Product Rule and Quotient Rule (PRAQR) - Solutions

Problem 1 *Differentiate the functions using product or quotient rule.*

(a) $h(u) = 7ue^u$

Solution: *Applying the power rule:*

$$\begin{aligned} h(u) &= 7ue^u \\ \implies h'(u) &= (7ue^u)' \\ &= 7 \cdot e^u + 7u \cdot e^u \end{aligned}$$

(b) $s(t) = \dfrac{\sqrt{t}}{e^t}$

Solution: *Applying the quotient rule:*

$$\begin{aligned} s(t) &= \frac{\sqrt{t}}{e^t} \\ \implies s'(t) &= \left(\frac{\sqrt{t}}{e^t}\right)' \\ &= \frac{\frac{1}{2}t^{-1/2} \cdot e^t - \sqrt{t} \cdot e^t}{e^{2t}} \end{aligned}$$

Problem 2 *Differentiate the function f defined by $f(x) = 1/x^8$ in two different ways.*

Solution: *Applying the quotient rule:*

$$\begin{aligned} f'(x) &= \frac{(1)' \cdot x^8 - (1 \cdot (x^8)')}{(x^8)^2} \\ &= \frac{(0 \cdot x^8) - (1 \cdot 8x^7)}{x^{16}} \\ &= \frac{-8x^7}{x^{16}} \\ &= \frac{-8}{x^9}. \end{aligned}$$

Applying the power rule:

$$\begin{aligned} f(x) &= \frac{1}{x^8} = x^{-8} \\ \implies f'(x) &= (x^{-8})' \\ &= -8 \cdot x^{-8-1} \\ &= -8x^{-9} \end{aligned}$$

Problem 3 *Suppose that $f(5) = 7$, $f'(5) = 8$, $g(5) = 3$, and $g'(5) = -4$. Find:*

(a) $(fg)'(5)$.

Solution:

$$\begin{aligned}(fg)'(5) &= (f'(5)\cdot g(5)) + (f(5)\cdot g'(5))\\ &= (8)(3) + (7)(-4)\\ &= 24 - 28 = -4.\end{aligned}$$

(b) $\left[\dfrac{d}{dx}\left(\dfrac{f}{g}\right)\right]_{x=5}$

Solution:

$$\begin{aligned}\left[\frac{d}{dx}\left(\frac{f}{g}\right)\right](5) &= \frac{(g(5)\cdot\left[\frac{d}{dx}f\right](5)) - (f(5)\cdot\left[\frac{d}{dx}g\right](5))}{(g(5))^2}\\ &= \frac{(3)(8) - (7)(-4)}{3^2}\\ &= \frac{24+28}{9} = \frac{52}{9}.\end{aligned}$$

(c) $\left(\dfrac{g}{f}\right)'(5)$

Solution:

$$\begin{aligned}\left(\frac{g}{f}\right)'(5) &= \frac{(f(5)\cdot g'(5)) - (g(5)\cdot f'(5))}{(f(5))^2}\\ &= \frac{(7)(-4) - (3)(8)}{7^2}\\ &= \frac{-28-24}{49} = -\frac{52}{49}.\end{aligned}$$

(d) $\left[\dfrac{d}{dx}\left(\dfrac{g(x)}{x+2}\right)\right]_{x=5}$

Solution:

$$\begin{aligned}\left[\frac{d}{dx}\left(\frac{g(x)}{x+2}\right)\right]_{x=5} &= \left[\frac{(x+2)\cdot g'(x)) - (g(x)\cdot(x+2)'}{(x+2)^2}\right]_{x=5}\\ &= \frac{(5+2)(-4) - (3)(1)}{(5+2)^2}\\ &= \frac{(7)(-4) - (3)(1)}{7^2}\\ &= \frac{-28-3}{49} = -\frac{31}{49}\end{aligned}$$

(e) $\left[\dfrac{d}{dx}\left(\dfrac{xf(x)}{g(x)}\right)\right]_{x=5}$

Solution:

$$\begin{aligned}
\left[\frac{d}{dx}\left(\frac{xf(x)}{g(x)}\right)\right]_{x=5} &= \left[\frac{(((xf(x))' \cdot g(x)) - (g'(x) \cdot (xf(x)))}{(g(x))^2}\right]_{x=5} \\
&= \left[\frac{((x)' \cdot f(x) + x \cdot f'(x)) \cdot g(x)) - (g'(x) \cdot (xf(x))}{(g(x))^2}\right]_{x=5} \\
&= \frac{(1 \cdot 7 + 5 \cdot 8)(3) - (-4)(5)(7)}{(3)^2} \\
&= \frac{141 + 140}{9} \\
&= \frac{281}{9}
\end{aligned}$$

(f) $\displaystyle\lim_{x\to 5} \frac{x^2 g(x) - 75}{x - 5}$. ***EXPLAIN***.

Solution: *We first note that this limit has form* $\frac{0}{0}$, *but with no formula for* $g(x)$ *there is no simplification we can perform. Notice that if we set* $w(x) = x^2 g(x)$, *then* $w(5) = (5)^2 g(5) = 75$. *This limit is the definition of the derivative,* $w'(5)$.

$$\begin{aligned}
\lim_{x\to 5} \frac{x^2 g(x) - 75}{x - 5} &= w'(5) \\
&= \left[\frac{d}{dx}(x^2 g(x))\right]_{x=5} \\
&= \left[2xg(x) + x^2 g'(x)\right]_{x=5} \\
&= 2(5)g(5) + (5)^2 g'(5) \\
&= -70
\end{aligned}$$

$\displaystyle\lim_{x\to 5} \frac{x^2 g(x) - 75}{x - 5} = -70$ *by the product rule since this is the definition of the derivative of* $x^2 g(x)$ *at* $x = 5$.

(g) $\displaystyle\lim_{x\to 5} \frac{\frac{x+2}{f(x)} - 1}{x - 5}$. ***EXPLAIN***.

Solution: *We first note that this limit has form* $\frac{0}{0}$. *Notice that if we set* $k(x) = \frac{x+2}{f(x)}$, *then*

$k(5) = \dfrac{5+2}{f(5)} = 1$. *This limit is the definition of the derivative, $k'(5)$.*

$$\begin{aligned}
\lim_{x\to 5} \frac{\frac{x+2}{f(x)} - 1}{x-5} &= k'(5) \\
&= \left[\frac{d}{dx}\left(\frac{x+2}{f(x)}\right)\right]_{x=5} \\
&= \left[\frac{f(x) - (x+2)f'(x)}{(f(x))^2}\right]_{x=5} \\
&= \frac{f(5) - (5+2)f'(5)}{(f(5))^2} \\
&= \frac{7 - 7(8)}{(7)^2} \\
&= -1
\end{aligned}$$

$\lim_{x\to 5} \dfrac{\frac{x+2}{f(x)} - 1}{x-5} = -1$ *by the quotient rule since this is the definition of the derivative of* $\dfrac{x+2}{f(x)}$ *at* $x = 5$.

Problem 4 *Use the given information to find the equation of the tangent line.*

(a) *Given $g(x) = x^3 f(x)$, $f(2) = 4$, and $f'(2) = 7$, find the equation of the tangent line to the graph of g at $x = 2$.*

Solution: *Slope of tangent line to the graph of g at the point where $x = 2$:*

$$\begin{aligned}
g'(x) &= 3x^2 f(x) + x^3 f'(x) \\
&\implies g'(2) = 12(4) + 8(7) = 48 + 56 = 104.
\end{aligned}$$

Point on tangent line where $x = 2$: $(2, g(2)) = (2, 8f(2)) = (2, 32)$.

Equation of tangent line:

$$\begin{aligned}
y - 32 &= 104(x - 2) \\
&\implies y = 104x - 176.
\end{aligned}$$

(b) *Given $h(z) = \dfrac{zs(z)}{z-3}$, $s(2) = 4$, and $s'(2) = 7$, find the equation of the tangent line to the graph of h at $z = 2$.*

Solution: *Slope of tangent line to the graph of h at the point where $z = 2$:*

$$\begin{aligned}
h'(z) &= \frac{(z-3)(zs(z))' - zs(z)(z-3)'}{(z-3)^2} \\
&= \frac{(z-3)(s(z) + zs'(z)) - zs(z)(1)}{(z-3)^2} \\
&\implies h'(2) = \frac{(2-3)(s(2) + 2s'(2)) - 2s(2)}{(2-3)^2} = -26.
\end{aligned}$$

Point on tangent line where $z = 2$*:* $(2, h(2)) = (2, \frac{2s(2)}{2-3}) = (2, -8)$.

Equation of tangent line to the graph of h *at the point where* $z = 2$*:*

$$\begin{aligned} y - (-8) &= -26(z-2) \\ \implies y &= -26z + 44. \end{aligned}$$

(c) *Given*

x	1	2	3	4	5
$f(x)$	5	3	0	-4	3
$f'(x)$	-3	-5	-2	6	-4
$g(x)$	6	9	-8	13	15
$g'(x)$	8	5	-10	7	6

find the equation of the tangent line of

$$\frac{f(x)}{e^x g(x)}$$

at $x = 2$.

Solution: *Slope of tangent line to the graph of* $\frac{f(x)}{e^x g(x)}$ *at the point where* $x = 2$*:*

$$\begin{aligned} \left(\frac{f(x)}{e^x g(x)}\right)' &= \frac{e^x g(x) f'(x) - f(x)(e^x g(x) + e^x g'(x))}{(e^x g(x))^2}. \\ \implies \frac{d}{dx}\left(\frac{f(x)}{e^x g(x)}\right)\bigg|_{x=2} &= \frac{e^2 g(2) f'(2) - f(2)(e^2 g(2) + e^2 g'(2))}{(e^2 g(2))^2} \\ &= \frac{e^2(9)(-5) - (3)(9e^2 + 5e^2)}{(9e^2)^2} \\ &= \frac{-45e^2 - 42e^2}{81e^4} \\ &= \frac{-87e^2}{81e^4} \\ &= \frac{-87}{81e^2} \end{aligned}$$

Point on tangent line where $x = 2$*:* $(2, f(2)/(e^2 g(2))) = (2, 3/(9e^2)) = (2, 1/(3e^2))$.

Equation of tangent line where $x = 2$*:*

$$\begin{aligned} y - \frac{1}{3e^2} &= \frac{-87}{81e^2}(x-2) \\ \implies y &= \frac{-87}{81e^2}x + \frac{67}{27e^2}. \end{aligned}$$

Problem 5 *Differentiate the following functions:*

(a) *Given* $f(x) = (x^2 + 4x - 7)e^{-x}$*, show* $f'(x) = (11 - x^2 - 2x)e^{-x}$.

Solution:

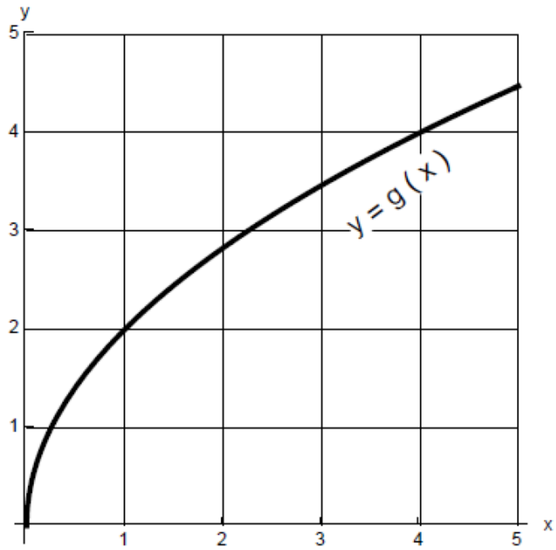
$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{x^2+4x-7}{e^x}\right) \\
&= \frac{(2x+4)e^x - (x^2+4x-7)e^x}{e^{2x}} \\
&= \frac{e^x(2x+4-x^2-4x+7)}{e^{2x}} \\
&= \frac{11-x^2-2x}{e^x} \\
&= (11-x^2-2x)e^{-x}
\end{aligned}$$

(b) *Given* $g(x) = \dfrac{x^2+4x-7}{e^{-x}}$, *show* $g'(x) = \dfrac{x^2+6x-3}{e^{-x}}$

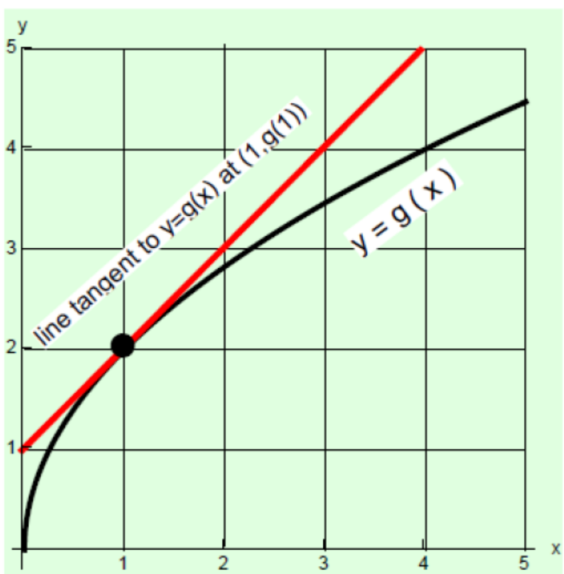
Solution:

$$\begin{aligned}
g'(x) &= \frac{d}{dx}\left(\frac{x^2+4x-7}{e^{-x}}\right) \\
&= \frac{d}{dx}\Big((x^2+4x-7)e^x\Big) \\
&= \frac{d}{dx}(x^2+4x-7)e^x + (x^2+4x-7)\frac{d}{dx}e^x \\
&= (2x+4)e^x + (x^2+4x-7)e^x \\
&= e^x(2x+4+x^2+4x-7) \\
&= e^x(x^2+6x-3) \\
&= \frac{x^2+6x-3}{e^{-x}}
\end{aligned}$$

Problem 6 *The graph of a function g is given below. Using the graph, estimate the derivative at the given point:*
$\dfrac{d}{dx}\left(\dfrac{xg(x)}{x+3}\right)$ at $x = 1$



Solution: *From the graph it appears that* $g(1) = 2$*. We can draw the line tangent to the curve* $y = g(x)$ *at the point* $(1, g(1))$



Since $g'(1)$ *is the slope of the line tangent to the curve* $y = g(x)$ *at the point* $(1, g(1))$*, we can estimate it from the figure above. The tangent line seems to pass through the points* $(1, 2)$ *and* $(2, 3)$*, so its slope is* $\dfrac{3-2}{2-1} = 1$.

$$
\begin{aligned}
\left[\frac{d}{dx}\left(\frac{xg(x)}{x+3}\right)\right]_{x=1} &= \left[\frac{(g(x)+xg'(x))(x+3)-xg(x)}{(x+3)^2}\right]_{x=1} \\
&= \frac{(g(1)+1g'(1))(1+3)-(1)g(1)}{(1+3)^2} \\
&\approx \frac{(2+1(1))(4)-1(2)}{4^2} \\
&= \frac{(2+1)(4)-2}{16} \\
&= \frac{3(4)-2}{16} \\
&= \frac{12-2}{16} \\
&= \frac{10}{16} \\
&= \frac{5}{8}
\end{aligned}
$$