Approximating the area under a curve (ATAUAC) - Solutions

Problem 1 *Evaluate the sum.*

(a) $\sum_{k=1}^{n} 5$

Solution: $\sum_{k=1}^{n} 5 = n \cdot 5$

(b) $\sum_{k=1}^{10} 5$

Solution: *Using the previous result for* $n = 10$, *we get* $\sum_{k=1}^{10} 5 = 10 \cdot 5 = 50$
or

$$\sum_{k=1}^{10} 5 = 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 = 50$$

(c) $\sum_{k=1}^{n} 8k$

Solution: $\sum_{k=1}^{n} 8k = 8 \sum_{k=1}^{n} k = 8 \frac{n(n+1)}{2} = 4n(n+1)$

(d) $\sum_{k=1}^{4} 8k$

Solution: *Using the previous result for* $n = 4$, *we get* $\sum_{k=1}^{4} 8k = 4(4)(4+1) = 80$
or

$$\sum_{k=1}^{4} 8k = 8(1) + 8(2) + 8(3) + 8(4) = 8 + 16 + 24 + 32 = 80$$

(e) $\sum_{k=1}^{n} (6k^2 - 8k - 5)$

Solution: $\sum_{k=1}^{n}(6k^2-8k-5)=\sum_{k=1}^{n}6k^2+\sum_{k=1}^{n}(-8k)+\sum_{k=1}^{n}(-5)=$

$=6\sum_{k=1}^{n}k^2-8\sum_{k=1}^{n}k-\sum_{k=1}^{n}5=$

Now we use the formulas.

$=6\frac{n(n+1)(2n+1)}{6}-8\frac{n(n+1)}{2}-\sum_{k=1}^{n}5=n(n+1)(2n+1)-4n(n+1)-5n=$

$=2n^3-n^2-8n$

(f) $\sum_{k=0}^{2}\cos\left(\frac{\pi}{2}k\right)$

Solution:

$$\begin{aligned}\sum_{k=0}^{2}\cos\left(\frac{\pi}{2}k\right)&=\cos\left(\frac{\pi}{2}(0)\right)+\cos\left(\frac{\pi}{2}(1)\right)+\cos\left(\frac{\pi}{2}(2)\right)\\&=\cos(0)+\cos\left(\frac{\pi}{2}\right)+\cos(\pi)\\&=1+0+-1=0.\end{aligned}$$

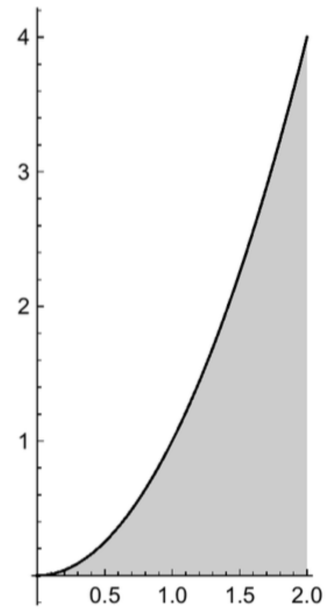
Problem 2 (a) *If a function is positive and decreasing on an interval $[a,b]$, will a right Riemann sum underestimate or overestimate the area of the region under the graph of the function? Justify your answer.*

Solution: *Because the function is decreasing, a right Riemann sum will be an underestimate. By definition, $f(x_k)<f(x)$ for all x in the interval $[x_{k-1},x_k]$.*

(b) *If a function is positive and decreasing on an interval $[a,b]$, will a left Riemann sum underestimate or overestimate the area of the region under the graph of the function? Justify your answer.*

Solution: *Because the function is decreasing, a right Rieimann sum will be an overestimate. By definition, $f(x_{k-1})>f(x)$ for all x in the interval $[x_{k-1},x_k]$.*

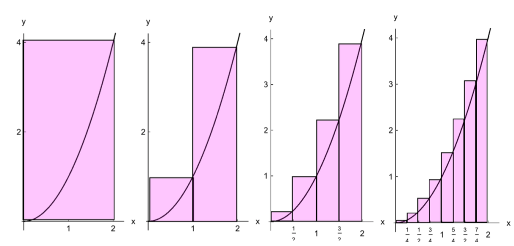
Problem 3 *The graph of the function $f(x)=x^2$ is given in the figure.*



Will a right Riemann sum approximation, for any value of n, be an underestimate or overestimate?

Solution: *Since the function f is positive and increasing on this interval, the right Riemann sum will always give an overestimate of the actual area.*

Approximate the shaded area using a right Riemann sum with $n = 1, 2, 4$, and 8 rectangles, as illustrated in the figure below.



(a) *Approximate the shaded area using a right Riemann sum with $n = 1$ rectangles.*

Solution: $\Delta x = \dfrac{2}{1} = 2$
$x_1^* = x_1 = 2$

$A \approx f(2)\Delta x = 4(2) = 8$

(b) *Approximate the shaded area using a right Riemann sum with $n = 2$ rectangles.*

Solution: $\Delta x = \dfrac{2}{2} = 1$
$x_k^* = x_k = k\Delta x = k(1)$, *for* $k = 1, 2$

$$A \approx \sum_{k=1}^{2} f(x_k^*)\Delta x = f(1)\Delta x + f(2)\Delta x = 1(1) + 4(1) = 5$$

(c) *Approximate the shaded area using a right Riemann sum with $n = 4$ rectangles.*

Solution: $\Delta x = \dfrac{2}{4} = \dfrac{1}{2}$

$x_k^* = x_k = k\Delta x = k\left(\dfrac{1}{2}\right) = \dfrac{k}{2}$, *for* $k = 1, 2, 3, 4$

$$A \approx \sum_{k=1}^{4} f(x_k^*)\Delta x = f\left(\frac{1}{2}\right)\Delta x + f(1)\Delta x + f\left(\frac{3}{2}\right) + f(2)\Delta x = \Delta x\left(f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2)\right)$$

$$A \approx \frac{1}{2}\left(\frac{1}{4} + 1 + \frac{9}{4} + 4\right) = \frac{5}{2} + 5 = \frac{15}{2}$$

(d) *Approximate the shaded area using a right Riemann sum with $n = 8$ rectangles.*

Solution: $\Delta x = \dfrac{2}{8} = \dfrac{1}{4}$

$x_k^* = x_k = k\Delta x = k\left(\dfrac{1}{4}\right) = \dfrac{k}{4}$, *for* $k = 1, ..., 8$

$$A \approx \sum_{k=1}^{8} f(x_k^*)\Delta x = \Delta x \cdot \sum_{k=1}^{8} f(x_k^*) = \frac{1}{4} \cdot \sum_{k=1}^{8} f\left(\frac{k}{4}\right) = \frac{1}{4} \cdot \sum_{k=1}^{8} \left(\frac{k}{4}\right)^2 = \frac{1}{4} \cdot \sum_{k=1}^{8} \frac{k^2}{16} = \frac{1}{64} \cdot \sum_{k=1}^{8} k^2$$

Recall: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ *Therefore*

$$A \approx \frac{1}{64} \cdot \sum_{k=1}^{8} k^2 = \frac{1}{64} \cdot \frac{8(9)(17)}{6} = \frac{1}{8} \cdot \frac{3(17)}{2} = \frac{51}{16}$$

Problem 4 *A positive continuous function will have area approximated on the interval $[1, 6]$ using n rectangles.*

(a) *Find a formula for the grid point, x_k.*

Solution: *Here* $\Delta x = \dfrac{b-a}{n} = \dfrac{5}{n}$.

The basic grid point formula is $x_k = a + k\Delta x = 1 + \dfrac{5k}{n}$.

(b) *Find a formula for the sample point x_k^* if using a right Riemann sum.*

Solution: *For right endpoints,* $x_k^* = x_k = 1 + \dfrac{5k}{n}$.

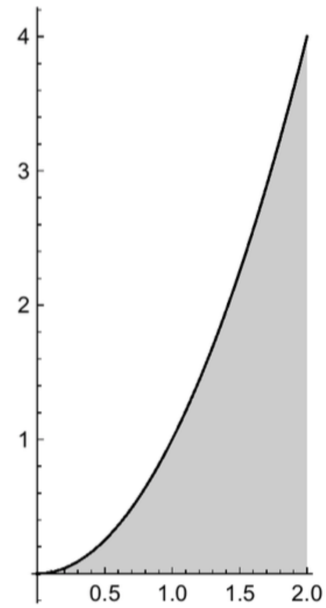
(c) *Find a formula for the sample point x_k^* if using a left Riemann sum.*

Solution: *For left endpoints,* $x_k^* = x_{k-1} = 1 + \dfrac{5(k_1)}{n}$.

(d) *Find a formula for the sample point x_k^* if using a midpoint Riemann sum.*

Solution: *For midpoints,* $x_k^* = \dfrac{x_k + x_{k-1}}{2} = 1 + \dfrac{5(k-\frac{1}{2})}{n}$.

Problem 5 *The graph of the function $f(x) = x^2$ is given in the figure.*



Find the exact value of the area of the shaded region. HINT: Use a right Riemann sum with n rectangles, and then take the limit as $n \to \infty$.

Solution: $\Delta x = \dfrac{2}{n}$

$x_k^* = x_k = k\Delta x = k\left(\dfrac{2}{n}\right) = \dfrac{2k}{n}$, *for* $k = 1, ..., n$

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x = \Delta x \cdot \sum_{k=1}^{n} f(x_k^*) = \frac{2}{n} \cdot \sum_{k=1}^{n} f\left(\frac{2k}{n}\right) = \frac{2}{n} \cdot \sum_{k=1}^{n} \left(\frac{2k}{n}\right)^2 = \frac{2}{n} \cdot \sum_{k=1}^{n} \frac{4k^2}{n^2} = \frac{8}{n^3} \cdot \sum_{k=1}^{n} k^2$$

Recall: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Therefore

$$A \approx \frac{8}{n^3} \cdot \sum_{k=1}^{n} k^2 = \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{4}{3} \cdot \frac{(n+1)(2n+1)}{n^2} = \frac{4}{3} \cdot \frac{2n^2+3n+1}{n^2}.$$

Let's take the limit of Riemann sums as $n \to \infty$.

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty} \frac{4}{3} \cdot \frac{2n^2+3n+1}{n^2} = \lim_{n\to\infty} \frac{4}{3} \cdot \left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{8}{3}.$$

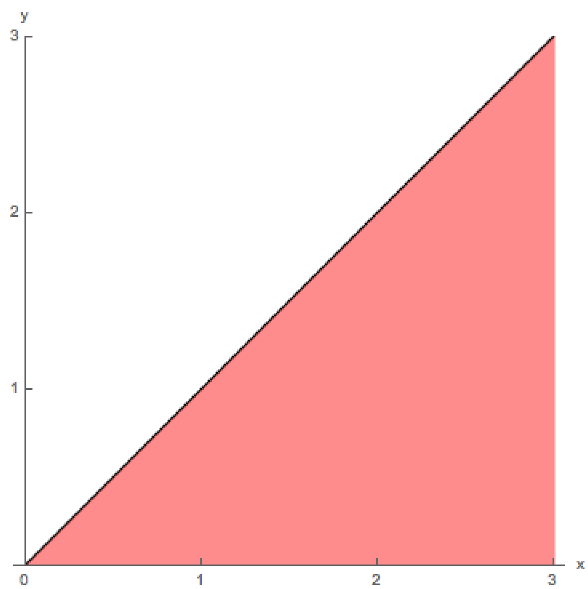
Problem 6 *Consider a Riemann sum with* n *rectangles for the function* f *on the interval* $[a, b]$. *Use geometry to find the limit of Riemann sums as* $n \to \infty$.

$\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$?

(a) $f(x) = x$, $[0, 3]$;

Solution: $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} x_k^*\Delta x = \frac{9}{2}$,

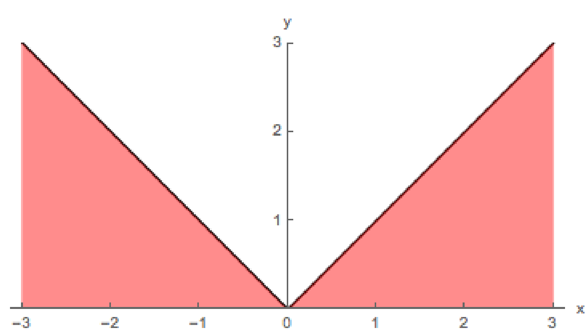
since $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = A$, *the area of the shaded region in the figure.*



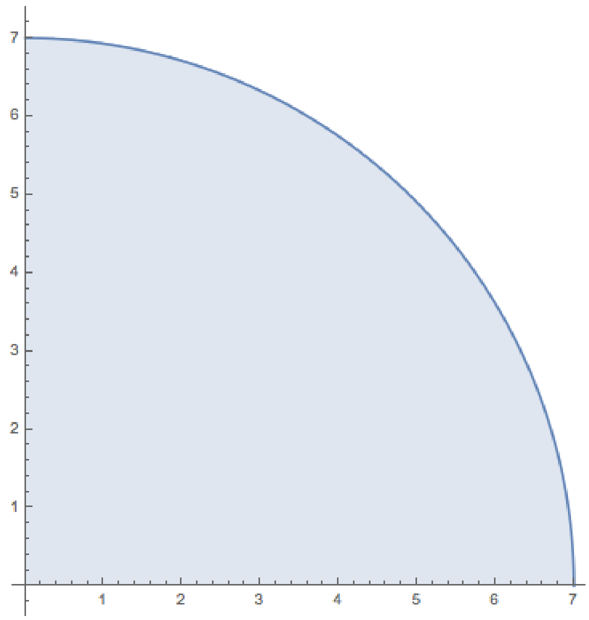
(b) $f(x) = |x|$, $[-3, 3]$;

Solution: $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n} |x_k^*|\Delta x = 9$,

since $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = A$, *the area of the shaded region in the figure.*



Problem 7 *A part of a circle is shown in the figure.*



(a) *If we express the area of the shaded region in the figure as the limit of Riemann sums*

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x,$$

find the function f and the interval $[a, b]$.

Solution: $f(x) = \sqrt{49 - x^2}$, *and the interval* $= [0, 7]$.

(b) *Compute the limit.*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{49 - (x_k^*)^2} \cdot \frac{7}{n}$$

Solution: $$\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{49 - (x_k^*)^2} \cdot \frac{7}{n} = \frac{49\pi}{4}$$

Problem 8 *We want to approximate the area under the curve using a right Riemann Sum with the given value of n. Write the sum in summation notation and evaluate it.*

(a) $\sin(x)$, $\left[0, \frac{\pi}{2}\right]$, $n = 3$

Solution: $$\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{3} = \frac{\pi}{6}.$$

$x_k^* = x_k = a + k\Delta x = 0 + \frac{\pi}{6}k = \frac{\pi}{6}k.$

$$\begin{aligned}
\sum_{k=1}^{3} f(x_k^*)\Delta x &= \sum_{k=1}^{3} \left(f\left(\frac{\pi}{6}k\right) \cdot \frac{\pi}{6} \right) \\
&= \frac{\pi}{6} \sum_{k=1}^{3} \sin\left(\frac{\pi}{6}k\right) \\
&= \frac{\pi}{6} \left(\sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{2\pi}{6}\right) + \sin\left(\frac{3\pi}{6}\right) \right) \\
&= \frac{\pi}{6} \left(\sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{2}\right) \right) \\
&= \frac{\pi}{6} \left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 \right) \\
&= \frac{\pi}{6} \left(\frac{3}{2} + \frac{\sqrt{3}}{2} \right) \\
&= \frac{\pi}{12}(3 + \sqrt{3})
\end{aligned}$$

(b) $f(x) = x^2 - 9x + 18, \quad [7, 10], \quad n = 6$

Solution: $\quad \Delta x = \frac{b-a}{n} = \frac{10-7}{6} = \frac{1}{2}.$
$x_k^* = x_k = a + k\Delta x = 7 + \frac{1}{2}k.$

$$\begin{aligned}
\sum_{k=1}^{6} f(x_k^*)\Delta x &= f(7 + \frac{1}{2}) \cdot \frac{1}{2} + f(7+1) \cdot \frac{1}{2} + f(7 + \frac{3}{2}) \cdot \frac{1}{2} + f(7+2) \cdot \frac{1}{2} + f(7 + \frac{5}{2}) \cdot \frac{1}{2} + f(7+3) \cdot \frac{1}{2} \\
&= f(7.5) \cdot \frac{1}{2} + f(8) \cdot \frac{1}{2} + f(8.5) \cdot \frac{1}{2} + f(9) \cdot \frac{1}{2} + f(9.5) \cdot \frac{1}{2} + f(10) \cdot \frac{1}{2} \\
&= \frac{1}{2} \left(f(7.5) + f(8) + f(8.5) + f(9) + f(9.5) + f(10) \right) \\
&= \frac{1}{2} \big((7.5^2 - 9(7.5) + 18) + (8^2 - 9(8) + 18) + (8.5^2 - 9(8.5) + 18) + (9^2 - 9(9) + 18) + (9.5^2 - 9(9.5) + 18 \\
&= \frac{1}{2} \big((7.5^2 + 8^2 + 8.5^2 + 9^2 + 9.5^2 + 10^2) - 9(7.5 + 8 + 8.5 + 9 + 9.5 + 10) + 6(18) \big) \\
&= \frac{397}{8}
\end{aligned}$$

Or in sigma notation:

$$\begin{aligned}
\sum_{k=1}^{6} f(a+k\Delta x)\Delta x &= \sum_{k=1}^{6}\left(f\left(7+\frac{1}{2}k\right)\cdot\frac{1}{2}\right)\\
&= \frac{1}{2}\sum_{k=1}^{6}\left(\left(7+\frac{1}{2}k\right)^2-9\left(7+\frac{1}{2}k\right)+18\right)\\
&= \frac{1}{2}\sum_{k=1}^{6}\left(49+7k+\frac{1}{4}k^2-63-\frac{9}{2}k+18\right)\\
&= \frac{1}{2}\sum_{k=1}^{6}\left(4+\frac{5}{2}k+\frac{1}{4}k^2\right)\\
&= \frac{1}{2}\left(4\sum_{k=1}^{6}1+\frac{5}{2}\sum_{k=1}^{6}k+\frac{1}{4}\sum_{k=1}^{6}k^2\right)\\
&= \frac{1}{2}\left(4(6)+\frac{5}{2}\cdot\frac{(6)(7)}{2}+\frac{1}{4}\cdot\frac{(6)(7)(13)}{6}\right)\\
&= \frac{1}{2}\left(24+\frac{105}{2}+\frac{91}{4}\right)\\
&= \frac{1}{2}\cdot\frac{96+210+91}{4}=\frac{397}{8}
\end{aligned}$$