Chain Rule (CR) - Solutions

Problem 1 *For the following problems, the derivative is given. Determine which function was the original function.*

(a) *The derivative is* $f'(x) = \cos(x)e^{\sin(x)}$. *Which is the original function?*

(i) $f(x) = (\sin(x))(e^x)$
(ii) $f(x) = \sin(e^x)$
(iii) $f(x) = e^{\sin(x)}$
(iv) $f(x) = e^{x\sin(x)}$

Solution: *Since*

$$\frac{d}{dx}\left(e^{\sin(x)}\right) = e^{\sin(x)} \cdot \cos(x) = \cos(x)e^{\sin(x)}$$

the correct answer is (iii).

(b) *The derivative is* $g'(x) = 4\left(\tan(x^4 - 5x)\right)^3 \sec^2(x^4 - 5x)(4x^3 - 5)$. *Which is the original function?*

(i) $g(x) = (\tan(x) - 5x)^4$
(ii) $g(x) = \tan^4(x) - 5x^4$
(iii) $g(x) = \tan(x^4 - 5x)$
(iv) $g(x) = \tan^4(x^4 - 5x)$

Solution: *Since*

$$\frac{d}{dx}\left(\tan^4(x^4 - 5x)\right) = 4\tan^3(x^4 - 5x)\sec^2(x^4 - 5x)(4x^3 - 5)$$

the correct answer is (iv). Notice that $\tan^3(x^4 - 5x) = \left(\tan(x^4 - 5x)\right)^3$ *are (slightly) different notations for the exact same expression.*

Problem 2 *A table of values for* $f(x)$ *and* $f'(x)$ *is shown below:*

x	$f(x)$	$f'(x)$
1	3	4
2	2	3
3	4	5
4	6	3

- *Evaluate the limit* $\lim\limits_{x\to 2} \dfrac{f(x^2) - 6}{x - 2}$. ***EXPLAIN***.

Solution: *If we set* $g(x) = f(x^2)$*, this limit is* $\lim_{x\to 2} \frac{g(x)-g(2)}{x-2}$*. In other words, we're being asked to evaluate* $\frac{d}{dx} f(x^2)$ *at* $x = 2$*. By the Chain Rule,* $\frac{d}{dx} f(x^2) = 2xf'(x^2)$*. When evaluated at* $x = 2$ *this is* $2(2)f'(4) = 4(3) = 12$*. The value of this limit is* 12*.*

Our full explanation is then: $\lim_{x\to 2} \frac{f(x^2)}{x-2} = \left[\frac{d}{dx} f(x^2)\right]_{x=2}$ *by the limit definition of the derivative. Our calculation above shows that this equals* 12*.*

- *Evaluate* $\frac{d}{dx} f(f(x))$ *at* $x = 3$*.*

  (a) *6*

  (b) *25*

  (c) *5*

  (d) *15*

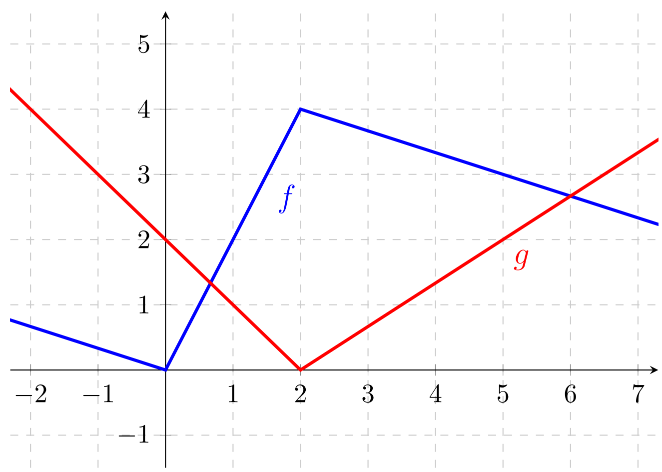
  (e) *DNE*

  (f) *None of the previous answers.*

  Solution: *The answer is (d):*

$$\begin{aligned}\left[\frac{d}{dx} f(f(x))\right]_{x=3} &= f'(f(3)) \cdot f'(3) \\ &= f'(4) \cdot 5 \\ &= 3 \cdot 5 = 15\end{aligned}$$

Problem 3 *Given the following graphs of* f *and* g *(both piecewise linear functions), define new functions* $u(x) = f(g(x))$ *and* $v(x) = f(x)g(x)$*. Find:*



(a) $u'(1)$

Solution: $u'(x) = \frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$. *So,*

$$\begin{aligned} u'(1) &= f'(g(1)) \cdot g'(1) \\ &= f'(1) \cdot (-1) \\ &= (2)(-1) = -2 \end{aligned}$$

(b) $v'(1)$

Solution: $v'(x) = \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$. *So,*

$$\begin{aligned} v'(1) &= f'(1)g(1) + f(1)g'(1) \\ &= (2)(1) + (2)(-1) \\ &= 0 \end{aligned}$$

(c) $\lim_{x \to 1} \frac{\sqrt{g(x)} - 1}{x - 1}$

Solution: *This is asking for* $\left[\frac{d}{dx}(\sqrt{g(x)})\right]_{x=1}$. *By the Chain Rule,* $\frac{d}{dx}(\sqrt{g(x)}) = \frac{g'(x)}{2\sqrt{g(x)}}$ *so*

$$\begin{aligned} \left[\frac{d}{dx}(\sqrt{g(x)})\right]_{x=1} &= \frac{g'(1)}{2\sqrt{g(1)}} \\ &= \frac{-1}{2\sqrt{1}} \\ &= -\frac{1}{2}. \end{aligned}$$

Problem 4 *Suppose the line tangent to the graph of f at $x = 1$ is $y = 6x - 7$. Find an equation of the line tangent to the following curves at $x = 1$:*

(a) $y = g(x) = 5(f(x))^4$

Solution: *First, in the equation of the tangent line to $f(x)$ at $x = 1$, when $x = 1$ we have that $y = 6(1) - 7 = -1$. Thus $f(1) = -1$ and therefore $g(1) = 5(-1)^4 = 5$. Hence, a point on our line is $(1, 5)$. Also, since the slope of the given tangent line is $m = 6$, we know that $f'(1) = 6$.*

Now, by the chain rule we have that $g'(x) = 5 \cdot 4(f(x))^3 \cdot f'(x)$. So $g'(1) = 20(f(1))^3 \cdot f'(1) = 20(-1)^3(6) = -120$. Thus, the equation of the line tangent to the graph of $y = g(x)$ at $x = 1$ is

$$y - 5 = -120(x - 1)$$

$$y = -120x + 125$$

(b) $y = h(x) = x^2(f(x^3))$

Solution: $h(1) = 1^2 \cdot f(1^3) = f(1) = -1$. *By the product and chain rules:*

$$h'(x) = 2x(f(x^3)) + x^2(f'(x^3) \cdot 3x^2) = 2x(f(x^3)) + 3x^4(f'(x^3))$$

Thus, $h'(1) = 2f(1) + 3f'(1) = 2(-1) + 3(6) = -2 + 18 = 16$. So, the equation of the line tangent to the graph of h at $x = 1$ is

$$y - (-1) = 16(x - 1)$$

$$y = 16x - 17$$

Problem 5 *Differentiate each function (with respect to x)*

(a) $\cos\left(\sqrt{x+7}\right)$

Solution:

$$\begin{aligned} \frac{d}{dx}\left(\cos\left(\sqrt{x+7}\right)\right) &= -\sin\left(\sqrt{x+7}\right)\cdot\frac{1}{2}(x+7)^{\frac{-1}{2}}(1) \\ &= \frac{-\sin\left(\sqrt{x+7}\right)}{2\sqrt{x+7}} \end{aligned}$$

(b) $\sqrt{\cos(x)+7}$

Solution:

$$\begin{aligned} \frac{d}{dx}\left(\sqrt{\cos(x)+7}\right) &= \frac{1}{2}\left(\cos(x)+7\right)^{\frac{-1}{2}}\left(-\sin(x)\right) \\ &= \frac{-\sin(x)}{2\sqrt{\cos(x)+7}} \end{aligned}$$

(c) $\sqrt{\cos(x)}+7$

Solution:

$$\begin{aligned} \frac{d}{dx}\left(\sqrt{\cos(x)}+7\right) &= \frac{1}{2}\left(\cos(x)\right)^{\frac{-1}{2}}\left(-\sin(x)\right)+0 \\ &= \frac{-\sin(x)}{2\sqrt{\cos(x)}} \end{aligned}$$

(d) $\cos\left(\sqrt{x}+7\right)$

Solution:

$$\begin{aligned} \frac{d}{dx}\left(\cos\left(\sqrt{x}+7\right)\right) &= -\sin\left(\sqrt{x}+7\right)\cdot\frac{1}{2}x^{\frac{-1}{2}} \\ &= \frac{-\sin\left(\sqrt{x}+7\right)}{2\sqrt{x}} \end{aligned}$$

(e) $\cos(x)\cdot\left(\sqrt{x}+7\right)$

Solution:

$$\begin{aligned} \frac{d}{dx}\left(\cos(x)\cdot\left(\sqrt{x}+7\right)\right) &= -\sin(x)\cdot\left(\sqrt{x}+7\right)+\cos(x)\cdot\frac{1}{2}x^{\frac{-1}{2}} \\ &= -\sin(x)\cdot\left(\sqrt{x}+7\right)+\frac{\cos(x)}{2\sqrt{x}} \end{aligned}$$

Problem 6 *Find the derivative of the following functions:*

(a) $f(x) = \sin(x)\cos(x)$

Solution:

$$f'(x) = (\cos(x))(\cos(x)) + (\sin(x))(-\sin(x)) = \cos^2(x) - \sin^2(x).$$

(b) $f(x) = \dfrac{e^x \tan(x)}{\sec(x) + 2}$

Solution:

$$\begin{aligned} f'(x) &= \frac{(\sec(x)+2)(e^x\tan(x) + e^x\sec^2(x)) - e^x\tan(x)(\sec(x)\tan(x))}{(\sec(x)+2)^2} \\ &= \frac{e^x[(\sec(x)+2)(\tan(x)+\sec^2(x)) - \sec(x)\tan^2(x)]}{(\sec(x)+2)^2}. \end{aligned}$$

(c) $f(x) = e^{x\tan(x)}$

Solution:

$$f'(x) = e^{x\tan(x)}\left(\tan(x) + x\sec^2(x)\right)$$

(d) $f(x) = \sin(x)\cos(x)e^{3x}$

Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[\sin(x)\cos(x)]e^{3x} + (\sin(x)\cos(x))\frac{d}{dx}(e^{3x}) \\ &= \left(\cos^2(x) - \sin^2(x)\right)e^{3x} + 3e^{3x}\sin(x)\cos(x) \\ &= e^{3x}\left(\cos^2(x) + 3\sin(x)\cos(x) - \sin^2(x)\right). \end{aligned}$$

(e) $f(x) = \dfrac{x+5}{7x^6 + \cot(x)}$

Solution:

$$\begin{aligned} f'(x) &= \frac{(7x^6 + \cot(x))(1) - (x+5)(42x^5 - \csc^2(x))}{(7x^6+\cot(x))^2} \\ &= \frac{7x^6 + \cot(x) - (x+5)(42x^5 - \csc^2(x))}{(7x^6+\cot(x))^2}. \end{aligned}$$

(f) $f(x) = \sin(2x)\sec^3\left(x^2+4x\right)$

Solution:

$$\begin{aligned} f'(x) &= 2\cos(2x)\sec^3\left(x^2+4x\right) + \sin(2x)3\sec^2\left(x^2+4x\right)\sec\left(x^2+4x\right)\tan\left(x^2+4x\right)(2x+4) \\ &= 2\cos(2x)\sec^3\left(x^2+4x\right) + 3\sin(2x)\sec^3\left(x^2+4x\right)\tan\left(x^2+4x\right)(2x+4). \end{aligned}$$

Problem 7 *Find values for a, b, and c so that the following function is differentiable everywhere.*

$$f(x) = \begin{cases} a\sin(x) + b\cos(x) & \text{if } x < 0 \\ ax^2 + bx + c & \text{if } x \geq 0 \end{cases}$$

Solution: *The function f is differentiable for $x < 0$ since f is a combination of trigonometric functions. f is also differentiable $x > 0$ since f is a polynomial on that interval. We need to focus on $x = 0$. Since f is differentiable everywhere, f must also be differentiable at $x = 0$ and therefore continuous at $x = 0$. Therefore:*

We need that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = f(0)$. *Observe that*

- $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (a\sin(x) + b\cos(x)) = b(1) = b.$
- $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (ax^2 + bx + c) = c.$

Thus, we must have that $b = c$

Next, if $f'(0)$ *exists, then:*

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

$$\text{since } f(0) = a \cdot 0^2 + b \cdot 0 + c = c$$

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x) - c}{x}$$

Since this limit exists, the left and right limits must be equal.

$$\begin{aligned}
\lim_{x\to 0^+} \frac{f(x) - c}{x} &= \lim_{x\to 0^+} \frac{ax^2 + bx + c - c}{x} \\
&= \lim_{x\to 0^+} \frac{ax^2 + bx}{x} \\
&= \lim_{x\to 0^+} (ax + b) \\
&= b
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0^-} \frac{f(x) - c}{x} &= \lim_{x\to 0^-} \frac{a\sin(x) + b\cos(x) - c}{x} \\
&= \lim_{x\to 0^-} \frac{a\sin(x)}{x} + \lim_{x\to 0^-} \frac{b\cos(x) - c}{x}
\end{aligned}$$

we already found $b = c$ *so we have*

$$\lim_{x\to 0^-} \frac{a\sin(x)}{x} + \lim_{x\to 0^-} \frac{c \cdot \cos(x) - c}{x}$$

$$= a \cdot \lim_{x\to 0^-} \frac{\sin(x)}{x} + c \cdot \lim_{x\to 0^-} \frac{\cos(x) - 1}{x}$$

since $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ *and* $\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$ *we have*

$$a \cdot \lim_{x\to 0^-} \frac{\sin(x)}{x} + c \cdot \lim_{x\to 0^-} \frac{\cos(x) - 1}{x} = a$$

This means in order for the derivative to exist, $a = b = c$.