L'Hôpital's Rule (LHR) - Solutions

Problem 1 *State the form of the limit. Determine whether the form is determinate or indeterminate. Evaluate each limit.*

(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x) - \cos(x) + 1}{x^2 - x}$

Solution: *The form is* $\frac{0}{0}$*; indeterminate. we can apply L.R.*

$\displaystyle\lim_{x\to 0} \frac{\sin(x) - \cos(x) + 1}{x^2 - x} = \lim_{x\to 0} \frac{\cos(x) + \sin(x)}{2x - 1} = \frac{\cos(0) + \sin(0)}{2(0) - 1} = \frac{1}{-1} = -1$

(b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

Solution: *The form is* $\frac{0}{0}$*; indeterminate.*

$\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x}$*, by L.R.*

The last limit also has the form $\frac{0}{0}$*, so, we can apply L.R. again.*

$\displaystyle\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$

(c) $\displaystyle\lim_{x\to \infty} \frac{e^x}{x^4}$

Solution: *The form is* $\frac{\infty}{\infty}$*; indeterminate.*

$\displaystyle\lim_{x\to \infty} \frac{e^x}{x^4} = \lim_{x\to \infty} \frac{e^x}{4x^3}$ *, by L.R.*

The last limit also has the form $\frac{\infty}{\infty}$*. It turns out that we can apply L.R. again, and again, and again:*

$\displaystyle\lim_{x\to \infty} \frac{e^x}{x^4} = \lim_{x\to \infty} \frac{e^x}{4x^3} = \lim_{x\to \infty} \frac{e^x}{12x^2} = \lim_{x\to \infty} \frac{e^x}{24x} = \lim_{x\to \infty} \frac{e^x}{24} = \infty$

This result shows that the function e^x *grows much faster than* x^4*, as* x *goes to* ∞*.*

Problem 2 *State the form of the limit. Determine whether the form is determinate or indeterminate. Evaluate each limit.*

(a) $\displaystyle\lim_{x\to \infty} \left(\ln(1 + e^{-x})\right)^x$

Solution: *Since* $\lim_{x\to\infty}(1+e^{-x}) = 1+0 = 1$ *and* $\ln(1) = 0$, *this limit is of the form* 0^{∞}. *This is a determinate form.* $\lim_{x\to\infty}\left(\ln(1+e^{-x})\right)^x = 0$.

(b) $\lim_{x\to\infty}\left(\frac{1}{x}+1\right)^{\frac{1}{x}}$

Solution: *This limit is of the form* 1^0, *which is a determinate form.*
Thus, $\lim_{x\to\infty}\left(\frac{1}{x}+1\right)^{\frac{1}{x}} = 1$

(c) $\lim_{x\to\infty}\left(\frac{\arctan(x)}{x}\right)$

Solution: *Since* $\lim_{x\to\infty}\arctan(x) = \frac{\pi}{2}$, *this limit is of the form* $\frac{\#}{\infty}$, *which is a determinate form.*

Thus, $\lim_{x\to\infty}\left(\frac{\arctan(x)}{x}\right) = 0$

(d) $\lim_{x\to\infty}(x-\ln(x))$

Solution: *This limit is of the form* $\infty-\infty$, *which is an indeterminate form. We can rewrite this as:*

$$\lim_{x\to\infty}(x-\ln(x)) = \lim_{x\to\infty}\left(x\left(1-\frac{\ln(x)}{x}\right)\right)$$

We can see that $\lim_{x\to\infty}\left(\frac{\ln(x)}{x}\right)$ *is of the form* $\frac{\infty}{\infty}$, *so we can use L'Hopital's Rule.*

$$\overset{L.R.}{\Longrightarrow}\lim_{x\to\infty}\left(\frac{1/x}{1}\right) = \lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$$

Now we have:

$$\lim_{x\to\infty}\left(x\left(1-\frac{\ln(x)}{x}\right)\right)$$

This limit has the form $\infty\cdot 1$. *This is a determinate form, and, therefore,* $\lim_{x\to\infty}(x-\ln(x)) = \infty$

(e) $\lim_{x\to\infty}\left(x\ln\left(\frac{1}{x}\right)\right)$

Solution: *As x approaches ∞, $\frac{1}{x}$ approaches 0 from the right. So*

$$\lim_{x\to\infty} \ln\left(\frac{1}{x}\right) = -\infty$$

Therefore, the limit in question is of the form $\infty \cdot -\infty$, which is a determinate form.
Thus,

$$\lim_{x\to\infty}\left(x\ln\left(\frac{1}{x}\right)\right) = -\infty$$

(f) $\lim_{x\to 0^+}(\sin(x)\cot(x))$

Solution: *Since $\lim_{x\to 0^+}\cot(x) = \infty$, this limit is of the form $0\cdot\infty$. This is an indeterminate form.*
Note: $\cot(x) = \frac{\cos(x)}{\sin(x)}$. *So*

$$\lim_{x\to 0^+}(\sin(x)\cot(x)) = \lim_{x\to 0^+}\cos(x) = 1.$$

Problem 3 *Circle the correct answer in each part:*

(a) *Consider the limit* $\lim_{x\to 0}(\cos(x))^{\sin(x)}$.

(i) *Evaluate the limit.*

i. *the limit DNE*
ii. e
iii. 1
iv. ∞
v. $-\infty$
vi. 0
vii. *none of the previous answers is correct*

Solution: ***The correct choice is (iii).***
Evaluation of limit:

$$\lim_{x\to 0}\underbrace{(\cos(x))^{\sin(x)}}_{\text{form } 1^0} = 1$$

(ii) *What Limit Law, rule or technique did you use to find this limit?*

i. *The Squeeze Theorem;*

ii. *L'Hôpital's Rule;*
iii. *The Product Law;*
iv. *evaluated the function at $x = 0$, since the function is continuous at $x = 0$;*
v. *none of the previous answers is correct*

Solution: *The correct choice is (iv).*

(b) *Evaluate the limit* $\displaystyle\lim_{x\to 4^-} \frac{\ln(x)}{x-4}$.

(i) *the limit DNE*
(ii) e
(iii) 1
(iv) ∞
(v) $-\infty$
(vi) 0
(vii) *none of the previous answers is correct*

Solution: *The correct choice is (v).*

Evaluation of limit: $\displaystyle\lim_{x\to 4^-} \frac{\ln(x)}{x-4} = -\infty$,

since the limit is of the form $\dfrac{\#}{0}$, *and since the numerator is positive and the denominator negative.*

(c) *Evaluate the limit* $\displaystyle\lim_{x\to \infty} \frac{\ln(x)}{x-4}$.

(i) *the limit DNE*
(ii) e
(iii) 1
(iv) ∞
(v) $-\infty$
(vi) 0
(vii) *none of the previous answers is correct*

Solution: *The correct choice is (vi).*

Evaluation of limit:

$$\lim_{x\to\infty} \underbrace{\frac{\ln(x)}{x-4}}_{\text{form } \infty/\infty} \overset{L.H.}{=} \lim_{x\to\infty} \frac{1/x}{1}$$
$$= 0$$

(d) *Consider the limit*

$$\lim_{h\to 0} \frac{(2+h)^3 - 8}{h} = f'(2).$$

Determine the function f.

(i) *such a function DNE;*
(ii) $f(x) = x^3$;
(iii) $f(x) = (2+x)^3$;
(iv) $f(x) = \dfrac{(2+x)^3}{x}$;
(v) *none of the previous answers is correct*

Solution: *The correct choice is (ii).*

(e) *Consider the limit:* $\displaystyle\lim_{x\to 0^+} \left(\frac{\sin(x)}{x}\right)^{|\ln(x)|}$.

Determine the form of this limit.

(i) $\dfrac{0}{0}$;
(ii) $\dfrac{\infty}{\infty}$;
(iii) 1^0;
(iv) 0^0;
(v) 1^∞;
(vi) ∞^∞;
(vii) *none of the previous answers is correct*

Solution: *The correct choice is (v).*

Problem 4 *Determine the following limits. Use L'Hôpital's Rule if applicable.*

(a) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

Solution: *This limit is of the form:* $\frac{\infty}{\infty}$. *L.R. is applicable, but we don't need to apply it.*

$$\begin{aligned}
\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty}\frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}} \\
&= \lim_{x\to\infty}\frac{x}{|x|\sqrt{1+\frac{1}{x^2}}} \\
&= \lim_{x\to\infty}\frac{x}{x\sqrt{1+\frac{1}{x^2}}} \\
&= \lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}} \\
&= \frac{1}{\sqrt{1+0}} = 1
\end{aligned}$$

(b) $\lim\limits_{x\to-\infty} x^2e^x$

Solution: *This limit is of the form:* $\infty\cdot 0$

$$\begin{aligned}
\lim_{x\to-\infty} x^2e^x &= \lim_{x\to-\infty}\frac{x^2}{e^{-x}} \quad \left(\textit{of the form } \frac{\infty}{\infty}\right) \\
&\overset{L.R.}{=} \lim_{x\to-\infty}\frac{2x}{-e^{-x}} \quad \left(\textit{of the form } \frac{\infty}{\infty}\right) \\
&\overset{L.R.}{=} \lim_{x\to-\infty}\frac{2}{e^{-x}} \\
&= 0
\end{aligned}$$

where "L.R." above an equals sign means that that equality is due to "L'Hôpital's Rule".

(c) $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$

Solution: *This limit is of the form:* ∞^0

$$\begin{aligned}
\lim_{x\to\infty} x^{\frac{1}{x}} &= \lim_{x\to\infty} e^{\ln\left(x^{\frac{1}{x}}\right)} \\
&= \lim_{x\to\infty} e^{\frac{1}{x}\ln x} \\
&= e^{\lim\limits_{x\to\infty}\frac{\ln x}{x}} \quad \left(\textit{limit is of the form } \frac{\infty}{\infty}\right) \\
&\overset{L.R.}{=} e^{\lim\limits_{x\to\infty}\frac{\frac{1}{x}}{1}} \\
&= e^{\lim\limits_{x\to\infty}\frac{1}{x}} \\
&= e^0 = 1
\end{aligned}$$

(d) $\displaystyle\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}$

Solution: *This limit is of the form:* 1^{∞}

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x} &= \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{2}{x}\right)^{x}\right)} \\
&= \lim_{x\to\infty} e^{x\ln\left(1+\frac{2}{x}\right)} \\
&= e^{\lim\limits_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{1/x}} \quad \left(\textit{limit is of the form } \frac{0}{0}\right) \\
&\overset{L.R.}{=} e^{\lim\limits_{x\to\infty}\frac{-2x^{-2}\cdot\frac{1}{(1+2/x)}}{-x^{-2}}} \\
&= e^{\lim\limits_{x\to\infty}\left(2\left(\frac{1}{1+\frac{2}{x}}\right)\right)} \\
&= e^{2}
\end{aligned}$$

(e) $\displaystyle\lim_{\theta\to 0^{+}}(\sin(\theta))^{\tan(\theta)}$

Solution: *This limit is of the form:* 0^{0}

$$\begin{aligned}
\lim_{\theta\to 0^{+}}(\sin(\theta))^{\tan\theta} &= \lim_{\theta\to 0^{+}} e^{\ln\left((\sin(\theta))^{\tan(\theta)}\right)} \\
&= \lim_{\theta\to 0^{+}} e^{\tan(\theta)\ln(\sin(\theta))} \\
&= e^{\lim\limits_{\theta\to 0^{+}}\frac{\ln(\sin(\theta))}{\cot(\theta)}} \quad \left(\textit{limit is of the form } \frac{\infty}{\infty}\right) \\
&\overset{L.R.}{=} e^{\lim\limits_{\theta\to 0^{+}}\frac{\cos(\theta)\cdot\frac{1}{\sin(\theta)}}{-\csc^{2}(\theta)}} \\
&= e^{\lim\limits_{\theta\to 0^{+}}\left(\frac{\cos(\theta)}{\sin(\theta)}\cdot\frac{-\sin^{2}(\theta)}{1}\right)} \\
&= e^{\lim\limits_{\theta\to 0^{+}}(-\cos(\theta)\cdot\sin(\theta))} \\
&= e^{0} = 1
\end{aligned}$$

Problem 5 *Let f and g be two functions such that $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.*

We say that f **grows faster** *than g as x goes to ∞ if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, *or, equivalently, if*

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

If the limit exists, namely, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = M$, *for some positive number M,*

then we say that the functions f and g have **comparable growth rates**.

In other words, we compare the growth rates of functions f and g by computing the limit of the form $\frac{\infty}{\infty}$: $\lim_{x\to\infty} \frac{f(x)}{g(x)}$.

For each of the following pairs of functions, determine which of the pair grows faster or state that they have comparable growth rates. Justify your answer by computing the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$.

(a) $b^x; x^x; b > 1$

Solution: $\lim_{x\to\infty} \frac{b^x}{x^x} = \lim_{x\to\infty} \left(\frac{b}{x}\right)^x = 0$. *Since this limit is of the form 0^∞, which is a determinate form. Therefore, x^x grows faster.*

(b) $x^x; \left(\frac{x}{e}\right)^x$

Solution: $\lim_{x\to\infty} \frac{x^x}{(x/e)^x} = \lim_{x\to\infty} \frac{x^x}{\frac{x^x}{e^x}} = \lim_{x\to\infty} x^x \cdot \frac{e^x}{x^x} = \lim_{x\to\infty} e^x = \infty$ *Therefore, x^x grows faster.*

(c) $x^3; x^3 \cdot \ln(x)$

Solution: $\lim_{x\to\infty} \frac{x^3}{x^3 \cdot \ln(x)} = \lim_{x\to\infty} \frac{1}{\ln(x)} = 0$. *Therefore, $x^3 \cdot \ln(x)$ grows faster.*

(d) $a^x; b^x; 0 < a < b$

Solution: $\lim_{x\to\infty} \frac{a^x}{b^x} = \lim_{x\to\infty} \left(\frac{a}{b}\right)^x = 0$ *Therefore, b^x grows faster*

(e) $\log_a(x); \log_b(x); 1 < a < b$

Solution: $\displaystyle\lim_{x\to\infty}\frac{\log_a(x)}{\log_b(x)}$

By L'Hopital's Rule: $\displaystyle\lim_{x\to\infty}\frac{\frac{1}{x\ln(a)}}{\frac{1}{x\ln(b)}} = \lim_{x\to\infty}\left(\frac{1}{x\ln(a)}\cdot\frac{x\ln(b)}{1}\right) = \frac{\ln b}{\ln a}$

Therefore, $\log_a(x)$ *and* $\log_b(x)$ *grow at comparable rates.*

(f) $\ln^3(x); x^{1/2}$

Solution: $\displaystyle\lim_{x\to\infty}\frac{\ln^3(x)}{x^{1/2}}$

By L'Hopital's Rule: $\displaystyle\lim_{x\to\infty}\frac{\ln^3(x)}{x^{1/2}} = \lim_{x\to\infty}\frac{3\ln^2(x)\cdot(1/x)}{(1/2)x^{-1/2}} = \lim_{x\to\infty}\frac{6\cdot\ln^2(x)}{x^{1/2}}$

By L'Hopital's Rule: $\displaystyle\lim_{x\to\infty}\frac{12\ln(x)\cdot(1/x)}{(1/2)x^{-1/2}} = \lim_{x\to\infty}\frac{24\cdot\ln(x)}{x^{1/2}}$

By L'Hopital's Rule: $\displaystyle\lim_{x\to\infty}\frac{48}{x^{1/2}} = 0$ *Therefore,* $x^{1/2}$ *grows faster*

(g) $x; \ln(x)\sqrt{x}$

Solution: $\displaystyle\lim_{x\to\infty}\frac{x}{\ln(x)\sqrt{x}} = \lim_{x\to\infty}\frac{x^{1/2}}{\ln(x)}$

By L'Hopital's Rule: $\displaystyle\lim_{x\to\infty}\frac{(1/2)x^{-1/2}}{x^{-1}} = \lim_{x\to\infty}(1/2)x^{1/2} = \infty$

Therefore, x *grows faster*

(h) *Challenge:* $x^{40}; 1.004^x$ *(Hint: Use the substitution* $x = \ln(t)$*.)*

Solution:

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^{40}}{1.004^x} &= \lim_{t\to\infty}\frac{[\ln(t)]^{40}}{1.004^{\ln(t)}}\\
&= \lim_{t\to\infty}\frac{\ln(t)^{40}}{e^{\ln(1.004)^{\ln(t)}}}\\
&= \lim_{t\to\infty}\frac{\ln(t)^{40}}{t^{\ln(1.004)}}\\
&= \lim_{t\to\infty}\left(\frac{\ln(t)}{t^{1.004/40}}\right)^{40}\\
&= \left[\lim_{t\to\infty}\left(\frac{\ln(t)}{t^{1.004/40}}\right)\right]^{40}\\
&\overset{L.R.}{=} \left[\lim_{t\to\infty}\frac{\frac{1}{t}}{(1.004/40)t^{(1.004/40)-1}}\right]^{40}\\
&= \left[\lim_{t\to\infty}\frac{1}{(1.004/40)t^{((1.004/40)-1+1)}}\right]^{40}\\
&= 0^{40}\\
&= 0
\end{aligned}$$

Therefore, 1.004^x *grows faster.*

(i) *Put the following functions in order of growth rate.*

$$x^3 \cdot \ln(x), \ln^3(x), x^x, 1.004^x, x^{40}, x^3$$

Solution:

$$\ln^3(x) << x^3 << x^3 \cdot \ln(x) << x^{40} << 1.004^x << x^x$$