Antiderivatives (A) - Solutions

Problem 1 *Find the most general antiderivative of the function*

$$g(t) = e^{-2t} - 5 + 6\sqrt{t} - \frac{7}{t} + \frac{5}{1+t^2}$$

Solution: *The family of all antiderivatives of g is given by*

$$\int \left(e^{-2t} - 5 + 6\sqrt{t} - \frac{7}{t} + \frac{5}{1+t^2} \right) dt.$$

We can apply the Sum Rule and Constant Multiple Rule.

$$\int \left(e^{-2t} - 5 + 6\sqrt{t} - \frac{7}{t} + \frac{5}{1+t^2} \right) dt = \int e^{-2t}\, dt - 5 \int dt + 6 \int \sqrt{t}\, dt - 7 \int \frac{1}{t}\, dt + 5 \int \frac{1}{1+t^2}\, dt$$

And finally,

$$\int \left(e^{-2t} - 5 + 6\sqrt{t} - \frac{7}{t} + \frac{5}{1+t^2} \right) dt = -\frac{1}{2} e^{-2t} - 5t + 4t^{\frac{3}{2}} - 7 \ln |t| + 5 \arctan t + C.$$

Problem 2 *Determine the following indefinite integrals.*

(a)

$$\int \left(\sec^2(x) + 5 \right) dx$$

Solution:

$$\int \left(\sec^2(x) + 5 \right) dx = \tan(x) + 5x + C$$

(b)

$$\int \left(\sec^2(x) - \sec(x)\tan(x) \right) dx$$

Solution: *We apply the Sum Rule.*

$$\int \left(\sec^2(x) - \sec(x)\tan(x)\right)\, dx = \int \sec^2(x)\, dx - \int \sec(x)\tan(x)\, dx = \tan(x) - \sec(x) + C.$$

(c)

$$\int \frac{x+3x^5}{x^3}\, dx$$

Solution: *First, we express the function in a more convenient way.*

$$\int \frac{x+3x^5}{x^3}\, dx = \int \left(\frac{x}{x^3} + \frac{3x^5}{x^3}\right)\, dx = \int \left(x^{-2} + 3x^2\right)\, dx.$$

Now we apply the Sum Rule and the Constant Multiple Rule.

$$\int \frac{x+3x^5}{x^3}\, dx = \int \left(x^{-2} + 3x^2\right)\, dx = \int x^{-2}\, dx + 3\int x^2\, dx.$$

Now we determine each integral.

$$\int \frac{x+3x^5}{x^3}\, dx = \int x^{-2}\, dx + 3\int x^2\, dx = -\frac{1}{x} + x^3 + C.$$

(d)

$$\int \frac{1+2x}{1+x^2}\, dx$$

(HINT: Think about the derivative of $\ln(f(x))$*).*

Solution: *First, we express the function in a more convenient way.*

$$\int \frac{1+2x}{1+x^2}\, dx = \int \left(\frac{1}{1+x^2} + \frac{2x}{1+x^2}\right)\, dx.$$

Now we apply the Sum Rule.

$$\int \frac{1+2x}{1+x^2}\, dx = \int \frac{1}{1+x^2}\, dx + \int \frac{2x}{1+x^2}\, dx.$$

Now we determine each integral.

$$\int \frac{1+2x}{1+x^2}\, dx = \int \frac{1}{1+x^2}\, dx + \int \frac{2x}{1+x^2}\, dx = \arctan(x) + \ln\left(1+x^2\right) + C.$$

(e)

$$\int \frac{2+x^2}{1+x^2}\,dx$$

Solution: *First, we express the function in a more convenient way.*

$$\int \frac{2+x^2}{1+x^2}\,dx = \int \frac{1+1+x^2}{1+x^2}\,dx = \int \left(\frac{1}{1+x^2} + \frac{1+x^2}{1+x^2}\right)dx = \int \left(\frac{1}{1+x^2} + 1\right)dx.$$

Now we apply the Sum Rule and the Constant Multiple Rule.

$$\int \frac{2+x^2}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int dx = \arctan(x) + x + C.$$

Problem 3 *Assume that $f'(t) = 4t^3 + 2t$ and $f(3) = 5$. Find $f(t)$.*

Solution: *This is an Initial Value Problem. First we find the general solution of the differential equation.*

$$f(t) = t^4 + t^2 + C.$$

Then, we find the unique solution of the IVP.

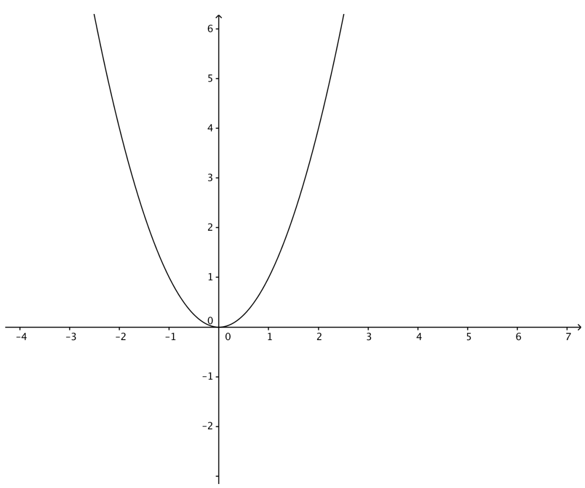
$$5 = f(3) = 3^4 + 3^2 + C = 81 + 9 + C = 90 + C$$

$$\implies \quad C = -85$$

and so

$$f(t) = t^4 + t^2 - 85.$$

Problem 4 *The graph of a function f is the parabola given in the figure below.*

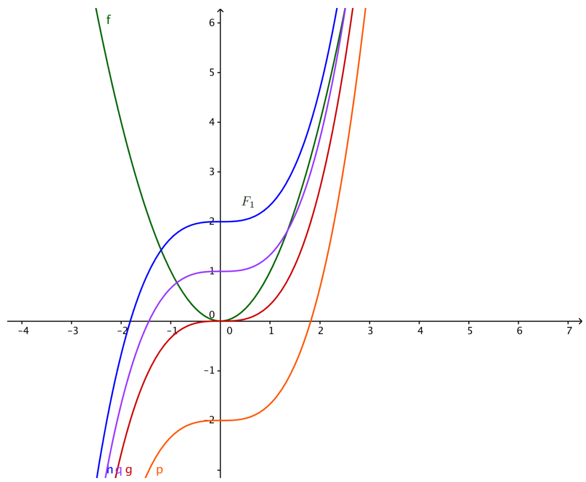


(a) *Find a formula for $f(x)$.*

Solution: $f(x) = x^2$

(b) *Suppose F_1 is an anti-derivatives of f satisfying $F_1(0) = 2$. Sketch and label the graph of F_1. Sketch the graphs of three more antiderivatives of f.*

Solution:



(c) *Using the expression you found in part (a) and your sketches in part (b), find the algebraic representation of F_1.*

Solution: *The general antiderivative is* $F(x) = \frac{1}{3}x^3 + C$. *Therefore,*
$F_1(x) = \frac{1}{3}x^3 + C$, *for some constant* C.
Since $F_1(0) = 2$, *it follows that* $2 = \frac{1}{3}(0)^3 + C = C$. *Therefore,* $F_1(x) = \frac{1}{3}x^3 + 2$.

(d) *Suppose we're given* $h(x) = (1/3)x^3 + 17$, *what is* $h'(x)$*?*

Solution: $h'(x) = x^2$.

(e) *What is the relationship between f, F_1, h, and h'?*

Solution: *We have* $F_1' = f = h'$ *and* $h = F_1 + C$, *where C is some constant. Determine this constant!*

Problem 5 *Given the acceleration function, the initial velocity and initial position of an object moving along a line, find the* ***position function***.

$a(t) = 4\cos t$, $v(0) = 2$, $s(0) = 6$.

Solution: $v(t) = 4\sin t + C$.
Letting $t = 0$, *we get*
$v(0) = 4\sin 0 + C$ *and*
$2 = C$.
Therefore, $v(t) = 4\sin t + 2$.

So, $s(t) = -4\cos t + 2t + C$.
Letting $t = 0$*, we get*

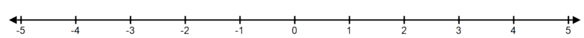
$s(0) = -4\cos 0 + 2(0) + C$.
So,
$6 = -4 + C$,
and $C = 10$.
Therefore,
$s(t) = -4\cos t + 2t + 10$.

Problem 6 *Consider an object moving along a line with velocity* $v(t) = \pi\sin(\pi t)$ *on* $[0, 2]$ *and initial position* $s(0) = 0$*. Time is measured in seconds and velocity in m/s.*

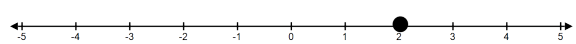
(a) *Determine the position function,* $s(t)$*, on* $[0, 2]$.

Solution: *General antiderivative of* $v(t)$*:* $s(t) = -\cos(\pi t) + C$
Apply initial position: $s(t) = -\cos(\pi t) + C \implies 0 = s(0) = -\cos(0) + C \implies C = 1$
Position function: $s(t) = -\cos(\pi t) + 1$

(b) *Mark the position of the object at the time* $t = 1$ *on the line below.*



Solution: $s(1) = -\cos(\pi) + 1 = -(-1) + 1 = 2$



(c) *Determine the average velocity,* v_{av}*, of the object during the interval* $[0, 2]$.

Solution:

$$\begin{aligned} v_{av} &= \frac{s(2) - s(0)}{2 - 0} \\ &= \frac{-\cos(2\pi) + 1 - (-\cos(0) + 1)}{2} \\ &= \frac{-\cos(2\pi) + \cos(0)}{2} \\ &= \frac{-1 + 1}{2} = 0 \end{aligned}$$

(d) *Determine when the motion is in the positive direction.*

Solution: *Motion is in the positive direction when* $v(t) > 0$.

$$\begin{aligned} v(t) > 0 &\iff \pi\sin(\pi t) > 0 \\ &\iff \sin(\pi t) > 0 \\ &\implies 0 < \pi t < \pi \\ &\iff 0 < t < 1 \end{aligned}$$

(e) *At what time (or times) is the object farthest from the origin?*

Solution: *Object is farthest from the origin when* $\mid s(t) - 0 \mid = \mid 1 - \cos(\pi t) \mid$ *is maximized. Since* $s(t) \geq t$ *for* t *in* $[0, 2]$ *we need to maximize* $s(t)$*. Finding critical points on the open interval* $(0, 2)$*.*

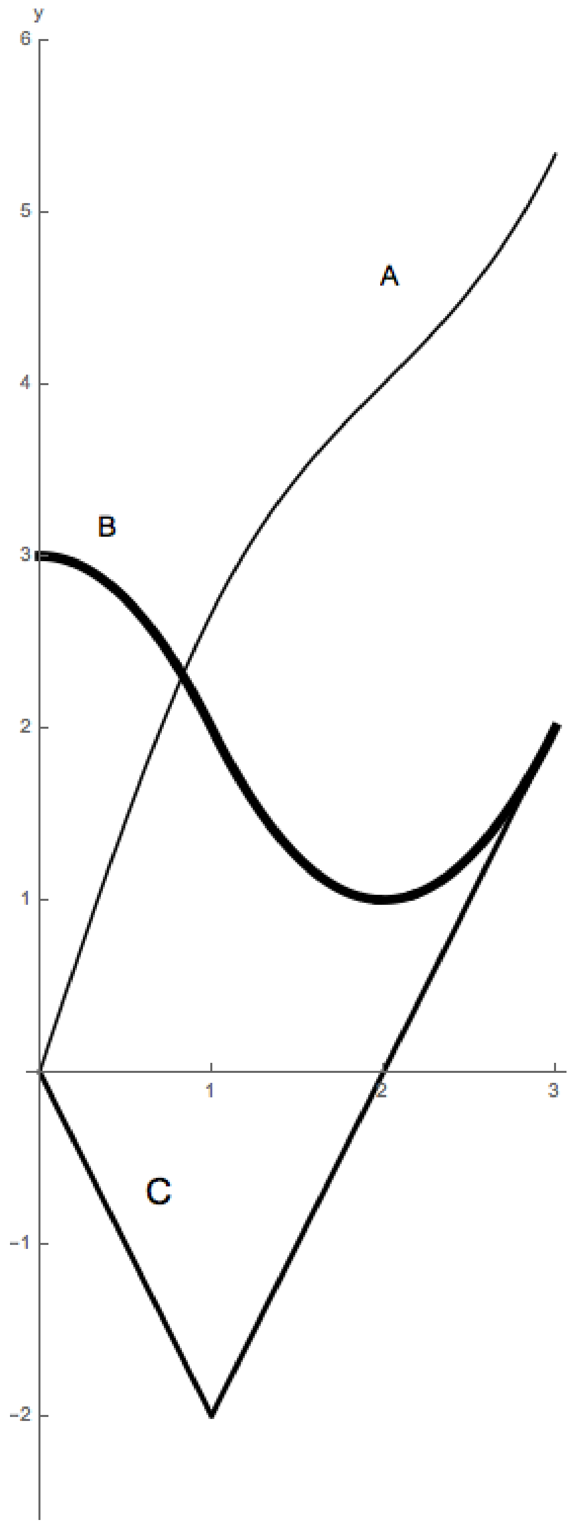
$$\begin{aligned} v(t) = 0 &\iff \pi \sin(\pi t) = 0 \\ &\implies \pi t = \pi \\ &\iff t = 1 \end{aligned}$$

Finding the absolute maximum, we have to check the endpoints and the critical points:

$$\begin{aligned} s(0) &= 0 \\ s(1) &= 2 \\ s(2) &= 0 \end{aligned}$$

Hence the object is farthest from the origin at $t = 1$*.*

Problem 7 *Consider an object moving along a straight line. The graphs of acceleration function (in* m/s^2*), the velocity function and the position function of the object are given in the figure below.*



Find the initial velocity, $v(0)$, and initial position, $s(0)$, of the object. (HINT: Start by determining which graph corresponds to each of $s(t)$, $v(t)$, and $a(t)$.)

Solution: *It is clear that the function B is the velocity. Therefore, $v(0) = 3$. It is clear that the function A is the position function. Therefore, $s(0) = 0$.*