Applications of Integrals (AOI) - Solutions

Problem 1 *Solve the following word problems:*

(a) *The velocity function for an object moving along a line east/west is given by $v(t) = -t^2 + 4t - 3$ feet per minute.*

(i) *Find the total displacement the object traveled from 2 minutes to 6 minutes (assume east is positive).*

Solution: *The object's total displacement is given by* $\int_2^6 v(t)\,dt$. *So we compute:*

$$\begin{aligned}\int_2^6 v(t)\,dt &= \int_2^6 (-t^2 + 4t - 3)\,dt \\ &= \left[-\frac{1}{3}t^3 + 2t^2 - 3t\right]_2^6 \\ &= (-72 + 72 - 18) - \left(-\frac{8}{3} + 8 - 6\right) \\ &= \frac{8}{3} - 20 = -\frac{52}{3}.\end{aligned}$$

So the object's displacement is $\frac{52}{3}$ *feet west of its original location.*

(ii) *Find the total distance the object traveled from 2 minutes to 6 minutes.*

Solution: *First notice that* $v(t) = -(t^2 - 4t + 3) = -(t-1)(t-3)$. *So we can see that*

$$\begin{aligned}&v(t) > 0 \text{ when } 2 \le t < 3. \\ &v(t) < 0 \text{ when } 3 < t \le 6.\end{aligned}$$

Thus, the total distance that the object traveled from 2 minutes to 6 minutes is:

$$\begin{aligned}\int_2^6 |v(t)|\,dt &= \int_2^3 |v(t)|\,dt + \int_3^6 |v(t)|\,dt \\ &= \int_2^3 v(t)\,dt + \int_3^6 -v(t)\,dt \\ &= \int_2^3 (-t^2 + 4t - 3)\,dt - \int_3^6 (-t^2 + 4t - 3)\,dt \\ &= \left[-\frac{1}{3}t^3 + 2t^2 - 3t\right]_2^3 - \left[-\frac{1}{3}t^3 + 2t^2 - 3t\right]_3^6 \\ &= \left((-9 + 18 - 9) - (-\frac{8}{3} + 8 - 6)\right) - \\ &((-72 + 72 - 18) - (-9 + 18 - 9)) \\ &= \left(0 - 2 + \frac{8}{3}\right) - (-18 - 0) \\ &= 16 + \frac{8}{3} = \frac{56}{3}.\end{aligned}$$

So, the total distance that the object traveled is $\dfrac{56}{3}$ *feet.*

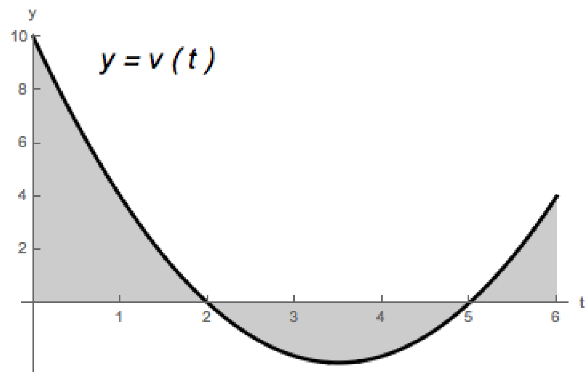
(iii) *Suppose that the object's position 2 minutes into the trip is 5 feet of a placement marker. What is its position (relative to the placement marker) at 6 minutes.*

Solution: $s(6) = s(2) + \displaystyle\int_2^6 v(t)\,dt = 5 + \left(-\frac{52}{3}\right) = -\frac{37}{3}.$

So the object's position at 6 minutes is $\dfrac{37}{3}$ *feet west of the placement marker.*

(b) *Sammy the Snail sets up camp in the median of I-70 and, starting at noon and ending at 6pm, hikes back and forth along the highway. He starts his hike at his campsite. His velocity at time t hours (after noon) is given by $v(t) = (t-2)(t-5)$ inches per hour. Find the total distance Sammy travelled on his hike.*

Solution: *The total distance that Sammy travels is* $\displaystyle\int_0^6 |v(t)|\,dt$. *The following picture indicates where $v(t)$ is positive and negative:*



So we compute:

$$\begin{aligned}
\int_0^6 |v(t)|\,dt &= \int_0^2 |v(t)|\,dt + \int_2^5 |v(t)|\,dt + \int_5^6 |v(t)|\,dt \\
&= \int_0^2 v(t)\,dt - \int_2^5 v(t)\,dt + \int_5^6 v(t)\,dt \\
&= \int_0^2 (t^2-7t+10)\,dt - \int_2^5 (t^2-7t+10)\,dt + \int_5^6 (t^2-7t+10)\,dt \\
&= \left[\frac{1}{3}t^3 - \frac{7}{2}t^2 + 10t\right]_0^2 - \left[\frac{1}{3}t^3 - \frac{7}{2}t^2 + 10t\right]_2^5 + \left[\frac{1}{3}t^3 - \frac{7}{2}t^2 + 10t\right]_5^6 \\
&= \left(\left(\frac{8}{3} - 14 + 20\right) - 0\right) - \left(\left(\frac{125}{3} - \frac{175}{2} + 50\right) - \left(\frac{8}{3} + 6\right)\right) + \\
&\left((72 - 126 + 60) - \left(\frac{125}{3} - \frac{175}{2} + 50\right)\right) \\
&= -82 + 175 - 78 = 15.
\end{aligned}$$

So Sammy has traveled a distance of 15 inches.

Problem 2 *Suppose that $r(t) = r_0 e^{-kt}$ (with $k > 0$) is the rate at which a nation extracts oil. The current rate of extraction is $r(0) = 10^7$ barrels/yr. Also assume that the estimate of the total oil reserve (ie, the amount of oil remaining beneath the ground in this country) is 2×10^9 barrels.*

(a) *Find $A(t)$, the total amount of oil extracted by the nation after t years.*

Solution: $A(t) = \int_0^t r(s)\,ds$

$$\begin{aligned} A(t) &= \int_0^t r(s)\,ds \\ &= \int_0^t r_0 e^{-ks}\,ds \\ &= -\frac{r_0}{k}\left[e^{-ks}\right]_0^t \\ &= -\frac{r_0}{k}\left(e^{-kt} - 1\right) \\ &= -\frac{1}{k}10^7\left(e^{-kt} - 1\right) \end{aligned}$$

(b) *Evaluate $\lim_{t\to\infty} A(t)$ and explain the meaning of this limit.*

Solution:

$$\begin{aligned} \lim_{t\to\infty} A(t) &= \lim_{t\to\infty} -\frac{1}{k}10^7\left(e^{-kt} - 1\right) \\ &= -\frac{1}{k}10^7(0 - 1) \\ &= \frac{1}{k}10^7. \end{aligned}$$

(c) *Find the minimum constant k for which the total oil reserves will last forever.*

Solution: *For the oil reserves to last forever, we need that*

$$\begin{aligned} &\lim_{t\to\infty} A(t) \le 2 \times 10^9 \\ \iff \quad & \frac{1}{k}10^7 \le 2 \times 10^9 \\ \iff \quad & \frac{1}{k} \le 2 \times 10^2 = 200 \\ \iff \quad & \frac{1}{200} \le k. \end{aligned}$$

So the minimum value for k is $\frac{1}{200}$.

(d) *Suppose that the constant k is half the minimum value found in part (c). When will the nation deplete its oil reserve?*

Solution: *First note that* $k = \frac{1}{2} \cdot \frac{1}{200} = \frac{1}{400}$. *We want to find the value of* t *such that:*

$$\begin{aligned}
A(t) &= 2 \times 10^9 \\
-400 \times 10^7 \left(e^{-\frac{1}{400}t} - 1\right) &= 2 \times 10^9 \\
\left(e^{-\frac{1}{400}t} - 1\right) &= \frac{2 \times 10^9}{-400 \times 10^7} = -\frac{1}{2} \\
e^{-\frac{1}{400}t} &= \frac{1}{2} \\
-\frac{1}{400}t &= \ln\left(\frac{1}{2}\right) = -\ln(2) \\
t &= 400\ln(2) \approx 277.259 \text{ years.}
\end{aligned}$$

Problem 3 *Assume that the rate of change (in dollars per day) of the price of shares of stock in the WeSaySo Company (with* t *in days) is modeled by the equation* $r(t) = -3t^2 + 30t - 63$ *(note that this is technically a discrete function, but prices change so often with stocks that modeling this with a continuous function makes sense). Assume also that the price of a share of stock on day 1 (i.e.,* $t = 1$*) is \$51. Answer the following questions:*

(a) *Find the rate of change of price at* $t = 5$.

Solution: $r(5) = -3(25) + 30(5) - 63 = -75 + 150 - 63 = 12$ *dollars/day.*

(b) *Find the price of a share of stock at* $t = 5$.

Solution: *Let* $p(t)$ *denote the price of a share of stock at any time* t*. Then notice that* $r(t) = p'(t)$*. We will first find* $p(t)$ *for general* t*, and then substitute* $t = 5$ *to solve this problem.*

$$\begin{aligned}
p(t) &= \int_1^t r(s)\,ds + p(1) \\
&= \int_1^t \left(-3s^2 + 30s - 63\right)\,ds + 51 \\
&= \Big[-s^3 + 15s^2 - 63s \Big]_1^t + 51 \\
&= \left(-t^3 + 15t^2 - 63t\right) - (-1 + 15 - 63) + 51 \\
&= -t^3 + 15t^2 - 63t + 100.
\end{aligned}$$

So, $p(5) = -125 + 375 - 315 + 100 = 35\$$.

(c) *How fast is the rate of change of price changing at* $t = 5$*?*

Solution: $r'(t) = -6t + 30$. *So,* $r'(5) = -30 + 30 = 0$.

(d) *How much did the price of a share of stock change in the first 6 days (i.e., on* $[1, 6]$*)?*

Solution: $p(6) - p(1) = (-216 + 540 - 378 + 100) - 51 = -5\$$.

(e) *What was the greatest rate of change of price during the first 6 days (i.e., on $[1,6]$)?*

Solution: *This question wants us to maximize $r(t)$ on the closed interval $[1,6]$. So we need to find all critical points of $r(t)$ on $[1,6]$.*

$$r'(t) = -6t + 30 := 0 \quad \Longrightarrow \quad t = 5$$

Then, using the closed interval method, we simply plug $t = 1, 5, 6$ into $r(t)$ and check which has the greatest output.

$$\begin{aligned} r(1) &= -3 + 30 - 63 = -36 \\ r(5) &= -75 + 150 - 63 = 12 \\ r(6) &= -108 + 180 - 63 = 9. \end{aligned}$$

Thus, the greatest rate of change of price is 12 dollars/day when $t = 5$.

(f) *What was the greatest price of a share of stock during the first 6 days (i.e., on $[1,6]$)?*

Solution: *This question wants us to maximize $p(t)$ on the closed interval $[1,6]$. So we need to find all critical points of $p(t)$ on $[1,6]$. But note that $p'(t) = r(t)$. So critical points of $p(t)$ are just roots of $r(t)$.*

$$\begin{aligned} r(t) &= 0 \\ -3t^2 + 30t - 63 &= 0 \\ -3(t^2 - 10t + 21) &= 0 \\ -3(t-3)(t-7) &= 0 \\ t = 3, 7 \quad &\Longrightarrow \quad t = 3 \end{aligned}$$

since 7 is not in the interval $[1,6]$. So we use the Closed Interval Method:

$$\begin{aligned} p(1) &= -1 + 15 - 63 + 100 = 51 \\ p(3) &= -27 + 135 - 189 + 100 = 19 \\ p(6) &= -216 + 540 - 378 + 100 = 46 \end{aligned}$$

Thus, the greatest price of the stock is \$51 when $t = 1$.

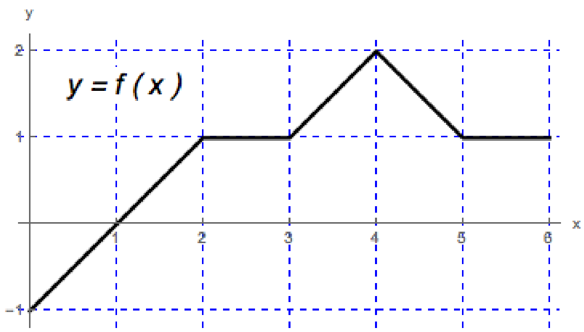
Problem 4 *A cup of coffee has temperature $20 + 75e^{-.02t}$ degrees (Celsius) t minutes after being poured into a cup. What is the average temperature of the coffee during the first half hour?*

Solution: *Let $T(t) := 20 + 75e^{-.02t}$. We want the average value of T on $[0,30]$.*

$$\begin{aligned} T_{\text{avg}} &= \frac{1}{30-0} \int_0^{30} \left(20 + 75e^{-.02t}\right)\, dt \\ &= \frac{1}{30}\left[20t - \frac{75}{.02}e^{-.02t}\right]_0^{30} \\ &= \frac{1}{30}\left[20t - 3750e^{-.02t}\right]_0^{30} \\ &= \frac{1}{30}\left[\left(20(30) - 3750e^{-0.6}\right) - (0 - 3750)\right] \\ &= \frac{1}{30}\left(4350 - 3750e^{-0.6}\right) \\ &= 145 - 125e^{-0.6} \end{aligned}$$

So the average temperature of the coffee during the first half hour is $145 - 125e^{-0.6} \approx 76.4$ *degrees Celsius.*

Problem 5 *The graph of a function f defined on the interval $[0, 6]$ is given in the figure.*

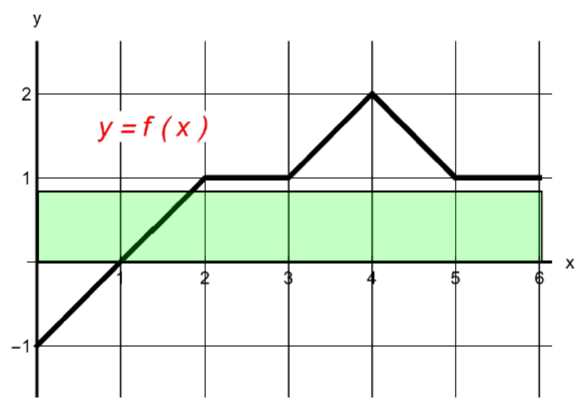


(a) *Compute the net area of the region between the graph of f and the x-axis, on the interval $[0, 6]$.*

Solution: The net area $= -1/2 + 1/2 + 1 + 3 + 1 = 5$

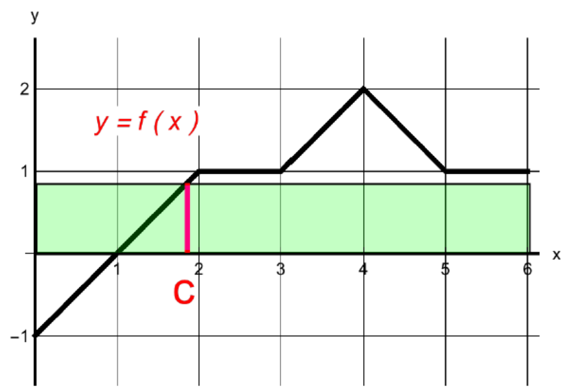
(b) *Draw a rectangle with base on the x-axis, $0 \leq x \leq 6$, whose area is equal to the net area in part (a).*

Solution: *Let A be the area of the rectangle. Then $A =$(base)(height)$= 6h = 5$.*
So, $6h = 5 \implies h = \dfrac{5}{6}$



(c) *In the figure, mark a point c in $(0, 6)$ such that $f(c)$ is the height of the rectangle from part (b).*

Solution: $f(c) = h = \dfrac{5}{6}$



Note: In this example, there is only one such point c. In some cases there may be more than one.

(d) *Using the figure and parts (a-c), what is the relationship between the rectangle from part (b), the net area from part (a), and the average value of f on $[0, 6]$?*

Solution: *The Mean Value Theorem for integrals states that there exists a point d in (a, b) such that*

$$f(d) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

The right-hand side of this equation is the average value of f on $[a, b]$.

Rewriting we have

$$f(d) \cdot (b-a) = \int_a^b f(x)\, dx.$$

The right-hand side of this equation is the net area found in part (a): $\displaystyle\int_0^6 f(x)\, dx = 5$.

The left-hand side of this equation is the area of the rectangle in part (b). The point d is the point c we found in part (c). From (b) and (c), $f(c) = \dfrac{5}{6}$ is the average value of f on $[0, 6]$.

Problem 6 *Find all points at which the given function equals its average value on the given interval.*

(a) $f(x) = e^x \qquad [0, 4]$

Solution: *First, we need to find f_{avg}:*

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{4-0} \int_0^4 e^x\, dx \\ &= \frac{1}{4} \Big[e^x \Big]_0^4 \\ &= \frac{1}{4} \left(e^4 - 1 \right) \end{aligned}$$

So we are looking for all values $c \in [0,4]$ such that:

$$\begin{aligned} f(c) &= \frac{1}{4}(e^4 - 1) \\ \implies \quad e^c &= \frac{1}{4}(e^4 - 1) \\ \implies \quad c &= \ln\left(\frac{1}{4}(e^4 - 1)\right) \end{aligned}$$

Therefore, our answer is $\ln\left(\frac{1}{4}(e^4 - 1)\right)$.

(b) $g(x) = \frac{\pi}{4}\sin(x) \qquad [0, \pi]$

Solution: *First, we need to find g_{avg}:*

$$\begin{aligned} g_{\text{avg}} &= \frac{1}{\pi - 0}\int_0^{\pi} \frac{\pi}{4}\sin(x)\,dx \\ &= \frac{1}{4}\Big[-\cos(x)\Big]_0^{\pi} \\ &= \frac{1}{4}\left(-(-1-1)\right) \\ &= \frac{1}{2} \end{aligned}$$

So we are looking for all values $c \in [0, \pi]$ such that:

$$\begin{aligned} & g(c) = \frac{1}{2} \\ \implies \quad & \frac{\pi}{4}\sin(c) = \frac{1}{2} \\ \implies \quad & \sin(c) = \frac{2}{\pi} \\ \implies \quad & c = \arcsin\left(\frac{2}{\pi}\right), \pi - \arcsin\left(\frac{2}{\pi}\right) \end{aligned}$$

Therefore, our two answers are $\arcsin\left(\frac{2}{\pi}\right), \pi - \arcsin\left(\frac{2}{\pi}\right)$.

Problem 7 *Find the average value of the function $g(t) = 4te^{-t^2}$ on the interval $[0, 3]$.*

Solution: *We know $g_{\text{avg}} = \frac{1}{b-a}\int_a^b g(t)\,dt$ with $[a, b]$ as $[0, 3]$. We'll make a substitution, $w = t^2$ with*

$dw = 2tdt$. *Notice that when* $t = 0$, $w = (0)^2 = 0$ *and when* $t = 3$, $w = (3)^2 = 9$.

$$
\begin{aligned}
g_{\text{avg}} &= \frac{1}{3-0} \int_0^3 4te^{-t^2}\, dt \\
&= \frac{1}{3} \int_0^9 2e^{-w}\, dw \\
&= \frac{1}{3} \Big[-2e^{-w} \Big]_0^9 \\
&= \frac{1}{3} \big[(-2e^{-9}) - (-2e^{0}) \big] \\
&= \frac{1}{3} \left(2 - \frac{2}{e^9} \right)
\end{aligned}
$$