Definite integrals (DI)

SUMMARY of Definite Integrals:

Definition
If function f is continuous on interval $[a,b]$ then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Note 1: When $f(x) \geq 0$, on the interval $[a,b]$, the definite integral, $\displaystyle\int_a^b f(x)\,dx$, gives the **area** of the region between the graph of f and the interval $[a,b]$ on the $x-$axis.

Note 2: When $f(x) < 0$, on some interval in $[a,b]$, then the definite integral, $\displaystyle\int_a^b f(x)\,dx$, gives the **net area** of the region between the graph of f and the interval $[a,b]$ on the $x-$axis.

Properties of Definite Integrals

(a) $\displaystyle\int_a^a f(x)\,dx = 0$

(b) $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

(c) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(d) $\displaystyle\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$

(e) $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;\ \int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Note 3: If the function f is **odd**, i.e, if $f(-x) = -f(x)$, for all x in $[-a,a]$, then

$$\int_{-a}^{a} f(x)\,dx = 0$$

Note 4: If the function f is **even**, i.e, if $f(-x) = f(x)$, for all x in $[-a,a]$, then

$$\int_{-a}^{a} f(x)\,dx = 2\cdot\int_0^a f(x)\,dx$$

Recitation Questions

Problem 1 *Consider the following limit of Riemann sums of a function g on $[a, b]$:*

$$\lim_{n\to\infty} \sum_{k=1}^{n} (x_k^* + \cos(x_k^*))\Delta x \ , \ [0, \pi].$$

Express the limit as a definite integral. Use geometry to evaluate the resulting definite integral.

Problem 2 *Let $f(x)$ and $g(x)$ be functions for which we only know the following:*

$$\int_1^4 f(x)\,dx = 7 \qquad \int_2^4 f(x)\,dx = 5 \qquad \int_1^4 g(x)\,dx = 2$$

Compute the following integrals, if possible. If it is not possible, give examples explaining why not.

(a) $\displaystyle\int_1^4 (8f(x) - 7g(x))\,dx$

(b) $\displaystyle\int_1^2 (-f(x))\,dx$

(c) $\displaystyle\int_1^4 |f(x)|\,dx$

(d) $\displaystyle\int_1^4 (2 - x + f(x))\,dx$

Problem 3 *Evaluate the following sums:*

(a) $\sum_{k=1}^{4} k^5$

(b) $\sum_{k=1}^{400} (5(k+1)^2 + 3)$

Problem 4 *Use geometry to evaluate the definite integral. Sketch the graph of the function and shade the relevant regions.*

(a) $\displaystyle\int_1^3 (2x-4)\,dx$

(b) $\displaystyle\int_1^3 |2x-4|\,dx$

(c) $\displaystyle\int_0^1 (2x-4)\,dx$

(d) $\displaystyle\int_{-1}^3 \sqrt{4-(x-1)^2}\,dx$

Problem 5 (a) *If f is an odd function, why is it true that* $\int_{-a}^{a} f(x)\,dx = 0$*? Support your reasoning with a picture.*

(b) *If f is an even function, why is it true that* $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$*? Support your reasoning with a picture.*

Problem 6 (a) *Find the following definite integral:*

$$\int_{-4}^{4} \frac{x^2 \sin^3(x)}{\sqrt{x^4+1}}\, dx$$

(b) *Suppose that f is an even function. Given that* $\displaystyle\int_0^6 f(x)\, dx = 13$, *find* $\displaystyle\int_{-6}^{6} (5f(x)+14)\, dx$.

Problem 7 *Evaluate the following integrals using symmetry arguments.*

(a) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin(t)\, dt$

(b) $\displaystyle\int_{-2}^{2} (1 + x + 3x^7 - x^9)\, dx$

(c) $\displaystyle\int_{-\pi}^{\pi} x\cos(x)\, dx$