Limit Laws (LL) - Solutions

Problem 1 *The following argument shows*

$$\lim_{x\to 3} \frac{5x^3 - 4\sqrt{x}}{\sqrt{x^5 - 87}} = \frac{135 - 4\sqrt{3}}{\sqrt{156}}.$$

State which limit law is used to justify each step. (A particular step may have more than one limit law as a justification.)

$$\begin{aligned}
\lim_{x\to 3} \frac{5x^3 - 4\sqrt{x}}{\sqrt{x^5 - 87}} &= \frac{\lim\limits_{x\to 3}\left(5x^3 - 4\sqrt{x}\right)}{\lim\limits_{x\to 3}\sqrt{x^5 - 87}} \\
&= \frac{\left(\lim\limits_{x\to 3} 5\right)\left(\lim\limits_{x\to 3}(x^3)\right) - \left(\lim\limits_{x\to 3} 4\right)\left(\lim\limits_{x\to 3}\sqrt{x}\right)}{\sqrt{\lim\limits_{x\to 3}(x^5 - 87)}} \\
&= \frac{5\lim\limits_{x\to 3}(x^3) - 4\lim\limits_{x\to 3}\sqrt{x}}{\sqrt{\lim\limits_{x\to 3}(x^5 - 87)}} \\
&= \frac{5(\lim\limits_{x\to 3} x)^3 - 4\sqrt{3}}{\sqrt{\lim\limits_{x\to 3}(x^5) - \lim\limits_{x\to 3}(87)}} \\
&= \frac{5(3)^3 - 4\sqrt{3}}{\sqrt{3^5 - 87}} \\
&= \frac{135 - 4\sqrt{3}}{\sqrt{156}}
\end{aligned}$$

Solution:

$$\begin{aligned}
\lim_{x\to 3} \frac{5x^3 - 4\sqrt{x}}{\sqrt{x^5 - 87}} &= \frac{\lim\limits_{x\to 3}\left(5x^3 - 4\sqrt{x}\right)}{\lim\limits_{x\to 3}\sqrt{x^5 - 87}} \quad \textit{Quotient law} \\
&= \frac{\left(\lim\limits_{x\to 3} 5\right)\left(\lim\limits_{x\to 3}(x^3)\right) - \left(\lim\limits_{x\to 3} 4\right)\left(\lim\limits_{x\to 3}\sqrt{x}\right)}{\sqrt{\lim\limits_{x\to 3}(x^5 - 87)}} \quad \textit{Product law, Composition} \\
&= \frac{5\lim\limits_{x\to 3}(x^3) - 4\lim\limits_{x\to 3}\sqrt{x}}{\sqrt{\lim\limits_{x\to 3}(x^5 - 87)}} \quad \textit{Continuity} \\
&= \frac{5(\lim\limits_{x\to 3} x)^3 - 4\sqrt{3}}{\sqrt{\lim\limits_{x\to 3}(x^5) - \lim\limits_{x\to 3}(87)}} \quad \textit{Difference,} \quad \textit{Continuity} \\
&= \frac{5(3)^3 - 4\sqrt{3}}{\sqrt{3^5 - 87}} \quad \textit{Continuity} \\
&= \frac{135 - 4\sqrt{3}}{\sqrt{156}}
\end{aligned}$$

Problem 2 *Find the limit and justify your answer.*

(a) $\lim\limits_{x\to 0}|x|$

Solution: *Remember* $|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

We first compute the one-sided limit: $\lim\limits_{x\to 0^+}|x| = \lim\limits_{x\to 0^+} x = 0.$

Next, we compute the other one-sided limit:

$\lim\limits_{x\to 0^-}|x| = \lim\limits_{x\to 0^-}(-x) = \left(\lim\limits_{x\to 0^-}(-1)\right)\left(\lim\limits_{x\to 0^-} x\right) = (-1)(0) = 0.$

Therefore $\lim\limits_{x\to 0^-}|x| = 0 = \lim\limits_{x\to 0^+}|x|$. *Hence,* $\lim\limits_{x\to 0}|x| = 0.$

(b) $\lim\limits_{x\to 2}\ln\left(\sin(x-2) + e^x\sin\left(\frac{\pi x}{4}\right)\right)$

Solution:

$$\begin{aligned}
\lim_{x\to 2}\ln\left(\sin(x-2) + e^x\sin\left(\frac{\pi x}{4}\right)\right) &= \ln\left(\lim_{x\to 2}\left(\sin(x-2) + e^x\sin\left(\frac{\pi x}{4}\right)\right)\right) \\
&= \ln\left(\lim_{x\to 2}\sin(x-2) + \lim_{x\to 2}\left(e^x\sin\left(\frac{\pi x}{4}\right)\right)\right) \\
&= \ln\left(\sin\left(\lim_{x\to 2}(x-2)\right) + \left(\lim_{x\to 2} e^x\right)\left(\lim_{x\to 2}\sin\left(\frac{\pi x}{4}\right)\right)\right) \\
&= \ln\left(\sin(0) + \left(\lim_{x\to 2} e^x\right)\sin\left(\lim_{x\to 2}\frac{\pi x}{4}\right)\right) \\
&= \ln\left(0 + e^2\sin\left(\frac{2\pi}{4}\right)\right) \\
&= \ln\left(e^2\sin\left(\frac{\pi}{2}\right)\right) = \ln\left(e^2(1)\right) \\
&= \ln\left(e^2\right) = 2
\end{aligned}$$

(c) $\lim\limits_{x\to 1}\dfrac{3 + 2\cos\left(\frac{\pi x}{3}\right)}{4x - 2x^3}$

Solution:

$$\begin{aligned}
\lim_{x\to 1}\frac{3 + 2\cos\left(\frac{\pi x}{3}\right)}{4x - 2x^3} &= \frac{\lim\limits_{x\to 1}\left(3 + 2\cos\left(\frac{\pi x}{3}\right)\right)}{\lim\limits_{x\to 1}\left(4x - 2x^3\right)} \\
&= \frac{\lim\limits_{x\to 1}(3) + \lim\limits_{x\to 1}\left(2\cos\left(\frac{\pi x}{3}\right)\right)}{4(1) - 2(1)^3} \\
&= \frac{3 + \left(\lim\limits_{x\to 1} 2\right)\left(\lim\limits_{x\to 1}\cos\left(\frac{\pi x}{3}\right)\right)}{4 - 2(1)} \\
&= \frac{3 + 2\cos\left(\lim\limits_{x\to 1}\frac{\pi x}{3}\right)}{4 - 2} \\
&= \frac{3 + 2\cos\left(\frac{\pi}{3}\right)}{2} = \frac{3 + 2\left(\frac{1}{2}\right)}{2} \\
&= \frac{3+1}{2} = 2.
\end{aligned}$$

Problem 3 *Suppose* $f(x) = \begin{cases} x^2 - ax & \text{if } x < 3 \\ \\ a2^x + 7 + a & \text{if } x > 3 \end{cases}$

Find a *so that* $\lim_{x \to 3} f(x)$ *exists.*

Solution: *Remember, it's not enough to find a value. We must also justify that the limit exists for the value chosen.*

To find a number a *for which* $\lim_{x \to 3} f(x)$ *exists we find* a *such that* $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$.

To find the left-sided limit:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - ax) = 9 - 3a.$$

To find the right-sided limit:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (a2^x + 7 + a) = a2^3 + 7 + a = 9a + 7.$$

In order for $\lim_{x \to 3} f(x)$ *to exists we must have the two one-sided limits equal. That is,*

$$\begin{aligned} \lim_{x \to 3^-} f(x) &= \lim_{x \to 3^+} f(x) \\ 9 - 3a &= 9a + 7 \\ 2 &= 12a \\ a &= 1/6. \end{aligned}$$

Problem 4 *Determine the value of* $\lim_{x \to 0} \left(x^2 \cos\left(\frac{1}{x}\right)\right)$. ***EXPLAIN***.

Solution: $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ *does not exist, so we cannot apply the Product law.*

We know that for any $x \neq 0$ *that* $-1 \leq \cos\left(\frac{1}{x}\right) \leq 1$. *Multiplying this inequality by* x^2 *(which is positive) we find that for all* x *near* 0,

$$-x^2 \leq x^2 \cdot \cos\left(\frac{1}{x}\right) \leq x^2$$

The limits of the functions on the left and right sides of this compound inequality are:

$$\begin{aligned} \lim_{x \to 0} (-x^2) &= 0 \\ \lim_{x \to 0} x^2 &= 0 \end{aligned}$$

Our explanation:

$\lim_{x \to 0} \left(x^2 \cos\left(\frac{1}{x}\right)\right) = 0$ *by the Squeeze Theorem since* $-x^2 \leq x^2 \cdot \cos\left(\frac{1}{x}\right) \leq x^2$ *and* $\lim_{x \to 0} (-x^2) = \lim_{x \to 0} x^2 = 0$.

Problem 5 *For all x near 0, the inequalities $1 - \frac{x^2}{6} \le \frac{\sin(x)}{x} \le 1$ are true. Use these inequalities to find $\lim_{x\to 0} \frac{\sin(x)}{x}$.* ***EXPLAIN.***

Solution: *$1 - x^2/6$ is a polynomial, so it is continuous at $x = 0$. That means $\lim_{x\to 0}\left(1 - \frac{x^2}{6}\right) = 1$. The function 1 is continuous since it is a constant, so $\lim_{x\to 0} 1 = 1$.*

That means $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ by the Squeeze Theorem, since $1 - \frac{x^2}{6} \le \frac{\sin(x)}{x} \le 1$ and $\lim_{x\to 0}\left(1 - \frac{x^2}{6}\right) = \lim_{x\to 0} 1 = 1$.

Problem 6 *Two functions, h and g, are given*

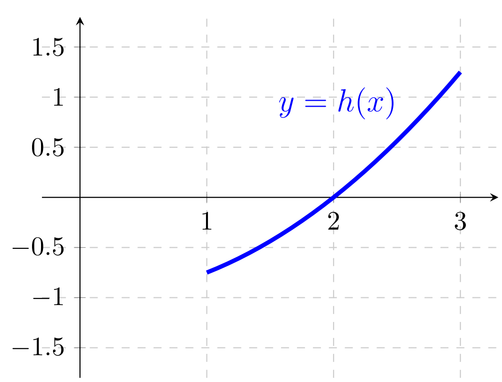
$$h(x) = \frac{x^2 - 4}{4},\ 1 < x < 3$$

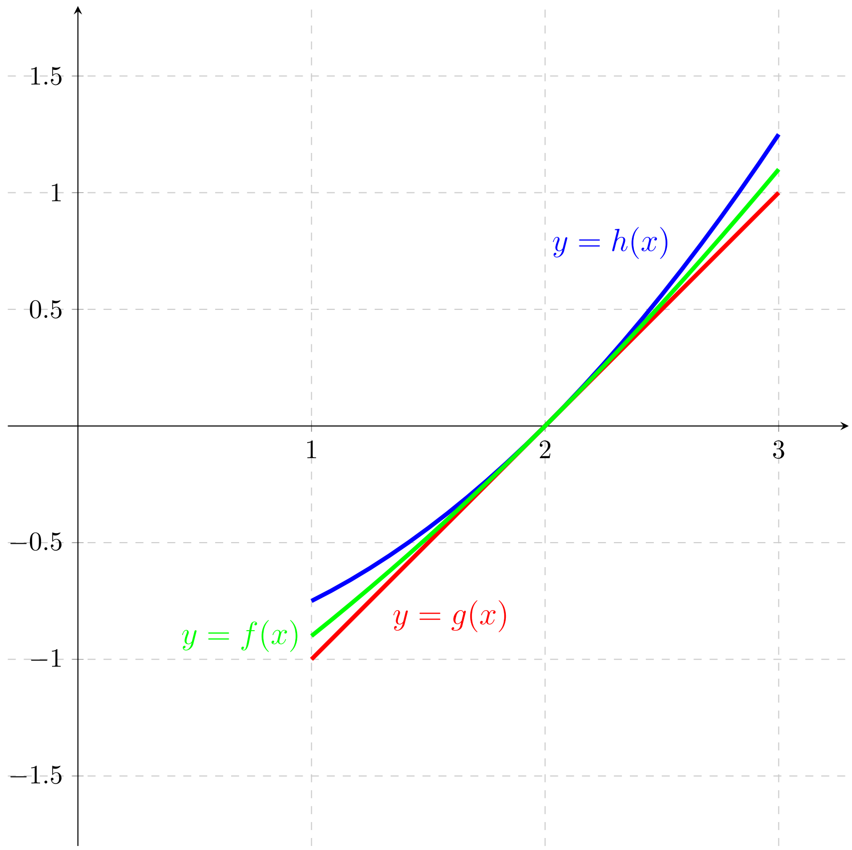
$$g(x) = x - 2,\ 1 < x < 3$$

The graph of the function h is given in the figure below. Let f be a function defined on the interval $(1, 3)$ that satisfies the following inequalities

$$g(x) \le f(x) \le h(x),\ 1 < x < 3$$

(a) *In the figure below, sketch and label the graph of g and a possible graph of f. (All three functions have a common domain $(1, 3)$.)*





Solution:

(b) *Evaluate the limit, or state that it does not exist. Justify your answer.*

(i) $\lim_{x\to 2} f(x)$

Solution: *g and h are both continuous function since they are both polynomials so we have:*

$$\lim_{x\to 2} g(x) = g(2) = 0$$

$$\lim_{x\to 2} h(x) = h(2) = 0$$

We have $g(x) \leq f(x) \leq h(x)$ *so by the squeeze theorem,* $\lim_{x\to 2} f(x) = 0$ *since* $g(x) \leq f(x) \leq h(x)$ *with* $\lim_{x\to 2} g(x) = \lim_{x\to 2} h(x) = 0$.

(ii) $\lim_{x\to 2} \dfrac{f(x)+2}{x-1}$

Solution:

$$\lim_{x\to 2} \frac{f(x)+2}{x-1} = \frac{\lim_{x\to 2}(f(x)+2)}{\lim_{x\to 2}(x-1)} = \frac{0+2}{2-1} = \frac{2}{1} = 2$$

(iii) $\lim_{x\to 2} g(1+e^{f(x)})$

Solution:

$$\lim_{x\to 2} g\Big(1+e^{f(x)}\Big) = g\Big(\lim_{x\to 2}(1+e^{f(x)})\Big) = g\Big(\lim_{x\to 2} 1 + \lim_{x\to 2} e^{f(x)}\Big) = g\Big(1+e^{\lim_{x\to 2} f(x)}\Big)$$

$$= g\Big(1+e^{0}\Big) = g(2) = 0$$