# Linear Approximation (LA)- Solutions

**Problem 1** (a) *Find the linearization, $L(x)$, of the function $f(x) = e^{2x}$ at $a = 0$.*

***Solution:*** *Recall that $L(x) = f(a) + f'(a)(x - a)$.*
*Hence $f'(x) = 2e^{2x} \implies f'(0) = 2e^0 = 2$ and $f(0) = e^0 = 1$.*
*Therefore $L(x) = 1 + 2 \cdot x$*

(b) *Using the linearization, $L(x)$, from the part (a), approximate $e$.*

***Solution:***

$$\begin{aligned} e &= e^{2\cdot(1/2)} \\ &= f(1/2) \approx L(1/2) \\ &\approx 1 + 2 \cdot (1/2) \\ &\approx 2 \end{aligned}$$

(c) *Is the estimation found in part (b) an overestimate or an underestimate? **EXPLAIN**.*

***Solution:*** *$f''(x) = 4e^{2x}$ is positive on the interval $\left(0, \frac{1}{2}\right)$. That means the graph of $f$ is concave up between $x = 0$ and $x = \frac{1}{2}$, so this estimate is an underestimate. (The graph of the linearization $L$ lies below the graph of the function $f$.)*

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**Problem 2** *Complete steps (i)-(vii) below in order to estimate the following values using linear approximation:*

(a) $\cos\left(\dfrac{31\pi}{180}\right)$

(b) $\sqrt[3]{8.13}$

(i) *Identify the function, $f(x)$.*

(ii) *Find the nearby value where the function can be easily calculated, $x = a$.*

(iii) *Find $\Delta x = dx$.*

(iv) *Find the linear approximation, $L(x)$.*

(v) *Compute the approximate value of the expression using the linear approximation.*

(vi) *Compare the approximated value to the value given by your calculator.*

(vii) *Compare $dy$ and $\Delta y$ using the value given by your calculator.*

***Solution:*** (a) $\cos\left(\frac{31\pi}{180}\right)$

(i) $f(x) = \cos x$

(ii) $a = \frac{30\pi}{180} = \frac{\pi}{6}$

(iii) $\Delta x = \frac{31\pi}{180} - \frac{\pi}{6} = \frac{\pi}{180}$

(iv)

$$\begin{aligned} L(x) &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) \\ &= \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right) \end{aligned}$$

(v)

$$\begin{aligned} L\left(\frac{31\pi}{180}\right) &= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{31\pi}{180} - \frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{\pi}{180}\right) \\ &= \frac{1}{2}\left(\sqrt{3} - \frac{\pi}{180}\right) \\ &\approx 0.857299 \end{aligned}$$

(vi) $\cos\left(\frac{31\pi}{180}\right) \approx 0.857167$

(vii)

$$dy = L\left(\frac{31\pi}{180}\right) - L\left(\frac{\pi}{6}\right) \approx -0.008727$$

$$\Delta y = \cos\left(\frac{31\pi}{180}\right) - \cos\left(\frac{\pi}{6}\right) \approx -0.008858$$

(b) $\sqrt[3]{8.13}$

(i) $f(x) = \sqrt[3]{x}$.

(ii) $a = 8$.

(iii) $\Delta x = 8.13 - 8 = 0.13$.

(iv)

$$\begin{aligned} L(x) &= f(8) + f'(8)(x-8) \\ &= \sqrt[3]{8} + \frac{1}{3(\sqrt[3]{8})^2}(x-8) \\ &= 2 + \frac{1}{12}(x-8) \end{aligned}$$

(v)

$$\begin{aligned} L(8.13) &= 2 + \frac{1}{12}(8.13-8) \\ &= 2 + \left(\frac{1}{12}\right)\left(\frac{13}{100}\right) \\ &= 2 + \frac{13}{1200} = \frac{2413}{1200} \\ &\approx 2.010833 \end{aligned}$$

(vi) $\sqrt[3]{8.13} \approx 2.010775$.

(vii)

$$\begin{aligned} dy &= L(8.13) - L(8) \approx 0.010833 \\ \Delta y &= \sqrt[3]{8.13} - \sqrt[3]{8} \approx 0.010775 \end{aligned}$$

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**Problem 3** *Estimate the value of* $\sin\left(\frac{178\pi}{180}\right)$. *Indicate whether your value is an overestimate or an underestimate.*

***Solution:*** *Set $f(x) = \sin(x)$. The problem is asking us to use linear approximation to estimate the value of $f\left(\frac{178\pi}{180}\right)$. Since $\frac{178\pi}{180}$ is very close to $\pi$, we will use $L$ as the linearization of $\sin(x)$ at $a = \pi$.*

*$f'(x) = \cos(x)$ so $f'(a) = \cos(\pi) = -1$. $f(a) = \sin(\pi) = 0$. The linear approximation is given by $L(x) = -(x-\pi) + 0$.*

$$\begin{aligned} \sin\left(\frac{178\pi}{180}\right) &\approx L\left(\frac{178\pi}{180}\right) \\ &= -\left(\frac{178\pi}{180} - \pi\right) \\ &= \frac{\pi}{90}. \end{aligned}$$

*$f''(x) = -\sin(x)$. That means $f''(\pi) = 0$, BUT $f''$ is negative for all $x$ in the interval $\left(\frac{178\pi}{180}, \pi\right)$. The graph of $f$ is concave down across this interval, so the estimate is an overestimate.*

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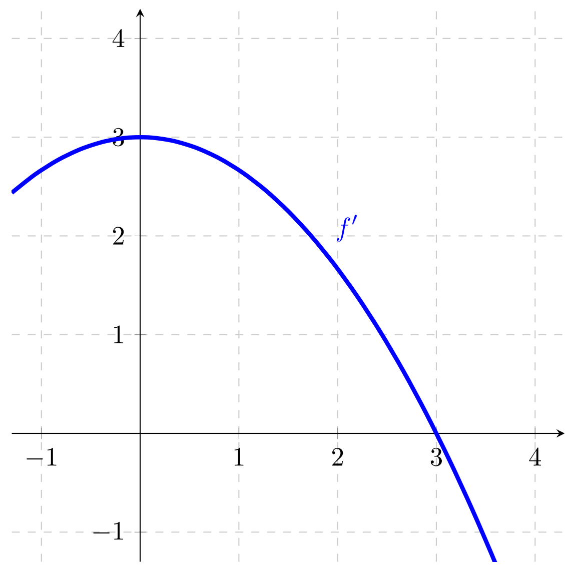
**Problem 4** *Consider the graph of $f'(x)$ given below. Suppose you know that $f(3) = 7$. Can you approximate $f(2.98)$ and $f(3.02)$? Explain your answer. Are these overestimates or underestimates?*




***Solution:*** $f(3) = 7$ *and* $f'(3) = 0$

$$L(x) = 7 + 0(x - 3) = 7$$

*This is a constant function, and so our approximations are $f(2.98) \approx 7$ and $f(3.02) \approx 7$. These are overestimates for the graph of $f$ is concave DOWN on the intervals $(2.98, 3)$ and $(3, 3.02)$.*

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**Problem 5** *Consider a square with a side $x$. Let $A$ be the area of the square.*

(a) *Compute $\Delta A$, the change in area if the side increases by $\Delta x = dx$.*

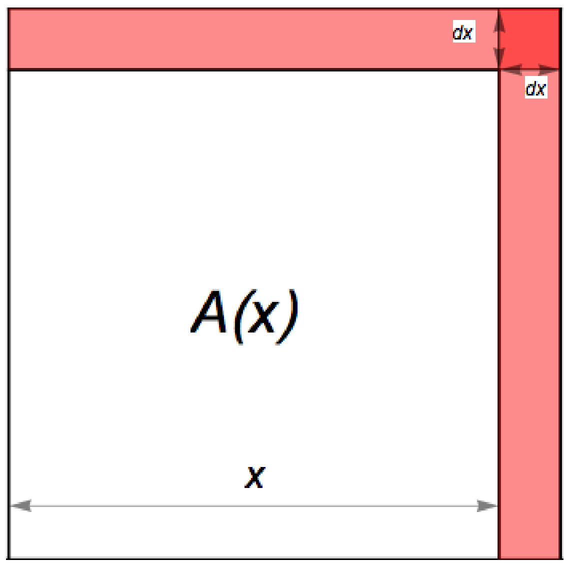
***Solution:*** $\Delta A = A(x + dx) - A(x) = (x + dx)^2 - x^2 = x^2 + 2x \cdot dx + (dx)^2 - x^2 = 2x \cdot dx + (dx)^2$

(b) *Compute $dA$, the differential of $A$ at $x$, and compare it to $\Delta A$.*

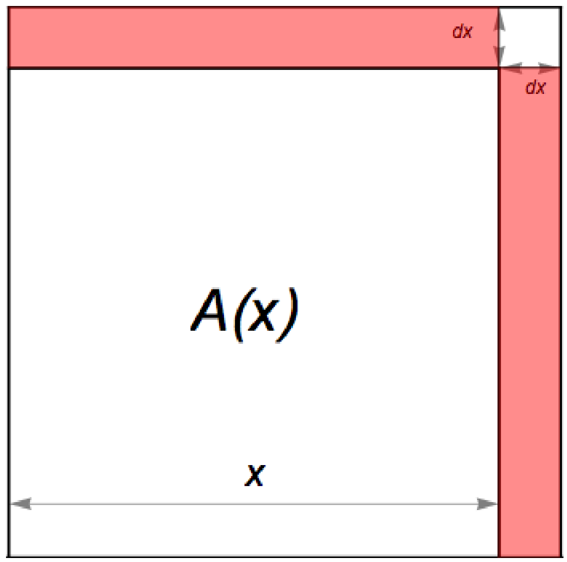
***Solution:*** $dA = A'(x)\,dx = 2x \cdot dx$

$\Delta A = [2x \cdot dx] + (dx)^2 = dA + (dx)^2$

(c) *In the figure below the shaded part represents the change $\Delta A$. Shade the part that represents $dA$.*




***Solution:*** *Since $dA = A'(x)\,dx = 2x \cdot dx$, we shade the part as shown in the figure below.*




*The picture illustrates that $\Delta A \approx dA$.*

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**Problem 6** *Estimate the amount of paint needed to apply a coat of paint .05 cm thick to a hemispherical dome with diameter 50m. Is this value an underestimate or an overestimate?*

***Solution:*** *The radius of the dome is $\frac{50}{2}m = 25m$. The paint increases this by 0.0005m (0.05cm to meters). The volume of a "hemispherical dome" (or half of a sphere) is $V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi r^3$. Then*

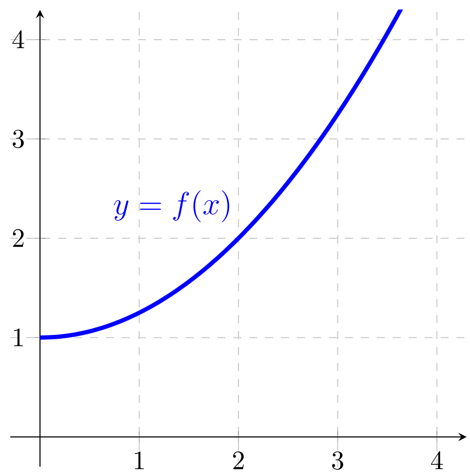
$$dV = 2\pi r^2\,dr$$

*The amount of paint needed is approximately the change in volume ($dV$). We also have that $r = 25m$ and $dr = (5 \times 10^{-4})m$. Thus, the amount of paint needed to paint the dome is approximately*

$$\Big[dV\Big]_{r=25,\,dr=0.0005} = 2\pi(25)^2(5\times 10^{-4}) = 5\pi(0.125) = 0.625\pi \approx 1.9635m^3$$

*The function $V(r) = \frac{2}{3}\pi r^3$ has derivatives $\frac{dV}{dr} = 2\pi r^2$, and $\frac{d^2V}{dr^2} = 4\pi r$. This second derivative is positive for $r$ in the interval $(25, 25.0005)$, so the graph of $V$ is concave up in that entire interval. This means the estimate is an underestimate.*

**Problem 7** *The graph of a function $f$ is given below.*

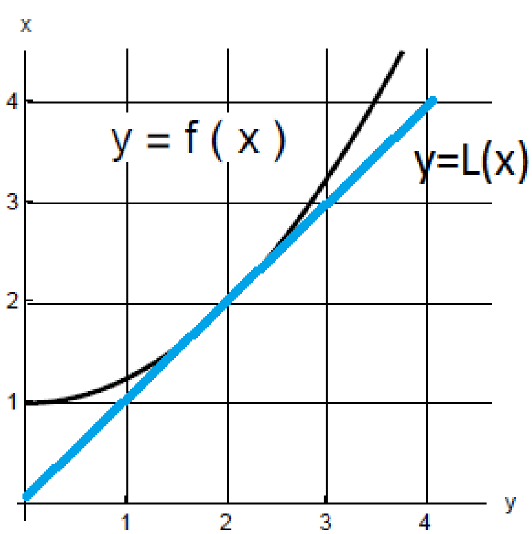



(a) *Given that $f'(2) = 1$, find the linear approximation $L$ to the function $f$ at $a = 2$.*

***Solution:*** $L(x) = f(2) + f'(2)(x-2) = 2 + (x-2) = x$

(b) *Sketch the graph of $L$ in the figure above.*

***Solution:***

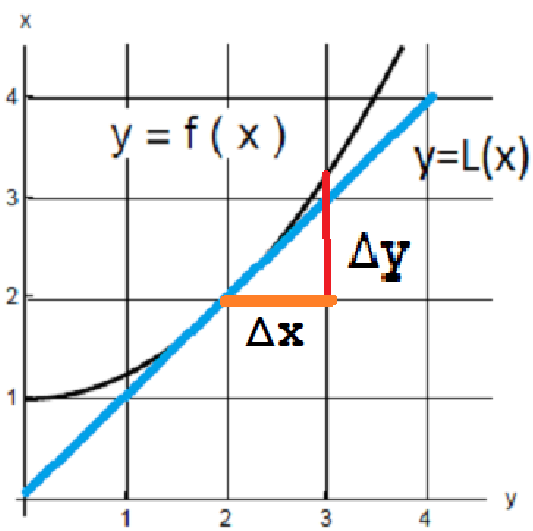


(c) *Use the linear approximation $L$ to estimate the value of $f(3)$. Is this an underestimate or overestimate?* ***EXPLAIN****.*

***Solution:*** $f(3) \approx L(3) = 3$. *It is an underestimate because $f$ is concave up on the interval $(2, 3)$.*

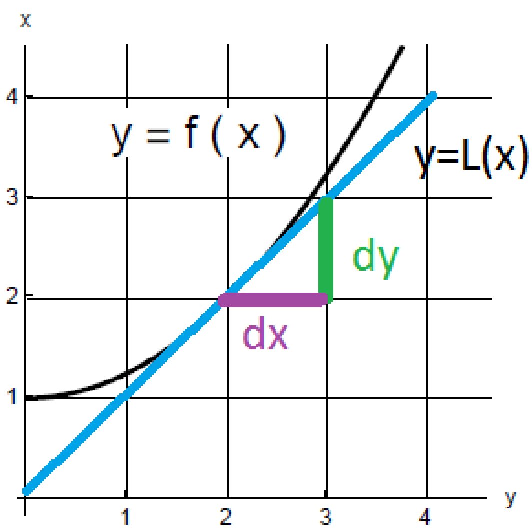
(d) *When $x$ changes from $a = 2$ to $a + \Delta x = 3$, the change in the* **function** $y = f(x)$, $\Delta y$, *is given by* $\Delta y = f(a + \Delta x) - f(a)$. *Draw and label $\Delta y$ and $\Delta x$ in the figure above.*

***Solution:***

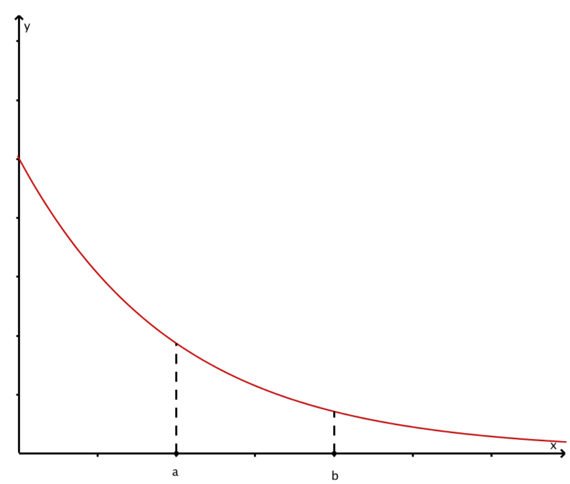



(e) *When $x$ changes from $a = 2$ to $a + \Delta x = 3$, the change in the* **linear approximaton**, *$dy$, is given by* $dy = L(a + \Delta x) - L(a) = f'(a)\Delta x$. *Draw and label $L(x)$, $dx$ and $dy$ (differential) in the figure above.*

***Solution:***




**Problem 8** *The figure shows the graph of a function $f$. Let $L_a(x)$ be the linear approximation of $f$ at $a$.*




*Circle ALL the correct statements below.*

(a) $L_a(b) < f(b)$

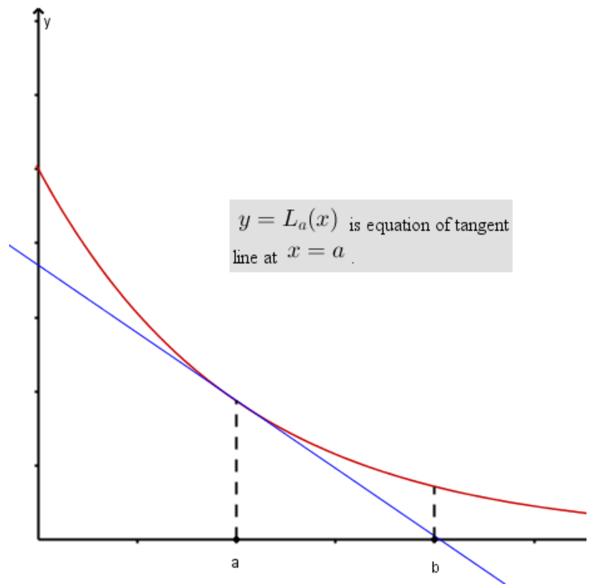
(b) $L_a(b) > f(b)$

(c) $L_a(a) < f(a)$

(d) $L_a(a) > f(a)$

(e) *No statement (a) – (d) is correct.*

***Solution:*** *From the graph of $f$ and the the graph of $L_a(x)$ we see that the correct statement is (a):*




**Problem 9** *By using linear approximation, determine which of the following is the best estimate of $e^{0.002}$.*

(a) *1.00100050016679834166*

(b) *1.00200200133400026675*

(c) *1.00300450450337702601*

(d) *1.02020134002675581016*

***Solution:*** *Let $f(x) = e^x$ and $a = 0$. Then since $f'(x) = e^x$, we have that*

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= f(0) + f'(0)(x-0) \\ &= e^0 + e^0 x \\ &= 1 + x \end{aligned}$$

*Then since $L(0.002) = 1 + 0.002 = 1.002$, the answer is (b).*

**Problem 10** *Find a formula for the differential of the following functions.*

(a) $y = 3x^6 e^x$.

***Solution:*** $\frac{dy}{dx} = 18x^5 e^x + 3x^6 e^x$ *so*

$$\begin{aligned} dy &= \frac{dy}{dx} dx \\ &= \left(18x^5 e^x + 3x^6 e^x\right) dx. \end{aligned}$$

(b) $z = \ln(1 + t^2)$.

***Solution:*** $dz = \frac{2t}{1 + t^2} dt$

(c) $\theta = \tan^{-1}(r^3)$.

***Solution:*** $d\theta = \frac{3r^2}{1 + r^6} dr$

**Problem 11** *In your own words, explain why $L_a(x)$ is a good approximation of the function $f$ for x values $x \approx a$.*

***Solution:*** *The graph of $L_a(x)$ is the line tangent to the graph of $f$ at $(a, f(a))$. For a differentiable function, the tangent line is very close to the graph of the function for x-values near that point.*