(ii) $(1, 2)$;

(iii) $(2, 4)$;

(iv) *none of the previous answers.*

Solution: *Correct choice: (iii) $(2, 4)$. A is concave down where $A' = g$ is decreasing. g is decreasing on $(2, 4)$.*

(j) *Circle the value (or values) of x where the function A has an inflection point.*

(i) $x = 1$;

(ii) $x = 2$;

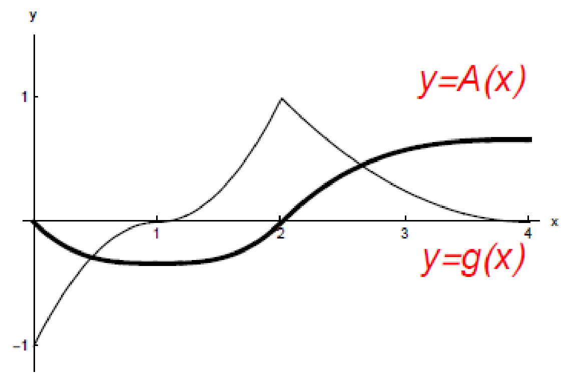
(iii) $x = 3$;

(iv) *none of the previous answers.*

Solution: *Correct choice: (ii) $x = 2$. $A' = g$ changes from increasing to decreasing.*

(k) *Sketch the graph of A, based on items (e)-(j)*

Solution:


y
1
-1
1
2
3
4
x
y=A(x)
y=g(x)