Derivatives as functions (DAF) - Solutions

Problem 1 (a) *Suppose $f'(2)$ exists. Which of the following must be true?*

(i) $\lim_{x\to 2} f(x)$ *must exist, but* $\lim_{x\to 2} f(x) \neq f(2)$

(ii) $\lim_{x\to 2} f(x) = f(2)$.

(iii) $\lim_{x\to 2} f(x) = f'(2)$

(iv) $\lim_{x\to 2} f(x)$ *need not exist.*

Solution: *The correct answer is (ii):*

$$\begin{aligned} f'(2) \text{ exists} &\iff f \text{ differentiable at } x = 2 \\ &\implies f \text{ continuous at } x = 2 \\ &\iff \lim_{x\to 2} f(x) = f(2) \end{aligned}$$

(b) *Assuming that* $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, *which of the following is true?*

(i) $\frac{0}{0} = 1$

(ii) *the tangent line to* $y = \sin(x)$ *at* $(0, 0)$ *has slope* 1.

(iii) *you can cancel the* x*'s.*

(iv) *for all* x *near* 0, $\sin(x) = x$.

(v) *for all* x *near* 0, $\sin(x) \approx x$.

Solution: *This problem has two correct answers: (ii) and (v).*

Statement (ii) is true:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{\sin(0+h) - \sin(0)}{h} \quad \text{Which has form } \frac{0}{0} \\ &= \lim_{h\to 0} \frac{\sin(h)}{h} \\ &= 1 \\ &\implies \text{slope of tangent line at } (0,0) \text{ is } 1 \end{aligned}$$

Statement (v) is true: When x is near 0 the tangent line $y = x$ is a good approximation to f.

Problem 2

(a) *Fill in the blanks*

$$f'(x) = \lim_{???} \frac{????}{h}$$

if the limit exists.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

if the limit exists.

(b) *Let*

$$f(x) = \frac{1}{x+4}.$$

Use the (limit) definition *of derivative in (a) to* find *$f'(x)$.*

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{x+h+4} - \frac{1}{x+4}}{h} \quad \textit{which has form } \frac{0}{0} \\
&= \lim_{h\to 0} \frac{\frac{x+4-(x+h+4)}{(x+h+4)(x+4)}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h+4)(x+4)} \\
&= \lim_{h\to 0} \frac{-1}{(x+h+4)(x+4)} = \frac{-1}{(x+4)^2}
\end{aligned}$$

Problem 3 *Let $f(x) = |5-x|$.*

(a) *For $a < 5$,* find *$f'(a)$.*

Solution: *Recall that*

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

When $a < 5$, $0 < 5-a$ and so $f(a) = |5-a| = 5-a$. Since we are considering the limit as x approaches a, (and therefore interested in values of x really close to a), we may also assume that $x < 5$ and therefore $f(x) = 5-x$. Thus

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{(5-x)-(5-a)}{x-a} \quad \textit{which has form } \frac{0}{0} \\
&= \lim_{x\to a} \frac{-(x-a)}{x-a} \\
&= \lim_{x\to a} -1 = -1.
\end{aligned}$$

(b) *For $a > 5$,* find *$f'(a)$.*

Solution: *When $a > 5$, $0 > 5 - a$ and so $f(a) = |5 - a| = -(5 - a) = a - 5$. Since we are considering the limit as x approaches a, we may also assume that $x > 5$ and therefore $f(x) = x - 5$. Thus*

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{(x - 5) - (a - 5)}{x - a} \quad \textit{which has form } \frac{0}{0} \\ &= \lim_{x \to a} \frac{x - a}{x - a} \\ &= \lim_{x \to a} 1 = 1. \end{aligned}$$

(c) *Determine whether $f'(5)$ exists.*

Solution: *If $f'(5)$ exists then $\lim_{x \to 5} \frac{f(x) - f(5)}{x - 5}$ exists. But when x is close to 5 it may be that $x > 5$ or $x < 5$. So in order to find this limit, we have to check whether the two one-sided limits are equal. This means, $\lim_{x \to 5^-} \frac{f(x) - f(5)}{x - 5} = \lim_{x \to 5^+} \frac{f(x) - f(5)}{x - 5}$*
Each of these has form $\frac{0}{0}$.

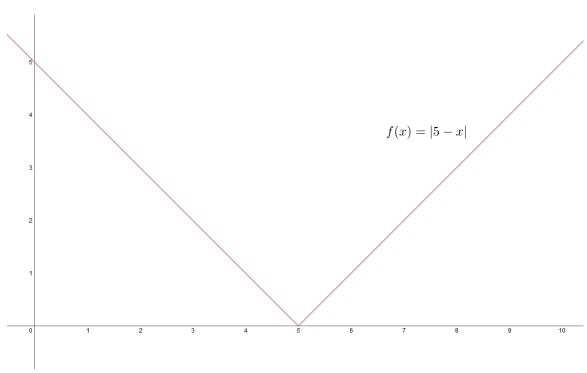
$$\begin{aligned} \lim_{x \to 5^-} \frac{f(x) - f(5)}{x - 5} &= \lim_{x \to 5^-} \frac{-(x - 5) - (0)}{x - 5} = -1 \\ \lim_{x \to 5^+} \frac{f(x) - f(5)}{x - 5} &= \lim_{x \to 5^+} \frac{(x - 5) - (0)}{x - 5} = 1 \\ \lim_{x \to 5^-} \frac{f(x) - f(5)}{x - 5} &\neq \lim_{x \to 5^+} \frac{f(x) - f(5)}{x - 5} \end{aligned}$$

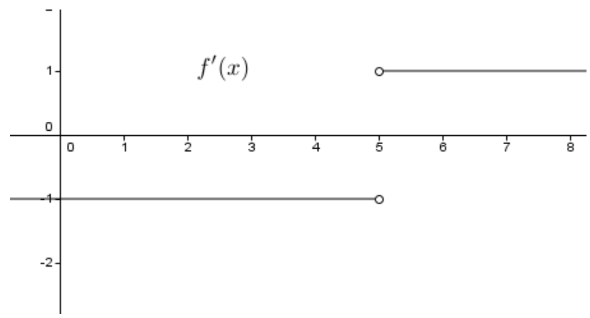
Therefore, $f'(5)$ does not exist.

$$f'(a) = \begin{cases} 1 & \text{if } a > 5 \\ \text{undefined} & \text{if } a = 5 \\ -1 & \text{if } a < 5 \end{cases}$$

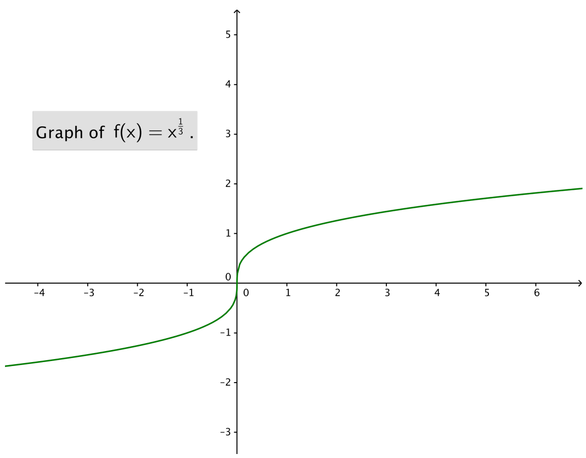
(d) *Sketch a graph of the function $f(x)$ and its derivative $f'(x)$*

Solution:





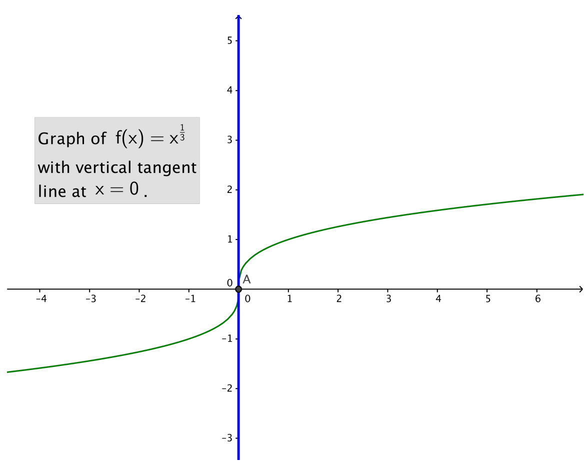
Problem 4 *Define the function f by $f(x) = x^{1/3}$ and consider the graph of this function:*



Which of the following two statements are true:

(a) *The graph of f has a tangent line at $x = 0$.*

Solution: *This statement is **true**! The function f has a vertical tangent at $x = 0$:*

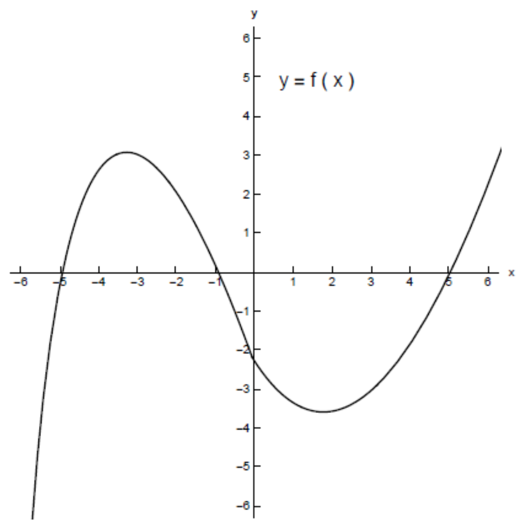


(b) *The derivative $f'(0)$ is defined.*

Solution: *This statement is* ***false****!*

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} &= \lim_{h\to 0^+} \frac{h^{1/3}}{h} \text{ which has form } \frac{0}{0} \\
&= \lim_{h\to 0^+} \frac{1}{h^{\frac{2}{3}}} \\
&= \infty \text{ because the limit is of the form: } \frac{pos}{0^+} \\
&\implies f'(0) \text{ is undefined}
\end{aligned}$$

Problem 5 *Suppose we are given the graph of a function f:*



(a) *Use this graph to find the following: (Assume all values will be integers or $+\infty$ or $-\infty$)*

(i) *all x where $f(x) = 0$,*

Solution: *$f(x)$ is zero when the function crosses the x-axis. Therefore $f(x) = 0$ when $x = -5, x = -1$, and $x = 5$.*

(ii) *all x where $f(x) > 0$,*

Solution: *$f(x)$ is positive when the graph of the function is above the x-axis. Therefore $f(x) > 0$ on $(-5, -1) \cup (5, \infty)$.*

(iii) *all x where $f(x) < 0$, and*

Solution: *$f(x)$ is negative when the graph of the function is below the x-axis. Therefore $f(x) < 0$ on $(-\infty, -5) \cup (-1, 5)$.*

(iv) *all x where $f(x)$ attains a local maximum and all x where f attains a local minimum.*

Solution: *$f(x)$ has a local maximum at $x = -3$. $f(x)$ has a local minimum at $x = 2$.*

Without sketching the graph of f' find

(b) (i) *all x where $f'(x) = 0$,*

Solution: *$f'(x)$ is zero when the tangent line has a slope of zero, which is approximately at $x = -3$ and $x = 2$. Note, for this question, these are the same answers as the (local) highest and lowest point for the graph of f.*

(ii) *all x where $f'(x) > 0$,*

Solution: *$f'(x)$ is positive when the slope of the tangent line is positive. Observe that f is increasing on $(-\infty, -3), (2, \infty)$ and this same set of intervals is where the tangent lines have positive slope. Therefore $f'(x) > 0$ on $(-\infty, -3), (2, \infty)$.*

(iii) *all x where $f'(x) < 0$, and*

Solution: *$f'(x)$ is negative when the slope of the tangent line is negative. Observe that f is decreasing on $(-3, 2)$ and this interval is where the tangent lines have negative slope. Therefore $f'(x) < 0$ on $(-3, 2)$.*

(iv) *On the following intervals, does $f'(x)$ seem to be increasing or decreasing?*

i. $(-\infty, 0)$

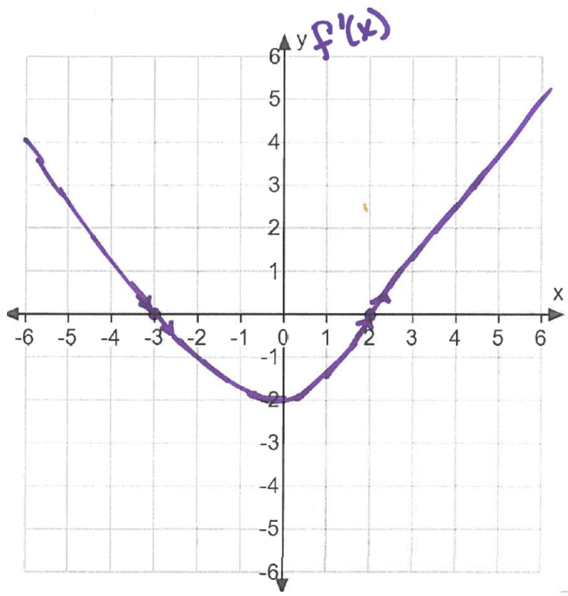
Solution: *decreasing*

ii. $(0, \infty)$

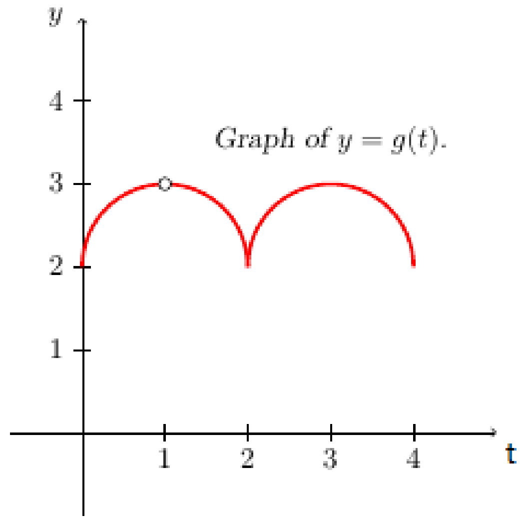
Solution: *increasing*

(c) *Sketch a graph of f'.*

Solution: *The graph of f' is approximately*



Problem 6 *Use the graph of g*



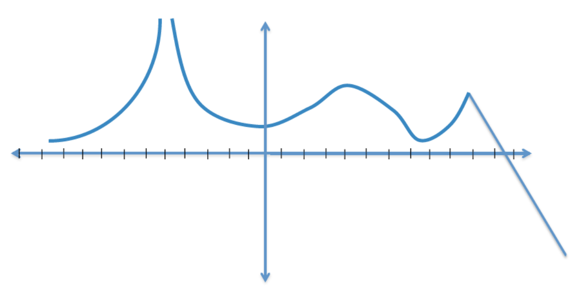
(a) *Find the values of t in $(0, 4)$ at which g is not continuous.*

Solution: *g is not continuous at $t = 1$.*

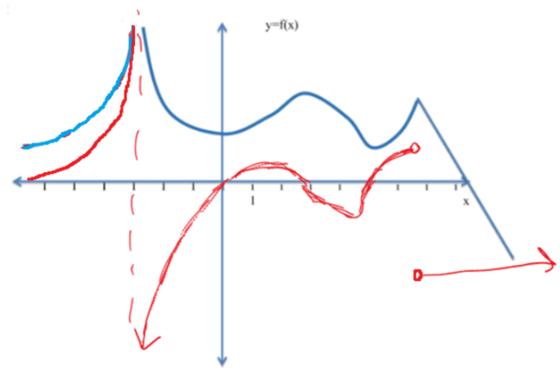
(b) *Find the values of t in $(0, 4)$ at which g is not differentiable.*

Solution: *g is not differentiable at $t = 1$, because g is not continuous there, and at $t = 2$, because g has a "cusp" there.*

Problem 7 *Given the following graph of a function h sketch a graph of the derivative h'.*



Solution: *The graph of the derivative is in red.* ***Important Note:*** *Despite being drawn on the same graph, the "units" for f and f' are not the same!*


y=f(x)
1
x