Approximating the area under a curve (ATAUAC)

SUMMARY of Sigma Notation:

Useful formulas: (1) $\sum_{k=1}^{n} C = n \cdot C$; (2) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$; (3) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

Useful example of application of rules for sums:

$$\sum_{k=1}^{n}(ak+b) = \sum_{k=1}^{n} ak + \sum_{k=1}^{n} b = a\sum_{k=1}^{n} k + \sum_{k=1}^{n} b = a\frac{n(n+1)}{2} + nb, \text{ where a and b constants.}$$

SUMMARY of Riemann Sums:

Riemann sum: $\sum_{k=1}^{n} f(x_k^*)\Delta x$,

Right Riemann sum: $x_k^* = x_k$; Left Riemann sum: $x_k^* = x_{k-1}$; Midpoint Riemann sum: $x_k^* = \frac{x_{k-1}+x_k}{2}$.

Width of each of n rectangles on the interval $[a,b]$: $\Delta x = \frac{b-a}{n}$

Grid points for interval $[a,b]$:

$x_0 = a$,

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$x_{k-1} = a + (k-1)\Delta x = a + (k-1)\frac{b-a}{n}$,

$x_k = a + k\Delta x = a + k\frac{b-a}{n}$,

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$x_n = a + n\Delta x = a + n\frac{b-a}{n} = b$

Recitation Questions

Problem 1 *Evaluate the sum.*

(a) $\displaystyle\sum_{k=1}^{n} 5$

(b) $\displaystyle\sum_{k=1}^{10} 5$

(c) $\displaystyle\sum_{k=1}^{n} 8k$

(d) $\displaystyle\sum_{k=1}^{4} 8k$

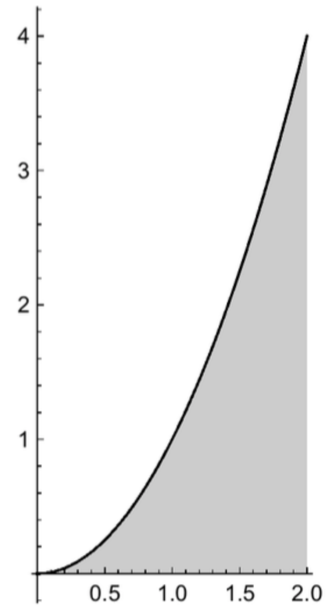
(e) $\displaystyle\sum_{k=1}^{n} (6k^2 - 8k - 5)$

(f) $\displaystyle\sum_{k=0}^{2} \cos\left(\frac{\pi}{2}k\right)$

Problem 2 (a) *If a function is positive and decreasing on an interval $[a, b]$, will a right Riemann sum underestimate or overestimate the area of the region under the graph of the function? Justify your answer.*

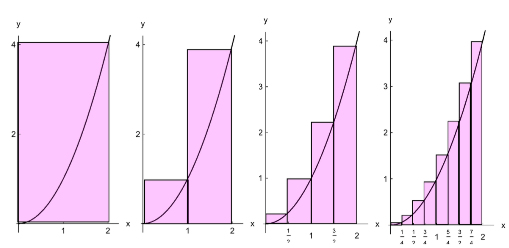
(b) *If a function is positive and decreasing on an interval $[a, b]$, will a left Riemann sum underestimate or overestimate the area of the region under the graph of the function? Justify your answer.*

Problem 3 *The graph of the function $f(x) = x^2$ is given in the figure.*



Will a right Riemann sum approximation, for any value of n, be an underestimate or overestimate?

Approximate the shaded area using a right Riemann sum with $n = 1, 2, 4$, and 8 rectangles, as illustrated in the figure below.



(a) *Approximate the shaded area using a right Riemann sum with $n = 1$ rectangles.*

(b) *Approximate the shaded area using a right Riemann sum with $n = 2$ rectangles.*

(c) *Approximate the shaded area using a right Riemann sum with $n = 4$ rectangles.*

(d) *Approximate the shaded area using a right Riemann sum with $n = 8$ rectangles.*

Problem 4 *A positive continuous function will have area approximated on the interval* $[1, 6]$ *using* n *rectangles.*

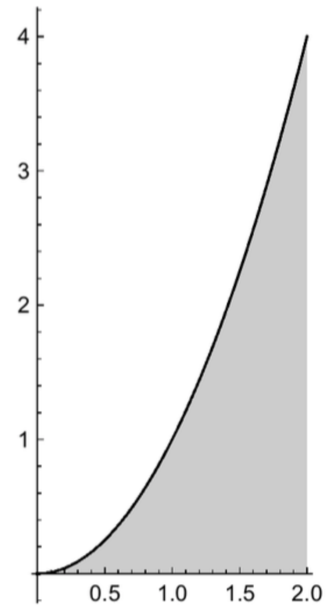
(a) *Find a formula for the grid point,* x_k.

(b) *Find a formula for the sample point* x_k^* *if using a right Riemann sum.*

(c) *Find a formula for the sample point* x_k^* *if using a left Riemann sum.*

(d) *Find a formula for the sample point* x_k^* *if using a midpoint Riemann sum.*

Problem 5 *The graph of the function $f(x) = x^2$ is given in the figure.*



Find the exact value of the area of the shaded region. HINT: Use a right Riemann sum with n rectangles, and then take the limit as $n \to \infty$.

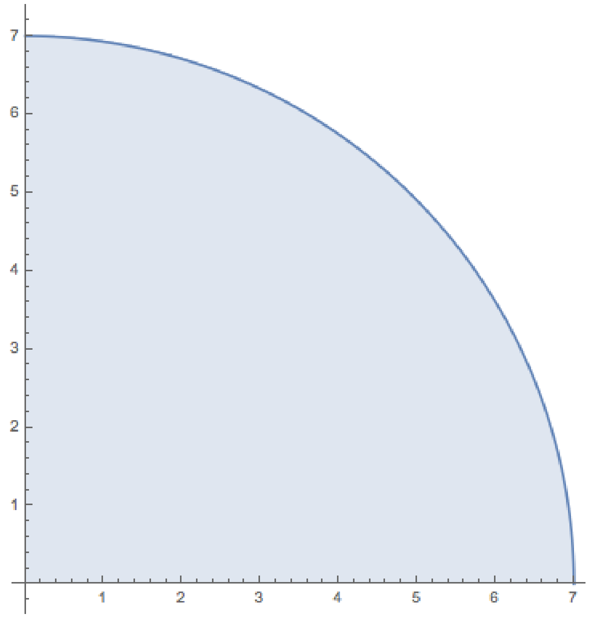
Problem 6 *Consider a Riemann sum with n rectangles for the function f on the interval $[a, b]$. Use geometry to find the limit of Riemann sums as $n \to \infty$.*

$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$?

(a) $f(x) = x$, $[0, 3]$;

(b) $f(x) = |x|$, $[-3, 3]$;

Problem 7 *A part of a circle is shown in the figure.*



(a) *If we express the area of the shaded region in the figure as the limit of Riemann sums*

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x,$$

find the function f and the interval $[a, b]$.

(b) *Compute the limit.*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{49 - (x_k^*)^2} \cdot \frac{7}{n}$$

Problem 8 *We want to approximate the area under the curve using a right Riemann Sum with the given value of n. Write the sum in summation notation and evaluate it.*

(a) $\sin(x), \quad \left[0, \frac{\pi}{2}\right], \quad n = 3$

(b) $f(x) = x^2 - 9x + 18, \quad [7, 10], \quad n = 6$