Derivatives of inverse functions (DOIF)

Derivatives of Inverse Trigonometric Functions

- $\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$
- $\frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}}$ for $|x| \geq 1$
- $\frac{d}{dx}\left[\cos^{-1}(x)\right] = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left[\cot^{-1}(x)\right] = \frac{-1}{1+x^2}$
- $\frac{d}{dx}\left[\csc^{-1}(x)\right] = \frac{-1}{|x|\sqrt{x^2-1}}$ for $|x| \geq 1$

Theorem 1 (The Inverse Function Theorem)**.** If f is a differentiable function that is one-to-one near a and $f'(a) \neq 0$, then:

(a) $f^{-1}(x)$ is **defined** for x near $b = f(a)$,

(b) $f^{-1}(x)$ is **differentiable** for x near $b = f(a)$,

(c) last, but not least: $\left[\frac{d}{dx}f^{-1}(x)\right]_{x=b} = \frac{1}{f'(a)}$ where $b = f(a)$.

Recitation Questions

Problem 1 *Explain what each of the following means:*

(a) $\sin^{-1}(x)$

(b) $(\sin(x))^{-1}$

(c) $\sin\left(x^{-1}\right)$

Problem 2 *Without using a calculator, determine if the statement below is true or false.*

$$\cos^{-1}\big(\cos(7\pi/6)\big) = 7\pi/6$$

Problem 3 ***True or False:*** $\sin^{-1}(0) = \pi$.

Problem 4 *Simplify each of the following expressions.*

(a) $\cos^{-1}\big(\sin(\pi/2)\big)$

(b) $\tan\big(\sin^{-1}(x/4)\big)$

Problem 5 *A table of values for f and f' is shown below. Suppose that f is a one-to-one function and f^{-1} is its inverse.*

x	$f(x)$	$f'(x)$
1	3	4
3	4	5
4	6	3

(a) *Evaluate* $f^{-1}(f(x))$ *at* $x = 3$.

(b) *Evaluate* $\dfrac{d}{dx} f(f(x))$ *at* $x = 3$.

(c) *Evaluate* $\dfrac{d}{dx} \ln((f(x))$ *at* $x = 3$.

(d) *Evaluate* $f^{-1}(x)$ *at* $x = 3$.

(e) *Evaluate* $\dfrac{d}{dx} f^{-1}(x)$ *at* $x = 3$.

(f) *Evaluate* $\lim\limits_{x \to 4} \dfrac{f(x) - f(4)}{x - 4}$

Problem 6 *An object is moving along a horizontal line. Its position (in meters) at the time t (in seconds) is given by $s(t)$.*
What does the value $s^{-1}(5)$ represent?

Problem 7 *Given the expression for $f(x)$, find the derivative of f^{-1} at the given point on the graph of f^{-1}, without solving for f^{-1}.*

(a) $f(x) = x^2 + 1$ *(for $x \geq 0$); the point on the graph of f^{-1}:* $(5, 2)$.
Verify your answer by evaluating the derivative of f^{-1} at the given point.

(b) $f(x) = x^2 - 2x - 3$ *(for $x \leq 1$); the point on the graph of f^{-1}:* $(12, -3)$.

Problem 8 *Find the slope of the tangent line to the curve* $y = f^{-1}(x)$ *at* $(4, 7)$ *if the slope of the tangent line to the curve* $y = f(x)$ *at* $(7, 4)$ *is* $\frac{2}{3}$.

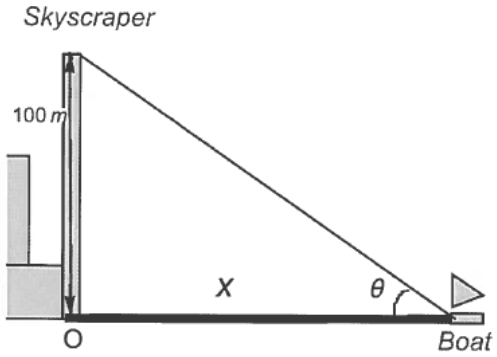
Problem 9 *Find the derivatives of the following functions:*

(a) $f(x) = \sec^{-1}(\sqrt{x})$.

(b) $g(x) = \ln(\sin^{-1}(x))$.

(c) $h(x) = \dfrac{1}{\tan^{-1}(x^2+4)}$.

Problem 10 *A boat sails directly toward a 100-meter skyscraper that stands on the edge of a harbor. The angular size θ of the building is the angle formed by lines from the top and bottom of the building to the observer on the boat (see figure below).*



(a) *Express the angle θ as the function of x, the distance of the boat from the building.*

(b) *The boat is sailing directly toward the skyscraper at 3 m/s. Find* $\dfrac{d\theta}{dt}$ *when the boat is* $x = 300$m *from the building.*