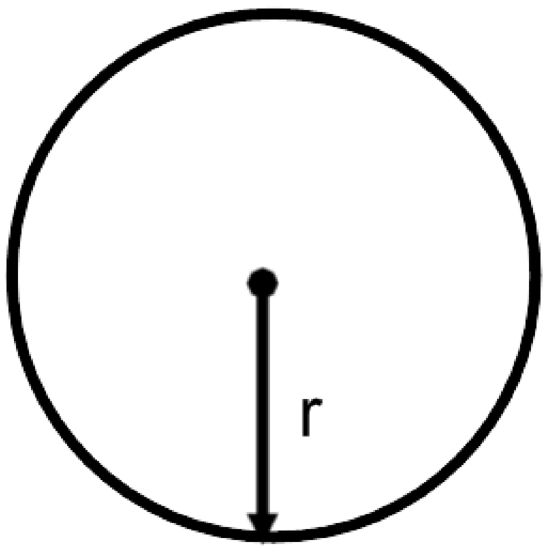
More Than One Rate- Solutions

Problem 1 *The radius of a circle is increasing at a rate of 2 inches per minute.*



(a) *At what rate is the circumference of the circle changing when the radius is 10 inches?*

Solution: *We know:* $\frac{dr}{dt} = 2$ *inches per minute and we want to find* $\frac{dc}{dt}$ *when* $r = 10$.

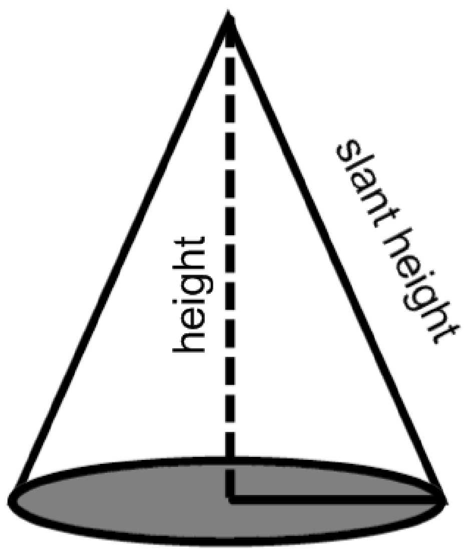
$$\begin{aligned} c &= 2\pi r \\ \frac{dc}{dt} &= 2\pi \frac{dr}{dt} \\ \left[\frac{dc}{dt}\right]_{r=10} &= 2\pi(2) \\ \left[\frac{dc}{dt}\right]_{r=10} &= 4\pi \textit{ inches per minute} \end{aligned}$$

(b) *At what rate is the area of the circle changing when the radius is 12 inches?*

Solution: *We know:* $\frac{dr}{dt} = 2$ *inches per minute and we want to find* $\frac{dA}{dt}$ *when* $r = 12$.

$$\begin{aligned} A &= \pi r^2 \\ \frac{dA}{dt} &= 2\pi r \frac{dr}{dt} \\ \left[\frac{dA}{dt}\right]_{r=12} &= 2\pi(12)(2) \\ \left[\frac{dA}{dt}\right]_{r=12} &= 48\pi \textit{ inches squared per minute} \end{aligned}$$

Problem 2 *A right cone has a fixed slant height (see figure below) of 9 ft. The cone's height is shrinking at a rate of 0.5 ft/sec. At what rate is the* ***volume*** *of the cone changing when the height is 6 ft? Be sure to label the picture.*

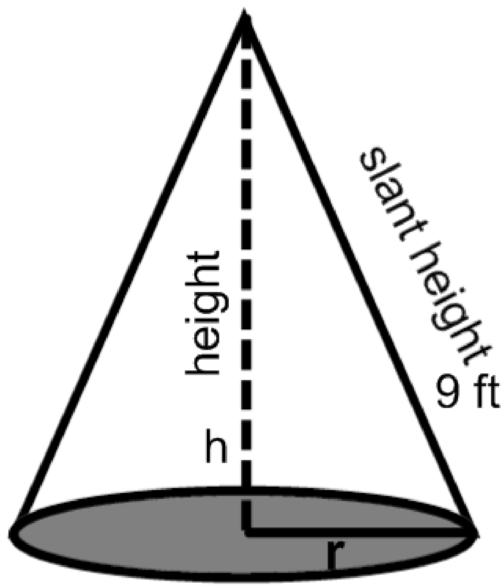


Solution: *We introduce the variables V, for the volume, r for the radius and h for the height of the cone.*

The given rate is $\frac{dh}{dt} = -0.5$ *ft/s;*

the rate to be determined is $\left[\frac{dV}{dt}\right]_{h=6}$.

We label the picture.



The equation relating all relevant variables is

$$V = \frac{1}{3}\pi r^2 \cdot h$$

Before we differentiate, we will express r^2 in terms of h.

$$r^2 = 81 - h^2$$

and we substitute this into the equation above

$$V = \frac{1}{3}\pi(81 - h^2) \cdot h$$

Then we simplify it.

$$V = \frac{1}{3}\pi(81h - h^3)$$

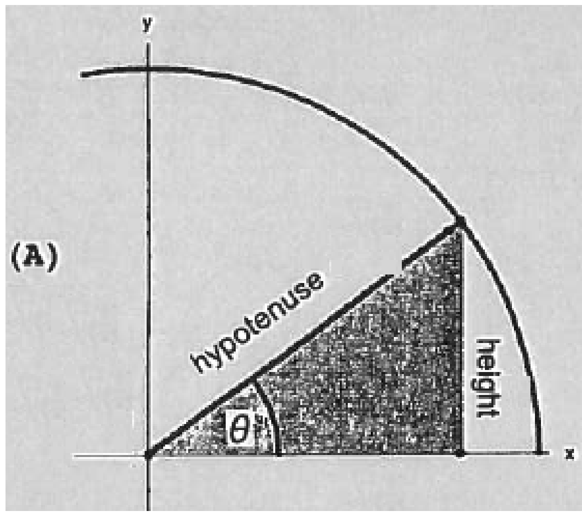
Now, we differentiate with respect to t.

$$\frac{dV}{dt} = \frac{\pi}{3}(81 - 3h^2) \cdot \frac{dh}{dt}$$

Now, we evaluate.

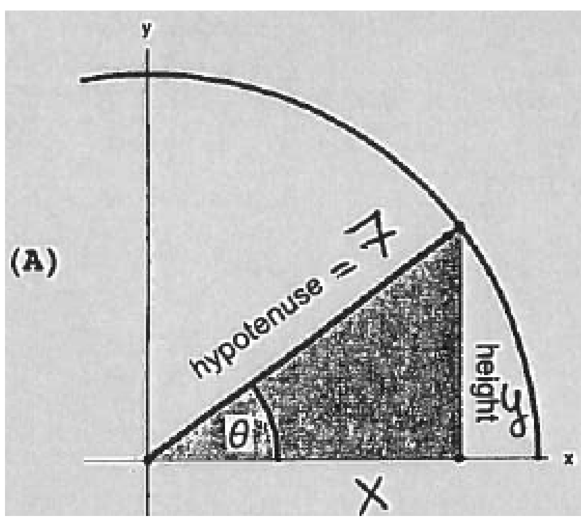
$$\left[\frac{dV}{dt}\right]_{h=6} = \frac{\pi}{3}(81 - 3(6)^2)(-0.5)$$
$$\left[\frac{dV}{dt}\right]_{h=6} = \frac{\pi}{3}(-27)(-0.5)$$
$$\left[\frac{dV}{dt}\right]_{h=6} = 4.5\pi \ \mathit{ft}^3/s$$

Problem 3 *A part of a circle centered at the origin with radius $r = 7$ cm is given in the figure (A) below. A right triangle is formed in the first quadrant (see Figure (A)). One of its sides lies on the x-axis. Its hypotenuse runs from the origin to a point on the circle. The hypotenuse makes an angle θ with the x-axis. Assume that the angle θ changes at the rate $\frac{d\theta}{dt} = 0.2$ radians per second.*

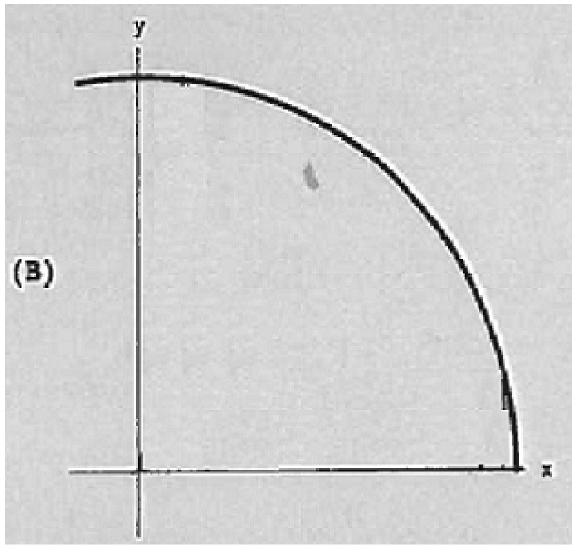


(a) *Label Figure A.*

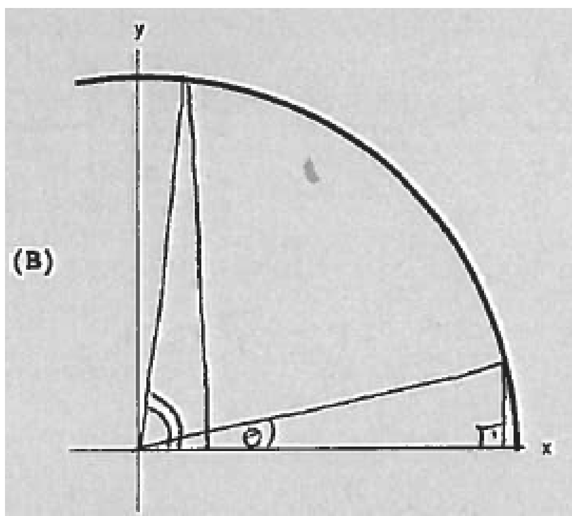
Solution:



(b) *In Figure B, draw the the triangle twice; once when θ is small and once more, when θ is close to $\frac{\pi}{2}$.*



Solution:



(c) *Find the rate of change of the height of the triangle when* $\theta = \dfrac{\pi}{3}$

Solution: $y = 7\sin\theta$

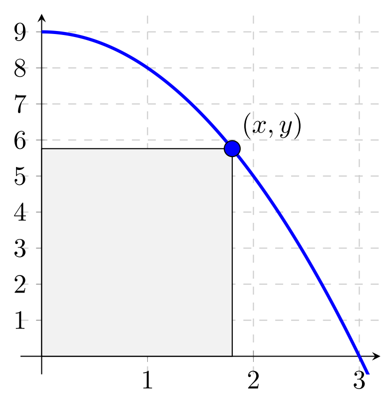
$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{d\theta}\cdot\frac{d\theta}{dt} \\
\frac{dy}{dt} &= 7\cos\theta\cdot\frac{d\theta}{dt} \\
\left[\frac{dy}{dt}\right]_{\theta=\frac{\pi}{3}} &= 7(1/2)(0.2) \\
&= 0.7\text{cm/sec}
\end{aligned}$$

(d) *Find the rate of change of the area of the triangle when* $\theta = \dfrac{\pi}{3}$

Solution: *First, we need to find x: $x = 7\cos\theta$. From part c, we know $y = 7\sin\theta$*

$$\begin{aligned}
A &= \frac{1}{2}xy \\
A &= \frac{49}{2}\cos(\theta)\sin(\theta) \\
\frac{dA}{dt} &= \frac{49}{2}(-\sin^2(\theta) + \cos^2(\theta)) \cdot \frac{d\theta}{dt} \\
\left[\frac{dA}{dt}\right]_{\theta=\frac{\pi}{3}} &= \frac{49}{2}\left(-\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right) \cdot 0.2 \\
&= \frac{49}{2} \cdot \frac{-1}{2} \cdot 0.2 \\
&= \frac{-4.9}{2} cm^2/sec
\end{aligned}$$

Problem 4 *A **rectangle in the first quadrant** is constructed by taking a point (x, y) on the graph of the function $f(x) = 9 - x^2$, drawing a line segment vertically downward to the x-axis and a line segment horizontally leftward to the y-axis, as in the picture below. Denote the **length of the base** (along the x-axis) of the rectangle by x, the **height of the rectangle** (along the y-axis) by y (in meters), and the **PERIMETER** of the rectangle by P. When $x = 2$ m (and only at that moment), the height y is shrinking at a rate of $\frac{1}{5}$ m/s. Find the value of $\left[\frac{dP}{dt}\right]_{x=2m}$ **by performing the steps below**.*



(a) ***Find a formula*** *for the height, y, of the rectangle as a function of x. (This is denoted as $y(x)$.)*

Solution: *The height of the rectangle is the y-coordinate of the point in the upper-right corner. Since this point lies on the graph of $f(x) = 9 - x^2$, we know $y = 9 - x^2$.*

(b) ***Find the value*** *of* $\left[\frac{dx}{dt}\right]_{x=2m}$.

Solution: *Since* $y = 9 - x^2$, *we know* $\frac{dy}{dt} = -2x\frac{dx}{dt}$. *Solving for* $\frac{dx}{dt}$ *gives:*

$$\begin{aligned}
\frac{dx}{dt} &= -\frac{\frac{dy}{dt}}{2x} \\
\left[\frac{dx}{dt}\right]_{x=2m} &= -\frac{\left[\frac{dy}{dt}\right]_{x=2m}}{2(2)} \\
&= -\frac{(-1/5)}{4} \\
&= \frac{1}{20}
\end{aligned}$$

(c) ***Find a formula*** *for the perimeter, P, of the rectangle as a function of x. (This is denoted as $P(x)$.)*

Solution:

$$\begin{aligned}
P(x) &= 2x + 2y \\
&= 2x + 2(9 - x^2) \\
&= -2x^2 + 2x + 18
\end{aligned}$$

(d) ***Find the value*** *of* $\left[\frac{dP}{dt}\right]_{x=2m}$.

Solution:

$$\begin{aligned}
P(x) &= -2x^2 + 2x + 18 \\
\frac{dP}{dt} &= (-4x + 2)\frac{dx}{dt}
\end{aligned}$$

$$\begin{aligned}
eval\frac{dP}{dt}_{x=2m} &= \left[(-4x + 2)\frac{dx}{dt}\right]_{x=2m} \\
&= (-4(2) + 2)\left(\frac{1}{20}\right) \\
&= -\frac{3}{10}
\end{aligned}$$

(e) ***Write a sentence to explain*** *what the value found in (d) means about the rectangle. (Don't forget UNITS.)*

Solution: *At the instant when $x = 2$ m, the perimeter of the rectangle is shrinking at the rate of $\frac{3}{10}$ m/s.*