Logarithmic differentiation (LD) - Solutions

Problem 1 (a) *Write as an exponential with base 5.* 7^{3x}.

Solution: $7 = 5^{\log_5(7)}$, *so*

$$7^{3x} = \left(5^{\log_5(7)}\right)^{3x} = 5^{3x\log_5(7)}$$

(b) *Write in terms of the natural logarithm.* $\log_3(4)$.

Solution: *The change of base formula says* $\log_a(u) = \dfrac{\log_b(u)}{\log_b(a)}$, *so*

$$\log_3(4) = \frac{\ln(4)}{\ln(3)}$$

(c) *Expand the following:* $\log_{1/2}\left(\dfrac{6x^5(2+\tan(x))^x}{\sqrt[5]{e^{4x}+1}}\right)$.

Solution:

$$\log_{1/2}\left(\frac{6x^5(2+\tan(x))^x}{\sqrt[5]{e^{4x}+1}}\right) = \log_{1/2}(6) + 5\log_{1/2}(x) + x\log_{1/2}(2+\tan(x)) - \frac{1}{5}\log_{1/2}(e^{4x}+1)$$

Problem 2 *Find all real numbers x which satisfy each of the following equations.*

(a) $\log_x(25) = 2$.

Solution: *Recall that, by definition of the inverse to an exponential function,*

$$\log_b(x) = y \iff x = b^y.$$

Using this relationship we have $\log_x(25) = 2 \iff x^2 = 25$. *Therefore*

$$\begin{aligned} x^2 = 25 &\implies x = \pm 5, \\ &\implies x = 5 \quad \textit{(base of log is always} > 0\textit{)}. \end{aligned}$$

Therefore $x = 5$ is the only solution to $\log_x(25) = 2$.

(b) $7^x = 15$

Solution: *Similar to the previous problem,*

$$\log_b(x) = y \iff x = b^y.$$

Using this relationship we have $7^x = 15 \iff x = \log_7(15)$. *Therefore $x = \log_7(15)$ is the only solution to $7^x = 15$.*

(c) $\ln(x) + 1 = 0$.

Solution: *Similar to the previous two problems we'll use the relationship*

$$\ln(x) = y \iff x = e^y.$$

Before applying this relationship we perform a bit of algebra first:

$$\ln(x) + 1 = 0 \implies \ln(x) = -1.$$

Now we have $\ln(x) = -1 \iff x = e^{-1}$. *Therefore* $x = e^{-1} (= 1/e)$ *is the only solution to* $\ln(x) + 1 = 0$.

Problem 3 *True or False:*

(1) *If* $f(x) = (x-2)^x$, *then* $f'(x) = x(x-2)^{x-1}$.

Solution: *False. Any time that you have a function of* x *raised to a function of* x, *in order to compute the derivative you need to use logarithmic differentiation (or something equivalent).*

Correct derivative of f:

$$\begin{aligned} f(x) = (x-2)^x &\implies f(x) = e^{x\ln(x-2)} \\ &\implies f'(x) = e^{x\ln(x-2)} \cdot \left(1 \cdot \ln(x-2) + x \cdot \frac{1}{x-2}\right) \\ &\implies f'(x) = (x-2)^x \cdot \left(\ln(x-2) + \frac{x}{x-2}\right) \end{aligned}$$

(2) *If* $f(x) = (3x)^x$, *then* $f'(x) = (3x)^x \ln(3x)$.

Solution: *False. Same as part (1).*

Correct derivative of f:

$$\begin{aligned} f(x) = (3x)^x &\implies f(x) = e^{x\cdot\ln(3x)} \\ &\implies f'(x) = e^{x\cdot\ln(3x)} \left(1 \cdot \ln(3x) + x \cdot \frac{3}{3x}\right) \\ &\implies f'(x) = (3x)^x \left(\ln(3x) + 1\right) \end{aligned}$$

Problem 4 *Find the derivatives of the following functions:*

(a) $f(x) = x^{e^x} + 7x$

Solution: $f'(x) = \frac{d}{dx}\left(x^{e^x}\right) + \frac{d}{dx}(7x) = \frac{d}{dx}\left(x^{e^x}\right) + 7$. *So the real problem is to find* $\frac{d}{dx}\left(x^{e^x}\right)$.

We will use logarithmic differentiation.

First, we take natural logarithm of both sides of the equation $g(x) = x^{e^x}$.

$\ln(g(x)) = e^x \ln(x)$.

Now, we differentiate both sides.

$\dfrac{g'(x)}{g(x)} = e^x \ln(x) + \dfrac{e^x}{x}.$

Next, we solve for $g'(x)$.

$g'(x) = g(x)\Big(e^x \ln(x) + \dfrac{e^x}{x}\Big).$

Now, we substitute.

$g'(x) = x^{e^x}\left(e^x \ln(x) + \dfrac{e^x}{x}\right).$

ALTERNATIVE APPROACH

$$\begin{aligned}
\frac{d}{dx}\left(x^{e^x}\right) &= \frac{d}{dx}\left(e^{\ln(x)^{e^x}}\right) \\
&= \frac{d}{dx}\left(e^{e^x \ln(x)}\right) \\
&= e^{e^x \ln(x)}\left(e^x \ln(x) + \frac{e^x}{x}\right) \\
&= x^{e^x}\left(e^x \ln(x) + \frac{e^x}{x}\right).
\end{aligned}$$

Thus, $f'(x) = x^{e^x}\left(e^x \ln(x) + \dfrac{e^x}{x}\right) + 7.$

(b) $g(x) = (\ln(x) + 9)^{\sec(x^4)}$

Solution: *We will use logarithmic differentiation.*

First, we take natural logarithm .

$\ln(g(x)) = \sec(x^4)\ln(\ln(x) + 9)$

Then, we differentiate.

$\dfrac{g'(x)}{g(x)} = \sec(x^4)\tan(x^4)4x^3 \ln(\ln(x) + 9) + \dfrac{\sec(x^4)}{x(\ln(x) + 9)}$

We multiply by $g(x)$.

$g'(x) = g(x)\Big(4x^3 \sec(x^4)\tan(x^4)\ln(\ln(x) + 9) + \dfrac{\sec(x^4)}{x(\ln(x) + 9)}\Big)$

We substitute.

$g'(x) = (\ln(x) + 9)^{\sec(x^4)}\Big(4x^3 \sec(x^4)\tan(x^4)\ln(\ln(x) + 9) + \dfrac{\sec(x^4)}{x(\ln(x) + 9)}\Big)$

ALTERNATIVE APPROACH

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}\left((\ln(x)+9)^{\sec(x^4)}\right) \\
&= \frac{d}{dx}\left(e^{\sec(x^4)\ln(\ln(x)+9)}\right) \\
&= e^{\sec(x^4)\ln(\ln(x)+9)}\left(4x^3\sec(x^4)\tan(x^4)\ln(\ln(x)+9)+\sec(x^4)\frac{\frac{1}{x}}{\ln(x)+9}\right) \\
&= (\ln(x)+9)^{\sec(x^4)}\left(4x^3\sec(x^4)\tan(x^4)\ln(\ln(x)+9)+\frac{\sec(x^4)}{x(\ln(x)+9)}\right).
\end{aligned}
$$

(c) $f(x) = \dfrac{(x+1)^5(\sin(x)+5)^4}{(x^2+5)\sqrt{x-3}}$

Solution:

$$
\begin{aligned}
\ln(f(x)) &= \ln(x+1)^5 + \ln(\sin(x)+5)^4 - \ln(x^2+5) - \ln\sqrt{x-3} \\
&= 5\ln(x+1) + 4\ln(\sin(x)+5) - \ln(x^2+5) - \frac{1}{2}\ln(x-3)
\end{aligned}
$$

Now we differentiate.

$\dfrac{f'(x)}{f(x)} = \dfrac{5}{x+1} + \dfrac{4\cos(x)}{\sin(x)+5} - \dfrac{2x}{x^2+5} - \dfrac{1}{2(x-3)}$

Then we solve for $f'(x)$.

$f'(x) = f(x)\Big(\dfrac{5}{x+1} + \dfrac{4\cos(x)}{\sin(x)+5} - \dfrac{2x}{x^2+5} - \dfrac{1}{2(x-3)}\Big)$ *Then we substitute.*

$f'(x) = \dfrac{(x+1)^5(\sin(x)+5)^4}{(x^2+5)\sqrt{x-3}}\Big(\dfrac{5}{x+1} + \dfrac{4\cos(x)}{\sin(x)+5} - \dfrac{2x}{x^2+5} - \dfrac{1}{2(x-3)}\Big)$

(d) $h(x) = \dfrac{(x^2-7)^5}{\cos^7(x^2-5)}$

Solution: *Since $h(x)$ changes sign on its domain, we write :*

$$
\begin{aligned}
\ln|h(x)| &= \ln\left(\frac{|x^2-7|^5}{|cos^7(x^2-5)|}\right) \\
&= 5\cdot\ln(|x^2-7|) - 7\cdot\ln(|\cos(x^2-5)|)
\end{aligned}
$$

Differentiate both sides with respect to x, (explanation given on the next page):

$\dfrac{h'(x)}{h(x)} = 5\dfrac{2x}{x^2-7} - 7\dfrac{-\sin(x^2-5)2x}{\cos(x^2-5)}$

$h'(x) = h(x)\big(5\dfrac{2x}{x^2-7} - 7\dfrac{-\sin(x^2-5)2x}{\cos(x^2-5)}\big) =$

$= \dfrac{(x^2-7)^5}{\cos^7(x^2-5)}\big(\dfrac{10x}{x^2-7} + \dfrac{14x\sin(x^2-5)}{\cos(x^2-5)}\big) =$

$$= \frac{(x^2-7)^4}{\cos^7(x^2-5)}\big(10x + 14x(x^2-7)\tan(x^2-5)\big)$$

EXPLANATION: The function $h(x)$ *assumes negative values on some intervals. On those intervals*

$$\big(\ln(|h(x)|)\big)' = \big(\ln(-h(x))\big)' = \frac{-h'(x)}{-h(x)} = \frac{h'(x)}{h(x)},$$

Also, e.g., if $\quad x^2 - 7 < 0$

$$\big(\ln(|x^2-7|)\big)' = \big(\ln(-(x^2-7))\big)' = \frac{-2x}{-(x^2-7)} = \frac{2x}{x^2-7}\ .$$