Rules of differentiation (ROD) - Solutions

Problem 1 *For each of the following functions, use the "short cut derivative rules" to compute their derivative.*

(a) $f(x) = \sqrt{x}$

Solution:

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}.$$

$$\begin{aligned} f'(x) &= \frac{1}{2}x^{\frac{1}{2}-1} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

(b) $s(u) = \dfrac{5}{u^2}$

Solution:

$$s(u) = \frac{5}{u^2} = 5u^{-2}.$$

$$\begin{aligned} s'(u) &= 5(-2)u^{-2-1} \\ &= -10u^{-3} \\ &= \frac{-10}{u^3} \end{aligned}$$

(c) $p(t) = t^5 + 4t^3 + \pi$

Solution:

$$\begin{aligned} p'(t) &= 5t^{5-1} + 4(3)t^{3-1} + 0 \\ &= 5t^4 + 12t^2 \end{aligned}$$

Note that $\dfrac{d}{dx}(\pi) = 0$ *because* π *is a constant.*

Problem 2 *Given the polynomial function* q *defined by* $q(v) = 2v^3 - 5v^2 + 7v - 9$ *find:*

(a) *The slope of the tangent line to the graph of* q *at the point where* $v = 3$ *using the limit definition of a derivative.*

Solution: *To compute $q'(3)$ from the limit definition we use either*

$$\lim_{v\to 3}\frac{q(v)-q(3)}{v-3} \quad \text{or} \quad \lim_{h\to 0}\frac{q(3+h)-q(3)}{h}.$$

Using the second formula:

$$\begin{aligned}
q'(3) &= \lim_{h\to 0}\frac{q(3+h)-q(3)}{h} \text{ which has form } \frac{0}{0}\\
&= \lim_{h\to 0}\frac{(2(3+h)^3-5(3+h)^2+7(3+h)-9)-(54-45+21-9)}{h}\\
&= \lim_{h\to 0}\frac{(2(27+27h+9h^2+h^3)-5(9+6h+h^2)+21+7h-9)-21}{h}\\
&= \lim_{h\to 0}\frac{54+54h+18h^2+2h^3-45-30h-5h^2+21+7h-9-21}{h}\\
&= \lim_{h\to 0}\frac{31h+13h^2+2h^3}{h}\\
&= \lim_{h\to 0}(31+13h+2h^2)\\
&= 31.
\end{aligned}$$

(b) *The slope of the tangent line to the graph of q at the point where $v=3$ using the "short-cut derivative rules" to find a formula for q' and evaluating $q'(3)$.*

Solution:

$$\begin{aligned}
q'(v) &= (2v^3-5v^2+7v-9)'\\
&= (2v^3)'+(-5v^2)'+(7v)'+(-9)'\\
&= 3\cdot 2\cdot v^{3-1}+-5\cdot(2)\cdot v^{2-1}+7\cdot 1\cdot v^{1-1}+0\\
&= 6v^2-10v+7\\
&\implies q'(3)=6\cdot 3^2-10\cdot 3+7=31.
\end{aligned}$$

(c) *The equation of the tangent line to the graph of q at $v=3$.*

Solution: *Slope of tangent line to the graph of q at the point where $v=3$:* $q'(3)=6\cdot 3^2-10\cdot 3+7=31$.

Point on tangent line at the point where $v=3$: $(3,q(3))=(3,21)$.

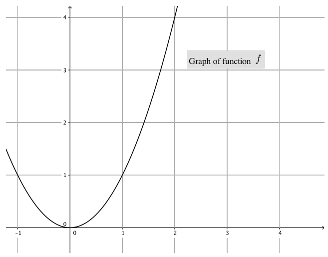
Equation of tangent line to the graph of q at the point where $v=3$:

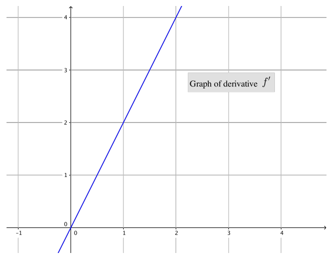
$$\begin{aligned}
y-21 &= 31(x-3)\\
&\implies y=31x-72.
\end{aligned}$$

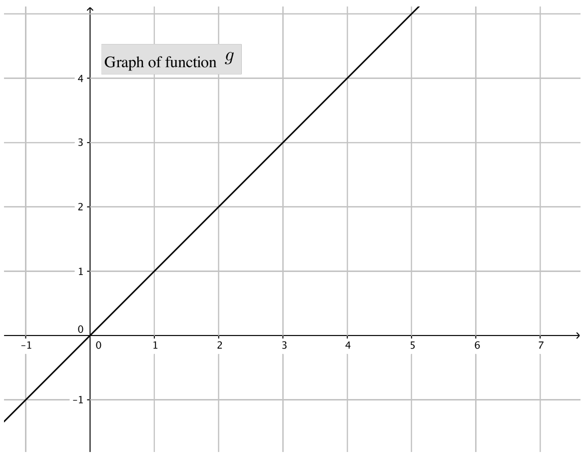
Problem 3 *Find s' of the function s defined by $s(t)=3t^2+5e^t-\dfrac{1}{t}$.*

Solution: $s'(t)=6t+5e^t+\dfrac{1}{t^2}$.

Problem 4 *Use the given graphs of f and g and their accompanying derivatives to answer the following questions.*





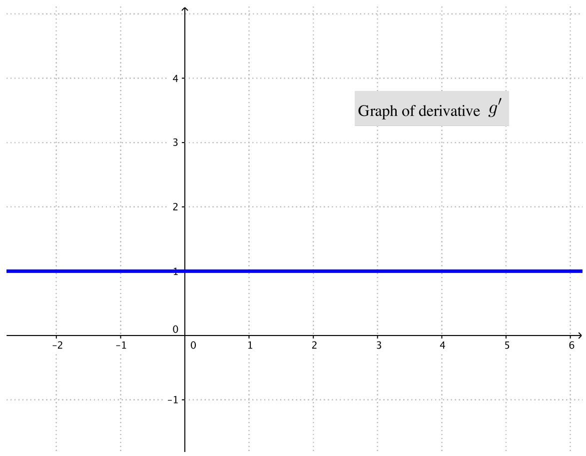


(a) *Write an equation for the tangent line to f at $x = 2$.*

Solution: $f(2) = 4, \quad f'(2) = 4, \quad y - 4 = 4(x - 2), \quad y = 4x - 4$

(b) *Draw the graph of g'*

Solution:



(c) *Compute the value of the $(5f + 3g)'(2)$.*

Solution:

$$
\begin{aligned}
(5f+3g)'(2) &= 5\cdot f'(2) + 3\cdot g'(2)\\
&= 5\cdot 4 + 3\cdot 1\\
&= 20 + 3 = 23.
\end{aligned}
$$

(d) *Find the equation of the tangent line to the graph of $(5f+3g)$ at the point where $x=2$.*

Solution:

$$
\begin{aligned}
(5f+3g)(2) &= 5\cdot f(2) + 3\cdot g(2)\\
&= 5\cdot 4 + 3\cdot 2\\
&= 20 + 6 = 26
\end{aligned}
$$

$$
\begin{aligned}
y - 26 &= 23(x-2)\\
y &= 23x - 20
\end{aligned}
$$

(e) *Use the given graph of f' and g' to find the following:*

(i) *A formula for f'*

Solution: $f'(x) = 2x$. *The graph shows a linear function through the point $(0,0)$ and $(1,2)$.*

(ii) *A formula for g'*

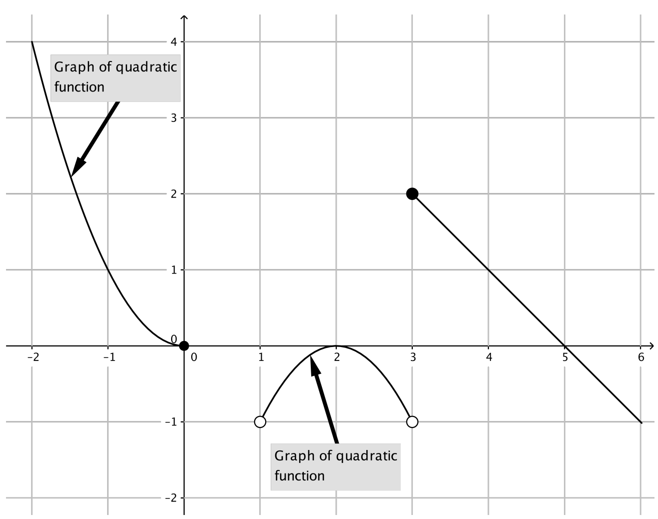
Solution: $g'(x) = 1$. *The graph shows a linear function with slope 0 and through the point $(0,1)$.*

(f) *Find the expression for $(5f+3g)'(x)$.*

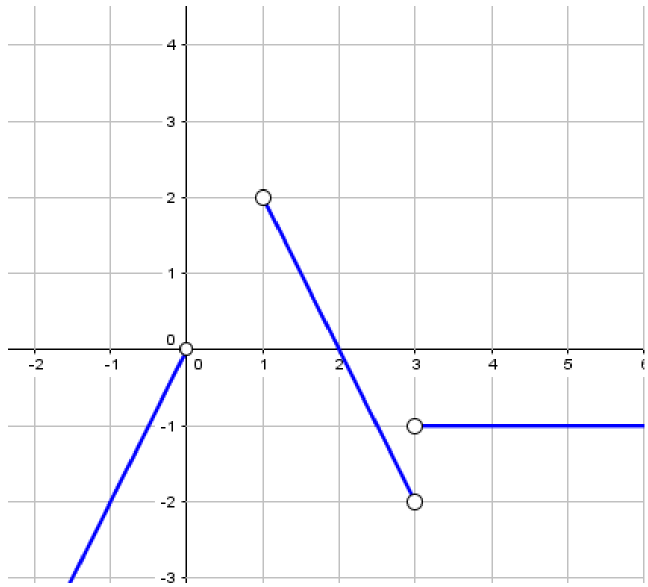
Solution:

$$
\begin{aligned}
(5f+3g)'(x) &= 5\cdot f'(x) + 3\cdot g'(x)\\
&= 5\cdot 2x + 3\cdot 1\\
&= 10x + 3
\end{aligned}
$$

Problem 5 *Sketch the graph of the derivative of the given function:*



Solution:



Problem 6 *A company is producing cell phones. The cost of producing x cell phones is given by $C(x)$, defined by*

$$C(x) = -0.01x^2 + 40x + 400, \text{ for } 0 \leq x < 1000.$$

AVERAGE cost of producing the first x cell phones is given by

$$C_{AVG} = \frac{C(x)}{x}$$

If the company has produced x cell phones, the cost of producing one more item is given by

$$\text{COST of producing one more item} = C(x+1) - C(x)$$

MARGINAL COST is approximation of the cost of producing one more cell phone

$$\text{MARGINAL COST} = C'(x)$$

(a) *Compute the average cost of the first 300 cellphones that the company produces.*

Solution: *The first 300 cell phones costs, on average,*

$$C_{AVG} = \frac{C(300)}{300} = 38.33$$

(b) *Compute the cost of producing one more cell phone, if the company has produced 300 cell phones.*

Solution:

$$C(301) - C(300) = -0.01(301)^2 + 40(301) + 400 - (-0.01(300)^2 + 40(300) + 400) = 33.99$$

(c) *Compute the marginal cost, if 300 cell phones have been produced.*

Solution:

$$\textit{MARGINAL COST} = C'(300) = \big[0.02x + 40\big]_{x=300} = 34$$

(d) *Why is the marginal cost a good approximation of the cost of producing one more item? Explain!*

Solution: *By the definition of the derivative we have that*

$$C'(300) = \lim_{h \to 0} \frac{C(300+h) - C(300)}{h} \approx \frac{C(300+h) - C(300)}{h}$$

for any h near 0. In particular, for $h = 1$, we obtain

$$C'(300) \approx \frac{C(300+1) - C(300)}{1} = C(301) - C(300),$$

Therefore, marginal cost at $x = 300 \approx$ the cost of producing one more item!