What is a Limit? - Solutions

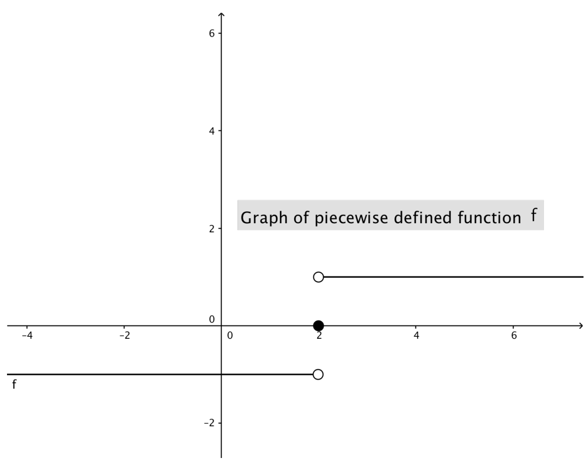
Problem 1

(a) *True or False: To find* $\lim_{x\to 2} f(x)$*, it's enough to know the values of* $f(2.1)$, $f(2.01)$, $f(2.001)$*, and so on.*

Solution: *False. These values will only help us make a guess at* $\lim_{x\to 2^+} f(x)$*, the right hand limit of* f *as* x *approaches* 2*. To determine* $\lim_{x\to 2} f(x)$*, we also need to know* $\lim_{x\to 2^-} f(x)$ *which we cannot determine from the above values. For example, consider the function*

$$f(x) = \begin{cases} -1 & \text{if } x < 2, \\ 0 & \text{if } x = 2, \text{ and} \\ 1 & \text{if } 2 < x. \end{cases}$$

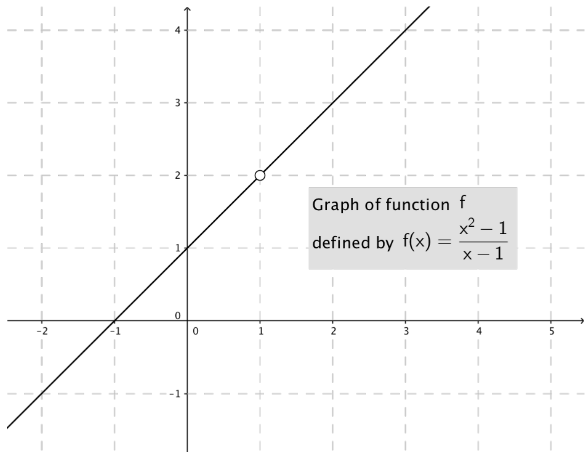
Looking at the graph of this function below, we can see that $\lim_{x\to 2^+} f(x) = 1$ *and* $\lim_{x\to 2^-} f(x) = -1$*. Thus, since* $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, $\lim_{x\to 2} f(x)$ *does not exist.*

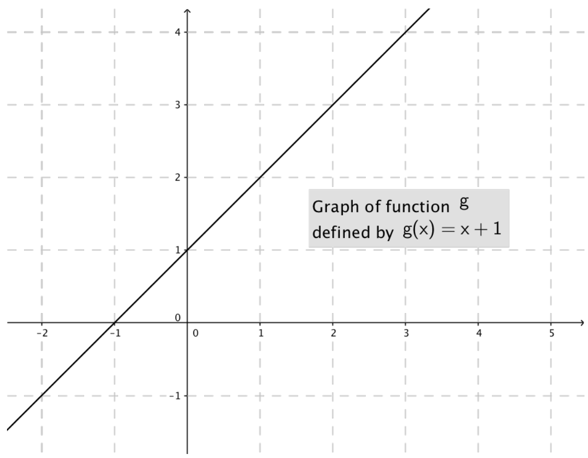


(b) *True or False: If we know the value of* $f(2)$*, then can we conclude that* $\lim_{x\to 2} f(x) = f(2)$*?*

Solution: *False. In the above example, we have that* $f(2) = 0$*, while* $\lim_{x\to 2} f(x)$ *does not exist.*

Problem 2 *Use the graphs and the given definitions of the following two functions to answer the questions below.*





(a) *Find the domain of f and the domain of g.*

Solution: *The domain of f is $(-\infty, 1) \cup (1, \infty)$ (all real numbers except 1). The domain of g is $(-\infty, \infty)$ (all real numbers).*

(b) *Is $f = g$? (Why or why not?)*

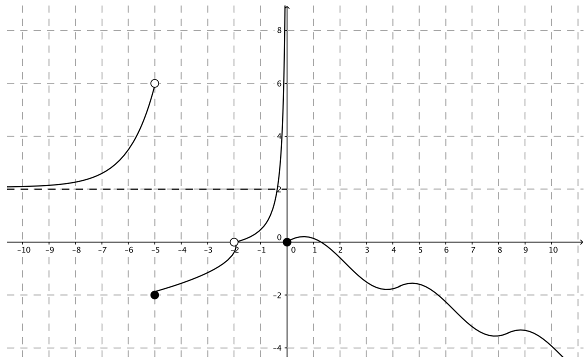
Solution: *No, these two functions are not equal. Two functions are equal if and only if they have identical domains and their values agree on all points in the domain.*

Since f and g have different domains, by part (a), they cannot be equal functions.

(c) *Looking at the graphs, find $\lim_{x\to 1} f(x)$ and $\lim_{x\to 1} g(x)$.*

Solution: *From the graphs we have that $\lim_{x\to 1} f(x) = 2$ and $\lim_{x\to 1} g(x) = 2$.*

Problem 3 *The graph of a function f is given below. Use this graph to answer the following questions.*



(a) *Find the domain of f.*

Solution: *The domain of f is $(-\infty, -2) \cup (-2, \infty)$.*

(b) *Find the range of f.*

Solution: *The range of f is $(-\infty, \infty)$.*

(c) *Find the following values.*

(i) $\lim_{x \to -2} f(x) =$

Solution: $\lim_{x \to -2} f(x) = 0$

(ii) $f(-2) =$

Solution: $f(-2)$ *is undefined*

(iii) $f(-5) =$

Solution: $f(-5) = -2$

(iv) $\lim_{x \to 0^+} f(x) =$

Solution: $\lim_{x \to 0^+} f(x) = 0$

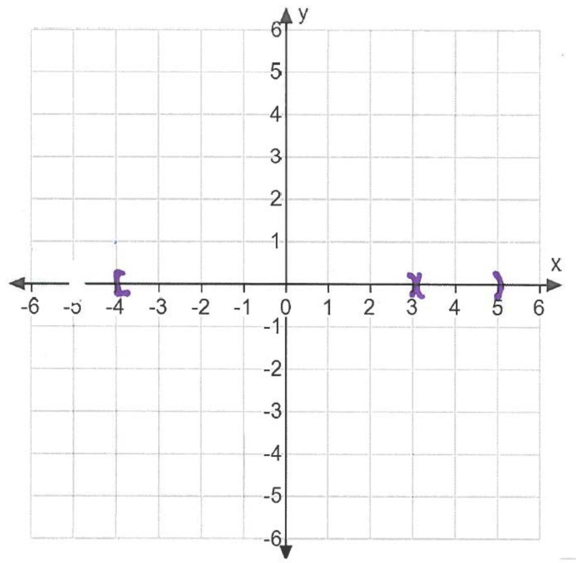
(v) $\lim_{x \to 0} f(x) =$

Solution: $\lim_{x \to 0} f(x)$ *is undefined*

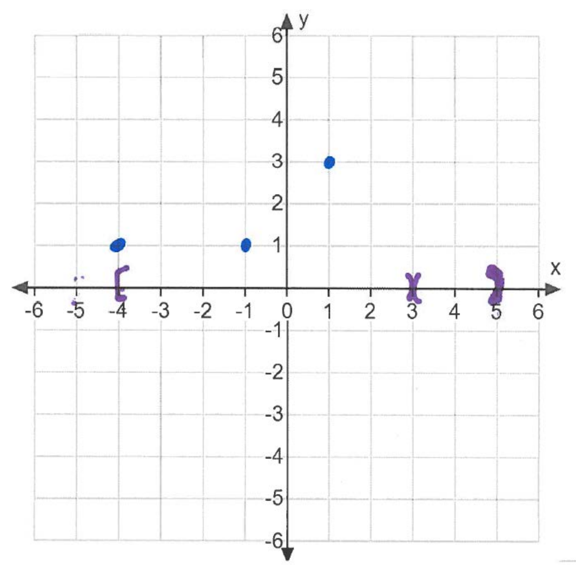
Problem 4 *Sketch a possible graph of a function that satisfies all of the given properties. (You do not need to find a formula for the function.)*

Domain: $[-4, 3) \cup (3, 5)$	$f(1) = 3$	$f(-1) = 1$	$f(-4) = 1$
$\lim_{x \to -1^-} f(x) = -2$	$\lim_{x \to -1^+} f(x) = 1$	$\lim_{x \to 1} f(x) = 2$	$\lim_{x \to 3^+} f(x) = -1$
$\lim_{x \to 3^-} f(x) = 1$		$\lim_{x \to -4^+} f(x) = 1$	$\lim_{x \to 5^-} f(x) = 1$

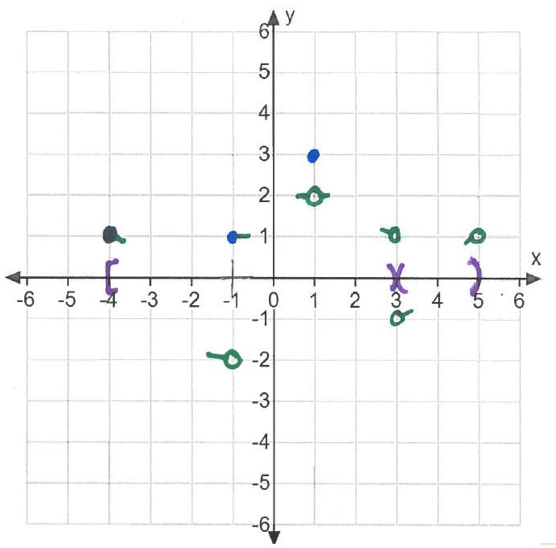
Solution: *One way to go about sketching the graph is as follows. First, on the x-axis, indicate the domain (seen here in purple)*



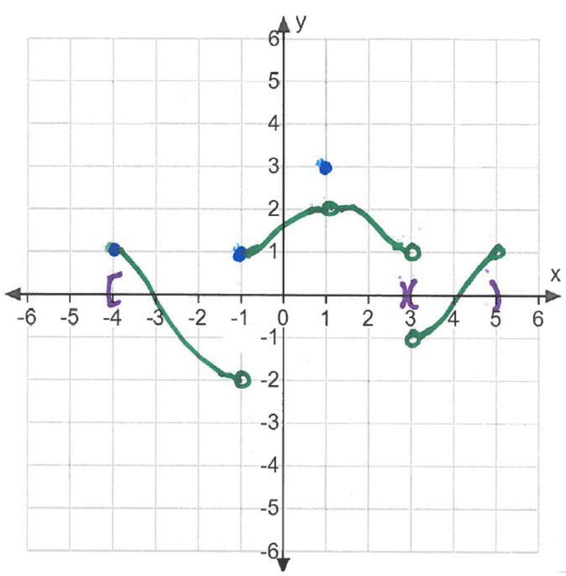
Next plot the given points (seen here in blue)



Then indicate the limits using open circles and tails to indicate if the limit is the value as the function approaches from the left or right or both(seen here in green)



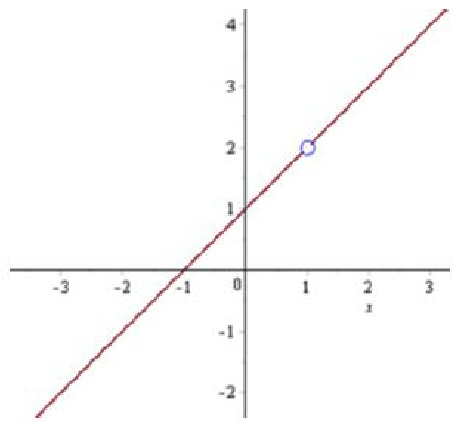
Finally connect the graph.



Problem 5 *True/False: Give an explanation or counterexample. Assume a and L are finite numbers.*

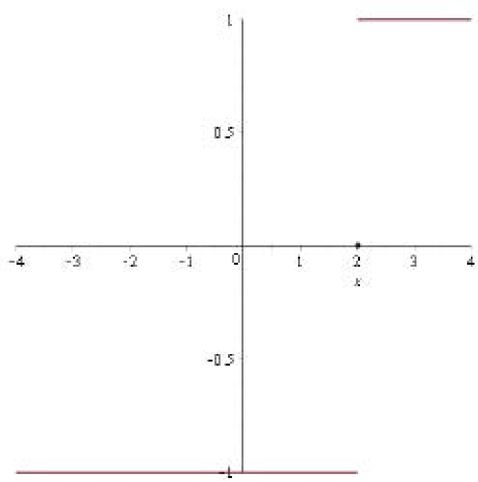
(a) *If* $\lim_{x\to a} f(x) = L$, *then* $f(a) = L$.

Solution: *False. In the graph below* $\lim_{x\to 1} f(x) = 2$, *but* $f(1)$ *does not exist.*



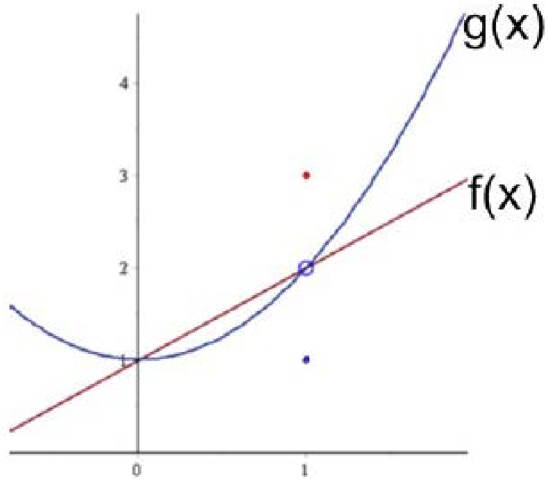
(b) *If* $\lim_{x\to a^-} f(x) = L$*, then* $\lim_{x\to a^+} f(x) = L$*.*

Solution: *False. In the graph below* $\lim_{x\to 2^-} f(x) = -1$ *but* $\lim_{x\to 2^+} f(x) = 1$*.*



(c) *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = L$*, then* $f(a) = g(a)$*.*

Solution: *False. If we let*



we see that $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x) = 2$, *but* $f(1) \neq g(1)$.

(d) $\lim_{x\to a} \frac{f(x)}{g(x)}$ *does not exist if* $g(a) = 0$.

Solution: *False. If* $f(x) = 5x^2$ *and* $g(x) = x^2$, *then* $g(0) = 0$ *but*

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{5x^2}{x^2} = \lim_{x\to 0} 5 = 5.$$