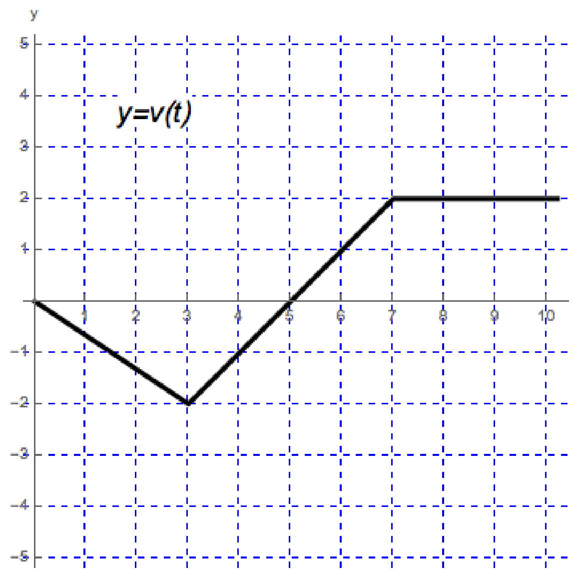
Antiderivatives and area (AAA) - Solutions

Problem 1 *The graph of the velocity (t in min, v in ft/min) of a particle moving along a straight line is given in the figure. Assume that the particle was at the origin initially.*



(a) *Evaluate the displacement of the particle over the following intervals.*

(i) $[0, 10]$

Solution:

$$s(10) - s(0) = \int_0^{10} v(t)\,dt = 3ft$$

(ii) $[0, 7]$

Solution:

$$s(7) - s(0) = -3ft$$

(iii) $[0, 5]$

Solution:

$$s(5) - s(0) = -5ft$$

(b) *When was the particle farthest from the origin?*

Solution: *We check the end points: $s(0) = 0$, $s(10) = s(0) + 3 = 3$, and critical points. Since the only critical point of s is at $t = 5$, since $s'(5) = v(5) = 0$ there, we evaluate $s(5) = s(0) - 5 = -5$. It follows that the particle was farthest from the origin at $t = 5$ min.*

(c) *What was the total distance the particle has travelled over the time interval $[0, 10]$?*

Solution: *The total distance travelled is given by the definite integral*

$$\int_0^{10} |v(t)|\,dt = \int_0^5 (-v(t))\,dt + \int_5^{10} v(t)\,dt = 5 + 8 = 13ft$$

Problem 2 *The velocity of an object moving along a straight line is given by $v(t)$ (in ft/s) and we only know the following: $\int_0^3 v(t)\,dt = -3$, and $\int_3^4 v(t)\,dt = 5$.*
Compute the following displacements, if possible. If it is not possible, give examples explaining why not.

(a) $s(3) - s(0)$

Solution:

$$s(3) - s(0) = \int_0^3 v(t)\,dt = -3ft$$

(b) $s(4) - s(0)$

Solution: *First notice that*

$$s(4) - s(0) = \int_0^4 v(t)\,dt$$

Therefore,

$$s(4) - s(0) = \int_0^3 v(t)\,dt + \int_3^4 v(t)\,dt = -3 + 5 = 2ft$$

(c) *Find the displacement during the interval $[0, 4]$ if the velocity at the time t, $0 \le t \le 4$, was $v(t) + 2$ ft/s instead?*

Solution: *In that case, we would have that*

$s(4) - s(0) = \int_0^4 (v(t) + 2)\,dt = \int_0^4 v(t)\,dt + \int_0^4 2\,dt$

We use the result in part (b) for the first integral, and geometry for the second interval.

$s(4) - s(0) = 2 + 2(4 - 0) = 2 + 8 = 10$ *ft*

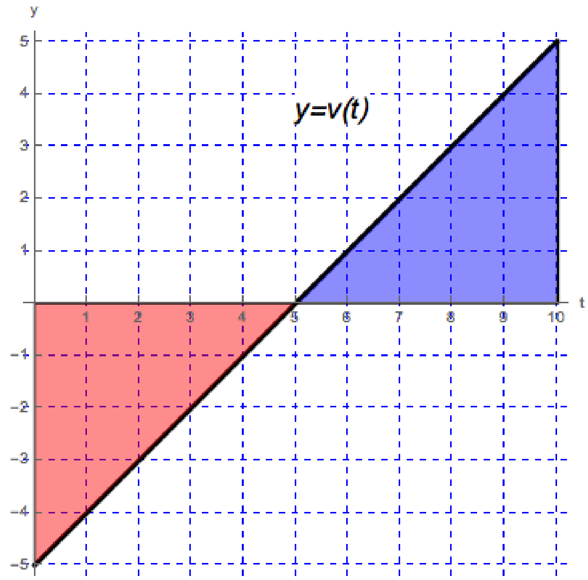
(d) *Find the displacement during the interval $[0, 4]$ if the velocity at the time t, $0 \le t \le 4$, was $5v(t)$ ft/s instead?*

Solution: $s(4) - s(0) = \int_0^4 5v(t)\,dt = 5\int_0^4 v(t)\,dt = 5(2) = 10$ *ft*

Problem 3 *The velocity of the object moving along a straight line is given by $v(t) = t - 5$, $0 \le t \le 10$, where t is in seconds and v in ft/s.*

(a) *Use geometry to evaluate the displacement of the object on the time interval $[0, 10]$. Sketch the graph of the velocity and shade the relevant region.*

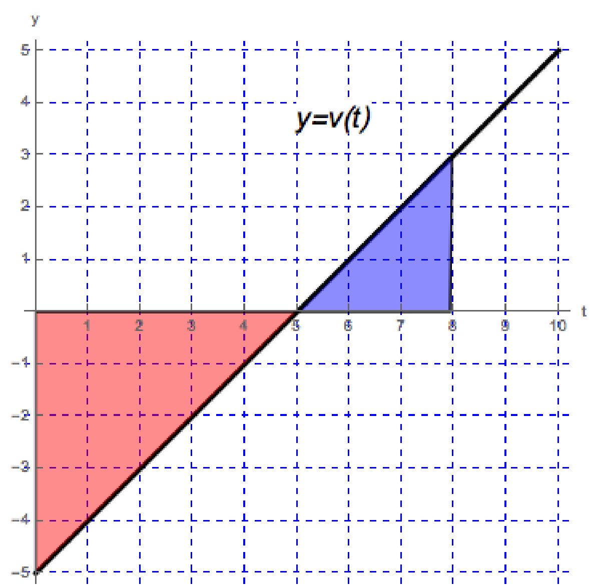
Solution: $s(10) - s(0) = \int_0^{10} (t - 5)\,dt =$ *area under the curve and the interval $[0, 10]$ on the $t-$axis.*
Therefore $s(10) - s(0) = 0$ ft.



(b) *Compute the displacement on the interval* $[0, 10]$ *using antiderivatives of the function* v.

Solution: *Since* $s(t) = \dfrac{t^2}{2} - 5t + s(0)$, *it follows that* $s(10) - s(0) = \dfrac{10^2}{2} - 5(10) + s(0) - s(0) = 50 - 50 = 0$ *ft.*

(c) *Use geometry to evaluate the displacement of the object on the time interval* $[0, 8]$. *Sketch the graph of the velocity and shade the relevant region.*



Solution:

$s(8) - s(0) = \displaystyle\int_0^8 (t-5)\, dt =$ *area under the curve and the interval* $[0, 8]$ *on the* $t-$*axis. Therefore* $s(8) - s(0) = -\dfrac{25}{2} + \dfrac{9}{2} = -8$ *ft.*

(d) *Compute the displacement on the interval* $[0, 8]$ *using antiderivatives of the function* v.

Solution: *Since* $s(t) = \frac{t^2}{2} - 5t + s(0)$, *it follows that*

$s(8) - s(0) = \frac{8^2}{2} - 5(8) + s(0) - s(0) = 32 - 40 = -8$ *ft.*

Problem 4 *The* ***velocity*** *function for a man walking along a straight road which runs east and west is given by* $v(t) = -t^2 + 4t - 3$ *ft/min.*

(a) *Set up a definite integral for the man's* ***displacement*** *during the time interval from 2 minutes to 6 minutes after he began running.*

Solution:

$$\int_2^6 v(t)\,dt = \lim_{n\to\infty} \sum_{k=1}^{n} v(t_k)\Delta t$$

Where:
$\Delta t = \frac{b-a}{n} = \frac{6-2}{n} = \frac{4}{n}$.
$t_k = a + k\Delta t = 2 + k\frac{4}{n} = 2 + \frac{4k}{n}$.

(b) ***At home:*** *Evaluate the definite integral using the limit of a right Riemann sum.*

Solution:

$$\begin{aligned} v(t_k) &= -\left(2 + \frac{4k}{n}\right)^2 + 4\left(2 + \frac{4k}{n}\right) - 3 \\ &= -\left(4 + \frac{16k}{n} + \frac{16k^2}{n^2}\right) + 8 + \frac{16k}{n} - 3 \\ &= 1 - \frac{16k^2}{n^2} \end{aligned}$$

So we compute:

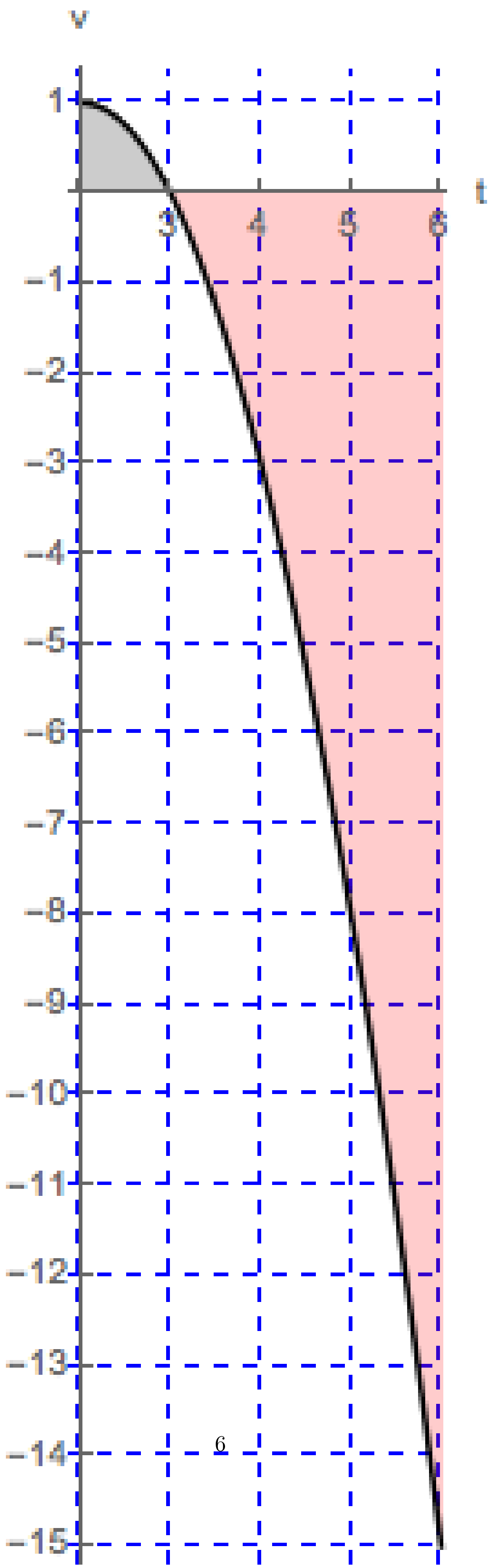
$$\begin{aligned} \int_2^6 v(t)\,dt &= \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(1 - \frac{16k^2}{n^2}\right)\left(\frac{4}{n}\right)\right] \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4}{n} - \frac{64k^2}{n^3}\right) \\ &= \lim_{n\to\infty} \left[\frac{4}{n}\sum_{k=1}^{n} 1 - \frac{64}{n^3}\sum_{k=1}^{n} k^2\right] \\ &= \lim_{n\to\infty} \left[\frac{4}{n}(n) - \frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right] \\ &= 4 - \frac{64}{3} = \frac{12-64}{3} = -\frac{52}{3}. \end{aligned}$$

(c) *Is this the same as the total* ***distance*** *the man walked from 2 minutes to 6 minutes? Why or why not?*

Solution: *This number is not the same as the total distance. The man starts his walk by going east (the positive direction) but eventually ends his walk west of where he started.*

The total distance that the man walks would be measured by computing

$$\int_2^6 |v(t)|\ dt$$


v
t
1
3
4
5
6
−1
−2
−3
−4
−5
−6
−7
−8
−9
−10
−11
−12
−13
−14
−15

$$\int_2^6 |v(t)|\ dt = \int_2^3 |v(t)|\ dt + \int_3^6 |v(t)|\ dt = \int_2^3 v(t)\,dt + \int_3^6 (-v(t))\,dt = \int_2^3 v(t)\,dt - \int_3^6 v(t)\,dt$$