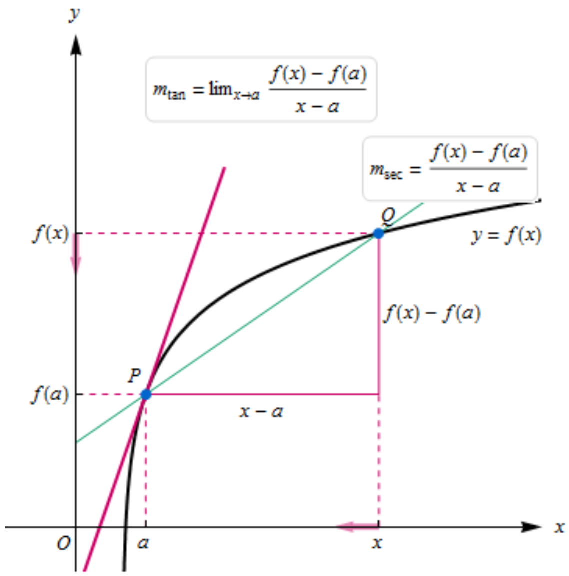
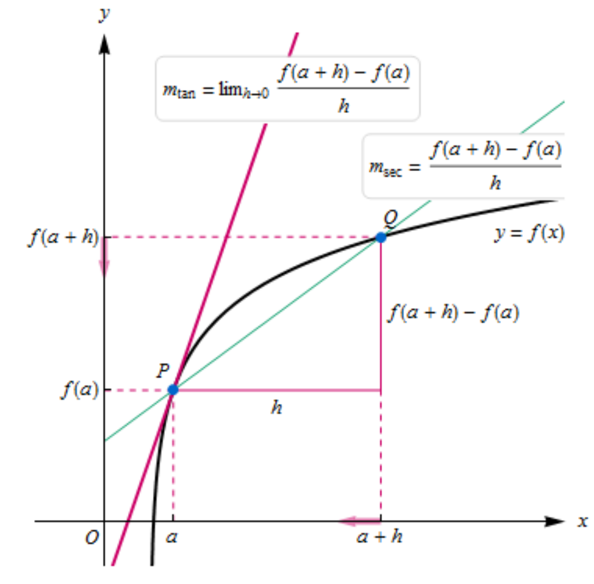
Definition of the derivative (DOTD) - Solutions

Problem 1 *Consider the following two figures depicting the same graph $y = f(x)$ and the same two lines, and the same two points P and Q:*





(a) *In Figure 1 on the left, what are the coordinates of P and Q? In Figure 2 on the right, what are the expressions for the coordinates of P and Q?*

Solution: *Figure 1 on the left: $P(a, f(a))$ and $Q(x, f(x))$*
Figure 2 on the right: $P(a, f(a))$ and $Q((a+h), f(a+h))$

(b) *Express the slope of the secant line through P and Q in terms of the above coordinates for each Figure 1 and Figure 2.*

Solution: *Figure 1:* $\dfrac{f(x) - f(a)}{x - a}$ *Figure 2:* $\dfrac{f(a+h) - f(a)}{h}$

(c) *Express the slope of the tangent line at the point P in terms of the above coordinates for each Figure 1 and Figure 2.*

Solution: *Figure 1:* $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ *Figure 2:* $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

(d) *What is the difference?*

Solution: *Just the variable we use, x or h where $x = a + h \iff h = x - a$*

(e) *For each of the two graphs, which lines are the secant lines?*

Solution: *The green line is the secant line in each of the two graphs.*

(f) *For each of the two graphs, which lines are the tangent lines?*

Solution: *The red line is the tangent line in each of the two graphs.*

Problem 2 *For each of the following functions find an equation of the tangent line at the given point.*

(a) $f(x) = -5x^2 + 7x - 9$ at $x = 3$.

Solution: *The slope of tangent line is given by this limit:* $f'(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3}$. *Notice that this limit has form* $\frac{0}{0}$, *so it is indeterminate.*

$$\begin{aligned}
f'(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} &= \lim_{x\to 3} \frac{(-5x^2 + 7x - 9) - (-45 + 21 - 9)}{x-3} \\
&= \lim_{x\to 3} \frac{-5x^2 + 7x - 9 + 33}{x-3} \\
&= \lim_{x\to 3} \frac{-5x^2 + 7x + 24}{x-3} \\
&= \lim_{x\to 3} \frac{(x-3)(-5x-8)}{x-3} \\
&= \lim_{x\to 3} (-5x-8) \\
&= -5(3) - 8 = -23.
\end{aligned}$$

Point on tangent line: $(3, f(3)) = (3, -33)$.
An equation of tangent line is:

$$y + 33 = -23(x - 3)$$

(b) $g(u) = \sqrt{5u - 4}$ at $u = 3$.

Solution: *The slope of tangent line is given by:* $g'(3) = \lim_{h\to 0} \frac{g(3+h) - g(3)}{h}$. *Notice that this limit* *has form* $\dfrac{0}{0}$*, so it is indeterminate.*

$$\begin{aligned}
g'(3) = \lim_{h\to 0}\frac{g(3+h)-g(3)}{h} &= \lim_{h\to 0}\frac{\sqrt{5(3+h)-4}-\sqrt{11}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{5(3+h)-4}-\sqrt{11}}{h}\cdot\frac{\sqrt{5(3+h)-4}+\sqrt{11}}{\sqrt{5(3+h)-4}+\sqrt{11}}\\
&= \lim_{h\to 0}\frac{5(3+h)-4-11}{h\left(\sqrt{5(3+h)-4}+\sqrt{11}\right)}\\
&= \lim_{h\to 0}\frac{5h+15-15}{h\left(\sqrt{5(3+h)-4}+\sqrt{11}\right)}\\
&= \lim_{h\to 0}\frac{5}{\left(\sqrt{5(3+h)-4}+\sqrt{11}\right)}\\
&= \frac{5}{\sqrt{5(3+0)-4}+\sqrt{11}}\\
&= \frac{5}{2\sqrt{11}}.
\end{aligned}$$

Point on tangent line: $(3, g(3)) = (3, \sqrt{11})$.

Equation of tangent line:

$$y-\sqrt{11} = \frac{5}{2\sqrt{11}}\,(u-3)$$

(c) $s(z) = \dfrac{z}{z-5}$ *at* $z=3$.

Solution: *The slope of tangent line is given by:* $s'(3) = \lim_{z\to 3}\dfrac{s(z)-s(3)}{z-3}$*. Notice that this limit has* *form* $\dfrac{0}{0}$*, so it is indeterminate.*

$$\begin{aligned}
s'(3) &= \lim_{z\to 3} \frac{s(z)-s(3)}{z-3}\\
&= \lim_{z\to 3} \frac{\frac{z}{z-5}-\frac{3}{3-5}}{z-3}\\
&= \lim_{z\to 3} \frac{\frac{z}{z-5}+\frac{3}{2}}{z-3}\\
&= \lim_{z\to 3} \frac{\frac{2z}{2(z-5)}+\frac{3(z-5)}{2(z-5)}}{z-3}\\
&= \lim_{z\to 3} \frac{\frac{2z+3z-15}{2(z-5)}}{z-3}\\
&= \lim_{z\to 3} \frac{5z-15}{2(z-5)}\cdot\frac{1}{z-3}\\
&= \lim_{z\to 3} \frac{5(z-3)}{2(z-5)}\cdot\frac{1}{z-3}\\
&= \lim_{z\to 3} \frac{5}{2(z-5)}\\
&= \frac{5}{2(3-5)} = -\frac{5}{4}.
\end{aligned}$$

Point on tangent line: $(3, s(3)) = (3, -3/2)$.

Equation of tangent line:

$$y+\frac{3}{2} = -\frac{5}{4}(z-3)$$

Problem 3 *Find an equation of the tangent line at the given point. Then graph the function and the tangent line on the same plot.* $f(x)=\sqrt{x+1}$ *at* $x=3$

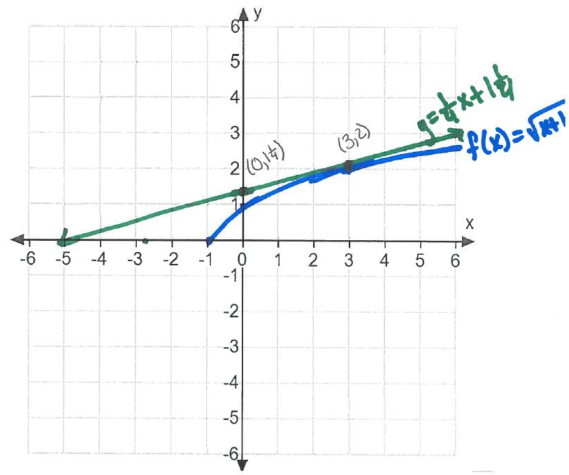
Solution: *The slope of tangent line is given by:* $f'(3)=\lim_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$. *This limit has form* $\dfrac{0}{0}$*, so it is an indeterminate form.*

$$\begin{aligned}
f'(3) = \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} &= \lim_{h\to 0}\frac{\sqrt{3+h+1}-\sqrt{4}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h}\cdot\frac{\sqrt{4+h}+\sqrt{4}}{\sqrt{4+h}+\sqrt{4}}\\
&= \lim_{h\to 0}\frac{4+h-4}{h(\sqrt{4+h}+\sqrt{4})}\\
&= \lim_{h\to 0}\frac{h}{h(\sqrt{4+h}+2)}\\
&= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}\\
&= \frac{1}{\sqrt{4+0}+2}\\
&= \frac{1}{4}
\end{aligned}$$

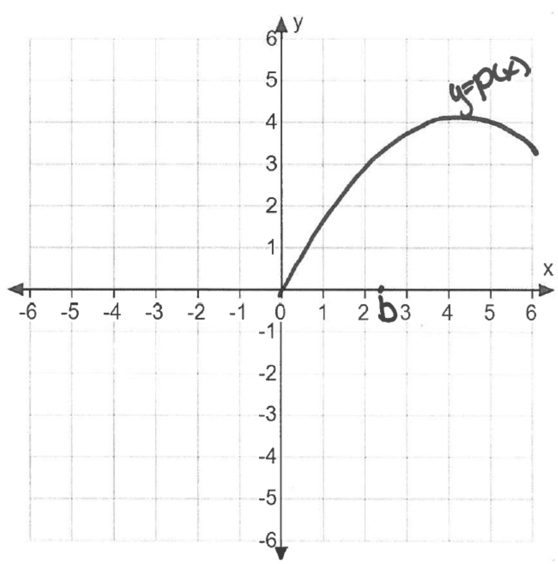
Point on tangent line: $(3, f(3)) = (3, 2)$.

Equation of tangent line:

$$y - f(3) = \frac{1}{4}(x-3) \implies y - 2 = \frac{1}{4}x - \frac{3}{4}$$
$$\implies y = \frac{1}{4}x + 1\frac{1}{4}.$$

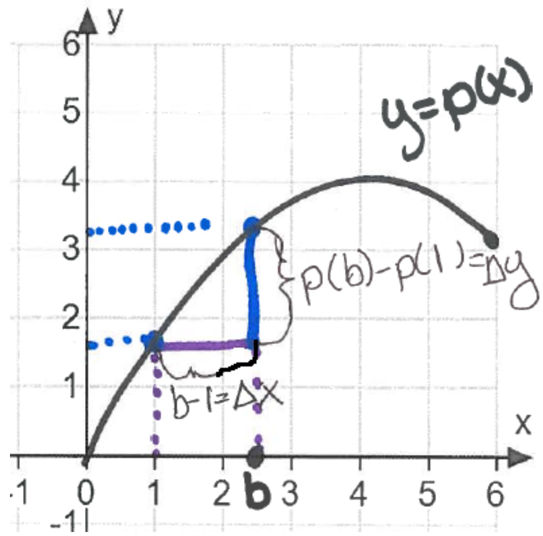


Problem 4 *The graph of a function p and a point in its domain, b, are shown in the figure below.*



(a) *In the figure above, draw and mark clearly the quantity $\Delta y = p(b) - p(1)$ and the quantity $\Delta x = b - 1$.*

Solution:



(b) *Complete the sentence. The quotient* $\dfrac{p(b)-p(1)}{b-1}$ *is the slope...*

Solution: *The quotient* $\dfrac{p(b)-p(1)}{b-1}$ *is the slope of the secant line through the points* $(b, p(b))$ *and* $(1, p(1))$

(c) *Complete the sentence. Provided it exists, the limit* $\lim\limits_{x\to 1} \dfrac{p(x)-p(1)}{x-1}$ *is the slope...*

Solution: *Provided it exists, the limit* $\lim\limits_{x\to 1} \dfrac{p(x)-p(1)}{x-1}$ *is the slope of the tangent line to the curve* $y = p(x)$ *at the point* $(1, p(1))$

Problem 5 *An object moving along a straight line has a position given by* $s(t) = \dfrac{1}{t-4}$, *where* s *is measured in meters and* t *in seconds. Find the velocity of the object at time* $t = 6$.

Solution: $v(6) = s'(6) = \lim\limits_{t\to 6} \dfrac{s(t)-s(6)}{t-6}$. *When we check the form of this limit, we see that it has form* $\dfrac{0}{0}$. $s'(6) = \lim\limits_{t\to 6} \dfrac{\frac{1}{t-4} - \frac{1}{2}}{t-6} = \lim\limits_{t\to 6} \dfrac{\frac{2-(t-4)}{2(t-4)}}{t-6} = \lim\limits_{t\to 6} \dfrac{(2-t+4)}{2(t-4)(t-6)} =$

$= \lim\limits_{t\to 6} \dfrac{(6-t)}{2(t-4)(t-6)} = \lim\limits_{t\to 6} \dfrac{-1}{2(t-4)} = \dfrac{-1}{2(6-4)} = \dfrac{-1}{4}$ *m/s.*
