calculus with review 1

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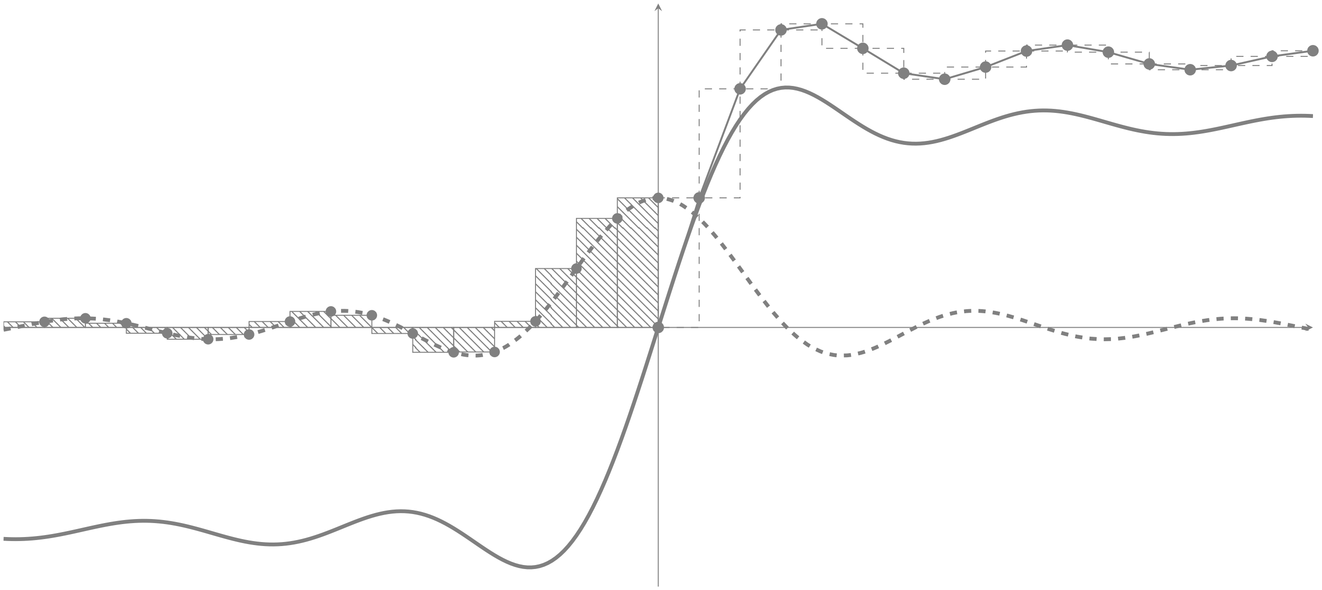

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We will be glad to receive corrections and suggestions for improvement at: ximera@math.osu.edu

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Part I

Content for the First Exam

1 Equations and Inequalities

After completing this section, students should be able to do the following.

- Solve linear equations.
- Solve quadratic equations.
- Solve equations by factoring.
- Solve linear inequalities.
- Solve nonlinear inequalities using a sign chart.

Break-Ground:

1.1 Needed Score?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I have a question.
Riley: What's on your mind?
Devyn: Suppose we have three exams.
Riley: Ok.
Devyn: On the first, I scored 74/100, and on the second I scored 84/100.
Riley: Not bad! What's the question, though?
Devyn: What do I need to get on the third exam to average an 82% overall?
Riley: To find the average of three numbers, you add them and divide the sum by three.
Devyn: Yes, but how do I figure out how to get an 82%?

Let's call Devyn's score on midterm 3 as x. According to Riley, Devyn's exam average would be

$$\frac{74+84+x}{3}.$$

In order to determine what score Devyn needs on his last exam,they will need to setup and solve an *equation* involving this expression.

Problem 1. *Is the following an equation:* $\frac{74+82+x}{3}$*?*

Multiple Choice:

(a) *yes*

(b) *no*

Dig-In:

1.2 Equations

An equation is a statement expressing the equality between two quantities. This means that an equation will always have a single equals sign.

Example 1. *From Devyn's question above,* $\dfrac{74+84+x}{3}$ *is the average of the three exam scores, with the variable x representing the third unknown exam score. What does the equation* $\dfrac{74+84+x}{3} = 82$ *mean in this setting?*

> This is an equation, stating that the average of Devyn's three exam scores is 82.

Usually, when dealing with an equation, we will be looking for values that make the equation true. A solution to an equation is a value that, when substituted in for the variables, yield a true number statement.

Example 2. $\dfrac{74+84+88}{3} = 82$*, so $x = 88$ is a solution of the equation* $\dfrac{74+84+x}{3} = 82$.

When we are asked to *solve an equation*, we are being asked to find all solutions. Exactly how to do that depends on the particular equation involved.

A *linear equation in x* is an equation which is equivalent to one with the form $ax + b = 0$, where a and b are constants, with $a \neq 0$.

To solve a linear equation, we isolate the x-term and divide by the coefficient.

Example 3. *Solve the linear equation* $\dfrac{x}{3} - 9 = 4\left(2x + \dfrac{5}{2}\right)$.

> This doesn't really look like $ax + b = 0$ when written like this. Let's start by distributing the 4 into the parentheses, then combining like terms.
>
> $$\frac{x}{3} - 9 = 4\left(2x + \frac{5}{2}\right)$$
> $$\frac{x}{3} - 9 = 8x + 10$$
>
> That looks better. From here, we'll move all the x-terms to the right side of the equation, and all of the constant terms to the left.
>
> $$-9 - 10 = 8x - \frac{x}{3}$$
> $$-19 = \frac{23}{3}x$$
> $$x = -19 \cdot \frac{3}{23} = -\frac{57}{23}$$

Problem 2. *Solve the linear equation* $\dfrac{4}{5}(x+2) - 3 = \dfrac{x-1}{2}$.

Dig-In:
1.3 Inequalities

Devyn asked what score was needed on the third exam to have an average of 82. This gave us the equation $\frac{74+84+x}{3} = 82$. The solutions only tell us what will give an average *exactly* 82. It may have been more advantageous to ask what score was needed to have an average of at least 82, yielding not an equation, but an *inequality*, $\frac{74+84+x}{3} \geq 82$.

As with equations, there are various types of inequalities.

A *linear inequality in x* is an inequality which is equivalent to $ax+b > 0$, $ax+b \geq 0$, $ax+b < 0$, or $ax+b \leq 0$. We solve the inequality in much the same manner as with a linear equation. The main differences come from changing the direction of the inequality when multiplying/dividing by a negative quantity and expressing our answers in interval notation.

Example 4. *Solve the inequality* $\frac{2x}{3} - 4 < 3(2-x)$.

As with equations, we'll start by simplifying and combining like terms.

$$\begin{aligned}
\frac{2x}{3} - 4 &< 3(2-x) \\
\frac{2x}{3} - 4 &< 6 - 3x \\
\frac{2x}{3} + 3x &< 6 + 4 \\
\frac{11}{3}x &< 10 \\
x &< \frac{30}{11}
\end{aligned}$$

All x less than $\frac{30}{11}$ satisfy the inequality.

The solution is $\left(-\infty, \frac{30}{11}\right)$.

Nonlinear inequalities are more complicated. To solve them, we will use a tool called a *sign chart*. The process requires us to move all the nonzero terms of the inequality to one side, and factor.

Example 5. *Solve the inequality*

$$\frac{x^3(2x-5)^2}{x+1} \geq 0.$$

Since the right-hand side is zero, we are really asking where the left-hand side will be positive. The only way a nice formula like the left-hand side can switch from positive to negative, or vice-versa, is if it is either zero or undefined at the point where it changes.

Notice where the left-hand side would be either zero or undefined. Those occur at $x = -1, 0, \frac{5}{2}$. We'll use those points to split the number line into four regions. Inside each of those regions, our factors are either always positive or always negative.

x		-1	0	$\frac{5}{2}$	
x^3	$-$		$-$	$+$	$+$
$(2x-5)^2$	$+$		$+$	$+$	$-$
$x+1$	$-$		$+$	$+$	$+$

For $x < -1$, we find $x^3 < 0$, $(2x-5)^2 > 0$, and $x+1 < 0$. Together, that means the left-hand side of our inequality has the form $\frac{\text{negative} \cdot \text{positive}}{\text{negative}}$. It is, therefore, positive in that region, and makes up part of our solution. We cannot include the endpoint $x = -1$ in the solution, as the fraction is undefined there.

In the same way, we see that in the interval $\left(0, \frac{5}{2}\right)$, the left-hand side of the inequality is also positive.

The solution is $(-\infty, -1) \cup \left[0, \frac{5}{2}\right]$.

Example 6. *Solve the inequality*

$$\frac{2x}{x-1} \le \frac{5-x}{x-1}.$$

Let's start by moving all the terms to one side of the inequality.

$$\begin{aligned} \frac{2x}{x-1} &\le \frac{5-x}{x-1} \\ \frac{2x}{x-1} - \frac{5-x}{x-1} &\le 0 \\ \frac{3x-5}{x-1} &\le 0 \end{aligned}$$

x		1	$\frac{5}{3}$
$3x-5$	$-$	$-$	$+$
$x-1$	$-$	$+$	$+$

The solution is $\left(1, \frac{5}{3}\right]$.

Problem 3. *Find the solution of the inequality*

$$\frac{x}{x-4} \ge \frac{2x-1}{x-4}.$$

Multiple Choice:

(a) $[1, 4)$

(b) $(-\infty, 1] \bigcup (4, \infty)$

(c) $(1, 4)$

(d) $(-\infty, 1) \bigcup (4, \infty)$

(e) *None of the above*

2 Understanding functions

After completing this section, students should be able to do the following.

- State the definition of a function.
- Find the domain and range of a function.
- Distinguish between functions by considering their domains.
- Determine where a function is positive or negative.
- Plot basic functions.
- Perform basic operations and compositions on functions.
- Work with piecewise defined functions.
- Determine if a function is one-to-one.
- Recognize different representations of the same function.
- Define and work with inverse functions.
- Plot inverses of basic functions.
- Find inverse functions (algebraically and graphically).
- Find the largest interval containing a given point where the function is invertible.
- Determine the intervals on which a function has an inverse.

Break-Ground:

2.1 Same or different?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I have a pressing question.
Riley: Tell me. Tell me everything.
Devyn: Think about the function

$$f(x) = \frac{x^2 - 3x + 2}{x - 2}.$$

Riley: OK.
Devyn: Is this function equal to $g(x) = x - 1$?
Riley: Well if I plot them with my calculator, they look the same.
Devyn: I know!
Riley: And I suppose if I write

$$\begin{aligned} f(x) &= \frac{x^2 - 3x + 2}{x - 2} \\ &= \frac{(x-1)(x-2)}{x-2} \\ &= x - 1 \\ &= g(x). \end{aligned}$$

Devyn: Sure! But what about when $x = 2$? In this case

$$g(2) = 1 \qquad \text{but} \qquad f(2) \text{ is undefined!}$$

Riley: Right, $f(2)$ is undefined because we cannot divide by zero. Hmm. Now I see the problem. Yikes!

Problem 1. *In the context above, are f and g the same function?*

Multiple Choice:

(a) *yes*

(b) *no*

Problem 2. *Suppose f and g are functions but the domain of f is different from the domain of g. Could it be that f and g are actually the same function?*

Multiple Choice:

(a) *yes*

(b) *no*

Problem 3. *Can the same function be represented by different formulas?*

Multiple Choice:

(a) *yes*

(b) *no*

Problem 4. *Are $f(x) = |x|$ and $g(x) = \sqrt{x^2}$ the same function?*

Multiple Choice:

(a) *These are the same function although they are represented by different formulas.*

(b) *These are different functions because they have different formulas.*

Problem 5. *Let $f(x) = \sin^2(x)$ and $g(u) = \sin^2(u)$. The domain of each of these functions is all real numbers. Which of the following statements are true?*

Multiple Choice:

(a) *There is not enough information to determine if $f = g$.*

(b) *The functions are equal.*

(c) *If $x \neq u$, then $f \neq g$.*

(d) *We have $f \neq g$ since f uses the variable x and g uses the variable u.*

Dig-In:

2.2 For each input, exactly one output

Life is complex. Part of this complexity stems from the fact that there are many relationships between seemingly unrelated events. Armed with mathematics, we seek to understand the world. Perhaps the most relevant "real-world" relation is

the position of an object with respect to time.

Our observations seem to indicate that every instant in time is associated to a unique positioning of the objects in the universe. You may have heard the saying,

you cannot be two places at the same time,

and it is this fact that motivates our definition for functions.

Definition. A **function** is a relation between sets where for each input, there is exactly one output.

Question 1. *If our function is the "position with respect to time" of some object, then the input is*

Multiple Choice:

(a) *position*

(b) *time*

(c) *none of the above*

and the output is

Multiple Choice:

(a) *position*

(b) *time*

(c) *none of the above*

Something as simple as a dictionary could be thought of as a relation, as it connects *words* to *definitions*. However, a dictionary is not a function, as there are words with multiple definitions. On the other hand, if each word only had a single definition, then a dictionary would be a function.

Question 2. *Which of the following are functions?*

Select All Correct Answers:

(a) *Mapping words to their definition in a dictionary.*

(b) *Mapping social security numbers of living people to actual living people.*

(c) *Mapping people to their birth date.*

(d) *Mapping mothers to their children.*

What we are hoping to convince you is that the following are true:

(a) The definition of a function is well-grounded in a real context.

(b) The definition of a function is flexible enough that it can be used to model a wide range of phenomena.

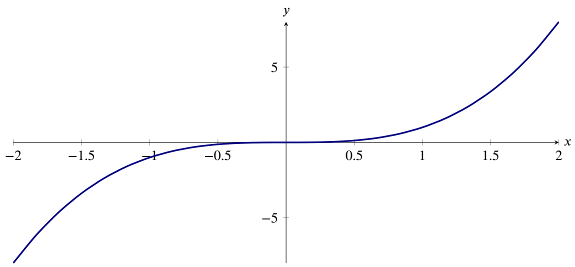
Whenever we talk about functions, we should explicitly state what type of things the inputs are and what type of things the outputs are. In calculus, functions often define a relation from (a subset of) the real numbers (denoted by $\mathbb{R}$) to (a subset of) the real numbers.

Definition. We call the set of the inputs of a function the **domain**, and we call the set of the outputs of a function the **range**.

Example 7. *Consider the function f that maps from the real numbers to the real numbers by taking a number and mapping* *it to its cube:*

$$\begin{aligned} 1 &\mapsto 1 \\ -2 &\mapsto -8 \\ 1.5 &\mapsto 3.375 \end{aligned}$$

and so on. This function can be described by the formula $f(x) = x^3$ or by the graph shown in the plot below:



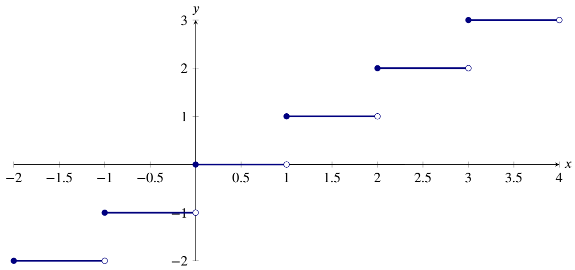
Warning. *A function is a relation (such that for each input, there is exactly one output) between sets. The formula and the graph are merely descriptions of this relation.*

- *A formula **describes** the relation using symbols.*
- *A graph **describes** the relation using pictures.*

*The **function is the relation itself**, and is independent of how it is described.*

Our next example may be a function that is new to you. It is the *greatest integer function*.

Example 8. *Consider the **greatest integer function**. This function maps any real number x to the greatest integer less than or equal to x. People sometimes write this as $f(x) = \lfloor x \rfloor$, where those funny symbols mean exactly the words above describing the function. For your viewing pleasure, here is a graph of the greatest integer function:*



Observe that here we have multiple inputs that give the same output. This is not a problem! To be a function, we merely need to check that for each input, there is exactly one output, and this condition is satisfied.

Question 3. *Compute:*

$$\lfloor 2.4 \rfloor$$

Question 4. *Compute:*

$$\lfloor -2.4 \rfloor$$

Notice that both the functions described above pass the so-called *vertical line test*.

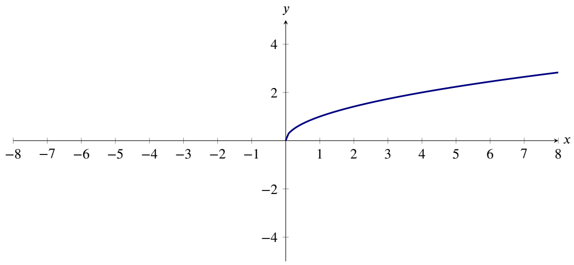
Theorem 1. *The curve $y = f(x)$ represents y as a function of x at $x = a$ if and only if the vertical line $x = a$ intersects the curve $y = f(x)$ at exactly one point. This is called the **vertical line test**.*

Sometimes the domain and range are the *entire* set of real numbers, denoted by $\mathbb{R}$. In our next examples we show that this is not always the case.

Example 9. *Consider the function that maps non-negative real numbers to their positive square root. This function can be described by the formula*

$$f(x) = \sqrt{x}.$$

The domain is $0 \le x < \infty$, which we prefer to write as $[0, \infty)$ in interval notation. The range is $[0, \infty)$. Here is a graph of $y = f(x)$:



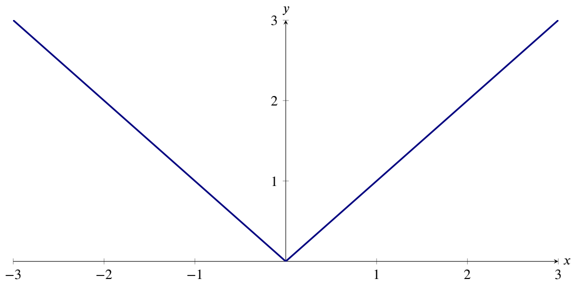
To really tease out the difference between a function and its description, let's consider an example of a function with two different descriptions.

Example 10. *Explain why* $\sqrt{x^2} = |x|$.

Although $\sqrt{x^2}$ may appear to simplify to just x, let's see what happens when we plug in some values.

$$\begin{aligned}\sqrt{3^2} &= \sqrt{9}\\ &= 3,\end{aligned} \qquad \text{and} \qquad \begin{aligned}\sqrt{(-3)^2} &= \sqrt{9}\\ &= 3.\end{aligned}$$

In an entirely similar way, we see that for any positive x, $f(-x) = x$. Hence $\sqrt{x^2} \neq x$. Rather we see that $\sqrt{x^2} = |x|$. The domain of $f(x) = \sqrt{x^2}$ is $(-\infty, \infty)$ and the range is $[0, \infty)$. For your viewing pleasure we've included a graph of $y = f(x)$:



Finally, we will consider a function whose domain is all real numbers except for a single point.

Example 11. *Are*

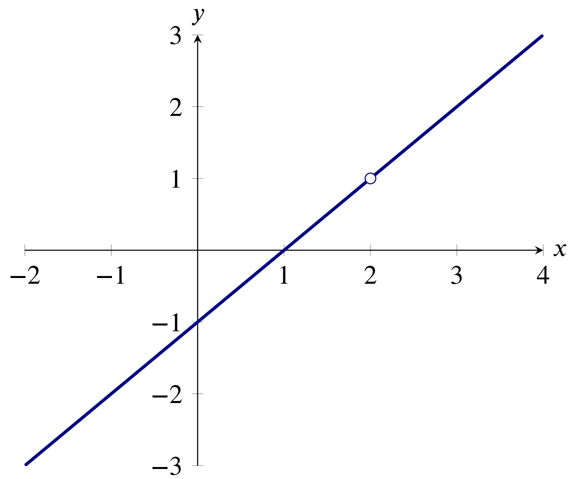
$$f(x) = \frac{x^2 - 3x + 2}{x - 2} = \frac{(x-2)(x-1)}{(x-2)}$$

and

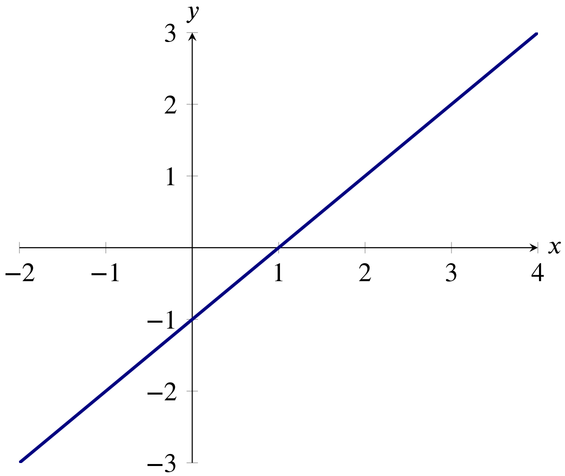
$$g(x) = x - 1$$

the same function?

Let's use a series of steps to think about this question. First, what if we compare graphs? Here we see a graph of f:



On the other hand, here is a graph of g:



Second, what if we compare the domains? We cannot evaluate f at $x = 2$. This is where f is undefined. On the other hand, there is no value of x where we cannot evaluate g. In other words, the domain of g is $(-\infty, \infty)$.

Since these two functions do not have the same graph, and they do not have the same domain, they must not be the same function.

However, if we look at the two functions everywhere except at $x = 2$, we can say that $f(x) = g(x)$. In other words,

$$f(x) = x - 1 \qquad \text{when} \qquad x \neq 2.$$

From this example we see that it is critical to consider the domain and range of a function.

Dig-In:

2.3 Compositions of functions

Given two functions, we can compose them. Let's give an example in a "real context."

Example 12. *Let*

$$g(m) = \textit{the amount of gas one can buy with } m \textit{ dollars,}$$

and let

$$f(g) = \textit{how far one can drive with } g \textit{ gallons of gas.}$$

What does $f(g(m))$ represent in this setting?

With $f(g(m))$ we first relate how far one can drive with g gallons of gas, and this in turn is determined by how much money m one has. Hence $f(g(m))$ represents how far one can drive with m dollars.

Composition of functions can be thought of as putting one function inside another. We use the notation

$$(f \circ g)(x) = f(g(x)).$$

Warning. *The composition $f \circ g$ only makes sense if*

$\{$*the range of g*$\}$ *is contained in or equal to* $\{$*the domain of f*$\}$

Example 13. *Suppose we have*

$$\begin{aligned} f(x) &= x^2 + 5x + 4 && \textit{for } -\infty < x < \infty, \\ g(x) &= x + 7 && \textit{for } -\infty < x < \infty. \end{aligned}$$

Find $f(g(x))$ and state its domain.

The range of g is $-\infty < x < \infty$, which is equal to the domain of f. This means the domain of $f \circ g$ is $-\infty < x < \infty$. Next, we substitute $x + 7$ for each instance of x found in

$$f(x) = x^2 + 5x + 4$$

and so

$$\begin{aligned} f(g(x)) &= f(x+7) \\ &= (x+7)^2 + 5(x+7) + 4. \end{aligned}$$

Now let's try an example with a more restricted domain.

Example 14. *Suppose we have:*

$$\begin{aligned} f(x) &= x^2 && \textit{for } -\infty < x < \infty, \\ g(x) &= \sqrt{x} && \textit{for } 0 \leq x < \infty. \end{aligned}$$

Find $f(g(x))$ and state its domain.

The domain of g is $0 \leq x < \infty$. From this we can see that the range of g is $0 \leq x < \infty$. This is contained in the domain of f.

This means that the domain of $f \circ g$ is $0 \leq x < \infty$. Next, we substitute $\sqrt{x}$ for each instance of x found in

$$f(x) = x^2$$

and so

$$\begin{aligned} f(g(x)) &= f(\sqrt{x}) \\ &= \left(\sqrt{x}\right)^2. \end{aligned}$$

We can summarize our results as a piecewise function, which looks somewhat interesting:

$$(f \circ g)(x) = \begin{cases} x & \text{if } 0 \leq x < \infty \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Example 15. *Suppose we have:*

$$\begin{aligned} f(x) &= \sqrt{x} && \textit{for } 0 \leq x < \infty, \\ g(x) &= x^2 && \textit{for } -\infty < x < \infty. \end{aligned}$$

Find $f(g(x))$ and state its domain.

> While the domain of g is $-\infty < x < \infty$, its range is only $0 \leq x < \infty$. This is exactly the domain of f. This means that the domain of $f \circ g$ is $-\infty < x < \infty$. Now we may substitute x^2 for each instance of x found in
>
> $$f(x) = \sqrt{x}$$
>
> and so
>
> $$\begin{aligned} f(g(x)) &= f(x^2) \\ &= \sqrt{x^2}, \\ &= |x|. \end{aligned}$$

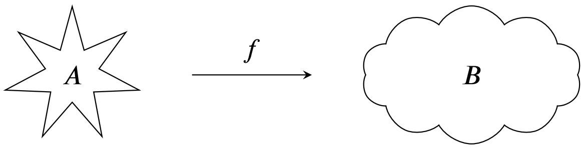
Compare and contrast the previous two examples. We used the same functions for each example, but composed them in different ways. The resulting compositions are not only different, they have different domains!

Dig-In:

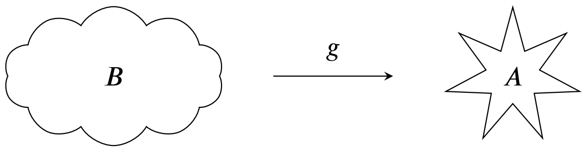
2.4 Inverses of functions

If a function maps every "input" to exactly one "output," an inverse of that function maps every "output" to exactly one "input." We need a more formal definition to actually say anything with rigor.

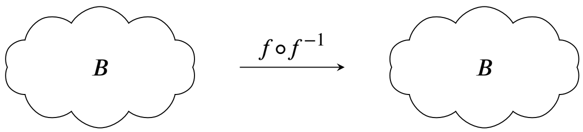
Definition. Let f be a function with domain A and range B:



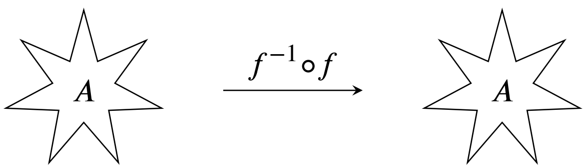
Let g be a function with domain B and range A:



We say that f and g are **inverses** of each other if $f(g(b)) = b$ for all b in B, and also $g(f(a)) = a$ for all a in A. Sometimes we write $g = f^{-1}$ in this case.



and



So, we could rephrase these conditions as

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

These two simple equations are somewhat more subtle than they initially appear.

Question 5. *Let f be a function. If the point $(1, 9)$ is on the graph of f, what point must be the the graph of f^{-1}?*

Warning. *This notation can be very confusing. Keep a watchful eye:*

$$f^{-1}(x) = \text{the inverse function of } f(x).$$
$$f(x)^{-1} = \frac{1}{f(x)}.$$

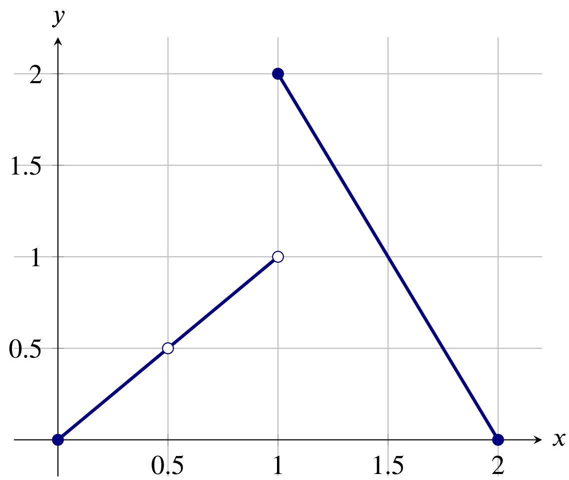
Question 6. *Which of the following is notation for the inverse of the function $\sin(\theta)$ on the interval $[-\pi/2, \pi/2]$?*

Multiple Choice:

(a) $\sin^{-1}(\theta)$

(b) $\sin(\theta)^{-1}$

Question 7. *Consider the graph of $y = f(x)$ below*



Is $f(x)$ invertible at $x = 0.5$?

Multiple Choice:

(a) *yes*

(b) *no*

Question 8.

$$f^{-1}(1) = \boxed{?}$$

So far, we've only dealt with abstract examples. Let's see if we can ground this in a real-life context.

Example 16. *The function*

$$f(t) = \left(\frac{9}{5}\right) t + 32$$

takes a temperature t in degrees Celsius, and converts it into Fahrenheit. The domain of this function is $-\infty < t < \infty$. What does the inverse of this function tell you? What is the inverse of this function?

> If f converts Celsius measurements to Fahrenheit measurements of temperature, then f^{-1} converts Fahrenheit measurements to Celsius measurements of temperature.
>
> To find the inverse function, first note that
>
> $$f(f^{-1}(t)) = t \qquad \text{by the definition of inverse functions.}$$
>
> Now write out the left-hand side of the equation
>
> $$f(f^{-1}(t)) = \left(\frac{9}{5}\right) f^{-1}(t) + 32 \qquad \text{by the rule for } f$$
>
> and solve for $f^{-1}(t)$.
>
> $$\begin{aligned} \left(\frac{9}{5}\right) f^{-1}(t) + 32 &= t \qquad \text{by the rule for } f \\ \left(\frac{9}{5}\right) f^{-1}(t) &= t - 32 \\ f^{-1}(t) &= \left(\frac{5}{9}\right)(t - 32). \end{aligned}$$
>
> So $f^{-1}(t) = \left(\frac{5}{9}\right)(t - 32)$ is the inverse function of f, which converts a Fahrenheit measurement back into a Celsius measurement. The domain of this inverse function is $-\infty < t < \infty$.
>
> Finally, we could check our work again using the definition of inverse functions. We have already guaranteed that
>
> $$f(f^{-1}(t)) = t,$$
>
> since we solved for f^{-1} in our calculation. On the other hand,
>
> $$\begin{aligned} f^{-1}(f(t)) &= \left(\frac{5}{9}\right)(f(t) - 32) \\ &= \left(\frac{5}{9}\right)(f(t) - 32) \end{aligned}$$
>
> which you should simplify to check that $f^{-1}(f(t)) = t$.

We have examined several functions in order to determine their inverse functions, but there is still more to this story. Not every function has an inverse function, so we must learn how to check for this situation.

Question 9. *Let f be a function, and imagine that the points $(2, 3)$ and $(7, 3)$ are both on its graph. Could f have an inverse function?*

Multiple Choice:

(a) *yes*

(b) *no*

Look again at the last question. If two different inputs for a function have the same output, there is no hope of that function having an inverse function. Why? This is because the inverse function must also be a function, and a function can only have one output for each input. More specifically, we have the next definition.

Definition. A function is called **one-to-one** if each output value corresponds to exactly one input value.

Question 10. *Which of the following are functions that are also one-to-one?*

Select All Correct Answers:

(a) *Mapping words to their meaning in a dictionary.*

(b) *Mapping social security numbers of living people to actual living people.*

(c) *Mapping people to their birthday.*

(d) *Mapping mothers to their children.*

Question 11. *Which of the following functions are one to one? Select all that apply.*

Select All Correct Answers:

(a) $f(x) = x$

(b) $f(x) = x^2$

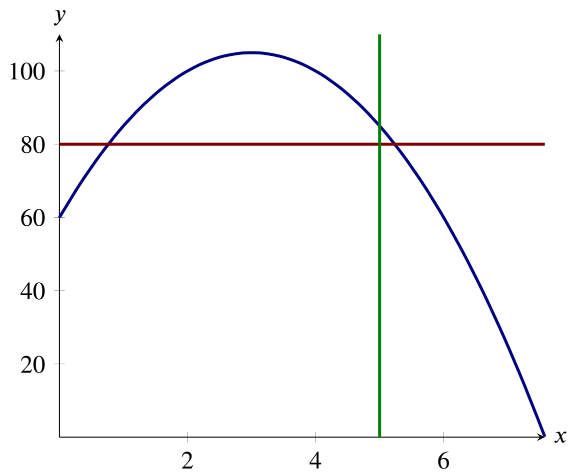
(c) $f(x) = x^3 - 4x$

(d) $f(x) = x^3 + 4$

You may recall that a plot gives y as a function of x if every vertical line crosses the plot at most once, and we called this the **vertical line test**. Similarly, a function is one-to-one if every horizontal line crosses the plot at most once, and we call this the **horizontal line test**.

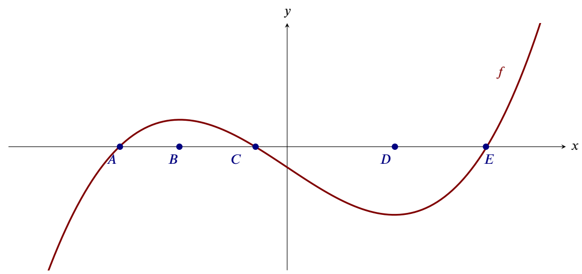
Theorem 2. *A function is one-to-one at $x = a$ if the horizontal line $y = f(a)$ intersects the curve $y = f(x)$ in exactly one point. This is called the* ***horizontal line test****.*

Below, we give a graph of $f(x) = -5x^2 + 30x + 60$. While this graph passes the vertical line test, and hence represents y as a function of x, it does not pass the horizontal line test, so the function is not one-to-one.



As we have discussed, we can only find an inverse of a function when it is one-to-one. If a function is not one-to-one, but we still want an inverse, we must restrict the domain. Let's see what this means in our next examples.

Question 12. *Consider the graph of the function f below:*



On which of the following intervals is f one-to-one?

Select All Correct Answers:

(a) $[A, B]$

(b) $[A, C]$

(c) $[B, D]$

(d) $[C, E]$

(e) $[C, D]$

This idea of restricting the domain is critical for understanding functions like $f(x) = \sqrt{x}$.

Warning. *We define $f(x) = \sqrt{x}$ to be the positive square-root, so that we can be sure that f is a function. Thinking of the square-root as the inverse of the squaring function, we can see the issue a little more clearly. There are two x-values that square to 9.*

$$x^2 = 9 \qquad \textit{means } x = \pm 3$$

*Since we require that **square-root is a function**, we must have only one output value when we plug in 9. We choose the positive square-root, meaning that*

$$\sqrt{9} = 3.$$

Example 17. *Consider the function*

$$f(x) = x^2.$$

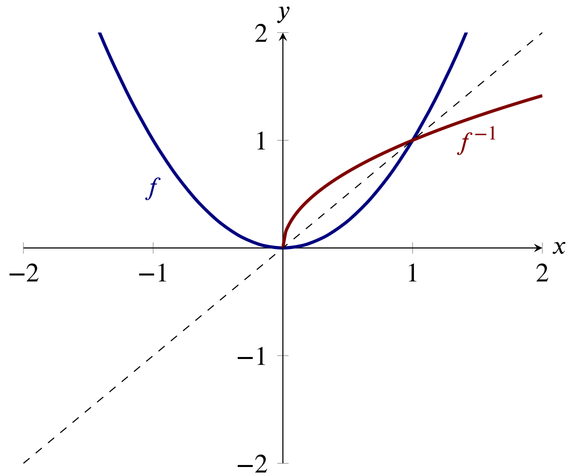
Does f have an inverse? If so, what is it? If not, attempt to restrict the domain of f and find an inverse on the restricted domain.

In this case f is not one-to-one. However, it is one-to-one on the interval $[0, \infty)$. Hence we can find an inverse of $f(x) = x^2$ on this interval. We plug $f^{-1}(x)$ into f and write

$$\begin{aligned} f(f^{-1}(x)) &= \left(f^{-1}(x)\right)^2 \\ x &= \left(f^{-1}(x)\right)^2. \end{aligned}$$

Since the domain of f is $[0, \infty)$, we know that x is positive. This means we can take the square-root of each side of the equation to find that

$$\sqrt{x} = f^{-1}(x).$$



Example 18. *Consider the function*

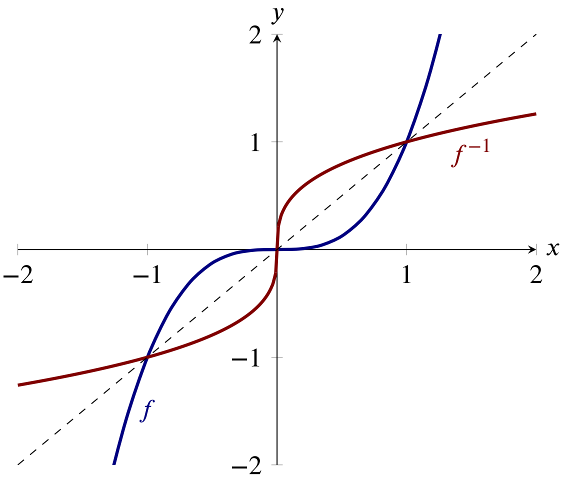
$$f(x) = x^3.$$

Does $f(x)$ have an inverse? If so, what is it? If not, attempt to restrict the domain of $f(x)$ and find an inverse on the restricted domain.

In this case $f(x)$ is one-to-one. We may write

$$\begin{aligned} f(f^{-1}(x) &= \left(f^{-1}(x)\right)^3 \\ x &= \left(f^{-1}(x)\right)^3 \\ \sqrt[3]{x} &= f^{-1}(x). \end{aligned}$$

For your viewing pleasure we give a graph of $y = f(x) = x^3$ and $y = f^{-1}(x) = \sqrt[3]{x}$. Note, the graph of f^{-1} is the image of f after being flipped over the line $y = x$.


y
2
1
−1
−2
−2
−1
1
2
x
f^{-1}
f

3 What is a limit?

After completing this section, students should be able to do the following.

- Consider function values nearer and nearer to a given input value.
- Understand the concept of a limit.
- Limits as understanding local behavior of functions.
- Calculate limits from a graph (or state that the limit does not exist).
- Understand possible issues when estimating limits using nearby values.
- Define a one-sided limit.
- Explain the relationship between one-sided and two-sided limits.
- Distinguish between limit values and function values.
- Identify when a limit does not exist.

Break-Ground:

3.1 Stars and functions

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, did you know I like looking at the stars at night?
Riley: Stars are freaking awesome balls of nuclear fire whose light took thousands of years to reach us.
Devyn: I know! But did you know that the best way to see a very dim star is to look **near** it but **not exactly at** it? It's because then you can use the "rods" in your eye, which work better in low light than the "cones" in your eyes.
Riley: That's amazing! Hey, that reminds me of when we were talking about the two functions

$$f(x) = \frac{x^2 - 3x + 2}{x - 2} \qquad \text{and} \qquad g(x) = x - 1,$$

which we now know are completely different functions.
Devyn: Whoa. How are you seeing a connection here?
Riley: If we want to understand what is happening with the function

$$f(x) = \frac{x^2 - 3x + 2}{x - 2},$$

at $x = 2$, we can't do it by setting $x = 2$. Instead we need to look **near** $x = 2$ but **not exactly at** $x = 2$.
Devyn: Ah ha! Because if we are **not exactly at** $x = 2$, then

$$\frac{x^2 - 3x + 2}{x - 2} = x - 1.$$

Problem 1. *Let* $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ *and* $g(x) = x - 1$. *Which of the following is true?*

Multiple Choice:

(a) $f(x) = g(x)$ *for every value of* x.

(b) *There is no* x*-value where* $f(x) = g(x)$.

(c) $f(x) = g(x)$ *when* $x \neq 2$.

Problem 2. *When you evaluate*

$$f(x) = \frac{x^2 - 3x + 2}{x - 2},$$

at x*-values approaching (but not equal to)* 2, *what happens to the value of* $f(x)$*?*

Problem 3. *Just from checking some values, can you be absolutely certain that your answer to the previous problem is correct?*

Multiple Choice:

(a) *yes*

(b) *no*

Dig-In:

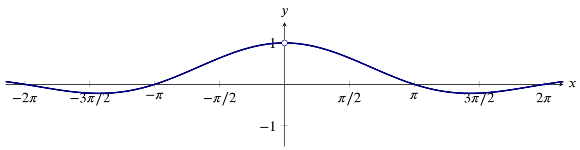
3.2 What is a limit?

The basic idea

Consider the function

$$f(x) = \frac{\sin(x)}{x}.$$

While $f(x)$ is undefined at $x = 0$, we can still plot $f(x)$ at other values near $x = 0$.



Question 13. *Use the graph of* $f(x) = \frac{\sin(x)}{x}$ *above to answer the following question: What is* $f(0)$*?*

Multiple Choice:

(a) 0

(b) $f(0)$

(c) 1

(d) $f(0)$ *is undefined*

(e) *it is impossible to say*

Nevertheless, we can see that as x approaches zero, $f(x)$ approaches one. From this setting we come to our definition of a limit.

Definition. Intuitively,

the **limit** of $f(x)$ as x approaches a is L, written

$$\lim_{x\to a} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all x sufficiently close, but not equal, to a.

Question 14. *Use the graph of* $f(x) = \frac{\sin(x)}{x}$ *above to finish the following statement: "A good guess is that…"*

Multiple Choice:

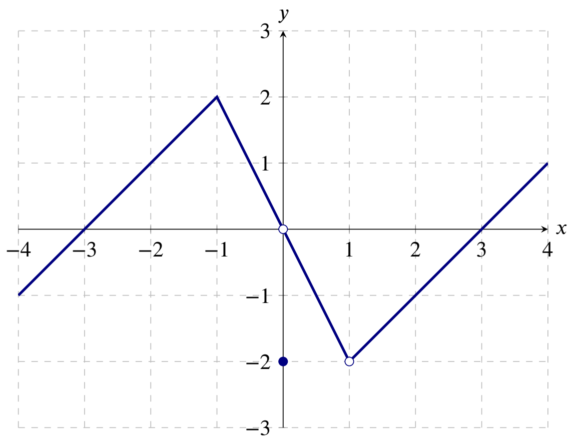
(a) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$

(b) $\displaystyle\lim_{x\to 1} \frac{\sin(x)}{x} = 0.$

(c) $\displaystyle\lim_{x\to 1} f(x) = \frac{\sin(1)}{1}.$

(d) $\displaystyle\lim_{x\to 0} f(x) = \frac{\sin(0)}{0} = \infty.$

Question 15. *Consider the following graph of* $y = f(x)$



Use the graph to evaluate the following. Write DNE if the value does not exist.

(a) $f(-2)$

(b) $\lim_{x \to -2} f(x)$

(c) $f(-1)$

(d) $\lim_{x \to -1} f(x)$

(e) $f(0)$

(f) $\lim_{x \to 0} f(x)$

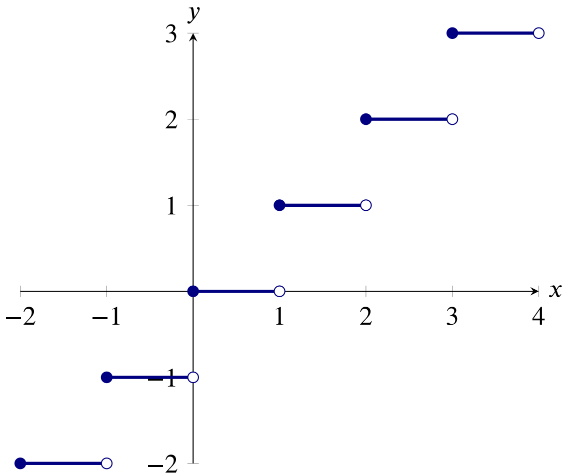
(g) $f(1)$

(h) $\lim_{x \to 1} f(x)$

Limits might not exist

Limits might not exist. Let's see how this happens.

Example 19. *Consider the graph of $f(x) = \lfloor x \rfloor$.*



Explain why the limit

$$\lim_{x \to 2} f(x)$$

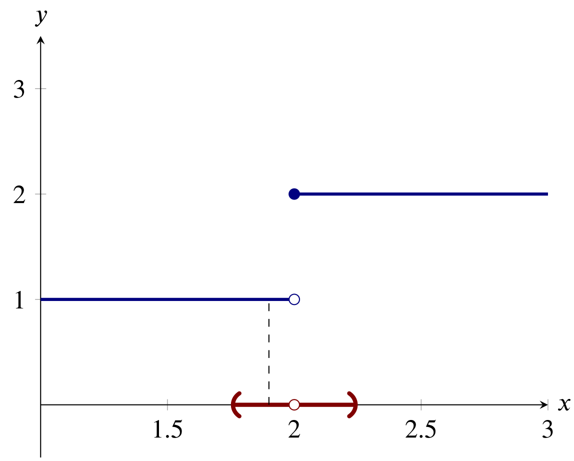
does not exist.

The function $\lfloor x \rfloor$ is the function that returns the greatest integer less than or equal to x. Recall that

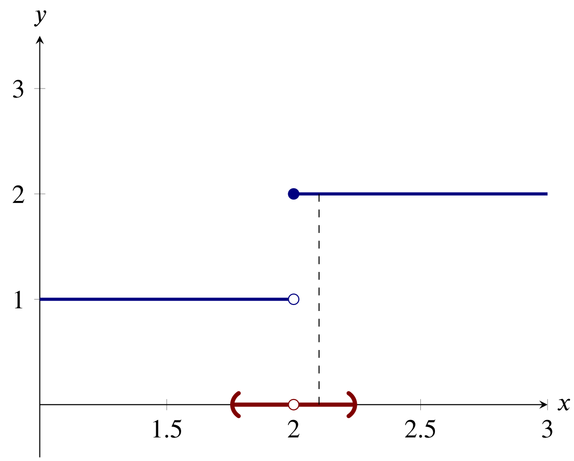
$$\lim_{x \to 2} \lfloor x \rfloor = L$$

if $\lfloor x \rfloor$ can be made arbitrarily close to L by making x sufficiently close, but not equal to, 2. So let's examine x near, but not equal to, 2. Now the question is: What is L?

If this limit exists, then we should be able to look sufficiently close, but not at, $x = 2$, and see that f is approaching some number. Let's look at a graph:



If we look closer and closer to $x = 2$ (on the left of $x = 2$) we see that $f(x) = 1$. However, if we look closer and closer to $x = 2$ (on the right of $x = 2$) we see



So just to the right of $x = 2$, $f(x) = 2$. We cannot find a single number that $f(x)$ approaches as x approaches $x = 2$, and so the limit does not exists.

Tables can be used to help guess limits, but one must be careful.

Question 16. *Consider* $f(x) = \sin\left(\frac{\pi}{x}\right)$. *Fill in the tables below:*

x	$f(x)$
0.1	
0.01	
0.001	
0.0001	

and

x	$f(x)$
0.3	
0.03	
0.003	
0.0003	

What do the tables tell us about

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)?$$

Multiple Choice:

(a) $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 0$

(b) $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = 1$

(c) $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = -.866$

(d) $\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right) = -.433$

(e) *it is unclear what the tables are telling us about the limit*

One-sided limits

While we have seen that $\lim_{x\to 2} \lfloor x \rfloor$ does not exist, more can still be said.

Definition. Intuitively,

the **limit from the right** of f as x approaches a is L,

written

$$\lim_{x\to a^+} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x > a$ sufficiently close, but not equal to, a.

Similarly,

the **limit from the left** of $f(x)$ as x approaches a is L,

written

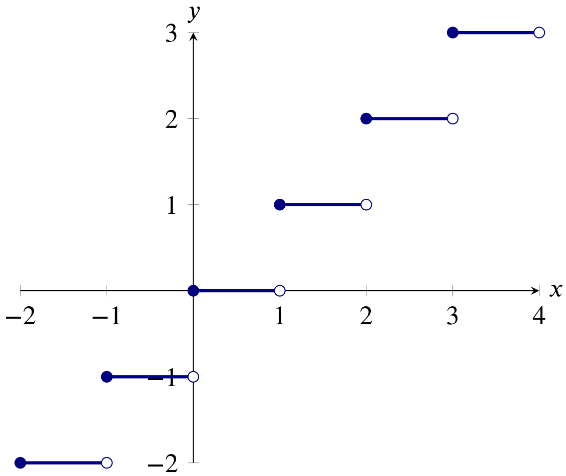
$$\lim_{x\to a^-} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all $x < a$ sufficiently close, but not equal to, a.

Example 20. *Compute:*

$$\lim_{x\to 2^-} f(x) \qquad \textit{and} \qquad \lim_{x\to 2^+} f(x)$$

by using the graph below



From the graph we can see that as x approaches 2 from the left, $\lfloor x \rfloor$ remains at $y = 1$ up until the exact point that $x = 2$. Hence

$$\lim_{x\to 2^-} f(x) = 1.$$

Also from the graph we can see that as x approaches 2 from the right, $\lfloor x \rfloor$ remains at $y = 2$ up to $x = 2$. Hence

$$\lim_{x\to 2^+} f(x) = 2.$$

When you put this all together

One-sided limits help us talk about limits.

Theorem 3. *A limit*

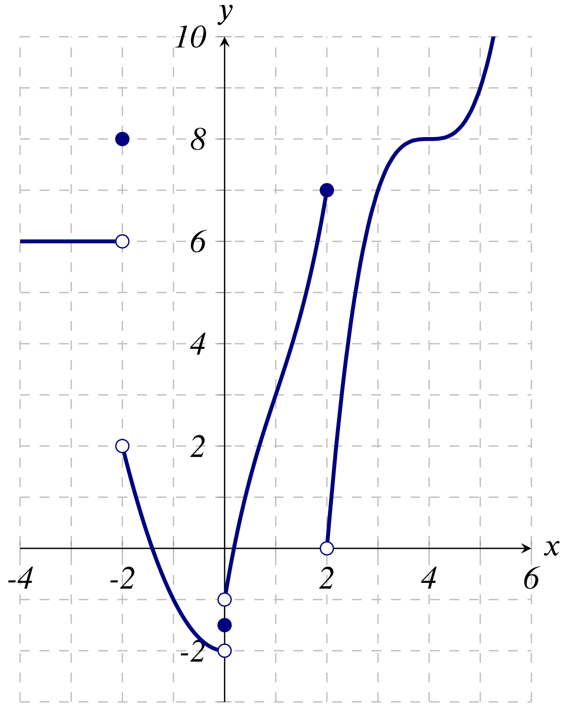
$$\lim_{x\to a} f(x)$$

exists if and only if

- $\lim_{x\to a^-} f(x)$ *exists*
- $\lim_{x\to a^+} f(x)$ *exists*
- $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

In this case, $\lim_{x\to a} f(x)$ *is equal to the common value of the two one sided limits.*

Question 17. *Evaluate the expressions by referencing the graph below. Write DNE if the limit does not exist.*



(a) $\lim_{x\to 4} f(x)$

(b) $\lim_{x\to -3} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $\lim_{x\to 0^-} f(x)$

(e) $\lim_{x\to 0^+} f(x)$

(f) $f(-2)$

(g) $\lim_{x\to 2^-} f(x)$

(h) $\lim_{x\to -2^-} f(x)$

(i) $\lim_{x\to 0} f(x+1)$

(j) $f(0)$

(k) $\lim_{x\to 1^-} f(x-4)$

(l) $\lim_{x\to 0^+} f(x-2)$

4 Polynomial functions

After completing this section, students should be able to do the following.

- Multiply polynomials.
- Factor polynomials.
- Solve polynomial equations.

Break-Ground:

4.1 How crazy could it be?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, do you remember when we first starting graphing functions? Like with a "T-chart?"

Riley: I remember everything.

Devyn: I used to get so excited to plot stuff! I would wonder: "What crazy curve would be drawn this time? What crazy picture will I see?"

Riley: Then we learned about the slope-intercept form of a line. Good-old

$$y = mx + b.$$

Devyn: Yeah, but lines are really boring. What about polynomials? What could you tell me about

$$y = 5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$$

just by looking at the equation?

Riley: Hmmmm. I'm not sure…

Problem 1. *When x is a large number (furthest from zero), which term of* $5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ *is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 2. *When x is a small number (near zero), which term of* $5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ *is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 3. *Very roughly speaking, what does the graph of* $y = 5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ *look like?*

Multiple Choice:

(a) *The graph starts in the lower left and ends in the upper right of the plane.*

(b) *The graph starts in the lower right and ends in the upper left of the plane.*

(c) *The graph looks something like the letter "U."*

(d) *The graph looks something like an upside down letter "U."*

Dig-In:

4.2 Working with polynomials

The functions you are most familiar with are probably polynomial functions.

What are polynomial functions?

Definition. A **polynomial function** in the variable x is a function which can be written in the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i's are all constants (called the **coefficients**) and n is a whole number (called the **degree** when $n \neq 0$). The domain of a polynomial function is $(-\infty, \infty)$.

Question 18. *Which of the following are polynomial functions?*

Select All Correct Answers:

(a) $f(x) = 0$

(b) $f(x) = -9$

(c) $f(x) = 3x + 1$

(d) $f(x) = x^{1/2} - x + 8$

(e) $f(x) = -4x^{-3} + 5x^{-1} + 7 - 18x^2$

(f) $f(x) = (x + 1)(x - 1) + e^x - e^x$

(g) $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$

(h) $f(x) = x^7 - 32x^6 - \pi x^3 + 45/84$

The phrase above "in the variable x" can actually change.

$$y^2 - 4y + 1$$

is a polynomial in y, and

$$\sin^2(x) + \sin(x) - 3$$

is a polynomial in $\sin(x)$.

Multiplying

Multiplying polynomials is based on the familiar property of arithmetic, **distribution**: $a \cdot (b + c) = ab + ac$.

Example 21. *Multiply* $2x^3$ *by* $x^5 - 4x^4 + 7x - 2$.

Use distribution.

$$\begin{aligned} 2x^3\left(x^5 - 4x^4 + 7x - 2\right) &= 2x^3 \cdot x^5 - 2x^3 \cdot 4x^4 + 2x^3 \cdot 7x \\ &\quad - 2x^3 \cdot 2 \\ &= 2x^8 - 8x^7 + 14x^4 - 4x^3 \end{aligned}$$

Example 22. *Multiply* $3x^2 + 2x - 1$ *by* $2x^4 + 5x^3 + x + 1$.

We'll start by distributing the first polynomial into the second. Then we'll distribute back, and finish by combining like terms.

$$\begin{aligned} (3x^2 + 2x - 1)(2x^4 + 5x^3 + x + 1) &= 2x^4(3x^2 + 2x - 1) \\ &\quad + 5x^3(3x^2 + 2x - 1) \\ &\quad + x(3x^2 + 2x - 1) \\ &\quad + 1(3x^2 + 2x - 1) \\ &= (6x^6 + 4x^5 - 2x^4) \\ &\quad + (15x^5 + 10x^4 - 5x^3) \\ &\quad + (3x^3 + 2x^2 - x) \\ &\quad + (3x^2 + 2x - 1) \end{aligned}$$

After combining like-terms, we find

$$6x^6 + 19x^5 + 8x^4 - 2x^3 + 5x^2 + x - 1.$$

The result is that we have multiplied every term of the first polynomial by every term of the second, then added the results together. In the case of two **binomials** (polynomials with only two terms), this is frequently referred to as **FOIL**.

There are several product formulas that arise repeatedly when working with binomials. You will likely have seen most of these before.

$$
\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a-b)^2 &= a^2 - 2ab + b^2 \\
(a+b)(a-b) &= a^2 - b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \\
(a-b)(a^2+ab+b^2) &= a^3 - b^3 \\
(a+b)(a^2-ab+b^2) &= a^3 + b^3
\end{aligned}
$$

Example 23. *If f is the polynomial function given by $f(x) = x^3 - 2x^2 + 5$, find $f(x+1)$. Find $f(2+h)$.*

$f(x+1)$ means to replace x in the formula for $f(x)$ by $x+1$.

$$
\begin{aligned}
f(x+1) &= (x+1)^3 - 2(x+1)^2 + 5 \\
&= (x^3 + 3x^2 + 3x + 1) - 2(x^2 + 2x + 1) + 5 \\
&= x^3 + x^2 - x + 4
\end{aligned}
$$

In the same way, $f(2+h)$ asks us to replace x by $2+h$.

$$
\begin{aligned}
f(2+h) &= (2+h)^3 - 2(2+h)^2 + 5 \\
&= (8 + 12h + 6h^2 + h^3) - 2(4 + 4h + h^2) + 5 \\
&= h^3 + 4h^2 + 4h + 5
\end{aligned}
$$

Problem 4. *If $f(x) = 4x^2 - 3x + 1$, find $f(x+h) - f(x)$.*

Multiple Choice:

(a) h

(b) $4h^2 - 3h + 1$

(c) $8xh + 4h^2 - 3h$

(d) $4x^2 + 8xh + 4h^2 - 3x - 3h + 1$

Factoring

Factoring is a bit like an inverse operation of multiplying polynomials. We start with the multiplied out polynomial, and ask what the individual factors were.

The easiest factors to deal with are common factors.

Example 24. *Factor* $8x^5 + 40x^4 + 40x^3$.

The coefficients of this polynomial have a greatest common divisor of 8, and the least degree of the terms is 3, so $8x^3$ is a common factor. Taking this out leaves: $8x^3(x^2 + 5x + 5)$.

For **trinomials** (polynomials with three terms) of the form $ax^2 + bx + c$, we try to factor as $(px + r)(qx + s)$. We'll start with an example with $a = 1$.

Example 25. *Factor* $x^2 + 2x - 24$.

Since the leading coefficient is 1, we want to factor $x^2 + 2x - 24$ as $(x+r)(x+s)$. If we multiply out $(x+r)(x+s)$ we get $x^2 + (r+s)x + rs$. We need to find two numbers that add to 2, and multiply to -24. Look at the different ways of multiplying two numbers to get -24: $-1 \cdot 24$, $-2 \cdot 12$, $-3 \cdot 8$, $-4 \cdot 6$... That last one, -4 and 6 are two numbers that add to 2 and multiply to give -24.

Our factorization is $x^2 + 2x - 24 = (x-4)(x+6)$.

The process is slightly more complicated when $a \neq 1$.

Example 26. *Factor* $6x^2 + 11x + 3$.

In this case, want to find $(px + r)(qx + s) = pqx^2 + (ps + qr)x + rs$. Ignore that middle term for now, and just focus on the leading term and the constant term. We need to find numbers that multiply to 6 for our coefficients, and that multiply to 3 for the constants in each factor. That means we're looking at things like $(6x+1)(x+3)$, $(6x+3)(x+1)$, $(2x+3)(3x+1)$, $(2x+1)(3x+3)$. Actually, that's it, as 6 can only factor as $1 \cdot 6$ or $2 \cdot 3$, and 3 only factors as $1 \cdot 3$.

All of these possibilities will give the right x^2 term and the right constant term. The only difference is the x-term they give. Do any of them give an x-coefficient of 11? Yes!

$6x^2 + 11x + 3 = (2x+3)(3x+1)$.

Remember, a **root** of a polynomial function is an x-value where the polynomial is zero. There is a close relationship between roots of a polynomial and factors of that polynomial. Precisely:

Theorem 4. *If $x = c$ is a root of a polynomial, then $x - c$ is a factor of that polynomial.*

Example 27. *Factor* $3x^3 - 13x^2 + 2x + 8$.

Notice that if we substitute $x = 1$ into the polynomial, we get zero out. That means $x - 1$ is a factor of $3x^3 - 12x^2 + 2x + 8$. Polynomial long division gives:

$$
\begin{array}{r|rrrr}
 & 3x^2 & -10x & -8 & \\
\hline
x-1 & 3x^3 & -13x^2 & +2x & +8 \\
 & -3x^3 & +3x^2 & & \\
\hline
 & & -10x^2 & +2x & \\
 & & 10x^2 & -10x & \\
\hline
 & & & -8x & +8 \\
 & & & 8x & -8 \\
\hline
 & & & & 0
\end{array}
$$

The quotient was $3x^2 - 10x - 8$, so $3x^3 - 13x^2 + 2x + 8 = (x-1)(3x^2 - 10x - 8)$. It remains to factor the quotient.

The leading coefficient 3 factors as $1 \cdot 3$, and 8 factors as $1 \cdot 8$ and $2 \cdot 4$. Examining the possibilities, we find the factorization of $3x^2 - 10x - 8$ as $(3x+2)(x-4)$. The entire polynomial factors as $(x-1)(3x+2)(x-4)$.

Equations

A *quadratic equation in x* is an equation which is equivalent to one with the form $ax^2 + bx + c = 0$, where a, b, and c are constants, with $a \neq 0$.

There are three major techniques you are probably familiar with for solving quadratic equations:

- Factoring.
- Completing the Square.
- Quadratic Formula.

Each of these methods are important. Factoring is vital, because it is a valid approach to solve nearly any type of equation. Completing the Square is a technique that becomes useful when we need to rewrite certain types of expressions. The Quadratic Formula will always work, but has some limitations.

Example 28. *Solve the quadratic equation*

$$2x^2 + 5x = 27x - 60.$$

We'll start by writing this in it's standard form: $2x^2 - 22x + 60 = 0$. Notice how there is a common factor of 2? Let's divide by 2.

$x^2 - 11x + 30 = 0$. Any of the above methods will work here, so let's try factoring. What numbers add to -11 and multiply to 30? -5 and -6 do. That gives us:

$$
\begin{aligned}
x^2 - 11x + 30 &= 0 \\
(x-5)(x-6) &= 0
\end{aligned}
$$

Either $x-5=0$ (giving us $x=5$) or $x-6=0$ (giving us $x=6$). The two solutions are $x=5,6$.

Example 29. *Solve the quadratic equation*

$$x^2 = 6x - 4.$$

Again, we'll write it in standard form: $x^2-6x+4=0$. The quadratic x^2-6x+4 does not factor, so we'll complete the square instead.

Start by moving the constant term to the other side

$$x^2 - 6x = -4.$$

This has $b=-6$. That means $\frac{b}{2}=-3$ and $\left(\frac{b}{2}\right)^2=9$. Add 9 to both sides.

$$x^2 - 6x + 9 = -4 + 9 = 5.$$

The left-hand side is now a perfect square, $(x-3)^2 = x^2-6x+9$. We can then solve using square roots.

$$\begin{aligned} x^2 - 6x + 9 &= 5 \\ (x-3)^2 &= 5 \\ x - 3 &= \pm\sqrt{5} \\ x &= 3 \pm \sqrt{5}. \end{aligned}$$

If you had used the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$, instead of factoring or completing the square above, you would have found the same solutions.

Problem 5. *Solve the quadratic equation* $x^2+4=4(x+2)$.

Multiple Choice:

(a) $x=\pm 2$

(b) $x = 2 \pm 2\sqrt{2}$

(c) $x=-2,-4$

(d) *none of the above*

Building on these methods, we'll consider more general types of equations. A polynomial equation is an equation which is equivalent to one of the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$.

Example 30. *Solve the equation* $x^4 = 6x^2 - 4$.

Start by rewriting as $x^4-6x^2+4=0$. This is not a quadratic equation in x, but it IS a quadratic in x^2. We continue by completing the square.

$$\begin{aligned} x^4 - 6x^2 &= 4 \\ x^4 - 6x^2 + 9 &= 4 + 9 \\ (x^2-3)^2 &= 13 \\ x^2 - 3 &= \pm\sqrt{13} \\ x^2 &= 3 \pm \sqrt{13} \end{aligned}$$

Either $x^2 = 3+\sqrt{13}$, or $x^2 = 3-\sqrt{13}$. Since $3-\sqrt{13}$ is negative, only $x^2 = 3+\sqrt{13}$ is possible.

The only real solutions are $x = \pm\sqrt{3+\sqrt{13}}$.

Factoring is a process that helps us solve more than just quadratic equations, as long as we first get one side of the equation equal to zero.

Example 31. *Solve the equation*

$$2x^2(x-4)(2x-5)^3 = 0.$$

The only way for a product of real numbers to be zero, is for one of the factors to have been zero. Here, that means either $2x^2 = 0$, or $x - 4 = 0$ or $(2x - 5)^3 = 0$. This gives us three equations, each substantially less complicated than the original one, to solve. By solving them, we find $x = 0, 4, \frac{5}{2}$.

Example 32. *Solve* $2x(2x^2 + 1) = x^2(x + 13) - 8$.

Start by rewriting the equation as $3x^3 - 13x^2 + 2x + 8 = 0$. This is not a quadratic-type equation, so we resort to factoring.

$$\begin{aligned} 3x^3 - 13x^2 + 2x + 8 &= 0 \\ (x-1)(3x^2 - 10x - 8) &= 0 \\ (x-1)(3x+2)(x-4) &= 0 \end{aligned}$$

Thus, either $x - 1 = 0$ (so $x = 1$), or $3x + 2 = 0$ (so $x = -2/3$), or $x - 4 = 0$ (so $x = 4$).

The solutions are $x = 1, 4, -3/2$.

The next theorem above is a deep fact of mathematics. The great mathematician Gauss proved the theorem in 1799.

Theorem 5 (The Fundamental Theorem of Algebra)**.** *Every polynomial of the form*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where the a_i*'s are real (or even complex!) numbers and* $a_n \neq 0$ *has exactly* n *(possibly repeated) complex roots.*

Recall that the **multiplicity** of a root indicates how many times that particular root is repeated. The Fundamental Theorem of Algebra tells us that a polynomial equation of degree n, will have exactly n complex solutions, once multiplicity is taken into account. As non-real solutions appear in complex-conjugate pairs (if our equation has real coefficients), we will always have an even number of non-real solutions. That means, taking multiplicity into account, a quadratic equation will have either 2 or 0 real solutions. A cubic equation will have either 3 or 1 real solution.

Problem 6. *The equation* $x^3 = 3x - 2$ *has* $x = 1$ *as one solution. Find another solution.*

What can the graphs look like?

Polynomial functions with degree 1 are referred to as linear polynomials. This is due to the fact that such a function can be written as $f(x) = mx + b$. The graph of such a function is a straight line with slope m and y-intercept at $(0, b)$.

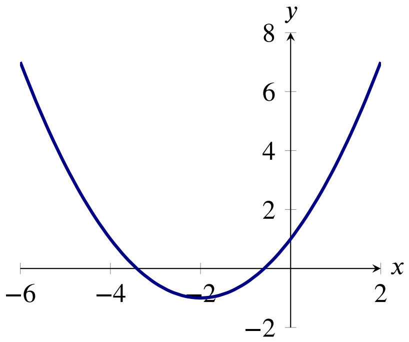
Quadratic functions, written as $f(x) = ax^2 + bx + c$ with $a \neq 0$, have parabolas as their graphs.

Example 33. *Sketch the graph of the function* $f(x) = \frac{1}{2}x^2 + 2x + 1$.

To plot the parabola, we first complete the square to write our function in the form $f(x) = a(x - h)^2 + k$.

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 + 2x + 1 \\ &= \frac{1}{2}\left(x^2 + 4x\right) + 1 \\ &= \frac{1}{2}\left(x^2 + 4x + 4\right) + 1 - \frac{1}{2} \cdot 4 \\ &= \frac{1}{2}(x+2)^2 - 1 \end{aligned}$$

The parabola we are looking for has vertex at $(-2, -1)$, opens upward (since $1/2 > 0$), and is wider than the standard $y = x^2$ parabola.

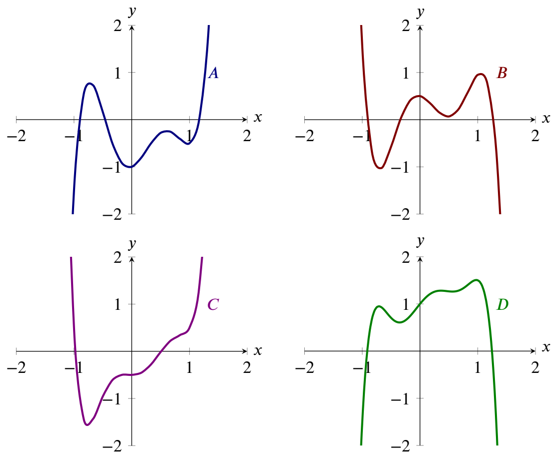


In terms of graph transformations, we can think of this as the graph of $g(x) = x^2$, which has been shifted horizontally 2 units to the left, vertically compressed by a factor of $\frac{1}{2}$, then shifted vertically 1 unit downward.

Problem 7. *Use the graph of $f(x) = 2x^2 - 4x + 3$ to estimate the value of $\lim_{x \to 2} f(x)$.*

The upshot of the Fundamental Theorem of Algebra (in terms of graphing) is that when we plot a polynomial of degree n, its graph will cross the x-axis at most n times. Each crossing corresponds to a root of that polynomial.

Example 34. *Here we see the the graphs of four polynomial functions.*



For each of the curves, determine if the polynomial has ***even*** *or* ***odd*** *degree, and if the leading coefficient (the one next to the highest power of x) of the polynomial is* ***positive*** *or* ***negative***.

- Curve A is defined by an odd degree polynomial with a positive leading term.
- Curve B is defined by an odd degree polynomial with a negative leading term.
- Curve C is defined by an even degree polynomial with a positive leading term.
- Curve D is defined by an even degree polynomial with a negative leading term.

5 Rational functions

After completing this section, students should be able to do the following.

- Determine the domain of a rational function.
- Simplify rational expressions.
- Solve equations involving rational functions.
- Solve inequalities involving rational functions.

Break-Ground:

5.1 Will it divide?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I'm wondering about polynomials.
Riley: What about them?
Devyn: If we have $x^2 + x$ and $x^2 - 4$, I know how to add them. I know how to subtract them. I know how to multiply them. I even know how to factor each of them.
Riley: Sure. To add or subtract, we combine like terms. To multiply, we use distribution or FOIL.
Devyn: But we can't just divide $x^2 + x$ by $x^2 - 4$ and arrive at a polynomial! If we divide polynomials, we don't necessarily end up with another polynomial.
Riley: Sometimes we can, though. $x^2 + x$ divided by x is basically just the polynomial $x + 1$.
Devyn: But if we don't get a polynomial back, what kind of function DO we get?

Problem 1. *If we divide* $x^3 + 3x^2 + 3x + 1$ *by* $x + 1$*, do we end up with a polynomial?*

Multiple Choice:

(a) *yes*

(b) *no*

Problem 2. *If we divide* $x^3 + 3x^2 + 3x + 4$ *by* $x + 1$*, do we end up with a polynomial?*

Multiple Choice:

(a) *yes*

(b) *no*

Dig-In:

5.2 Working with rational functions

What are rational functions?

In algebra, polynomials play the same role as the integers do in arithmetic. We add them, subtract them, multiply them, and factor them. We cannot divide them, however, if we want an integer answer. Since 4 is not a factor of 7, $\frac{7}{4}$ is not an integer. If we want to be able to divide integers, we have to move to the *rational numbers*, which are fractions $\frac{p}{q}$ where p and q are integers, and $q \neq 0$.

The same idea holds for polynomials. We can add them, subtract them, multiply them, and factor them. However, to divide them we have to move to rational functions.

Definition. A **rational function** in the variable x is a function the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions, and q is not the constant zero function. The domain of a rational function is all real numbers except for where the denominator is equal to zero.

Question 19. *Which of the following are rational functions?*

Select All Correct Answers:

(a) $f(x) = 0$

(b) $f(x) = \dfrac{3x+1}{x^2-4x+5}$

(c) $f(x) = e^x$

(d) $f(x) = \dfrac{\sin(x)}{\cos(x)}$

(e) $f(x) = -4x^{-3} + 5x^{-1} + 7 - 18x^2$

(f) $f(x) = x^{1/2} - x + 8$

(g) $f(x) = \dfrac{\sqrt{x}}{x^3 - x}$

Working with rational functions

Example 35. *Find the domain of the rational function* $f(x) = \dfrac{4x^3+1}{6x^2-7x-3}$

We start by setting the denominator equal to zero.

$$\begin{aligned} 6x^2 - 7x - 3 &= 0 \\ (3x+1)(2x-3) &= 0 \\ x &= -\frac{1}{3}, \frac{3}{2} \end{aligned}$$

The domain of f is all x except these two values. Thus:

$$\left(-\infty, \tfrac{1}{3}\right) \bigcup \left(-\tfrac{1}{2}, \tfrac{3}{2}\right) \bigcup \left(\tfrac{3}{2}, \infty\right).$$

When we need to simplify the form of a rational expression, our approach depends on the particular form we are presented with. If it consists of only a single fraction, we divide out the common factors.

Example 36. *Simplify the expression* $\dfrac{2x^2-3x-2}{4x^2+6x+2}$.

We begin by factoring. The numerator factors as $2x^2 - 3x - 2 = (2x+1)(x-2)$, and the denominator factors as $4x^2 + 6x + 2 = 2(2x + 1)(x + 1)$. That means,

$$\begin{aligned} \frac{2x^2 - 3x - 2}{4x^2 + 6x + 2} &= \frac{(2x+1)(x-2)}{2(2x+1)(x+1)} \\ &= \frac{2x+1}{2x+1} \cdot \frac{x-2}{2(x+1)} \\ &= \frac{x-2}{2(x+1)}, \end{aligned}$$

where we used the fact that $\dfrac{2x+1}{2x+1} = 1$.

When dealing with a sum/difference of two fractions, we must first convert to a common denominator. After the addition, we can divide away any common factors that are still present.

Example 37. *Simplify the expression*

$$\frac{x-6}{x^2+2x} + \frac{x+3}{x^2+x}.$$

We'll start by factoring the denominators. $x^2 + 2x = x(x + 2)$, and $x^2 + x = x(x + 1)$. These two fractions have a common denominator of $x(x + 1)(x + 2)$.

$$\begin{aligned} \frac{x-6}{x^2+2x} + \frac{x+3}{x^2+x} &= \frac{x-6}{x(x+2)} + \frac{x+3}{x(x+1)} \\ &= \frac{(x-6)(x+1)}{x(x+2)(x+1)} + \frac{(x+3)(x+2)}{x(x+1)(x+2)} \\ &= \frac{x^2-5x-6}{x(x+2)(x+1)} + \frac{x^2+5x+6}{x(x+2)(x+1)} \\ &= \frac{2x^2}{x(x+2)(x+1)} \\ &= \frac{2x}{(x+2)(x+1)}. \end{aligned}$$

Once we convert each fraction to the common denominator, we added numerators, then simplified.

If we have a complex fraction, involving a fraction in the numerator or the denominator, we can start by multiplying the numerator and denominator of the big fraction, by the common denominator of the smaller fractions. That eliminates the smaller fractions leaving the outer one to deal with.

Example 38. *Simplify the expression*

$$\frac{\frac{x+2}{x-1} + x}{\frac{x}{x-1} + \frac{2x+1}{x+1}}.$$

The smaller fractions in the numerator and denominator have a common denominator $(x - 1)(x + 1)$. We begin my multiplying both the numerator and denominator by $(x - 1)(x + 1)$.

$$\begin{aligned} \frac{\frac{x+2}{x-1} + x}{\frac{x}{x-1} + \frac{2x+1}{x+1}} &= \frac{\frac{x+2}{x-1} + x}{\frac{x}{x-1} + \frac{2x+1}{x+1}} \cdot \frac{(x-1)(x+1)}{(x-1)(x+1)} \\ &= \frac{(x+2)(x+1) + x(x+1)(x-1)}{x(x+1) + (2x+1)(x-1)} \\ &= \frac{(x+1)\left((x+2) + (x^2-x)\right)}{(x^2+x) + (2x^2-x-1)} \\ &= \frac{(x+1)\left(x^2+2\right)}{3x^2-1} \end{aligned}$$

There are no common factors between the numerator and denominator, so this cannot be simplified any further.

Example 39. *For the rational function $f(x) = \dfrac{2x}{x-3}$, find and simplify the following:*

$$\frac{f(x+h) - f(x)}{h}$$

$f(x+h)$ means replace x in the formula for f with $x+h$. This gives:

$$\begin{aligned}\frac{\frac{2(x+h)}{(x+h)-3}-\frac{2x}{x-3}}{h} &= \frac{\frac{2x+2h}{x+h-3}-\frac{2x}{x-3}}{h}\\ &= \frac{\frac{(2x+2h)(x-3)}{(x+h-3)(x-3)}-\frac{2x(x+h-3)}{(x-3)(x+h-3)}}{h}\\ &= \frac{\frac{2x^2-6x+2xh-6h}{(x+h-3)(x-3)}-\frac{2x^2+2xh-6x}{(x+h-3)(x-3)}}{h}\\ &= \frac{\frac{-6h}{(x+h-3)(x-3)}}{h}\\ &= \frac{-6h}{h(x+h-3)(x-3)}\\ &= \frac{-6}{(x+h-3)(x-3)}.\end{aligned}$$

To solve equations involving rational expressions, we have the freedom to clear out fractions before proceeding. After multiplying both sides by the common denominator, we are left with a polynomial equation.

Example 40. *Solve the equation*

$$\frac{2}{x}+\frac{3x}{x+1}=4.$$

The common denominator is $x(x+1)$. We multiply both sides by $x(x+1)$ to clear out the fractions.

$$\begin{aligned}\frac{2}{x}+\frac{3x}{x+1} &= 4\\ x(x+1)\left(\frac{2}{x}+\frac{3x}{x+1}\right) &= x(x+1)(4)\\ x(x+1)\cdot\frac{2}{x}+x(x+1)\cdot\frac{3x}{x+1} &= 4x(x+1)\\ 2(x+1)+3x(x) &= 4x^2+4x\\ 3x^2+2x+2 &= 4x^2+4x\\ x^2+2x-2 &= 0.\end{aligned}$$

The quadratic formula gives solutions as $x = \dfrac{-2\pm\sqrt{12}}{2} = -1\pm\sqrt{3}$.

If we look back at the original equation, we notice that there are some numbers that we are not allowed to plug in for x. When $x=0$ or $x=-1$, the left-hand side of the equation is not defined due to a division by zero issue. Since neither $-1+\sqrt{3}$ nor $-1-\sqrt{3}$ have such an issue, they are both solutions.

Question 20. *One solution of the equation*

$$\frac{2}{x+1}+\frac{1}{x+2}=1$$

is $x=\sqrt{3}$. *Find another solution.*

Inequalities

When faced with nonlinear inequalities, such as those involving general rational functions, we make use of a sign chart. The inequality in the following example is not given in factored form, so we have some work to do.

Example 41. *Solve the inequality* $x^2+5x \leq -10-\dfrac{16}{x-2}$.

We'll begin by moving everything to one side, then combining them all together into a single fraction.

$$\begin{aligned}
x^2 + 5x &\leq -10 - \frac{16}{x-2} \\
x^2 + 5x + 10 + \frac{16}{x-2} &\leq 0 \\
\left(x^2 + 5x + 10\right) \cdot \left(\frac{x-2}{x-2}\right) + \frac{16}{x-2} &\leq 0 \\
\frac{x^3 + 3x^2 - 20}{x-2} + \frac{16}{x-2} &\leq 0 \\
\frac{x^3 + 3x^2 - 4}{x-2} &\leq 0 \\
\frac{(x-1)(x+2)^2}{x-2} &\leq 0
\end{aligned}$$

Now that the inequality is in a better form for us to work with, we'll build a sign chart like we did in the last example.

x		-2	1	2
$x-1$	$-$	$-$	$+$	$+$
$(x+2)^2$	$+$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$

We see from the chart that $\dfrac{(x-1)(x+2)^2}{x-2}$ will be negative in $(1, 2)$. At $x = -2$ and $x = 1$ it is zero. The solution is then: $\{-2\} \cup [1, 2)$.

Problem 3. *Find the solution of the inequality:* $x + \dfrac{9}{x-1} > 5$.

Multiple Choice:

(a) $(-1, \infty)$

(b) $[-1, \infty)$

(c) $(-1, 2) \cup (2, \infty)$

(d) $\{-1\} \cup [2, \infty)$

(e) *none of the above*

Example 42. *Solve the inequality*

$$\frac{1}{x-2} - \frac{4}{x+1} \leq 3$$

We'll start by moving everything to the left-hand side and combining them into a single fraction.

$$\begin{aligned}
\frac{1}{x-2} - \frac{4}{x+1} &\leq 3 \\
\frac{1}{x-2} - \frac{4}{x+1} - 3 &\leq 0 \\
\frac{1(x+1)}{(x-2)(x+1)} - \frac{4(x-2)}{(x+1)(x-2)} - \frac{3(x+1)(x-2)}{(x+1)(x-2)} &\leq 0 \\
\frac{(x+1) - (4x-8) - (3x^2 - 3x - 6)}{(x+1)(x-2)} &\leq 0 \\
\frac{-3x^2 + 15}{(x+1)(x-2)} &\leq 0 \\
\frac{-3(x^2 - 5)}{(x+1)(x-2)} &\leq 0
\end{aligned}$$

To solve this, we'll construct a sign chart. We start by noticing that $x^2 - 5 = 0$ only if $x = \pm\sqrt{5}$.

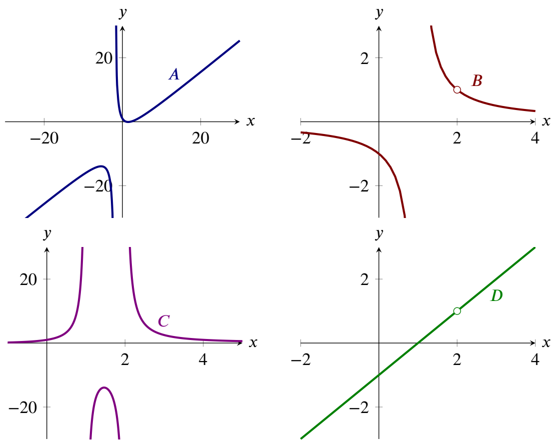
x	$-\sqrt{5}$ $\quad -1 \quad 2 \quad \sqrt{5}$				
-3	$-$	$-$	$-$	$-$	$-$
x^2-5	$+$	$-$	$-$	$-$	$+$
$x+1$	$-$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$	$+$

We see from the sign chart, that the solution is $\left(-\infty,-\sqrt{5}\right] \bigcup (-1,2) \bigcup \left[\sqrt{5},\infty\right)$.

What can the graphs look like?

There is a somewhat wide variation in the graphs of rational functions.

Example 43. *Here we see the the graphs of four rational functions.*



Match the curves A, B, C, and D with the functions

$$\frac{x^2-3x+2}{x-2}, \qquad \frac{x^2-3x+2}{x+2},$$
$$\frac{x-2}{x^2-3x+2}, \qquad \frac{x+2}{x^2-3x+2}.$$

Consider $\dfrac{x^2-3x+2}{x-2}$. This function is undefined only at $x=2$. Of the curves that we see above, D is undefined exactly at $x=2$.

Now consider $\dfrac{x^2-3x+2}{x+2}$. This function is undefined only at $x=-2$. The only function above that undefined exactly at $x=-2$ is curve A.

Now consider $\dfrac{x-2}{x^2-3x+2}$. This function is undefined at the roots of

$$x^2-3x+2=(x-2)(x-1).$$

Hence it is undefined at $x=2$ and $x=1$. It looks like both curves B and C would work. Distinguishing between these two curves is easy enough if we evaluate at $x=-2$. Check it out.

$$\begin{aligned}\left[\frac{x-2}{x^2-3x+2}\right]_{x=-2} &= \frac{-2-2}{(-2)^2-3(-2)+2}\\ &= \frac{-4}{4+6+2}\\ &= \frac{-4}{12}.\end{aligned}$$

Since this is negative, we see that $\dfrac{x-2}{x^2-3x+2}$ corresponds to curve B.

Finally, it must be the case that curve C corresponds to $\dfrac{x+2}{x^2-3x+2}$. We should note that if this function is evaluated at $x = -2$, the output is zero, and this corroborates our work above.

6 Limit laws

After completing this section, students should be able to do the following.

- Define continuity in terms of limits.
- Calculate limits using the limit laws.
- Famous functions are continuous on their domains.
- Calculate limits by replacing a function with a continuous function.
- Understand the Squeeze Theorem and how it can be used to find limit values.
- Calculate limits using the Squeeze Theorem.

Break-Ground:

6.1 Equal or not?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I've been thinking about limits.
Riley: So awesome!
Devyn: Think about

$$\lim_{x\to a}(f(x)+g(x)).$$

This is the number that $f(x)+g(x)$ gets nearer and nearer to, as x gets nearer and nearer to a.
Riley: You know it!
Devyn: So I think it is the same as

$$\lim_{x\to a} f(x)+\lim_{x\to a} g(x).$$

Riley: Yeah, that does make sense, since when you add two numbers, say

(a number near 6) + (a number near 7)

you get

(a number near 13)

Riley: Right! And I think the same reasoning will work for multiplication! So we should be able to say

$$\lim_{x\to a}(f(x)\cdot g(x)) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right).$$

Devyn: Yes, I think that's right! But what about *division*? Can we use similar reasoning to conclude

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

Problem 1. *Give an argument (similar to the one above) supporting the idea that*

$$\lim_{x\to a}(f(x)\cdot g(x)) = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right).$$

For the next problems, we use large to represent large positive numbers, and small to represent small positive values near 0.

Problem 2. *Using the context above,*

$$\frac{\textit{large}}{\textit{small}} = ?$$

Multiple Choice:

(a) *"large"*

(b) *"small"*

(c) *impossible to say*

Problem 3. *Using the context above,*

$$\frac{\textit{small}}{\textit{small}} = ?$$

Multiple Choice:

(a) *"large"*

(b) *"small"*

(c) *impossible to say*

Dig-In:

6.2 Continuity

Limits are simple to compute when they can be found by plugging the value into the function. That is, when

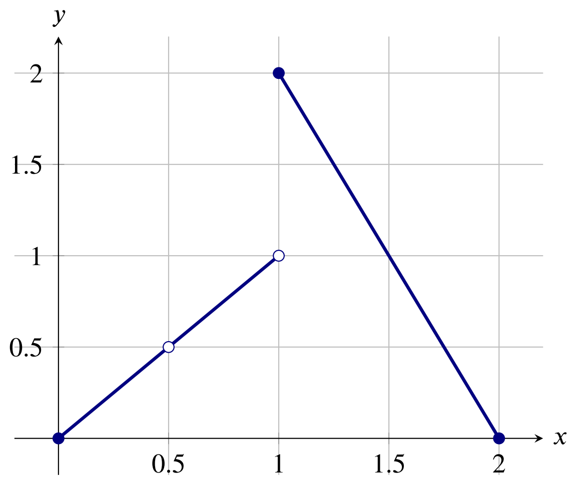
$$\lim_{x\to c} f(x) = f(c).$$

We call this property *continuity*.

Definition. A function f is **continuous at a point** a if

$$\lim_{x\to a} f(x) = f(a).$$

Question 21. *Consider the graph of $y = f(x)$ below*



Which of the following are true?

Multiple Choice:

(a) *f is continuous at $x = 0.5$*

(b) *f is continuous at $x = 1$*

(c) *f is continuous at $x = 1.5$*

It is very important to note that saying

"a function f is continuous at a point a"

is really making **three** statements:

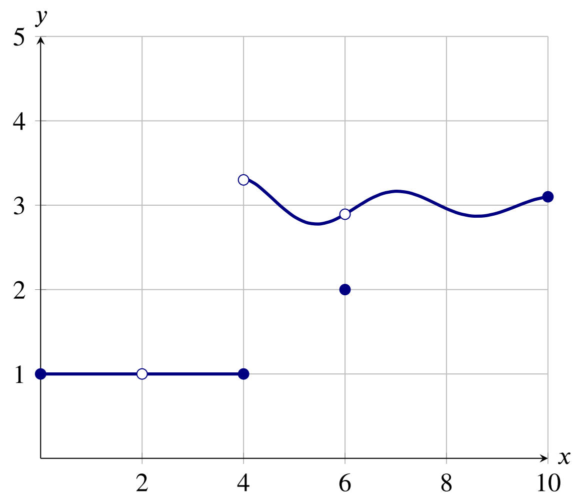
(a) $f(a)$ is defined. That is, a is in the domain of f.

(b) $\lim_{x\to a} f(x)$ exists.

(c) $\lim_{x\to a} f(x) = f(a)$.

The first two of these statements are implied by the third statement.

Example 44. *Find the discontinuities (the points x where a function is not continuous) for the function described below:*



To start, f is not even defined at $x = 2$, hence f cannot be continuous at $x = 2$.

Next, from the plot above we see that $\lim_{x\to 4} f(x)$ does not

> exist because
>
> $$\lim_{x\to 4^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 4^+} f(x) \approx 3.5$$
>
> Since $\lim_{x\to 4} f(x)$ does not exist, f cannot be continuous at $x = 4$.
>
> We also see that $\lim_{x\to 6} f(x) \approx 3$ while $f(6) = 2$. Hence $\lim_{x\to 6} f(x) \neq f(6)$, and so f is not continuous at $x = 6$.

Building from the definition of *continuity at a point*, we can now define what it means for a function to be *continuous* on an open interval.

Definition. A function f is **continuous on an open interval** I if $\lim_{x\to a} f(x) = f(a)$ for all a in I.

Loosely speaking, a function is continuous on an interval I if you can draw the function on that interval without any breaks in the graph. This is often referred to as being able to draw the graph "without picking up your pencil."

Theorem 6 (Continuity of Famous Functions)**.** *The following functions are continuous on the given intervals for k a real number and b a positive real number:*

Constant function $f(x) = k$ *is continuous on* $-\infty < x < \infty$.

Identity function $f(x) = x$ *is continuous on* $-\infty < x < \infty$.

Power function $f(x) = x^b$ *is continuous on* $-\infty < x < \infty$.

Exponential function $f(x) = b^x$ *is continuous on* $-\infty < x < \infty$.

Logarithmic function $f(x) = \log_b(x)$ *is continuous on* $0 < x < \infty$.

Sine and cosine *Both* $\sin(x)$ *and* $\cos(x)$ *are continuous on* $-\infty < x < \infty$.

In essence, we are saying that the functions listed above are continuous wherever they are defined, that is, on their natural domains.

Question 22. *Compute:* $\lim_{x\to 3} x^{\pi}$

Left and right continuity

At this point we have a small problem. For functions such as $\sqrt{x}$, the natural domain is $0 \leq x < \infty$. This is not an open interval. What does it mean to say that $\sqrt{x}$ is continuous at 0 when $\sqrt{x}$ is not defined for $x < 0$? To get us out of this quagmire, we need a new definition:

Definition. A function f is **left continuous** at a point a if $\lim_{x\to a^-} f(x) = f(a)$.

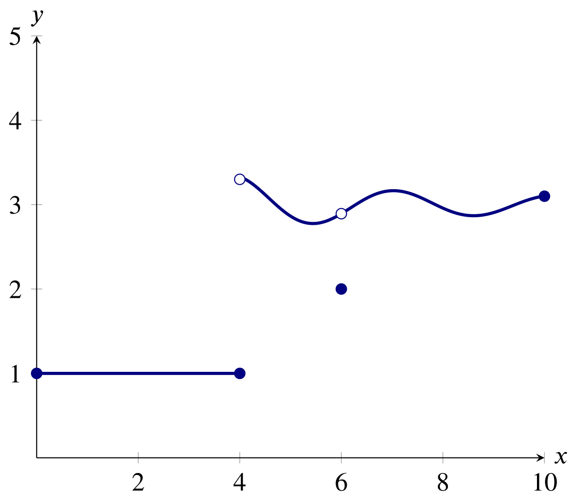
A function f is **right continuous** at a point a if $\lim_{x\to a^+} f(x) = f(a)$.

Now we can say that a function is continuous at a left endpoint of an interval if it is right continuous there, and a function is continuous at the right endpoint of an interval if it is left continuous there. This allows us to talk about continuity on closed intervals.

Definition. A function f is

- **continuous on a closed interval** $[a, b]$ if f is continuous on (a, b), right continuous at a, and left continuous at b;
- **continuous on a half-closed interval** $[a, b)$ if f is continuous on (a, b) and right continuous at a;
- **continuous on a half-closed interval** $(a, b]$ if f is f is continuous on (a, b) and left continuous at b.

Question 23. *Here we give the graph of a function defined on* $[0, 10]$.



What are the largest intervals of continuity for this function?

Multiple Choice:

(a) $[0, 10]$

(b) $[0, 4)$ *and* $(4, 10]$

(c) $[0, 4]$, $[4, 6]$, *and* $[6, 10]$

(d) $(0, 4)$, $(4, 6)$, *and* $(6, 10)$

(e) $[0, 4]$, $(4, 6)$, *and* $[6, 10]$

(f) $[0, 4]$, $(4, 6)$, *and* $(6, 10]$

(g) $[0, 4)$, $(4, 6)$, *and* $(6, 10]$

(h) $(0, 4]$, $[4, 6]$, *and* $[6, 10)$

Dig-In:

6.3 The limit laws

In this section, we present a handful of rules called the *Limit Laws* that allow us to find limits of various combinations of functions.

Theorem 7 (Limit Laws)**.** *Suppose that* $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$.

Sum/Difference Law $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$.

Product Law $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM$.

Quotient Law $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, *if* $M \neq 0$.

Question 24. *True or false: If f and g are continuous functions on an interval I, then $f \pm g$ is continuous on I.*

Multiple Choice:

(a) *True*

(b) *False*

Question 25. *True or false: If f and g are continuous functions on an interval I, then f/g is continuous on I.*

Multiple Choice:

(a) *True*

(b) *False*

Example 45. *Compute the following limit using limit laws:*

$$\lim_{x\to 1}(5x^2 + 3x - 2)$$

Well, get out your pencil and write with me:

$$\lim_{x\to 1}(5x^2 + 3x - 2) = \lim_{x\to 1} 5x^2 + \lim_{x\to 1} 3x - \lim_{x\to 1} 2$$

by the Sum/Difference Law. So now

$$= 5\lim_{x\to 1} x^2 + 3\lim_{x\to 1} x - \lim_{x\to 1} 2$$

by the Product Law. Finally by continuity of x^k and k,

$$= 5(1)^2 + 3(1) - 2 = 6.$$

We can generalize the example above to get the following theorems.

Theorem 8 (Continuity of Polynomial Functions)**.** *All polynomial functions, meaning functions of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a whole number and each a_i is a real number, are continuous for all real numbers.

Theorem 9 (Continuity of Rational Functions)**.** *Let f and g be polynomials. Then a rational function, meaning an expression of the form*

$$h = \frac{f}{g}$$

is continuous for all real numbers except where $g(x) = 0$. That is, rational functions are continuous wherever they are defined.

Let a be a real number such that $g(a) \neq 0$. Then, since $g(x)$ is continuous at a, $\lim_{x\to a} g(x) \neq 0$. Therefore, write with me,

$$\lim_{x\to a} h(x) = \lim_{x\to a} \frac{f(x)}{g(x)}$$

and now by the Quotient Law,

$$\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$$

and by the continuity of polynomials we may now set $x = a$

$$\frac{f(a)}{g(a)} = h(a).$$

Since we have shown that $\lim\limits_{x\to a} h(x) = h(a)$, we have shown that h is continuous at $x = a$.

Question 26. *Where is* $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$ *continuous?*

Multiple Choice:

(a) *for all real numbers*

(b) *at* $x = 2$

(c) *for all real numbers, except* $x = 2$

(d) *impossible to say*

Back in Theorem 6 we mentioned a big list of functions that were continuous. We mention them again in the following statement. We will study some of these functions in more detail in later sections. For now, we focus only on the fact that they are continuous.

Theorem 10 (Continuity of Other Functions)**.** *Polynomials, the trigonometric functions* $\sin x$ *and* $\cos x$*, and exponential functions* b^x *are continuous everywhere. Rational functions, logarithms, and the other trigonometric functions are continuous in their domain.*

Example 46. *Compute the limit:*

$$\lim_{x\to\pi/4} \sin x \cos x.$$

By the limit laws,

$$\lim_{x\to\pi/4} \sin x \cos x = \left(\lim_{x\to\pi/4} \sin x\right)\left(\lim_{x\to\pi/4} \cos x\right).$$

$\lim\limits_{x\to\pi/4} \sin x = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$. $\lim\limits_{x\to\pi/4} \cos x = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$.

Putting these together, $\lim\limits_{x\to\pi/4} \sin x \cos x = \dfrac{1}{2}$.

Now, we give basic rules for how limits interact with composition of functions.

Theorem 11 (Composition Limit Law)**.** *If* $f(x)$ *is continuous at* $x = \lim\limits_{x\to a} g(x)$*, then*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)).$$

Because the limit of a continuous function is the same as the function value, we can now pass limits inside continuous functions.

Corollary 1 (Continuity of Composite Functions)**.** *If* g *is continuous at* $x = a$*, then* $f(g(x))$ *is continuous at* $x = a$*.*

Example 47. *Compute the following limit using limit laws:*

$$\lim_{x\to 0} \sqrt{\cos(x)}$$

By continuity of x^k, assuming $\lim\limits_{x\to 0} \cos(x) > 0$,

$$\lim_{x\to 0} \sqrt{\cos(x)} = \sqrt{\lim_{x\to 0} \cos(x)},$$

and now since cosine is continuous for all real numbers,

$$\begin{aligned}\sqrt{\lim_{x\to 0}\cos(x)} &= \sqrt{\cos(0)}\\ &= \sqrt{1}\\ &= 1.\end{aligned}$$

Many of the Limit Laws and theorems about continuity in this section might seem like they should be obvious. You may be wondering why we spent an entire section on these theorems. The answer is that these theorems will tell you exactly when it is easy to find the value of a limit, and exactly what to do in those cases.

The most important thing to learn from this section is whether the limit laws can be applied for a certain problem, and when we need to do something more interesting. We will begin discussing those more interesting cases in the next section. For now, we end this section with a question:

A list of questions

Let's try this out.

Question 27. *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 2}\frac{x^2+3x+2}{x+2}$$

Multiple Choice:

(a) *yes*

(b) *no*

Question 28. *Compute:*

$$\lim_{x\to 2}\frac{x^2+3x+2}{x+2}$$

Question 29. *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 2}\frac{x^2-3x+2}{x-2}$$

Multiple Choice:

(a) *yes*

(b) *no*

Question 30. *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0} x\sin(1/x)$$

Multiple Choice:

(a) *yes*

(b) *no*

Question 31. *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0}\cot(x^3)$$

Multiple Choice:

(a) *yes*

(b) *no*

Question 32. *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 0}\sec^2(5^x-1)$$

Multiple Choice:

(a) *yes*

(b) *no*

Question 33. *Compute:*

$$\lim_{x\to 0}\sec^2(5^x-1)$$

Question 34. *Can this limit be directly computed by limit laws?*

$$\lim_{x\to 1}(x-1)\cdot\csc(\sqrt{x^2-1})$$

Multiple Choice:

(a) *yes*

(b) *no*

Question 35. *Can this limit be directly computed by limit laws?*

$$\lim_{x \to 0} x \cot^2 x$$

Multiple Choice:

(a) *yes*

(b) *no*

Question 36. *Can this limit be directly computed by limit laws?*

$$\lim_{x \to 0} \frac{2^x - 1}{3^{x-1}}$$

Multiple Choice:

(a) *yes*

(b) *no*

Question 37. *Compute:*

$$\lim_{x \to 0} \frac{2^x - 1}{3^{x-1}}$$

Question 38. *Can this limit be directly computed by limit laws?*

$$\lim_{x \to 0} (1 + x)^{1/x}$$

Multiple Choice:

(a) *yes*

(b) *no*

Dig-In:

6.4 The Squeeze Theorem

In mathematics, sometimes we can study complex functions by exchanging them for simplier functions. The *Squeeze Theorem* tells us one situation where this is possible.

Theorem 12 (Squeeze Theorem)**.** *Suppose that*

$$g(x) \leq f(x) \leq h(x)$$

for all x close to a but not necessarily equal to a. If

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x),$$

then $\lim_{x\to a} f(x) = L$.

Question 39. *I'm thinking of a function f. I know that for all x*

$$0 \leq f(x) \leq x^2.$$

What is $\lim_{x\to 0} f(x)$*?*

Multiple Choice:

(a) $f(x)$

(b) $f(0)$

(c) 0

(d) *impossible to say*

Example 48. *An continuous function f satisfies the property that* $8x-13 \leq f(x) \leq x^2+2x-4$. *What is* $\lim_{x\to 3} f(x)$*?*

$$\lim_{x\to 3} (8x-13) = 11.$$

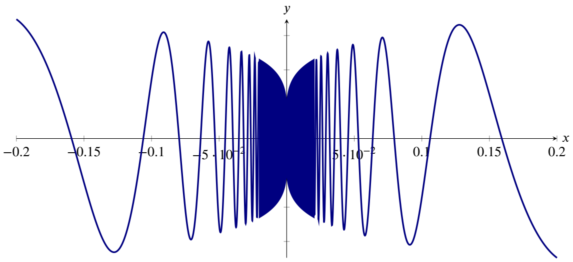
$$\lim_{x\to 3} \left(x^2+2x-4\right) = 11.$$

Then

$$\lim_{x\to 3} f(x) = 11.$$

Example 49. *Consider the function*

$$f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\frac{1}{x}\right) & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0, \end{cases}$$



Is this function continuous at $x = 0$?

We must show that $\lim_{x\to 0} f(x) = 0$. Note

$$-|\sqrt[5]{x}| \leq f(x) \leq |\sqrt[5]{x}|.$$

Since

$$\lim_{x\to 0} -|\sqrt[5]{x}| = 0 = \lim_{x\to 0} |\sqrt[5]{x}|,$$

we see by the Squeeze Theorem, Theorem, that $\lim_{x\to 0} f(x) = 0$. Hence $f(x)$ is continuous.

Here we see how the informal definition of continuity being that you can "draw it" without "lifting your pencil" differs from the formal definition.

7 (In)determinate forms

After completing this section, students should be able to do the following.

- Understand what is meant by the form of a limit.
- Calculate limits of the form zero over zero.
- Calculate limits of the form nonzero number over zero.
- Identify determinate and indeterminate forms.
- Distinguish between determinate and indeterminate forms.
- Discuss why infinity is not a number.

Break-Ground:

7.1 Could it be anything?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Hey Riley, what is

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}?$$

Riley: I've been looking at the graph of $f(x) = \dfrac{\sin x}{x}$, and it looks like the limit should be 1!

Devyn: But why don't we actually calculate it? I mean, just plug in zero.

$$\left[\frac{\sin(\theta)}{\theta}\right]_{\theta=0} = \frac{\sin(0)}{0} = \frac{0}{0} \cdots$$

Riley: You were going to say "1," right?

Devyn: Yeah, but now I'm not sure I was right.

Riley: Dividing by zero is usually a bad idea.

Devyn: You are right. I will never do it again! Also, don't tell anyone about this conversation.

Riley: What conversation?

Devyn: Exactly.

Problem 1. *Consider the function*

$$f(x) = \frac{x}{x}.$$

$$f(0) = \boxed{?} \qquad \lim_{x \to 0} f(x) = \boxed{?}.$$

Problem 2. *Consider the function*

$$f(x) = \frac{4x}{x}.$$

$$f(0) = \boxed{?} \qquad \lim_{x \to 0} f(x) = \boxed{?}.$$

Problem 3. *Consider the function*

$$f(x) = \frac{x}{-3x}.$$

$$f(0) = \boxed{?} \qquad \lim_{x \to 0} f(x) = \boxed{?}.$$

Dig-In:

7.2 Limits of the form zero over zero

In the last section, we were interested in the limits we could compute using continuity and the limit laws. What about limits that cannot be directly computed using these methods? Let's think about an example. Consider

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2}.$$

Here

$$\lim_{x\to 2}\left(x^2-3x+2\right)=0 \qquad \text{and} \qquad \lim_{x\to 2}(x-2)=0$$

in light of this, you may think that the limit is one or zero. **Not so fast**. This limit is of an *indeterminate form*. What does this mean? Read on, young mathematician.

Definition. A limit

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

is said to be of the form $\frac{\mathbf{0}}{\mathbf{0}}$ if

$$\lim_{x\to a} f(x)=0 \qquad \text{and} \qquad \lim_{x\to a} g(x)=0.$$

Question 40. *Which of the following limits are of the form* $\frac{\mathbf{0}}{\mathbf{0}}$*?*

Select All Correct Answers:

(a) $\lim_{x\to 0}\frac{\sin(x)}{x}$

(b) $\lim_{x\to 0}\frac{\cos(x)}{x}$

(c) $\lim_{x\to 0}\frac{x^2-3x+2}{x-2}$

(d) $\lim_{x\to 2}\frac{x^2-3x+2}{x-2}$

(e) $\lim_{x\to 3}\frac{x^2-3x+2}{x-3}$

Warning. *The symbol* $\frac{\mathbf{0}}{\mathbf{0}}$ *is* ***not*** *the number* 0 *divided by* 0*. It is simply short-hand and means that a limit* $\lim_{x\to a}\frac{f(x)}{g(x)}$ *has the property that*

$$\lim_{x\to a} f(x)=0 \qquad \textit{and} \qquad \lim_{x\to a} g(x)=0.$$

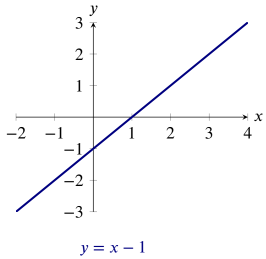
Let's consider an example with the function above:

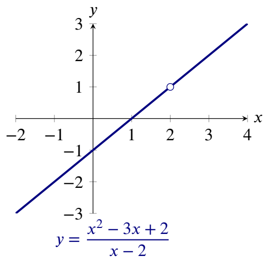
Example 50. *Compute:*

$$\lim_{x\to 2}\frac{x^2-3x+2}{x-2}$$

This limit is of the form $\frac{\mathbf{0}}{\mathbf{0}}$. However, note that if we assume $x\neq 2$, then we can write

$$\frac{x^2-3x+2}{x-2}=\frac{(x-2)(x-1)}{(x-2)}=x-1.$$





This means that

$$\lim_{x\to 2}\frac{x^2-3x+2}{x-2}=\lim_{x\to 2}(x-1).$$

But now, the limit is in a form on which we can use the limit laws! We have $\lim_{x\to 2}(x-1) = 1$. Hence

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2} = 1.$$

Let's consider some more examples of the form $\frac{\mathbf{0}}{\mathbf{0}}$.

Example 51. *Compute:*

$$\lim_{x\to 1} \frac{x-1}{x^2+2x-3}.$$

First note that

$$\lim_{x\to 1}(x-1) = 0 \qquad \text{and} \qquad \lim_{x\to 1}\left(x^2+2x-3\right) = 0$$

Hence this limit is of the form $\frac{\mathbf{0}}{\mathbf{0}}$, which tells us we can likely cancel a factor going to 0 out of the numerator and denominator. Since $(x-1)$ is a factor going to 0 in the numerator, let's see if we can factor a $(x-1)$ out of the denominator as well.

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{x^2+2x-3} &= \lim_{x\to 1} \frac{x-1}{(x-1)(x+3)} \\ &= \lim_{x\to 1} \frac{1}{x+3} \\ &= \frac{1}{4}.\end{aligned}$$

Example 52. *Compute:*

$$\lim_{x\to 1} \frac{\frac{1}{x+1}-\frac{3}{x+5}}{x-1}.$$

We find the form of this limit by looking at the limits of the numerator and denominator separately

$$\lim_{x\to 1}\left(\frac{1}{x+1}-\frac{3}{x+5}\right) = 0 \qquad \text{and} \qquad \lim_{x\to 1}(x-1) = 0.$$

Our limit is therefore of the form $\frac{\mathbf{0}}{\mathbf{0}}$ and we can probably factor a term going to 0 out of both the numerator and denominator.

$$\lim_{x\to 1} \frac{\frac{1}{x+1}-\frac{3}{x+5}}{x-1}$$

When looking at the denominator, we hope that this term is $(x-1)$. Unfortunately, it is not immediately obvious how to factor an $(x-1)$ out of the numerator. In order to simplify the numerator, we will "clear denominators." by multiplying by

$$1 = \frac{(x+1)(x+5)}{(x+1)(x+5)}$$

this will allow us to cancel immediately

$$\begin{aligned}&\lim_{x\to 1} \frac{\frac{1}{x+1}-\frac{3}{x+5}}{x-1} \cdot \frac{(x+1)(x+5)}{(x+1)(x+5)} \\ &\quad = \lim_{x\to 1} \frac{(x+5)-3(x+1)}{(x+1)(x+5)(x-1)}.\end{aligned}$$

Now we will multiply out the numerator. Note that we do not want to multiply out the denominator because we already have an $(x-1)$ factored out of the denominator and that was the goal.

$$\begin{aligned}\lim_{x\to 1} \frac{(x+5)-3(x+1)}{(x+1)(x+5)(x-1)} &= \lim_{x\to 1} \frac{x+5-3x-3}{(x+1)(x+5)(x-1)} \\ &= \lim_{x\to 1} \frac{-2x+2}{(x+1)(x+5)(x-1)} \\ &= \lim_{x\to 1} \frac{-2(x-1)}{(x+1)(x+5)(x-1)} \\ &= \lim_{x\to 1} \frac{-2}{(x+1)(x+5)}.\end{aligned}$$

We now have canceled, and can apply the usual Limit

Laws. Hence

$$\lim_{x\to 1} \frac{\frac{1}{x+1} - \frac{3}{x+5}}{x-1} = \lim_{x\to 1} \frac{-2}{(x+1)(x+5)}$$
$$= \frac{-2}{((1)+1)((1)+5)}$$
$$= \frac{-1}{6}.$$

Finally, we'll look at one more example.

Example 53. *Compute:*

$$\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}.$$

Note that

$$\lim_{x\to -1} \left(\sqrt{x+5}-2\right) = 0 \qquad \text{and} \qquad \lim_{x\to -1} (x+1) = 0.$$

Our limit is therefore of the form $\frac{\mathbf{0}}{\mathbf{0}}$ and we can probably factor a term going to 0 out of both the numerator and denominator. We suspect from looking at the denominator that this term is $(x+1)$. Unfortunately, it is not immediately obvious how to factor an $(x+1)$ out of the numerator.

We will use an algebraic technique called **multiplying by the conjugate**. This technique is useful when you are trying to simplify an expression that looks like

$$\sqrt{\text{something} \pm \text{something else}.}$$

It takes advantage of the difference of squares rule

$$a^2 - b^2 = (a-b)(a+b).$$

In our case, we will use $a = \sqrt{x+5}$ and $b = 2$. Write

$$\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} = \lim_{x\to -1} \frac{\left(\sqrt{x+5}-2\right)}{(x+1)} \cdot \frac{\left(\sqrt{x+5}+2\right)}{\left(\sqrt{x+5}+2\right)}$$
$$= \lim_{x\to -1} \frac{\left(\sqrt{x+5}\right)^2 - 2^2}{(x+1)\left(\sqrt{x+5}+2\right)}$$
$$= \lim_{x\to -1} \frac{x+5-4}{(x+1)\left(\sqrt{x+5}+2\right)}$$
$$= \lim_{x\to -1} \frac{(x+1)}{(x+1)\left(\sqrt{x+5}+2\right)}$$
$$= \lim_{x\to -1} \frac{1}{\sqrt{x+5}+2}$$
$$= \frac{1}{\sqrt{-1+5}+2}$$
$$= \frac{1}{4}.$$

All of the examples in this section are limits of the form $\frac{\mathbf{0}}{\mathbf{0}}$. When you come across a limit of the form $\frac{\mathbf{0}}{\mathbf{0}}$, you should try to use algebraic techniques to come up with a continuous function whose limit you can evaluate.

Notice that we solved multiple examples of limits of the form $\frac{\mathbf{0}}{\mathbf{0}}$ and we got different answers each time. This tells us that just knowing that the form of the limit is $\frac{\mathbf{0}}{\mathbf{0}}$ is not enough to compute the limit. The moral of the story is

Limits of the form $\frac{\mathbf{0}}{\mathbf{0}}$ can take any value.

Definition. A form that give us no information about the value of the limit is called an **indeterminate form**.

A forms that give information about the value of the limit is called a **determinate form**.

Finally, you may find it distressing that we introduced a form, namely $\frac{\mathbf{0}}{\mathbf{0}}$, only to end up saying they give no information on the value of the limit. But this is precisely what makes indeterminate forms interesting… they're a mystery!

Dig-In:

7.3 Limits of the form nonzero over zero

Let's cut to the chase:

Definition. A limit
$$\lim_{x\to a} \frac{f(x)}{g(x)}$$
is said to be of the form $\frac{\#}{\mathbf{0}}$ if
$$\lim_{x\to a} f(x) = k \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0.$$
where k is some nonzero constant.

Question 41. *Which of the following limits are of the form* $\frac{\#}{\mathbf{0}}$*?*

Select All Correct Answers:

(a) $\displaystyle\lim_{x\to -1} \frac{1}{(x+1)^2}$

(b) $\displaystyle\lim_{x\to 2} \frac{x^2-3x+2}{x-2}$

(c) $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x}$

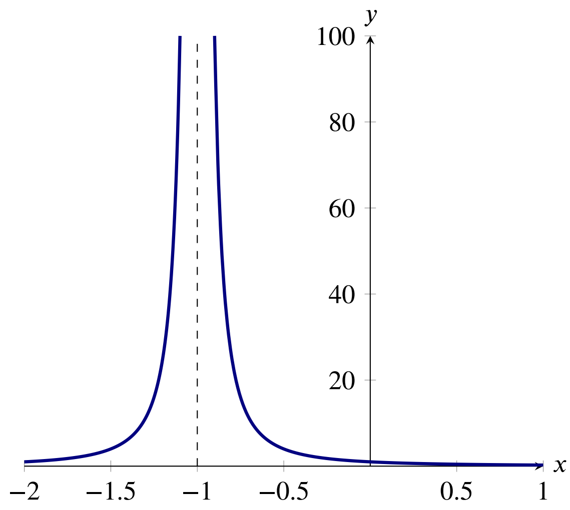
(d) $\displaystyle\lim_{x\to 2} \frac{x^2-3x-2}{x-2}$

(e) $\displaystyle\lim_{x\to 1} \frac{e^x}{\ln(x)}$

Let's see what is going on with limits of the form $\frac{\#}{\mathbf{0}}$. Consider the function
$$f(x) = \frac{1}{(x+1)^2}.$$
While the $\lim_{x\to -1} f(x)$ does not exist, something can still be said. First note that
$$\lim_{x\to -1} \frac{1}{(x+1)^2} \qquad \text{is of the form } \frac{\#}{\mathbf{0}}$$
as
$$\lim_{x\to -1} 1 = 1 \qquad \text{and} \qquad \lim_{x\to -1} (x+1)^2 = 0.$$
Moreover, as x approaches -1:

- The numerator is positive.
- The denominator approaches zero and is positive.

Hence
$$\lim_{x\to -1} \frac{1}{(x+1)^2}$$
will become arbitrarily large, as we can see in the next graph.



We are now ready for our next definition.

Definition. If $f(x)$ grows arbitrarily large as x approaches a, we write
$$\lim_{x\to a} f(x) = \infty$$
and say that the limit of $f(x)$ **approaches infinity** as x goes to a.

If $|f(x)|$ grows arbitrarily large as x approaches a and $f(x)$ is negative, we write

$$\lim_{x\to a} f(x) = -\infty$$

and say that the limit of $f(x)$ **approaches negative infinity** as x goes to a.

Let's consider a few more examples.

Example 54. *Compute:*

$$\lim_{x\to -2} \frac{e^x}{(x+2)^4}$$

First let's look at the form of this limit, we do this by taking the limits of both the numerator and denominator:

$$\lim_{x\to -2} e^x = \frac{1}{e^2} \qquad \text{and} \lim_{x\to -2} \left((x+2)^4\right) = 0$$

so this limit is of the form $\frac{\#}{\mathbf{0}}$. As x approaches -2:

- The numerator is a positive number.
- The denominator is positive and is approaching zero.

This means that

$$\lim_{x\to -2} \frac{e^x}{(x+2)^4} = \infty.$$

Example 55. *Compute:*

$$\lim_{x\to 3^+} \frac{x^2-9x+14}{x^2-5x+6}$$

First let's look at the form of this limit, which we do by taking the limits of both the numerator and denominator.

$$\lim_{x\to 3^+} \left(x^2-9x+14\right) = -4 \qquad \text{and} \lim_{x\to 3^+} \left(x^2-5x+6\right) = 0$$

This limit is of the form $\frac{\#}{\mathbf{0}}$. Next, we should factor the numerator and denominator to see if we can simplify the problem at all.

$$\begin{aligned}\lim_{x\to 3^+} \frac{x^2-9x+14}{x^2-5x+6} &= \lim_{x\to 3^+} \frac{(x-2)(x-7)}{(x-2)(x-3)}\\ &= \lim_{x\to 3^+} \frac{x-7}{x-3}\end{aligned}$$

Canceling a factor of $x-2$ from the numerator and denominator means we can more easily check the behavior of this limit. As x approaches 3 from the right:

- The numerator is a negative number.
- The denominator is positive and approaching zero.

This means that

$$\lim_{x\to 3^+} \frac{x^2-9x+14}{x^2-5x+6} = -\infty.$$

Here is our final example.

Example 56. *Compute:*

$$\lim_{x\to 3} \frac{x^2-9x+14}{x^2-5x+6}$$

We've already considered part of this example, but now we consider the two-sided limit. We already know that

$$\lim_{x\to 3} \frac{x^2-9x+14}{x^2-5x+6} = \lim_{x\to 3} \frac{x-7}{x-3},$$

and that this limit is of the form $\frac{\#}{\mathbf{0}}$. We also know that as x approaches 3 from the right,

- The numerator is a negative number.
- The denominator is positive and approaching zero.

Hence our function is approaching $-\infty$ from the right.

As x approaches 3 from the left,

- The numerator is negative.
- The denominator is negative and approaching zero.

Hence our function is approaching ∞ from the left. This means

$$\lim_{x\to 3} \frac{x^2 - 9x + 14}{x^2 - 5x + 6} = DNE.$$

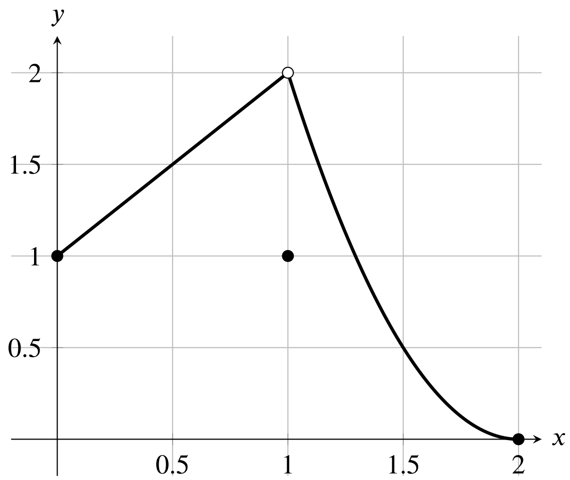
Some people worry that the mathematicians are passing into mysticism when we talk about infinity and negative infinity. However, when we write

$$\lim_{x\to a} f(x) = \infty \qquad \text{and} \qquad \lim_{x\to a} g(x) = -\infty$$

all we mean is that as x approaches a, $f(x)$ becomes arbitrarily large and $|g(x)|$ becomes arbitrarily large, with $g(x)$ taking negative values.

7.4 Practice

Exercise 4 *Let f be defined on the interval $[0, 2]$, and no where else, whose graph is:*



Find

(a) $\lim_{x \to 1^-} f(x)$

(b) $\lim_{x \to 1^+} f(x)$

(c) $\lim_{x \to 1} f(x)$

(d) $f(1)$

(e) $\lim_{x \to 0^-} f(x)$

(f) $\lim_{x \to 0^+} f(x)$

Exercise 5

$$\lim_{v \to -5} \frac{v^2 + 6v + 5}{v + 5}$$

Exercise 6

$$\lim_{z \to -4} \frac{z^2 + 6z + 8}{z + 4}$$

Exercise 7

$$\lim_{n \to -5} \frac{\sqrt{4 - n} - 3}{n + 5}$$

Exercise 8

$$\lim_{z \to -4} \frac{\sqrt{-z - 3} - 1}{z + 4}$$

Exercise 9

$$\lim_{x \to -1} \frac{\frac{1}{x+3} + \frac{3}{x-5}}{x + 1}$$

Exercise 10

$$\lim_{\theta \to 4} \frac{\frac{1}{\theta-2} - \frac{7}{2(\theta+3)}}{\theta - 4}$$

Exercise 11 *Let* $A(x) = \dfrac{1}{x + 4}$. *Compute*

$$\lim_{x \to 2} \frac{A(x) - A(2)}{x - 2}$$

Exercise 12 *Let*

$$g(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2} & \text{if } x < 1, \\ x^3 + 1 & \text{if } x > 1. \end{cases}$$

Does $\lim_{x\to 2} g(x)$ *exist? If it does, give its value. Otherwise write DNE.*

Exercise 13 *Let* $S(x) = \dfrac{|x|}{x}$. *Does* $\lim_{x\to -4} S(x)$ *exist? If it does, give its value. Otherwise write DNE.*

Exercise 14 *Consider:*

$$\lim_{t\to 0}\left(t\cos\left(\frac{3}{t}\right)\right)$$

A good way to compute this limit would be to use the Squeeze Theorem.

Exercise 15 *List two functions g and h such that*

$$g(t) \leq t\cos\left(\frac{3}{t}\right) \leq h(t)$$

for all t except for $t = \boxed{?}$ *on some interval containing* $t = 0$.

$$g(t) = \boxed{?} \qquad h(t) = \boxed{?}$$

Exercise 16 *Compute:*

$$\lim_{t\to 0} -|t| = \boxed{?} \qquad \lim_{t\to 0} |t| = \boxed{?}$$

Exercise 17 *By the Squeeze Theorem:*

$$\lim_{t\to 0}\left(t\cos\left(\frac{3}{t}\right)\right) = \boxed{?}$$

8 Using limits to detect asymptotes

After completing this section, students should be able to do the following.

- Recognize when a limit is indicating there is a vertical asymptote.
- Evaluate the limit as x approaches a point where there is a vertical asymptote.
- Match graphs of functions with their equations based on vertical asymptotes.
- Discuss what it means for a limit to equal ∞.
- Define a vertical asymptote.
- Find horizontal asymptotes using limits.
- Produce a function with given asymptotic behavior.
- Recognize that a curve can cross a horizontal asymptote.
- Understand the relationship between limits and vertical asymptotes.
- Calculate the limit as x approaches $\pm\infty$ of common functions algebraically.
- Find the limit as x approaches $\pm\infty$ from a graph.
- Define a horizontal asymptote.
- Define a slant asymptote.
- Approximate a slant asymptote from the graph of a function.
- Find slant asymptotes algebraically and graphically.
- Compute limits at infinity of famous functions.
- Find vertical asymptotes of famous functions.
- Identify horizontal asymptotes by looking at a graph.
- Identify vertical asymptotes by looking at a graph.
- Identify slant asymptotes by looking at a graph.

Break-Ground:

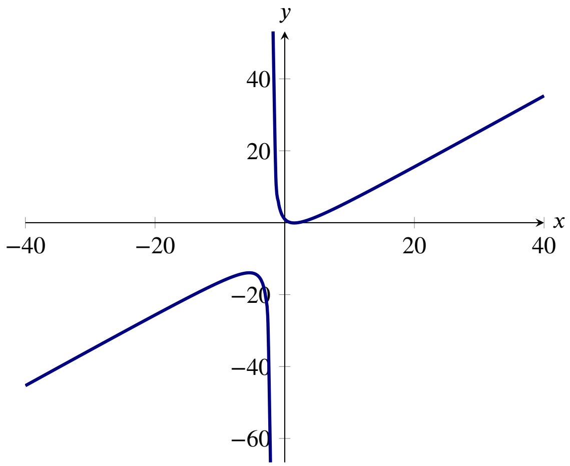
8.1 Zoom out

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, think about this function:

$$f(x) = \frac{x^2 - 3x + 2}{x + 2}.$$

Riley: Hmmm. If you plot it, the graph looks like this:



Devyn: Right! What I've noticed is that if x gets big, then our function looks like a line.

Riley: I wonder how we find the line?

Problem 1. *Devyn and Riley have noticed that the function* $f(x) = \dfrac{x^2 - 3x + 2}{x + 2}$ *looks like a line when we zoom out. Guess the slope of this line. Come back and check your answer after reading the Dig-In!*

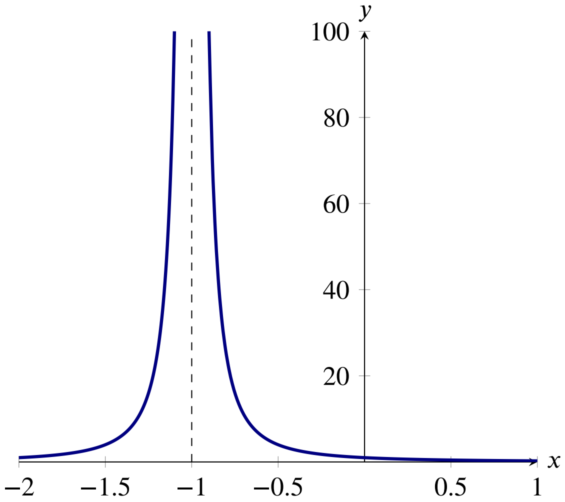
Problem 2. *Guess the y-intercept of this line. Come back and check your answer after reading the Dig-In!*

Dig-In:

8.2 Vertical asymptotes

Consider the function

$$f(x) = \frac{1}{(x+1)^2}.$$



While the $\lim_{x\to -1} f(x)$ does not exist, something can still be said.

Definition. If $f(x)$ grows arbitrarily large as x approaches a, we write

$$\lim_{x\to a} f(x) = \infty$$

and say that the limit of $f(x)$ **approaches infinity** as x goes to a.

If $|f(x)|$ grows arbitrarily large as x approaches a and $f(x)$ is negative, we write

$$\lim_{x\to a} f(x) = -\infty$$

and say that the limit of $f(x)$ **approaches negative infinity** as x goes to a.

Question 42. *Which of the following are correct?*

Select All Correct Answers:

(a) $\lim_{x\to -1} \frac{1}{(x+1)^2} = \infty$

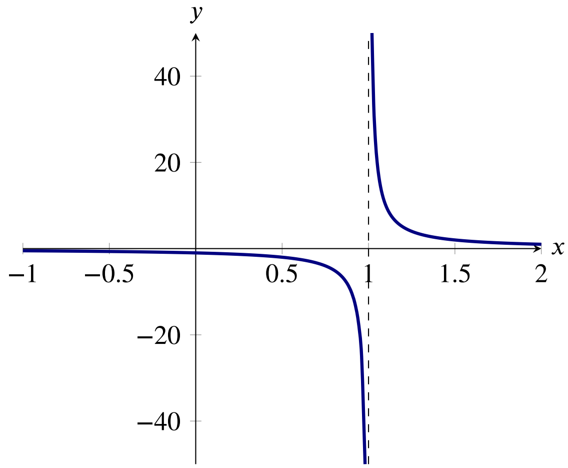
(b) $\lim_{x\to -1} \frac{1}{(x+1)^2} \to \infty$

(c) $f(x) = \frac{1}{(x+1)^2}$, *so* $f(-1) = \infty$

(d) $f(x) = \frac{1}{(x+1)^2}$, *so as* $x \to -1$, $f(x) \to \infty$

On the other hand, consider the function

$$f(x) = \frac{1}{(x-1)}.$$



While the two sides of the limit as x approaches 1 do not agree, we can still consider the one-sided limits. We see $\lim_{x\to 1^+} f(x) = \infty$ and $\lim_{x\to 1^-} f(x) = -\infty$.

Definition. If at least one of the following hold:

- $\lim_{x\to a} f(x) = \pm\infty$,
- $\lim_{x\to a^+} f(x) = \pm\infty$,
- $\lim_{x\to a^-} f(x) = \pm\infty$,

then the line $x = a$ is a **vertical asymptote** of f.

Example 57. *Find the vertical asymptotes of*

$$f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6}.$$

Start by factoring both the numerator and the denominator:

$$\frac{x^2 - 9x + 14}{x^2 - 5x + 6} = \frac{(x-2)(x-7)}{(x-2)(x-3)}$$

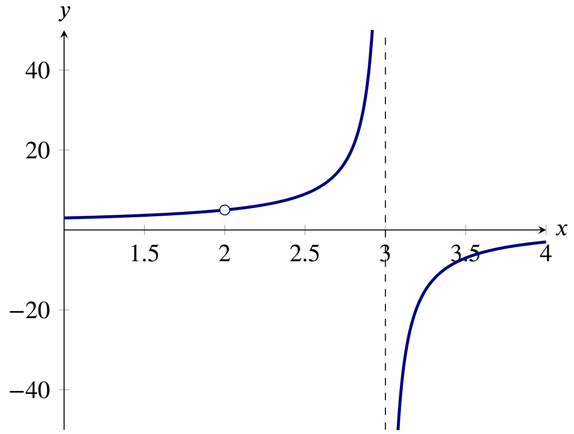
Using limits, we must investigate when $x \to 2$ and $x \to 3$. Write

$$\begin{aligned}\lim_{x\to 2}\frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x\to 2}\frac{(x-7)}{(x-3)}\\ &= \frac{-5}{-1}\\ &= 5.\end{aligned}$$

Now write

$$\begin{aligned}\lim_{x\to 3}\frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x\to 3}\frac{(x-7)}{(x-3)}\\ &= \lim_{x\to 3}\frac{-4}{x-3}.\end{aligned}$$

Consider the one-sided limits separately. Since $\lim_{x\to 3^+}(x-3)$ approaches 0 from the right and the numerator is negative, $\lim_{x\to 3^+} f(x) = -\infty$. Since $\lim_{x\to 3^-}(x-3)$ approaches 0 from the left and the numerator is negative, $\lim_{x\to 3^-} f(x) = \infty$.



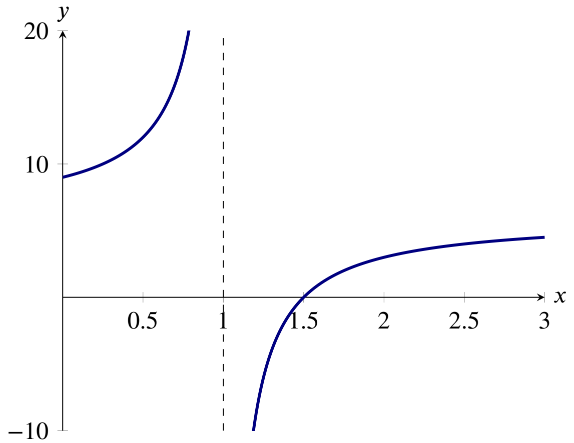
Hence we have a vertical asymptote at $x = 3$.

Dig-In:

8.3 Horizontal asymptotes

Consider the function:

$$f(x) = \frac{6x-9}{x-1}$$



As x approaches infinity, it seems like $f(x)$ approaches a specific value. Such a limit is called a *limit at infinity*.

Definition. If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large, we write

$$\lim_{x\to\infty} f(x) = L$$

and we say, the **limit at infinity** of $f(x)$ is L.

If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large and negative, we write

$$\lim_{x\to-\infty} f(x) = L$$

and we say, the **limit at negative infinity** of $f(x)$ is L.

Example 58. *Compute*

$$\lim_{x\to\infty} \frac{6x-9}{x-1}.$$

Write

$$\begin{aligned}\lim_{x\to\infty} \frac{6x-9}{x-1} &= \lim_{x\to\infty} \frac{6x-9}{x-1} \cdot \frac{1/x}{1/x}\\ &= \lim_{x\to\infty} \frac{\frac{6x}{x} - \frac{9}{x}}{\frac{x}{x} - \frac{1}{x}}\\ &= \lim_{x\to\infty} \frac{6}{1}\\ &= 6.\end{aligned}$$

Sometimes one must be careful, consider this example.

Example 59. *Compute*

$$\lim_{x\to-\infty} \frac{x^3+1}{\sqrt{x^6+5}}$$

In this case we multiply the numerator and denominator by $-1/x^3$, which is a positive number as since $x \to -\infty$, x^3 is a negative number.

$$\begin{aligned}\lim_{x\to-\infty} \frac{x^3+1}{\sqrt{x^6+5}} &= \lim_{x\to-\infty} \frac{x^3+1}{\sqrt{x^6+5}} \cdot \frac{-1/x^3}{-1/x^3}\\ &= \lim_{x\to-\infty} \frac{-1-1/x^3}{\sqrt{x^6/x^6+5/x^6}}\\ &= -1.\end{aligned}$$

Note, since

$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} f\left(\frac{1}{x}\right)$$

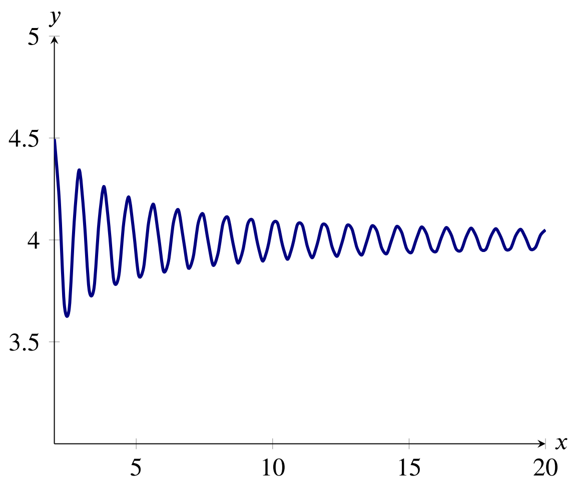
and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} f\left(\frac{1}{x}\right)$$

we can also apply the Squeeze Theorem when taking limits at infinity. Here is an example of a limit at infinity that uses the Squeeze Theorem, and shows that functions can, in fact, cross their horizontal asymptotes.

Example 60. *Compute:*

$$\lim_{x\to\infty} \frac{\sin(7x)+4x}{x}$$



We can bound our function

$$\frac{-1+4x}{x} \leq \frac{\sin(7x)+4x}{x} \leq \frac{1+4x}{x}.$$

Now write with me

$$\lim_{x\to\infty} \frac{-1+4x}{x}\cdot\frac{1/x}{1/x} = \lim_{x\to\infty} \frac{-1/x+4}{1}$$
$$= 4.$$

And we also have

$$\lim_{x\to\infty} \frac{1+4x}{x}\cdot\frac{1/x}{1/x} = \lim_{x\to\infty} \frac{1/x+4}{1}$$
$$= 4.$$

Since

$$\lim_{x\to\infty} \frac{-1+4x}{x} = 4 = \lim_{x\to\infty} \frac{1+4x}{x}$$

we conclude by the Squeeze Theorem, $\lim_{x\to\infty} \frac{\sin(7x)}{x}+4 = 4$.

Definition. If

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L,$$

then the line $y = L$ is a **horizontal asymptote** of $f(x)$.

Example 61. *Give the horizontal asymptotes of*

$$f(x) = \frac{6x-9}{x-1}$$

From our previous work, we see that $\lim_{x\to\infty} f(x) = 6$, and upon further inspection, we see that $\lim_{x\to-\infty} f(x) = 6$. Hence the horizontal asymptote of $f(x)$ is the line $y = 6$.

It is a common misconception that a function cannot cross an asymptote. As the next example shows, a function can cross a horizontal asymptote, and in the example this occurs an infinite number of times!

Example 62. *Give a horizontal asymptote of*

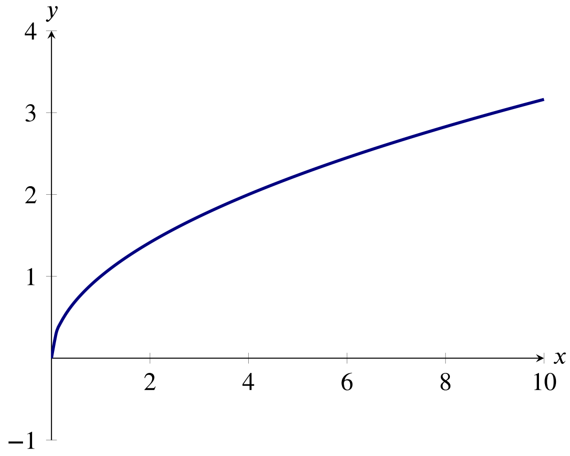
$$f(x) = \frac{\sin(7x)+4x}{x}.$$

Again from previous work, we see that $\lim_{x\to\infty} f(x) = 4$. Hence $y = 4$ is a horizontal asymptote of $f(x)$.

We conclude with an infinite limit at infinity.

Example 63. *Compute*

$$\lim_{x\to\infty} \sqrt{x}.$$



The function $\sqrt{x}$ grows slowly, and seems like it may have a horizontal asymptote, see the graph above. However, if we consider the square root as the inverse of the function $f(x) = x^2, \quad x \geq 0$.

$$\sqrt{x} = y \text{ means that } y^2 = x \text{ and that } y \geq 0.$$

We see that we may square higher and higher values to obtain larger outputs. This means that $\sqrt{x}$ is unbounded, and hence $\lim_{x\to\infty} \sqrt{x} = \infty$.

Dig-In:

8.4 Slant asymptotes

If we think of an asymptote as a "line that a function resembles when the input or output is large," then there are three types of asymptotes, just as there are three types of lines:

Vertical Asymptotes	$\leftrightarrow$	Vertical Lines
Horizontal Aymptotes	$\leftrightarrow$	Horizontal Lines
Slant Asymptotes	$\leftrightarrow$	Slant Lines

Here we've made up a new term "slant" line, meaning a line whose slope is neither zero, nor is it undefined. Let's do a quick review of the different types of asymptotes:

Vertical asymptotes Recall, a function f has a vertical asymptote at $x = a$ if at least one of the following hold:

- $\lim_{x\to a} f(x) = \pm\infty$,
- $\lim_{x\to a^+} f(x) = \pm\infty$,
- $\lim_{x\to a^-} f(x) = \pm\infty$.

In this case, the asymptote is the vertical line

$$x = a.$$

Horizontal asymptotes We have also seen that a function f has a horizontal asymptote if

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L,$$

and in this case, the asymptote is the horizontal line

$$\ell(x) = L.$$

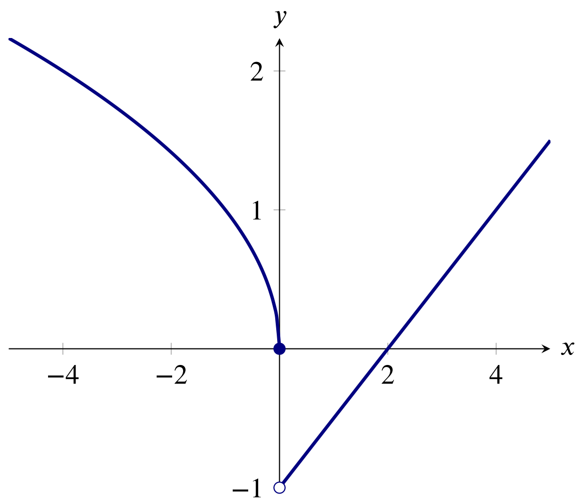
Slant asymptotes On the other hand, a *slant asymptote* is a somewhat different beast.

Definition. If there is a nonhorizontal line $\ell(x) = m \cdot x + b$ such that

$$\lim_{x\to\infty} (f(x) - \ell(x)) = 0 \qquad \text{or} \qquad \lim_{x\to-\infty} (f(x) - \ell(x)) = 0,$$

then ℓ is a **slant asymptote** for f.

Question 43. *Consider the graph of the following function.*



What is the slant asymptote of this function?

To analytically find slant asymptotes, one must find the required information to determine a line:

- The slope.
- The y-intercept.

While there are several ways to do this, we will give a method that is fairly general.

Example 64. *Find the slant asymptote of*

$$f(x) = \frac{3x^2 + x + 2}{x + 2}.$$

We are looking to see if there is a line ℓ such that

$$\lim_{x\to\infty} (f(x) - \ell(x)) = 0 \qquad \text{or} \qquad \lim_{x\to-\infty} (f(x) - \ell(x)) = 0.$$

First, let's consider the limit as x approaches positive infinity. We will imagine that we have such a line

$$\ell(x) = m \cdot x + b$$

and attempt to find the correct values for m and b. Let's look again at our limit. We are assuming:

$$\lim_{x\to\infty} \left(\frac{3x^2 + x + 2}{x + 2} - (mx + b) \right) = 0.$$

We know that $f(x)$ is continuous everywhere except at $x = -2$ and $\ell(x)$ is continuous everywhere, so we can apply our limit laws away from $x = -2$. We're looking at large values of x, so this is no problem. We use the fact that the sum of the limits is the limit of the sums.

$$\begin{aligned} \lim_{x\to\infty} \left(\frac{3x^2 + x + 2}{x + 2} \right) - \lim_{x\to\infty} (mx + b) &= 0 \\ \lim_{x\to\infty} \frac{3x^2 + x + 2}{x + 2} &= \lim_{x\to\infty} (mx + b) \end{aligned}$$

We are assuming these two limits are equal. Dividing by x on the right hand side makes the limit equal to m:

$$m = \lim_{x\to\infty} \left(\frac{mx}{x} + \frac{b}{x} \right) = \lim_{x\to\infty} \left(\frac{mx + b}{x} \right).$$

To find the value of m, then, we can divide the left hand side by x and evaluate the limit. We see the following.

$$\begin{aligned} m &= \lim_{x\to\infty} \frac{\frac{3x^2+x+2}{x+2}}{x} \\ &= \lim_{x\to\infty} \frac{\frac{3x^2+x+2}{x+2}}{x} \\ &= \lim_{x\to\infty} \frac{3x^2 + x + 2}{x^2 + 2x} \\ &= \lim_{x\to\infty} \left(\frac{3x^2 + x + 2}{x^2 + 2x} \cdot \frac{1/x^2}{1/x^2} \right) \\ &= \lim_{x\to\infty} \frac{3 + 1/x + 2/x^2}{1 + 2/x} \\ &= 3. \end{aligned}$$

So $m = 3$. We now know that

$$\lim_{x\to\infty} \frac{3x^2 + x + 2}{x + 2} = \lim_{x\to\infty} (3x + b)$$

for some value of b. To find the y-intercept b, we use a similar method. Notice that

$$\lim_{x\to\infty} (3x + b - 3x) = \lim_{x\to\infty} b = b,$$

so if we subtract $3x$ from the right hand side, we are left with just b. Since the two sides are equal, subtracting $3x$ from the left hand side and evaluating the limit will give us the value for b. We write the following.

$$\begin{aligned}
b &= \lim_{x\to\infty}\left(\frac{3x^2+x+2}{x+2} - 3x\right) \\
&= \lim_{x\to\infty}\left(\frac{3x^2+x+2}{x+2} - \frac{3x^2+6x}{x+2}\right) \\
&= \lim_{x\to\infty}\frac{3x^2+x+2-3x^2-6x}{x+2} \\
&= \lim_{x\to\infty}\frac{-5x+2}{x+2} \\
&= \lim_{x\to\infty}\left(\frac{-5x+2}{x+2}\cdot\frac{1/x}{1/x}\right) \\
&= \lim_{x\to\infty}\frac{-5+2/x}{1+2/x} \\
&= -5.
\end{aligned}$$

By this method, we have determined that

$$\lim_{x\to\infty}\left(\frac{3x^2+x+2}{x+2} - (3x-5)\right) = 0.$$

In other words, $\ell(x) = 3x - 5$ is a slant asymptote for our function f. You should check that we get the same slant asymptote $\ell(x) = 3x - 5$ when we take the limit to negative infinity as well.

9 Continuity and the Intermediate Value Theorem

After completing this section, students should be able to do the following.

- Identify where a function is, and is not, continuous.
- Understand the connection between continuity of a function and the value of a limit.
- Make a piecewise function continuous.
- State the Intermediate Value Theorem including hypotheses.
- Determine if the Intermediate Value Theorem applies.
- Sketch pictures indicating why the Intermediate Value Theorem is true, and why all hypotheses are necessary.
- Explain why certain points exist using the Intermediate Value Theorem.
- Bisection method.

Break-Ground:

9.1 Roxy and Yuri like food

Check out this dialogue between two calculus students (based on a true story):

Devyn: Yo Riley, I was watching my two cats *Roxy* and *Yuri* eat their dry cat food last night.

Riley: Cats love food! It's so weird that they swallow the pieces whole!

Devyn: I know! I noticed something else kinda funny though: Both Roxy and Yuri start and finish eating at the same times; and while I gave Roxy a little more food than Yuri, less food was left in Roxy's bowl when they stopped eating.
I wonder, is there is a point in time when Roxy and Yuri have the exact same amount of **dry cat food** in their bowls?

Riley: Hmmmmm. Do Roxy and Yuri both start and finish drinking their water at the same times? And does Roxy start with a little more water than Yuri, and finish with less water left than Yuri?

Devyn: Yes!

Riley: Interesting. I wonder, is there is a point in time when Roxy and Yuri have the exact same amount of **water** in their bowls?

Problem 1. *Is there a time when Roxy and Yuri have the same amount of dry cat food in their bowls assuming:*

- *They start and finish eating at the same times.*
- *Roxy starts with more food than Yuri, and leaves less food uneaten than Yuri.*

Multiple Choice:

(a) *yes*

(b) *no*

(c) *There is no way to tell.*

Problem 2. *Is there a time when Roxy and Yuri have the same amount of water in their bowls assuming:*

- *They start and finish drinking at the same times.*
- *Roxy starts with more water than Yuri, and leaves less water left in her bowl than Yuri.*

Multiple Choice:

(a) *yes*

(b) *no*

(c) *There is no way to tell.*

Problem 3. *Within the context of the two problems above, what is the difference between "dry cat food" and "water?"*

Dig-In:

9.2 Continuity of piecewise functions

In this section we will work a couple of examples involving limits, continuity and piecewise functions.

Example 65. *Consider the following piecewise defined function*

$$f(x) = \begin{cases} \dfrac{x}{x-1} & \text{if } x < 0 \text{ and } x \neq 1, \\ \dfrac{x+1}{x^2+1} + c & \text{if } x \geq 0. \end{cases}$$

Find c so that f is continuous at $x = 0$.

To find c such that f is continuous at $x = 0$, we need to find c such that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0).$$

In this case

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} \frac{x}{x-1} \\ &= \frac{0}{-1} \\ &= 0. \end{aligned}$$

On there other hand

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left(\frac{x+1}{x^2+1} + c \right) \\ &= \frac{0+1}{0^2+1} + c \\ &= 1 + c \end{aligned}$$

Hence for our function to be continuous, we need

$$1 + c = 0 \qquad \text{so} \qquad c = -1.$$

Now, $\lim_{x \to 0} f(x) = f(0)$, and so f is continuous.

Consider the next, more challenging example.

Example 66. *Consider the following piecewise defined function*

$$f(x) = \begin{cases} x + 4 & \text{if } x < 1, \\ ax^2 + bx + 2 & \text{if } 1 \leq x < 3, \\ 6x + a - b & \text{if } x \geq 3. \end{cases}$$

Find a and b so that f is continuous at both $x = 1$ and $x = 3$.

This problem is more challenging because we have more unknowns. However, be brave intrepid mathematician. To find a and b that make f is continuous at $x = 1$, we need to find a and b such that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1).$$

Looking at the limit from the left, we have

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (x + 4) \\ &= 5. \end{aligned}$$

Looking at the limit from the right, we have

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} \left(ax^2 + bx + 2 \right) \\ &= a + b + 2. \end{aligned}$$

Hence for this function to be continuous at $x = 1$, we must have that

$$\begin{aligned} 5 &= a + b + 2 \\ 3 &= a + b. \end{aligned}$$

Hmmmm. More work needs to be done.

To find a and b that make f is continuous at $x = 3$, we need to find a and b such that

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3).$$

Looking at the limit from the left, we have

$$\begin{aligned}\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} \left(ax^2 + bx + 2\right)\\ &= a\cdot 9 + b\cdot 3 + 2.\end{aligned}$$

Looking at the limit from the right, we have

$$\begin{aligned}\lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} (6x + a - b)\\ &= 18 + a - b.\end{aligned}$$

Hence for this function to be continuous at $x = 3$, we must have that

$$\begin{aligned}a\cdot 9 + b\cdot 3 + 2 &= 18 + a - b\\ a\cdot 8 + b\cdot 4 - 16 &= 0\\ a\cdot 2 + b - 4 &= 0\end{aligned}$$

So now we have two equations and two unknowns:

$$3 = a + b \qquad \text{and} \qquad a\cdot 2 + b - 4 = 0.$$

Set $b = 3 - a$ and write

$$\begin{aligned}0 &= a\cdot 2 + (3 - a) - 4\\ &= a - 1,\end{aligned}$$

hence

$$a = 1 \qquad \text{and so} \qquad b = 2.$$

Let's check, so now plugging in values for both a and b we find

$$f(x) = \begin{cases} x + 4 & \text{if } x < 1,\\ x^2 + 2x + 2 & \text{if } 1 \le x < 3,\\ 6x - 1 & \text{if } x \ge 3.\end{cases}$$

Now

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 5,$$

and

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 17.$$

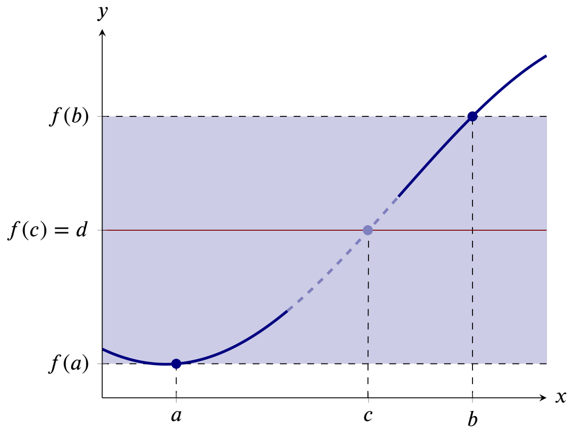
So setting $a = 1$ and $b = 2$ makes f continuous at $x = 1$ and $x = 3$.

Dig-In:

9.3 The Intermediate Value Theorem

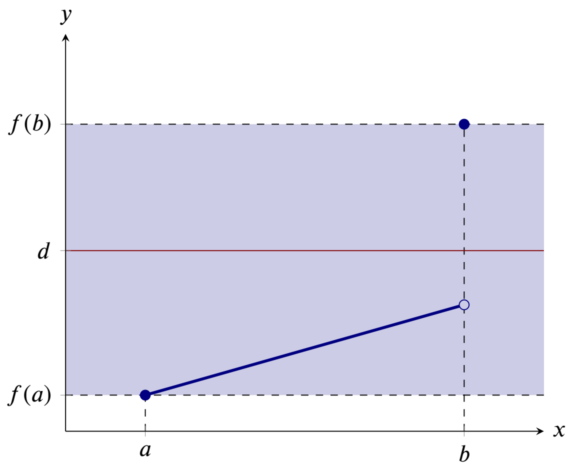
The *Intermediate Value Theorem* should not be brushed off lightly. Once it is understood, it may seem “obvious,” but mathematicians should not underestimate its power.

Theorem 13 (Intermediate Value Theorem)**.** *If f is a continuous function for all x in the closed interval $[a,b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ such that $f(c)=d$.*



Now, let’s contrast this with a time when the conclusion of the Intermediate Value Theorem does not hold.

Question 44. *Consider the following situation,*



and select all that are true:

Select All Correct Answers:

(a) *f is continuous on (a,b).*

(b) *f is continuous on $[a,b)$.*

(c) *f is continuous on $(a,b]$.*

(d) *f is continuous on $[a,b]$.*

(e) *There is a point c in $[a,b]$ with $f(c)=d$.*

Building on the question above, it is not difficult to see that each of the hypothesis of the Intermediate Value Theorem are necessary.

Let’s see the Intermediate Value Theorem in action.

Example 67. *Explain why the function $f(x)=x^3+3x^2+x-2$ has a root between 0 and 1.*

> Since f is a polynomial, we see that f is continuous for all real numbers. Since $f(0)=-2$ and $f(1)=3$, and 0

is between -2 and 3, by the Intermediate Value Theorem, there is a point c in the interval $[0, 1]$ such that $f(c) = 0$.

This example also points the way to a simple method for approximating roots.

Example 68. *Approximate a root of $f(x) = x^3 + 3x^2 + x - 2$ between 0 and 1 to within one decimal place.*

Again, since f is a polynomial, we see that f is continuous for all real numbers. Compute

x	$f(x)$
0.1	−1.869
0.2	−1.672
0.3	−1.403
0.4	−1.056
0.5	−0.625
0.6	−0.104
0.7	0.513

By the Intermediate Value Theorem, f has a root between 0.6 and 0.7. Repeating the process

x	$f(x)$
0.61	−0.046719
0.62	0.011528

so by the Intermediate Value Theorem, f has a root between 0.61 and 0.62, and the root is 0.6 rounded to one decimal place.

The Intermediate Value Theorem can be use to show that curves cross:

Example 69. *Explain why the functions*

$$f(x) = x^2\sqrt[3]{x}$$
$$g(x) = 2(x+1)\cos(\sqrt[3]{x})$$

intersect on the interval $\left[0, \left(\frac{\pi}{2}\right)^3\right]$.

To start, note that both f and g are continuous functions, and hence $h = f - g$ is also a continuous function. Now

$$\begin{aligned} h(0) &= f(0) - g(0) \\ &= (0)^2 \cdot \sqrt[3]{(0)} - 2 \cdot (0+1) \cdot \cos\left(\sqrt[3]{(0)}\right) \\ &= -2. \end{aligned}$$

and in a similar fashion

$$\begin{aligned} h\left(\left(\frac{\pi}{2}\right)^3\right) &= f\left(\left(\frac{\pi}{2}\right)^3\right) - g\left(\left(\frac{\pi}{2}\right)^3\right) \\ &= \left(\left(\frac{\pi}{2}\right)^3\right)^2 \cdot \sqrt[3]{\left(\frac{\pi}{2}\right)^3} \\ &\quad - 2 \cdot \left(\left(\frac{\pi}{2}\right)^3 + 1\right) \cdot \cos\left(\sqrt[3]{\left(\frac{\pi}{2}\right)^3}\right) \end{aligned}$$

Since $\cos\left(\frac{\pi}{2}\right) = 0$ we see that the expression above is positive. Hence by the Intermediate Value Theorem, f and g intersect on the interval $\left[0, \left(\frac{\pi}{2}\right)^3\right]$.

Now we move on to a more subtle example:

Example 70. *Suppose you have two cats, Roxy and Yuri. Is there a time when Roxy and Yuri have the same amount of water in their bowls assuming:*

- *They start and finish drinking at the same times.*

- *Roxy starts with more water than Yuri, and leaves less water left in her bowl than Yuri.*

To solve this problem, consider two functions:

- $W_{\text{Roxy}}(t)$ = the amount of water in Roxy's bowl at time t.
- $W_{\text{Yuri}}(t)$ = the amount of water in Yuri's bowl at time t.

Now if t_{start} is the time the cats start drinking and t_{finish} is the time the cats finish drinking. Then we have

$$W_{\text{Roxy}}(t_{\text{start}}) - W_{\text{Yuri}}(t_{\text{start}}) > 0$$

and

$$W_{\text{Roxy}}(t_{\text{finish}}) - W_{\text{Yuri}}(t_{\text{finish}}) < 0.$$

Since the amount of water in a bowl at time t is a continuous function, as water is "lapped" up in continuous amounts,

$$W_{\text{Roxy}} - W_{\text{Yuri}}$$

is a continuous function, and hence the Intermediate Value Theorem applies. Since $W_{\text{Roxy}} - W_{\text{Yuri}}$ is positive when at t_{start} and negative at t_{finish}, there is some time t_{equal} when the value is zero, meaning

$$W_{\text{Roxy}}(t_{\text{equal}}) - W_{\text{Yuri}}(t_{\text{equal}}) = 0$$

meaning there is the same amount of water in each of their bowls.

And finally, an example when the Intermediate Value Theorem *does not* apply.

Example 71. *Suppose you have two cats, Roxy and Yuri. Is there a time when Roxy and Yuri have the same amount of dry cat food in their bowls assuming:*

- *They start and finish eating at the same times.*
- *Roxy starts with more food than Yuri, and leaves less food uneaten than Yuri.*

Here we could try the same approach as before, setting:

- $F_{\text{Roxy}}(t)$ = the amount of dry cat food in Roxy's bowl at time t.
- $F_{\text{Yuri}}(t)$ = the amount of dry cat food in Yuri's bowl at time t.

However in this case, the amount of food in a bowl at time t is **not** a continuous function! This is because dry cat food consists of discrete kibbles, and is not eaten in a continuous fashion. Hence the Intermediate Value Theorem **does not apply**, and we can make no definitive statements concerning the question above.

9.4 Practice

Exercise 4 *Consider*

$$a(x) = \frac{x^2 + x - 6}{x^2 - x - 12}.$$

Find all vertical asymptotes.

Exercise 5 *Consider*

$$s(x) = \frac{x^2 - 2x - 15}{x^2 + 3x + 2}.$$

Find all vertical asymptotes.

Exercise 6 *Give intervals on which each of the following functions are continuous. Write combinations of intervals going from left to right on the number line.*

(a) $\frac{1}{e^x + 1}$ *is continuous on* ($\boxed{?}$, $\boxed{?}$).

(b) $\frac{1}{x^2 - 1}$ *is continuous on* ($\boxed{?}$, $\boxed{?}$) *and* ($\boxed{?}$, $\boxed{?}$) *and* ($\boxed{?}$, $\boxed{?}$).

(c) $\sqrt{5 - x}$ *is continuous on* ($\boxed{?}$, $\boxed{?}$).

(d) $\sqrt{5 - x^2}$ *is continuous on* ($\boxed{?}$, $\boxed{?}$).

Exercise 7 *Let*

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ x + 5, & x \geq 3. \end{cases}$$

Is f continuous everywhere?

Multiple Choice:

(a) *Yes*

(b) *No*

Exercise 8 *Let*

$$g(x) = \begin{cases} -2e^x & \text{if } x < 0 \\ \dfrac{x+6}{x-3} & \text{if } x \geq 0. \end{cases}$$

Find

(a) $\lim_{x \to 0^-} g(x)$

(b) $\lim_{x \to 0^+} g(x)$

(c) $\lim_{x \to 3^-} g(x)$

(d) $\lim_{x \to 3^+} g(x)$

(e) $\lim_{x \to -\infty} g(x)$

(f) $\lim_{x \to +\infty} g(x)$

Exercise 9 *The Intermediate Value Theorem states: If f is a continuous function for all x in the closed interval $[a, b]$ and r is between f(a) and f(b), then there is a number u in $[a, b]$ such that f(u) = r.*

Exercise 10 *Is the function*

$$f(x) = \begin{cases} \dfrac{x^2 - 64}{x^2 - 11x + 24}, & x \neq 8 \\ 5, & x = 8 \end{cases}$$

continuous at $x = 0$ or $x = 8$?

Exercise 11 *Let f be continuous on $[1, 5]$ where $f(1) = -2$ and $f(5) = -10$. Does a value $1 < c < 5$ exist such that $f(c) = -9$?*

Multiple Choice:

(a) *There does not exist a value.*

(b) *Yes, by the Intermediate Value Theorem*

(c) *Yes, by the Mean Value Theorem*

(d) *There does not necessarily exist such a value*

Part II

Content for the Second Exam

10 An application of limits

After completing this section, students should be able to do the following.

- Compute limits of families of functions.
- Compute average velocity.
- Approximate instantaneous velocity.
- Compare average and instantaneous velocity.
- Plot difference quotients for varying approximations of the instantaneous rate of change.

Break-Ground:

10.1 Limits and velocity

Check out this dialogue between two calculus students (based on a true story):

Devyn: Hey Riley, I've been thinking about limits.
Riley: That is awesome.
Devyn: I know! You know limits remind me of something... How a GPS or a phone computes velocity!
Riley: Huh. A GPS can calculate our location. Then, to compute velocity from position, it must look at

$$\frac{\text{change in position}}{\text{change in time}}$$

Devyn: And then we study this as the change in time gets closer and closer to zero.
Riley: Just like with limits at zero, we can study something by looking **near** a point, but **not exactly at** a point.
Devyn: O.M.G. Life's a rich tapestry.
Riley: Poet, you know it.

Suppose you take a road trip from Columbus Ohio to Urbana-Champaign Illinois. Moreover, suppose your position is modeled by

$$s(t) = 36t^2 - 4.8t^3 \qquad \text{(miles West of Columbus)}$$

where t is measured in hours and runs from 0 to 5 hours.

Problem 1. *What is the average velocity for the entire trip?*

Problem 2. *Use a calculator to estimate the instantaneous velocity at $t = 2$.*

Problem 3. *Considering the work above, when we want to compute instantaneous velocity, we need to compute*

$$\frac{\textit{change in position}}{\textit{change in time}}$$

when (choose all that apply):

Select All Correct Answers:

(a) *The "change in time" is zero.*

(b) *The "change in time" gets closer and closer to zero.*

(c) *The "change in time" approaches zero.*

(d) *The "change in time" is near zero.*

(e) *The "change in time" goes to zero.*

Computing average velocities for smaller, and smaller, values of Δt as we did above is tedious. Nevertheless, this is exactly how a GPS determines velocity from position! To avoid these tedious calculations, we would really like to have a formula.

Dig-In:

10.2 Instantaneous velocity

When one computes average velocity, we look at

$$\frac{\text{change in displacement}}{\text{change in time}}.$$

To obtain the (instantaneous) velocity, we want the change in time to "go to" zero. By this point we should know that "go to" is a buzz-word for a *limit*. The change in time is often given as an interval whose length goes to zero. However, intervals must always be written

$$[a, b] \qquad \text{where } a < b.$$

Given

$$I = [a, a + h],$$

we see that h cannot be negative, or else it violates the notation for intervals. Hence, if we want smaller, and smaller, intervals around a point a, and we want h to be able to be negative, we write

$$I_h = \begin{cases} [a + h, a] & \text{if } h < 0, \\ [a, a + h] & \text{if } 0 < h. \end{cases}$$

Question 45. *Let* $a = 3$ *and* $h = 0.1$

$$I_h = \left[\boxed{?}, \boxed{?}\right]$$

Question 46. *Let* $a = 3$ *and* $h = -0.1$

$$I_h = \left[\boxed{?}, \boxed{?}\right]$$

Regardless of the value of h, the average velocity on the interval I_h is computed by

$$\frac{\text{change in displacement}}{\text{change in time}} = \frac{s(t + h) - s(t)}{h}.$$

We will be most interested in this ratio when h approaches zero. Let's put all of this together by working an example.

Example 72. *A group of young mathematicians recently took a road trip from Columbus Ohio to Urbana-Champaign Illinois. The position (west of Columbus, Ohio) of van they drove in is roughly modeled by*

$$s(t) = 36t^2 - 4.8t^3 \qquad \textit{(miles West of Columbus)}$$

on the interval $[0, 5]$, *where* t *is measured in hours. What is the average velocity on the interval* $[0, 5]$*?*

Additionally, let

$$I_h = \begin{cases} [1 + h, 1] & \textit{if } h < 0, \\ [1, 1 + h] & \textit{if } 0 < h. \end{cases}$$

What is the average velocity on I_h *when* $h = 0.1$*? What is the average velocity on* I_h *when* $h = -0.1$*?*

The average velocity on the interval $[0, 5]$ is

$$\begin{aligned} \frac{s(5) - s(0)}{5 - 0} &= \frac{36 \cdot 5^2 - 4.8 \cdot 5^3 - (36 \cdot 0^2 - 4.8 \cdot 0^3)}{5} \\ &= \frac{300}{5} \\ &= 60 \qquad \text{miles per hour.} \end{aligned}$$

On the other hand, consider the interval

$$I_h = \begin{cases} [1 + h, 1] & \text{if } h < 0, \\ [1, 1 + h] & \text{if } 0 < h. \end{cases}$$

When $h = 0.1$, the average velocity is

$$\frac{s(1 + 0.1) - s(1)}{0.1}$$

$$
\begin{aligned}
&= \frac{36\cdot(1.1)^2 - 4.8\cdot(1.1)^3 - (36\cdot 1^2 - 4.8\cdot 1^3)}{0.1}\\
&= \frac{5.9712}{0.1}\\
&= 59.712 \qquad \text{miles per hour.}
\end{aligned}
$$

On the other hand, when $h = -0.1$, the average velocity is

$$\frac{s(1-0.1)-s(1)}{-0.1}$$

$$
\begin{aligned}
&= \frac{36\cdot(0.9)^2 - 4.8\cdot(0.9)^3 - (36\cdot 1^2 - 4.8\cdot 1^3)}{-0.1}\\
&= \frac{-5.5392}{-0.1}\\
&= 55.392 \qquad \text{miles per hour.}
\end{aligned}
$$

In our previous example, we computed *average velocity* on three different intervals. If we let the size of the interval go to zero, we get **instantaneous velocity**. Limits will allow us to compute instantaneous velocity. Let's use the same setting as before.

Example 73. *The position of van (west of Columbus, Ohio) our young mathematicians drove to Urbana-Champaign, Illinois is roughly modeled by*

$$s(t) = 36t^2 - 4.8t^3 \qquad \textit{for } 0 \le t \le 5,$$

Find a formula for the (instantaneous) velocity of this van.

Again, we are working with the interval

$$I_h = \begin{cases} [t+h, t] & \text{if } h < 0,\\ [t, t+h] & \text{if } 0 < h. \end{cases}$$

To compute the average velocity, we write

$$\frac{s(t+h)-s(t)}{h}$$

but this time, we will let h go to zero. Write with me

$$
\begin{aligned}
&\lim_{h\to 0}\frac{s(t+h)-s(t)}{h}\\
&\quad = \lim_{h\to 0}\frac{36(t+h)^2 - 4.8(t+h)^3 - \left(36t^2 - 4.8t^3\right)}{h}
\end{aligned}
$$

Now expand the numerator of the fraction and combine like-terms:

$$= \lim_{h\to 0}\frac{72th + 36h^2 - 14.4t^2h - 14.4th^2 - 4.8h^3}{h}$$

Factor an h from every term in the numerator:

$$= \lim_{h\to 0}\frac{h\left(72t + 36h - 14.4t^2 - 14.4th - 4.8h^2\right)}{h}$$

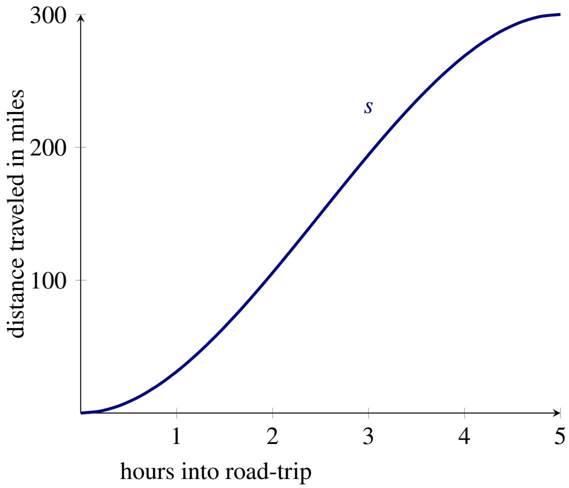
Cancel h from the numerator and denominator:

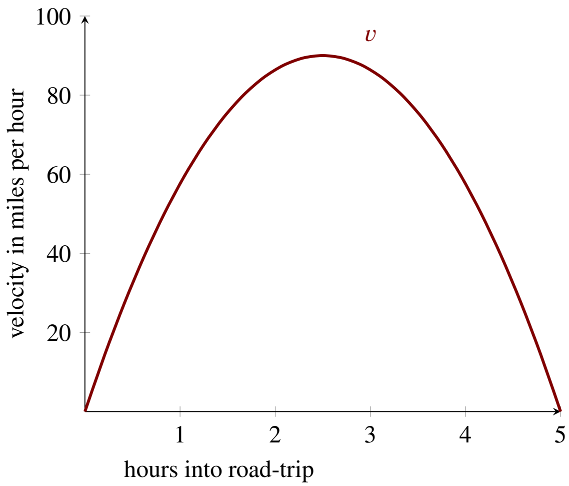
$$= \lim_{h\to 0}\left(72t + 36h - 14.4t^2 - 14.4th - 4.8h^2\right)$$

Plug in $h = 0$:

$$= 72t - 14.4t^2$$

This gives us a formula for our instantaneous velocity, $v(t) = 72t - 14.4t^2$. For your viewing enjoyment, check out graphs of both $y = s(t)$ and $y = v(t)$:


300
200
100
distance traveled in miles
s
1
2
3
4
5
hours into road-trip


100
80
60
40
20
velocity in miles per hour
v
1
2
3
4
5
hours into road-trip

11 Definition of the derivative

After completing this section, students should be able to do the following.

- Use limits to find the slope of the tangent line at a point.
- Understand the definition of the derivative at a point.
- Compute the derivative of a function at a point.
- Estimate the slope of the tangent line graphically.
- Write the equation of the tangent line to a graph at a given point.
- Recognize and distinguish between secant and tangent lines.
- Recognize different notation for the derivative.
- Recognize the the tangent line as a local approximation for a differentiable function.

Break-Ground:

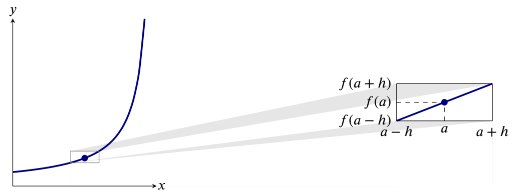
11.1 Slope of a curve

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, do you remember "slope?'
Riley: Most definitely. "Rise over run."
Devyn: You know it.
Riley: "Change in y over change in x."
Devny: That's right.
Riley: Brought to you by the letter "m."
Devny: Enough! My important question is: could we define "slope" for a curve that's not a straight line?
Riley: Well, maybe if we "zoom in" on a curve, it would look like a line, and then we could call it "the slope at that point."
Devyn: Ah! And this "zoom in" idea sounds like a limit!
Riley: This is so awesome. We just made math!

The concept introduced above, of the "slope of a curve at a point," is in fact one of the central concepts of calculus. It will, of course, be completely explained. Let's explore Devyn and Riley's ideas a little more, first.

To find the "slope of a curve at a point," Devyn and Riley spoke of "zooming in" on a curve until it looks like a line. When you zoom in on a *smooth* curve, it will eventually look like a line. This line is called the tangent line.



Problem 1. *Which of the following approximate the slope of the "zoomed line"?*

Select All Correct Answers:

(a) $\dfrac{(f(a)+h)-f(a)}{(a+h)-a}$

(b) $\dfrac{f(a+h)-f(a)}{(a+h)-a}$

(c) $\dfrac{(f(a)-h)-f(a)}{(a-h)-a}$

(d) $\dfrac{f(a-h)-f(a)}{(a-h)-a}$

(e) $\dfrac{f(a)-(f(a)+h)}{a-(a+h)}$

(f) $\dfrac{f(a)-f(a+h)}{a-(a+h)}$

(g) $\dfrac{f(a)-(f(a)-h)}{a-(a-h)}$

(h) $\dfrac{f(a)-f(a-h)}{a-(a-h)}$

Problem 2. *Let $f(x) = 3x - 1$. Zoom in on the curve around $a = -2$ so that $h = 0.1$. Use one of the formulations in the problem above to approximate the slope of the curve. The slope of the curve at $a = -2$ is approximately…*

Problem 3. *Repeat the previous problem for $f(x) = x^2 - 1$, $a = 0$, and $h = 0.2$. Choose a formulation that will give you a positive answer for the slope. The (positive) slope of the curve at $a = 0$ is approximately…*

Problem 4. *Zoom in on the curve $f(x) = x^2 - 1$ near $x = 0$ again. By looking at the graph, what is your best guess for the actual slope of the curve at zero?*

Multiple Choice:

(a) *impossible to say*

(b) *zero*

(c) *one*

(d) *infinity*

Dig-In:

11.2 The definition of the derivative

Given a function, it is often useful to know the rate at which the function changes. To give you a feeling why this is true, consider the following:

- If $s(t)$ represents the **displacement** (position relative to an origin) of an object with respect to time, the rate of change gives the **velocity** of the object.
- If $v(t)$ represents the **velocity** of an object with respect to time, the rate of change gives the **acceleration** of the object.
- If $R(x)$ represents the revenue generated by selling x objects, the rate of change gives us the **marginal revenue**, meaning the additional revenue generated by selling one additional unit. Note, there is an implicit assumption that x is quite large compared to 1.
- If $C(x)$ represents the cost to produce x objects, the rate of change gives us the **marginal cost**, meaning the additional cost generated by selling one additional unit. Again, there is an implicit assumption that x is quite large compared to 1.
- If $P(x)$ represents the profit gained by selling x objects, the rate of change gives us the **marginal profit**, meaning the additional cost generated by selling one additional unit. Again, there is an implicit assumption that x is quite large compared to 1.
- The rate of change of a function can help us approximate a complicated function with a simple function.
- The rate of change of a function can be used to help us solve equations that we would not be able to solve via other methods.

From slopes of secant lines to slopes of tangent lines

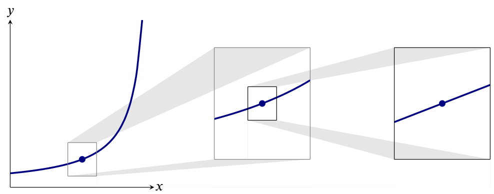
You've been computing average rates of change for a while now, the computation is simply

$$\frac{\text{change in the function}}{\text{change in the input to the function}}.$$

However, the question remains: Given a function that represents an amount, how exactly does one find the function that will give the instantaneous rate of change? Recall that the instantaneous rate of change of a line is the slope of the line. Hence the instantaneous rate of change of a function is the slope of the tangent line. For now, consider the following informal definition of a *tangent line*:

> Given a function f and a number a in the domain of f, if one can "zoom in" on the graph at $(a, f(a))$ sufficiently so that it appears to be a straight line, then that line is the **tangent line** to $f(x)$ at the point $(a, f(a))$.

We illustrate this informal definition with the following diagram:



The *derivative* of a function f at a, is the instantaneous rate of change, and hence is the slope of the tangent line at $(a, f(a))$.

Question 47. *What is the instantaneous rate of change of $f(x) = 4x - 3$?*

Unfortunately, if f is not a straight line we cannot use the slope formula to calculate this rate of change, since $(a, f(a))$

is the only point on this line that we know. In order to deal with this problem, we consider **secant** lines, lines that locally intersect the curve at two points. One of these points will be $(a, f(a))$, the point at which we are trying to find the rate of change. If we call h the difference between the x-coordinates of the two points, then the second point for our secant line is $(a+h, f(a+h))$. The slope of any secant line that passes through the points $(a, f(a))$ and $(a+h, f(a+h))$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

Example 74. *If $f(x) = x^2 - 2x$, find the slope of the secant line through $x = 2$ and $x = 2 + h$, in terms of h.*

Start with the slope formula we just found,

$$\frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{h}.$$

Now substitute in for the function we know,

$$\frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 2(2+h) - 0}{h}.$$

Now expand the numerator of the fraction,

$$\frac{\Delta y}{\Delta x} = \frac{4 + 4h + h^2 - 4 - 2h}{h}.$$

Now combine like-terms,

$$\frac{\Delta y}{\Delta x} = \frac{2h + h^2}{h}.$$

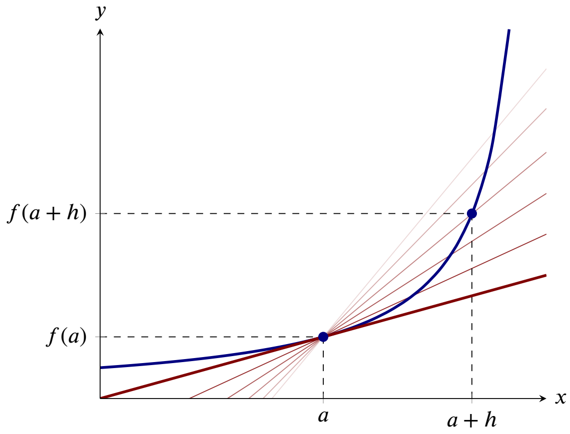
Factor an h from every term in the numerator,

$$\frac{\Delta y}{\Delta x} = \frac{h(2+h)}{h}.$$

Cancel h from the numerator and denominator,

$$\frac{\Delta y}{\Delta x} = 2 + h.$$

The following diagram shows the secant lines for several values of h, as well as the tangent line at $(a, f(a))$.



Notice that as $a + h$ approaches a, the slopes of the secant lines are approaching the slope of the tangent line. This leads to the *definition of the derivative*:

Definition. The **derivative** of f at a is

$$\left[\frac{d}{dx} f(x)\right]_{x=a} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

If this limit exists, then we say that f is **differentiable** at a. If this limit does not exist for a given value of a, then f is **non-differentiable** at a.

Question 48. *Which of the following computes the derivative,* $\left[\frac{d}{dx} f(x)\right]_{x=a}$*?*

Select All Correct Answers:

(a) $\lim_{h\to 0} \frac{(f(a) + h) - f(a)}{(a+h) - a}$

(b) $\lim_{h\to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$

(c) $\lim_{h\to 0} \dfrac{(f(a)-h)-f(a)}{(a-h)-a}$

(d) $\lim_{h\to 0} \dfrac{f(a-h)-f(a)}{(a-h)-a}$

(e) $\lim_{h\to 0} \dfrac{f(a)-(f(a)+h)}{a-(a+h)}$

(f) $\lim_{h\to 0} \dfrac{f(a)-f(a+h)}{a-(a+h)}$

(g) $\lim_{h\to 0} \dfrac{f(a)-(f(a)-h)}{a-(a-h)}$

(h) $\lim_{h\to 0} \dfrac{f(a)-f(a-h)}{a-(a-h)}$

Definition. There are several different notations for the derivative. The two we'll mainly be using are

$$\left[\frac{d}{dx}f(x)\right]_{x=a} = f'(a).$$

Now we will give a number of examples.

Example 75. *If $f(x) = x^2 - 2x$, find the derivative of f at* 2.

Start with the definition of the derivative,

$$\left[\frac{d}{dx}f(x)\right]_{x=2} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}.$$

Now substitute in for the function we know,

$$\left[\frac{d}{dx}f(x)\right]_{x=2} = \lim_{h\to 0} \frac{(2+h)^2-2(2+h)-0}{h}.$$

Now expand the numerator of the fraction,

$$\left[\frac{d}{dx}f(x)\right]_{x=2} = \lim_{h\to 0} = \frac{4+4h+h^2-4-2h}{h}.$$

Now combine like-terms,

$$\left[\frac{d}{dx}f(x)\right]_{x=2} = \lim_{h\to 0} \frac{2h+h^2}{h}.$$

Factor an h from every term in the numerator,

$$\left[\frac{d}{dx}f(x)\right]_{x=2} = \lim_{h\to 0} \frac{h(2+h)}{h}.$$

Cancel h from the numerator and denominator,

$$\left[\frac{d}{dx}f(x)\right]_{x=2} = \lim_{h\to 0} 2+h.$$

Take the limit as h goes to 0,

$$\left[\frac{d}{dx}f(x)\right]_{x=2} = 2.$$

Example 76. *Find an equation for the line tangent to $f(x) = \dfrac{1}{3-x}$ at the point* (2, 1).

To find an equation for a line, we need two pieces of information. We need to know a point on the line, and we need to know the slope. In this question, we are given that (2, 1) is on the line. That means we need to find the slope of the tangent line. Finding the slope of the tangent line at the point (2, 1) means finding $f'(2)$.

Start by writing out the definition of the derivative,

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{1-h}-1}{h}.$$

Multiply by $\dfrac{1-h}{1-h}$ to clear the fraction in the numerator,

$$f'(2) = \lim_{h\to 0} \frac{1-(1-h)}{h(1-h)}.$$

Combine like-terms in the numerator,

$$f'(2) = \lim_{h \to 0} \frac{h}{h(1-h)},$$

Cancel h from the numerator and denominator,

$$f'(2) = \lim_{h \to 0} \frac{1}{1-h},$$

Take the limit as h goes to 0,

$$f'(2) = 1.$$

We are looking for an equation of the line through the point $(2, 1)$ with slope $m = f'(2) = 1$. The point-slope formula tells us that the line has equation given by

$$y = (x-2) + 1.$$

Example 77. *An object moving along a straight line has displacement given by $s(t) = \sqrt{t+3}$. Find the velocity of the object at time $t = 6$.*

Velocity is the rate of change of displacement with respect to time. We are being asked to find $\left[\frac{d}{dt}s(t)\right]_{t=6}$. The definition of the derivative gives

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \lim_{h \to 0} \frac{s(6+h) - s(6)}{h} = \lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h}.$$

Multiply by $\dfrac{\sqrt{9+h}+3}{\sqrt{9+h}+3}$,

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \lim_{h \to 0} \left(\frac{\sqrt{9+h} - 3}{h}\right)\left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right).$$

Now expand the numerator,

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \lim_{h \to 0} \frac{9+h-9}{h\left(\sqrt{9+h}+3\right)}.$$

Combine like-terms,

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{9+h}+3\right)}.$$

Cancel h from the numerator and denominator,

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \lim_{h \to 0} \frac{1}{\sqrt{9+h}+3}.$$

Take the limit as h tends to 0,

$$\left[\frac{d}{dt}s(t)\right]_{t=6} = \frac{1}{6}.$$

The object has velocity $\dfrac{1}{6}$ at time $t = 6$.

12 Derivatives as functions

After completing this section, students should be able to do the following.

- Understand the derivative as a function related to the original definition of a function.
- Find the derivative function using the limit definition.
- Relate the derivative function to the derivative at a point.
- Explain the relationship between differentiability and continuity.
- Relate the graph of the function to the graph of its derivative.
- Determine whether a piecewise function is differentiable.

Break-Ground:

12.1 Wait for the right moment

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I might be a calculus genius.
Riley: Yeah? Explain this one to me.
Devyn: Let me first ask you a question. Say you have a function, like $f(x) = x^2$, and you want to know $f'(3)$. Do you plug in the number 3 before or after you find the derivative?
Riley: Hmmmm. Well, my next step is usually

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}.$$

So I guess before.
Devyn: Aha! I think you're wasting time. You see I write

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

and it means that I can look at the derivative of my function at *any* point. So, I plug in the 3 *after* I've found the derivative.
Riley: That does seem like a pretty genius move. But doesn't working with x, instead of numbers, make all of this more difficult?
Devyn: Not at all. Let's do the problems both ways, at the same time:

$$\underbrace{\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{(3+h)^2 - 9}{h} \\ &= \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h} \\ &= \lim_{h\to 0} \frac{6h + h^2}{h} \\ &= \lim_{h\to 0} (6+h) \\ &= 6. \end{aligned}}_{\text{plugging in}} \qquad \underbrace{\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x+h) \\ &= 2x, \\ \text{so } f'(3) &= 6. \end{aligned}}_{\text{working with } x}$$

Riley: Whoa. So now the derivative is a function. Wait, what's its domain? Its range?

Problem 1. *Suppose you have a function f. Which of the following are true?*

Select All Correct Answers:

(a) *The domain of f' is equal to the domain of f.*

(b) *The range of f' is equal to the range of f.*

(c) *The domain of f' is a subset of the real numbers.*

(d) *The range of f' is a subset of the real numbers.*

(e) *The domain of f' is functions from the real numbers to the real numbers.*

(f) *The range of f' is functions from the real numbers to the real numbers.*

Problem 2. *Find $g'(2)$ for $g(x) = x^2 + 1$ using both methods described above.*

Dig-in:

12.2 The derivative as a function

The derivative of a function, as a function

We know that to find the derivative of a function at a point $x = a$ we write

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

However, if we replace the given number a with a variable x, we now have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

This tells us the instantaneous rate of change at any given point x.

Warning. *The notation:*

> *$f'(a)$ means take the derivative of f first, then evaluate at $x = a$.*

In other words, given f a function of x

$$f'(a) = \left[\frac{d}{dx} f(x)\right]_{x=a}.$$

Given a function f from the real numbers to the real numbers, the derivative f' is also a function from the real numbers to the real numbers. Understanding the relationship between the *functions* f and f' helps us understand any situation (real or imagined) involving changing values.

Question 49. *Let $f(x) = 3x + 2$. What is $f'(-1)$?*

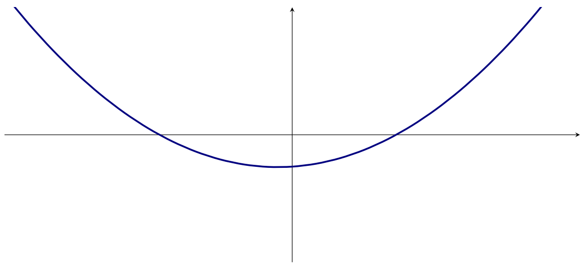
Multiple Choice:

(a) *$f'(-1) = 0$ because $f'(3)$ is a number, and a number corresponds to a horizontal line, which has a slope of zero.*

(b) *$f'(-1) = 3$ because $y = f(x)$ is a line with slope 3.*

(c) *We cannot solve this problem yet.*

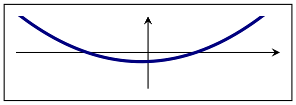
Question 50. *Here we see the graph of f'.*

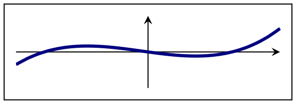


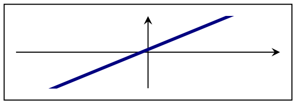
Describe $y = f(x)$ when f' is positive. Describe $y = f(x)$ when f' is negative.

Question 51. *Which of the following graphs could be $y = f(x)$?*

Multiple Choice:

(a) 

(b) 

(c) 

The derivative as a function of functions

While writing f' is viewing the derivative of f as a function in its own right, the derivative itself

$$\frac{d}{dx}$$

is in fact a function that maps functions to functions,

$$\begin{aligned}\frac{d}{dx}x^2 &= 2x\\ \frac{d}{dx}f(x) &= f'(x).\end{aligned}$$

Question 52. *As a function, is*

$$\frac{d}{dx}$$

one-to-one?

Multiple Choice:

(a) *yes*

(b) *no*

Dig-In:

12.3 Differentiability implies continuity

There are connections between continuity and differentiability.

Theorem 14 (Differentiability Implies Continuity)**.** *If f is a differentiable function at $x = a$, then f is continuous at $x = a$.*

To explain why this is true, we are going to use the following definition of the derivative

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Assuming that $f'(a)$ exists, we want to show that $f(x)$ is continuous at $x = a$, hence we must show that

$$\lim_{x\to a} f(x) = f(a).$$

Starting with

$$\lim_{x\to a} (f(x) - f(a))$$

we multiply and divide by $(x - a)$ to get

$$\begin{aligned}
&= \lim_{x\to a} \left((x-a)\frac{f(x) - f(a)}{x - a}\right) \\
&= \left(\lim_{x\to a}(x-a)\right)\left(\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right) \qquad \text{Limit Law.} \\
&= 0 \cdot f'(a) = 0.
\end{aligned}$$

Since

$$\lim_{x\to a} (f(x) - f(a)) = 0$$

we see that $\lim_{x\to a} f(x) = f(a)$, and so f is continuous at $x = a$.

This theorem is often written as its contrapositive:

> If $f(x)$ is not continuous at $x = a$, then $f(x)$ is not differentiable at $x = a$.

Thus from the theorem above, we see that all differentiable functions on $\mathbb{R}$ are continuous on $\mathbb{R}$. Nevertheless there are continuous functions on $\mathbb{R}$ that are not differentiable on $\mathbb{R}$.

Question 53. *Which of the following functions are continuous but not differentiable on $\mathbb{R}$?*

Multiple Choice:

(a) x^2

(b) $\lfloor x \rfloor$

(c) $|x|$

(d) $\dfrac{x^2}{x}$

Example 78. *Consider*

$$f(x) = \begin{cases} x^2 & \textit{if } x < 3, \\ mx + b & \textit{if } x \geq 3. \end{cases}$$

What values of m and b make f differentiable at $x = 3$?

To start, we know that we must make f both continuous and differentiable. We will start by showing f is continuous at $x = 3$. Write with me:

$$\begin{aligned}
\lim_{x\to 3^-} f(x) &= 9 \\
\lim_{x\to 3^+} f(x) &= m3 + b \\
f(3) &= m3 + b
\end{aligned}$$

So for the function to be continuous, we must have

$$m \cdot 3 + b = 9.$$

We also must ensure that the value of the derivatives of both pieces of f agree at $x = 3$. Write with me

$$\begin{aligned}
\frac{d}{dx}x^2 &= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0}\frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0}\frac{2xh + h^2}{h} \\
&= \lim_{h\to 0}(2x + h) \\
&= 2x.
\end{aligned}$$

Moreover, by the definition of a tangent line

$$\frac{d}{dx}(mx + b) = m$$

Hence we must have

$$\begin{aligned}
\left[\frac{d}{dx}x^2\right]_{x=3} &= \left[\frac{d}{dx}(mx+b)\right]_{x=3} \\
\left[2x\right]_{x=3} &= \left[m\right]_{x=3} \\
6 &= m.
\end{aligned}$$

Ah! So now

$$\begin{aligned}
9 &= m \cdot 3 + b \\
9 &= 6 \cdot 3 + b \\
9 &= 18 + b,
\end{aligned}$$

so $b = -9$. Thus setting $m = 6$ and $b = -9$ will give us a differentiable (and hence continuous) piecewise function.

13 Rules of differentiation

After completing this section, students should be able to do the following.

- Use the definition of the derivative to develop shortcut rules to find the derivatives of constants and constant multiples.
- Use the definition of the derivative to develop shortcut rules to find the derivatives of powers of x.
- Use the definition of the derivative to develop shortcut rules to find the derivatives of sums and differences of functions.
- Compute the derivative of polynomials.

Break-Ground:

13.1 Patterns in derivatives

Check out this dialogue between two calculus students (based on a true story):

Devyn: I hate the limit definition of derivative. I wish there were a shorter way.
Riley: I think I might have found a pattern for taking derivatives.
Devyn: Really? I love patterns!
Riley: I know! Check this out, I've made a chart

$f(x)$	$f'(x)$
x^2	$2 \cdot x^1$
x^3	$3 \cdot x^2$
x^4	$4 \cdot x^3$

So maybe if we have a function

$$f(x) = x^n \quad \text{then} \qquad f'(x) = n \cdot x^{n-1}.$$

Devyn: Hmmm does it work with square roots?
Riley: Oh that's right, a square root is a power, just write

$$f(x) = \sqrt{x} = x^{1/2}.$$

So a square root is of the form x^n.

Devyn: Let's check it. If $f(x) = \sqrt{x}$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\ &= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2} \cdot x^{-1/2}. \end{aligned}$$

Riley: Holy Cat Fur! It works! In this case $f'(x) = n \cdot x^{n-1}$.
Devyn: I wonder if it *always* works? If so I want to know *why* it works! I wonder what other patterns we can find?

The pattern

$$\text{if} \qquad f(x) = x^n \quad \text{then} \qquad f'(x) = n \cdot x^{n-1}$$

holds whenever n is a constant. Explaining why it works in generality will take some time. For now, let's see if we can use the problem to squash some derivatives with ease.

Problem 1. *Using the pattern found above, compute:*

$$\frac{d}{dx} x^{101}$$

Problem 2. *Using the pattern found above, compute:*

$$\frac{d}{dx} \frac{1}{x^{77}}$$

Problem 3. *Using the pattern found above, compute:*

$$\frac{d}{dx}\sqrt[5]{x}$$

Problem 4. *Using the pattern found above, compute:*

$$\frac{d}{dx}x^e$$

Dig-In:

13.2 Basic rules of differentiation

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to quickly compute the derivative of almost any function we are likely to encounter. We will start simply and build up to more complicated examples.

The constant rule

The simplest examples of functions, hence the best place to start our investigation, are constant functions. Recall that derivatives measure the rate of change of a function at a given point. This means the derivative of a constant function is zero. Here are some ways to think about this situation.

- The constant function plots a horizontal line—so the slope of the tangent line is 0.
- If $s(t)$ represents the position of an object with respect to time and $s(t)$ is constant, then the object is not moving, so its velocity is zero. Hence $\frac{d}{dt}s(t) = 0$.
- If $v(t)$ represents the velocity of an object with respect to time and $v(t)$ is constant, then the object's acceleration is zero. Hence $\frac{d}{dt}v(t) = 0$.

The examples above lead us to our next theorem. To gain intuition, you should compute the derivative of $f(x) = 6$ using the limit definition of the derivative.

Theorem 15 (The constant rule)**.** *Given a constant c,*

$$\frac{d}{dx}c = 0.$$

From the limit definition of the derivative, write

$$\begin{aligned}\frac{d}{dx}c &= \lim_{h\to 0}\frac{c-c}{h}\\ &= \lim_{h\to 0}\frac{0}{h}\\ &= \lim_{h\to 0} 0 = 0.\end{aligned}$$

Question 54. *What is*

$$\frac{d}{dx}e$$

equal to?

The power rule

Next, let's examine derivatives of powers of a single variable. To gain intuition, you should compute the derivative of $f(x) = x^3$ using the limit definition of the derivative. Before computing this derivative, we should recall the *Binomial Theorem*.

Theorem 16 (Binomial Theorem)**.** *If n is a nonnegative integer, then*

$$(a+b)^n = a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + a^0 b^n$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The Binomial Theorem tells us the pattern of the coefficients for the expanded form of

$$(a+b)^n,$$

which will help us prove our next derivative rule.

Example 79. *Expand $(x+h)^3$ using the Binomial Theorem.*

We apply the Binomial Theorem to the expression in question, then simplify.

$$\begin{aligned}(x+h)^3 &= x^3h^0 + \binom{3}{1}x^{3-1}h^1 + \binom{3}{2}x^{3-2}h^2 + x^0h^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3\end{aligned}$$

Theorem 17 (The power rule)**.** *For any real number n,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

At this point we will only prove this theorem for values of n which are positive integers. Later we will give the complete explanation. From the limit definition of the derivative, write with me

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

Expand the term $(x+h)^n$:

$$= \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n - x^n}{h}$$

Note that we are using the Binomial Theorem to write $\binom{n}{k}$ for the coefficients. Canceling the terms x^n and $-x^n$, and noting $\binom{n}{1} = n$, write with me

$$\begin{aligned}&= \lim_{h\to 0}\frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n}{h} \\ &= \lim_{h\to 0}\left(nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1}\right).\end{aligned}$$

Since every term but the first has a factor of h, we see

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Let's consider several examples. We begin with something basic.

Example 80. *Compute:*

$$\frac{d}{dx}x^{13}$$

Applying the power rule, we write

$$\frac{d}{dx}x^{13} = 13x^{12}.$$

Sometimes, it is not as obvious that one should apply the power rule.

Example 81. *Compute:*

$$\frac{d}{dx}\frac{1}{x^4}$$

Applying the power rule, we write

$$\frac{d}{dx}\frac{1}{x^4} = \frac{d}{dx}x^{-4} = -4x^{-5}.$$

The power rule also applies to radicals once we rewrite them as exponents.

Example 82. *Compute:*

$$\frac{d}{dx}\sqrt[5]{x}$$

Applying the power rule, we write

$$\frac{d}{dx}\sqrt[5]{x} = \frac{d}{dx}x^{1/5} = \frac{x^{-4/5}}{5}.$$

The sum rule

We want to be able to take derivatives of functions "one piece at a time." The *sum rule* allows us to do exactly this. The sum rule says that we can add the rates of change of two functions to obtain the rate of change of the sum of both functions. For example, viewing the derivative as the velocity of an object, the sum rule states that to find the velocity of a person walking on a moving bus, we add the velocity of the bus and the velocity of the walking person.

Theorem 18 (The sum rule)**.** *If $f(x)$ and $g(x)$ are differentiable and c is a constant, then*

(a) $\dfrac{d}{dx}\big(f(x)+g(x)\big) = f'(x)+g'(x)$,

(b) $\dfrac{d}{dx}\big(f(x)-g(x)\big) = f'(x)-g'(x)$,

(c) $\dfrac{d}{dx}\big(c\cdot f(x)\big) = c\cdot f'(x)$.

We will only prove part (a) above; the rest are similar. Write with me

$$\begin{aligned}
\frac{d}{dx}\big(f(x)+g(x)\big) &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x).
\end{aligned}$$

Question 55. *Using different notation for the derivative can be confusing. Which of the following are the same as part (a) of the Sum Rule?*

Select All Correct Answers:

(a) $(f(x)+g(x))' = f'(x)+g'(x)$

(b) $(f(x)+g(x))' = \dfrac{d}{dx}f(x)+\dfrac{d}{dx}g(x)$

(c) $\dfrac{d}{dx}f(x)+\dfrac{d}{dx}g(x) = f'(x)+g'(x)$

(d) $(f'(x)+g(x))' = \dfrac{d}{dx}(f(x)+g(x))$

Example 83. *Suppose we have two functions, f, and g, and we know that $f'(2)=6$ and $g'(2)=-3$. What is the slope of $f(x)+g(x)$ at $x=2$?*

Using the sum rule, the slope of $f(x)+g(x)$ at $x=2$ is the sum $f'(2)+g'(2)$. In this case, we have $f'(x)+g'(x)=3$.

We now have the tools to work some more complicated examples.

Example 84. *Compute:*

$$\frac{d}{dx}\left(x^5+\frac{1}{x}\right)$$

> Write with me
>
> $$\frac{d}{dx}\left(x^5 + \frac{1}{x}\right) = \frac{d}{dx}x^5 + \frac{d}{dx}x^{-1}$$
> $$= 5x^4 - x^{-2}.$$

Example 85. *Compute:*

$$\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right)$$

> Write with me
>
> $$\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right)$$
> $$= 3\frac{d}{dx}x^{-1/3} - 2\frac{d}{dx}x^{1/2} + \frac{d}{dx}x^{-7}$$
> $$= -x^{-4/3} - x^{-1/2} - 7x^{-8}.$$

14 Product rule and quotient rule

After completing this section, students should be able to do the following.

- Identify products of functions.
- Use the product rule to calculate derivatives.
- Identify quotients of functions.
- Use the quotient rule to calculate derivatives.
- Combine derivative rules to take derivatives of more complicated functions.
- Explain the signs of the terms in the numerator of the quotient rule.
- Multiply tangent lines to justify the product rule.
- Use the product and quotient rule to calculate derivatives from a table of values.

Break-Ground:

14.1 Derivatives of products are tricky

Check out this dialogue between two calculus students (based on a true story):

Devyn: Hey Riley, remember the sum rule for derivatives?
Riley: You know I do.
Devyn: What do you think that the "product rule" will be?
Riley: Let's give this a spin:

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f'(x)\cdot g'(x)?$$

Devyn: Hmmm, let's give this theory an acid test. Let's try

$$f(x) = x^2 + 1 \qquad \text{and} \qquad g(x) = x^3 - 3x$$

Now

$$\begin{aligned} f'(x)g'(x) &= (2x)(3x^2 - 3) \\ &= 6x^3 - 6x. \end{aligned}$$

Riley: On the other hand,

$$\begin{aligned} f(x)g(x) &= (x^2 + 1)(x^3 - 3x) \\ &= x^5 - 3x^3 + x^3 - 3x \\ &= x^5 - 2x^3 - 3x. \end{aligned}$$

Devyn: And so,

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = 5x^4 - 6x^2 - 3.$$

Riley: Wow. Hmmm. It looks like our guess was incorrect.
Devyn: I've got a feeling that the so-called "product rule" might be a bit tricky.

Problem 1. *Above, our intrepid young mathematicians guess that the "product rule" might be:*

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f'(x)\cdot g'(x)?$$

*Does this **ever** hold true?*

Dig-In:

14.2 The Product rule and quotient rule

The product rule

Consider the product of two simple functions, say

$$f(x) \cdot g(x)$$

where $f(x) = x^2 + 1$ and $g(x) = x^3 - 3x$. An obvious guess for the derivative of $f(x)g(x)$ is the product of the derivatives:

$$\begin{aligned} f'(x)g'(x) &= (2x)(3x^2 - 3) \\ &= 6x^3 - 6x. \end{aligned}$$

Is this guess correct? We can check by rewriting f and g and doing the calculation in a way that is known to work. Write with me

$$\begin{aligned} f(x)g(x) &= (x^2 + 1)(x^3 - 3x) \\ &= x^5 - 3x^3 + x^3 - 3x \\ &= x^5 - 2x^3 - 3x. \end{aligned}$$

Hence

$$\frac{d}{dx} f(x)g(x) = \frac{d}{dx}(x^5 - 2x^3 - 3x) = 5x^4 - 6x^2 - 3,$$

so we see that

$$\frac{d}{dx} f(x)g(x) \neq f'(x)g'(x).$$

So the derivative of $f(x)g(x)$ is **not** as simple as $f'(x)g'(x)$. Never fear, we have a rule for exactly this situation.

Theorem 19 (The product rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

Let's return to the example with which we started.

Example 86. *Let $f(x) = (x^2 + 1)$ and $g(x) = (x^3 - 3x)$. Compute:*

$$\frac{d}{dx} f(x)g(x)$$

Write with me

$$\begin{aligned} \frac{d}{dx} f(x)g(x) &= f(x)g'(x) + f'(x)g(x) \\ &= (x^2 + 1)(3x^2 - 3) + (2x)(x^3 - 3x). \end{aligned}$$

We could stop here, but we should show that expanding this out recovers our previous result. Write with me

$$\begin{aligned} &(x^2 + 1)(3x^2 - 3) + 2x(x^3 - 3x) \\ &\quad = 3x^4 - 3x^2 + 3x^2 - 3 + 2x^4 - 6x^2 \\ &\quad = 5x^4 - 6x^2 - 3, \end{aligned}$$

which is precisely what we obtained before.

Now that we are pros, let's try one more example.

Example 87. *Suppose f is a function whose values are given in the following table.*

x	$f(x)$	$f'(x)$
1	-3	4
2	5	-1
3	2	0

Compute $\left[\frac{d}{dx}\left((x^2 + 1)f(x)\right)\right]_{x=2}$.

Using the product rule, write with me.

$$\begin{aligned}\frac{d}{dx}\left((x^2+1)f(x)\right) &= \frac{d}{dx}\left(x^2+1\right)\cdot f(x) + (x^2+1)\cdot\frac{d}{dx}(f(x))\\ &= (2x)\cdot f(x) + (x^2+1)\cdot\frac{d}{dx}(f(x))\end{aligned}$$

Plugging in $x = 2$ yields:

$$\begin{aligned}\left[\frac{d}{dx}\left((x^2+1)f(x)\right)\right]_{x=2} &= (4)\cdot f(2) + (5)\cdot f'(2)\\ &= 4\cdot 5 + 5\cdot -1\\ &= 15\end{aligned}$$

The quotient rule

We'd like to have a formula to compute

$$\frac{d}{dx}\frac{f(x)}{g(x)}$$

if we already know $f'(x)$ and $g'(x)$. Instead of attacking this problem head-on, let's notice that we've already done part of the problem: $f(x)/g(x) = f(x)\cdot(1/g(x))$, that is, this is really a product, and we can compute the derivative if we know $f'(x)$ and $(1/g(x))'$. This brings us to our next derivative rule.

Theorem 20 (The quotient rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example 88. *Compute:*

$$\frac{d}{dx}\frac{x^2+1}{x^3-3x}$$

Write with me

$$\begin{aligned}\frac{d}{dx}\frac{x^2+1}{x^3-3x} &= \frac{2x(x^3-3x)-(x^2+1)(3x^2-3)}{(x^3-3x)^2}\\ &= \frac{-x^4-6x^2+3}{(x^3-3x)^2}.\end{aligned}$$

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler.

Example 89. *Compute:*

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}}$$

in two ways. First using the quotient rule and then using the product rule.

First, we'll compute the derivative using the quotient rule. Write with me

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}} = \frac{(-2x)\left(\sqrt{x}\right)-(625-x^2)\left(\frac{1}{2}x^{-1/2}\right)}{x}.$$

Second, we'll compute the derivative using the product rule:

$$\begin{aligned}\frac{d}{dx}\frac{625-x^2}{\sqrt{x}} &= \frac{d}{dx}\left(625-x^2\right)x^{-1/2}\\ &= \left(625-x^2\right)\left(\frac{-x^{-3/2}}{2}\right) + (-2x)\left(x^{-1/2}\right).\end{aligned}$$

With a bit of algebra, both of these simplify to

$$-\frac{3x^2+625}{2x^{3/2}}.$$

15 Chain rule

After completing this section, students should be able to do the following.

- Recognize a composition of functions.
- Take derivatives of compositions of functions using the chain rule.
- Take derivatives that require the use of multiple derivative rules.
- Use the chain rule to calculate derivatives from a table of values.
- Understand rate of change when quantities are dependent upon each other.
- Use order of operations in situations requiring multiple derivative rules.
- Justify the chain rule via the composition of linear approximations.
- Apply chain rule to relate quantities expressed with different units.

Break-Ground:

15.1 An unnoticed composition

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley! Something is bothering me.
Riley: What is it?
Devyn: It's about rates of change.
Riley: That's just a derivative. What's the issue?
Devyn: It might take a bit of setup. Suppose my car has a 10 gallon tank. The distance D I can drive, without filling up, is $D(m) = 10m$ miles, where m is my car's fuel efficiency, in miles per gallon.
Riley: Sure. So if you're getting 35 miles per gallon, that is $D(35) = 350$ miles, so you can drive 350 miles before running out of gas.
Devyn: Right! Now, the fuel efficiency depends on how fast I'm driving. If I'm driving 55 miles per hour, I can get 40 miles per gallon, but if I'm driving 70 miles per hour on the interstate, I only get 30 miles per gallon.
Riley: Oh! So your distance function D depends on m, but m depends on your velocity v. That means D is really a function of v.
Devyn: Exactly! Finding the derivative $\frac{dD}{dm}$ is easy, but how do I find $\frac{dD}{dv}$?
Riley: If only we could find the derivative of a composite function...

Question 56. *What is* $\frac{dD}{dm}$*?*

$$\frac{dD}{dm} = \boxed{?}.$$

Question 57. *Suppose that the fuel efficiency m is a linear between $v = 50$ mph to $v = 70$ mph. What is* $\frac{dm}{dv}$*?*

Multiple Choice:

(a) 10

(b) -10

(c) -1.5

(d) 1.5

Dig-In:

15.2 The chain rule

So far we have seen how to compute the derivative of a function built up from other functions by addition, subtraction, multiplication and division. There is another very important way that we combine functions: composition. The *chain rule* allows us to deal with this case. Consider

$$h(x) = \left(3x^2+1\right)^{18}.$$

While there are several different ways to differentiate this function, if we let $f(x) = x^{18}$ and $g(x) = 3x^2+1$, then we can express $h(x) = f(g(x))$. The question is, can we compute the derivative of a composition of functions using the derivatives of the constituents $f(x)$ and $g(x)$? To do so, we need the *chain rule*.

Theorem 21 (Chain Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

It will take a bit of practice to make the use of the chain rule come naturally, it is more complicated than the earlier differentiation rules we have seen. Let's return to our motivating example.

Example 90. *Compute:*

$$\frac{d}{dx}\left(3x^2+1\right)^{18}$$

Set $f(x) = x^{18}$ and $g(x) = 3x^2+1$, now

$$f'(x) = 18x^{17} \qquad \text{and} \qquad g'(x) = 6x.$$

Hence

$$\begin{aligned}\frac{d}{dx}\left(3x^2+1\right)^{18} &= \frac{d}{dx} f(g(x))\\ &= f'(g(x))g'(x)\\ &= \left(3x^2+1\right)^{18}\cdot 6x\\ &= 6x\left(3x^2+1\right).\end{aligned}$$

Let's see a more complicated chain of compositions.

Example 91. *Compute:*

$$\frac{d}{dx}\sqrt{1+\sqrt{x}}$$

Set $f(x) = \sqrt{x}$ and $g(x) = 1+x$. Hence,

$$\sqrt{1+\sqrt{x}} = f(g(f(x)))$$

and by the chain rule we know

$$\frac{d}{dx} f(g(f(x))) = f'(g(f(x)))g'(f(x))f'(x).$$

Since

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad \text{and} \qquad g'(x) = 1$$

We have that

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2\sqrt{1+\sqrt{x}}}\cdot 1\cdot\frac{1}{2\sqrt{x}}.$$

The chain rule allows to differentiate compositions of functions that would otherwise be difficult to get our hands on.

Example 92. *Compute:*

$$\frac{d}{dx}\sqrt[3]{\left(3x^6+4x^2-1\right)^{10}+1}$$

Set $f(x) = \sqrt[3]{x}$, $g(x) = x^{10}+1$,and $h(x) = 3x^6+4x^2-1$ so that $f(g(h(x))) = \sqrt[3]{\left(3x^6+4x^2-1\right)^{10}+1}$. Now

$$\begin{aligned}\frac{d}{dx}\sqrt[3]{\left(3x^6+4x^2-1\right)^{10}+1} &= \frac{d}{dx}f(g(h(x)))\\ &= f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)\\ &= \frac{1}{3}\left(\left(3x^6+4x^2-1\right)^{10}+1\right)^{-2/3}\\ &\cdot 10\left(3x^6+4x^2-1\right)\cdot\left(18x^2+8x\right).\end{aligned}$$

Using the chain rule, the power rule, and the product rule it is possible to avoid using the quotient rule entirely.

Example 93. *Compute:*

$$\frac{d}{dx}\frac{x^3}{x^2+1}s$$

Rewriting this as

$$\frac{d}{dx}x^3(x^2+1)^{-1},$$

set $f(x) = x^{-1}$ and $g(x) = x^2+1$ so that $f(g(x)) = (x^2+1)^{-1}$. Now

$$x^3(x^2+1)^{-1} = x^3 f(g(x)),$$

and by the product and chain rules

$$\frac{d}{dx}x^3 f(g(x)) = 3x^2\cdot f(g(x)) + x^3\cdot f'(g(x))g'(x).$$

Since $f'(x) = \dfrac{-1}{x^2}$ and $g'(x) = 2x$, write

$$\frac{d}{dx}\frac{x^3}{x^2+1} = \frac{3x^2}{x^2+1} - \frac{2x^4}{(x^2+1)^2}.$$

Now that we are getting comfortable with chain rule, try one other.

Example 94. *Suppose f is a function whose values are given in the following table.*

x	f(x)	f'(x)
1	-3	4
2	5	-1
3	2	0

Find $\left[\frac{d}{dx}\left((4x-3)f\left(x^2+1\right)\right)\right]_{x=1}$

At it's outer most level, this is a product of $4x-3$ and $f\left(x^2+1\right)$, so we start with product rule.

$$\begin{aligned}\frac{d}{dx}\left((4x-3)f\left(x^2+1\right)\right) &= (4)\cdot f\left(x^2+1\right)\\ &+ (4x-3)\cdot\frac{d}{dx}\left(f\left(x^2+1\right)\right)\\ &= 4f(x^2+1) + (4x-3)f'(x^2+1)\cdot(2x)\\ &= 4f(x^2+1) + (4x-3)f'(x^2+1)\cdot 2x.\end{aligned}$$

Evaluating at $x = 1$ gives:

$$\begin{aligned}\left[\frac{d}{dx}\left((4x-3)f\left(x^2+1\right)\right)\right]_{x=1} &= 4f(2) + (1)f'(2)\cdot 2\\ &= 4\cdot(5) + 2\cdot(-1)\\ &= 18.\end{aligned}$$

Part III

Content for the Third Exam

16 Exponential and Logarithmic Functions

After completing this section, students should be able to do the following.

- Understand the relationship between exponential and logarithmic functions.
- Know and use the properties of exponential and logarithmic functions.
- Solve equations using exponential and logarithmic functions.

Break-Ground:

16.1 An interesting situation

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I've been thinking about the interest on my bank account.

Riley: So, like the compound interest formula,

$$A = P\left(1+\frac{r}{n}\right)^{nt}?$$

Devyn: Yes! To make it easier to work with, suppose I deposit just \$1, so $P = 1$.

Riley: A \$1 bank account balance? I can imagine that!

Devyn: If the interest is compounded monthly, $n = 12$, and if the deposit is at 3% interest, then $r = 0.03$.

Riley: Since $\frac{0.03}{12} = .0025$, that brings you to

$$A = (1.0025)^{12t}.$$

Devyn: My question is: How long does it take the account to double in value? How long until the account balance is \$2?

Riley: Can't we just setup

$$(1.0025)^{12t} = 2?$$

Devyn: Of course, but how to solve that for t? The variable is up in an exponent.

Riley: Hmmmm. I'm not sure…

Question 58. *What kind of operation will allow us to solve for t?*

Multiple Choice:

(a) *Take the square root of both sides.*

(b) *Take the cosecent of both sides.*

(c) *Raise both sides to the* 1.0025*-th power.*

(d) *Take a logarithm of both sides.*

Question 59. *Take the compound interest formula above $A = P\left(1+\frac{r}{n}\right)^{nt}$ with $P = 1$, $r = 1$, and $t = 1$:*

$$A = \left(1+\frac{1}{n}\right)^{n}.$$

Plug in larger and larger values of n and see what happens to the values of A.

Multiple Choice:

(a) $\frac{1}{n}$ *tends to* 0*, so* $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = 1$.

(b) *The exponent is getting bigger and bigger, so* $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \infty$.

(c) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = 3$.

(d) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = 2.71828\ldots$.

Dig-In:

16.2 Exponential and logarithmetic functions

Exponential and logarithmic functions may seem somewhat esoteric at first, but they model many phenomena in the real-world.

What are exponential and logarithmic functions?

Definition. An **exponential function** is a function of the form

$$f(x) = b^x$$

where $b \neq 1$ is a positive real number. The domain of an exponential function is $(-\infty, \infty)$ and the range is $(0, \infty)$.

Question 60. *Is b^{-x} an exponential function?*

Multiple Choice:

(a) *yes*

(b) *no*

Definition. A **logarithmic function** is a function defined as follows

$$\log_b(x) = y \qquad \text{means that} \qquad b^y = x$$

where $b \neq 1$ is a positive real number. The domain of a logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$.

In either definition above b is called the **base**.

Remember that with exponential and logarithmic functions, there is one very special base:

$$e = 2.7182818284590\dots$$

This is an irrational number that you will see frequently. The exponential with base e, $f(x) = e^x$ is often called the 'natural exponential' function. For the logarithm with base e, we have a special notation, $\ln x$ is 'natural logarithm' function. We'll talk about where e comes from when we talk about derivatives.

Connections between exponential functions and logarithms Let b be a positive real number with $b \neq 1$.

- $b^{\log_b(x)} = x$ for all positive x
- $\log_b(b^x) = x$ for all real x

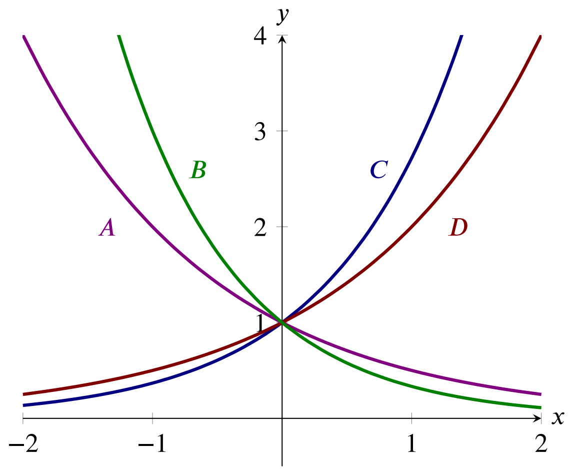
Question 61. *What exponent makes the following expression true?*

$$3^x = e^{\left(x \cdot \boxed{?}\right)}.$$

What can the graphs look like?

Graphs of exponential functions

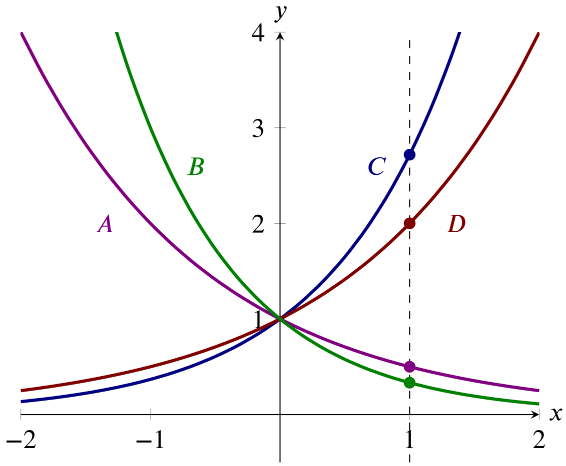
Example 95. *Here we see the the graphs of four exponential functions.*



Match the curves A, B, C, and D with the functions

$$e^x, \qquad \left(\frac{1}{2}\right)^x, \qquad \left(\frac{1}{3}\right)^x, \qquad 2^x.$$

One way to solve these problems is to compare these functions along the vertical line $x = 1$,



Note

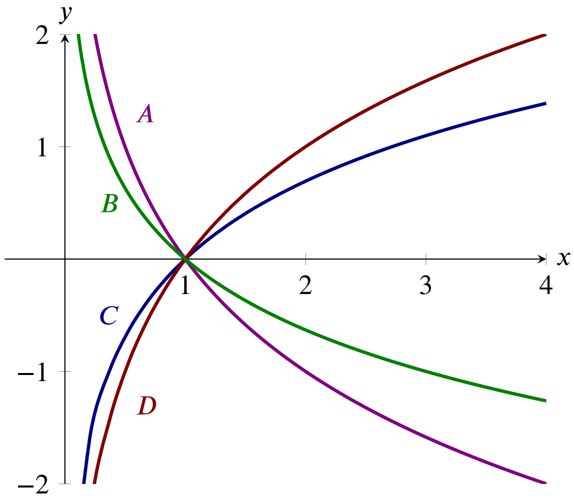
$$\left(\frac{1}{3}\right)^1 < \left(\frac{1}{2}\right)^1 < 2^1 < e^1.$$

Hence we see:

- $\left(\frac{1}{3}\right)^x$ corresponds to B.
- $\left(\frac{1}{2}\right)^x$ corresponds to A.
- 2^x corresponds to D.
- e^x corresponds to C.

Graphs of logarithmic functions

Example 96. *Here we see the the graphs of four logarithmic functions.*



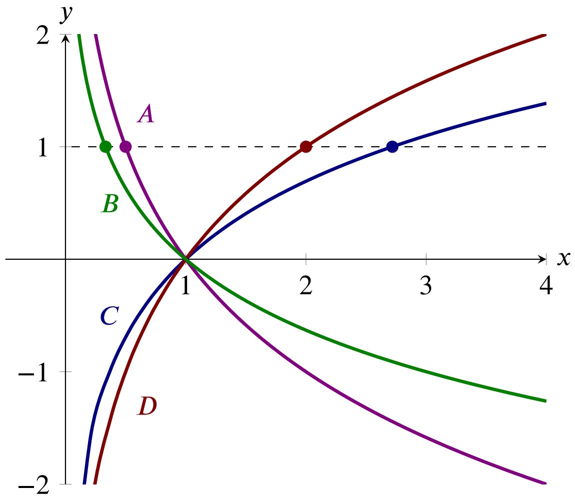
Match the curves A, B, C, and D with the functions

$$\ln(x), \qquad \log_{1/2}(x), \qquad \log_{1/3}(x), \qquad \log_2(x).$$

First remember what $\log_b(x) = y$ means:

$$\log_b(x) = y \qquad \text{means that} \qquad b^y = x.$$

Moreover, $\ln(x) = \log_e(x)$ where $e = 2.71828\ldots$. So now examine each of these functions along the horizontal line $y = 1$



Note again (this is from the definition of a logarithm)

$$\left(\frac{1}{3}\right)^1 < \left(\frac{1}{2}\right)^1 < 2^1 < e^1.$$

Hence we see:

- $\log_{1/3}(x)$ corresponds to B.
- $\log_{1/2}(x)$ corresponds to A.
- $\log_2(x)$ corresponds to D.
- $\ln(x)$ corresponds to C.

Properties of exponential functions and logarithms

Working with exponential and logarithmic functions is often simplified by applying properties of these functions. These properties will make appearances throughout our work.

Properties of exponents Let b be a positive real number with $b \neq 1$.

- $b^m \cdot b^n = b^{m+n}$
- $b^{-1} = \dfrac{1}{b}$
- $(b^m)^n = b^{mn}$

Question 62. *What exponent makes the following true?*

$$2^4 \cdot 2^3 = 2^{\boxed{?}}$$

Properties of logarithms Let b be a positive real number with $b \neq 1$.

- $\log_b(m \cdot n) = \log_b(m) + \log_b(n)$
- $\log_b(m^n) = n \cdot \log_b(m)$
- $\log_b\left(\dfrac{1}{m}\right) = \log_b(m^{-1}) = -\log_b(m)$
- $\log_a(m) = \dfrac{\log_b(m)}{\log_b(a)}$

Question 63. *What value makes the following expression true?*

$$\log_2\left(\frac{8}{16}\right) = 3 - \boxed{?}$$

Question 64. *What makes the following expression true?*

$$\log_3(x) = \frac{\ln(x)}{\boxed{?}}$$

Exponential equations

Let's look into solving equations involving these functions. We'll start with a straightforward example.

Example 97. *Solve the equation:* $4^x = 8$.

We know 4 and 8 are each powers of 2, we start by rewriting in terms of this base.

$$4^x = 2^{2x} \quad \text{and} \quad 8 = 2^{\boxed{?}} \quad \text{so } 2^{2x} = 2^{\boxed{?}}$$

Since exponential functions are one-to-one, the only way for $a^m = a^n$ is if $m = n$. In this case, that means $2x = 3$.

The solution is: $x = \boxed{?}$.

Of course, if we couldn't rewrite both sides with the same base, we can still use the properties of logarithms to solve.

Example 98. *Solve the equation:* $5^{2x-3} = 7$.

Since we can't easily rewrite both sides as exponentials with the same base, we'll use logarithms instead. Above we said that $\log_b(x) = y$ means that $b^y = x$. That statement means that each exponential equation has an equivalent logarithmic form and vice-versa. We'll convert to a logarithmic equation and solve from there.

$$\begin{aligned} 5^{2x-3} &= 7 \\ \log_{\boxed{?}}\left(\boxed{?}\right) &= 2x-3 \end{aligned}$$

From here, we can solve for x directly.

$$\begin{aligned} 2x &= \log_5(7) + 3 \\ x &= \frac{\log_5(7)+3}{2} \end{aligned}$$

Example 99. *Solve the equation:* $e^{2x} = e^x + 6$.

Immediately taking logarithms of both sides will not help here, as the right side has multiple terms. We know that logarithms do not behave well with sums, but with products/quotients. Instead, we notice that $e^{2x} = (e^x)^2$. (This is a common trick that you will likely see many times.)

$$\begin{aligned} e^{2x} &= e^x + 6 \\ (e^x)^2 &= e^x + 6 \\ (e^x)^2 - e^x - 6 &= 0 \end{aligned}$$

Our equation is really a quadratic equation in e^x! The left-hand side factors as $\left(e^x - \boxed{?}\right)\left(e^x + \boxed{?}\right)$, so we are dealing with

$$e^x - \boxed{?} = 0 \qquad \text{and} \qquad e^x + \boxed{?} = 0.$$

For the first:

$$\begin{aligned} e^x &= \boxed{?} \\ x &= \ln\left(\boxed{?}\right). \end{aligned}$$

From the second: $e^x = \boxed{?}$. Look back at the graph of $y = e^x$ above. What was the range of the exponential function? It didn't include any negative numbers, so $e^x = -2$ has no solutions.

The solution to $e^{2x} = e^x + 6$ is $x = \boxed{?}$.

Problem 1. *Solve the equation:* $2\left(5^{2x} + 6\right) = 11 \cdot 5^x$.

Select All Correct Answers:

(a) $\log_5\left(\frac{3}{2}\right)$

(b) $\dfrac{\ln\left(\frac{3}{2}\right)}{5}$

(c) $\log_4(5)$

(d) $\log_5(4)$

(e) *The equation has no solutions.*

Example 100. *Solve the inequality:* $\dfrac{6^x - 7\cdot 3^x}{4^x - 15} \geq 0.$

Since this isn't a linear inequality, we'll solve it using a sign-chart. Luckily, the right-side is already 0. Let's factor the numerator on the left:

$$\begin{aligned} 6^x - 7\cdot 3^x &= (2\cdot 3)^x - 7\cdot 3^x \\ &= 2^x\cdot 3^x - 7\cdot 3^x \\ &= 3^x\left(2^x - 7\right). \end{aligned}$$

That means we need to construct a sign chart for $\dfrac{3^x\left(2^x-7\right)}{4^x - 15}$. (Note: $\log_2(7)$ is about 2.81 and $\log_4(15)$ is about 1.95.)

x		$\log_4(15)$	$\log_2(7)$
3^x	+	+	+
$2^x - 7$	−	−	+
$4^x - 15$	−	+	+

The solution is:

$$\left(-\infty, \boxed{?}\right) \bigcup \left[\boxed{?}, \infty\right)$$

Logarithmic equations

Example 101. *Solve the equation:* $\log_5(2x+1) = 3$.

Our first step will be to rewrite this logarithmic equation into its exponential form.

$$\log_5(2x+1) = 3 \qquad \text{means} \qquad 2x+1 = 5^{\boxed{?}}$$

From here we solve directly.

$$\begin{aligned} 2x+1 &= \boxed{?} \\ 2x &= \boxed{?} \\ x &= \boxed{?}. \end{aligned}$$

Example 102. *Solve the equation:*

$$\log_3(2x+1) = 1 - \log_3(x+2).$$

With more than one logarithm, we'll typically try to use the properties of logarithms to combine them into a single term.

$$\begin{aligned} \log_3(2x+1) &= 1 - \log_3(x+2) \\ \log_3(2x+1) + \log_3(x+2) &= 1 \\ \log_3\left((2x+1)(x+2)\right) &= 1 \\ \log_3\left(2x^2+5x+2\right) &= 1 \\ 2x^2+5x+2 &= 3 \\ 2x^2+5x-1 &= 0 \end{aligned}$$

Let's use quadratic formula to solve this.

$$x = \frac{-5 \pm \sqrt{5^2 - 4\cdot 2\cdot -1}}{2\cdot 2} = \frac{-5 \pm \sqrt{\boxed{?}}}{4}.$$

What happens if we try to plug $x = \dfrac{-5-\sqrt{33}}{4}$ into the equation? Both $2\left(\dfrac{-5-\sqrt{33}}{4}\right)+1$ and $\dfrac{-5-\sqrt{33}}{4}+2$ are negative. That means, the logarithms of these values is not defined.

It turns out that $\dfrac{-5-\sqrt{33}}{4}$ is a solution of the equation

$2x^2 + 5x - 1 = 0$, but not a solution of the original equation $\log_3(2x + 1) = 1 - \log_3(x + 2)$.

When working with logarithmic equations, we must always check that the solutions we find actually satisfy the original equation.

The only solution is $x = \dfrac{-5 + \sqrt{33}}{4}$.

17 Derivatives of exponential functions

After completing this section, students should be able to do the following.

- State the derivative of the natural exponential function.
- State the derivative of exponential functions with other bases.

Break-Ground:

17.1 Interesting changes

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, remember when I asked you about compound interest?
Riley: Yes I do! You were dealing with a formula like

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Devyn: I was. With monthly compoundings and 3% interest with a principal of \$1, we this to just

$$A = (1.0025)^{12t}.$$

Riley: I remember that. What about it?
Devyn: As time passes, the account is worth more and more.
Riley: Makes sense. You are gaining interest each month.
Devyn: But how much? How fast is it growing?
Riley: You are thinking about rates of change!
Devyn: Right! How do we find the derivative $\frac{dA}{dt}$?
Riley: It isn't a polynomial, so we can't use Power Rule. What can we do?

Problem 1. *What is the average rate of change of*

$$A = (1.0025)^{12t}$$

over the first 2 years (so from $t = 0$ to $t = 2$)? (ROUND TO 2 DECIMAL PLACES)

Multiple Choice:

(a) 0.03.

(b) 0.06.

(c) 1.06.

(d) 1.

Problem 2. *What are the units of that average rate of change?*

Multiple Choice:

(a) *dollars*

(b) *years*

(c) *dollars/year*

(d) *years/dollar*

(e) *It does not have a unit*

Dig-In:

17.2 The derivative of the natural exponential function

We don't know anything about derivatives that allows us to compute the derivatives of exponential functions without getting our hands dirty. Let's do a little work with the definition of the derivative:

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}\\ &= \lim_{h\to 0}\frac{a^x a^h - a^x}{h}\\ &= \lim_{h\to 0} a^x\frac{a^h-1}{h}\\ &= a^x \lim_{h\to 0}\frac{a^h-1}{h}\\ &= a^x\cdot \underbrace{(\text{constant})}_{\lim_{h\to 0}\frac{a^h-1}{h}}\end{aligned}$$

There are two interesting things to note here: We are left with a limit that involves h but not x, which means that whatever $\lim_{h\to 0}(a^h-1)/h$ is, we know that it is a number, or in other words, a constant. This means that a^x has a remarkable property:

> **The derivative of an exponential function is a constant times itself.**

Unfortunately it is beyond the scope of this text to compute the limit

$$\lim_{h\to 0}\frac{a^h-1}{h}.$$

However, we can look at some examples. Consider $(2^h-1)/h$ and $(3^h-1)/h$:

h	$(2^h-1)/h$
-1	$.5$
-0.1	≈ 0.6700
-0.01	≈ 0.6910
-0.001	≈ 0.6929
-0.0001	≈ 0.6931
-0.00001	≈ 0.6932

h	$(2^h-1)/h$
1	1
0.1	≈ 0.7177
0.01	≈ 0.6956
0.001	≈ 0.6934
0.0001	≈ 0.6932
0.00001	≈ 0.6932

h	$(3^h-1)/h$
-1	≈ 0.6667
-0.1	≈ 1.0404
-0.01	≈ 1.0926
-0.001	≈ 1.0980
-0.0001	≈ 1.0986
-0.00001	≈ 1.0986

h	$(3^h-1)/h$
1	2
0.1	≈ 1.1612
0.01	≈ 1.1047
0.001	≈ 1.0992
0.0001	≈ 1.0987
0.00001	≈ 1.0986

While these tables don't prove that we have a pattern, it turns out that

$$\lim_{h\to 0}\frac{2^h-1}{h}\approx .7 \qquad \text{and} \qquad \lim_{h\to 0}\frac{3^h-1}{h}\approx 1.1.$$

Moreover, if you do more examples, choosing other values for the base a, you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a=2$ and $a=3$ the limit will be exactly 1. This happens when

$$a = e = 2.718281828459045\dots.$$

We will define the number e by this property, see the next definition:

Definition. The number denoted by e, called **Euler's number**, is defined to be the number satisfying the following relation

$$\lim_{h\to 0}\frac{e^h-1}{h}=1.$$

Using this definition, we see that the function e^x has the following truly remarkable property.

Theorem 22 (The derivative of the natural exponential function)**.** *The derivative of the natural exponential function is the natural exponential function itself. In other words,*

$$\frac{d}{dx}e^x = e^x.$$

From the limit definition of the derivative, write

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^xe^h-e^x}{h}\\ &= \lim_{h\to 0} e^x\frac{e^h-1}{h}\\ &= e^x\lim_{h\to 0}\frac{e^h-1}{h}\\ &= e^x.\end{aligned}$$

Hence e^x is its own derivative. In other words, the slope of the plot of e^x is the same as its height, or the same as its second coordinate. Said another way, the function $f(x) = e^x$ goes through the point (a, e^a) and has slope e^a at that point, no matter what a is.

Question 65. *What is the slope of the tangent line to the function $f(x) = e^x$ at $x = 5$?*

Example 103. *Compute:*

$$\frac{d}{dx}\left(8\sqrt{x}+7e^x\right)$$

Write with me:

$$\begin{aligned}\frac{d}{dx}\left(8\sqrt{x}+7e^x\right) &= 8\frac{d}{dx}x^{1/2}+7\frac{d}{dx}e^x\\ &= 4x^{-1/2}+7e^x.\end{aligned}$$

Example 104. *Compute:*

$$\frac{d}{dx}e^{4x\sqrt{x+1}}$$

We know the derivative of e^x, but we're being asked the derivative of $e^{4x\sqrt{x+1}}$. The x has been replaced by a function of x. We think of $e^{4x\sqrt{x+1}}$ as a composite function with $f(x) = e^x$ and $g(x) = 4x\sqrt{x+1}$. To take the derivative of a composite function, we have chain rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$.

We know $f'(x) = \frac{d}{dx}e^x = e^x$ and $g'(x) = \frac{d}{dx}\left(4x\sqrt{x+1}\right)$ which we calculate by product rule as $4\sqrt{x+1}+4x\frac{1}{2\sqrt{x+1}}\cdot 1$ (where we used chain rule AGAIN inside the square root). Our answer is then:

$$e^{4x\sqrt{x+1}}\left(4\sqrt{x+1}+\frac{2x}{\sqrt{x+1}}\right).$$

Dig-In:

17.3 Derivatives of exponential and logarithmetic functions

The Derivative of the Natural Logarithm

We do not yet have a shortcut formula for the derivative of the natural logarithm, so let's start from the definition. Set $f(x) = \ln x$.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\ln(x+h)-\ln(x)}{h}\\ &= \frac{\ln\left(\frac{x+h}{x}\right)}{h}\\ &= \frac{1}{h}\cdot\ln\left(\frac{x+h}{x}\right)\\ &= \ln\left(\left(\frac{x+h}{x}\right)^{1/h}\right).\end{aligned}$$

That is,

$$f'(x) = \lim_{h\to 0}\ln\left(\left(\frac{x+h}{x}\right)^{1/h}\right) = \ln\left(\lim_{h\to 0}\left(\frac{x+h}{x}\right)^{1/h}\right).$$

The limit inside the logarithm is a bit beyond what we can deal with right now, so unless we can come up with a different strategy, we're stuck.

What do we know about this logarithm? We know that the natural logarithm function $f(x) = \ln x$ is the inverse of the exponential function e^x That is,

$$y = \ln(x) \qquad \text{means} \qquad e^y = x.$$

Since we're trying to find the derivative of $\ln x$, that means we're trying to find y'. Rather than working with the logarithmic version of $y = \ln x$, let's try to work with its exponential version $e^y = x$. We'll start by taking the derivative of both sides

$$\frac{d}{dx}\left(e^y\right) = \frac{d}{dx}(x).$$

The right-hand side is easy, but what about the left-hand side? If we think of y as just a function of x, then the left-hand side is the exponential e^x with the x replaced by a function. It's a Chain Rule problem, when we think of e^y as having an outside function $g(x) = e^x$ and an inside function $f(x) = y$. By Chain Rule, $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x) = e^y y'$. Let's put all this together.

$$\begin{aligned}e^y &= x\\ \frac{d}{dx}\left(e^y\right) &= \frac{d}{dx}(x)\\ e^y y' &= 1\\ y' &= \frac{1}{e^y}.\end{aligned}$$

We notice once again that $e^y = x$, so $\frac{1}{e^y} = \frac{1}{x}$. This gives our derivative formula.

Theorem 23 (The derivative of the natural logarithm function)**.** *The derivative of the natural logarithm function is given by*

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

for $x > 0$.

Example 105. *Compute* $\frac{d}{dx}\ln(2x+1)$.

> This is a Chain Rule question with outer function $f(x) = \ln x$ and inner function $g(x) = 2x+1$. We know $f'(x) = \frac{1}{x}$ and $g'(x) = 2$, so that
>
> $$\frac{d}{dx}\ln(2x+1) = \frac{2}{2x+1}.$$

Question 66. *Compute* $\frac{d}{dx}e^{3x}\ln\left(4x^2+3\right)$.

Multiple Choice:

(a) $e^{3x}\frac{1}{4x^2+3}$

(b) $3e^{3x}\dfrac{8x}{4x^2+3}$

(c) $3e^{3x}\ln(4x^2+3)+e^{3x}\dfrac{8x}{4x^2+3}$

(d) $3e^{3x}+\dfrac{8x}{4x^2+3}$

The Derivative of Exponentials and Logarithms with Other Bases

We have found derivative formulas for the natural exponential function e^x and the natural logarithm function $\ln x$, but we have not yet explored other bases. That will be our focus for the rest of the section.

For exponentials, we remember that any number $b > 0$ can be written in the form e^x for some specific value of x. To determine the x, we solve the equation $b = e^x$ so $x = \ln b$. That is, $b = e^{\ln b}$.

$$\begin{aligned} b^x &= (b)^x \\ &= \left(e^{\ln b}\right)^x \\ &= e^{x \ln b}. \end{aligned}$$

To find $\dfrac{d}{dx}b^x$ we are finding $\dfrac{d}{dx}e^{x\ln b}$, which we know by Chain Rule.

$$\begin{aligned} \frac{d}{dx}b^x &= \frac{d}{dx}e^{x\ln b} \\ &= e^{x\ln b}\cdot\frac{d}{dx}(x\ln b) \\ &= e^{x\ln b}\cdot(\ln b). \end{aligned}$$

Rewriting $e^{x\ln b}$ as b^x we find our derivative formula.

Theorem 24. *The derivative of the exponential function b^x, for $b > 0$, $b \neq 1$ is given by:*

$$\frac{d}{dx}b^x = b^x \ln b.$$

Example 106. *Compute* $\dfrac{d}{dx}\sqrt{2}^x$.

> This is directly from our formula with $b = \sqrt{2}$. We get $\sqrt{2}^x \ln\sqrt{2}$.

Example 107. *Compute* $\dfrac{d}{dx}3^{x^2+2x}$.

> This is the $b = 3$ situation, but the x has been replaced by x^2+2x. That means we'll need Chain Rule, too.
>
> $$\begin{aligned} \frac{d}{dx}3^{x^2+2x} &= 3^{x^2+2x}\cdot\ln(3)\cdot\frac{d}{dx}\left(x^2+2x\right) \\ &= 3^{x^2+2x}\ln(3)(2x+2). \end{aligned}$$

To deal with logarithms of other bases, we rely on the change of base formula:

Theorem 25 (The change of base formula)**.** *Let $b > 0$ with $b \neq 1$. Then for any $a > 0$ and $a \neq 1$ and any $x > 0$,*

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

This formula allows us to replace a logarithm with one base with a logarithm with whatever base we want. There is one base that we like more than the rest, base e. This means $\log_b x = \dfrac{\ln x}{\ln b}$.

$$\begin{aligned} \frac{d}{dx}\log_b x &= \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) \\ &= \frac{1}{\ln b}\cdot\frac{d}{dx}(\ln x) \\ &= \frac{1}{\ln b}\cdot\frac{1}{x}. \end{aligned}$$

This gives our derivative formula.

Theorem 26. *The derivative of the logarithmic function $\log_b x$, for $b > 0$, $b \neq 1$ is given by:*

$$\frac{d}{dx}log_b x = \frac{1}{x\ln b}.$$

Example 108. *Compute* $\dfrac{d}{dx}\log_7(2x-3)$.

> This is the formula for $b = 7$, with x replaced by $2x - 3$. We will need Chain Rule with inner function $g(x) = 2x - 3$ and outer function $f(x) = \log_7 x$. We know $g'(x) = 2$ and $f'(x) = \dfrac{1}{x\ln(7)}$. Then The derivative is
>
> $$\frac{2}{(2x-3)\ln 7}.$$

Example 109. *Compute* $\dfrac{d}{dx}5^{2x}\log_{\frac{1}{3}}\left(6x^3-2x\right)$.

> The outer most operation in this function is multiplication, so this is a product rule question. We know the derivative of 5^{2x} is $5^{2x}\cdot 2$, and the derivative of $\log_{\frac{1}{3}}\left(6x^3-2x\right)$ is $\dfrac{1}{(6x^3-2x)\ln\left(\frac{1}{3}\right)}\cdot\left(18x^2-2\right))$. The whole derivative is
>
> $$2\cdot 5^{2x}\cdot\log_{\frac{1}{3}}\left(6x^3-2x\right)+5^{2x}\cdot\frac{18x^2-2}{(6x^3-2x)\ln\left(\frac{1}{3}\right)}.$$

Question 67. *Compute* $\dfrac{d}{dx}6^{x^2}\log_5(x+3)$.

Multiple Choice:

(a) $6^{2x}\dfrac{1}{(x+3)\ln 5}$

(b) $2x\cdot 6^{x^2}\log_5(x+3)+6^{x^2}\dfrac{1}{(x+3)\ln 5}$

(c) $x^2\cdot 6^{x^2-1}\log_5(x+3)+6^{x^2}\dfrac{1}{(x+3)\ln 5}$

(d) $2x\cdot 6^{x^2}\log_5(x+3)+6^{x^2}\dfrac{5}{x+3}$

18 Higher order derivatives and graphs

After completing this section, students should be able to do the following.

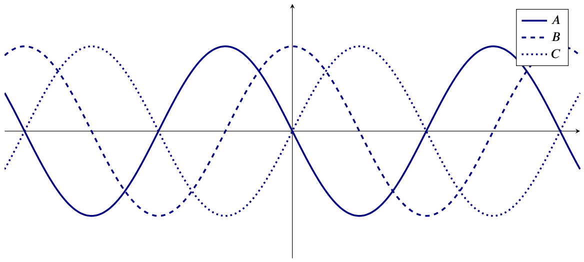
- Use the first derivative to determine whether a function is increasing or decreasing.
- Define higher order derivatives.
- Compare differing notations for higher order derivatives.
- Identify the relationships between the function and its first and second derivatives.
- Sketch a graph of the second derivative, given the original function.
- Sketch a graph of the original function, given the graph of its first and second derivatives.
- Sketch a graph of a function satisfying certain constraints on its higher-order derivatives.
- State the relationship between concavity and the second derivative.
- Interpret the second derivative of a position function as acceleration.
- Calculate higher order derivatives.

Break-Ground:

18.1 Rates of rates

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, check out this plot:



Riley: Whoa, that looks cool!
Devyn: I know! Anyway, someone told me that this was a graph of a function, its derivative, and the derivative of the derivative. Which one is which?
Riley: Hmmmmm. I'm not sure.

Problem 1. *Which of the following is true?*

Multiple Choice:

(a) *Curve A is increasing when curve B is positive.*

(b) *Curve A is increasing when curve C is positive.*

(c) *None of the above.*

Problem 2. *Which of the following is true?*

Multiple Choice:

(a) *Curve B is increasing when curve A is positive.*

(b) *Curve B is increasing when curve C is positive.*

(c) *None of the above.*

Problem 3. *Which of the following is true?*

Multiple Choice:

(a) *Curve C is increasing when curve A is positive.*

(b) *Curve C is increasing when curve B is positive.*

(c) *None of the above.*

Dig-In:

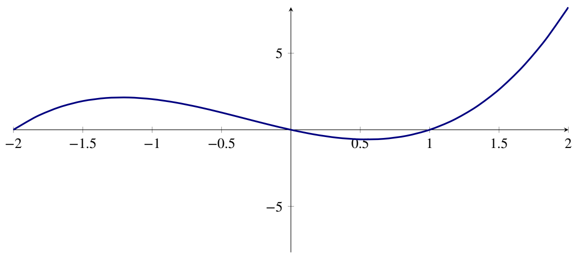
18.2 Higher order derivatives and graphs

Since the derivative gives us a formula for the slope of a tangent line to a curve, we can gain information about a function purely from the sign of the derivative. In particular, we have the following theorem

Theorem 27. *If f is differentiable on an interval, then*

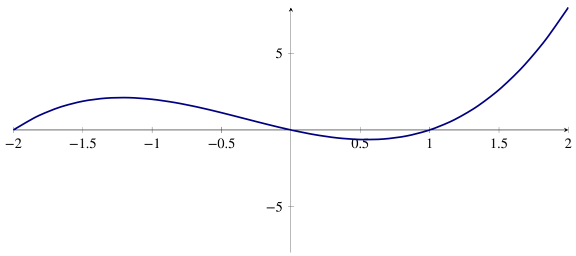
- *$f'(x) > 0$ on that interval whenever f is increasing as x increases on that interval.*
- *$f'(x) < 0$ on that interval whenever f is decreasing as x increases on that interval.*

Question 68. *Below we have graphed $y = f(x)$:*



Is the first derivative positive or negative on the interval $-1 < x < 1/2$?

Question 69. *Below we have graphed $y = f'(x)$:*



Is the graph of $f(x)$ increasing or decreasing as x increases on the interval $-1 < x < 0$?

We call the derivative of the derivative the **second derivative**, the derivative of the derivative of the derivative the **third derivative**, and so on. We have special notation for higher derivatives, check it out:

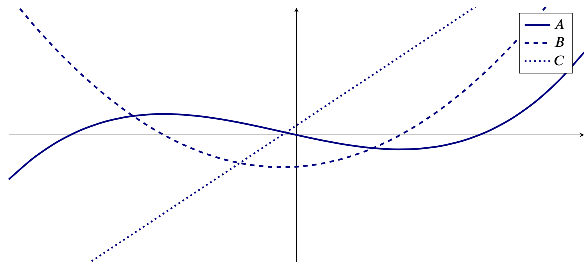
First derivative: $\dfrac{d}{dx} f(x) = f'(x) = f^{(1)}(x).$

Second derivative: $\dfrac{d^{\,2}}{dx^2} f(x) = f''(x) = f^{(2)}(x).$

Third derivative: $\dfrac{d^{\,3}}{dx^3} f(x) = f'''(x) = f^{(3)}(x).$

We use the facts above in our next example.

Example 110. *Here we have unlabeled graphs of f, f', and f'':*

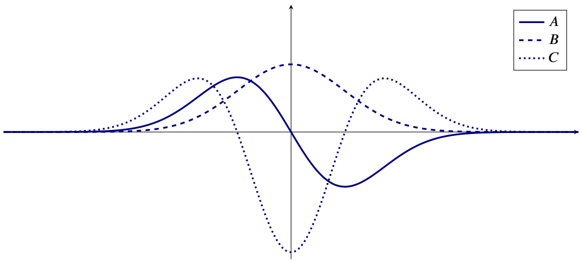


Identify each curve above as a graph of f, f', or f''.

> Here we see three curves, A, B, and C. Since A is increasing when B is positive and decreasing when B is negative, we see
> $$A' = B.$$
> Since B is increasing when C is positive and decreasing when C is negative, we see
> $$B' = C.$$

Hence $f = A$, $f' = B$, and $f'' = C$.

Example 111. *Here we have unlabeled graphs of f, f', and f'':*



Identify each curve above as a graph of f, f', or f''.

Here we see three curves, A, B, and C. Since B is increasing when A is positive and decreasing when A is negative, we see

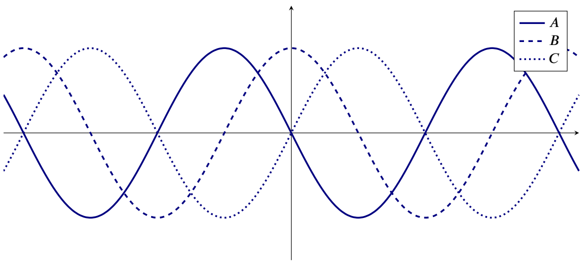
$$B' = A.$$

Since A is increasing when C is positive and decreasing when C is negative, we see

$$A' = C.$$

Hence $f = B$, $f' = A$, and $f'' = C$.

Example 112. *Here we have unlabeled graphs of f, f', and f'':*



Identify each curve above as a graph of f, f', or f''.

Here we see three curves, A, B, and C. Since C is increasing when B is positive and decreasing when B is negative, we see

$$C' = B.$$

Since B is increasing when A is positive and decreasing when A is negative, we see

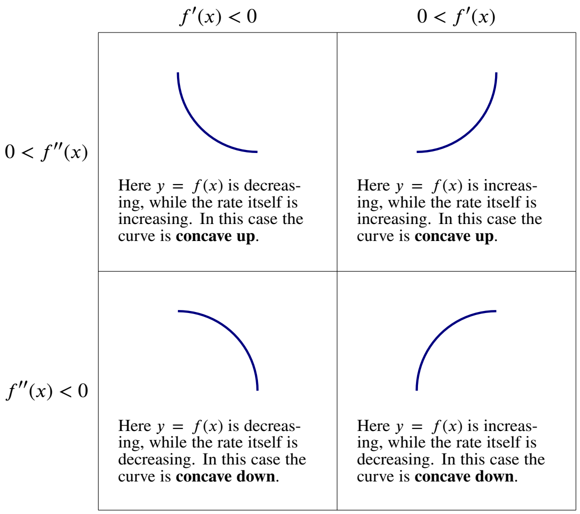
$$B' = A.$$

Hence $f = C$, $f' = B$, and $f'' = A$.

Dig-In:

18.3 Concavity

We know that the sign of the derivative tells us whether a function is increasing or decreasing at some point. Likewise, the sign of the second derivative $f''(x)$ tells us whether $f'(x)$ is increasing or decreasing at x. We summarize the consequences of this seemingly simple idea in the table below:



If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 28 (Test for Concavity)**.** *Suppose that $f''(x)$ exists on an interval.*

(a) *$f''(x) > 0$ on that interval whenever $y = f(x)$ is concave up on that interval.*

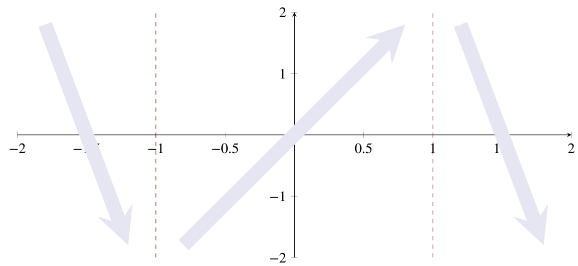
(b) *$f''(x) < 0$ on that interval whenever $y = f(x)$ is concave down on that interval.*

Example 113. *Let f be a continuous function and suppose that:*

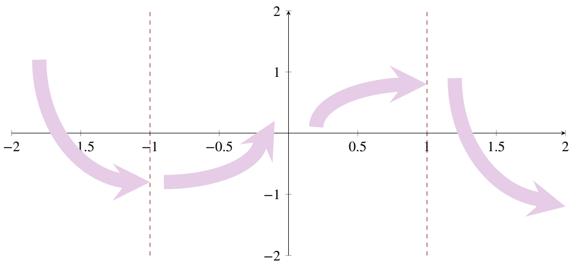
- *$f'(x) > 0$ for $-1 < x < 1$.*
- *$f'(x) < 0$ for $-2 < x < -1$ and $1 < x < 2$.*
- *$f''(x) > 0$ for $-2 < x < 0$ and $1 < x < 2$.*
- *$f''(x) < 0$ for $0 < x < 1$.*

Sketch a possible graph of f.

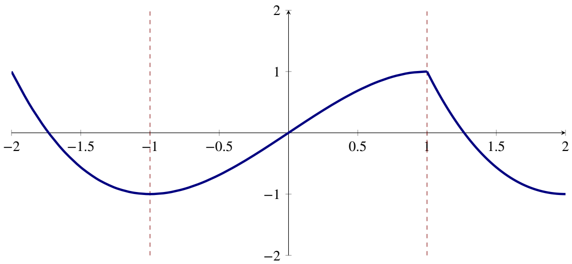
Start by marking where the derivative changes sign and indicate intervals where f is increasing and intervals f is decreasing. The function f has a negative derivative from -2 to $x = -1$. This means that f is decreasing on this interval. The function f has a positive derivative from $x = -1$ to $x = 1$. This means that f is increasing on this interval. Finally, The function f has a negative derivative from $x = 1$ to 2. This means that f is decreasing on this interval.



Now we should sketch the concavity: concave up when the second derivative is positive, concave down when the second derivative is negative.



Finally, we can sketch our curve:



Dig-In:

18.4 Position, velocity, and acceleration

Studying functions and their derivatives might seem somewhat abstract. However, consider this passage from a physics book:

> Assuming acceleration a is constant, we may write velocity and position as
>
> $$v(t) = v_0 + at,$$
> $$x(t) = x_0 + v_0 t + (1/2)at^2,$$
>
> where a is the (constant) acceleration, v_0 is the position at time zero, and x_0 is the position at time zero.

These equations model the position and velocity of any object with constant acceleration. In particular these equations can be used to model the motion a falling object, since the acceleration due to gravity is constant.

Calculus allows us to see the connection between these equations. First note that the derivative of the formula for position with respect to time, is the formula for velocity with respect to time.

$$x'(t) = v_0 + at = v(t).$$

Moreover, the derivative of formula for velocity with respect to time, is simply a, the acceleration.

Question 70. *Suppose that s represents the position of a ball tossed at time $t = 0$. Recalling that the acceleration for $t > 0$ is only due to gravity, and knowing that the acceleration due to gravity is $-9.8\ \mathrm{m/s^2}$, what is $s''(t)$?*

Example 114. *You recently took a road trip from Columbus Ohio to Urbana-Champaign Illinois. The distance traveled from Columbus Ohio is roughly modeled by:*

$$s(t) = 36t^2 - 4.8t^3 \qquad \textit{(miles West of Columbus)}$$

where t is measured in hours, and is between 0 and 6. Find a formula for your acceleration.

Here we simply need to find the second derivative:

$$s'(t) = 72t - 14.4t^2 \qquad \textit{and} \qquad s''(t) = 72 - 28.8t$$

Hence our acceleration is $72 - 28t$ miles/hour^2.

19 Trigonometric Functions

After completing this section, students should be able to do the following.

- Understand the properties of trigonometric functions.
- Evaluate expressions and solve equations involving trigonometric functions.
- Evaluate limits involving trigonometric functions.

Break-Ground:

19.1 Trig Function BreakGround

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, do you remember when we first starting graphing functions? Like with a "T-chart?"
Riley: I remember everything.
Devyn: ...
Riley: ...
Devyn:
Riley:

Problem 1. *statement of problem*

Multiple Choice:

(a) *choice 1*

(b) *choice 2*

(c) *choice 3*

(d) *choice 4*

(e) *correct choice 5*

Problem 2. *another question*

Multiple Choice:

(a) *The graph starts in the lower left and ends in the upper right of the plane.*

(b) *The graph starts in the lower right and ends in the upper left of the plane.*

(c) *The graph looks something like the letter "U."*

(d) *The graph looks something like an upside down letter "U."*

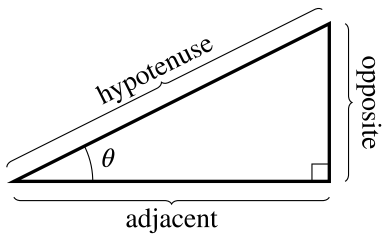
Dig-In:

19.2 Trigonometric functions

What are trigonometric functions?

Definition. A **trigonometric function** is a function that relates a measure of an angle of a right triangle to a ratio of the triangle's sides.

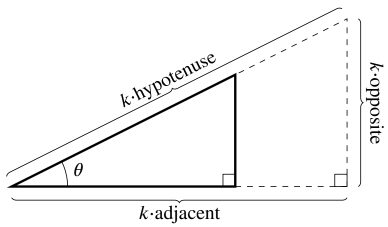
The basic trigonometric functions are cosine and sine. They are called "trigonometric" because they relate measures of angles to measurements of triangles. Given a right triangle



we define

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \text{and} \qquad \sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}.$$

Note, the values of sine and cosine do not depend on the scale of the triangle. Being very explicit, if we scale a triangle by a scale factor k,

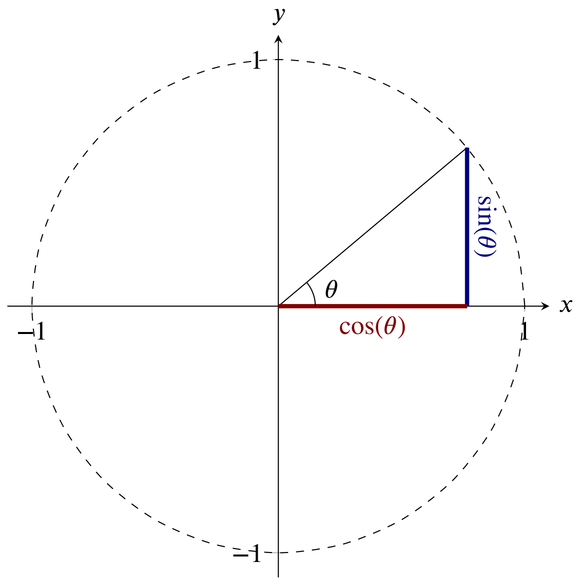


$$\cos(\theta) = \frac{k \cdot \text{adjacent}}{k \cdot \text{hypotenuse}} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

and

$$\sin(\theta) = \frac{k \cdot \text{opposite}}{k \cdot \text{hypotenuse}} = \frac{\text{opposite}}{\text{hypotenuse}}.$$

At this point we could simply assume that whenever we draw a triangle for computing sine and cosine, that the hypotenuse will be 1. We can do this because we are simply scaling the triangle, and as we see above, this makes absolutely no difference when computing sine and cosine. Hence, when the hypotenuse is 1, we find that a convenient way to think about sine and cosine is via the unit circle:



If we consider all possible combinations of ratios of

$$\text{adjacent}, \qquad \text{opposite}, \qquad \text{hypotenuse},$$

(allowing the adjacent and opposite to be negative, as on the unit circle) we obtain all of the trigonometric functions.

Definition. The trigonometric functions are:

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \sin(\theta) = \frac{\text{opp}}{\text{hyp}}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

where the domain of sine and cosine is all real numbers, and the other trigonometric functions are defined precisely when their denominators are nonzero.

Question 71. *Which of the following expressions are equal to* $\sec(\theta)$*?*

Select All Correct Answers:

(a) $\dfrac{1}{\cos(\theta)}$

(b) $\dfrac{1}{\sin(\theta)}$

(c) $\dfrac{adj}{hyp}$

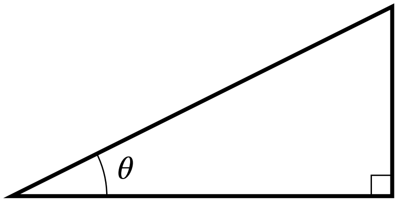
(d) $\dfrac{hyp}{adj}$

(e) $\dfrac{\tan(\theta)}{\sin(\theta)}$

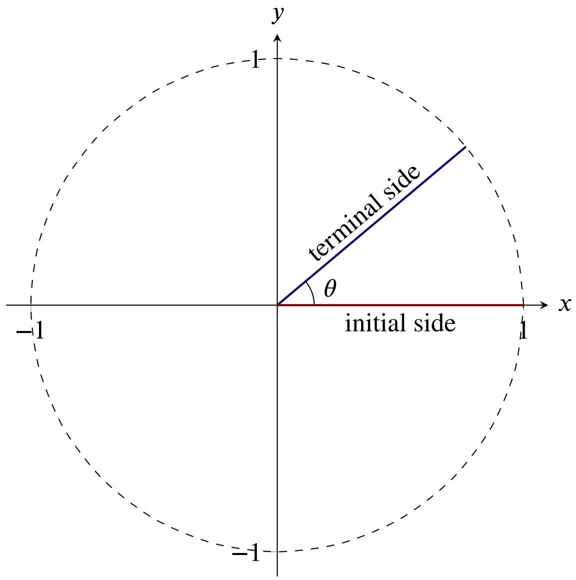
(f) $\dfrac{1}{\sin(\theta)\cdot\cot(\theta)}$

Not all angles come from triangles.

Given a right triangle like



the angle θ cannot exceed $\frac{\pi}{2}$ radians. That means to talk about trigonometric functions for *other* angles, we need to be able to describe the trigonometric functions a little more generally. To do this, we use the unit circle from the previous section. Given an angle θ, we construct the angle with initial side along the positive x-axis and vertex at the origin.



As the angle θ grows larger and larger, the terminal side of that angle spins around the circle. The trigonometric functions of the angle θ are defined in terms of the terminal side.

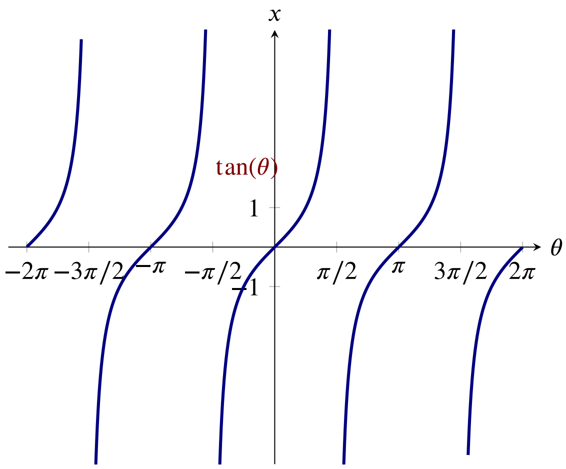
Definition. Suppose (x, y) is the point at which the terminal side of the angle with measure θ intersects the unit circle. The

the trigonometric functions are defined to be

$$\cos(\theta) = x \qquad \sin(\theta) = y$$
$$\sec(\theta) = \frac{1}{x} \qquad \csc(\theta) = \frac{1}{y}$$
$$\tan(\theta) = \frac{y}{x} \qquad \cot(\theta) = \frac{x}{y}$$

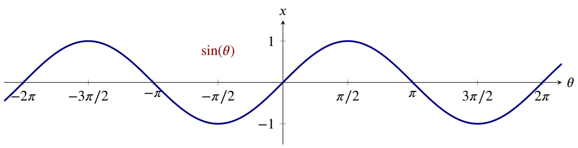
provided these values exist.

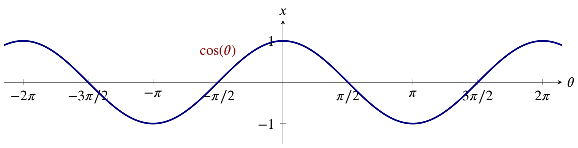
From the picture, you see that this agrees with what you know about trigonometry for triangles, but it allows us to extend the definition of sine and cosine to all real numbers, instead of only the interval $\left(0, \frac{\pi}{2}\right)$

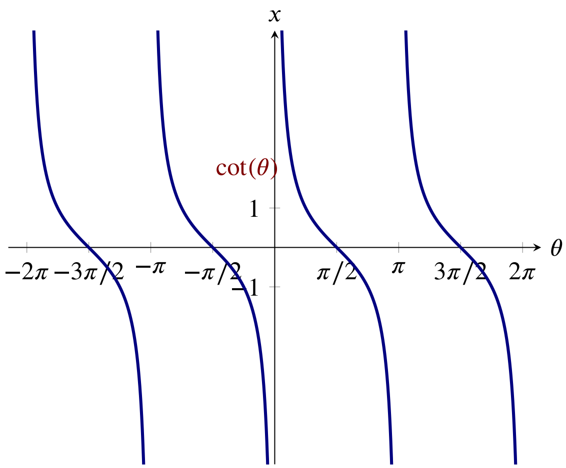


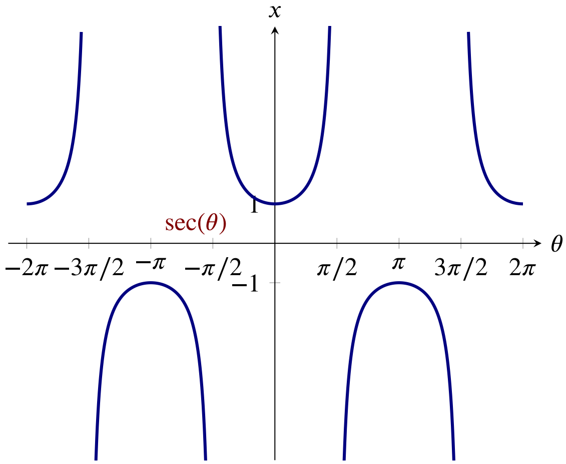
Graphs

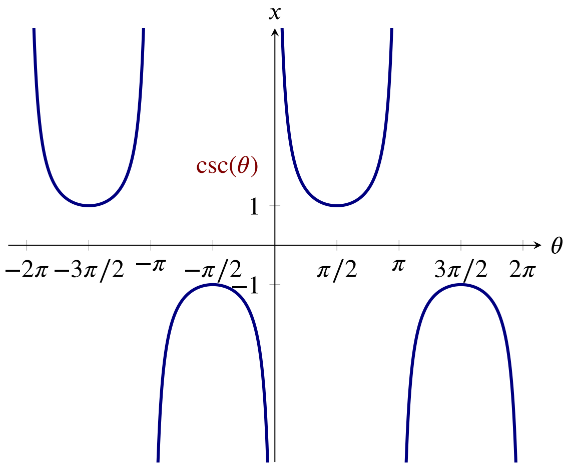
As a reminder, we include the graphs here.







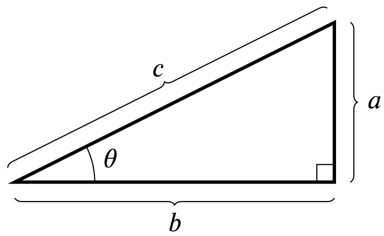




The power of the Pythagorean Theorem

The Pythagorean Theorem is probably the most famous theorem in all of mathematics.

Theorem 29 (Pythagorean Theorem)**.** *Given a right triangle:*



We have that:

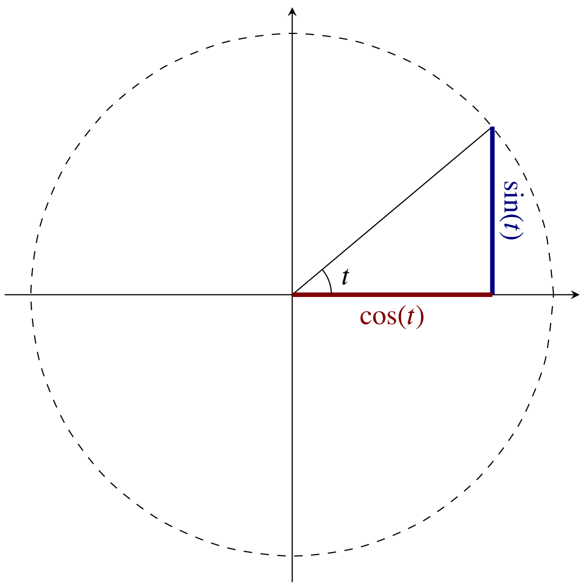
$$a^2 + b^2 = c^2$$

The Pythagorean Theorem gives several key trigonometric identities.

Theorem 30 (Pythagorean Identities)**.** *The following hold:*

$$\cos^2\theta + \sin^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

From the unit circle we can see



via the Pythagorean Theorem that

$$\cos^2(t) + \sin^2(t) = 1.$$

If we divide this expression by $\cos^2(t)$ we obtain

$$1 + \tan^2(t) = \sec^2(t)$$

and if we divide $\cos^2(t)+\sin^2(t) = 1$ by $\sin^2(t)$ we obtain

$$\cot^2(t) + 1 = \csc^2(t).$$

There several other trigonometric identities that appear on occasion.

Theorem 31 (Angle Addition Formulas)**.**

$$\begin{aligned}\sin(s+t) &= \sin(s)\cos(t) + \cos(s)\sin(t)\\ \cos(s+t) &= \cos(s)\cos(t) - \sin(s)\sin(t)\end{aligned}$$

If we plug $s = t = \theta$ into the angle addition formulas, we find the double-angle identities.

Theorem 32 (Double-Angle Identities)**.**

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= 2\cos^2(\theta) - 1\\ &= 1 - 2\sin^2(\theta)\end{aligned}$$

Solving the bottom two formulas for $\cos^2(\theta)$ and $\sin^2(\theta)$ gives the half-angle identities.

Theorem 33 (Half-Angle Identities)**.**

$$\begin{aligned}\sin^2(\theta) &= \frac{1-\cos(2\theta)}{2}\\ \cos^2(\theta) &= \frac{1+\cos(2\theta)}{2}\end{aligned}$$

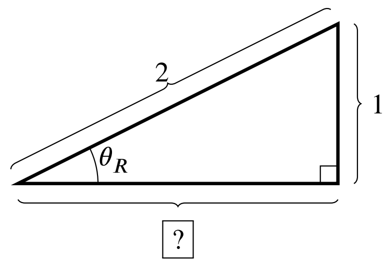
Trigonometric equations

Frequently we are in the situation of having to determine precisely which angles satisfy a particular equation. The most basic example is probably like this one.

Example 115. *Solve the equation:*

$$\sin\theta = -\frac{1}{2}.$$

We'll start by finding the reference angle, θ_R, the acute angle between the terminal side of θ and the x-axis. The reference angle satisfies $\sin\theta_R = \dfrac{1}{2}$.



From the picture we see $\theta_R = \dfrac{\pi}{6}$. In one period $[0, 2\pi]$, there are two angles that have reference angle $\dfrac{\pi}{6}$ and have negative sine value. One is in quadrant 3, and one in quadrant 4. That means the solutions in the interval $[0, 2\pi]$ are $\dfrac{5\pi}{6}$ and $\dfrac{11\pi}{6}$.

To find all solutions, we have to add all multiples of 2π to these. The solutions are then

$$\theta = \frac{5\pi}{6} + 2\pi k,\ \ \frac{11\pi}{6} + 2\pi k,\ \ k \text{ any integer.}$$

Problem 3. *Solve the equation:*

$$\tan\theta = -\sqrt{3}.$$

Multiple Choice:

(a) $\theta = \dfrac{\pi}{3} + \pi k$

(b) $\theta = \dfrac{\pi}{6} + \pi k$

(c) $\theta = \dfrac{2\pi}{3} + \pi k$

(d) $\theta = \dfrac{5\pi}{6} + \pi k$

(e) *None of the above*

Let's try one a bit more complicated.

Example 116. *Solve the equation:*

$$\cos\theta\,(\cos\theta + 1) = \sin^2\theta.$$

We'll start by simplifying a bit.

$$\begin{aligned}
\cos\theta\,(\cos\theta + 1) &= \sin^2\theta \\
\cos^2\theta + \cos\theta &= \sin^2\theta \\
\cos^2\theta - \sin^2\theta + \cos\theta &= 0 \\
\cos^2\theta - \left(1 - \cos^2\theta\right) + \cos\theta &= 0 \\
2\cos^2\theta + \cos\theta - 1 &= 0.
\end{aligned}$$

Notice that this equation is quadratic in $\cos\theta$. We can factor it like we try to do to solve any other quadratic equation:

$$(\cos\theta + 1)(2\cos\theta - 1) = 0.$$

On the interval $[0, 2\pi]$, $\cos\theta = -1$ has only one solution, $\theta = \pi$. For $\cos\theta = \dfrac{1}{2}$, we see that the reference angle $\theta_R = \dfrac{\pi}{6}$. Since cosine is positive in quadrants 1 and 4, we find solutions $\theta = \dfrac{\pi}{6}$ and $\dfrac{11\pi}{6}$.

All solutions are:

$$\theta = \pi + 2\pi k,\ \frac{\pi}{6} + 2\pi k,\ \frac{11\pi}{6} + 2\pi k,\ k \text{ any integer.}$$

Limits involving trigonometric functions

Back when we introduced continuity we mentioned that each trigonometric function is continuous on its domain.

Example 117. *Compute the limit:*

$$\lim_{\theta\to 2\pi/3} \theta\tan\theta.$$

The multiplicative limit law allows us to split this into

$$\left(\lim_{\theta\to 2\pi/3} \theta\right)\left(\lim_{\theta\to 2\pi/3} \tan\theta\right).$$

The function $f(\theta) = \theta$ is continuous everywhere, so $\displaystyle\lim_{\theta\to 2\pi/3} \theta = \frac{2\pi}{3}$. Since $\dfrac{2\pi}{3}$ is in the domain of $\tan\theta$, we have $\displaystyle\lim_{\theta\to 2\pi/3} \tan\theta = \tan\left(\frac{2\pi}{3}\right) = -\sqrt{3}$. Putting these together we find

$$\lim_{\theta\to 2\pi/3} \theta\tan\theta = -\frac{2\pi\sqrt{3}}{3}.$$

Example 118. *Compute the limit:*

$$\lim_{\theta\to\pi^-} \cot\theta.$$

Recall that $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, so that

$$\lim_{\theta\to\pi^-} \cot\theta = \lim_{\theta\to\pi^-} \frac{\cos\theta}{\sin\theta}.$$

Since sine and cosine are continuous, $\displaystyle\lim_{\theta\to\pi^-} \cos\theta =$

$\cos(\pi) = -1$ and $\lim_{\theta\to\pi^-} \sin\theta = \sin(\pi) = 0$. That is, $\lim_{\theta\to\pi^-} \cot\theta$ is of the form $\frac{\#}{\mathbf{0}}$.

The numerator is negative for θ near π. From the graph of $\sin\theta$, we know that the denominator is negative and approaching 0 as $\theta \to \pi^-$. That means

$$\lim_{\theta\to\pi^-} \cot\theta = -\infty.$$

Question 72. *Compute the limit:*

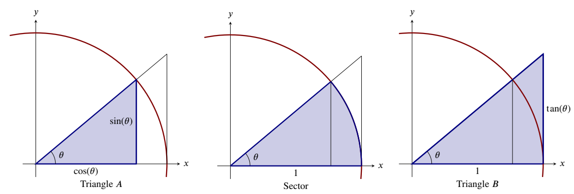
$$\lim_{\theta\to\frac{\pi}{12}} \frac{2\cos(4\theta)\sin(6\theta)}{\theta} = \boxed{?}$$

We'll end with a couple very involved limits where the Squeeze Theorem makes a surprising return.

Example 119. *Compute:*

$$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$$

To compute this limit, use the Squeeze Theorem. First note that we only need to examine $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and for the present time, we'll assume that θ is positive. Consider the diagrams below:



From our diagrams above we see that

Area of Triangle $A \leq$ Area of Sector $\leq$ Area of Triangle B

and computing these areas we find

$$\frac{\cos(\theta)\sin(\theta)}{2} \leq \frac{\theta}{2} \leq \frac{\tan(\theta)}{2}.$$

Multiplying through by 2, and recalling that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ we obtain

$$\cos(\theta)\sin(\theta) \leq \theta \leq \frac{\sin(\theta)}{\cos(\theta)}.$$

Dividing through by $\sin(\theta)$ and taking the reciprocals (reversing the inequalities), we find

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{\cos(\theta)}.$$

Note, $\cos(-\theta) = \cos(\theta)$ and $\frac{\sin(-\theta)}{-\theta} = \frac{\sin(\theta)}{\theta}$, so these inequalities hold for all $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Additionally, we know

$$\lim_{\theta\to 0} \cos(\theta) = 1 = \lim_{\theta\to 0} \frac{1}{\cos(\theta)},$$

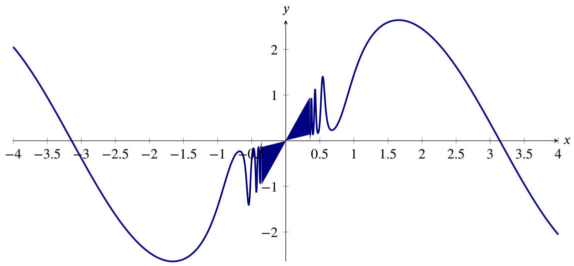
and so we conclude by the Squeeze Theorem, $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

When solving a problem with the Squeeze Theorem, one must write a sort of mathematical poem. You have to tell your friendly reader exactly which functions you are using to "squeeze-out" your limit.

Example 120. *Compute:*

$$\lim_{x\to 0} \left(\sin(x) e^{\cos\left(\frac{1}{x^3}\right)}\right)$$

Let's graph this function to see what's going on:



The function $\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}$ has two factors:

$$\overbrace{\sin(x)}^{\text{goes to zero as } x \to 0} \cdot \underbrace{e^{\cos\left(\frac{1}{x^3}\right)}}_{\text{bounded between } e^{-1} \text{ and } e}$$

Hence we have that when $x > 0$

$$\sin(x)e^{-1} \le \sin(x)e^{\cos\left(\frac{1}{x^3}\right)} \le \sin(x)e$$

and we see

$$\lim_{x\to 0^+} \sin(x)e^{-1} = 0 = \lim_{x\to 0^+} \sin(x)e$$

and so by the Squeeze theorem,

$$\lim_{x\to 0^+} \left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = 0.$$

In a similar fashion, when $x < 0$,

$$\sin(x)e \le \sin(x)e^{\cos\left(\frac{1}{x^3}\right)} \le \sin(x)e^{-1}$$

and so

$$\lim_{x\to 0^-} \sin(x)e = 0 = \lim_{x\to 0^-} \sin(x)e^{-1},$$

and again by the Squeeze Theorem $\lim_{x\to 0^-} \left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = 0$. Hence we see that

$$\lim_{x\to 0} \left(\sin(x)e^{\cos\left(\frac{1}{x^3}\right)}\right) = 0.$$

20 Derivatives of trigonometric functions

After completing this section, students should be able to do the following.

- Calculate the derivatives of trigonometric functions.

Break-Ground:

20.1 Trig Derivative BreakGround

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, do you remember when we first starting graphing functions? Like with a "T-chart?"
Riley: I remember everything.
Devyn: ...
Riley: ...
Devyn: ...
Riley: Hmmmm. I'm not sure…

Problem 1. *When x is a large number (furthest from zero), which term of $5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 2. *When x is a small number (near zero), which term of $5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 3. *Very roughly speaking, what does the graph of $y = 5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ look like?*

Multiple Choice:

(a) *The graph starts in the lower left and ends in the upper right of the plane.*

(b) *The graph starts in the lower right and ends in the upper left of the plane.*

(c) *The graph looks something like the letter "U."*

(d) *The graph looks something like an upside down letter "U."*

Dig-In:

20.2 The derivative of sine

It is now time to visit our two friends who concern themselves periodically with triangles and circles. In particular, we want to show that

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta).$$

Before we tackle this monster, let's remember a fact, and derive a new fact. You may initially be uncomfortable because you can't quite see why we need these results, but this style of exposition is a fact of technical writing; it is best to get used to it.

First, recall the fact that

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1.$$

Next, we will use this fact to derive our new fact:

Example 121.

$$\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = 0.$$

Write with me:

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} &= \lim_{\theta\to 0}\left(\frac{\cos(\theta)-1}{\theta}\cdot\frac{\cos(\theta)+1}{\cos(\theta)+1}\right)\\
&= \lim_{\theta\to 0}\frac{\cos^2(\theta)-1}{\theta(\cos(\theta)+1)}\\
&= \lim_{\theta\to 0}\frac{-\sin^2(\theta)}{\theta(\cos(\theta)+1)}\\
&= -\lim_{\theta\to 0}\left(\frac{\sin(\theta)}{\theta}\cdot\frac{\sin(\theta)}{(\cos(\theta)+1)}\right)\\
&= -\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin(\theta)}{(\cos(\theta)+1)}\\
&= -1\cdot\frac{0}{2} = 0.
\end{aligned}$$

After these delicious appetizers, we are now ready for the main course.

Theorem 34 (The derivative of sine)**.** *For any angle θ measured in radians, the derivative of* $\sin(\theta)$ *with respect to θ is* $\cos(\theta)$*. In other words,*

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta).$$

Using the definition of the derivative, write with me

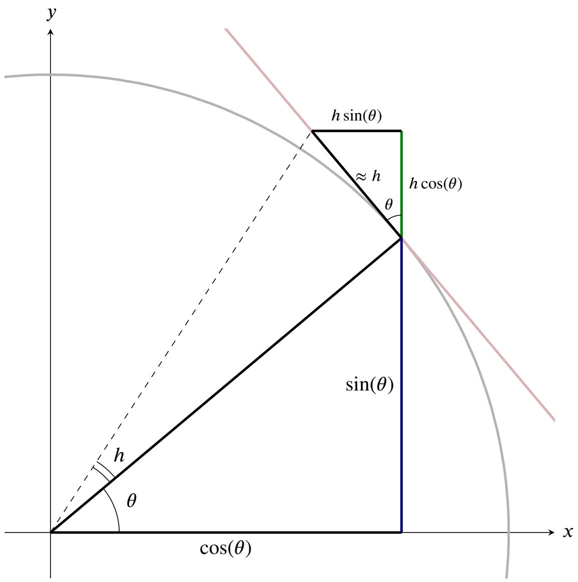
$$\frac{d}{d\theta}\sin(\theta) = \lim_{h\to 0}\frac{\sin(\theta+h)-\sin(\theta)}{h}$$

Now we get sneaky and apply the trigonometric addition formula for sine, that says $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$, to write

$$\begin{aligned}
&= \lim_{h\to 0}\frac{\sin(\theta)\cos(h)+\sin(h)\cos(\theta)-\sin(\theta)}{h}\\
&= \lim_{h\to 0}\left(\frac{\sin(\theta)\cos(h)-\sin(\theta)}{h}+\frac{\sin(h)\cos\theta}{h}\right)\\
&= \lim_{h\to 0}\left(\sin(\theta)\frac{\cos(h)-1}{h}+\cos(\theta)\frac{\sin(h)}{h}\right)\\
&= \sin(\theta)\cdot 0+\cos(\theta)\cdot 1\\
&= \cos(\theta).
\end{aligned}$$

Question 73. *What is the slope of the line tangent to* $\sin(\theta)$ *at* $\theta = \pi/4$*?*

For your intellectual stimulation, consider the following geometric interpretation of the derivative of $\sin(\theta)$.

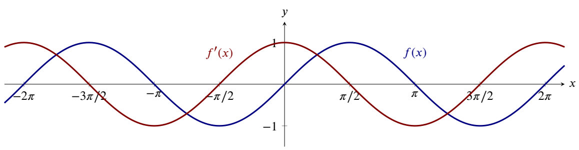


Pro-tip: When working with trigonometric functions, you should always keep their graphical representations in mind.

From this diagram, we see that increasing θ by a small amount h increases $\sin(\theta)$ by approximately $h\cos(\theta)$. Hence,

$$\frac{\Delta y}{\Delta \theta} \approx \frac{h\cos(\theta)}{h} = \cos(\theta).$$

With all of this said, the derivative of a function measures the slope of the plot of a function. If we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true.



Question 74. *Using the graph above, what is the value of x in the interval $[0, 2\pi]$ where the tangent to the graph of $f(x) = \sin(x)$ has slope -1?*

Dig-In:

20.3 Derivatives of trigonometric functions

Up until this point of the course we have been ignoring a large class of functions: Trigonometric functions other than $\sin(x)$. We know that

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Armed with this fact we will discover the derivatives of all of the standard trigonometric functions.

Theorem 35 (The derivative of cosine)**.**

$$\frac{d}{dx}\cos(x) = -\sin(x).$$

Recall that

- $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$, and
- $\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$.

Now

$$\begin{aligned}\frac{d}{dx}\cos(x) &= \frac{d}{dx}\sin\left(\frac{\pi}{2} - x\right)\\ &= -\cos\left(\frac{\pi}{2} - x\right)\\ &= -\sin(x).\end{aligned}$$

Example 122. *Compute:*

$$\left[\frac{d}{dx}\cos\left(\frac{x^3}{2}\right)\right]_{x=\sqrt[3]{\pi}}$$

Now that we know the derivative of cosine, we may combine this with the chain rule, so we have that

$$\frac{d}{dx}\cos\left(\frac{x^3}{2}\right) = \frac{3x^2}{2}\left(-\sin\left(\frac{x^3}{2}\right)\right)$$

and so

$$\begin{aligned}&\left[\frac{d}{dx}\cos\left(\frac{x^3}{2}\right)\right]_{x=\sqrt[3]{\pi}}\\ &= \left[\left(\frac{3}{2}x^2\left(-\sin\left(\frac{x^3}{2}\right)\right)\right)\right]_{x=\sqrt[3]{\pi}}\\ &= -\frac{3}{2}(\sqrt[3]{\pi})^2\sin\left(\frac{\pi}{2}\right)\\ &= -\frac{3}{2}\pi^{\frac{2}{3}}\cdot 1\\ &= \frac{-3\pi^{\frac{2}{3}}}{2}.\end{aligned}$$

Next we have:

Theorem 36 (The derivative of tangent)**.**

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

We'll rewrite $\tan(x)$ as $\dfrac{\sin(x)}{\cos(x)}$ and use the quotient rule. Write with me:

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)}\\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x).\end{aligned}$$

Example 123. *Compute:*

$$\frac{d}{dx}\left(\frac{5x\tan(x)}{x^2-3}\right)$$

Applying the quotient rule, and the product rule, and the derivative of cosine:

$$\begin{aligned}
&\frac{d}{dx}\left(\frac{5x\tan(x)}{x^2-3}\right)\\
&= \frac{(x^2-3)\cdot\frac{d}{dx}(5x\tan(x)) - 5x\tan(x)\cdot\frac{d}{dx}(x^2-3)}{(x^2-3)^2}\\
&= \frac{(x^2-3)(5\tan(x)+5x\sec^2(x)) - 5x\tan(x)\cdot 2x}{(x^2-3)^2}\\
&= \frac{5(x^2-3)(\tan(x)+x\sec^2(x)) - 10x^2\tan(x)}{(x^2-3)^2}
\end{aligned}$$

Finally, we have:

Theorem 37 (The derivative of secant)**.**

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x).$$

We'll rewrite $\sec(x)$ as $(\cos(x))^{-1}$ and use the power rule and the chain rule. Write

$$\begin{aligned}
\frac{d}{dx}\sec(x) &= \frac{d}{dx}(\cos(x))^{-1}\\
&= -1(\cos(x))^{-2}(-\sin(x))\\
&= \frac{\sin(x)}{\cos^2(x)}\\
&= \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}\\
&= \sec(x)\tan(x).
\end{aligned}$$

The derivatives of the cotangent and cosecant are similar and left as exercises. Putting this all together, we have:

Theorem 38 (The Derivatives of Trigonometric Functions)**.**

- $\dfrac{d}{dx}\sin(x) = \cos(x).$
- $\dfrac{d}{dx}\cos(x) = -\sin(x).$
- $\dfrac{d}{dx}\tan(x) = \sec^2(x).$
- $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x).$
- $\dfrac{d}{dx}\csc(x) = -\csc(x)\cot(x).$
- $\dfrac{d}{dx}\cot(x) = -\csc^2(x).$

Example 124. *Compute:*

$$\left[\frac{d}{dx}(\csc(x)\cot(x))\right]_{x=\frac{\pi}{3}}$$

Applying the product rule the facts above, we know that

$$\frac{d}{dx}(\csc(x)\cot(x)) = -\csc^3(x) - \cot^2(x)\csc(x)$$

and so

$$\begin{aligned}
&\left[\frac{d}{dx}(\csc(x)\cot(x))\right]_{x=\frac{\pi}{3}}\\
&= \left[-\csc^3(x) - \cot^2(x)\csc(x)\right]_{x=\frac{\pi}{3}}\\
&= -\frac{8}{3\sqrt{3}} - \frac{1}{3}\cdot 2/\sqrt{3}
\end{aligned}$$

Warning. *When working with derivatives of trigonometric functions, we suggest you use* ***radians*** *for angle measure. For* *example, while*

$$\sin\left((90°)^2\right) = \sin\left(\left(\frac{\pi}{2}\right)^2\right),$$

one must be careful with derivatives as

$$\left[\frac{d}{dx}\sin\left(x^2\right)\right]_{x=90°} \neq \underbrace{2 \cdot 90 \cdot \cos(90^2)}_{\textit{incorrect}}$$

Alternatively, one could think of $x°$ *as meaning* $\frac{x \cdot \pi}{180}$, *as then* $90° = \frac{90 \cdot \pi}{180} = \frac{\pi}{2}$. *In this case*

$$2 \cdot 90° \cdot \cos((90°)^2) = 2 \cdot \frac{\pi}{2} \cdot \cos\left(\left(\frac{\pi}{2}\right)^2\right).$$

21 Maximums and minimums

After completing this section, students should be able to do the following.

- Define a critical point.
- Find critical points.
- Define absolute maximum and absolute minimum.
- Find the absolute max or min of a continuous function on a closed interval.
- Define local maximum and local minimum.
- Compare and contrast local and absolute maxima and minima.
- Identify situations in which an absolute maximum or minimum is guaranteed.
- Classify critical points.
- State the First Derivative Test.
- Apply the First Derivative Test.
- State the Second Derivative Test.
- Apply the Second Derivative Test.
- Define inflection points.
- Find inflection points.

Break-Ground:

21.1 More coffee

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley!
Riley: Yes Devyn?
Devyn: Do you like coffee? I like coffee! Sometimes I feel really "bad," sluggish and tired. Then I drink coffee and I feel good! Sometimes I drink a lot of coffee!
Riley: Um?
Devyn: But here's the problem, see: If I drink too much, I become over excited and can't stop talking. I just drink coffee, then talk. Then drink more coffee. Then I start to feel sick. Ugh. I have a love-hate relationship with coffee.
Riley: If only there were a calculus solution to this problem!

Remember, calculus is about studying functions. If we can "see" a function in the work above, maybe we can figure out how to solve it.

Problem 1. *If we were to try to solve Devyn's coffee problem, what would be the best function to know?*

Multiple Choice:

(a) *How many donuts Devyn eats.*

(b) *How "good" Devyn feels after x cups of coffee.*

(c) *How many cups of coffee Devyn drinks when Devyn feels x "good."*

(d) *Impossible to say.*

Problem 2. *If we let $f(x)$ be "How 'good' Devyn feels after x cups of coffee," and we think about what Devyn says above, is there an amount Devyn can drink and feel maximally "good?"*

Multiple Choice:

(a) *yes*

(b) *no*

Dig-In

21.2 Maximums and minimums

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function.

Extrema

Local *extrema* on a function are points on the graph where the y-coordinate is larger (or smaller) than all other y-coordinates on the graph at points "close to" (x, y).

Definition.

(a) A function f has a **local maximum** at $x = a$, if $f(a) \geq f(x)$ for every x near a.

(b) A function f has a **local minimum** at $x = a$, if $f(a) \leq f(x)$ for every x near a.

A **local extremum** is either a local maximum or a local minimum.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well.

Critical points

If $(x, f(x))$ is a point where f reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

Theorem 39 (Fermat's Theorem)**.** *If f has a local extremum at $x = a$ and f is differentiable at a, then $f'(a) = 0$.*

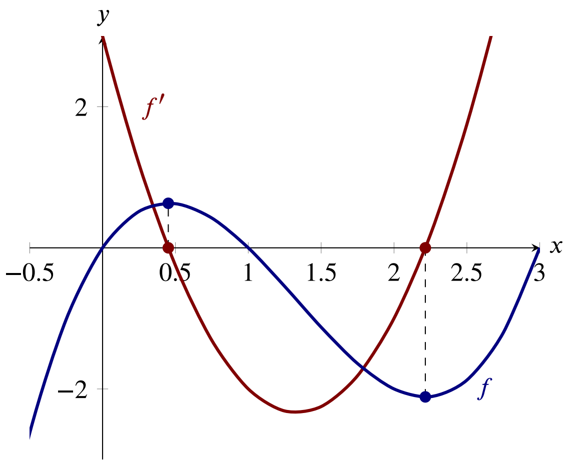
Question 75. *Does Fermat's Theorem say that if $f'(a) = 0$, then f has a local extrema at $x = a$?*

Multiple Choice:

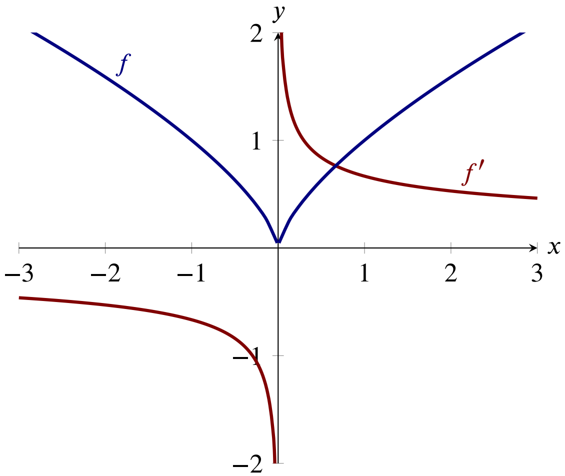
(a) *yes*

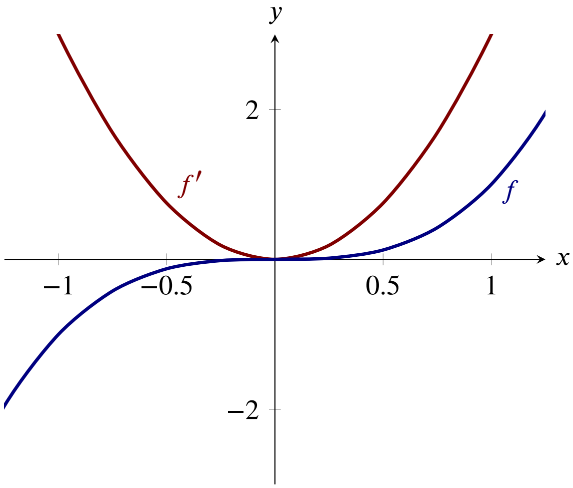
(b) *no*

Fermat's Theorem says that the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, consider the plots of $f(x) = x^3 - 4x^2 + 3x$ and $f'(x) = 3x^2 - 8x + 3$,



or the derivative is undefined, as in the plot of $f(x) = x^{2/3}$ and $f'(x) = \dfrac{2}{3x^{1/3}}$:





While $f'(0) = 0$, there is neither a maximum nor minimum at $(0, f(0))$.

This brings us to our next definition.

Definition. A function has a **critical point** at $x = a$ if

$$f'(a) = 0 \qquad \text{or} \qquad f'(a) \text{ does not exist.}$$

Warning. *When looking for local maximum and minimum points, you are likely to make two sorts of mistakes:*

- *You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere.*
- *You might assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true, consider the plots of $f(x) = x^3$ and $f'(x) = 3x^2$.*

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach is to test directly whether the y coordinates near the potential maximum or minimum are above or below the y coordinate at the point of interest.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks: they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

Example 125. *Find all local maximum and minimum points for the function $f(x) = x^3 - x$.*

> Write
> $$\frac{d}{dx} f(x) = 3x^2 - 1.$$

This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. Looking first at $x = \sqrt{3}/3$, we see that

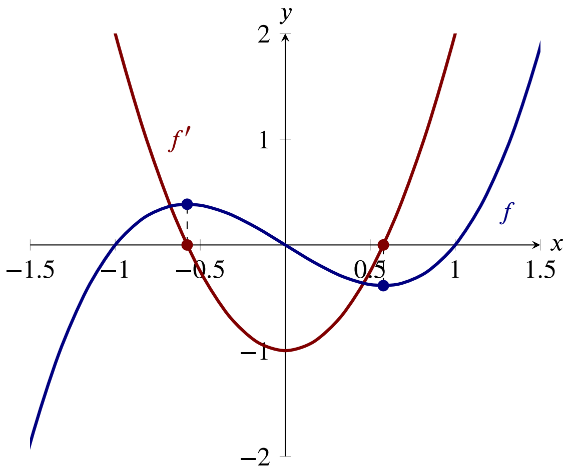
$$f(\sqrt{3}/3) = -2\sqrt{3}/9.$$

Now we test two points on either side of $x = \sqrt{3}/3$, making sure that neither is farther away than the nearest critical point; since $\sqrt{3} < 3$, $\sqrt{3}/3 < 1$ and we can use $x = 0$ and $x = 1$. Since

$$f(0) = 0 > -2\sqrt{3}/9 \qquad \text{and} \qquad f(1) = 0 > -2\sqrt{3}/9,$$

there must be a local minimum at $x = \sqrt{3}/3$.

For $x = -\sqrt{3}/3$, we see that $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. This time we can use $x = 0$ and $x = -1$, and we find that $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, so there must be a local maximum at $x = -\sqrt{3}/3$, see the plot below:



The first derivative test

The method of the previous section for deciding whether there is a local maximum or minimum at a critical point by testing “near-by” points is not always convenient. Instead, since we have already had to compute the derivative to find the critical points, we can use information about the derivative to decide. Recall that

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

So how exactly does the derivative tell us whether there is a maximum, minimum, or neither at a point? Use the *first derivative test.*

Theorem 40 (First Derivative Test)**.** *Suppose that f is continuous on an interval, and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.*
- *If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.*
- *If $f'(x)$ has the same sign to the left and right of a, then $f(a)$ is not a local extremum.*

Example 126. *Consider the function*

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Find the intervals on which f is increasing and decreasing and identify the local extrema of f.

Start by computing

$$\frac{d}{dx} f(x) = x^3 + x^2 - 2x.$$

Now we need to find when this function is positive and when it is negative. To do this, solve

$$f'(x) = x^3 + x^2 - 2x = 0.$$

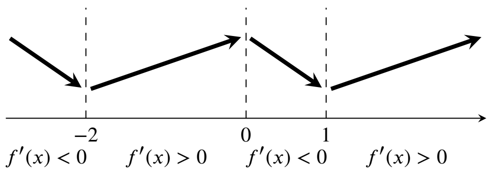
Factor $f'(x)$

$$\begin{aligned} f'(x) &= x^3 + x^2 - 2x \\ &= x(x^2 + x - 2) \\ &= x(x+2)(x-1). \end{aligned}$$

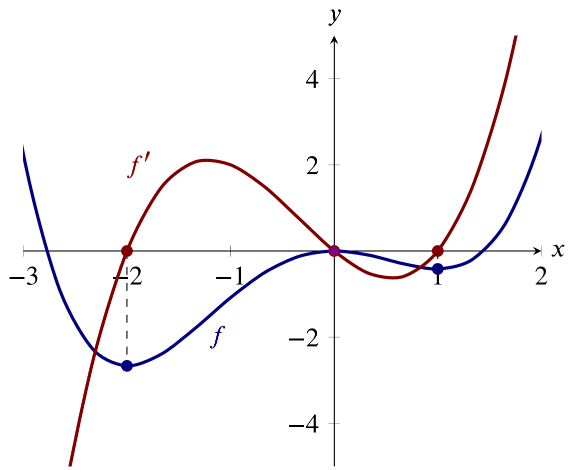
So the critical points (when $f'(x) = 0$) are when $x = -2$, $x = 0$, and $x = 1$. Now we can check points **between** the critical points to find when $f'(x)$ is increasing and decreasing:

$$\begin{aligned} f'(-3) &= -12, \\ f'(.5) &= -0.625, \\ f'(-1) &= 2, \\ f'(2) &= 8. \end{aligned}$$

From this we can make a sign table:



Hence f is increasing on $(-2, 0)$ and $(1, \infty)$ and f is decreasing on $(-\infty, -2)$ and $(0, 1)$. Moreover, from the first derivative test, the local maximum is at $x = 0$ while the local minima are at $x = -2$ and $x = 1$, see the graphs of $f(x) = x^4/4 + x^3/3 - x^2$ and $f'(x) = x^3 + x^2 - 2x$.



Hence we have seen that if f' is zero and increasing at a point, then f has a local minimum at the point. If f' is zero and decreasing at a point then f has a local maximum at the point. Thus, we see that we can gain information about f by studying how f' changes. This leads us to our next section.

Inflection points

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 41 (Test for Concavity)**.** *Suppose that $f''(x)$ exists on an interval.*

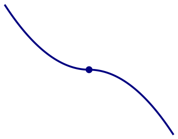
(a) *If $f''(x) > 0$ on an interval, then f is concave up on that interval.*

(b) *If $f''(x) < 0$ on an interval, then f is concave down on that interval.*

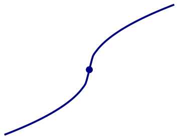
Of particular interest are points at which the concavity changes from up to down or down to up.

Definition. If f is continuous and its concavity changes either from up to down or down to up at $x = a$, then f has an **inflection point** at $x = a$.

It is instructive to see some examples of inflection points:



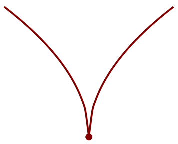
This is an inflection point. The concavity changes from concave up to concave down.



This is an inflection point. The concavity changes from concave up to concave down.

It is also instructive to see some nonexamples of inflection points:

This is **not** an inflection point. The curve is concave down on either side of the point.



This is **not** an inflection point. The curve is concave down on either side of the point.

We identify inflection points by first finding x such that $f''(x)$ is zero or undefined and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points.

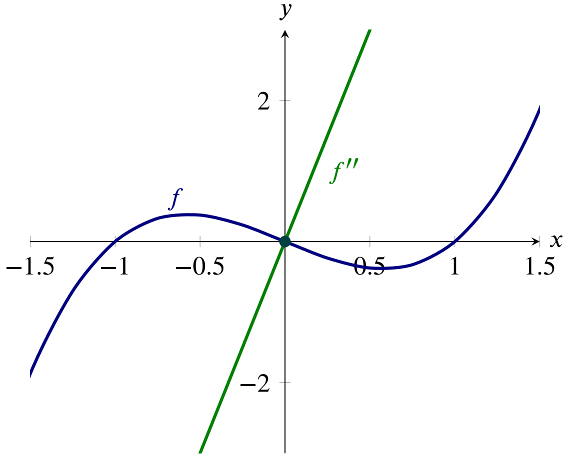
Warning. *Even if $f''(a) = 0$, the point determined by $x = a$ might **not** be an inflection point.*

Example 127. *Describe the concavity of $f(x) = x^3 - x$.*

To start, compute the first and second derivative of $f(x)$ with respect to x,

$$f'(x) = 3x^2 - 1 \qquad \text{and} \qquad f''(x) = 6x.$$

Since $f''(0) = 0$, there is potentially an inflection point at $x = 0$. Using test points, we note the concavity does change from down to up, hence there is an inflection point at $x = 0$. The curve is concave down for all $x < 0$ and concave up for all $x > 0$, see the graphs of $f(x) = x^3 - x$ and $f''(x) = 6x$.



Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

The second derivative test

Recall the first derivative test:

- If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.

- If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.

If f' changes from positive to negative it is decreasing. In this case, f'' might be negative, and if in fact f'' is negative then f' is definitely decreasing, so there is a local maximum at the point in question. On the other hand, if f' changes from negative to positive it is increasing. Again, this means that f'' might be positive, and if in fact f'' is positive then f' is definitely increasing, so there is a local minimum at the point in question. We summarize this as the *second derivative test*.

Theorem 42 (Second Derivative Test)**.** *Suppose that $f''(x)$ is continuous on an open interval and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f''(a) < 0$, then f has a local maximum at a.*
- *If $f''(a) > 0$, then f has a local minimum at a.*
- *If $f''(a) = 0$, then the test is inconclusive. In this case, f may or may not have a local extremum at $x = a$.*

The second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails and sometimes the second derivative is quite difficult to evaluate. In such cases we must fall back on one of the previous tests.

Example 128. *Once again, consider the function*

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Use the second derivative test, to locate the local extrema of f.

Start by computing

$$f'(x) = x^3 + x^2 - 2x \qquad \text{and} \qquad f''(x) = 3x^2 + 2x - 2.$$

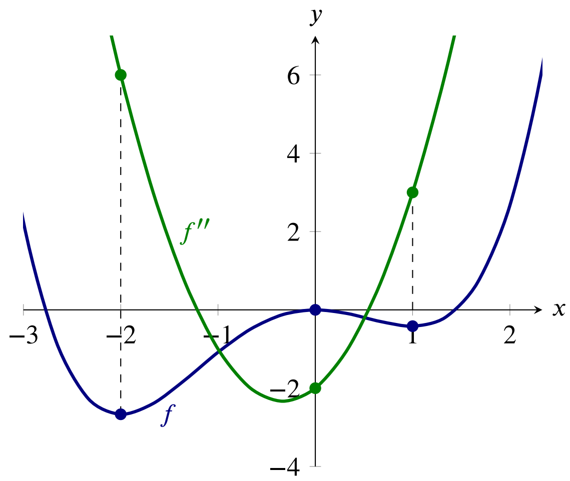
Using the same technique as we used before, we find that

$$f'(-2) = 0, \qquad f'(0) = 0, \qquad f'(1) = 0.$$

Now we'll attempt to use the second derivative test,

$$f''(-2) = 6, \qquad f''(0) = -2, \qquad f''(1) = 3.$$

Hence we see that f has a local minimum at $x = -2$, a local maximum at $x = 0$, and a local minimum at $x = 1$, see below for a plot of $f(x) = x^4/4 + x^3/3 - x^2$ and $f''(x) = 3x^2 + 2x - 2$:



Question 76. *If $f''(a) = 0$, what does the second derivative test tell us?*

Multiple Choice:

(a) *The function has a local extrema at $x = a$.*

(b) *The function does not have a local extrema at $x = a$.*

(c) *It gives no information on whether $x = a$ is a local extremum.*

Part IV

Additional content for Autumn Final Exam

22 Mean Value Theorem

After completing this section, students should be able to do the following.

- Understand the statement of the Extreme Value Theorem.
- Understand the statement of the Mean Value Theorem.
- Sketch pictures to illustrate why the Mean Value Theorem is true.
- Determine whether Rolle's Theorem or the Mean Value Theorem can be applied.
- Find the values guaranteed by Rolle's Theorem or the Mean Value Theorem.
- Use the Mean Value Theorem to solve word problems.
- Compare and contrast the Intermediate Value Theorem, Mean Value Theorem, and Rolle's Theorem.
- Identify calculus ideas which are consequences of the Mean Value Theorem.
- Use the Mean Value Theorem to bound the error in linear approximation.

Break-Ground:

22.1 Let's run to class

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I want to go to math class. Now. Let's race!
Riley: Yes. I love math class. Let's do race! On, your mark.... Ready. Steady. Go!
Devyn: You may think you're fast, but I'm catching up!
Riley: Noooooo!
Devyn: Now I'm winning! I've never won a foot race in my life!
Riley: Never...give...up!
Devyn: Whew! We both made it to math class at exactly the same time!
Riley: Wow. We should run to every class. Hey I have a question, was there a time during our race that we were running at exactly the same speed?

Problem 1. *Which of the following describes the race above?*

Multiple Choice:

(a) *Devyn was leading until the end, when the race finished in a tie.*

(b) *Riley was leading until the end, when the race finished in a tie.*

(c) *Devyn was leading, then Riley was leading until the end, when the race finished in a tie.*

(d) *Riley was leading, then Devyn was leading until the end, when the race finished in a tie.*

(e) *None of the above.*

Problem 2. *What can you say about Devyn's and Riley's average velocities?*

Multiple Choice:

(a) *Devyn has the larger average velocity.*

(b) *Riley has the larger average velocity.*

(c) *Their average velocities are equal.*

(d) *None of the above.*

Problem 3. *Record your guess to Riley's question: is there a moment during the race where Devyn and Riley were running at exactly the same speed?*

Dig-In:

22.2 The Extreme Value Theorem

Definition.

(a) A function f has an **global maximum** at $x = a$, if $f(a) \geq f(x)$ for every x in the domain of the function.

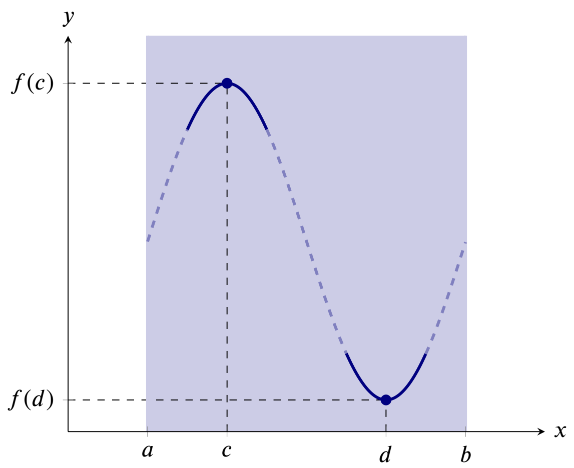
(b) A function f has an **global minimum** at $x = a$, if $f(a) \leq f(x)$ for every x in the domain of the function.

A **global extremum** is either a global maximum or a global minimum.

If we are working on an finite closed interval, then we have the following theorem.

Theorem 43 (Extreme Value Theorem)**.** *If f is a continuous function for all x in the closed interval $[a, b]$, then there are points c and d in $[a, b]$, such that $(c, f(c))$ is a global maximum and $(d, f(d))$ is a global minimum on $[a, b]$.*

Below, we see a geometric interpretation of this theorem.



Question 77. *Would this theorem hold if we were working on an open interval?*

Multiple Choice:

(a) *yes*

(b) *no*

Dig-In:

22.3 The Mean Value Theorem

Here are some interesting questions involving derivatives:

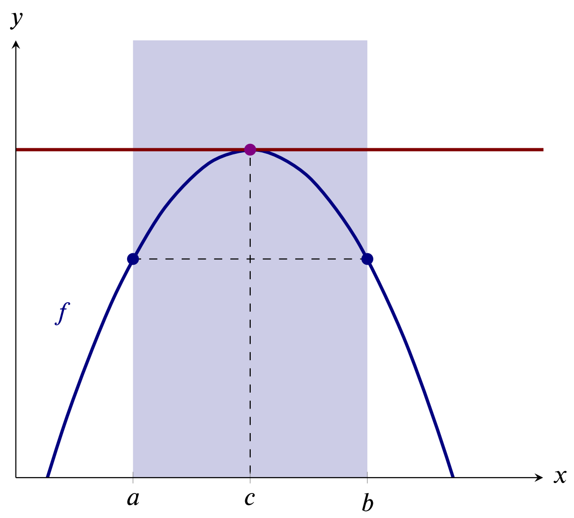
(a) Suppose you toss a ball into the air and then catch it. Must the ball's vertical velocity have been zero at some point?

(b) Suppose you drive a car from toll booth on a toll road to another toll booth 30 miles away in half of an hour. Must you have been driving at 60 miles per hour at some point?

(c) Suppose two different functions have the same derivative. What can you say about the relationship between the two functions?

While these problems sound very different, it turns out that the problems are very closely related. We'll start simply:

Theorem 44 (Rolle's Theorem)**.** *Suppose that f is differentiable on the interval (a, b), is continuous on the interval $[a, b]$, and $f(a) = f(b)$.*



Then

$$f'(c) = 0$$

for some $a < c < b$.

We can now answer our first question above.

Example 129. *Suppose you toss a ball into the air and then catch it. Must the ball's vertical velocity have been zero at some point?*

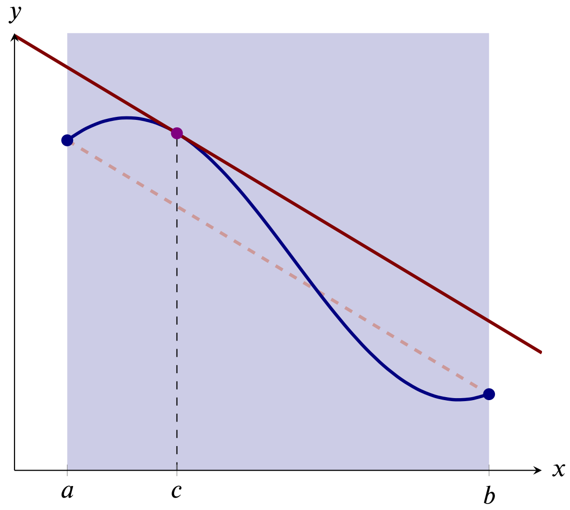
Let $p(t)$ be the position of the ball at time t. Our interval in question will be

$$[t_{\text{start}}, t_{\text{finish}}].$$

We may assume that p is continuous on $[a, b]$ and differentiable on (a, b). We may now apply Rolle's Theorem to see at some time c, $p'(c) = 0$. Hence the velocity must be zero at some point.

Rolle's Theorem is a special case of a more general theorem.

Theorem 45 (Mean Value Theorem)**.** *Suppose that f has a derivative on the interval (a, b) and is continuous on the interval $[a, b]$.*



Then

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $a < c < b$*.*

We can now answer our second question above.

Example 130. *Suppose you drive a car from toll booth on a toll road to another toll booth* 30 *miles away in half of an hour. Must you have been driving at* 60 *miles per hour at some point?*

If $p(t)$ is the position of the car at time t, and 0 hours is the starting time with $1/2$ hours being the final time, then we may assume that p is continuous on $[0, 1/2]$ and differentiable on $(0, 1/2)$. Now the Mean Value Theorem states there is a time c

$$p'(c) = \frac{30 - 0}{1/2} = 60 \qquad \text{where } 0 < c < 1/2.$$

Since the derivative of position is velocity, this says that the car must have been driving at 60 miles per hour at some point.

Now we will address the unthinkable: could there be a continuous function f on $[a, b]$ whose derivative is zero on (a, b) that is not constant? As we will see, the answer is "no."

Theorem 46. *If* $f'(x) = 0$ *for all* x *in an interval* I*, then* $f(x)$ *is constant on* I*.*

Let $a < b$ be two points in I. Since f is continuous on $[a, b]$ and differentiable on (a, b), by the Mean Value Theorem we know

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c in the interval (a, b). Since $f'(c) = 0$ we see that $f(b) = f(a)$. Moreover, since a and b were arbitrarily chosen, $f(x)$ must be the constant function.

Now let's answer our third question.

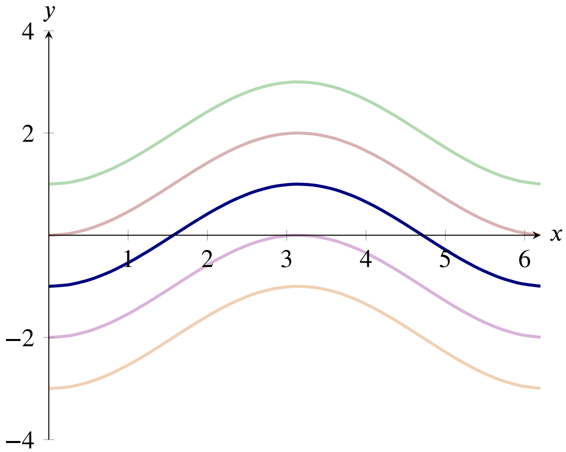
Example 131. *Suppose two different functions have the same derivative. What can you say about the relationship between the two functions?*

Set $h(x) = f(x) - g(x)$, so $h'(x) = f'(x) - g'(x)$. Now $h'(x) = 0$ on the interval (a, b). This means that $h(x) = k$ where k is some constant. Hence

$$g(x) = f(x) + k.$$

Example 132. *Describe all functions whose derivative is* $\sin(x)$.

One such function is $-\cos(x)$, so all such functions have the form $-\cos(x) + k$,



Finally, let us investigate two young mathematicians who run to class.

Example 133. *Two students Devyn and Riley raced to class. Was there a point during the race that Devyn and Riley were running at exactly the same velocity?*

Let P_{Devyn} represent Devyn's position with respect to time, and let P_{Riley} represent Riley's position with respect to time. Let t_{start} be the starting time of the race, and t_{finish} be the end of the race. Set

$$f(t) = P_{\text{Devyn}}(t) - P_{\text{Riley}}(t).$$

Note, we may assume that P_{Devyn} and $P_{\text{Riley}}(t)$ are continuous on $[t_{\text{start}}, t_{\text{finish}}]$ and that they are differentiable on $(t_{\text{start}}, t_{\text{finish}})$. Hence the same is true for f. Since both runners start and finish at the same place,

$$f(t_{\text{start}}) = P_{\text{Devyn}}(t_{\text{start}}) - P_{\text{Riley}}(t_{\text{start}}) = 0 \qquad \text{and}$$
$$f(t_{\text{finish}}) = P_{\text{Devyn}}(t_{\text{finish}}) - P_{\text{Riley}}(t_{\text{finish}}) = 0.$$

In fact, this shows us that the average rate of change of

$$f(t) = P_{\text{Devyn}}(t) - P_{\text{Riley}}(t) \qquad \text{on} \qquad [t_{\text{start}}, t_{\text{finish}}]$$

is 0. Hence by the mean value theorem, there is a point c

$$t_{\text{start}} \le c \le t_{\text{finish}}$$

with $f'(c) = 0$. However,

$$0 = f'(c) = P'_{\text{Devyn}}(c) - P'_{\text{Riley}}(c).$$

Hence at c,

$$P'_{\text{Devyn}}(c) = P'_{\text{Riley}}(c),$$

this means that there was a time when they were running at exactly the same velocity.

23 Optimization

After completing this section, students should be able to do the following.

- Describe the goals of optimization problems generally.
- Find all local maximums and minimums using the First and Second Derivative tests.
- Identify when we can find an absolute maximum or minimum on an open interval.
- Contrast optimization on open and closed intervals.
- Describe the objective function and constraints in a given optimization problem.
- Solve optimization problems by finding the appropriate extreme values.

Break-Ground:

23.1 A mysterious formula

Check out this dialogue between two calculus students:

Devyn: Riley, what do you think is the maximum value of

$$f(x) = \frac{10}{x^2 - 2.8x + 3}?$$

Riley: Where did that function come from?
Devyn: It's just some, um, random function.
Riley: Wait, does this have to do with coffee?
Devyn: Um, uh, no?
Riley: Well what interval are we on?
Devyn: Let's say $[0, 10]$, I mean there's no way I could possibly drink ten cups of coff...
Riley: I knew this was about coffee.

Here Devyn has made a function, that is supposed to record Devyn's "well-being" with respect to the number of cups of coffee consumed in one day.

Problem 1. *Graph Devyn's function. Where do you estimate the maximum on the interval* $[0, 10]$ *to be?*

Problem 2. *If you wanted to argue that this is the (global) maximum value on* $[0, 10]$ *without plotting, what arguments could you use?*

Dig-In:

23.2 Basic optimization

An **optimization problem** is a problem where you need to maximize or minimize some quantity given some constraints. This can be accomplished using the tools of differential calculus that we have already developed.

Perhaps the most basic optimization problems is generated by the following question:

> Among all rectangles of a fixed perimeter, which has the greatest area?

Let's not do this problem in the abstract, let's do it with numbers.

Example 134. *Of all rectangles of perimeter* 12*, which side lengths give the greatest area?*

If a rectangle has perimeter 12 and one side is length x, then the length of the other side is $6 - x$. Hence the area of a rectangle of perimeter 12 can be given by

$$A(x) = x(6 - x).$$

However, for the side lengths to be physically relevant, we must assume that x is in the interval $(0, 6]$.

So to maximize the area of the rectangle, we need to find the maximum value of $A(x)$ on the appropriate interval.

> **At this point, you should graph the function if you can.**

We'll continue on without the aid of a graph, and use the derivative. Write

$$A'(x) = 6 - 2x$$

Now we find the critical points, solving the equation

$$6 - 2x = 0,$$

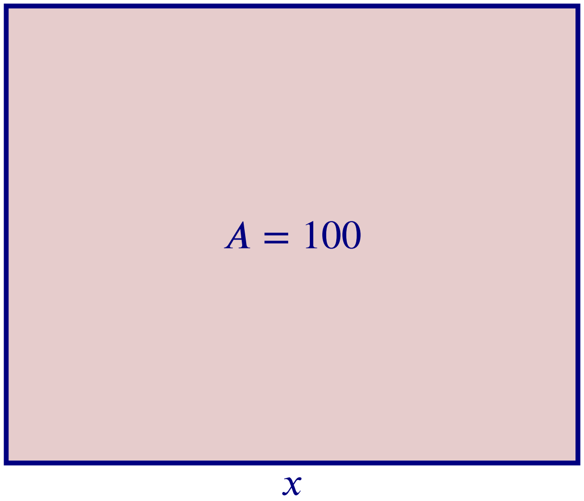
we see that the only critical point of A is at $x = 3$

Since $A'(x) = 6-2x$ is positive on $(0, 3)$ and negative on $(3, 6]$, $x = 3$ is where the maximum value of A happens. This is exactly when the rectangle is a square!

A key step to note, is when we explained why $x = 3$ is actually the maximum. Above we basically used facts about the derivative. Below we use a similar argument.

Example 135. *Of all rectangles of area* 100*, which has the smallest perimeter?*

First we draw a picture, Here is a rectangle with an area of 100.



If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$.

The perimeter of this rectangle is given by

$$P(x) = 2x + 2 \cdot 100/x.$$

We wish to minimize $P(x)$. Note, not all values of x make sense in this problem: lengths of sides of rectangles must be positive, so $x > 0$. If $x > 0$ then so is $100/x$, so we need no second condition on x.

At this point, you should graph the function if you can.

We next find $P'(x)$ and set it equal to zero. Write

$$P'(x) = 2 - 200/x^2 = 0.$$

Solving for x gives us $x = \pm 10$. We are interested only in $x > 0$, so only the value $x = 10$ is of interest. Since $P'(x)$ is defined everywhere on the interval $(0, \infty)$, there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at $x = 10$? The second derivative is

$$P''(x) = 400/x^3,$$

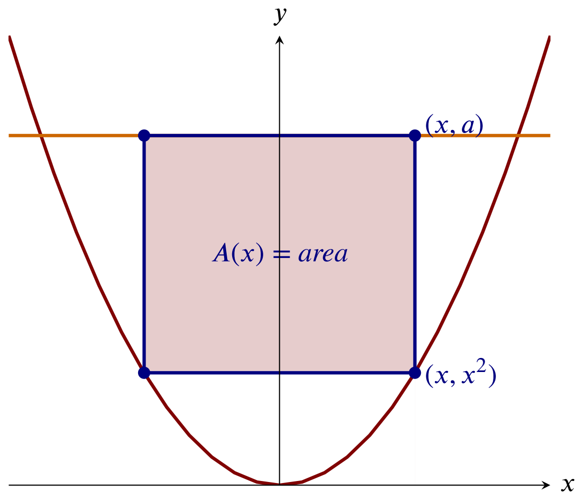
and $P''(10) > 0$, so there is a local minimum. Since there is only one critical point, this is also the global minimum, so the rectangle with smallest perimeter is the 10×10 square.

Hence, calculus gives a **reason** for **why** a square is the rectangle with both

- the largest area for a given perimeter.
- the smallest perimeter for a given area.

We may be done with rectangles, but they aren't done with us. Here is a problem where there are more constraints on the possible side lengths of the rectangle.

Example 136. *Find the rectangle with largest area that fits inside the graph of the parabola $y = x^2$ below the line $y = a$, where a is an unspecified constant value, with the top side of the rectangle on the horizontal line $y = a$. See the figure below:*



We want to maximize value of $A(x)$. The lower right corner of the rectangle is at (x, x^2), and once this is chosen the rectangle is completely determined. Then the area is

$$A(x) = (2x)(a - x^2) = -2x^3 + 2ax.$$

We want the maximum value of $A(x)$ when x is in $[0, \sqrt{a}]$. You might object to allowing $x = 0$ or $x = \sqrt{a}$, since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area as we may then apply the Extreme Value Theorem and see that we indeed have a maximum and minimum value.

At this point, you should graph the function if you can.

> Setting $0 = A'(x) = -6x^2 + 2a$ we find $x = \sqrt{a/3}$ as the only critical point. Testing this and the two endpoints (as the maximum could also be there), we have $A(0) = A(\sqrt{a}) = 0$ and $A(\sqrt{a/3}) = (4/9)\sqrt{3}a^{3/2}$. Hence, the maximum area occurs when the rectangle has dimensions $2\sqrt{a/3} \times (2/3)a$.

Again, note that above we used the Extreme Value Theorem to guarantee that we found the maximum.

24 Applied optimization

After completing this section, students should be able to do the following.

- Recognize optimization problems.
- Translate a word problem into the problem of finding the extreme values of a function.
- Solve basic word problems involving maxima or minima.
- Interpret an optimization problem as the procedure used to make a system or design as effective or functional as possible.
- Set up an optimization problem by identifying the objective function and appropriate constraints.
- Solve optimization problems by finding the appropriate absolute extremum.
- Identify the appropriate domain for functions which are models of real-world phenomena.

Break-Ground:

24.1 Volumes of aluminum cans

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, have you ever noticed aluminum cans?
Riley: So very recyclable!
Devyn: I know! But I've also noticed that there are some that are short and fat, and others that are tall and skinny, and yet they can still have the same volume!
Riley: So very observant!
Devyn: This got me wondering, if we want to make a can with volume V, what shape of can uses the least aluminum?
Riley: Ah! This sounds like a job for calculus! The volume of a cylindrical can is given by

$$V = \pi \cdot r^2 \cdot h$$

where r is the radius of the cylinder and h is the height of the cylinder. Also the surface area is given by

$$\begin{aligned} A &= \underbrace{\pi \cdot r^2}_{\text{bottom}} + \underbrace{2 \cdot \pi \cdot r \cdot h}_{\text{sides}} + \underbrace{\pi \cdot r^2}_{\text{top}} \\ &= 2 \cdot \pi \cdot r^2 + 2 \cdot \pi \cdot r \cdot h. \end{aligned}$$

Somehow we want to minimize the surface area, because that's the amount of aluminum used, but we also want to keep the volume constant.
Devyn: Whoa, we have way too many letters here.
Riley: Yeah, somehow we need only one variable. Yikes. Too many letters.

Problem 1. *Suppose we wish to construct an aluminum can with volume V that uses the least amount of aluminum. In the context above, what do we want to minimize?*

Select All Correct Answers:

(a) A

(b) V

(c) h

(d) r

Problem 2. *In the context above, what should be considered a constant?*

Select All Correct Answers:

(a) A

(b) V

(c) h

(d) r

As Devyn and Riley noticed, when we work out this type of problem, we need to reduce the problem to a single variable.

Problem 3. *Consider r to be the variable, and express A as a function of r.*

Problem 4. *Now consider h to be the variable, and express A as a function of h.*

Notice that we've reduced (one way or another) this function of two variables to a function of one variable. This process will be a key step in nearly every problem in this next section.

Dig-In:

24.2 Applied optimization

In this section, we will present several worked examples of optimization problems. Our method for solving these problems is essentially the following:

Draw a picture. If possible, draw a schematic picture with all the relevant information.

Determine your goal. We need identify what needs to be optimized.

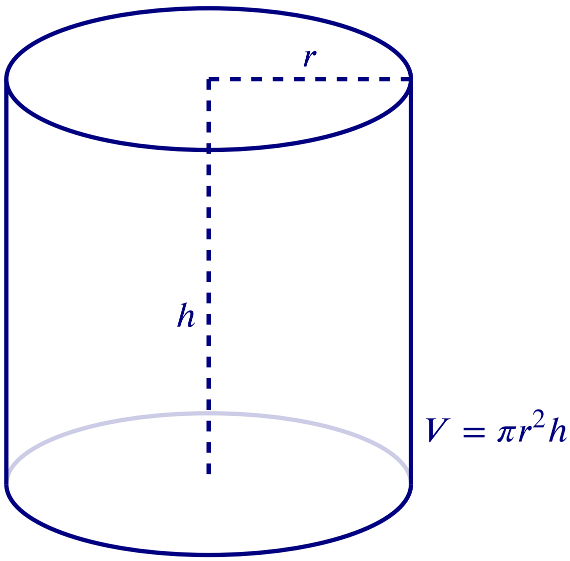
Find constraints. What limitations are set on our optimization?

Solve for a single variable. Now you should have a function to optimize.

Use calculus to find the extreme values. Be sure to check your answer!

Example 137. *You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.*

First we draw a picture:



Letting c represent the cost of the lateral side, we can write an expression for the cost of materials:

$$C = 2\pi c r h + 2\pi r^2 N c.$$

Since we know that $V = \pi r^2 h$, we can use this relationship to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). We find

$$\begin{aligned} C(r) &= 2c\pi r \frac{V}{\pi r^2} + 2Nc\pi r^2 \\ &= \frac{2cV}{r} + 2Nc\pi r^2. \end{aligned}$$

We want to know the minimum value of this function when r is in $(0, \infty)$. Setting

$$C'(r) = -2cV/r^2 + 4Nc\pi r = 0$$

we find $r = \sqrt[3]{V/(2N\pi)}$. Since $C''(r) = 4cV/r^3 + 4Nc\pi$ is positive when r is positive, there is a local minimum at the critical value, and hence a global minimum since there is only one critical value.

Finally, since $h = V/(\pi r^2)$,

$$\begin{aligned}\frac{h}{r} &= \frac{V}{\pi r^3}\\ &= \frac{V}{\pi(V/(2N\pi))}\\ &= 2N,\end{aligned}$$

so the minimum cost occurs when the height h is $2N$ times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius.

Example 138. *You want to sell a certain number n of items in order to maximize your profit. Market research tells you that if you set the price at* \$1.50*, you will be able to sell* 5000 *items, and for every* 10 *cents you lower the price below* \$1.50 *you will be able to sell another* 1000 *items. Suppose that your fixed costs ("start-up costs") total* \$2000*, and the per item cost of production ("marginal cost") is* \$0.50*. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.*

The first step is to convert the problem into a function maximization problem. The revenue for selling n items at x dollars is given by

$$r(x) = nx$$

and the cost of producing n items is given by

$$c(x) = 2000 + 0.5n.$$

However, from the problem we see that the number of items sold is itself a function of x,

$$n(x) = 5000 + \frac{1000(1.5 - x)}{0.10}$$

So profit is give by:

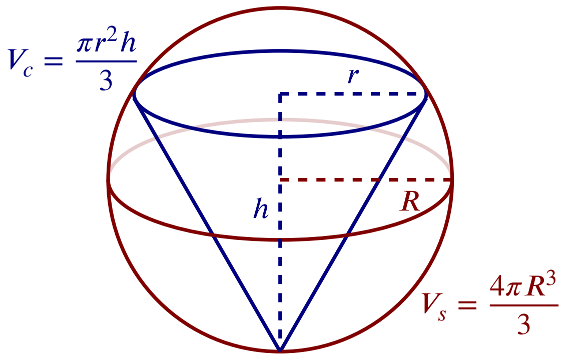
$$\begin{aligned}P(x) &= r(x) - c(x)\\ &= nx - (2000 + 0.5n)\\ &= -10000x^2 + 25000x - 12000.\end{aligned}$$

We want to know the maximum value of this function when x is between 0 and 1.5. The derivative is

$$P'(x) = -20000x + 25000,$$

which is zero when $x = 1.25$. Since $P''(x) = -20000 < 0$, there must be a local maximum at $x = 1.25$, and since this is the only critical value it must be a global maximum as well. Alternately, we could compute $P(0) = -12000$, $P(1.25) = 3625$, and $P(1.5) = 3000$ and note that $P(1.25)$ is the maximum of these. Thus the maximum profit is \$3625, attained when we set the price at \$1.25 and sell 7500 items.

Example 139. *If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)*



Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. Our goal is to maximize the volume of the cone: $V_c = \pi r^2 h/3$. The largest r could be is R and the largest h could be is $2R$.

Notice that the function we want to maximize, $\pi r^2 h/3$, depends on *two* variables. Our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure, as the upper corner of the triangle, whose coordinates are $(r, h-R)$, must be on the circle of radius R. Write

$$r^2 + (h-R)^2 = R^2.$$

Solving for r^2, since r^2 is found in the formula for the volume of the cone, we find

$$r^2 = R^2 - (h-R)^2.$$

Substitute this into the formula for the volume of the cone to find

$$\begin{aligned} V_c(h) &= \pi(R^2 - (h-R)^2)h/3 \\ &= -\frac{\pi}{3}h^3 + \frac{2}{3}\pi h^2 R \end{aligned}$$

We want to maximize $V_c(h)$ when h is between 0 and $2R$. We solve

$$V_c'(h) = -\pi h^2 + (4/3)\pi h R = 0,$$

finding $h = 0$ or $h = 4R/3$. We compute

$$V_c(0) = V_c(2R) = 0$$

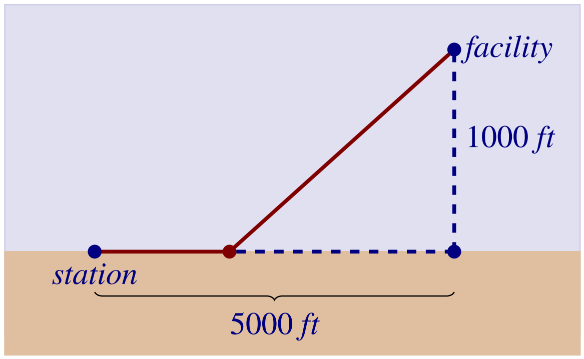
and

$$V_c(4R/3) = (32/81)\pi R^3.$$

The maximum is the latter. Since the volume of the sphere is $(4/3)\pi R^3$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3} = \frac{8}{27} \approx 30\%.$$

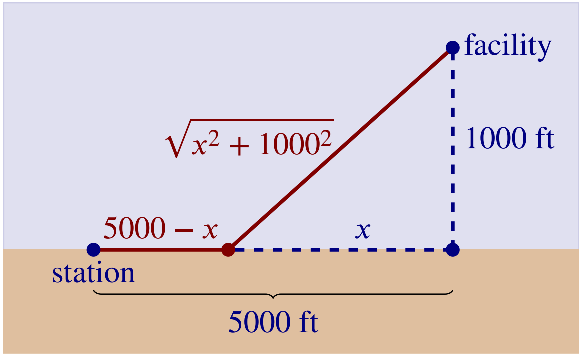
Example 140. *A power line needs to be run from an power station located on the beach to an offshore facility.*



It costs \$50/ft. to run a power line along the land, and \$130/ft. to run a power line under water. How much of the power line should be run along the land to minimize the overall cost? What is the minimal cost?

There are two immediate solutions that we could consider, each of which we will reject through "common sense." First, we could minimize the distance by directly connecting the two locations with a straight line. However, this requires that all the wire be laid underwater, the most costly option. Second, we could minimize the underwater length by running a wire all 5000 ft. along the beach, directly across from the offshore facility. This has the undesired effect of having the longest distance of all, probably ensuring a nonminimal cost.

The optimal solution likely has the line being run along the ground for a while, then underwater, as the figure implies. We need to label our unknown distances: the distance run along the ground and the distance run underwater. Recognizing that the underwater distance can be measured as the hypotenuse of a right triangle, we can label our figure as follows



By choosing x as we did, we make the expression under the square root simple. We now create the cost function:

$$\begin{array}{ccc} \text{Cost} = & & \\ \text{land cost} & + & \text{water cost} \\ \$50 \times \text{land distance} & + & \$130 \times \text{water distance} \\ 50(5000-x) & + & 130\sqrt{x^2+1000^2}. \end{array}$$

So we have

$$c(x) = 50(5000-x) + 130\sqrt{x^2+1000^2}.$$

This function only makes sense on the interval $[0, 5000]$. While we are fairly certain the endpoints will not give a minimal cost, we still evaluate $c(x)$ at each to verify.

$$c(0) = 380000 \qquad c(5000) \approx 662873.$$

We now find the critical points of $c(x)$. We compute $c'(x)$ as

$$c'(x) = -50 + \frac{130x}{\sqrt{x^2+1000^2}}.$$

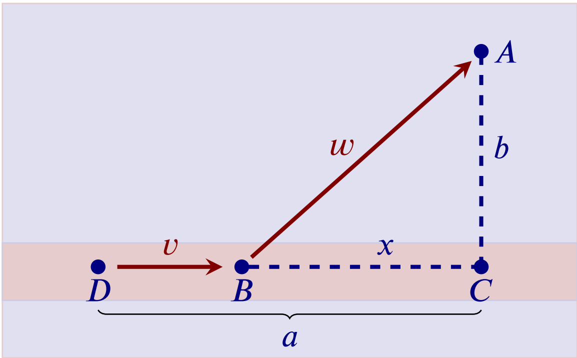
Recognize that this is never undefined. Setting $c'(x) = 0$ and solving for x, we have:

$$\begin{aligned} -50 + \frac{130x}{\sqrt{x^2+1000^2}} &= 0 \\ \frac{130x}{\sqrt{x^2+1000^2}} &= 50 \\ \frac{130^2x^2}{x^2+1000^2} &= 50^2 \\ 130^2x^2 &= 50^2(x^2+1000^2) \\ 130^2x^2 - 50^2x^2 &= 50^2 \cdot 1000^2 \\ (130^2-50^2)x^2 &= 50000^2 \\ x^2 &= \frac{50000^2}{130^2-50^2} \\ x &= \frac{50000}{\sqrt{130^2-50^2}} \\ x &= \frac{50000}{120}, \end{aligned}$$

Evaluating $c(x)$ at $x = 416.67$ gives a cost of about \$370000. The distance the power line is laid along land is $5000 - 416.67 = 4583.33$ ft and the underwater distance is $\sqrt{416.67^2 + 1000^2} \approx 1083$ ft.

We now work a similar problem without concrete numbers.

Example 141. *Suppose you want to reach a point A that is located across the sand from a nearby road.*



Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

Let x be the distance short of C where you turn off, the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.

You travel the distance from D to B at speed v, and then the distance from B to A at speed w. The distance from D to B is $a-x$. By the Pythagorean theorem, the distance from B to A is

$$\sqrt{x^2+b^2}.$$

Hence the total time for the trip is

$$T(x) = \frac{a-x}{v} + \frac{\sqrt{x^2+b^2}}{w}.$$

We want to find the minimum value of T when x is between 0 and a. As usual we set $T'(x)=0$ and solve for x. Write

$$T'(x) = -1/v + \frac{x}{w\sqrt{x^2+b^2}} = 0.$$

We find that

$$x = \frac{wb}{\sqrt{v^2-w^2}}$$

Notice that a does not appear in the last expression, but a is not irrelevant, since we are interested only in critical values that are in $[0,a]$, and $wb/\sqrt{v^2-w^2}$ is either in this interval or not. If it is, we can use the second derivative to test it:

$$T''(x) = \frac{b^2}{(x^2+b^2)^{3/2}w}.$$

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0,a]$ it is larger than a. In this case the minimum must occur at one of the endpoints. We can compute

$$T(0) = \frac{a}{v} + \frac{b}{w}$$
$$T(a) = \frac{\sqrt{a^2+b^2}}{w}$$

but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of v, w, a, and b, then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $T''(x)$ is always positive, so the derivative $T'(x)$ is always increasing. We know that at $wb/\sqrt{v^2-w^2}$ the derivative is zero, so for values of x less than that critical value, the derivative is negative. This means that $T(0) > T(a)$, so the minimum occurs when $x = a$.

> So the upshot is this: If you start farther away from C than $wb/\sqrt{v^2 - w^2}$ then you always want to cut across the sand when you are a distance $wb/\sqrt{v^2 - w^2}$ from point C. If you start closer than this to C, you should cut directly across the sand.

With optimization problems you will see a variety of situations that require you to combine problem solving skills with calculus. Focus on the *process*. One must learn how to form equations from situations that can be manipulated into what you need. Forget memorizing how to do "this kind of problem" as opposed to "that kind of problem."

Learning a process will benefit one far more than memorizing a specific technique.

Part V

Content for the First Exam

25 Review of Limits

After completing this section, students should be able to do the following.

- ...
- ...
- ...

Break-Ground:

25.1 Review Limits BreakGround Here

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, do you remember when we first starting graphing functions? Like with a “T-chart?”
Riley: I remember everything.
Devyn: ...
Riley: ...
Devyn: ...
Riley: ...
Devyn: ...
Riley: ...

Problem 1. *Question*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 2. *When x is a small number (near zero), which term of* $5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ *is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 3. *Very roughly speaking, what does the graph of* $y = 5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ *look like?*

Multiple Choice:

(a) *The graph starts in the lower left and ends in the upper right of the plane.*

(b) *The graph starts in the lower right and ends in the upper left of the plane.*

(c) *The graph looks something like the letter “U.”*

(d) *The graph looks something like an upside down letter “U.”*

Dig-In:

25.2 Review Limits.

Remember limits?

We started last semester with the idea of a 'limit'. Remember what that means.

Definition. Intuitively,

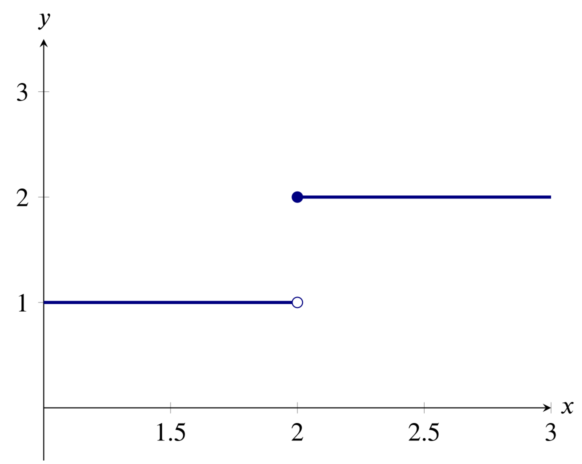
the **limit** of $f(x)$ as x approaches a is L,

written

$$\lim_{x\to a} f(x) = L,$$

if the value of $f(x)$ can be made as close as one wishes to L for all x sufficiently close, but not equal, to a.

If we look at the graph of a function f:



If x is really close to $a = 2.5$ (say, within 0.001), then the output value is $f(x) = 2$. As long as x is within a very small tolerance of 2.5, the output values are always 2. That is, $\lim_{x\to 2.5} f(x) = 2$.

What happens at $a = 2$? If x is really close to 2, but larger than 2, then the output values $f(x) = 2$. If x is really close to 2, but less than 2, then the output values $f(x) = 1$. This lead us to consider 'one-sided limits'. We have $\lim_{x\to 2^+} f(x) = 2$ and $\lim_{x\to 2^-} f(x) = 1$. (Does it matter that the circle at $(2,1)$ is not filled in?) What can we say about $\lim_{x\to 2} f(x)$?

Question 78. *What is* $\lim_{x\to 2} f(x)$*?*

Multiple Choice:

(a) 1

(b) 2

(c) 1.5

(d) *DNE*

We discussed several different methods for calculating limit values last semester, based on what we called the 'Limit Laws'. We'll recall them here.

Theorem 47 (Limit Laws)**.** *Suppose that* $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$.

Sum/Difference Law $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$.

Product Law $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM$.

Quotient Law $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, *if* $M \neq 0$.

These Limit Laws allow us to use limit values that we know, in order to calculate limit values of more complicated functions.

For example, if we know $\lim_{x\to\pi} x = \pi$ and $\lim_{x\to\pi} \cos(x) = -1$, then we can find something like $\lim_{x\to\pi} \left(3x^2 - 4x\cos(x)\right)$.

Example 142. *Calculate* $\lim_{x\to\pi}\left(3x^2-4x\cos(x)\right)$.

$$\begin{aligned}\lim_{x\to\pi}\left(3x^2-4x\cos(x)\right) &= \lim_{x\to\pi}3x^2-\lim_{x\to\pi}4x\cos(x)\\ &= 3\left(\lim_{x\to\pi}x\right)^2-4\left(\lim_{x\to\pi}x\right)\left(\lim_{x\to\pi}\cos(x)\right))\\ &= 3\left(\boxed{?}\right)^2-4\left(\boxed{?}\right)\left(\boxed{?}\right)\\ &= 3\pi^2+4\pi.\end{aligned}$$

From the Limit Laws, we saw that polynomial functions had a very important property for us. They were 'continuous'.

Definition. A function f is **continuous at a point** a if

$$\lim_{x\to a}f(x)=f(a).$$

That means to find the limit value we just have to 'plug in a'. Most of our favorite functions have this property. Polynomials and exponential functions are continuous everywhere. Rational functions and trigonometric functions are continuous on their domain. Inverse functions of invertible continuous functions are continuous on their domain, so logarithms and radical functions are continuous on their domains. If we take a composition of continuous functions, the result is also a continuous function! Continuity gives us an efficient way of calculating limits!

Example 143. *Evaluate:* $\lim_{x\to 0}\left(x^2\sin(x)+4x\cos(x)-3^{6x}\right)$

The functions given by the equations $4x$, x^2, $\sin(x)$, and $\cos(x)$ are basic continuous functions. The function 3^{4x} is a composition of the continuous functions 3^x and $6x$, so it is continuous. The function $x^2\sin(x)+4x\cos(x)-3^{6x}$ is itself a continuous function. That means

$$\lim_{x\to 0}\left(x^2\sin(x)+4x\cos(x)-3^{6x}\right)=\boxed{?}.$$

Try one on your own.

Problem 4. *Evaluate the limit.*

$$\lim_{x\to 3}\left(4(x-2)^2-3e^{2x-6}+\tan\left(\frac{x^2-8}{4}\pi\right)\right)$$

What if we get a function that isn't continuous?

Example 144. *Evaluate:* $\lim_{x\to 2}\dfrac{x^2-4}{x^2+x-6}$.

Notice that when $x=2$, the denominator x^2+x-6 becomes 0, so 2 is not in the domain of the function $\dfrac{x^2-4}{x^2+x-6}$, so we can't just plug into it. If we plug $x=2$ into the numerator, we also get 0, so this is an indeterminate form of type $\frac{\mathbf{0}}{\mathbf{0}}$. In order to evaluate this limit, we need to do some algebra to simplify. Let's factor!

$$\begin{aligned}\frac{x^2-4}{x^2+x-6} &= \frac{(x+2)(x-2)}{(x+3)(x-2)}\\ &= \frac{(x+2)\cancel{(x-2)}}{(x+3)\cancel{(x-2)}}\\ &= \frac{x+2}{x+3}\end{aligned}$$

$$\lim_{x\to 2}\frac{x-4}{x^2+x-6}$$

In this example, we were able to use our algebra skills to replace the function with a discontinuity at $x=2$ with a function that is continuous there. Try one on your own.

Problem 5. *Evaluate the limit.*

$$\lim_{x\to -1}\frac{x^2+5x+4}{2x^2+5x+3}$$

Algebra doesn't always help, though. The function in the next example is not continuous at $x=3$, but we cannot replace it by a function that is continuous there.

Example 145. *Evaluate:* $\lim_{x\to 3} \dfrac{2^x \ln(x-1)}{(x-3)^3 \cos(x)}$.

What happens if we try to plug in $x = 3$? The denominator becomes 0, so that won't work. What happens to the numerator? $2^3 \ln(3-1) = 8\ln 2$ is not zero, so this is a limit of form $\frac{\#}{0}$. The one-sided limits will be either ∞ or $-\infty$ (giving vertical asymptotes in the graph), and we need to check both.

Let's start with the right-hand limit $x \to 3^+$: The numerator $2^x \ln(x-1)$ is continuous and positive near 3. In the denominator, $\cos(x)$ is continuous and near $\cos(3)$ (and negative) for x near 3. The $(x-3)^3$ factor will be 0 at $x = 3$, but for $x > 3$, $(x-3)$ is positive. That means the fraction will be $\dfrac{\text{positive}}{\text{positive} \cdot \text{negative}}$, so

$$\lim_{x\to 3^+} \frac{2^x \ln(x-1)}{(x-3)^3 \cos(x)} = -\infty.$$

On to the left-hand limit $x \to 3^-$: The numerator $2^x \ln(x-1)$ is continuous and positive near 3. In the denominator, $\cos(x)$ is continuous and near $\cos(3)$ (and negative) for x near 3. The only possible change would come from the $(x-3)^3$ factor. Since we are looking at $x < 3$, we know $(x-3)$ is negative, so $(x-3)^3$ is negative. The fraction will be $\dfrac{\text{positive}}{\text{negative} \cdot \text{negative}}$. Then

$$\lim_{x\to 3^-} \frac{2^x \ln(x-1)}{(x-3)^3 \cos(x)} = \infty.$$

Putting these together we see that $\lim_{x\to 3} \dfrac{2^x \ln(x-1)}{(x-3)^3 \cos(x)}$ does not exist.

For $\frac{0}{0}$ forms, algebra will usually help us evaluate the limit we are interested in. For $\frac{\#}{0}$ forms, like in this previous example, we're expecting one-sided limits of $\pm\infty$. We use the signs of the individual factors to see whether each one-sided limit is either $+\infty$ or $-\infty$, then compare them to determine if we can say anything about the limit itself.

Problem 6. *Evaluate the limit.*

$$\lim_{x\to 1^-} \frac{(3+x)^2 \sin(x)}{x(1-x)^5 e^x}$$

In all of the limits we've been talking about so far, the input variable is tending to a specific value. If instead we allow it to grow (in either direction) without bound, we are talking about 'Limits at Infinity' rather than just 'Limits'. These limits at infinity are the tools we used to detect horizontal asymptotes.

Example 146. *Evaluate:* $\lim_{x\to\infty} \dfrac{4x^3 - 6x + 2}{5x^3 + 7x^2 - 3x + 1}$.

The numerator and denominator each tend to ∞ as x tends to ∞, so this limit has form $\frac{\infty}{\infty}$, another indeterminate form. We need to do some simplification to evaluate it. The function is rational, so remember our trick. Start by finding the largest degree term in the denominator. Here that's an x^3-term. Divide the numerator and denominator by x^3.

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x^3 - 6x + 2}{5x^3 + 7x^2 - 3x + 1} &= \lim_{x\to\infty} \frac{4x^3 - 6x + 2}{5x^3 + 7x^2 - 3x + 1} \cdot \frac{1/x^3}{1/x^3} \\
&= \lim_{x\to\infty} \frac{\frac{4x^3}{x^3} - \frac{6x}{x^3} + \frac{2}{x^3}}{\frac{5x^3}{x^3} + \frac{7x^2}{x^3} - \frac{3x}{x^3} + \frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{4 - \frac{6}{x^2} + \frac{2}{x^3}}{5 + \frac{7}{x} - \frac{3}{x^2} + \frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{4}{5} \\
&= \boxed{?}.
\end{aligned}$$

With a little work, this trick also helps with functions that are not rational.

Example 147. *Evaluate:* $\lim\limits_{x\to-\infty} \dfrac{3x+1}{\sqrt{x^2+9}}$.

In the square root, there is a polynomial x^2+9. Since x is tending toward $-\infty$, we can rewrite the radical in the denominator.

$$\begin{aligned}\sqrt{x^2+9} &= \sqrt{x^2\left(1+\frac{9}{x^2}\right)}\\ &= \sqrt{x^2}\sqrt{1+\frac{9}{x^2}}\\ &= |x|\sqrt{1+\frac{9}{x^2}}\\ &= -x\sqrt{1+\frac{9}{x^2}}.\end{aligned}$$

(Why $-x$ instead of x? When would we have gotten x instead?)

If the denominator were really $-x$, then we would divide the numerator and denominator by x. Let's see what happens if we do that anyway.

$$\begin{aligned}\lim_{x\to-\infty}\frac{3x+1}{\sqrt{x^2+9}} &= \lim_{x\to-\infty}\frac{3x+1}{\sqrt{x^2+9}}\cdot\frac{1/x}{1/x}\\ &= \lim_{x\to-\infty}\frac{3+\frac{1}{x}}{\frac{1}{x}\cdot\sqrt{x^2+9}}\\ &= \lim_{x\to-\infty}\frac{3+\frac{1}{x}}{\frac{1}{x}\cdot -x\sqrt{1+\frac{9}{x^2}}}\\ &= \lim_{x\to-\infty}\frac{3+\frac{1}{x}}{-\sqrt{1+\frac{9}{x^2}}}\\ &= \lim_{x\to-\infty}\frac{3}{-\sqrt{1}}\\ &= \boxed{?}.\end{aligned}$$

Let's try one that looks a bit different.

Example 148. *Evaluate:* $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2+1}\right)$.

It is clear that $\lim\limits_{x\to\infty} x = \infty$ and $\lim\limits_{x\to\infty}\sqrt{x^2+1} = \infty$. That means this limit has form $\infty - \infty$, which is indeterminate. To evaluate, we'll need to rewrite it a bit. We'll start by multiplying and dividing by the conjugate.

$$\begin{aligned}
\lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right) &= \lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right)\cdot\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}} \\
&= \lim_{x\to\infty}\frac{\left(x-\sqrt{x^2+1}\right)\left(x+\sqrt{x^2+1}\right)}{x+\sqrt{x^2+1}} \\
&= \lim_{x\to\infty}\frac{x^2-\left(x^2+1\right)}{x+\sqrt{x^2+1}} \\
&= \lim_{x\to\infty}\frac{-1}{x+\sqrt{x^2+1}}.
\end{aligned}$$

The numerator is a constant -1 and the denominator tends to ∞. That means

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right) = \boxed{?}.$$

Try one on your own.

Problem 7. *Evaluate the limit.*

$$\lim_{x\to-\infty}\frac{4x^3+x}{5x^3+2\sqrt{x^6+1}}$$

26 Review of differentiation

After completing this section, students should be able to do the following.

- ...
- ...
- ...

Break-Ground:

26.1 Review Derivatives BreakGround

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, do you remember when we first starting graphing functions? Like with a "T-chart?"
Riley: I remember everything.
Devyn: ...
Riley: ...
Devyn: ...
Riley: ...
Devyn: ...
Riley: ...

Problem 1. *Question*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 2. *When x is a small number (near zero), which term of* $5x^6 - 5x^5 - 5x^4 + 5x^3 + x^2 - 1$ *is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Dig-In:

26.2 Review Derivatives

What are derivatives?

Definition. A **rational function** in the variable x is a function the form

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions. The domain of a rational function is all real numbers except for where the denominator is equal to zero.

Question 79. *Which of the following are rational functions?*

Select All Correct Answers:

(a) $f(x) = 0$

(b) $f(x) = \dfrac{3x+1}{x^2-4x+5}$

(c) $f(x) = e^x$

(d) $f(x) = \dfrac{\sin(x)}{\cos(x)}$

(e) $f(x) = -4x^{-3} + 5x^{-1} + 7 - 18x^2$

(f) $f(x) = x^{1/2} - x + 8$

(g) $f(x) = \dfrac{\sqrt{x}}{x^3-x}$

27 Linear approximation

After completing this section, students should be able to do the following.

- Define linear approximation as an application of the tangent to a curve.
- Find the linear approximation to a function at a point and use it to approximate the function value.
- Identify when a linear approximation can be used.
- Label a graph with the appropriate quantities used in linear approximation.
- Find the error of a linear approximation.
- Compute differentials.
- Use the second derivative to discuss whether the linear approximation over or underestimates the actual function value.
- Contrast the notation and meaning of dy versus Δy.
- Understand that the error shrinks faster than the displacement in the input.
- Justify the chain rule via the composition of linear approximations.

Break-Ground:

27.1 Replacing curves with lines

Check out this dialogue between two calculus students (based on a true story):

Devyn: Hmmmm. Riley, I just thought of something...
Riley: What is it?
Devyn: When we compute derivatives, we are looking at the slope of tangent lines right?
Riley: You know it.
Devyn: Well, I wonder: Instead of studying curves, could we just study "zoomed-in" lines?
Riley: I'm not sure...

You read someplace that

$$\ell(x) = \frac{1}{4}(x-4)+2$$

is a good approximation for $f(x) = \sqrt{x}$ when x is close to 4.

Problem 1. *Plot $\ell(x)$ and $f(x)$. Explain how this shows that $\ell(x)$ is a good approximation when x is close to* 4.

Problem 2. *Explain (if you can) using concepts of calculus to explain why $\ell(x)$ is a good approximation for $f(x)$ when x is close to* 4.

Dig-In:

27.2 Linear approximation

Given a function, a *linear approximation* is a fancy phrase for something you already know:

The derivative is the slope of the tangent line.

Except in this section, the emphasis is on the **line**.

Definition. If f is a differentiable function at $x = a$, then a **linear approximation** for f at $x = a$ is given by

$$\ell(x) = f'(a)(x - a) + f(a).$$

Note that $\ell(x)$ is just the tangent line to $f(x)$ at $x = a$.

A linear approximation of f is a "good" approximation as long as x is "not too far" from a. If one "zooms in" on f sufficiently, then f and the linear approximation are nearly indistinguishable. As a first example, we will see how linear approximations allow us to make approximate "difficult" computations.

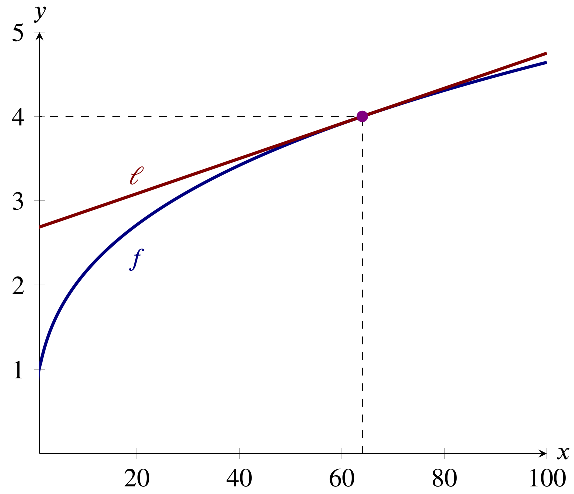
Example 149. *Use a linear approximation of* $f(x) = \sqrt[3]{x}$ *at* $x = 64$ *to approximate* $\sqrt[3]{50}$.

To start, write

$$\frac{d}{dx} f(x) = \frac{d}{dx} x^{1/3} = \frac{1}{3x^{2/3}}.$$

So our linear approximation is

$$\begin{aligned}\ell(x) &= \frac{1}{3 \cdot 64^{2/3}}(x - 64) + 4\\ &= \frac{1}{48}(x - 64) + 4\\ &= \frac{x}{48} + \frac{8}{3}.\end{aligned}$$



Now we evaluate $\ell(50) \approx 3.71$ and compare it to $\sqrt[3]{50} \approx 3.68$. From this we see that the linear approximation, while perhaps inexact, is computationally **easier** than computing the cube root.

With modern calculators and computing software it may not appear necessary to use linear approximations. In fact they are quite useful. In cases requiring an explicit numerical approximation, they allow us to get a quick rough estimate which can be used as a "reality check" on a more complex calculation. In some complex calculations involving functions, the linear approximation makes an otherwise intractable calculation possible, without serious loss of accuracy.

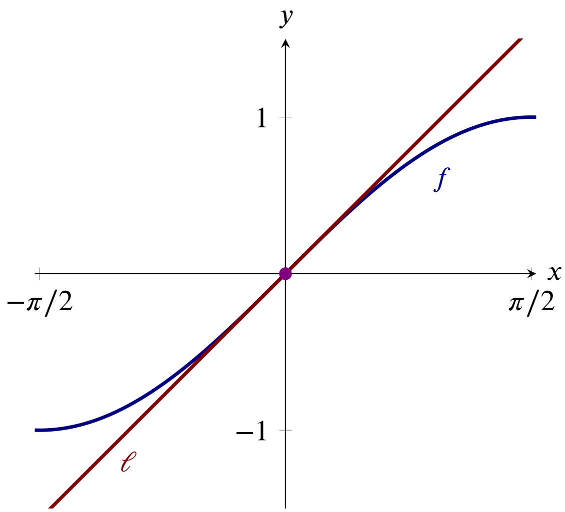
Example 150. *Use a linear approximation of* $f(x) = \sin(x)$ *at* $x = 0$ *to approximate* $\sin(0.3)$.

To start, write

$$\frac{d}{dx} f(x) = \cos(x),$$

so our linear approximation is

$$\begin{aligned}\ell(x) &= \cos(0) \cdot (x - 0) + 0 \\ &= x.\end{aligned}$$



Hence a linear approximation for $\sin(x)$ at $x = 0$ is $\ell(x) = x$, and so $\ell(0.3) = 0.3$. Comparing this to $\sin(.3) \approx 0.295$, we see that the approximation is quite good. For this reason, it is common to approximate $\sin(x)$ with its linear approximation $\ell(x) = x$ when x is near zero.

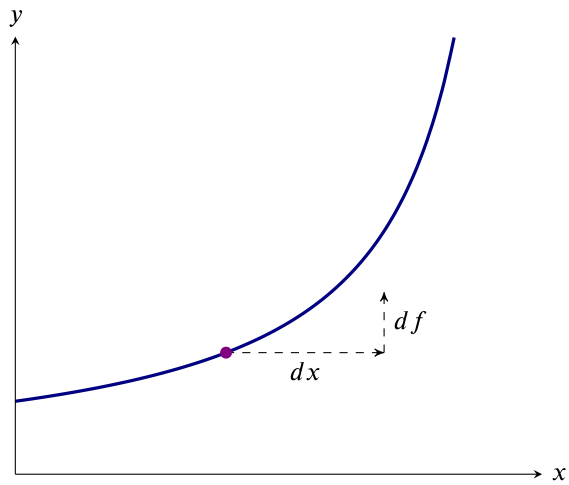
Differentials

The notion of a *differential* goes back to the origins of calculus, though our modern conceptualization of a differential is somewhat different than how they were initially understood.

Definition. Let f be a differentiable function. We define a new independent variable dx, and a new dependent variable

$$df = f'(x) \cdot dx.$$

The variables dx and df are called **differentials**. Geometrically, differentials can be interpreted via the diagram below



Note, it is now the case (by definition!) that

$$\frac{df}{dx} = f'(x).$$

Question 80. *The differential dx is:*

Multiple Choice:

(a) *d times x.*

(b) *A single variable.*

Question 81. *The differential df is:*

Multiple Choice:

(a) *d times f.*

(b) *A single variable that is dependent on dx.*

Essentially, differentials allow us to solve the problems presented in the previous examples from a slightly different point of view. Recall, when h is near but not equal zero,

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

hence,

$$f'(x)h \approx f(x+h)-f(x).$$

Since h is simply a variable, and dx is simply a variable, we can replace h with dx to write

$$\begin{aligned} f'(x)\cdot dx &\approx f(x+dx)-f(x) \\ df &\approx f(x+dx)-f(x). \end{aligned}$$

Adding $f(x)$ to both sides we see

$$f(x+dx) \approx f(x)+df.$$

While this is something of a "sleight of hand" with variables, there are contexts where the language of differentials is common. Here is the basic strategy:

$$\underbrace{f(x+dx)}_{\text{what you want}} \approx \overbrace{f(x)}^{\text{what you know}} + \underbrace{df}_{\text{what you compute}}$$

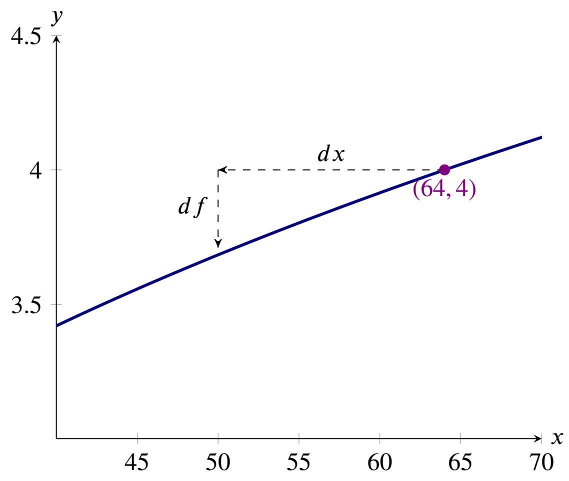
We will repeat our previous examples using differentials.

Example 151. *Use differentials to approximate* $\sqrt[3]{50}$.

Set $f(x)=\sqrt[3]{x}$. We want to know $\sqrt[3]{50}$. Since $4^3=64$, we set $x=64$. Setting $dx=-14$, we have

$$\begin{aligned} \sqrt[3]{50} = f(x+dx) &\approx f(x)+df \\ &\approx \sqrt[3]{64}+df. \end{aligned}$$

Here we see a plot of $y=\sqrt[3]{x}$ with the differentials above marked:



Now we must compute df:

$$\begin{aligned} df &= f'(x)\cdot dx \\ &= \frac{1}{3x^{2/3}}\cdot dx \\ &= \frac{1}{3\cdot 64^{2/3}}\cdot(-14) \\ &= \frac{1}{3\cdot 64^{2/3}}\cdot(-14) \\ &= \frac{-7}{24} \end{aligned}$$

Hence $f(50) \approx f(64) + \dfrac{-7}{24} \approx 3.71.$

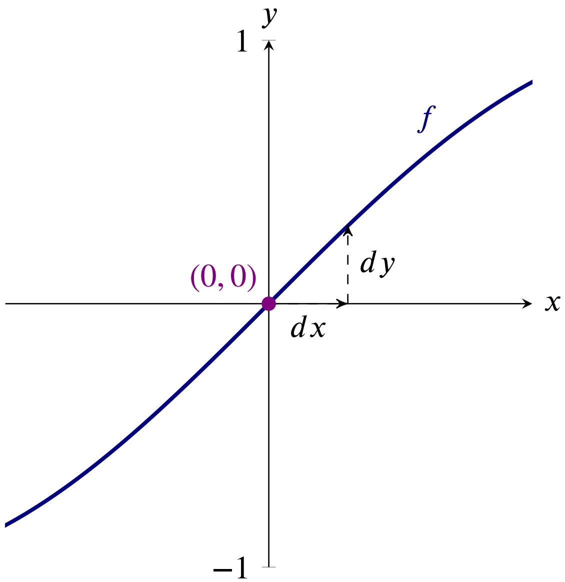
Example 152. *Use differentials to approximate* $\sin(0.3)$.

Set $y=\sin(x)$. We want to know $\sin(0.3)$. Since $\sin(0)=$

0, we will set $x = 0$ and $dx = 0.3$. Write with me

$$\begin{aligned}\sin(0.3) = \sin(x + dx) &\approx \sin(x) + dy \\ &\approx 0 + dy.\end{aligned}$$

Here we see a plot of $y = \sin(x)$ with the differentials above marked:



Now we must compute dy:

$$\begin{aligned}dy &= \left(\frac{d}{dx}\sin(x)\right)\cdot dx \\ &= \cos(0)\cdot dx \\ &= 1\cdot(0.3) \\ &= 0.3\end{aligned}$$

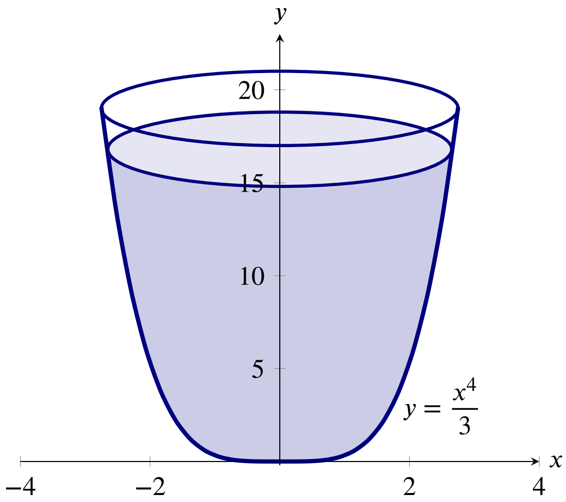
Hence $\sin(0.3) \approx \sin(0) + 0.3 \approx 0.3$.

The upshot is that linear approximations and differentials are simply two slightly different ways of doing the exact same thing.

Error approximation

Differentials also help us estimate error in real life settings.

Example 153. *The cross-section of a 250 ml glass can be modeled by the function* $r(x) = \dfrac{x^4}{3}$*:*



At 16.8 cm from the base of the glass, there is a mark indicating when the glass is filled to 250 ml. If the glass is filled within ±2 millimeters of the mark, what are the bounds on the volume? As a gesture of friendship, we will tell you that the volume in milliliters, as a function of the height of water in centimeters, y, is given by

$$V(y) = \frac{2\pi y^{3/2}}{\sqrt{3}}.$$

Note: If you persist in your quest to learn calculus, you will be able to derive the formula above like it's no-big-deal.

> We want to know what a small change in the height, y does to the volume V. These small changes can be modeled by the differentials dV and dy. Since
>
> $$dV = V'(y)\,dy$$
>
> and $V'(y) = \pi\sqrt{3y}$ we use the fact that $dy = \pm 0.2$ with $y = 16.8$ to see
>
> $$dV = \pi\sqrt{3 \cdot 16.8} \cdot 0.2.$$
>
> Hence the volume will vary by at most ± 4.46062 milliliters.

New and old friends

You might be wondering, given a plot $y = f(x)$,

> What's the difference between Δx and dx? What about Δy and dy?

Regardless, it is now a pressing question. Here's the deal:

$$\frac{\Delta y}{\Delta x}$$

is the **average rate of change** of $y = f(x)$ with respect to x. On the other hand:

$$\frac{dy}{dx}$$

is the **instantaneous rate of change** of $y = f(x)$ with respect to x. Essentially, Δx and dx are the same type of thing, they are (usually small) changes in x. However, Δy and dy are very different things.

- Δy is the change of y associated to Δx.
- dy is the change in y needed to make the following relation true:

  $$dy = f'(x)\,dx$$

Question 82. *Suppose $f(x) = x^2$. If we are at the point $x = 1$ and $\Delta x = dx = 0.1$, what is Δy? What is dy?*

Differentials can be confusing at first. However, when you master them, you will have a powerful tool at your disposal.

Dig-In:

27.3 Explanation of the product and chain rules

Now that we know about differentials, let's use them to give some intuition as to why the product and chain rules are true.

Explanation of the product rule

Linear approximations can help us explain why the product rule works.

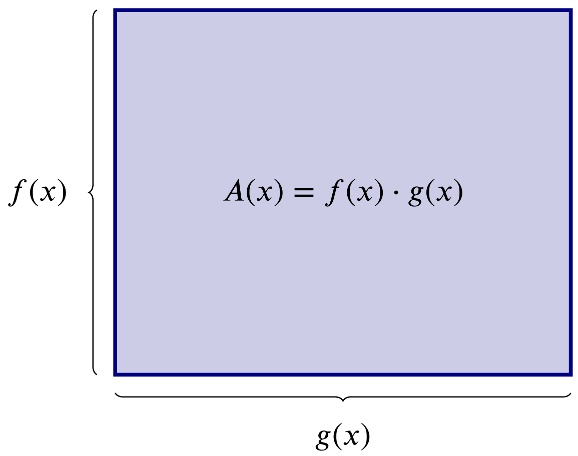
Theorem 48 (The product rule)**.** *If f and g are differentiable, then*

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

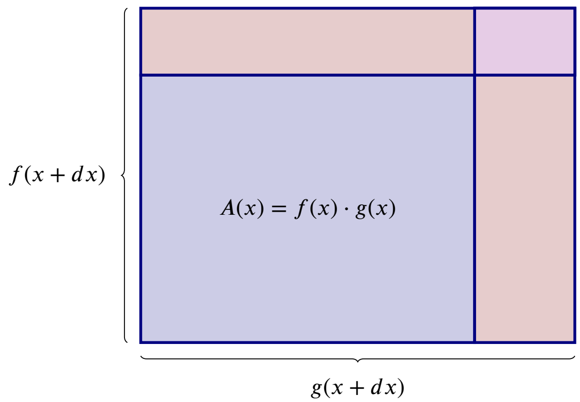
To start, we need some way to understand the function

$$A(x) = f(x) \cdot g(x).$$

One interpretation of multiplication is it is the area of a $f(x) \times g(x)$ rectangle:



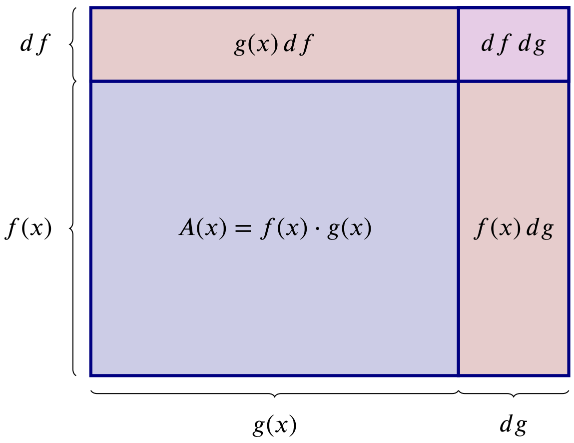
To understand the derivative of the product, we must understand how the area, A, changes as x changes. If we change the inputs of f and g by dx, then the size of the rectangle changes:



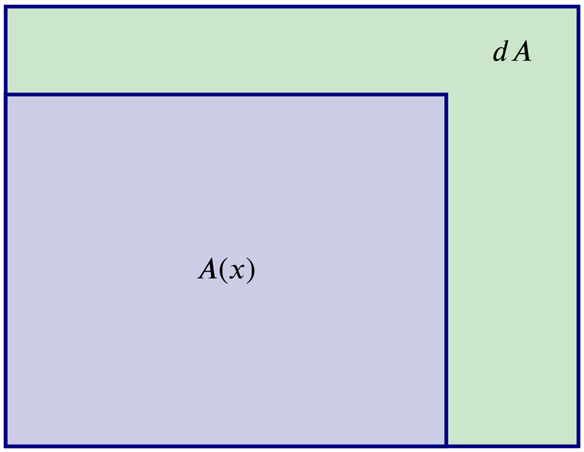
However, we know from our previous work that

$$\begin{aligned} f(x+dx) &\approx f(x) + df, \\ g(x+dx) &\approx g(x) + dg, \end{aligned}$$

so now our picture becomes:



Note, if we think of $A(x) = f(x) \cdot g(x)$, then we can also label our picture as follows:



Finally, from the pictures above and recalling that

$$\begin{aligned} d f &= f'(x)\,d x \\ d g &= g'(x)\,d x, \end{aligned}$$

we see that

$$\begin{aligned} d A &= f(x)\,d g + g(x)\,d f + d f\ d g \\ &= f(x)g'(x)\,d x + g(x)f'(x)\,d x + f'(x)g'(x)\,(d x)^2 \,. \end{aligned}$$

Dividing both sides by $d x$ we see

$$\frac{d A}{d x} = f(x)g'(x) + g(x)f'(x) + f'(x)g'(x)\,d x$$

and letting $d x$ go to zero we see

$$A'(x) = f(x)g'(x) + g(x)f'(x).$$

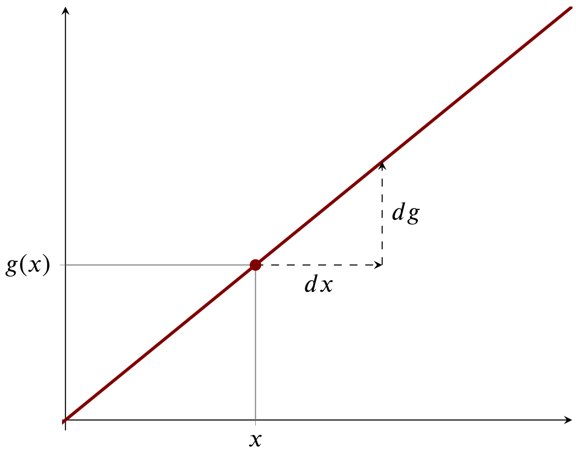
Explanation of the chain rule

Now we'll use linear approximations to help explain why the chain rule is true.

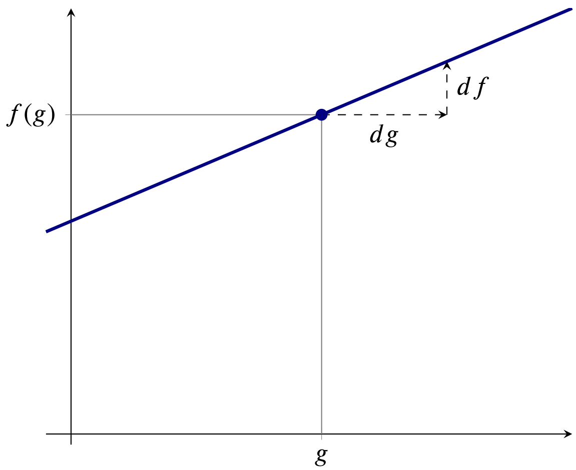
Theorem 49 (Chain Rule)**.** *If f and g are differentiable, then*

$$\frac{d}{d x} f(g(x)) = f'(g(x))g'(x).$$

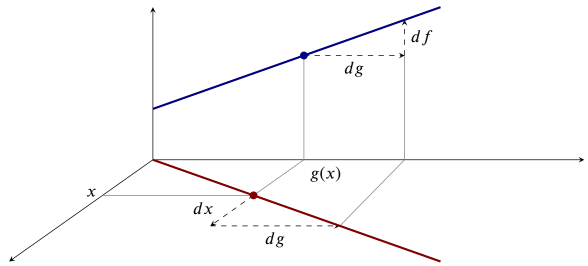
We'll try to understand this geometrically. In what follows, the functions f and g look like lines; however, the young mathematician should realize that we are **not** looking a true lines, instead we are looking at f and g sufficiently "zoomed-in" so that they appear to be lines. First consider a graph of g with respect to x:



Now consider a graph of f with respect to g:



If we combine these graphs, by laying the graph of g on its side, we obtain:



Ah! From this we see that

$$\begin{aligned} df &= f'(g)\,dg \\ &= f'(g(x))g'(x)\,dx, \end{aligned}$$

so

$$\frac{df}{dx} = f'(g(x))g'(x).$$

These "explanations" are not meant to be the end of the story for the product rule and chain rule, rather they are hopefully the beginning. As you learn more mathematics, these explanations will be refined and made precise.

28 Concepts of graphing functions

After completing this section, students should be able to do the following.

- Understand what information the derivative gives concerning when a function is increasing or decreasing.
- Understand what information the second derivative gives concerning concavity of a function.
- Interpret limits as giving information about functions.
- Determine how the graph of a function looks based on an analytic description of the function.

Break-Ground:

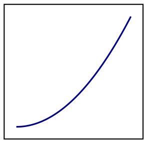
28.1 What's the graph look like?

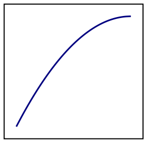
Check out this dialogue between two calculus students (based on a true story):

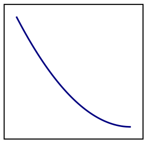
Devyn: Riley, I've been thinking about the derivative.
Riley: It's all about change. It's some "change-detector" tool for math.
Devyn: I know! What's crazy is that you can use it as a tool for sniffing out dirt on functions.
Riley: First f' tells us increasing or decreasing.
Devyn: Then f'' tells us concavity.
Riley: From just that we know all local maxes and mins.
Devyn: And if we use limits, we can find any asymptotes!
Riley: You know, I'd like to make up a procedure based on all these facts, that would tell me what the graph of any function would look like.
Devyn: Me too! Let's get to work!

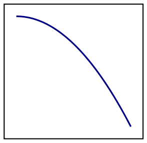
Problem 1. *On some interval, we know that $f'(x)$ is positive and $f''(x)$ is positive. Which of the following is the best option for the shape of the graph on that interval?*

Multiple Choice:

(a) 

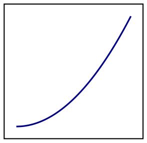
(b) 

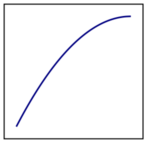
(c) 

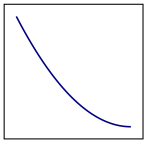
(d) 

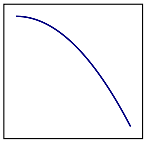
Problem 2. *On some interval, we know that $f'(x)$ is negative and $f''(x)$ is positive. Which of the following is the best option for the shape of the graph on that interval?*

Multiple Choice:

(a) 

(b) 

(c) 

(d) 

Dig-In:

28.2 Concepts of graphing functions

In this section, we review the graphical implications of limits, and the sign of the first and second derivative. You already know all this stuff: it is just important enough to hit it more than once, and put it all together.

Example 154. *Sketch the graph of a function f which has the following properties:*

- $f(0) = 0$
- $\lim_{x \to 10^+} f(x) = +\infty$
- $\lim_{x \to 10^-} f(x) = -\infty$
- $f'(x) < 0$ *on* $(-\infty, 0) \cup (6, 10) \cup (10, 14)$
- $f'(x) > 0$ *on* $(0, 6) \cup (14, \infty)$
- $f''(x) < 0$ *on* $(4, 10)$
- $f''(x) > 0$ *on* $(-\infty, 4) \cup (10, \infty)$

> Try this on your own first, then either check with a friend or check the online version.

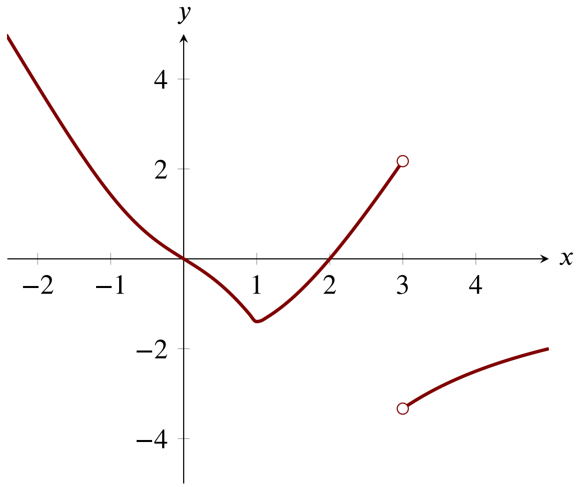
Example 155. *Sketch the graph of a function f which has the following properties:*

- $f(0) = 1$
- $f(6) = 2$
- $\lim_{x \to 6^+} f(x) = 3$
- $\lim_{x \to 6^-} f(x) = 1$
- $f'(x) < 0$ *on* $(-\infty, 1)$
- $f'(x) > 0$ *on* $(1, 6)$
- $f'(x) = -2$ *on* $(6, \infty)$
- $f''(x) < 0$ *on* $(2.5, 5)$
- $f''(x) > 0$ *on* $(-\infty, 2.5) \cup (5, 6)$

> Try this on your own first, then either check with a friend or check the online version.

Example 156. *The graph of f' (the derivative of f) is shown below.*

Assume f is continuous for all real numbers.



Question 83. *On which of the following intervals is f increasing?*

Select All Correct Answers:

(a) $(-\infty, 0)$

(b) $(0, 1)$

(c) $(1, 2)$

(d) $(2,3)$

(e) $(3,\infty)$

Question 84. *Which of the following are critical points of f?*

Select All Correct Answers:

(a) $x=0$

(b) $x=1$

(c) $x=2$

(d) $x=3$

Question 85. *Where do the local maxima occur?*

Select All Correct Answers:

(a) $x=0$

(b) $x=1$

(c) $x=2$

(d) $x=3$

Question 86. *Where does a point of inflection occur?*

Select All Correct Answers:

(a) $x=0$

(b) $x=1$

(c) $x=2$

(d) $x=3$

Question 87. *On which of the following intervals is f concave down?*

Select All Correct Answers:

(a) $(-\infty,0)$

(b) $(0,1)$

(c) $(1,2)$

(d) $(2,3)$

(e) $(3,\infty)$

Break-Ground:

28.3 Wanted: graphing procedure

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, OK I know how to plot something if I'm given a description.
Riley: Yes, it's kinda fun right?
Devyn: I know! But now I'm not sure how to get the information I need.
Riley: You know, I'd like to make up a procedure based on all these facts, that would tell me what the graph of any function would look like.
Devyn: Me too! Let's get to work!

Problem 3. *Below is a list of features of a graph of a function.*

(a) *Find any vertical asymptotes, these are points $x = a$ where $f(x)$ goes to infinity as x goes to a (from the right, left, or both).*

(b) *Find the critical points (the points where $f'(x) = 0$ or $f'(x)$ is undefined).*

(c) *Identify inflection points and concavity.*

(d) *Determine an interval that shows all relevant behavior.*

(e) *Find the candidates for inflection points, the points where $f''(x) = 0$ or $f''(x)$ is undefined.*

(f) *Compute f' and f''.*

(g) *Find the y-intercept, this is the point $(0, f(0))$. Place this point on your graph.*

(h) *Use either the first or second derivative test to identify local extrema and/or find the intervals where your function is increasing/decreasing.*

(i) *If possible, find the x-intercepts, the points where $f(x) = 0$. Place these points on your graph.*

(j) *Analyze end behavior: as $x \to \pm\infty$, what happens to the graph of f? Does it have horizontal asymptotes, increase or decrease without bound, or have some other kind of behavior?*

In what order should we take these steps? For example, one must compute f' before computing f''. Also, one must compute f' before finding the critical points. There is more than one correct answer.

Dig-In:

28.4 Computations for graphing functions

Let's get to the point. Here we use all of the tools we know to sketch the graph of $y = f(x)$:

- Compute f' and f''.
- Find the y-intercept, this is the point $(0, f(0))$. Place this point on your graph.
- Find any vertical asymptotes, these are points $x = a$ where $f(x)$ goes to infinity as x goes to a (from the right, left, or both).
- If possible, find the x-intercepts, the points where $f(x) = 0$. Place these points on your graph.
- Analyze end behavior: as $x \to \pm\infty$, what happens to the graph of f? Does it have horizontal asymptotes, increase or decrease without bound, or have some other kind of behavior?
- Find the critical points (the points where $f'(x) = 0$ or $f'(x)$ is undefined).
- Use either the first or second derivative test to identify local extrema and/or find the intervals where your function is increasing/decreasing.
- Find the candidates for inflection points, the points where $f''(x) = 0$ or $f''(x)$ is undefined.
- Identify inflection points and concavity.
- Determine an interval that shows all relevant behavior.

At this point you should be able to sketch the plot of your function.

Example 157. *Sketch the plot of* $2x^3 - 3x^2 - 12x$.

> Try this on your own first, then either check with a friend, a graphing calculator (like Desmos[a]) or check the online version.
>
> [a] See Desmos at http://www.desmos.com

Example 158. *Sketch the plot of*

$$f(x) = \begin{cases} xe^x + 2 & \textit{if } x < 0 \\ x^4 - x^2 + 3 & \textit{if } x \geq 0. \end{cases}$$

> Try this on your own first, then either check with a friend, a graphing calculator (like Desmos[a]), or check the online version.
>
> [a] See Desmos at http://www.desmos.com

29 Implicit differentiation

After completing this section, students should be able to do the following.

- Implicitly differentiate expressions.
- Find the equation of the tangent line for curves that are not plots of functions.
- Understand how changing the variable changes how we take the derivative.
- Understand the derivatives of expressions that are not functions or not solved for y.

Break-Ground:

29.1 Standard form

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I think we've been too explicit with each other. We should try to be more implicit.
Riley: I. Um. Don't really…
Devyn: I mean when plotting things!
Riley: Okay, but I still have no idea what you are talking about.
Devyn: Remember when we first learned the equation of a line, and the "standard form" was

$$ax + by = c$$

or something, which is totally useless for graphing. Also a circle is

$$x^2 + y^2 = r^2$$

or something, and here y isn't even a function of x.
Riley: Ah, I'm starting to remember. We can write the same thing in two ways. For example, if you write

$$y = mx + b,$$

then y is **explicity** a function of x but if you write

$$ax + by = c,$$

then y is **implicitly** a function of x.
Devyn: What I'm trying to say is that we need to learn how to work with these "implicit" functions.

Problem 1. *Consider the unit circle*

$$x^2 + y^2 = 1.$$

The point $P = (0, 1)$ is on this circle. Reason geometrically to determine the slope of the line tangent to $x^2 + y^2 = 1$ at P.

Problem 2. *Consider the unit circle*

$$x^2 + y^2 = 1.$$

The point

$$P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

is on this circle. Reason geometrically to determine the slope of the line tangent to $x^2 + y^2 = 1$ at P.

Dig-In:

29.2 Implicit differentiation

Review of the chain rule

Implicit differentiation is really just an application of the chain rule. So recall:

Theorem 50 (Chain Rule)**.** *If $f(x)$ and $g(x)$ are differentiable, then*

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

Of particular use in this section is the following. If y is a differentiable function of x and if f is a differentiable function, then

$$\frac{d}{dx}(f(y)) = f'(y) \cdot \frac{d}{dx}(y) = f'(y)\frac{dy}{dx}.$$

Implicit differentiation

The functions we've been dealing with so far have been *explicit functions*, meaning that the dependent variable is written in terms of the independent variable. For example:

$$y = 3x^2 - 2x + 1, \qquad y = e^{3x}, \qquad y = \frac{x-2}{x^2-3x+2}.$$

However, there is another type of function, called an *implicit function*. In this case, the dependent variable is not stated explicitly in terms of the independent variable. Some examples are:

$$x^2+y^2 = 4, \qquad x^3+y^3 = 9xy, \qquad x^4+3x^2 = x^{2/3}+y^{2/3} = 1.$$

Your inclination might be simply to solve each of these for y and go merrily on your way. However this can be difficult and it may require two *branches*, for example to explicitly plot $x^2 + y^2 = 4$, one needs both $y = \sqrt{4-x^2}$ and $y = -\sqrt{4-x^2}$. Moreover, it may not even be possible to solve for y. To deal with such situations, we use *implicit differentiation*. We'll start with a basic example.

Example 159. *Consider the curve defined by:*

$$x^2 + y^2 = 1$$

(a) *Compute* $\frac{dy}{dx}$.

(b) *Find the slope of the tangent line at* $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Starting with

$$x^2 + y^2 = 1$$

we apply the differential operator $\frac{d}{dx}$ to both sides of the equation to obtain

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}1.$$

Applying the sum rule we see

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0.$$

Let's examine each of these terms in turn. To start

$$\frac{d}{dx}x^2 = 2x.$$

On the other hand, $\frac{d}{dx}y^2$ is somewhat different. Here you imagine that $y = f(x)$, and hence by the chain rule

$$\begin{aligned}\frac{d}{dx}y^2 &= \frac{d}{dx}(f(x))^2\\ &= 2 \cdot f(x) \cdot f'(x)\\ &= 2y\frac{dy}{dx}.\end{aligned}$$

Putting this together we are left with the equation

$$2x + 2y\frac{dy}{dx} = 0$$

At this point, we solve for $\frac{dy}{dx}$. Write

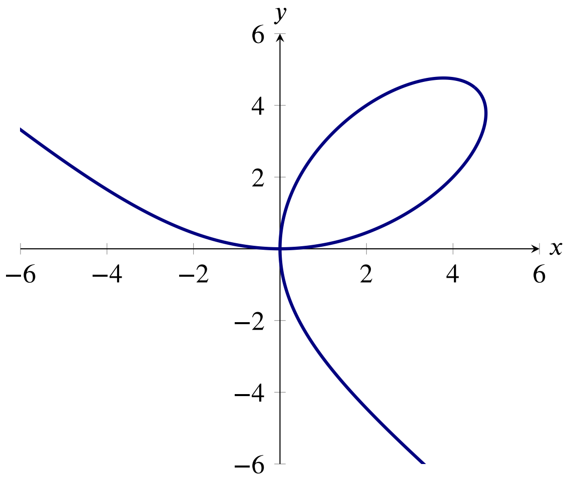
$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-x}{y}. \end{aligned}$$

For the second part of the problem, we simply plug $x = \frac{\sqrt{2}}{2}$ and $y = \frac{\sqrt{2}}{2}$ into the formula above, hence the slope of the tangent line at this point is -1.

Let's see another illustrative example:

Example 160. *Consider the curve defined by:*

$$x^3 + y^3 = 9xy$$



(a) *Compute* $\frac{dy}{dx}$.

(b) *Find the slope of the tangent line at* $(4, 2)$.

Starting with

$$x^3 + y^3 = 9xy,$$

we apply the differential operator $\frac{d}{dx}$ to both sides of the equation to obtain

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}9xy.$$

Applying the sum rule we see

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}9xy.$$

Let's examine each of these terms in turn. To start

$$\frac{d}{dx}x^3 = 3x^2.$$

On the other hand $\frac{d}{dx}y^3$ is somewhat different. Here you imagine that $y = f(x)$, and hence by the chain rule

$$\begin{aligned} \frac{d}{dx}y^3 &= \frac{d}{dx}(y(x))^3 \\ &= 3(f(x))^2 \cdot f'(x) \\ &= 3y^2\frac{dy}{dx}. \end{aligned}$$

Considering the final term $\frac{d}{dx}9xy$, we again imagine that $y = f(x)$. Hence

$$\begin{aligned} \frac{d}{dx}9xy &= 9\frac{d}{dx}x \cdot f(x) \\ &= 9\left(x \cdot y'(x) + f(x)\right) \\ &= 9x\frac{dy}{dx} + 9y. \end{aligned}$$

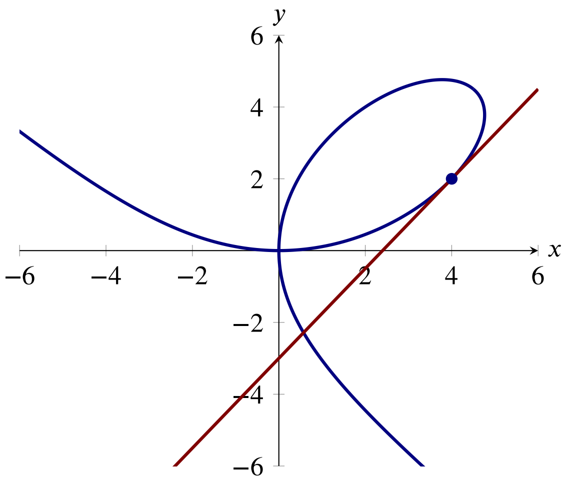
Putting this all together we are left with the equation

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y.$$

At this point, we solve for $\frac{dy}{dx}$. Write

$$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} &= 9x\frac{dy}{dx} + 9y\\ 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} &= 9y - 3x^2\\ \frac{dy}{dx}\left(3y^2 - 9x\right) &= 9y - 3x^2\\ \frac{dy}{dx} &= \frac{9y-3x^2}{3y^2-9x} = \frac{3y-x^2}{y^2-3x}.\end{aligned}$$

For the second part of the problem, we simply plug $x = 4$ and $y = 2$ into the formula above, hence the slope of the tangent line at $(4, 2)$ is $\frac{5}{4}$. We've included a plot for your viewing pleasure:



You might think that the step in which we solve for $\frac{dy}{dx}$ could sometimes be difficult. In fact, *this never happens*. All occurrences $\frac{dy}{dx}$ arise from applying the chain rule, and whenever the chain rule is used it deposits a single $\frac{dy}{dx}$ multiplied by some other expression. Hence our expression is linear in $\frac{dy}{dx}$, it will always be possible to group the terms containing $\frac{dy}{dx}$ together and factor out the $\frac{dy}{dx}$, just as in the previous examples.

One more last example:

Example 161. *Consider the curve defined by*

$$\cos(xy) - \frac{y}{x} = 4x^2y^3.$$

Compute $\frac{dx}{dy}$.

First, notice that the problem asks for $\frac{dx}{dy}$, **not** $\frac{dy}{dx}$. So we are considering x as a function of y. This means the variables have changed places! Not to worry, everything is exactly the same. We apply $\frac{d}{dy}$ to both sides of the equation to get

$$\frac{d}{dy}\left(\cos(xy) - \frac{y}{x}\right) = \frac{d}{dy}(4x^2y^3)$$

which gives us

$$-\sin(xy)\left(y\frac{dx}{dy} + x\right) - \frac{x - y\frac{dx}{dy}}{x^2} = 8xy^3\frac{dx}{dy} + 12x^2y^2.$$

Distributing and multiplying by x^2 yields

$$-x^2 y\sin(xy)\frac{dx}{dy} - x^3\sin(xy) - x + y\frac{dx}{dy} = 8x^3y^3\frac{dx}{dy} + 12x^4y^2.$$

Grouping terms, factoring, and dividing finally gives us

$$-x^2 y\sin(xy)\frac{dx}{dy} + y\frac{dx}{dy} - 8x^3y^3\frac{dx}{dy} = x^3\sin(xy) + x + 12x^4y^2$$

so,

$$\left(y - x^2 y\sin(xy) - 8x^3y^3\right)\frac{dx}{dy} = x^3\sin(xy)+x+12x^4y^2$$

and now we see

$$\frac{dx}{dy} = \frac{x^3\sin(xy) + x + 12x^4y^2}{y - x^2y\sin(xy) - 8x^3y^3}.$$

30 Logarithmic differentiation

After completing this section, students should be able to do the following.

- Identify situations where logs can be used to help find derivatives.
- Use logarithmic differentiation to simplify taking derivatives.
- Take derivatives of logarithms and exponents of all bases.
- Take derivatives of functions raised to functions.
- Recognize the difference between a variable as the base and a variable as the exponent.
- Work with the inverse properties of e^x and $\ln(x)$.

Break-Ground:

30.1 Multiplication to addition

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, why is the product rule so much harder than the sum rule?
Riley: Ever since 2nd grade, I've known that multiplication is *harder* than addition.
Devyn: I know! I was reading somewhere that a slide-rule somehow turns "multiplication into addition."
Riley: Wow! I wonder how that works?
Devyn: I *think* it has something to do with logs?
Riley: What? How does this work?

Devyn is right, logarithms are used (and were invented) to convert difficult multiplication problems into simpler addition problems.

Problem 1. *Let* $f(x) = \sin(x) \cdot \cos(x) \cdot e^x$. *Compute*

$$\frac{d}{dx} f(x)$$

Now, let's see what happens if we do the same problem but we take the natural log of both sides first:

$$\begin{aligned} f(x) &= \sin(x) \cdot \cos(x) \cdot e^x \\ \ln(f(x)) &= \ln(\sin(x) \cdot \cos(x) \cdot e^x) \\ \ln(f(x)) &= \ln(\sin(x)) + \ln(\cos(x)) + \ln(e^x) \end{aligned}$$

Now we'll take the derivative of both sides of the equation. By the chain rule

$$\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$$

Problem 2. *Compute*

$$\frac{d}{dx} \ln(\sin(x))$$

Problem 3. *Compute*

$$\frac{d}{dx} \ln(\cos(x))$$

Problem 4. *Compute*

$$\frac{d}{dx} \ln(e^x)$$

So we have

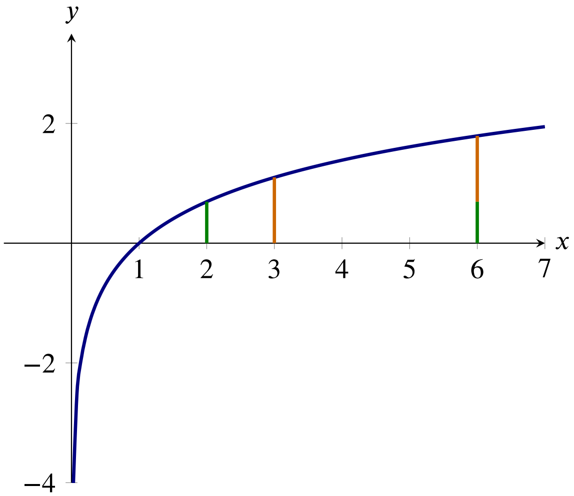
$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)} + 1 \\ f'(x) &= f(x) \left(\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)} + 1 \right) \\ &= \sin(x)\cos(x)e^x \left(\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)} + 1 \right) \end{aligned}$$

Dig-In:

30.2 Logarithmic differentiation

Logarithms were originally developed as a computational tool. The key fact that made this possible is that:

$$\log_b(xy) = \log_b(x) + \log_b(y).$$



Before the days of calculators and computers, this was critical knowledge for anyone in a computational discipline.

Example 162. *Compute* $138 \cdot 23.4$ *using logarithms.*

Start by writing both numbers in scientific notation

$$\left(1.38 \cdot 10^2\right) \cdot \left(2.34 \cdot 10^1\right).$$

Next we use a log-table, which gives $\log_{10}(N)$ for values of N ranging between 0 and 9. We've reproduced part of such a table below.

N	0	1	2	3	4	5	6	7	8	9
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784
3.2	0.5052	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172

From the table, we see that

$$\log_{10}(1.38) \approx 0.1399 \qquad \text{and} \qquad \log_{10}(2.34) \approx 0.3692$$

Add these numbers together to get 0.5091. Essentially, we know the following at this point:

$$\begin{array}{ccccc} \log_{10}(?) & = & \log_{10}(1.38) & + & \log_{10}(2.34) \\ \approx & & \approx & & \approx \\ 0.5091 & = & 0.1399 & + & 0.3692 \end{array}$$

Using the table again, we see that $\log_{10}(3.23) \approx 0.5091$. Since we were working in scientific notation, we need to multiply this by 10^3. Our final answer is

$$3230 \approx 138 \cdot 23.4$$

Since $138 \cdot 23.4 = 3229.2$, this is a good approximation.

The moral is:

> **Logarithms allow us to use addition in place of multiplication.**

Logarithmic differentiation

When taking derivatives, both the product rule and the quotient rule can be cumbersome to use. Logarithms will save the day. A key point is the following

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

which follows from the chain rule. Let's look at an illustrative example to see how this is actually used.

Example 163. *Compute:*

$$\frac{d}{dx}\frac{x^9 e^{4x}}{\sqrt{x^2+4}}$$

Recall the properties of logarithms:

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(x/y) = \log_b(x) - \log_b(y)$
- $\log_b(x^y) = y\log_b(x)$

While we could use the product and quotient rule to solve this problem, it would be tedious. Start by taking the logarithm of the function to be differentiated.

$$\begin{aligned}\ln\left(\frac{x^9e^{4x}}{\sqrt{x^2+4}}\right) &= \ln\left(x^9e^{4x}\right) - \ln\left(\sqrt{x^2+4}\right)\\ &= \ln\left(x^9\right) + \ln\left(e^{4x}\right) - \ln\left((x^2+4)^{1/2}\right)\\ &= 9\ln(x) + 4x - \frac{1}{2}\ln(x^2+4).\end{aligned}$$

Setting $f(x) = \dfrac{x^9e^{4x}}{\sqrt{x^2+4}}$*, we can write*

$$\ln(f(x)) = 9\ln(x) + 4x - \frac{1}{2}\ln(x^2+4).$$

Differentiating both sides, we find

$$\frac{f'(x)}{f(x)} = \frac{9}{x} + 4 - \frac{x}{x^2+4}.$$

Finally we solve for $f'(x)$*, write*

$$f'(x) = \left(\frac{9}{x} + 4 - \frac{x}{x^2+4}\right)\left(\frac{x^9e^{4x}}{\sqrt{x^2+4}}\right).$$

The process above is called *logarithmic differentiation*. Logarithmic differentiation allows us to compute new derivatives too.

Example 164. *Compute:*

$$\frac{d}{dx}x^x$$

The function x^x is tricky to differentiate. We cannot use the power rule, as the exponent is not a constant. However, if we set $f(x) = x^x$ we can write

$$\begin{aligned}\ln(f(x)) &= \ln\left(x^x\right)\\ &= x\ln(x).\end{aligned}$$

Differentiating both sides, we find

$$\frac{f'(x)}{f(x)} = 1 + \ln(x).$$

Now we can solve for $f'(x)$,

$$f'(x) = x^x + x^x\ln(x).$$

A general explanation of the power rule

Finally, recall that previously we only explained the power rule for positive exponents. Now we'll use logarithmic differentiation to give a explanation for all real-valued exponents. We restate the power rule for convenience sake:

Theorem 51 (Power Rule)**.** *For any real number n,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

We will use logarithmic differentiation. Set $f(x) = x^n$. Write

$$\begin{aligned}\ln(f(x)) &= \ln\left(x^n\right)\\ &= n\ln(x).\end{aligned}$$

> Now differentiate both sides, and solve for $f'(x)$
>
> $$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{n}{x}\\ f'(x) &= \frac{nf(x)}{x}\\ &= nx^{n-1}.\end{aligned}$$
>
> Thus we see that the power rule holds for all real-valued exponents.

While logarithmic differentiation might seem strange and new at first, with a little practice it will seem much more natural to you.

Part VI

Content for the Second Exam

31 Inverse Trigonometric Functions

After completing this section, students should be able to do the following.

- Know the domains and ranges of the inverse trigonometric functions.
- Evaluate inverse trigonometric functions of standard values.
- Understand the relationship between trigonometric and inverse trigonometric functions.
- Evaluate expressions and solve equations involving trigonometric functions and inverse trigonometric functions.

Break-Ground:

31.1 Inv Trig Function BreakGround

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, do you remember when we first starting graphing functions? Like with a "T-chart?"
Riley: I remember everything.
Devyn: ...
Riley: ...
Devyn: ...
Riley: ...
Devyn: ...
Riley: ...

Problem 1. *When x is a large number (furthest from zero), which term of $5x^6-5x^5-5x^4+5x^3+x^2-1$ is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 2. *When x is a small number (near zero), which term of $5x^6-5x^5-5x^4+5x^3+x^2-1$ is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 3. *Very roughly speaking, what does the graph of $y=5x^6-5x^5-5x^4+5x^3+x^2-1$ look like?*

Multiple Choice:

(a) *The graph starts in the lower left and ends in the upper right of the plane.*

(b) *The graph starts in the lower right and ends in the upper left of the plane.*

(c) *The graph looks something like the letter "U."*

(d) *The graph looks something like an upside down letter "U."*

Dig-In:

31.2 Inverse trigonometric functions

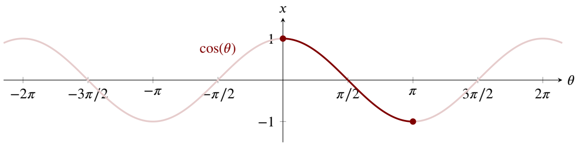
*** Need more in here ***

What are inverse trigonometric functions?

Trigonometric functions arise frequently in problems, and often we are interested in finding specific angle measures. For instance, you may want to find some angle θ such that

$$\cos(\theta) = .7$$

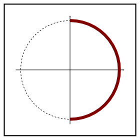
Hence we want to be able to "undo" trigonometric functions. However, since trigonometric functions are not one-to-one, meaning there are are infinitely many angles with $\cos(\theta) = .7$, it is impossible to find a true inverse function for $\cos(\theta)$. Nevertheless, it is useful to have something like an inverse to these functions, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values exactly once. If we "discard" all other angles, the resulting function has a proper inverse. In other words, we are restricting the domain of the trigonometric function in order to find an inverse. The function $\cos(\theta)$ takes on all values between -1 and 1 exactly once on the interval $[0, \pi]$.

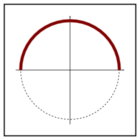


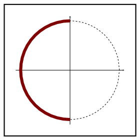
If we restrict the domain of $\cos(\theta)$ to this interval, then this restricted function is one-to-one and hence has an inverse.

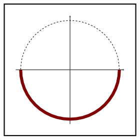
Question 88. *What arc on the unit circle corresponds to the restricted domain described above of* $\cos(\theta)$*?*

Multiple Choice:

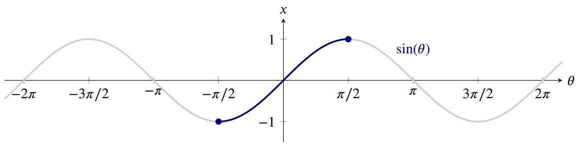
(a) 

(b) 

(c) 

(d) 

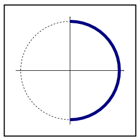
In a similar fashion, we need to restrict the domain of sine to be able to take an inverse. The function $\sin(\theta)$ takes on all values between -1 and 1 exactly once on the interval $[-\pi/2, \pi/2]$.

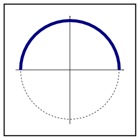


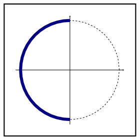
If we restrict the domain of $\sin(\theta)$ to this interval, then this restricted function is one-to-one and thus has an inverse.

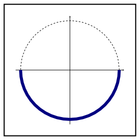
Question 89. *What arc on the unit circle corresponds to the restricted domain described above of* $\sin(\theta)$*?*

Multiple Choice:

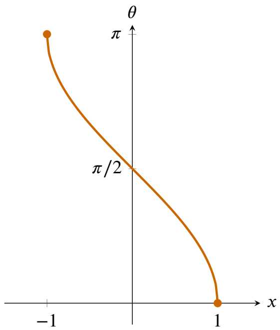
(a) 

(b) 

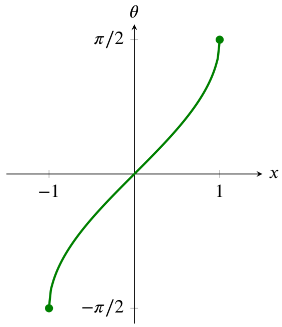
(c) 

(d) 

By examining both sine and cosine on restricted domains, we can now produce functions arcsine and arccosine:



Here we see a plot of $\arccos(x)$, the inverse function of $\cos(\theta)$ when the domain is restricted to the interval $[0, \pi]$.



Here we see a plot of $\arcsin(x)$, the inverse function of $\sin(\theta)$ when the domain is restricted to the interval $[-\pi/2, \pi/2]$.

The functions

$$\arccos(x) \qquad \text{and} \qquad \arcsin(x)$$

are called "arc" because they give the angle that cosine or sine used to produce their value. It is quite common to write

$$\arccos(x) = \cos^{-1}(x) \qquad \text{and} \qquad \arcsin(x) = \sin^{-1}(x).$$

However, this notation is misleading as $\cos^{-1}(x)$ and $\sin^{-1}(x)$ are not true inverse functions of cosine and sine. Recall that a function and its inverse undo each other in either order, for example,

$$\sqrt[3]{x^3} = x \qquad \text{and} \qquad \left(\sqrt[3]{x}\right)^3 = x.$$

Since arcsine is the inverse of sine restricted to the interval $[-\pi/2, \pi/2]$, this does not work with sine and arcsine, for example

$$\arcsin(\sin(\pi)) = 0.$$

though it is true that

$$\sin(\arcsin(x)) = x \qquad \text{and} \qquad \cos(\arccos(x)) = x.$$

Question 90. *Which of the following statements is true?*

Multiple Choice:

(a) $\sin^{-1}(x)$ *is the inverse function of* $\sin(x)$

(b) $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \frac{1}{2}$

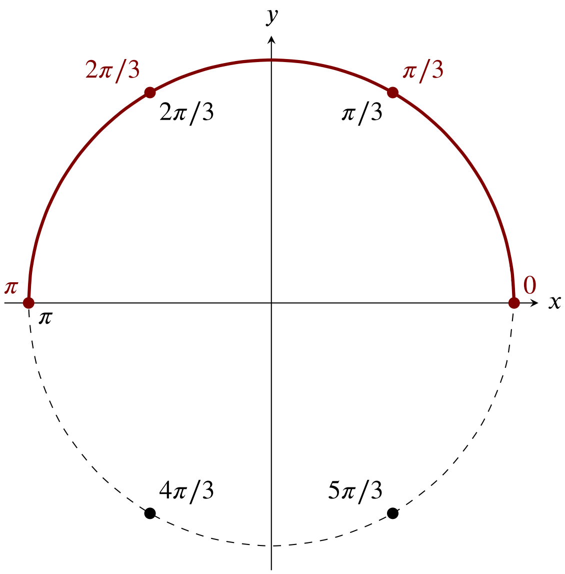
(c) $\sin^{-1}\left(\sin\left(\frac{5\pi}{2}\right)\right) = \frac{5\pi}{2}$

(d) $\sin^{-1}(x) = \dfrac{1}{\sin(x)}$

Example 165. *Compute:*

$$\arccos(\cos(5\pi/3))$$

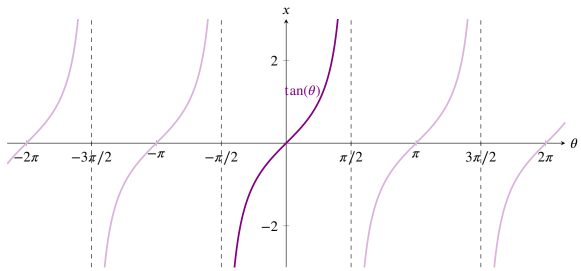
The issue here is that $5\pi/3$ might not be in the range of arccosine. To find our missing number, we'll check with a unit circle that we've decorated with the domain of arccosine:



Since the points $5\pi/3$ and $\pi/3$ have the same x-coordinate, $\cos(5\pi/3) = \cos(\pi/3)$. Hence

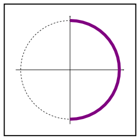
$$\arccos(\cos(5\pi/3)) = \pi/3.$$

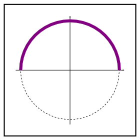
Now that you have a feel for how $\arcsin(x)$ and $\arccos(x)$ behave, let's examine tangent.

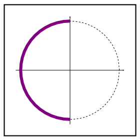


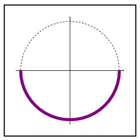
Question 91. *What arc on the unit circle corresponds to the restricted domain described above of* $\tan(\theta)$*?*

Multiple Choice:

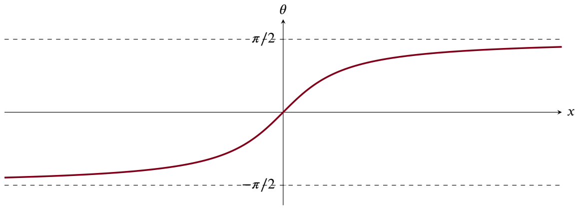
(a) 

(b) 

(c) 

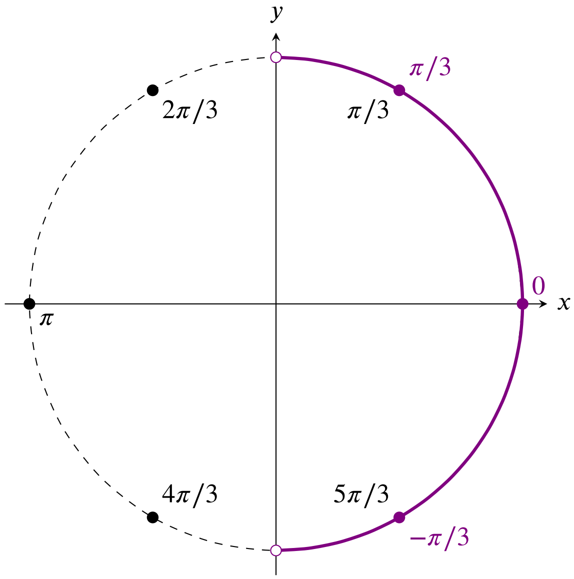
(d) 

Again, only working on a restricted domain of tangent, we can produce an inverse function, arctangent. Here we see a plot of $\arctan(x)$, the inverse function of $\tan(\theta)$ when its domain is restricted to the interval $(-\pi/2, \pi/2)$.



Example 166. *Compute:*

$$\arctan(\tan(5\pi/3))$$

> The issue here is that $5\pi/3$ might not be in the range of arctangent. To find our missing number, we'll check with a unit circle that we've decorated with the domain of arctangent:
>
> 
>
>
> Since the points $5\pi/3$ and $-\pi/3$ have the same x and y-coordinates,
>
> $$\arctan(\tan(5\pi/3)) = -\pi/3.$$

Now we give some facts of other trigonometric and "inverse" trigonometric functions.

Definition.

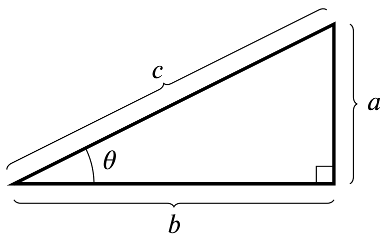
- $\arccos(x) = \theta$ means that $\cos(\theta) = x$ and $0 \leq \theta \leq \pi$. The domain of $\arccos(x)$ is $-1 \leq x \leq 1$.
- $\arcsin(x) = \theta$ means that $\sin(\theta) = x$ and $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$. The domain of $\arcsin(x)$ is $-1 \leq x \leq 1$.
- $\arctan(x) = \theta$ means that $\tan(\theta) = x$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. The domain of $\arctan(x)$ is $-\infty < x < \infty$.
- $\operatorname{arccot}(x) = \theta$ means that $\cot(\theta) = x$ and $0 < \theta < \pi$. The domain of $\operatorname{arccot}(x)$ is $-\infty < x < \infty$.

- $\operatorname{arcsec}(x) = \theta$ means that $\sec(\theta) = x$ and $0 \le \theta \le \pi$ with $\theta \ne \pi/2$. The domain of $\operatorname{arcsec}(x)$ is all x with absolute value greater than 1.
- $\operatorname{arccsc}(x) = \theta$ means that $\csc(\theta) = x$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ with $\theta \ne 0$. The domain of $\operatorname{arccsc}(x)$ is all x with absolute value greater than 1.

The power of the Pythagorean Theorem

The Pythagorean Theorem is probably the most famous theorem in all of mathematics.

Theorem 52 (Pythagorean Theorem)**.** *Given a right triangle:*



We have that:

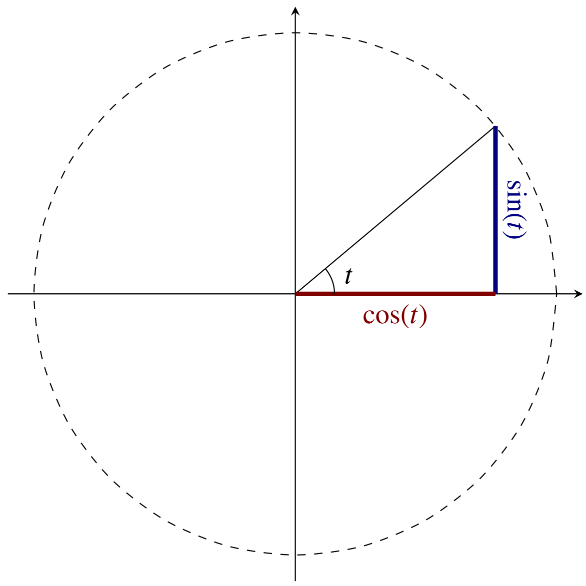
$$a^2 + b^2 = c^2$$

The Pythagorean Theorem gives several key trigonometric identities.

Theorem 53 (Pythagorean Identities)**.** *The following hold:*

$$\cos^2\theta + \sin^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \csc^2\theta$$

From the unit circle we can see



via the Pythagorean Theorem that

$$\cos^2(t) + \sin^2(t) = 1.$$

If we divide this expression by $\cos^2(t)$ we obtain

$$1 + \tan^2(t) = \sec^2(t)$$

and if we divide $\cos^2(t) + \sin^2(t) = 1$ by $\sin^2(t)$ we obtain

$$\cot^2(t) + 1 = \csc^2(t).$$

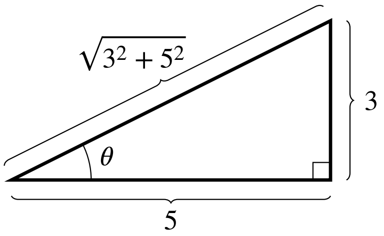
We can simplify expressions using the Pythagorean Theorem

Example 167. *Suppose that* $\arctan(3/5) = \theta$*. Compute* $\sin(\theta)$.

If $\arctan(3/5) = \theta$, then

$$\begin{aligned} \tan(\arctan(3/5)) &= \tan(\theta) \\ 3/5 &= \tan(\theta). \end{aligned}$$

Now we will use the Pythagorean Theorem to deduce $\sin(\theta)$. If $\tan(\theta) = 3/5$, the triangle in question must be similar to this triangle:



From this triangle and our work above, we see that

$$\sin(\theta) = 3/\sqrt{3^2 + 5^2}.$$

We'll also use the Pythagorean Theorem to help us simplify abstract expressions into ones we can compute with ease.

Example 168. *Simplify*

$$\tan(\arccos(x))$$

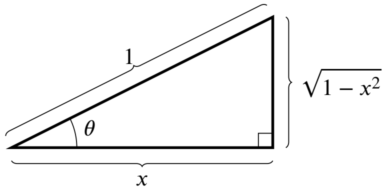
Start by saying

$$\theta = \arccos(x)$$

This means $\tan(\arccos(x)) = \tan(\theta)$. Apply cosine to both sides of the equation above,

$$\cos(\theta) = \cos(\arccos(x))$$
$$\cos(\theta) = x.$$

Now we will use the Pythagorean Theorem to deduce $\tan(\theta)$. If $\cos(\theta) = x$, the triangle in question must be similar to this triangle:



From this triangle and our work above, we see that

$$\tan(\arccos(x)) = \tan(\theta) = \frac{\sqrt{1 - x^2}}{x}.$$

32 Derivatives of inverse trigonometric functions

After completing this section, students should be able to do the following.

- Know and use the derivatives of the inverse trigonometric functions.
- Understand how the derivative of an inverse function relates to the original derivative.

Break-Ground:

32.1 Derivatives of inverse trigonometric functions BreakGround

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, do you remember when we first starting graphing functions? Like with a "T-chart?"
Riley: I remember everything.
Devyn: ...
Riley: Hmmmm. I'm not sure…

Problem 1. *When x is a large number (furthest from zero), which term of $5x^6-5x^5-5x^4+5x^3+x^2-1$ is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 2. *When x is a small number (near zero), which term of $5x^6-5x^5-5x^4+5x^3+x^2-1$ is largest (furthest from zero)?*

Multiple Choice:

(a) -1

(b) x^2

(c) $5x^3$

(d) $-5x^4$

(e) $-5x^5$

(f) $5x^6$

Problem 3. *Very roughly speaking, what does the graph of $y=5x^6-5x^5-5x^4+5x^3+x^2-1$ look like?*

Multiple Choice:

(a) *The graph starts in the lower left and ends in the upper right of the plane.*

(b) *The graph starts in the lower right and ends in the upper left of the plane.*

(c) *The graph looks something like the letter "U."*

(d) *The graph looks something like an upside down letter "U."*

Dig-In:

32.2 Derivatives of inverse trigonometric functions

Now we will derive the derivative of arcsine, arctangent, and arcsecant.

Theorem 54 (The derivative of arcsine)**.**

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

Recall

$$\arcsin(x) = \theta$$

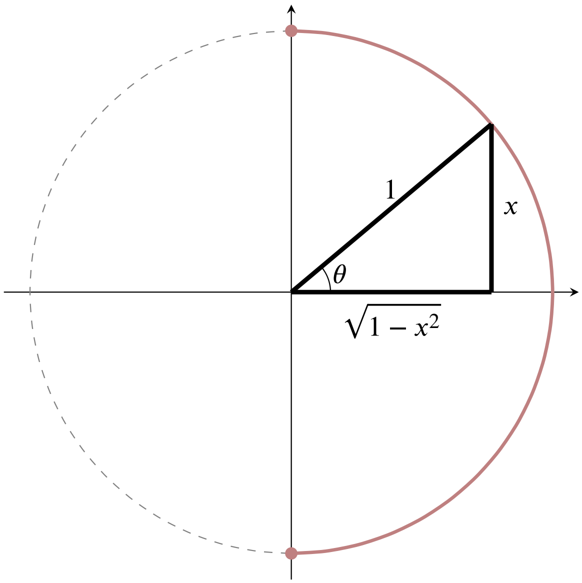
means that $\sin(\theta) = x$ and $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Implicitly differentiating with respect x we see

$$\begin{aligned} \sin(\theta) &= x \\ \frac{d}{dx}\sin(\theta) &= \frac{d}{dx}x && \text{Differentiate both sides.} \\ \cos(\theta)\cdot\theta' &= 1 && \text{Implicit differentiation.} \\ \theta' &= \frac{1}{\cos(\theta)} && \text{Solve for } \theta'. \end{aligned}$$

While $\theta' = \frac{1}{\cos(\theta)}$, we need our answer written in terms of x. Since we are assuming that

$$\sin(\theta) = x,$$

consider the following triangle with the unit circle:



From the unit circle above, we see that

$$\begin{aligned} \theta' &= \frac{1}{\cos(\theta)} \\ &= \frac{\text{hyp}}{\text{adj}} \\ &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

To be completely explicit,

$$\frac{d}{dx}\theta = \frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

Question 92. *Compute:*

$$\frac{d}{dx}\sin^{-1}(x)$$

We can do something similar with arctangent.

Theorem 55 (The derivative of arctangent)**.**

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

Recall

$$\arctan(x) = \theta$$

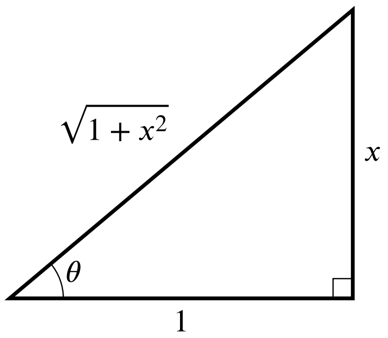
means that $\tan(\theta) = x$ and $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. Implicitly differentiating with respect x we see

$$\begin{aligned}
\tan(\theta) &= x \\
\frac{d}{dx}\tan(\theta) &= \frac{d}{dx}x && \text{Differentiate both sides.} \\
\sec^2(\theta)\cdot\theta' &= 1 && \text{Implicit differentiation.} \\
\theta' &= \frac{1}{\sec^2(\theta)} && \text{Solve for } \theta'. \\
\theta' &= \cos^2(\theta).
\end{aligned}$$

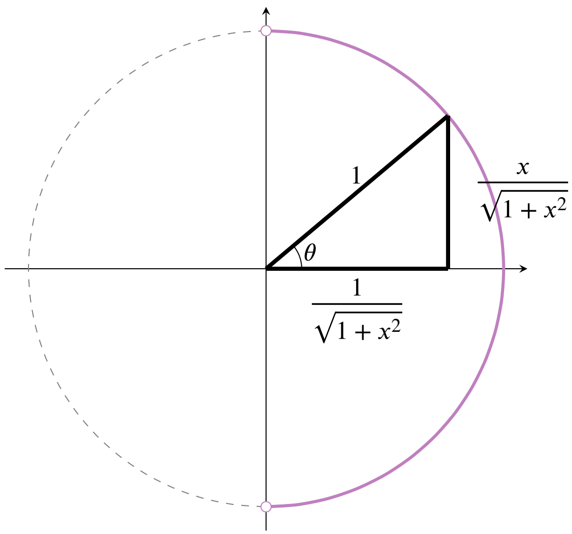
While $\theta' = \cos^2(\theta)$, we need our answer written in terms of x. Since we are assuming that

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = x,$$

we may consider the following triangle:



We may now scale this triangle by a factor of $\frac{1}{\sqrt{1+x^2}}$ to place it on the unit circle:



From the unit circle above, we see that

$$\begin{aligned}
\theta' &= \cos^2(\theta) \\
&= \left(\frac{\text{adj}}{\text{hyp}}\right)^2 \\
&= \frac{1}{1+x^2}.
\end{aligned}$$

To be completely explicit,

$$\frac{d}{dx}\theta = \frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

Question 93. *Compute:*

$$\frac{d}{dx}\tan^{-1}(\sqrt{x})$$

Finally, we investigate the derivative of arcsecant.

Theorem 56 (The derivative of arcsecant)**.**

$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \textit{for } |x| > 1.$$

Recall

$$\operatorname{arcsec}(x) = \theta$$

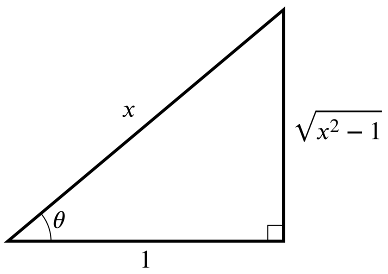
means that $\sec(\theta) = x$ and $0 \leq \theta \leq \pi$ with $\theta \neq \pi/2$. Implicitly differentiating with respect x we see

$$\begin{aligned} \sec(\theta) &= x \\ \frac{d}{dx}\sec(\theta) &= \frac{d}{dx}x && \text{Differentiate both sides.} \\ \sec(\theta)\tan(\theta)\cdot\theta' &= 1 && \text{Implicit differentiation.} \\ \theta' &= \frac{1}{\sec(\theta)\tan(\theta)} && \text{Solve for } \theta'. \\ \theta' &= \frac{\cos^2(\theta)}{\sin(\theta)}. \end{aligned}$$

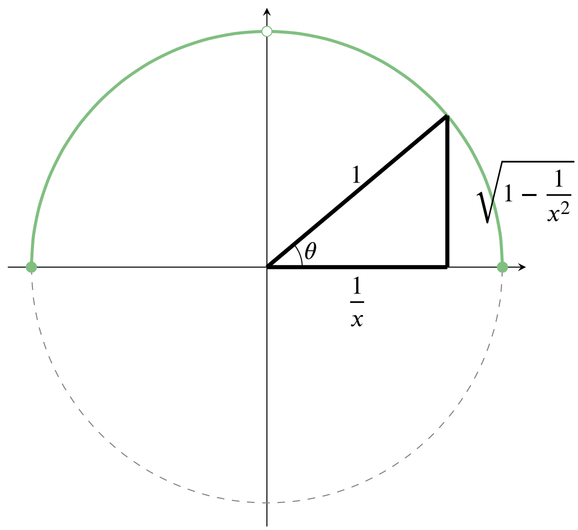
While $\theta' = \frac{\cos^2(\theta)}{\sin(\theta)}$, we need our answer written in terms of x. Since we are assuming that

$$\sec(\theta) = \frac{1}{\cos(\theta)} = x,$$

we may consider the following triangle:



We may now scale this triangle by a factor of $\frac{1}{x}$ to place it on the unit circle:



From the unit circle above, we see that

$$\begin{aligned} \theta' &= \frac{\cos^2(\theta)}{\sin(\theta)} \\ &= \frac{\left(\frac{\text{adj}}{\text{hyp}}\right)^2}{\frac{\text{opp}}{\text{hyp}}} \\ &= \frac{(\text{adj})^2}{\text{opp}} && \text{Note, hyp} = 1. \\ &= \frac{1/x^2}{\sqrt{1-1/x^2}} \\ &= \frac{1}{|x|\sqrt{x^2-1}}, \end{aligned}$$

> To be completely explicit,
>
> $$\frac{d}{dx}\theta = \frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \text{for } |x| > 1.$$

Question 94. *Compute:*

$$\frac{d}{dx}\sec^{-1}(3x)$$

We leave it to you, the reader, to investigate the derivatives of cosine, arccosecant, and arccotangent. However, as a gesture of friendship, we now present you with a list of derivative formulas for inverse trigonometric functions.

Theorem 57 (The Derivatives of Inverse Trigonometric Functions)**.**

- $\dfrac{d}{dx}\arcsin(x) = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}\arccos(x) = \dfrac{-1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}\arctan(x) = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx}\operatorname{arcsec}(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$ *for* $|x| > 1$
- $\dfrac{d}{dx}\operatorname{arccsc}(x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$ *for* $|x| > 1$
- $\dfrac{d}{dx}\operatorname{arccot}(x) = \dfrac{-1}{1+x^2}$

Dig-In:

32.3 The Inverse Function Theorem

There is one catch to all the explanations given above where we computed derivatives of inverse functions. To write something like

$$\frac{d}{dx}(e^y) = e^y \cdot y'$$

we need to know that the function y *has* a derivative. The *Inverse Function Theorem* guarantees this.

Theorem 58 (Inverse Function Theorem)**.** *If f is a differentiable function that is one-to-one near a and $f'(a) \neq 0$, then*

(a) $f^{-1}(x)$ *is **defined** for x near $b = f(a)$,*

(b) $f^{-1}(x)$ *is **differentiable** near $b = f(a)$,*

(c) *last, but not least:*

$$\left[\frac{d}{dx} f^{-1}(x)\right]_{x=b} = \frac{1}{f'(a)} \qquad \textit{where} \qquad b = f(a).$$

We will only explain the last result. We know

$$f(f^{-1}(x)) = x,$$

and now we use implicit differentiation (and the chain rule) to write

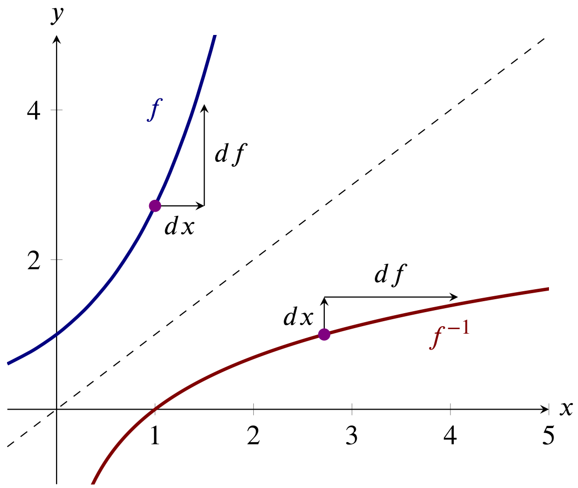
$$\begin{aligned}\frac{d}{dx} f(f^{-1}(x)) &= f'(f^{-1}(x))(f^{-1})'(x)\\ &= 1.\end{aligned}$$

Solving for $(f^{-1})'(x)$ we see

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

This is what we have written above.

It is worth giving one more piece of evidence for the formula above, this time based on differentials. Consider this plot of a function f and its inverse:



Since the inverse of a function is the reflection of the function over the line $y = x$, we see that the differentials are "switched" when reflected. Hence we see that

$$\frac{df^{-1}}{dx} = \frac{dx}{df}.$$

The inverse function theorem gives us a recipe for computing the derivatives of inverses of functions at points.

Example 169. *Let f be a differentiable function that has an inverse. In the table below we give several values for both f and f':*

x	f	f'
2	0	2
3	1	-2
4	-3	0

Compute

$$\frac{d}{dx} f^{-1}(x) \textit{ at } x = 1.$$

> From the table above we see that
>
> $$1 = f(3).$$
>
> Hence, by the inverse function theorem
>
> $$\left(f^{-1}(1)\right)' = \frac{1}{f'(3)} = \frac{-1}{2}.$$

If one example is good, two are better:

Example 170. *Let f be a differentiable function that has an inverse. In the table below we give several values for both f and f':*

x	f	f'
2	0	2
3	1	-2
4	-3	0

Compute

$$\left(f^{-1}(0)\right)'$$

> Note,
>
> $$\left(f^{-1}(0)\right)' = \frac{d}{dx}f^{-1}(x) \text{ at } x = 0.$$
>
> From the table above we see that
>
> $$0 = f(2).$$
>
> Hence, by the inverse function theorem
>
> $$\left(f^{-1}(0)\right)' = \frac{1}{f'(2)} = \frac{1}{2}.$$

Finally, let's see an example where the theorem does not apply.

Example 171. *Let f be a differentiable function that has an inverse. In the table below we give several values for both f and f':*

x	f	f'
2	0	2
3	1	-2
4	-3	0

Compute

$$\left[\frac{d}{dx}f^{-1}(x)\right]_{x=-3}$$

> From the table above we see that
>
> $$-3 = f(4).$$
>
> Ah! But here, $f'(4) = 0$, so we have no guarantee that the inverse exists near the point $x = -3$, but even if it did the inverse would not be differentiable there.

33 More than one rate

After completing this section, students should be able to do the following.

- Solve basic related rates word problems.
- Understand the process of solving related rates problems.
- Calculate derivatives of expressions with multiple variables implicitly.

Break-Ground:

33.1 A changing circle

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I've been thinking about calculus.
Riley: YOLO.
Devyn: Consider a circle of some radius r.
Riley: Ha! What else would we ever call the "radius?"
Devyn: Exactly. Now the formula for the perimeter of a circle is?
Riley: $P = 2 \cdot \pi \cdot r$ baby.
Devyn: And its area?
Riley: You know it's $A = \pi \cdot r^2$.
Devyn: Right, but here's what's bugging me: If I know r', what is P'? What's A'?
Riley: Oooh. Ouch. Hmmm. I wanna say it's

$$P' = 2 \cdot \pi \cdot r' \qquad \text{and} \qquad A' = \pi(r')^2$$

but I'm not sure that is right.
Devyn: Yeah…me too. But I'm not sure that's right either. Are we forgetting something?

Problem 1. *Do you think our young mathematicians above are correct?*

Multiple Choice:

(a) *Yes.* $P' = 2 \cdot \pi \cdot r'$ *and* $A' = \pi(r')^2$.

(b) *No. While* $P' = 2 \cdot \pi \cdot r'$, $A' \neq \pi(r')^2$.

(c) *No. While* $A' = \pi(r')^2$, $P' \neq 2 \cdot \pi \cdot r'$.

(d) *No.* $P' \neq 2 \cdot \pi \cdot r'$ *and* $A' \neq \pi(r')^2$.

(e) *There is no way to tell.*

Problem 2. *Set* $r(t) = 3 \cdot t$. *What is* $r'(t)$ *when* $r = 15$*?*

Problem 3. *Set* $r = 3 \cdot t$. *Now* $P(t) = 2 \cdot \pi \cdot 3 \cdot t$. *What is* $P'(t)$ *when* $r = 15$*?*

Problem 4. *Describe what* P' *means in this context.*

Problem 5. *Set* $r = 3 \cdot t$. *Now* $A(t) = \pi \cdot (3 \cdot t)^2$. *What is* $A'(t)$ *when* $r = 15$*?*

Problem 6. *Describe what* A' *means in this context. Does it make sense that* A' *is positive?*

Problem 7. *What, if anything, did our two young mathematicians forget about above?*

Dig-In:

33.2 More than one rate

Suppose we have two variables x and y which are both changing with respect to time. A *related rates* problem is a problem where we know one rate at a given instant, and wish to find the other.

Here the chain rule is key: If y is written in terms of x, and we are given $\frac{dx}{dt}$, then it is easy to find $\frac{dy}{dt}$ using the chain rule:

$$\frac{dy}{dt} = y'(x(t)) \cdot x'(t).$$

In many cases, particularly the interesting ones, our functions will be related in some other way. Nevertheless, in each case we'll use the power of the chain rule to help us find the desired rate. In this section, we will work several abstract examples, so we can emphasize the mathematical concepts involved. In each of the examples below, we will follow essentially the same plan of attack:

Draw a picture. If possible, draw a schematic picture with all the relevant information.

Find equations. We want equations that relate all relevant functions.

Differentiate the equations. Here we will often use implicit differentiation.

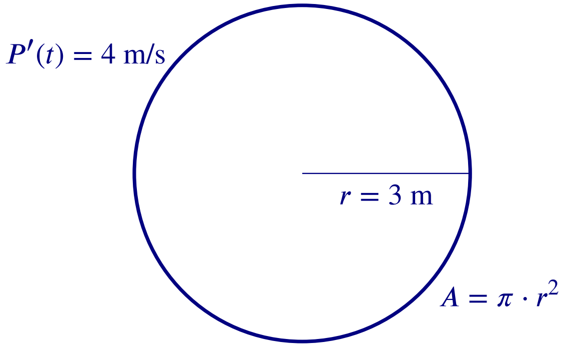
Evaluate and solve. Evaluate each equation at all known desired values and solve for the relevant rate.

Formulas

One way to combine several functions is with a known formula.

Example 172. *Imagine an expanding circle. If we know that the perimeter is expanding at a rate of* 4 *m/s, what rate is the area changing when the radius is* 3 *meters?*

To start, we **draw a picture**.



We must **find equations** that combine relevant functions. Here we use the common formulas for perimeter and area

$$P = 2 \cdot \pi \cdot r \qquad \text{and} \qquad A = \pi \cdot r^2.$$

Next we imagine that A, r, and P are functions of time

$$P(t) = 2 \cdot \pi \cdot r(t) \qquad \text{and} \qquad A(t) = \pi \cdot r(t)^2.$$

and we **differentiate the equations** using implicit differentiation, treating all functions as functions of t

$$P'(t) = 2 \cdot \pi \cdot r'(t) \qquad \text{and} \qquad A'(t) = 2 \cdot \pi \cdot r(t) \cdot r'(t).$$

Now we **evaluate and solve**. We know $P'(t) = 4$ and that $r(t) = 3$. Hence our equations become

$$4 = 2 \cdot \pi \cdot r'(t) \qquad \text{and} \qquad A'(t) = 2 \cdot \pi \cdot 3 \cdot r'(t).$$

We see that

$$\begin{aligned} 4 &= 2 \cdot \pi \cdot r'(t) \\ 2/\pi &= r'(t). \end{aligned}$$

and now that

$$\begin{aligned} A'(t) &= 2 \cdot \pi \cdot 3 \cdot 2/\pi \\ &= 12. \end{aligned}$$

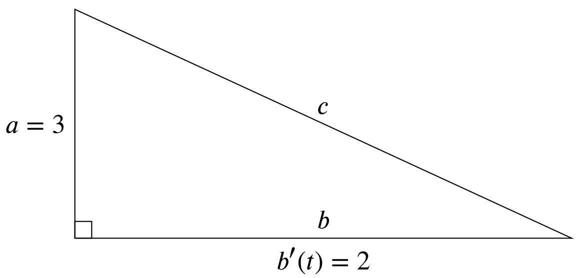
Hence the area is expanding at a rate of 12 m/s.

Right triangles

A common way to combine functions is through facts related to right triangles.

Example 173. *Imagine an expanding right triangle. If one leg has a fixed length of* 3 *m, one leg is increasing with a rate of* 2 *m/s, and the hypotenuse is expanding to accommodate the expanding leg, at what rate is the hypotenuse expanding when both legs are* 3 *m long?*

To start, we **draw a picture**.



We must **find equations** that combines relevant functions. Here we use the Pythagorean Theorem.

$$c^2 = a^2 + b^2$$

Imagining c and b as being functions of time

$$c(t)^2 = a^2 + b(t)^2$$

we are now able to **differentiate the equation** using implicit differentiation, treating all functions as functions of t, note a is constant,

$$2 \cdot c(t) \cdot c'(t) = 2 \cdot b(t) \cdot b'(t).$$

Now we **evaluate and solve**. We know that $b'(t) = 2$ and that $b(t) = 3$

$$2 \cdot c(t) \cdot c'(t) = 12$$

However, we still need to know $c(t)$ when $b(t) = 3$. Here we use the Pythagorean Theorem,

$$\begin{aligned} c(t)^2 &= 3^2 + 3^2 \\ &= 18, \end{aligned}$$

and so we see that $c(t) = 3\sqrt{2}$. We may now write

$$\begin{aligned} 6\sqrt{2} \cdot c'(t) &= 12 \\ c'(t) &= \sqrt{2}. \end{aligned}$$

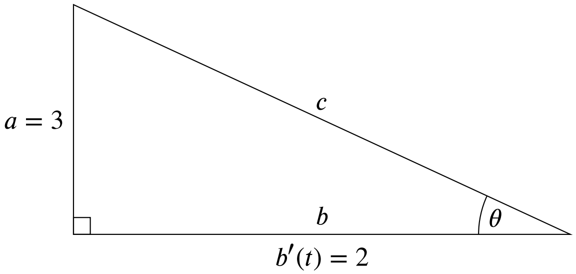
Hence $c(t)$ is growing at a rate of $\sqrt{2}$ m/s.

Angular rates

We can also investigate problems involving angular rates.

Example 174. *Imagine an expanding right triangle. If one leg has a fixed length of* 3 *m, one leg is increasing with a rate of* 2 *m/s, and the hypotenuse is expanding to accommodate the expanding leg, at what rate is the angle opposite the fixed leg changing when both legs are* 3 *m long?*

To start, we **draw a picture**.



We must **find equations** that combines relevant functions. Here we note that

$$\tan(\theta) = \frac{a}{b}$$

Imagining θ and b as being functions of time

$$\tan(\theta(t)) = \frac{a}{b(t)}$$

we are now able to **differentiate the equation** using implicit differentiation, treating all functions as functions of t, note a is constant,

$$\sec^2(\theta(t))\theta'(t) = \frac{-a \cdot b'(t)}{b(t)^2}.$$

Now we **evaluate and solve**. We know that $a = 3$, $b'(t) = 2$, and that $b(t) = 3$

$$\begin{aligned} \sec^2(\theta(t)) \cdot \theta'(t) &= \frac{-3 \cdot 2}{3^2} \\ &= \frac{-6}{9} \\ &= \frac{-2}{3}. \end{aligned}$$

However, we still need to know $\sec^2(\theta)$. Here we use the Pythagorean Theorem,

$$\begin{aligned} c^2(t) &= 3^2 + 3^2 \\ &= 18, \end{aligned}$$

and so we see that $c(t) = 3\sqrt{2}$. Now

$$\begin{aligned} \sec^2(\theta) &= \frac{\text{hypotenuse}^2}{\text{adjacent}^2} \\ &= \frac{\left(3\sqrt{2}\right)^2}{3^2} \\ &= 2. \end{aligned}$$

Hence

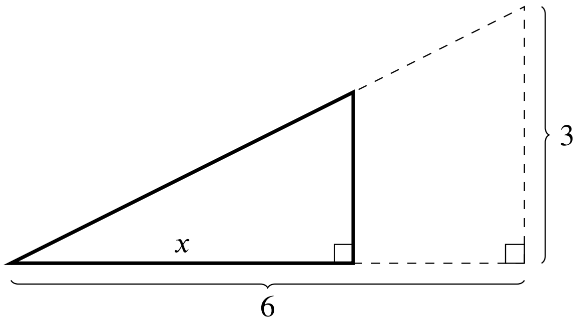
$$\begin{aligned} \sec^2(\theta(t)) \cdot \theta'(t) &= \frac{-2}{3} \\ 2 \cdot \theta'(t) &= \frac{-2}{3} \\ \theta'(t) &= \frac{-1}{3}. \end{aligned}$$

So when $a = b = 3$, the angle is changing at $-1/3$ radians per second.

Similar triangles

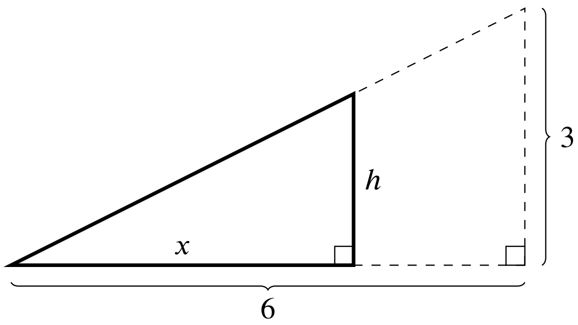
Finally, facts about similar triangles are often useful when solving related rates problems.

Example 175. *Imagine two right triangles that share an angle:*



If x is growing from the vertex with a rate of 3 m/s, what rate is the area of the smaller triangle changing when $x = 5$?

Despite the fact that a nice picture is given, we should start as we always do and **draw a picture**. Note, we've added information to the picture:



We must **find equations** that combines relevant functions. In this case there are two. The first is the formula for the area of a triangle:

$$A = (1/2) \cdot x \cdot h$$

The second uses the fact that the larger triangle is similar to the smaller triangle, meaning that the proportions of the sides are the same,

$$\frac{x}{h} = \frac{6}{3} \qquad \text{so} \qquad x = 2 \cdot h$$

Imagining A, x, and h as functions of time we may write

$$A(t) = (1/2) \cdot x(t) \cdot h(t) \qquad \text{and} \qquad x(t) = 2 \cdot h(t).$$

We are now able to **differentiate the equations** using implicit differentiation, treating all functions as functions of t,

$$\begin{aligned} &A'(t) = (1/2) \cdot x'(t) \cdot h(t) + (1/2) \cdot x(t) \cdot h'(t), \\ &x'(t) = 2 \cdot h'(t). \end{aligned}$$

Now we **evaluate and solve**. We know that $x(t) = 5$ and that $x'(t) = 3$. Since

$$5 = x(t) = 2 \cdot h(t) \qquad \text{and} \qquad 3 = x'(t) = 2 \cdot h'(t)$$

we see that $h(t) = 5/2$ and $h'(t) = 3/2$. Hence

$$\begin{aligned} A'(t) &= (1/2) \cdot 3 \cdot (5/2) + (1/2) \cdot 5 \cdot (3/2) \\ &= 15/4 + 15/4 \\ &= 15/2. \end{aligned}$$

Hence the area is changing at a rate of $15/2$ m^2/s.

34 Applied related rates

After completing this section, students should be able to do the following.

- Identify word problems as related rates problems.
- Solve related rates word problems.
- Translate word problems into mathematical expressions.

Break-Ground:

34.1 Pizza and calculus, so cheesy

Check out this dialogue between two calculus students (based on a true story):

Devyn: Hey Riley, do you know what I love?
Riley: Calculus?
Devyn: And pizza! Last night I made my own pizza crust from scratch!
Riley: Mmmmmmm. Calculus.
Devyn: I know! The best part of making the crust is tossing it. But during my pizza tossing, I noticed something. The dough is basically a **cylinder** with a **radius that is expanding**, and a **height that is getting smaller**. Then I started wondering: how are those two rates related?
Riley: Pizza. So delicious. So cheesy. So calculus.

The problem above is an example of a related rates word problem. To solve it, we will need the tools in the Dig-In. For now, let's see if we can reason through some questions related to this problem.

Problem 1. *In the context above, which of the following should be assumed to be functions that vary?*

Select All Correct Answers:

(a) *The radius of the cylinder.*

(b) *The height of the cylinder.*

(c) *The surface area of the cylinder.*

(d) *The volume of the cylinder.*

Problem 2. *In the context above, which of the following should be assumed to be constant?*

Select All Correct Answers:

(a) *The radius of the cylinder.*

(b) *The height of the cylinder.*

(c) *The surface area of the cylinder.*

(d) *The volume of the cylinder.*

Dig-In:

34.2 Applied related rates

Now we are ready to work related rates problems in context. Just as before, we are going to follow essentially the same plan of attack in each problem.

Draw a picture. If possible, draw a schematic picture with all the relevant information.

Find equations. We want equations that relate all relevant functions.

Differentiate the equations. Here we will often use implicit differentiation.

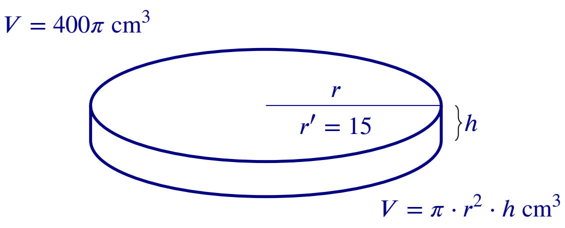
Evaluate and solve. Evaluate each equation at all known desired values and solve for the relevant rate.

Formulas

Example 176. *A hand-tossed pizza crust starts off as a ball of dough with a volume of* 400π *cm*^{3}*. First, the cook stretches the dough to the shape of a cylinder of radius* 12 *cm. Next the cook tosses the dough.*

If during tossing, the dough maintains the shape of a cylinder and the radius is increasing at a rate of 15 *cm/min, how fast is its thickness changing when the radius is* 20 *cm?*

To start, **draw a picture**. Here we see a cylinder that represents our pizza dough.



Next we need to **find equations**. We see that we have

$$400\pi = \pi \cdot r^2 \cdot h,$$

which immediately simplifies to

$$400 = r^2 \cdot h.$$

Imagining that r and h are functions of time, we now may write

$$400 = r(t)^2 \cdot h(t)$$

and so we may now **differentiate the equation** using implicit differentiation, treating all functions as functions of t,

$$0 = 2 \cdot r(t) \cdot r'(t) \cdot h(t) + r(t)^2 \cdot h'(t).$$

Now we'll **evaluate and solve**. We know that $r(t) = 20$ cm and that $r'(t) = 15$ cm/min. Moreover, we can now find $h(t)$ as we have

$$400 = r(t)^2 \cdot h(t) \qquad \textit{meaning} \qquad h(t) = \frac{400}{r(t)^2}.$$

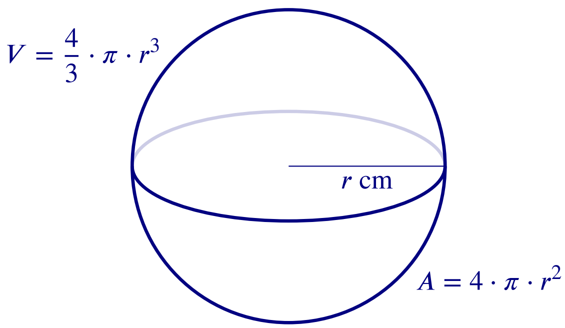
Since $20^2 = 400$, we see that $h(t) = 1$. Substituting in, we see

$$\begin{aligned} 0 &= 2 \cdot 20 \cdot 15 \cdot 1 + 20^2 \cdot h'(t), \\ -2 \cdot 20 \cdot 15 &= 20^2 \cdot h'(t), \\ \frac{-2 \cdot 20 \cdot 15}{20^2} &= h'(t) \\ -1.5 &= h'(t). \end{aligned}$$

Hence the thickness of the dough is changing at a rate of -1.5 cm/min.

Example 177. *Consider a melting snowball. We will assume that the rate that the snowball is melting is proportional to its surface area. Show that the radius of the snowball is changing at a constant rate.*

To start, **draw a picture**.



Next we need to **find equations**. The equations we'll use are

$$V = (4/3) \cdot \pi \cdot r^3 \qquad \text{and} \qquad A = 4 \cdot \pi \cdot r^2.$$

Now the key words are "the rate that the snowball is melting is proportional to its surface area." From this we have the following equation:

$$\underbrace{V'}_{\text{rate the snowball is melting}} = \overbrace{k}^{\text{is proportional to}} \cdot \underbrace{A}_{\text{its surface area}}$$

So we need to know V'. We know $V = \frac{4}{3} \cdot \pi \cdot r^3$. If we imagine r as a function of t, we can write volume as a function of t:

$$V(t) = \frac{4}{3} \cdot \pi \cdot r(t)^3$$

so

$$V'(t) = 4 \cdot \pi \cdot r(t)^2 \cdot r'(t).$$

Now we'll **evaluate and solve**. We now know

$$V'(t) = 4 \cdot \pi \cdot r(t)^2 \cdot r'(t) = k \cdot 4 \cdot \pi \cdot r(t)^2 = k \cdot A(t).$$

So

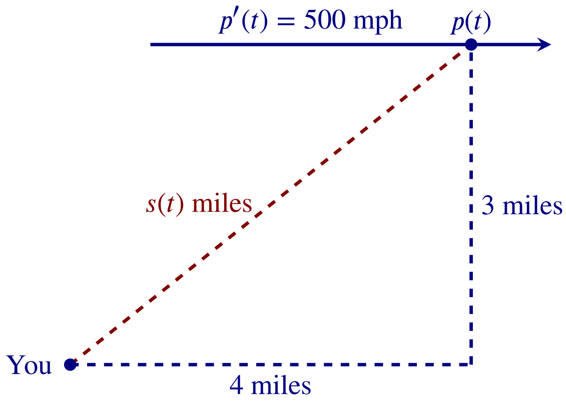
$$\begin{aligned} 4 \cdot \pi \cdot r(t)^2 \cdot r'(t) &= k \cdot 4 \cdot \pi \cdot r(t)^2 \\ r'(t) &= k. \end{aligned}$$

Hence the radius is changing at a constant rate.

Right triangles

Example 178. *A plane is flying directly away from you at* 500 *mph at an altitude of* 3 *miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground* 4 *miles from you?*

To start, **draw a picture**.



Next we need to **find equations**. By the Pythagorean Theorem we know that

$$p^2 + 3^2 = s^2.$$

Imagining that p and s are functions of time, we now **differentiate the equation**. Write

$$2 \cdot p(t) \cdot p'(t) = 2 \cdot s(t) \cdot s'(t).$$

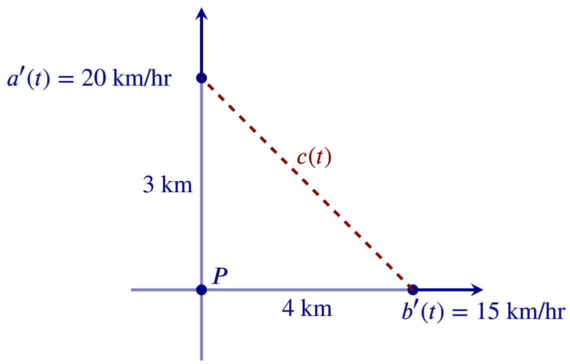
Now we'll **evaluate and solve**. We are interested in the time at which $p(t) = 4$ and $p'(t) = 500$. Additionally, at this time we know that $4^2 + 9 = s^2$, so $s(t) = 5$. Putting together all the information we get

$$2(4)(500) = 2(5)s'(t),$$

thus $s'(t) = 400$ mph.

Example 179. *A road running north to south crosses a road going east to west at the point P. Cyclist A is riding north along the first road, and cyclist B is riding east along the second road. At a particular time, cyclist A is 3 kilometers to the north of P and traveling at 20 km/hr, while cyclist B is 4 kilometers to the east of P and traveling at 15 km/hr. How fast is the distance between the two cyclists changing?*

We start the same way we always do, we **draw a picture**.



Here $a(t)$ is the distance of cyclist A north of P at time t, and $b(t)$ the distance of cyclist B east of P at time t, and $c(t)$ is the distance from cyclist A to cyclist B at time t.

We must **find equations**. By the Pythagorean Theorem,

$$c(t)^2 = a(t)^2 + b(t)^2.$$

Now we can **differentiate the equation**. Taking derivatives we get

$$2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t).$$

Now we can **evaluate and solve**. We know that $a(t) = 3$, $a'(t) = 20$, $b(t) = 4$ and $b'(t) = 15$. Hence by the Pythagorean Theorem, $c(t) = 5$. So

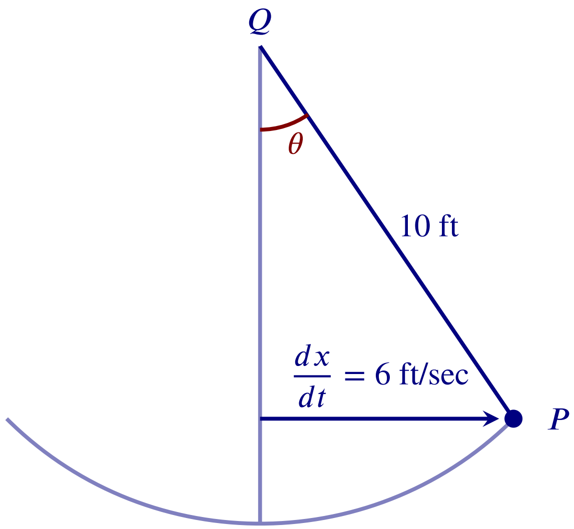
$$2 \cdot 5 \cdot c'(t) = 2 \cdot 3 \cdot 20 + 2 \cdot 4 \cdot 15$$

solving for $c'(t)$ we find $c'(t) = 24$ km/hr.

Angular rates

Example 180. *A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point P at the end of the rope, and let Q be the point of attachment at the other end. Suppose that the swing is directly below Q at time $t = 0$, and is being pushed by someone who walks at 6 ft/sec from left to right. What is the angular speed of the rope in rad/sec after 1 sec?*

To start, **draw a picture**.



Now we must **find equations**. From the right triangle in our picture, we see

$$\sin(\theta) = x/10.$$

We can now **differentiate the equation**. Taking derivatives we obtain

$$\cos(\theta) \cdot \theta'(t) = 0.1x'(t).$$

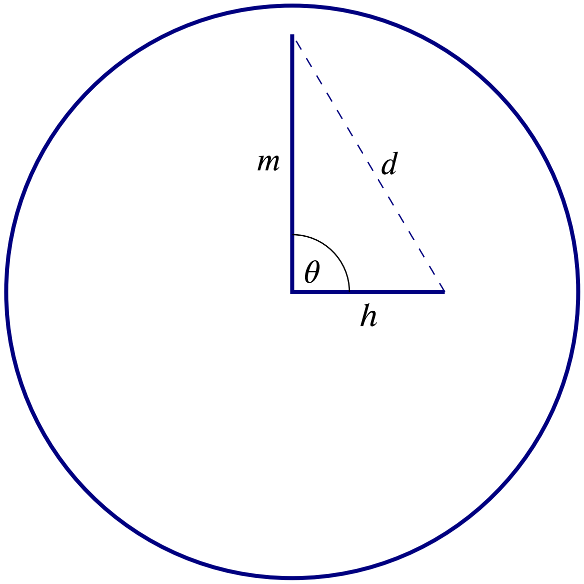
Now we can **evaluate and solve**. When $t = 1$ sec, the person was pushed by someone who walks 6 ft/sec. Hence we have a $6-8-10$ right triangle, with $x'(t) = 6$, and $\cos\theta = 8/10$. Thus

$$(8/10)\theta'(t) = 6/10,$$

and so $\theta'(t) = 3/4$ rad/sec.

Example 181. *The Palace of Westminster in London has a large clock tower. The minute hand is* 4.2 *meters long and the hour hand is* 2.7 *meters long. At what rate is the distance between the tip of the hands is changing when the clock strikes* 3 *pm?*

To start, we **draw a picture**.



Now we must **find equations** that combine relevant functions. Initially we might suppose that

$$d^2 = m^2 + h^2;$$

however, here θ is function of time, so this relationship only holds for certain times. Hence we must use the Law of Cosines to write

$$d^2 = m^2 + h^2 - 2mh\cos(\theta).$$

To find θ, imagine we are measuring the angle starting at "twelve o'clock" with t being measured in hours. Then letting θ_m be the angle made by the minute hand and θ_h

be the angle made by the hour hand we have

$$\theta_m(t) = 2\pi \cdot t,$$
$$\theta_h(t) = \frac{2\pi}{12} \cdot t = \frac{\pi}{6} \cdot t.$$

Finally since θ is decreasing, as the minute hand is traveling faster than the hour hand,

$$\theta(t) = \theta_h(t) - \theta_m(t).$$

On the other hand, m and h are constants. We may now write

$$d(t)^2 = m^2 + h^2 - 2mh\cos(\theta(t)).$$

If **differentiate the equations** using implicit differentiation we find

$$\theta'(t) = \frac{\pi}{6} - 2\pi$$

and

$$2 \cdot d(t) \cdot d'(t) = 2mh\sin(\theta(t)) \cdot \theta'(t).$$

Now we **evaluate and solve**. We know that $m = 4.2$, $h = 2.7$, $\theta'(t) = \frac{-11\pi}{6}$, and since the time is 3 pm, $\theta(t) = \pi/2$. Thus

$$d(t) \cdot d'(t) = 4.2 \cdot 2.7 \cdot \frac{-11\pi}{6}.$$

on the other hand

$$d(t) = \sqrt{4.2^2 + 2.7^2}$$

and so

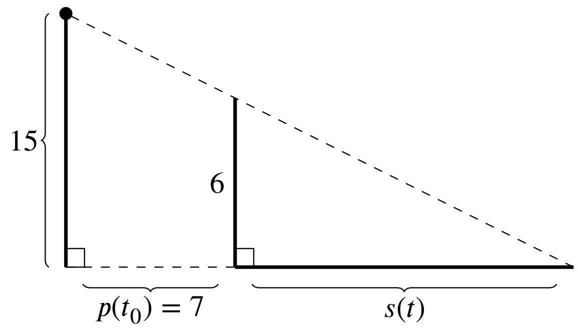
$$d'(t) = \frac{4.2 \cdot 2.7 \cdot \frac{-11\pi}{6}}{\sqrt{4.2^2 + 2.7^2}}.$$

This is the desired rate in units of meters per hour.

Similar triangles

Example 182. *It is night. Someone who is 6 feet tall is walking away from a street light at a rate of 3 feet per second. The street light is 15 feet tall. The person casts a shadow on the ground in front of them. How fast is the length of the shadow growing when the person is 7 feet from the street light?*

To start, **draw a picture**.



Here t is the variable and t_0 is the specific time when $p(t_0) = 7$.

Now we need to **find equations**. We use the fact that we have similar triangles to write:

$$\frac{s(t) + p(t)}{15} = \frac{s(t)}{6},$$
$$6 \cdot s(t) + 6 \cdot p(t) = 15 \cdot s(t),$$
$$6 \cdot p(t) = 9 \cdot s(t),$$
$$2 \cdot p(t) = 3 \cdot s(t).$$

Now we must **differentiate the equation**. We should use implicit differentiation, and treat each of the variables as functions of t. Write

$$2 \cdot p'(t) = 3 \cdot s'(t)$$

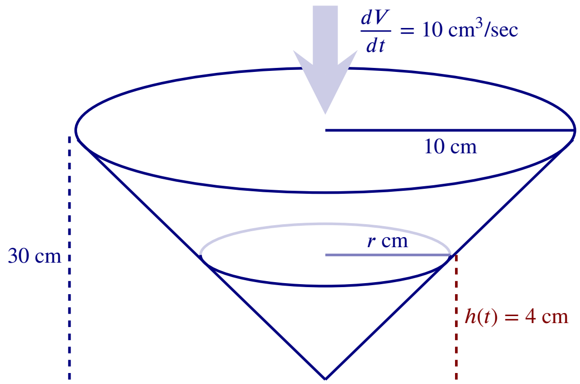
At this point we **evaluate and solve**. Since the person is waling at a rate of 3 feet per second, we may write

$$2 \cdot 3 = 3 \cdot s'(t),$$

and cancel to see that $s'(t) = 2$, meaning the shadow is growing at a rate of 2 feet per second.

Example 183. *Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep?*

To start, **draw a picture**.



Note, no attempt was made to draw this picture to scale, rather we want all of the relevant information to be available to the mathematician.

Now we need to **find equations**. The formula for the volume of a cone tells us that

$$V = \frac{\pi}{3} r^2 h.$$

Also the dimensions of the cone of water must have the same proportions as those of the container. That is, because of similar triangles,

$$\frac{r}{h} = \frac{10}{30} \qquad \text{so} \qquad r = h/3.$$

Now we must **differentiate the equation**. We should use implicit differentiation, and treat each of the variables as functions of t. Write

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) \qquad \text{and} \qquad \frac{dr}{dt} = \frac{1}{3} \cdot \frac{dh}{dt}.$$

At this point we **evaluate and solve**. We plug in $\frac{dV}{dt} = 10$, $r = 4/3$, $\frac{dr}{dt} = \frac{1}{3} \cdot \frac{dh}{dt}$ and $h = 4$. Write

$$\begin{aligned}
10 &= \frac{\pi}{3}\left(2 \cdot \frac{4}{3} \cdot 4 \cdot \frac{1}{3} \cdot \frac{dh}{dt} + \left(\frac{4}{3}\right)^2 \frac{dh}{dt}\right) \\
10 &= \frac{\pi}{3}\left(\frac{32}{9}\frac{dh}{dt} + \frac{16}{9}\frac{dh}{dt}\right) \\
10 &= \frac{16\pi}{9}\frac{dh}{dt} \\
\frac{90}{16\pi} &= \frac{dh}{dt}.
\end{aligned}$$

Thus, $\frac{dh}{dt} = \frac{90}{16\pi}$ cm/sec.

35 L'Hôpital's rule

After completing this section, students should be able to do the following.

- Recall how to find limits for forms that are not indeterminate.
- Define an indeterminate form.
- Determine if a form is indeterminate.
- Convert indeterminate forms to the form zero over zero or infinity over infinity.
- Define l'Hôpital's Rule and identify when it can be used.
- Use l'Hôpital's Rule to find limits.

Break-Ground:

35.1 A limitless dialogue

Check out this dialogue between two calculus students (based on a true story):

Devyn: Yo Riley, guess what I did last night?
Riley: What?
Devyn: I was doing some calculus.
Riley: That. Is. Awesome.
Devyn: I know! Anyway, I noticed something kinda funny. I think you can sometimes take limits by taking the derivative of the numerator and the denominator.
Riley: That's crazy.
Devyn: I know! But check it:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(x)}{x} &= \lim_{x\to 0}\frac{\frac{d}{dx}\sin(x)}{\frac{d}{dx}x}\\ &= \lim_{x\to 0}\frac{\cos(x)}{1}\\ &= 1.\end{aligned}$$

Riley: Woah. That. Is. Awes…weird. Hmmm, but it seems like cheating. Wait, it doesn't always work, check this out:

$$\lim_{x\to 0}\frac{x^2+1}{x+1} = 1,$$

but

$$\begin{aligned}\lim_{x\to 0}\frac{\frac{d}{dx}\left(x^2+1\right)}{\frac{d}{dx}(x+1)} &= \lim_{x\to 0}\frac{2x}{1}\\ &= 0.\end{aligned}$$

Problem 1. *Find* ***five*** *examples where this "trick" works, and* ***five*** *examples where it doesn't work.*

Problem 2. *What is the pattern for when the "trick" works and when it does not work?*

Dig-In:

35.2 L'Hôpital's rule

Derivatives allow us to take problems that were once difficult to solve and convert them to problems that are easier to solve. Let us consider L'Hôpital's rule:

Theorem 59 (L'Hôpital's Rule)**.** *Let $f(x)$ and $g(x)$ be functions that are differentiable near a. If*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \qquad \text{or} \pm\infty,$$

and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x near a, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, so we will not prove it here.

Remark. L'Hôpital's rule applies even when $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \mp\infty$.

L'Hôpital's rule allows us to investigate limits of *indeterminate form*.

Definition (List of Indeterminate Forms)**.**

$\frac{\mathbf{0}}{\mathbf{0}}$ This refers to a limit of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$.

$\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}}$ This refers to a limit of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$.

$\mathbf{0} \cdot \boldsymbol{\infty}$ This refers to a limit of the form $\lim_{x\to a} (f(x) \cdot g(x))$ where $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$.

$\boldsymbol{\infty} - \boldsymbol{\infty}$ This refers to a limit of the form $\lim_{x\to a} (f(x) - g(x))$ where $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$.

$\mathbf{1}^{\boldsymbol{\infty}}$ This refers to a limit of the form $\lim_{x\to a} f(x)^{g(x)}$ where $f(x) \to 1$ and $g(x) \to \infty$ as $x \to a$.

$\mathbf{0}^{\mathbf{0}}$ This refers to a limit of the form $\lim_{x\to a} f(x)^{g(x)}$ where $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$.

$\boldsymbol{\infty}^{\mathbf{0}}$ This refers to a limit of the form $\lim_{x\to a} f(x)^{g(x)}$ where $f(x) \to \infty$ and $g(x) \to 0$ as $x \to a$.

In each of these cases, the value of the limit is **not** immediately obvious. Hence, a careful analysis is required!

Basic indeterminant forms

Our first example is the computation of a limit that was somewhat difficult before.

Example 184. *Compute*

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

Set $f(x) = \sin(x)$ and $g(x) = x$. Since both $f(x)$ and $g(x)$ are differentiable functions at 0, and

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0,$$

this situation is ripe for L'Hôpital's Rule. Now

$$f'(x) = \cos(x)$$

and

$$g'(x) = 1.$$

L'Hôpital's rule tells us that

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{\cos(x)}{1} = 1.$$

Remark. Note, the astute mathematician will notice that in our example above, we are somewhat cheating. To apply L'Hôpital's rule, we need to know the derivative of sine; however, to know the derivative of sine we must be able to compute the limit:

$$\lim_{x\to 0}\frac{\sin(x)}{x}$$

Hence using L'Hôpital's rule to compute this limit is a circular argument! We encourage the gentle reader to view L'Hôpital's rule a "reminder" as to what is true, not as the formal derivation of the result.

Our next set of examples will run through the remaining indeterminate forms one is likely to encounter.

Example 185. *Compute*

$$\lim_{x\to \pi/2^+}\frac{\sec(x)}{\tan(x)}.$$

Set $f(x) = \sec(x)$ and $g(x) = \tan(x)$. Both $f(x)$ and $g(x)$ are differentiable near $\pi/2$. Additionally,

$$\lim_{x\to \pi/2^+} f(x) = \lim_{x\to \pi/2^+} g(x) = -\infty.$$

This situation is ripe for L'Hôpital's Rule. Now

$$f'(x) = \sec(x)\tan(x)$$

and

$$g'(x) = \sec^2(x).$$

L'Hôpital's rule tells us that

$$\begin{aligned}\lim_{x\to \pi/2^+}\frac{\sec(x)}{\tan(x)} &= \lim_{x\to \pi/2^+}\frac{\sec(x)\tan(x)}{\sec^2(x)}\\ &= \lim_{x\to \pi/2^+}\sin(x)\\ &= 1.\end{aligned}$$

Example 186. *Compute*

$$\lim_{x\to 0^+} x\ln x.$$

This doesn't appear to be suitable for L'Hôpital's Rule. As x approaches zero, $\ln x$ goes to $-\infty$, so the product looks like

(something very small)·(something very large and negative).

This product could be anything. A careful analysis is required. Write

$$x\ln x = \frac{\ln x}{x^{-1}}.$$

Set $f(x) = \ln(x)$ and $g(x) = x^{-1}$. Since both functions are differentiable near zero and

$$\lim_{x\to 0+}\ln(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} x^{-1} = \infty,$$

we may apply L'Hôpital's rule. Write with me

$$f'(x) = x^{-1}$$

and

$$g'(x) = -x^{-2},$$

so

$$\begin{aligned}\lim_{x\to 0^+} x\ln x &= \lim_{x\to 0^+} \frac{\ln x}{x^{-1}}\\ &= \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}}\\ &= \lim_{x\to 0^+} -x\\ &= 0.\end{aligned}$$

One way to interpret this is that since $\lim_{x\to 0^+} x\ln x = 0$, the function x approaches zero much faster than $\ln x$ approaches $-\infty$.

Indeterminate forms involving subtraction

There are two basic cases here, we'll do an example of each.

Example 187. *Compute*

$$\lim_{x\to 0} (\cot(x) - \csc(x)).$$

Here we simply need to write each term as a fraction,

$$\begin{aligned}\lim_{x\to 0} (\cot(x) - \csc(x)) &= \lim_{x\to 0} \left(\frac{\cos(x)}{\sin(x)} - \frac{1}{\sin(x)}\right)\\ &= \lim_{x\to 0} \frac{\cos(x)-1}{\sin(x)}\end{aligned}$$

Setting $f(x) = \cos(x) - 1$ and $g(x) = \sin(x)$, both functions are differentiable near zero and

$$\lim_{x\to 0}(\cos(x) - 1) = \lim_{x\to 0} \sin(x) = 0.$$

We may now apply L'Hôpital's rule. Write with me

$$f'(x) = -\sin(x)$$

and

$$g'(x) = \cos(x),$$

so

$$\begin{aligned}\lim_{x\to 0} (\cot(x) - \csc(x)) &= \lim_{x\to 0} \frac{\cos(x)-1}{\sin(x)}\\ &= \lim_{x\to 0} \frac{-\sin(x)}{\cos(x)}\\ &= 0.\end{aligned}$$

Sometimes one must be slightly more clever.

Example 188. *Compute*

$$\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right).$$

Again, this doesn't appear to be suitable for L'Hôpital's Rule. A bit of algebraic manipulation will help. Write with me

$$\begin{aligned}\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) &= \lim_{x\to\infty} \left(x\left(\sqrt{1+1/x} - 1\right)\right)\\ &= \lim_{x\to\infty} \frac{\sqrt{1+1/x}-1}{x^{-1}}\end{aligned}$$

Now set $f(x) = \sqrt{1+1/x} - 1$, $g(x) = x^{-1}$. Since both functions are differentiable for large values of x and

$$\lim_{x\to\infty}(\sqrt{1+1/x} - 1) = \lim_{x\to\infty} x^{-1} = 0,$$

we may apply L'Hôpital's rule. Write with me

$$f'(x) = (1/2)(1+1/x)^{-1/2}\cdot(-x^{-2})$$

and

$$g'(x) = -x^{-2}$$

so

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty}\frac{\sqrt{1+1/x}-1}{x^{-1}}\\ &= \lim_{x\to\infty}\frac{(1/2)(1+1/x)^{-1/2}\cdot(-x^{-2})}{-x^{-2}}\\ &= \lim_{x\to\infty}\frac{1}{2\sqrt{1+1/x}}\\ &= \frac{1}{2}.\end{aligned}$$

Exponential Indeterminate Forms

There is a standard trick for dealing with the indeterminate forms

$$\mathbf{1}^{\infty}, \quad \mathbf{0}^{\mathbf{0}}, \quad \boldsymbol{\infty}^{\mathbf{0}}.$$

Given $u(x)$ and $v(x)$ such that

$$\lim_{x\to a} u(x)^{v(x)}$$

falls into one of the categories described above, rewrite as

$$\lim_{x\to a} e^{v(x)\ln(u(x))}$$

and then examine the limit of the exponent

$$\lim_{x\to a} v(x)\ln(u(x)) = \lim_{x\to a}\frac{\ln(u(x))}{v(x)^{-1}}$$

using L'Hôpital's rule. Since these forms are all very similar, we will only give a single example.

Example 189. *Compute*

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.$$

Write

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}.$$

So now look at the limit of the exponent

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}}.$$

Setting $f(x) = \ln\left(1+\frac{1}{x}\right)$ and $g(x) = x^{-1}$, both functions are differentiable for large values of x and

$$\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} x^{-1} = 0.$$

We may now apply L'Hôpital's rule. Write

$$f'(x) = \frac{-x^{-2}}{1+\frac{1}{x}}$$

and

$$g'(x) = -x^{-2},$$

so

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} &= \lim_{x\to\infty}\frac{\frac{-x^{-2}}{1+\frac{1}{x}}}{-x^{-2}}\\ &= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}\\ &= 1.\end{aligned}$$

Hence,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^1 = e.$$

36 Antiderivatives

After completing this section, students should be able to do the following.

- Define an antiderivative.
- Compute basic antiderivatives.
- Compare and contrast finding derivatives and finding antiderivatives.
- Define initial value problems.
- Solve basic initial value problems.
- Use antiderivatives to solve simple word problems.
- Discuss the meaning of antiderivatives of a position function.

Break-Ground:

36.1 Jeopardy! Of calculus

Check out this dialogue between two calculus students (based on a true story):

Devyn: (Pretending to Alex Trebek) I've got a new costume.
Riley: Whoa! You look just like Trebek!
Devyn: In *Jeopardy!*, I, Trebek, give you an answer, and you must tell me the question.
Riley: Uh Alex, 'What are the rules of *Jeopardy!*?'
Devyn: Ha. Exactly! Let's play a different version where I'll tell you a derivative, and you tell me the function. Are you ready?
Riley: I'll take "Formulas for slope" for $200.
Devyn: $3 \cdot e^{3x}$
Riley: I've got an answer! Actually, I've got three different answers, I mean questions!

(a) "What's the derivative of e^{3x}?

(b) "What's the derivative of $e^{3x} + 1$?

(c) "What's the derivative of $e^{3x} - 1$?

Devyn: Hmmm. Now I'm not sure which one it was.
Riley: What about if you had given me $\dfrac{\sin(x)}{x}$?

Problem 1. *How many functions whose derivative is* $3 \cdot e^{3x}$ *are there?*

Multiple Choice:

(a) *Zero*

(b) *One*

(c) *Two*

(d) *Three*

(e) *Four*

(f) *Infinitely many*

Problem 2. *How many functions whose derivative is* $3 \cdot e^{3x}$ *that equal* 1 *at* $x = 0$ *are there?*

Multiple Choice:

(a) *Zero*

(b) *One*

(c) *Two*

(d) *Three*

(e) *Four*

(f) *Infinitely many*

Dig-In:

36.2 Basic antiderivatives

Computing derivatives is not too difficult. At this point, you should be able to take the derivative of almost any function you can write down. However, undoing derivatives is much harder. This process of undoing a derivative is called taking an *antiderivative*.

Definition. A function F is called an **antiderivative** of f on an interval if

$$F'(x) = f(x)$$

for all x in the interval.

Question 95. *How many antiderivatives does $f(x) = 2x$ have?*

Multiple Choice:

(a) *none*

(b) *one*

(c) *infinitely many*

There are two common ways to notate antiderivatives, either with a capital letter or with a funny symbol:

Definition. The antiderivative is denoted by

$$\int f(x)\,dx = F(x) + C,$$

where dx identifies x as the variable and C is a constant indicating that there are many possible antiderivatives, each varying by the addition of a constant. This is often called the **indefinite integral**.

Fill out these basic antiderivatives. Note each of these examples comes directly from our knowledge of basic derivatives.

Theorem 60 (Basic Antiderivatives)**.**

- $\displaystyle\int k\,dx = kx + C$
- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int a^x\,dx = \frac{a^x}{\ln(a)} + C$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int \cos(x)\,dx = \sin(x) + C$
- $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$
- $\displaystyle\int \tan(x)\,dx = -\ln|\cos(x)| + C$
- $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C$
- $\displaystyle\int \csc^2(x)\,dx = -\cot(x) + C$
- $\displaystyle\int \sec(x)\tan(x)\,dx = \sec(x) + C$
- $\displaystyle\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
- $\displaystyle\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$

It may seem that one could simply memorize these antiderivatives and antidifferentiating would be as easy as differentiating. This is **not** the case. The issue comes up when trying to combine these functions. When taking derivatives we have the *product rule* and the *chain rule*. The analogues of these two rules are much more difficult to deal with when taking antiderivatives. However, not all is lost. We have the following analogue of the sum rule for derivatives and the constant factor rule.

Theorem 61 (The Sum Rule for Antiderivatives)**.** *If F is an antiderivative of f and G is an antiderivative of g, then $F+G$ is an antiderivative of $f+g$.*

Theorem 62 (The Constant Factor Rule for Antiderivatives)**.** *If F is an antiderivative of f, and k is a constant, then kF is an antiderivative of kf.*

Let's put these rules and our knowledge of basic derivatives to work.

Example 190. *Find the antiderivative of $3x^7$.*

> By the theorems above , we see that
>
> $$\begin{aligned}\int 3x^7\,dx &= 3\int x^7\,dx\\ &= 3\cdot\frac{x^8}{8}+C.\end{aligned}$$

The sum rule for antiderivatives allows us to integrate term-by-term. Let's see an example of this.

Example 191. *Compute:*

$$\int \left(x^4+5x^2-\cos(x)\right)dx$$

> Let's start by simplifying the problem using the sum rule for antiderivatives,
>
> $$\begin{aligned}&\int \left(x^4+5x^2-\cos(x)\right)dx\\ &\qquad=\int x^4\,dx+5\int x^2\,dx-\int \cos(x)\,dx.\end{aligned}$$
>
> Now we may integrate term-by-term to find
>
> $$=\frac{x^5}{5}+\frac{5x^3}{3}-\sin(x)+C.$$

Warning. *While the sum rule for antiderivatives allows us to integrate term-by-term, we cannot integrate factor-by-factor, meaning that in general*

$$\int f(x)g(x)\,dx \neq \int f(x)\,dx\cdot\int g(x)\,dx.$$

Computing antiderivatives

Unfortunately, we cannot tell you how to compute every antiderivative. We advise that the mathematician view antiderivatives as a sort of *puzzle*. Later we will learn a hand-full of techniques for computing antiderivatives. However, a robust and simple way to compute antiderivatives is guess-and-check.

Tips for guessing antiderivatives

(a) Make a guess for the antiderivative.

(b) Take the derivative of your guess.

(c) Note how the above derivative is different from the function whose antiderivative you want to find.

(d) Change your original guess by **multiplying** by constants or by **adding** in new functions.

Template 1. *If the indefinite integral looks* something *like*

$$\int \text{stuff}' \cdot (\text{stuff})^n \, dx$$

guess

$$\text{stuff}^{n+1}$$

where $n \neq -1$.

Example 192. *Compute:*

$$\int \frac{x^3}{\sqrt{x^4 - 6}} \, dx$$

Start by rewriting the indefinite integral as

$$\int x^3 \left(x^4 - 6\right)^{-1/2} dx.$$

Now start with a guess of

$$\int x^3 \left(x^4 - 6\right)^{-1/2} dx \approx \left(x^4 - 6\right)^{1/2}.$$

Take the derivative of your guess to see if it is correct:

$$\frac{d}{dx}\left(x^4 - 6\right)^{1/2} = (4/2)x^3 \left(x^4 - 6\right)^{-1/2}.$$

We're off by a factor of $2/4$, so multiply our guess by this constant to get the solution,

$$\int \frac{x^3}{\sqrt{x^4 - 6}} \, dx = (2/4)(x^4 - 6)^{1/2} + C.$$

Template 2. *If the indefinite integral looks* something *like*

$$\int \text{junk} \cdot e^{\text{stuff}} \, dx$$

guess

$$e^{\text{stuff}} \textit{ or } \text{junk} \cdot e^{\text{stuff}}.$$

Example 193. *Compute:*

$$\int x e^x \, dx$$

We try to guess the antiderivative. Start with a guess of

$$\int x e^x \, dx \approx x e^x.$$

Take the derivative of your guess to see if it is correct:

$$\frac{d}{dx} x e^x = e^x + x e^x.$$

Ah! So we need only subtract e^x from our original guess. We now find

$$\int x e^x \, dx = x e^x - e^x + C.$$

Template 3. *If the indefinite integral looks* something *like*

$$\int \frac{\text{stuff}'}{\text{stuff}} \, dx$$

guess

$$\ln(\text{stuff}).$$

Example 194. *Compute:*

$$\int \frac{2x^2}{7x^3 + 3} \, dx$$

We'll start with a guess of

$$\int \frac{2x^2}{7x^3 + 3} \, dx \approx \ln(7x^3 + 3).$$

Take the derivative of your guess to see if it is correct:

$$\frac{d}{dx} \ln(7x^3 + 3) = \frac{21x^2}{7x^3 + 3}.$$

We are only off by a factor of $2/21$, so we need to multiply our original guess by this constant to get the solution,

$$\int \frac{2x^2}{7x^3+3}\,dx = (2/21)\ln(7x^3+3)+C.$$

Template 4. *If the indefinite integral looks* something *like*

$$\int \text{junk}\cdot\sin(\text{stuff})\,dx$$

guess

$$\cos(\text{stuff}) \textit{ or } \text{junk}\cdot\cos(\text{stuff}),$$

likewise if you have

$$\int \text{junk}\cdot\cos(\text{stuff})\,dx$$

guess

$$\sin(\text{stuff}) \textit{ or } \text{junk}\cdot\sin(\text{stuff}).$$

Example 195. *Compute:*

$$\int x^4\sin(3x^5+7)\,dx$$

Here we simply try to guess the antiderivative. Start with a guess of

$$\int x^4\sin(3x^5+7)\,dx \approx \cos(3x^5+7).$$

To see if your guess is correct, take the derivative of $\cos(3x^5+7)$,

$$\frac{d}{dx}\cos(3x^5+7) = -15x^4\sin(3x^5+7).$$

We are off by a factor of $-1/15$. Hence we should multiply our original guess by this constant to find

$$\int x^4\sin(3x^5+7)\,dx = \frac{-\cos(3x^5+7)}{15}+C.$$

Final thoughts

Computing antiderivatives is a place where insight and rote computation meet. We cannot teach you a method that will always work. Moreover, merely *understanding* the examples above will probably not be enough for you to become proficient in computing antiderivatives. You must practice, practice, practice!

Dig-In:

36.3 Falling objects

Remember, a **differential equation** is simply an equation with a derivative in it like this:

$$f'(x) = kf(x).$$

When a mathematician solves a differential equation, they are finding a *function* that satisfies the equation.

Recall that the acceleration due to gravity is about -9.8 m/s^2. Since the first derivative of the function giving the velocity of an object gives the acceleration of the object and the second derivative of a function giving the position of a falling object gives the acceleration, we have the differential equations

$$v'(t) = -9.8,$$
$$p''(t) = -9.8.$$

From these simple equation, we can derive equations for the velocity of the object and for the position using antiderivatives.

Example 196. *A ball is tossed into the air with an initial velocity of* 15 *m/s. What is the velocity of the ball after 1 second? How about after 2 seconds?*

> Knowing that the acceleration due to gravity is -9.8 m/s^2, we write
>
> $$v'(t) = -9.8.$$
>
> To solve this differential equation, take the antiderivative of both sides
>
> $$\int v'(t)\,dt = \int -9.8\,dt$$
> $$v(t) = -9.8t + C.$$
>
> Here C represents the initial velocity of the ball. Since it is tossed up with an initial velocity of 15 m/s,
>
> $$15 = v(0) = -9.8 \cdot 0 + C,$$
>
> and we see that $C = 15$. Hence $v(t) = -9.8t + 15$. Now when $t = 1$, $v(1) = 5.2$ m/s, and the ball is rising, and at $t = 2$, $v(2) = -4.6$ m/s, and the ball is falling.

Now let's do a similar problem, but instead of finding the velocity, we will find the position.

Example 197. *A ball is tossed into the air with an initial velocity of* 15 *m/s from a height of 2 meters. When does the ball hit the ground?*

> Knowing that the acceleration due to gravity is -9.8 m/s^2, we write
>
> $$p''(t) = -9.8.$$
>
> Start by taking the antiderivative of both sides of the equation
>
> $$\int p''(t)\,dt = \int -9.8\,dt$$
> $$p'(t) = -9.8t + C.$$
>
> Here C represents the initial velocity of the ball. Since it is tossed up with an initial velocity of 15 m/s, $C = 15$ and
>
> $$p'(t) = -9.8t + 15.$$
>
> Now let's take the antiderivative again.
>
> $$\int p'(t)\,dt = \int -9.8t + 15\,dt$$
> $$p(t) = \frac{-9.8t^2}{2} + 15t + D.$$
>
> Since we know the initial height was 2 meters, write
>
> $$2 = p(0) = \frac{-9.8 \cdot 0^2}{2} + 15 \cdot 0 + D.$$

Hence $p(t) = \frac{-9.8t^2}{2} + 15t + 2$. We need to know when the ball hits the ground, this is when $p(t) = 0$. Solving the equation

$$\frac{-9.8t^2}{2} + 15t + 2 = 0$$

we find two solutions $t \approx -0.1$ and $t \approx 3.2$. Discarding the negative solution, we see the ball will hit the ground after approximately 3.2 seconds.

The power of calculus is that it frees us from rote memorization of formulas and enables us to derive what we need.

37 Differential equations

After completing this section, students should be able to do the following.

- Define a differential equation.
- Verify solutions to differential equations.
- Find numerical solutions to differential equations using Euler's method.
- Understand how a slope field can be used to find solutions to a differential equation.

Break-Ground:

37.1 Modeling the spread of infectious diseases

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, check out this book, it has some cool applications of calculus:

> The following differential equation can be used to model the spread of an infectious disease:
>
> $$\mathrm{infect}'(t) = k \cdot \mathrm{infect}(t) \cdot (P - \mathrm{infect}(t))$$
>
> where k is a constant, $\mathrm{infect}(t)$ is the number of people infected by the disease on day t, and P is the size of the population vulnerable to the disease.

Riley: Whoa. That's like a formula for a derivative. Wow. Much calculus.

Devyn: I wonder how you solve equations like this?

Riley: I wonder if we can sometimes just use facts about the derivative to give us an approximation for the function that models the spread of infection?

Problem 1. *Suppose your calculus class has had a freak outbreak of the math-philia. Some facts: We have around* 200 *students in our class, we are now on the* 23*rd day of the outbreak, and currently* 100 *students are infected. Using the differential equation*

$$\mathrm{infect}'(t) = k \cdot \mathrm{infect}(t) \cdot (P - \mathrm{infect}(t))$$

we can model the spread of math-philia by setting $k = 0.001$. *What is* $\mathrm{infect}'(23)$*?*

Problem 2. *Do your best to explain why the equation*

$$\mathrm{infect}'(t) = k \cdot \mathrm{infect}(t) \cdot (P - \mathrm{infect}(t))$$

is reasonable.

Dig-In:

37.2 Differential equations

A *differential equation* is simply an equation with a derivative in it. Here is an example:

$$a \cdot f''(x) + b \cdot f'(x) + c \cdot f(x) = g(x).$$

Question 96. *What is a differential equation?*

Multiple Choice:

(a) *An equation that you take the derivative of.*

(b) *An equation that relates the rate of a function to other values.*

(c) *It is a formula for the slope of a tangent line at a given point.*

When a mathematician solves a differential equation, they are finding *functions* satisfying the equation.

Question 97. *Which of the following functions solve the differential equation*

$$f''(x) = -f(x)?$$

Select All Correct Answers:

(a) e^x

(b) $\sin(x)$

(c) $\cos(x)$

Exponential growth and decay

A function f exhibits **exponential growth** if its growth rate is proportional to its value. As a differential equation, this means

$$f'(x) = kf(x) \qquad \text{for some constant of proportionality } k.$$

We claim that this differential equation is solved by

$$f(x) = Ae^{kx},$$

where A and k are constants. Check it out, if $f(x) = Ae^{kx}$, then

$$\begin{aligned} f'(x) &= Ake^{kx} \\ &= k\left(Ae^{kx}\right) \\ &= kf(x). \end{aligned}$$

Example 198. *A culture of yeast starts with* 100 *cells. After* 160 *minutes, there are* 350 *cells. Assuming that the growth rate of the yeast is proportional to the number of yeast cells present, estimate when the culture will have* 1000 *cells.*

Since the growth rate of the yeast is proportional to the number of yeast cells present, we have the following differential equation

$$p'(t) = kp(t)$$

where $p(t)$ is the population of the yeast culture at time t with t measured in minutes. We know that this differential equation is solved by the function

$$p(t) = Ae^{kt}$$

where A and k are yet to be determined constants. Since

$$100 = p(0) = Ae^{k\cdot 0}$$

we see that $A = 100$. So

$$p(t) = 100e^{kt}.$$

Now we must find k. Since we know that

$$350 = p(160) = 100e^{k\cdot 160}$$

we need to solve for k. Write

$$\begin{aligned} 350 &= 100e^{k\cdot 160} \\ 3.5 &= e^{k\cdot 160} \\ \ln(3.5) &= k\cdot 160 \\ \ln(3.5)/160 &= k. \end{aligned}$$

Hence

$$p(t) = 100e^{t\ln(3.5)/160} = 100\cdot 3.5^{t/160}.$$

To find out when the culture has 1000 cells, write

$$\begin{aligned} 1000 &= 100\cdot 3.5^{t/160} \\ 10 &= 3.5^{t/160} \\ \ln(10) &= \frac{t\ln(3.5)}{160} \\ \frac{160\ln(10)}{\ln(3.5)} &= t. \end{aligned}$$

From this we find that after approximately 294 minutes, there are around 1000 yeast cells present.

It is worth seeing an example of exponential decay as well. Consider this: Living tissue contains two types of carbon, a stable isotope carbon-12 and a radioactive (unstable) isotope carbon-14. While an organism is alive, the ratio of one isotope of carbon to the other is always constant. When the organism dies, the ratio changes as the radioactive isotope decays. This is the basis of radiocarbon dating.

Example 199. *The half-life of carbon-14 (the time it takes for half of an amount of carbon-14 to decay) is about 5730 years. Moreover, the rate of decay of carbon-14 is proportional to the amount of carbon-14.*

If we find a bone with $1/70$th of the amount of carbon-14 we would expect to find in a living organism, approximately how old is the bone?

Since the rate of decay of carbon-14 is proportional to the amount of carbon-14 present, we can model this situation with the differential equation

$$f'(t) = kf(t).$$

We know that this differential equation is solved by the function defined by

$$f(t) = Ae^{kt}$$

where A and k are yet to be determined constants. Since the half-life of carbon-14 is about 5730 years we write

$$\frac{1}{2} = e^{k5730}.$$

Solving this equation for k, gives

$$k = \frac{-\ln(2)}{5730}.$$

Since we currently have $1/70$th of the original amount of carbon-14 we write

$$\frac{1}{70} = 1\cdot e^{\frac{-\ln(2)t}{5730}}.$$

Solving this equation for t, we find $t \approx -35121$. This means that the bone is approximately 35121 years old.

Infectious diseases

There are many models for the spread of infectious diseases. Perhaps the most basic is the following:

$$\mathrm{infect}'(t) = k\cdot \mathrm{infect}(t)\cdot(P - \mathrm{infect}(t))$$

where k is a constant, $\mathrm{infect}(t)$ is the number of people infected by the disease on day t, and P is the size of the population vulnerable to the disease.

What this is saying is that the rate that the infectious disease spreads is proportional to the product of the infected by the uninfected:

$$\underbrace{\operatorname{infect}'(t)}_{\text{rate the disease spreads}} = \overbrace{k}^{\text{is proportional to}} \cdot \underbrace{\operatorname{infect}(t)\cdot(P-\operatorname{infect}(t))}_{\text{this product}}$$

Why might this make a good model? We expect the rate that disease is spreading to be largest when

$$\operatorname{infect}(t) \approx P/2.$$

The product

$$\operatorname{infect}(t)\cdot(P-\operatorname{infect}(t))$$

is largest when $\operatorname{infect}(t) = P/2$. Finally we add the constant of proportionality as a scale factor.

Example 200. *Suppose your calculus class has had a freak outbreak of the math-philia. Some facts: We have around* 200 *students in our class, we are now on the* 23rd *day of the outbreak, and currently* 100 *students are infected. Using the differential equation*

$$\operatorname{infect}'(t) = k\cdot\operatorname{infect}(t)\cdot(P-\operatorname{infect}(t))$$

we can model the spread of math-philia by setting $k = 0.001$. *What is* $\operatorname{infect}'(23)$*?*

Here all we need to do is substitute all of the necessary information into the differential equation. We know

$$\begin{aligned} t &= 23,\\ P &= 200,\\ \operatorname{infect}(23) &= 100,\\ k &= 0.001. \end{aligned}$$

So

$$\begin{aligned} \operatorname{infect}'(t) &= k\cdot\operatorname{infect}(t)\cdot(P-\operatorname{infect}(t))\\ &= 0.001\cdot 100\cdot(200-100)\\ &= 10. \end{aligned}$$

Hence on day 23, we expect the disease to be spreading at a rate of 10 newly infected people per day.

Part VII

Content for the Third Exam

38 Approximating the area under a curve

After completing this section, students should be able to do the following.

- Define area.
- Understand the relationship between area under a curve and sums of rectangles.
- Approximate area under a curve.
- Compute left, right, and midpoint Riemann sums with 10 or fewer rectangles.

Break-Ground:

38.1 What is area?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I've have a troubling question.
Riley: Yes?
Devyn: What is area?
Riley: Oh, for a rectangle it is just

$$\text{height} \times \text{width}$$

Devyn: Right. But shouldn't it mean more?

Area is a funny thing. What is it really? Well the idea is this:

> Define something as having a "unit" area, and see how many times it "covers" something.

The most obivious thing to use for our "unit" area is a unit square. From this we can quickly move on to find the area of any rectangle as

$$\text{height} \times \text{width}$$

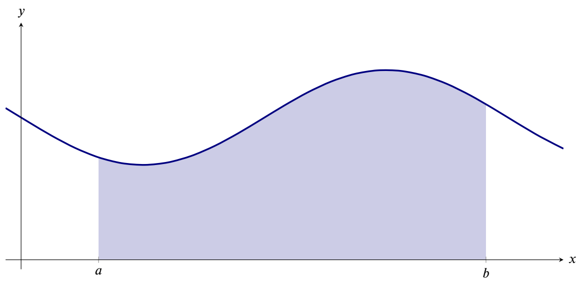
Problem 1. *Sometimes area is described as "how much paint will cover a surface." Is this accurate? What do you think?*

Dig-In:

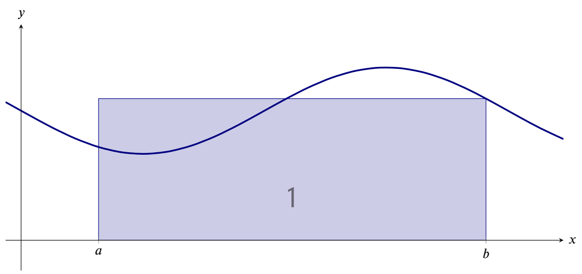
38.2 Approximating area with rectangles

Rectangles and areas

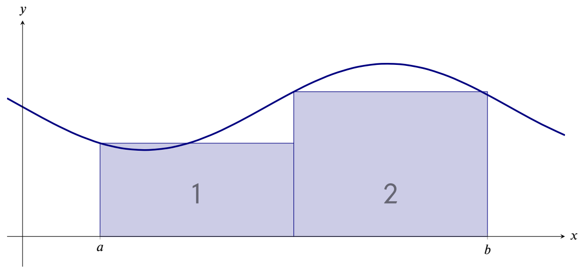
We want to compute the area between the curve $y = f(x)$ and the horizontal axis on the interval $[a, b]$:



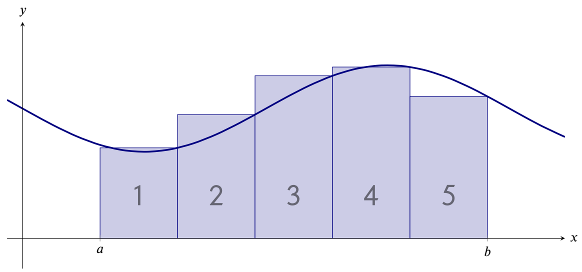
One way to do this would be to approximate the area with rectangles. With one rectangle we get a rough approximation:



Two rectangles might make a better approximation:

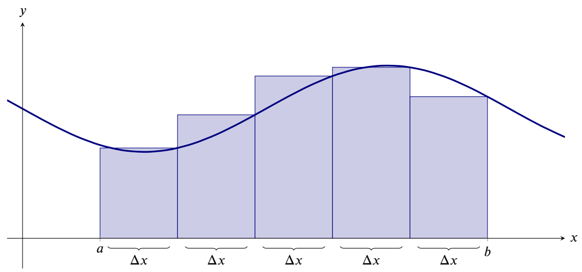


With even more, we get a closer, and closer, approximation:



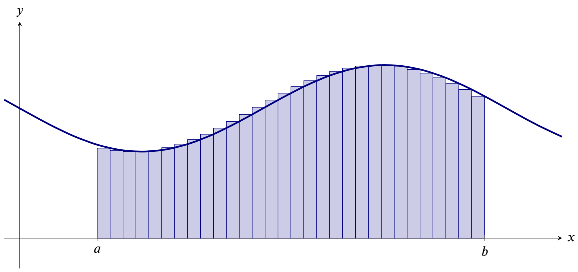
Definition. If we are approximating the area between a curve and the x-axis on $[a, b]$ with n rectangles of width Δx, then

$$\Delta x = \frac{b-a}{n}.$$



Question 98. *Suppose we wanted to approximate area between the curve $y = x^2 + 1$ and the x-axis on the interval $[-1, 1]$, with 8 rectangles. What is Δx?*

As we add rectangles, we are more closely approximating the area we are interested in:



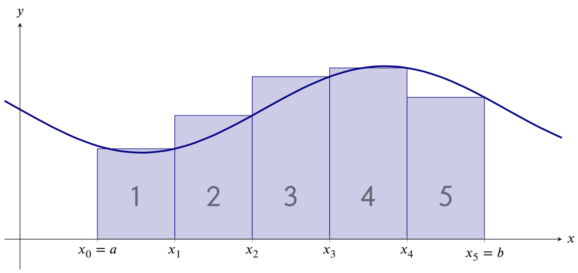
We could find the area exactly if we could compute the limit as the width of the rectangles goes to zero and the number of rectangles goes to infinity.

Let's setup some notation to help with these calculuations:

Definition. When approximating an area with n rectangles, the **grid points**

$$x_0, x_1, x_2, \ldots, x_n$$

are the x-coordinates that determine the edges of the rectangles. In the graph below, we've numbered the rectangles to help you see the relation between the indices of the grid points and the kth rectangle.



Note, if we are approximating the area between a curve and the horizontal axis on $[a, b]$ with n rectangles, then it is always the case that

$$x_0 = a \qquad \text{and} \qquad x_n = b.$$

Question 99. *If we are approximating the area between a curve and the horizontal axis with* 11 *rectangles, how many grid points will we have?*

But which set of rectangles?

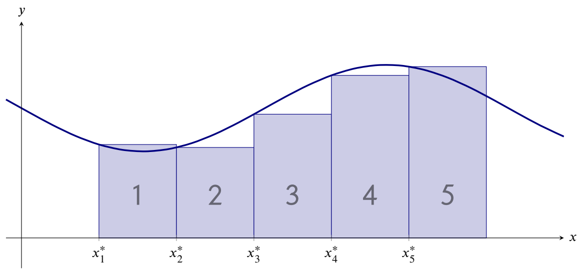
If we are going to try and actually use many small rectangles to compute the area under a curve, we should decide on exactly *which* rectangles we want to use. We need another definition:

Definition. When approximating an area with rectangles, a **sample point** is the x-coordinate that determines the relevant height of our rectangles. We denote a sample point as:

$$x_k^*$$

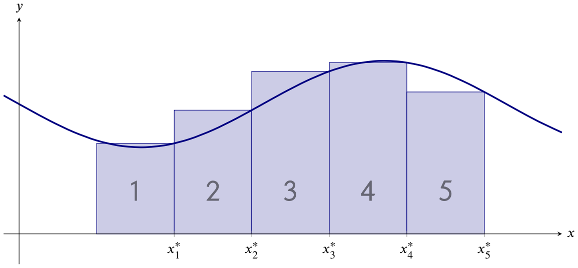
The sample point tells us where the rectangles touch the curve. Here are three options for sample points that we consider:

Rectangles defined by left-endpoints We can set the rectangles up so that the left-endpoint touches the curve.



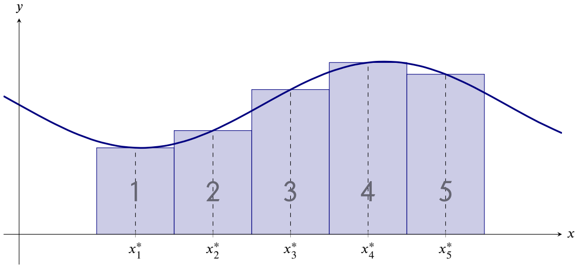
In the graph above, the kth rectangle's left-endpoint is touching the curve.

Rectangles defined by right-endpoints We can set the rectangles up so that the right-endpoint touches the curve.



In the graph above, the kth rectangle's right-endpoint is touching the curve.

Rectangles defined by midpoints We can set the rectangles up so that the midpoint of one of the horizontal sides touches the curve.



In the graph above, the midpoint of the horizontal side of the kth rectangle is touching the curve.

Riemann sums and approximating area

Once we know how to identify our rectangles, we can compute some approximate areas. If we are approximating area with n rectangles, then

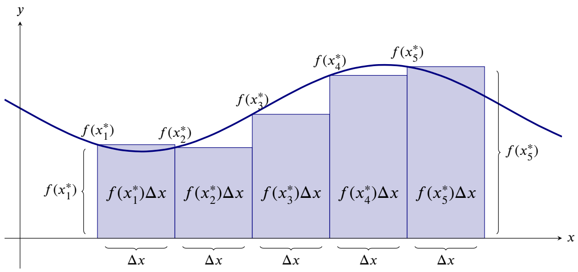
$$\begin{aligned}\text{Area} &\approx \sum_{k=1}^{n} (\text{height of } k\text{th rectangle}) \times (\text{width of } k\text{th rectangle}) \\ &= \sum_{k=1}^{n} f(x_k^*)\Delta x \\ &= f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \cdots + f(x_n^*)\Delta x.\end{aligned}$$

Definition. A sum of the form:

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

is called a **Riemann sum**, pronounced "ree-mahn" sum.

A Riemann sum computes an approximation of the area between a curve and the x-axis on the interval $[a, b]$. It can be defined several different ways. In our class, it will be defined via left-endpoints, right-endpoints, or midpoints. Here we see the explicit connection between a Riemann sum defined by left-endpoints and the area between a curve and the x-axis on the interval $[a, b]$:



and here is the associated Riemann sum

$$\sum_{k=1}^{5} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x.$$

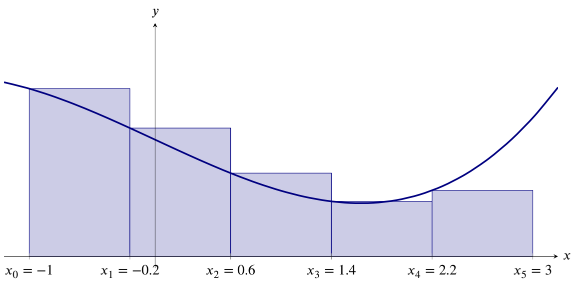
Left Riemann sums

Example 201. *Consider $f(x) = x^3/8 - x + 2$. Approximate the area between f and the x-axis on the interval $[-1, 3]$ using a left-endpoint Riemann sum with $n = 5$ rectangles.*

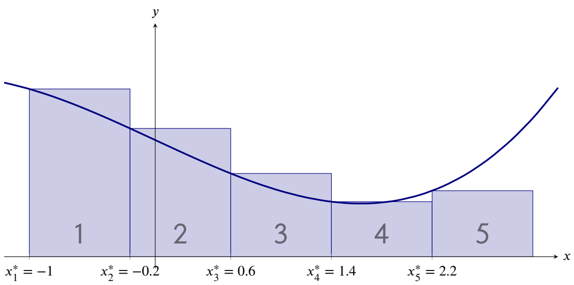
First note that the width of each rectangle is

$$\Delta x = \frac{3-(-1)}{5} = 4/5.$$

The grid points define the edges of the rectangle and are seen below:



On the other hand, the sample points identify which endpoints we use:



It is helpful to collect all of this data into a table:

k	x_k	x_k^*	$f(x_k^*)$
0	-1	NA	NA
1	-0.2	-1	2.875
2	0.6	-0.2	2.199
3	1.4	0.6	1.427
4	2.2	1.4	0.943
5	3	2.2	1.131

Now we may write a left Riemann sum and approximate the area

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x,$$

which evaluates to

$$\begin{aligned} &= 2.875 \cdot (4/5) + 2.199 \cdot (4/5) + 1.427 \cdot (4/5) \\ &\quad + 0.943 \cdot (4/5) + 1.131 \cdot (4/5) \end{aligned}$$

and we find

$$= 6.86.$$

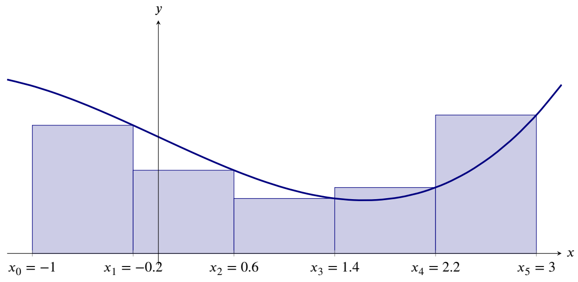
Right Riemann sums

Example 202. *Consider $f(x) = x^3/8 - x + 2$. Approximate the area between f and the x-axis on the interval $[-1, 3]$ using a right-endpoint Riemann sum with $n = 5$ rectangles.*

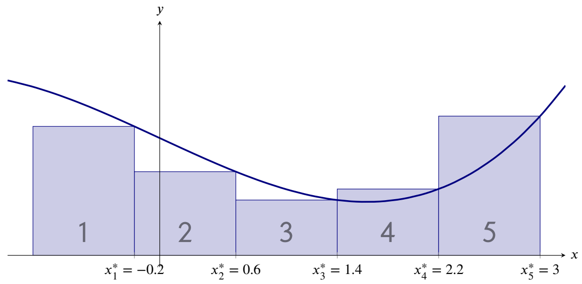
First note that the width of each rectangle is

$$\Delta x = \frac{3-(-1)}{5} = 4/5.$$

The grid points define the edges of the rectangle and are seen below:



On the other hand, the sample points identify which endpoints we use:



It is helpful to collect all of this data into a table:

k	x_k	x_k^*	$f(x_k^*)$
0	-1	NA	NA
1	-0.2	-0.2	2.199
2	0.6	0.6	1.427
3	1.4	1.4	0.943
4	2.2	2.2	1.131
5	3	3	2.375

Now we may write a right Riemann sum and approximate the area

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x,$$

which evaluates to

$$\begin{aligned} &= 2.199 \cdot (4/5) + 1.427 \cdot (4/5) + 0.943 \cdot (4/5) \\ &\qquad + 1.131 \cdot (4/5) + 2.375 \cdot (4/5) \end{aligned}$$

and we find

$$= 6.46.$$

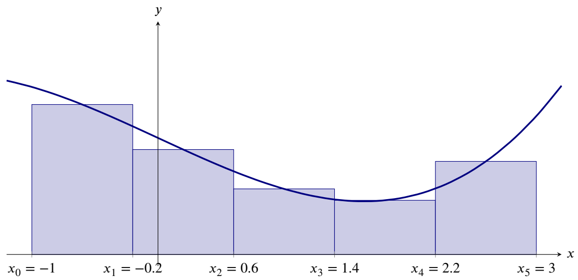
Midpoint Riemann sums

Example 203. *Consider $f(x) = x^3/8 - x + 2$. Approximate the area between f and the x-axis on the interval $[-1, 3]$ using a midpoint Riemann sum with $n = 5$ rectangles.*

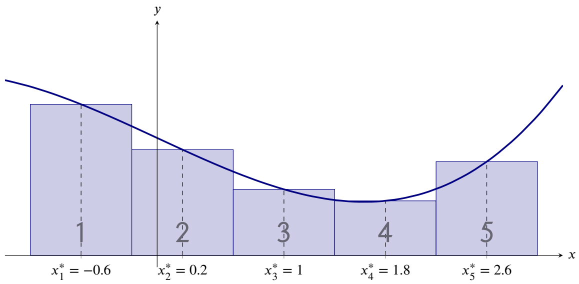
First note that the width of each rectangle is

$$\Delta x = \frac{3 - (-1)}{5} = 4/5.$$

The grid points define the edges of the rectangle and are seen below:



On the other hand, the sample points identify which endpoints we use:



It is helpful to collect all of this data into a table:

k	x_k	x_k^*	$f(x_k^*)$
0	-1	NA	NA
1	-0.2	-0.6	2.573
2	0.6	0.2	1.801
3	1.4	1	1.125
4	2.2	1.8	0.929
5	3	2.6	1.597

Now we may write a midpoint Riemann sum and approximate the area

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x + f(x_5^*)\Delta x,$$

which evaluates to

$$\begin{aligned} &= 2.573 \cdot (4/5) + 1.801 \cdot (4/5) + 1.125 \cdot (4/5) \\ &\quad + 0.929 \cdot (4/5) + 1.597 \cdot (4/5) \end{aligned}$$

and we find

$$= 6.42.$$

Summary Riemann sums approximate the area between curves and the x-axis via rectangles. When computing this area via rectangles, there are several things to know:

- What interval are we on? In our discussion above we call this $[a, b]$.
- How many rectangles will be used? In our discussion above we called this n.
- What is the width of each individual rectangle? In our discussion above we called this Δx.
- What points will determine the height of the rectangle? In our discussion above we called these sample points, x_k^*, and they can be left-endpoints, right-endpoints, or midpoints.
- What is the actual height of the rectangle? This will always be $f(x_k^*)$.

39 Definite integrals

After completing this section, students should be able to do the following.

- Use integral notation for both antiderivatives and definite integrals.
- Compute definite integrals using geometry.
- Compute definite integrals using the properties of integrals.
- Justify the properties of definite integrals using algebra or geometry.
- Understand how Riemann sums are used to find exact area.
- Define net area.
- Approximate net area.
- Split the area under a curve into several pieces to aid with calculations.
- Use symmetry to calculate definite integrals.
- Explain geometrically why symmetry of a function simplifies calculation of some definite integrals.

Break-Ground:

39.1 Computing areas

Check out this dialogue between two calculus students (based on a true story):

Devyn: I'm tired of Riemann sums.
Riley: Can't we use high school geometry to compute integrals?

Problem 1. *Which curves are good for using high school geometry to compute the area?*

Problem 2. *Which curves are good for using Riemann sums to compute the area?*

Dig-In:

39.2 The definite integral

Definite integrals, often simply called integrals, compute signed area.

Definition. The **definite integral**

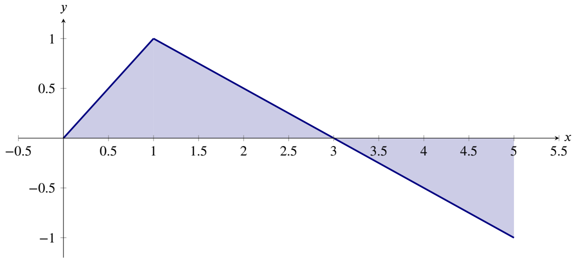
$$\int_a^b f(x)\,dx$$

computes the signed area between $y = f(x)$ and the x-axis on the interval $[a, b]$.

- If the region is above the x-axis, then the area has positive sign.
- If the region is below the x-axis, then the area has negative sign.

Note, when working with signed area, "positive" and "negative" area cancel each other out.

Question 100. *Consider the following graph of $y = f(x)$:*



Compute:

(a) $\int_0^3 f(x)\,dx$

(b) $\int_3^5 f(x)\,dx$

(c) $\int_0^5 f(x)\,dx$

(d) $\int_0^3 5 \cdot f(x)\,dx$

(e) $\int_1^1 5 \cdot f(x)\,dx$

Our previous question hopefully gives us enough insight that this next theorem is unsurprising.

Theorem 63 (Properties of the definite integral)**.** *Let f and g be defined on a closed interval $[a, b]$ that contains the value c, and let k be a constant. The following hold:*

(a) $\int_a^a f(x)\,dx = 0$

(b) $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

(c) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

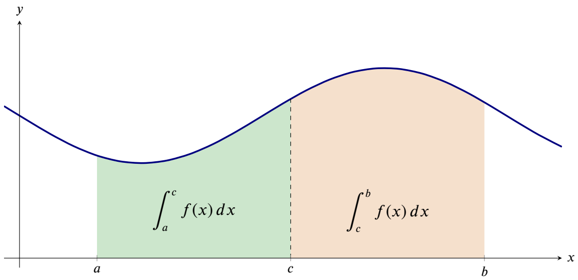
(d) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(e) $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$

We will address each property in turn:

(a) Here, there is no "area under the curve" when the region has no width; hence this definite integral is 0.

(b) This states that total area is the sum of the areas of subregions. Here a picture is worth a thousand words:



It is important to note that this still holds true even if $a < b < c$. We discuss this in the next point.

(c) For now, this property can be viewed a merely a convention to make other properties work well. However, later we will see how this property has a justification all its own.

(d) This states that when one scales a function by, for instance, 7, the area of the enclosed region also is scaled by a factor of 7.

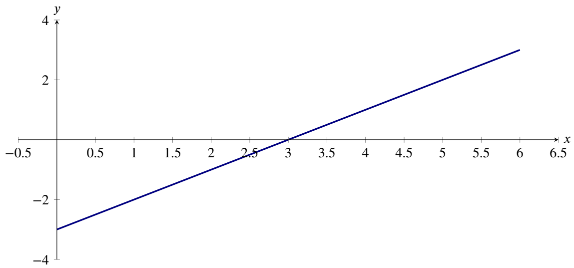
(e) This states that the integral of the sum is the sum of the integrals.

Due to the geometric nature of integration, geometric properties of functions can help us compute integrals.

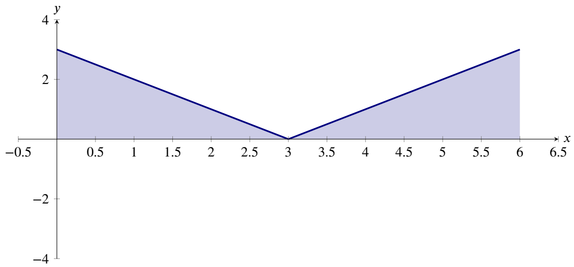
Example 204. *Compute:*

$$\int_0^6 |x-3|\,dx$$

This may seem difficult at first. Perhaps the first thing to do is look at a graph of $y = x - 3$:



Now we can graph $y = |x-3|$:



Now we see that we really have two triangles, each with base 3 and height 3. Hence

$$\begin{aligned}\int_0^6 |x-3|\,dx &= \int_0^3 3-x\,dx + \int_3^6 x-3\,dx\\ &= \frac{3\cdot 3}{2} + \frac{3\cdot 3}{2}\\ &= 9.\end{aligned}$$

Definition. A function f is an **odd** function if

$$f(-x) = -f(x),$$

and a function g is an **even** function if

$$g(-x) = g(x).$$

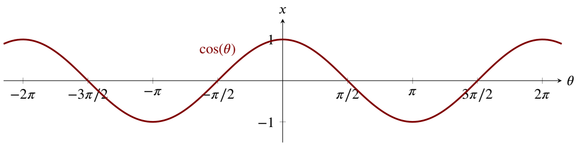
The names *odd* and *even* come from the fact that these properties are shared by functions of the form x^n where n is either odd or even. For example, if $f(x) = x^3$, then

$$f(-7) = -f(7),$$

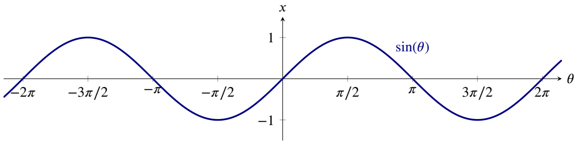
and if $g(x) = x^4$, then

$$g(-7) = g(7).$$

Geometrically, even functions have *horizontal symmetry*. Cosine is an even function:



On the other hand, odd functions have 180° *rotational symmetry* around the origin. Sine is an odd function:



Question 101. *Let f be an odd function defined for all real numbers. Compute:*

$$\int_{-2}^{2} f(x)\,dx$$

Question 102. *Let f be an odd function defined for all real numbers. Which of the following are equal to*

$$\int_{2}^{4} f(x)\,dx?$$

Select All Correct Answers:

(a) $\displaystyle\int_{4}^{2} f(x)\,dx$

(b) $\displaystyle\int_{-4}^{-2} f(x)\,dx$

(c) $\displaystyle\int_{-2}^{-4} f(x)\,dx$

(d) $\displaystyle\int_{-2}^{4} f(x)\,dx$

(e) $\displaystyle\int_{4}^{-2} f(x)\,dx$

(f) $\displaystyle\int_{2}^{-4} f(x)\,dx$

(g) $\displaystyle\int_{-4}^{2} f(x)\,dx$

(h) $\displaystyle -\int_{-4}^{2} f(x)\,dx$

(i) $\displaystyle -\int_{-4}^{-2} f(x)\,dx$

Signed verses geometric area

We know that the signed area between a curve $y = f(x)$ and the x-axis on $[a, b]$ is given by

$$\int_{a}^{b} f(x)\,dx.$$

On the other hand, if we want to know the *geometric area*, meaning the "actual" area, we compute

$$\int_{a}^{b} |f(x)|\,dx.$$

Question 103. *True or false:*

$$\int_{a}^{b} |f(x)|\,dx = \left|\int_{a}^{b} f(x)\,dx\right|$$

Multiple Choice:

(a) *true*

(b) *false*

Integrals and Riemann sums

Exactly how does an integral compute area? It depends on who you ask. If you ask Riemann, then you set

$$\Delta x = \frac{b-a}{n}$$

and look at the following limit of Riemann sums:

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)\,dx.$$

This says, take a curve, slice it up into n pieces on the interval $[a,b]$, add up all the areas of rectangles whose width is determined by the slices and the height is determined by a sample point in one of these pieces.

Example 205. *Compute this limit:*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{1-\left(-1+\frac{2k}{n}\right)^2}\right)\left(\frac{2}{n}\right)$$

This is a limit of Riemann sums! Specifically, it is a limit of Riemann sums of n rectangles, where

$$\Delta x = \frac{2}{n}$$

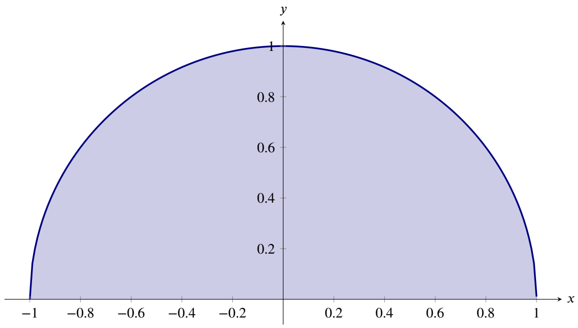
and

$$x_k^* = -1+\frac{2k}{n}.$$

Hence, we may rewrite this as

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{1-(x_k^*)^2}\right)\Delta x.$$

Now we see that this computes the area between the x-axis and the curve $y=\sqrt{1-x^2}$. Let's see it:



By geometry, we know that this semicircle has area $\pi/2$. Hence

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{1-(x_k^*)^2}\right)\Delta x = \pi/2.$$

40 Antiderivatives and area

After completing this section, students should be able to do the following.

- Interpert the product of rate and time as area.
- Approximate position from velocity.
- Recognize Riemann sums.

Break-Ground:

40.1 Meaning of multiplication

Check out this dialogue between two calculus students (based on a true story):

Devyn: Hey Riley, I was reading about the history of mathematics.
Riley: Really? Tell me about it!
Devyn: Apparently, back in the day, mathematicians worried about writing things like:

$$3 \times 4 + 5$$

Riley: Why? What's the matter here?
Devyn: Well, they thought of 3×4 as an **area**. But 5 was thought of as a **length**. Apparently they worried whether it made sense to add "areas" and "lengths."
Riley: Hmmm. We don't seem to worry about that now. I wonder why?

Problem 1. *What are some ways to interpret* 3×4*?*

Dig-In:

40.2 Relating velocity and position, antiderivatives and areas

A central theme of this course has been that we can often gain a better understanding of a function by looking at its derivative, and then working backwards. This has been our approach to max/min problems, curve sketching, linear approximation, and so on. So antiderivatives have really been important to us all along. We have a geometric interpretation of the derivative as the slope of a tangent line at a point. We have not yet found a geometric interpretation of antiderivatives.

More than one perspective

We'll start with a question:

Question 104. *Suppose you are in slow traffic moving at* 4 *mph from* 2*pm to* 5*pm. How far have you traveled?*

Now we move to a seemingly unrelated question:

Question 105. *What is the area of the region bounded by the graph of* $f(x) = 4$*, the horizontal axis, and the vertical lines* $x = 2$ *and* $x = 5$*?*

The fact that these two answers are the same is the germ of one of the most "fundamental" ideas in all of calculus. However, before we can step ahead, we might first look back to our even younger days of being mathematicians.

Recall that there are (at least!) two basic models of multiplication: A "rate times time" perspective and an "area" perspective. For instance, we could interpret

$$4 \times 3$$

as an answer to the question:

> If I am going 4mph for 3 hours, how far have I traveled?

or as the answer to the question

> What is the area of a rectangle with height 4 and width 3?

In what follows below, we will leverage these two different notions to gain insight into our study of functions.

From position to area

Suppose you have a continuous function $v(t)$ representing the velocity of some object at time t. We know that velocity is the change in position over time. If we wanted to recover the position of the object, we could approximate a graph by taking values of $v(t)$ and projecting the position via a small amount of time. Let's say this twice more:

$$\text{change in position} = \text{velocity} \times \text{change in time}$$

letting $s(t)$ be position, we see this translates **directly** into the language of differentials, since $s'(t) = v(t)$,

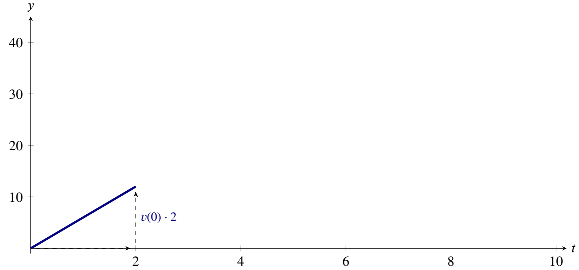
$$ds = v(t)\,dt.$$

Suppose we want to know how far we have traveled over the time interval $[0, 10]$. One way to proceed would be to cut the interval $[0, 10]$ into equal sized sections, we'll say five sections to keep things easy. Now we need to know the velocity of our object at those times. We'll describe $v(t)$ with the following table:

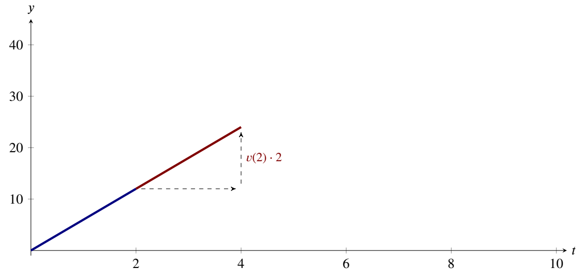
t	$v(t)$
0	6
2	6
4	4
6	0
8	-6

Example 206. *Assuming* $s(0) = 0$*, make an approximate graph of* $y = s(t)$ *with a piecewise linear function using the idea of differentials with* $dt = 2$*.*

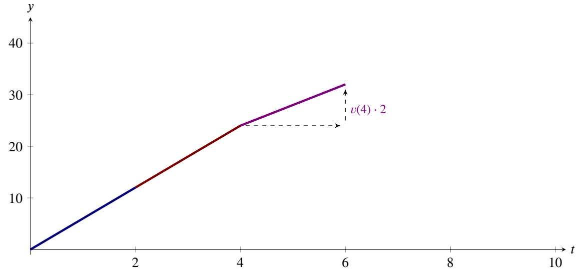
Our table from before gives us all the information we need! At $t = 0$, $s'(0) = v(0) = 6$, so starting with the point $(0, 0)$ we attach a segment of slope 6 over an interval of length 2:



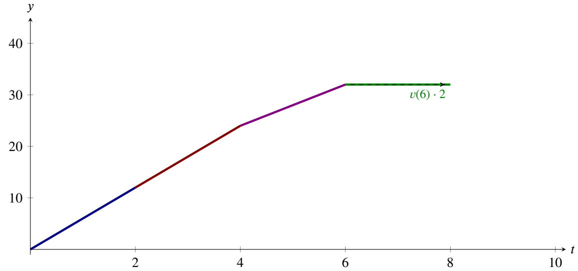
At $t = 2$, $s'(2) = v(2) = 6$, so we attach a segment of slope 6 over the next interval of length 2:



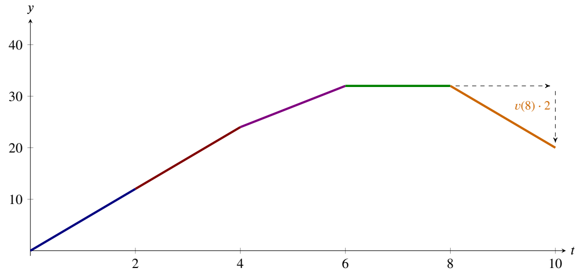
At $t = 4$, $s'(4) = v(4) = 4$, so we attach a segment of slope 4 over the next interval of length 2:



At $t = 6$, $s'(6) = v(6) = 0$, so we attach a segment of slope 0 over the next interval of length 2:



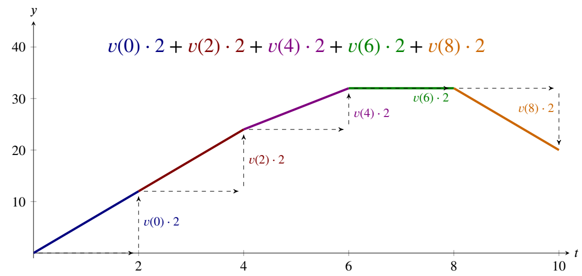
At $t = 8$, $s'(8) = v(8) = -6$, so we attach a segment of slope -6 over the next interval of length 2:



This gives us a plot of the approximate position of our object.

Example 207. *Use your approximation above with a step-size* *of $dt = 2$ to estimate $s(10)$.*

If we label our graph above, we will have a much easier time solving this problem:



So

$$\begin{aligned} s(10) &\approx v(0)\cdot 2 + v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2 + v(8)\cdot 2 \\ &= 6\cdot 2 + 6\cdot 2 + 4\cdot 2 + 0\cdot 2 + (-6)\cdot 2 \\ &= 20. \end{aligned}$$

Hmmm. Let's look at our answer from the previous question again:

$$v(0)\cdot 2 + v(2)\cdot 2 + v(4)\cdot 2 + v(6)\cdot 2 + v(8)\cdot 2$$

This is a Riemann sum!

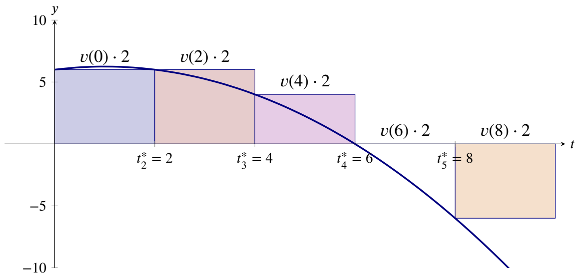
When we exclaim (with great excitement!) that "this is a Riemann sum," we are really saying that we are in a situation where we may view

$$\text{rate} \times \text{time}$$

as

$$\text{height} \times \text{width}$$

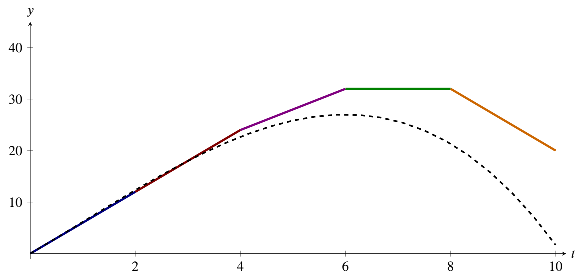
Check it out: here $n = 5$, $\Delta t = 2$, and we are looking at left-endpoints! Here is a suggestive graph of $y = v(t)$:



Here is the upshot: When we are computing values of antiderivatives, we are simultaneously computing areas between curves and the horizontal axis!

On the notation for antiderivatives

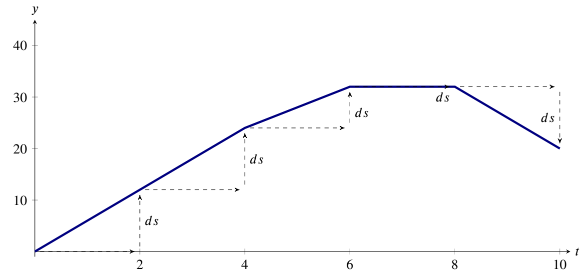
Let's look at our approximation of $s(t)$ we found above, we'll include the actual plot of $s(t)$ as well, shown as a dashed curve:



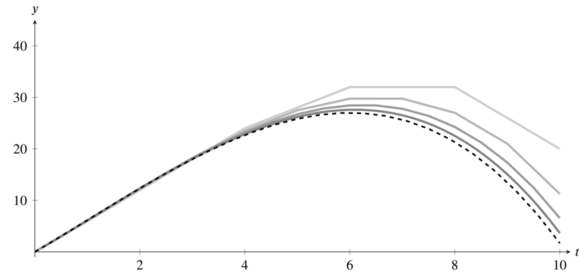
At each step we are computing

$$ds = v(t)\,dt$$

and adding it to the previous result:



If we wanted to, we could take a smaller step-size (our value for dt) and gain better, and better, approximations of $s(t)$:



At this point, we can explain the notation for antiderivatives. If we "sum" all of the "ds," we find the value of position. Hence

$$\begin{aligned} ds &= v(t)\,dt \\ \text{"sum" } ds &= \text{"sum" } v(t)\,dt \\ \int ds &= \int v(t)\,dt \\ s(t) &= \int v(t)\,dt, \end{aligned}$$

thus we see that an antiderivative is, essentially, a sum.

41 First Fundamental Theorem of Calculus

After completing this section, students should be able to do the following.

- Define accumulation functions.
- Calculate and evaluate accumulation functions.
- State the First Fundamental Theorem of Calculus.
- Take derivatives of accumulation functions using the First Fundamental Theorem of Calculus.
- Use accumulation functions to find information about the original function.
- Understand the relationship between the function and the derivative of its accumulation function.

Break-Ground:

41.1 What's in a calculus problem?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley! How do you know when you are doing a calculus problem?
Riley: Hmmm. I guess if you take a derivative you are doing a calculus problem.
Devyn: What else?

Problem 1. *Let's hear your answer to Devyn's question above.*

Dig-In:

41.2 The First Fundamental Theorem of Calculus

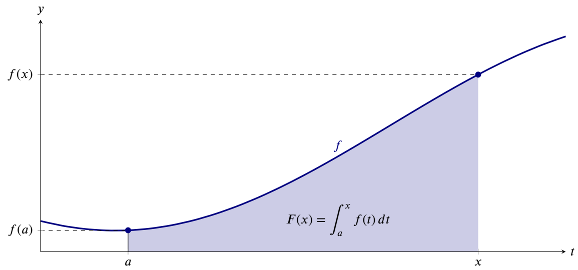
Accumulation functions

While the definite integral computes a signed area, which is a fixed number, there is a way to turn it into a function.

Definition. A given a function f, an **accumulation function** is

$$F(x) = \int_a^x f(t)\,dt.$$

One thing that you might notice is that an accumulation function seems to have two variables: x and t. Let's see if we can explain this. Consider the following graph:



An accumulation function F measures the signed area in the region $[a,x]$ between f and the t-axis. Hence t is playing the role of a "place-holder" that allows us to evaluate f. On the other hand, x is the **specific number** that we are using to bound the region that will determine the area between f and the t-axis, and hence the value of F.

Question 106. *Given*

$$F(x) = \int_{-3}^x 4\,dt,$$

what is $F(5)$?

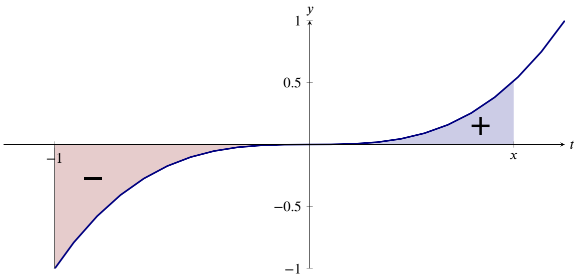
Question 107. *What is $F(-5)$?*

Example 208. *Consider the following accumulation function for $f(x) = x^3$.*

$$F(x) = \int_{-1}^x t^3\,dt.$$

Considering the interval $[-1,1]$, where is F increasing? Where is F decreasing? When does F have local extrema?

We can see a graph of f along with the signed area measured by the accumulation function below



The accumulation function starts off at zero, and then as x grows, F is decreasing as the function accumulates negatively signed area.

However when $x > 0$, F starts to accumulate positively signed area, and hence is increasing. Thus F is increasing on $(0,1)$, decreasing on $(-1,0)$ and hence has a local minimum at $(0,0)$.

Working with the accumulation function leads us to a question, what is

$$\int_a^x f(x)\,dx$$

when $x < a$? The general convention is that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

With this in mind, let's consider one more example.

Example 209. *Consider the following accumulation function for $f(x) = x^3$.*

$$F(x) = \int_{-1}^{x} t^3\,dt.$$

Where is F increasing? Where is F decreasing? When does F have local extrema?

> From our previous example, we know that F is increasing on $(0, 1)$. Since f continues to be positive at $t = 1$ and beyond, F is increasing on $(0, \infty)$. On the other hand, we know from our previous example that F is decreasing on $(-1, 0)$.
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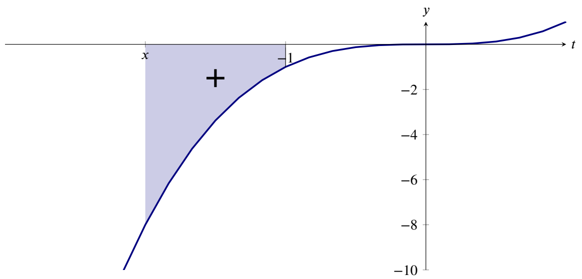
> 
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>
> For values to the left of $t = -1$, F is still decreasing, as less and less positively signed area is accumulated. Hence F is increasing on $(0, \infty)$, decreasing on $(-\infty, 0)$ and hence has an absolute minimum at $(0, 0)$.

The key point to take from these examples is that an accumulation function

$$F(x) = \int_{a}^{x} f(t)\,dt$$

is increasing precisely when f is positive and is decreasing precisely when f is negative. In short, it seems that f is behaving in a similar fashion to F'.

The First Fundamental Theorem of Calculus

Let f be a continuous function on the real numbers and consider

$$F(x) = \int_{a}^{x} f(t)\,dt.$$

From our previous work we know that F is increasing when f is positive and F is decreasing when f is negative. Moreover, with careful observation, we can even see that F is concave up when f' is positive and that F is concave down when f' is negative. Thinking about what we have learned about the relationship of a function to its first and second derivatives, it is not too hard to guess that there must be a connection between F' and the function f. This is a good guess, check out our next theorem:

Theorem 64 (First Fundamental Theorem of Calculus)**.** *Suppose that f is continuous on the real numbers and let*

$$F(x) = \int_{a}^{x} f(t)\,dt.$$

Then $F'(x) = f(x)$.

The First Fundamental Theorem of Calculus says that an accumulation function of f is an antiderivative of f. Another way of saying this is:

$$\frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x)$$

This could be read as:

> **The rate that accumulated area under a curve grows is described identically by that curve.**

Now that we are working with accumulation functions, let's see what happens when we compose them with other functions.

Example 210. *Find the derivative of*

$$F(x) = \int_2^{x^2} \ln t \, dt.$$

Consider

$$G(x) = \int_2^{x} \ln t \, dt$$

and set $h(x) = x^2$. Now

$$F(x) = G(h(x)).$$

The First Fundamental Theorem of Calculus states that $G'(x) = \ln x$. The chain rule gives us

$$\begin{aligned} F'(x) &= G'(h(x))h'(x) \\ &= \ln(h(x))h'(x) \\ &= \ln(x^2)2x. \end{aligned}$$

Let's practice this once more.

Example 211. *Find the derivative of*

$$F(x) = \int_{\cos x}^{5} t^3 \, dt.$$

Consider

$$G(x) = -\int_5^{x} t^3 \, dt$$

and set $h(x) = \cos(x)$. Now

$$F(x) = G(h(x)).$$

The First Fundamental Theorem of Calculus states that $G'(x) = -x^3$. The chain rule gives us

$$\begin{aligned} F'(x) &= G'(h(x))h'(x) \\ &= -h(x)^3 h'(x) \\ &= -\cos^3(x)(-\sin(x)). \end{aligned}$$

42 Second Fundamental Theorem of Calculus

After completing this section, students should be able to do the following.

- State the Second Fundamental Theorem of Calculus.
- Evaluate definite integrals using the Second Fundamental Theorem of Calculus.
- Understand how the area under a curve is related to the antiderivative.
- Understand the relationship between indefinite and definite integrals.

Break-Ground:

42.1 A secret of the definite integral

Check out this dialogue between two calculus students (based on a true story):

Devyn: Ah. So now we have a connection between derivatives and integrals.
Riley: Right, the derivative of the accumulation function is the "inside" function.
Devyn: So how do we use this to compute area?

Sometimes it helps to think about the most basic examples. Consider

$$\int_2^5 4\,dt$$

We know (by geometry) that this computes the area of a 3×4 rectangle which equals 12. On the other hand, if we consider the accumulation function

$$F(x) = \int_2^x 4\,dt,$$

we see that

$$F(5) = \int_2^5 4\,dt.$$

Problem 1. *What is $F(2)$?*

Problem 2. *On the other hand, the First Fundamental Theorem of Calculus says that if*

$$F(x) = \int_2^x 4\,dt,$$

then $F'(x) = 4$. Armed with this knowledge, and the fact that $F(2) = 0$, what must $F(x)$ be?

Dig-In:

42.2 The Second Fundamental Theorem of Calculus

There is a another common form of the Fundamental Theorem of Calculus:

Theorem 65 (Second Fundamental Theorem of Calculus)**.** *Let f be continuous on $[a, b]$. If F is* ***any*** *antiderivative of f, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Let $a \le c \le b$ and write

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\\ &= \int_c^b f(x)\,dx - \int_c^a f(x)\,dx.\end{aligned}$$

By the First Fundamental Theorem of Calculus, we have

$$F(b) = \int_c^b f(x)\,dx \qquad \text{and} \qquad F(a) = \int_c^a f(x)\,dx$$

for some antiderivative F of f. So

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

for this antiderivative. However, **any** antiderivative could have be chosen, as antiderivatives of a given function differ only by a constant, and this constant *always* cancels out of the expression when evaluating $F(b) - F(a)$.

From this you should see that the two versions of the Fundamental Theorem are very closely related. In reality, the two forms are **equivalent**, just differently stated. Hence people often simply call them both "The Fundamental Theorem of Calculus." One way of thinking about the Second Fundamental Theorem of Calculus is:

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

This could be read as:

The accumulation of a rate is given by the change in the amount.

When we compute a definite integral, we first find an antiderivative and then evaluate at the limits of integration. It is convenient to first display the antiderivative and then evaluate. A special notation is often used in the process of evaluating definite integrals using the Fundamental Theorem of Calculus. Instead of explicitly writing $F(b) - F(a)$, we often write

$$\Big[F(x)\Big]_a^b$$

meaning that one should evaluate $F(x)$ at b and then subtract $F(x)$ evaluated at a

$$\Big[F(x)\Big]_a^b = F(b) - F(a).$$

Let's see some examples of the fundamental theorem in action.

Example 212. *Compute:*

$$\int_{-2}^2 x^3\,dx$$

We start by finding an antiderivative of x^3. A correct choice is $\frac{x^4}{4}$, one could verify this by taking the derivative. Hence

$$\begin{aligned}\int_{-2}^{2} x^3\,dx &= \left[\frac{x^4}{4}\right]_{-2}^{2}\\ &= \frac{2^4}{4} - \frac{(-2)^4}{4}\\ &= 0.\end{aligned}$$

Example 213. *Compute:*

$$\int_0^{\pi} \sin\theta\,d\theta$$

We start by finding an antiderivative of $\sin\theta$. A correct choice is $-\cos\theta$, one could verify this by taking the derivative. Hence

$$\begin{aligned}\int_0^{\pi} \sin\theta\,d\theta &= \Big[-\cos\theta\Big]_0^{\pi}\\ &= -\cos(\pi) - (-\cos(0))\\ &= 2.\end{aligned}$$

This is interesting: It says that the area under one "hump" of a sine curve is 2.

Example 214. *Compute:*

$$\int_0^5 e^t\,dt$$

We start by finding an antiderivative of e^t. A correct choice is e^t, one could verify this by taking the derivative. Hence

$$\begin{aligned}\int_0^5 e^t\,dt &= \Big[e^t\Big]_0^5\\ &= e^5 - e^0\\ &= e^5 - 1.\end{aligned}$$

Example 215. *Compute:*

$$\int_1^2 \left(x^9 + \frac{1}{x}\right) dx$$

We start by finding an antiderivative of $x^9 + \frac{1}{x}$. A correct choice is $\frac{x^{10}}{10} + \ln(x)$, one could verify this by taking the derivative. Hence

$$\begin{aligned}\int_1^2 \left(x^9 + \frac{1}{x}\right) dx &= \left[\frac{x^{10}}{10} + \ln(x)\right]_1^2\\ &= \frac{2^{10}}{10} + \ln(2) - \frac{1}{10}.\end{aligned}$$

Understanding motion with the Fundamental Theorem of Calculus

We know that

- The derivative of a position function is a velocity function.
- The derivative of a velocity function is an acceleration function.

Now consider definite integrals of velocity and acceleration functions. Specifically, if $v(t)$ is a velocity function, what does $\int_a^b v(t)\,dt$ mean?

The Second Fundamental Theorem of Calculus states that

$$\int_a^b v(t)\,dt = V(b) - V(a),$$

where $V(t)$ is any antiderivative of $v(t)$. Since $v(t)$ is a velocity function, $V(t)$ must be a position function, and $V(b) - V(a)$ measures a **change in position**, or **displacement**.

Example 216. *A ball is thrown straight up with velocity given by $v(t) = -32t + 20$ft/s, where t is measured in seconds. Find, and interpret, $\displaystyle\int_0^1 v(t)\,dt$.*

> Using the Second Fundamental Theorem of Calculus, we have
>
> $$\begin{aligned}\int_0^1 v(t)\,dt &= \int_0^1 (-32t + 20)\,dt \\ &= \Big[-16t^2 + 20t\Big]_0^1 \\ &= 4.\end{aligned}$$
>
> Thus if a ball is thrown straight up into the air with velocity
>
> $$v(t) = -32t + 20,$$
>
> the height of the ball, 1 second later, will be 4 feet above the initial height. Note that the ball has *traveled* much farther. It has gone up to its peak and is falling down, but the difference between its height at $t = 0$ and $t = 1$ is 4ft.

Now we know that to solve certain kinds of problems, those that involve accumulation of some form, we "merely" find an antiderivative and substitute two values and subtract. Unfortunately, finding antiderivatives can be quite difficult. While there are a small number of rules that allow us to compute the derivative of any common function, there are no such rules for antiderivatives. There are some techniques that frequently prove useful, but we will never be able to reduce the problem to a completely mechanical process.

Dig-In:

42.3 A tale of three integrals

At this point we have three different "integrals." Let's see if we can sort out the differences.

Indefinite integrals

An indefinite integral, also called an **antiderivative** computes classes of functions:

$$\int f(x)\,dx = \text{"a class of functions whose derivative is } f\text{"}$$

Here there are no limits of integration, and your answer will have a "$+C$" at the end. Pay attention to the notation:

$$\int f(x)\,dx = F(x) + C$$

Where $F'(x) = f(x)$.

> Indefinite integrals do not have limits of integration, and they compute a class of antiderivatives.

Question 108. *Two students, say Devyn and Riley, are working with the following indefinite integral:*

$$\int \frac{2}{x\ln(x^2)}\,dx$$

Devyn computes the integral as

$$\int \frac{2}{x\ln(x^2)}\,dx = \ln|\ln|x^2|| + C$$

and Riley computes the integral as

$$\int \frac{2}{x\ln(x^2)}\,dx = \ln|\ln|x|| + C.$$

Which student is correct?

Multiple Choice:

(a) *Devyn is correct*

(b) *Riley is correct*

(c) *Both students are correct*

(d) *Neither student is correct*

Accumulation functions

An **accumulation function**, also called an **area function** computes accumulated area:

$$\int_a^x f(t)\,dt = \text{"a function } F \text{ whose derivative is } f\text{"}$$

This is a function of x whose derivative is f, with the additional property that $F(a) = 0$. Pay attention to the notation:

$$F(x) = \int_a^x f(t)\,dt$$

Where $F'(x) = f(x)$.

> Accumulation functions have limits of integration, and they compute an antiderivative.

Question 109. *True or false: There exists a function f such that*

$$\int_0^x f(t)\,dt = e^x$$

Multiple Choice:

(a) *true*

(b) *false*

Definite integrals

A **definite integral** computes signed area:

$$\int_a^b f(x)\,dx = \text{“the signed area between the } x\text{-axis and } f\text{”}$$

Here we always have limits of integration, both of which are numbers. Moreover, definite integrals have definite values, the signed area between f and the x-axis. Pay attention to the notation:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Where $F'(x) = f(x)$.

> Definite integrals have limits of integration, and they compute signed area.

Question 110. *Consider*

$$f(x) = \begin{cases} -2 & \text{if } x < 1, \\ 2 & \text{if } x \geq 1. \end{cases}$$

If we compute an antiderivative of f, we find

$$F(x) = \begin{cases} -2x & \text{if } x < 1, \\ 2x & \text{if } x \geq 1. \end{cases}$$

Is it correct to say

$$\begin{aligned} \int_0^1 f(x)\,dx &= \Big[F(x)\Big]_0^1 \\ &= F(1) - F(0) \\ &= 2? \end{aligned}$$

Multiple Choice:

(a) *yes*

(b) *no*

43 Applications of integrals

After completing this section, students should be able to do the following.

- Given a velocity function, calculate displacement and distance traveled.
- Given a velocity function, find the position function.
- Given an acceleration function, find the velocity function.
- Understand the difference between displacement and distance traveled.
- Understand the relationship between position, velocity and acceleration.

Break-Ground:

43.1 What could it represent?

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley, I like integrals.
Riley: I feel fancy when I make an integral sign.
Devyn: I know! An integral computes the signed area between a curve $y = f(x)$ and the x-axis. But why *signed* area? Maybe we should just compute plain old area.
Riley: Makes sense to me!
Deyvn: Unless... maybe there are other applications where "signed" area makes more sense.

One really great way to think about integrals is that they "accumulate rates."

Problem 1. *Write down as many examples of "rates" and "accumulated rates" as you can. For example:*

> 5 *miles per hour is a rate, and* 5 *miles is then an accumulated rate.*

Dig-In:

43.2 Applications of integrals

Velocity and displacement, speed and distance

Some values include "direction" that is relative to some fixed point.

Definition.

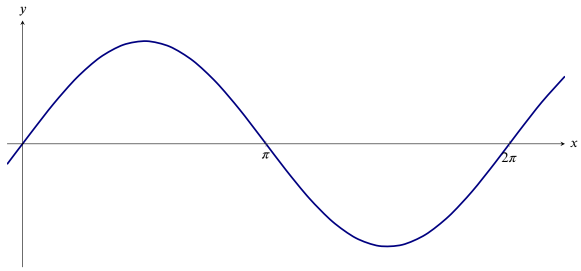
- $v(t)$ is the **velocity** of an object at time t. This represents the "change in position" at time t.
- $s(t)$ is the **position** of an object at time t. This gives location with respect to the origin. If we can assume that $s(a) = 0$, then
$$s(t) = \int_a^t v(x)\,dx.$$
- $s(b) - s(a)$ is the **displacement**, the distance between the starting and finishing locations.

On the other hand *speed* and *distance* are values without "direction."

Definition.

- $|v(t)|$ is the **speed**.
- $\displaystyle\int_a^b |v(t)|\,dt$ is the **distance** traveled.

Question 111. *Consider a particle whose velocity at time t is given by $v(t) = \sin(t)$.*



What is the displacement of the particle from $t = 0$ to $t = \pi$? That is, compute:
$$\int_0^\pi \sin(t)\,dt$$
What is the displacement of the particle from $t = 0$ to $t = 2\pi$? That is, compute:
$$\int_0^{2\pi} \sin(t)\,dt$$
What is the distance traveled by the particle from $t = 0$ to $t = \pi$? That is, compute:
$$\int_0^\pi |\sin(t)|\,dt$$
What is the distance traveled by the particle from $t = 0$ to $t = 2\pi$? That is, compute:
$$\int_0^{2\pi} |\sin(t)|\,dt$$

Average value

Conceptualizing definite integrals as "signed area" works great as long as one can actually visualize the "area." In some cases, a better metaphor for integrals comes from the idea of *average value*. Looking back to your days as an even younger mathematician, you may recall the idea of an *average*:
$$\frac{f_1 + f_2 + \cdots + f_n}{n} = \frac{1}{n}\sum_{k=1}^n f_i$$

If we want to know the average value of a function, a naive approach might be to partition the interval $[a, b]$ into n equally spaced subintervals,

$$a = x_0 < x_1 < \cdots < x_n = b$$

and choose any x_k^* in $[x_i, x_{i+1}]$. The average of $f(x_1^*)$, $f(x_2^*)$, $\ldots$, $f(x_n^*)$ is:

$$\frac{f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)}{n} = \frac{1}{n}\sum_{k=1}^{n} f(x_k^*).$$

Multiply this last expression by $1 = \dfrac{(b-a)}{(b-a)}$:

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} f(x_k^*)\frac{(b-a)}{(b-a)} &= \sum_{k=1}^{n} f(x_k^*)\frac{1}{n}\frac{(b-a)}{(b-a)}\\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\frac{b-a}{n}\\ &= \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x\end{aligned}$$

where $\Delta x = (b-a)/n$. Ah! On the right we have a Riemann Sum! Now take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

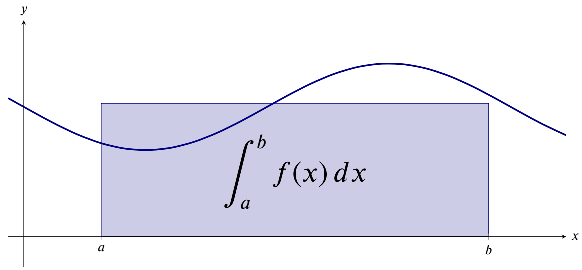
This leads us to our next definition:

Definition. Let f be continuous on $[a, b]$. The **average value** of f on $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

The average value of a function gives the height of a single rectangle whose area is equal to

$$\int_a^b f(x)\,dx$$



An application of this definition is given in the next example.

Example 217. *An object moves back and forth along a straight line with a velocity given by $v(t) = (t-1)^2$ on $[0, 3]$, where t is measured in seconds and $v(t)$ is measured in ft/s.*

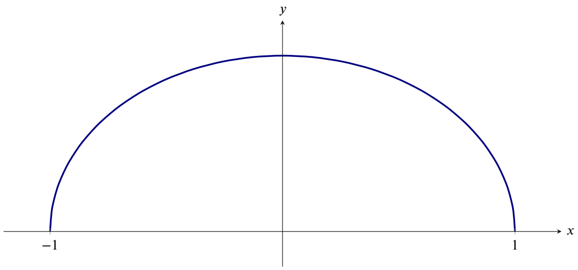
What is the average velocity of the object?

By our definition, the average velocity is:

$$\begin{aligned}\frac{1}{3-0}\int_0^3 (t-1)^2\,dt &= \frac{1}{3}\int_0^3 t^2 - 2t + 1\,dt\\ &= \frac{1}{3}\left[\frac{t^3}{3} - t^2 + t\right]_0^3\\ &= 1 \text{ ft/s}.\end{aligned}$$

When we take the average of a finite set of values, it does not matter how we order those values. When we are taking the average value of a function, however, we need to be more careful.

For instance, there are at least two different ways to make sense of a vague phrase like “The average height of a point on the unit semi circle”



One way we can make sense of "The average height of a point on the unit semi circle" is to compute the average value of the function

$$f(x) = \sqrt{1-x^2}$$

on the interval $[-1, 1]$.

Example 218. *Compute the average value of the function*

$$f(x) = \sqrt{1-x^2}$$

on the interval $[-1, 1]$.

By definition, we wish to compute

$$\frac{1}{2}\int_{-1}^{1} \sqrt{1-x^2}\,dx.$$

Computing this integral geometrically, we find the average value to be $\dfrac{\pi}{4}$.

Another way we can make sense of "The average height of a point on the unit semi circle" is the average value of the function

$$g(\theta) = \sin(\theta)$$

on $[0, \pi]$, since $\sin(\theta)$ is the height of the point on the unit circle at the angle θ.

Example 219. *Compute the average value of the function*

$$g(\theta) = \sin(\theta)$$

on the interval $[0, \pi]$.

By definition, we wish to compute

$$\frac{1}{\pi}\int_{0}^{\pi} \sin(\theta)\,d\theta.$$

Computing this integral geometrically, we find the average value to be $\dfrac{2}{\pi}$.

See if you can understand intuitively why the average using f should be larger than the average using g.

Mean value theorem for integrals

Just as we have a Mean Value Theorem for Derivatives, we also have a Mean Value Theorem for Integrals.

Theorem 66 (The Mean Value Theorem for integrals)**.** *Let f be continuous on $[a, b]$. There exists a value c in $[a, b]$ such that*

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

This is an *existential* statement. The Mean Value Theorem for Integrals tells us:

> **The average value of a continuous function is in the range of the function.**

We demonstrate the principles involved in this version of the Mean Value Theorem in the following example.

Example 220. *Consider* $\displaystyle\int_0^{\pi} \sin x\,dx$. *Find a value c guaranteed by the Mean Value Theorem.*

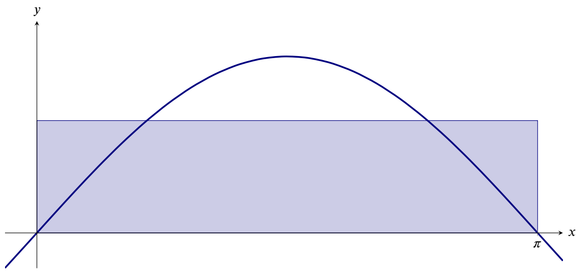
We first need to evaluate $\displaystyle\int_0^{\pi} \sin x\, dx$.

$$\int_0^{\pi} \sin x\, dx = \Big[-\cos x \Big]_0^{\pi} = 2.$$

Thus we seek a value c in $[0, \pi]$ such that $\pi \sin c = 2$.

$\pi \sin c = 2 \;\Rightarrow\; \sin c = 2/\pi \;\Rightarrow\; c = \arcsin(2/\pi) \approx 0.69.$

A graph of $\sin x$ is sketched along with a rectangle with height $\sin(0.69)$ are pictured below. The area of the rectangle is the same as the area under $\sin x$ on $[0, \pi]$.



Part VIII

Additional content for the Final Exam

44 The idea of substitution

After completing this section, students should be able to do the following.

- Undo the chain rule.
- Calculate indefinite integrals (antiderivatives) using basic substitution.
- Calculate definite integrals using basic substitution.

Break-Ground:

44.1 Geometry and substitution

Check out this dialogue between two calculus students (based on a true story):

Devyn: Riley! We should be able to figure some integrals geometrically using transformations of functions.

Riley: That sounds like a cool idea. Maybe, since we know the graph of $f(x) = \sqrt{1-x^2}$ is a semicircle, we get an ellipse defined on $[-2, 2]$ just by stretching the graph of f by a factor of 2 horizontally. The equation of this ellipse would be

$$g(x) = \sqrt{1 - \left(\frac{x}{2}\right)^2}$$

Devyn: Exactly! So since we know that

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}$$

geometrically…

Riley: And we know that the area under g from $[-2, 2]$ is twice the under f…

Devyn and Riley: We must have

$$\int_{-2}^{2} \sqrt{1-(x/2)^2}\, dx = \pi!$$

Devyn and Riley: Jinx!

Devyn: It is kind of like we just stretched out our whole coordinate system, and that helped us solve an integral.

Riley: In this case, everything got stretched out by a constant factor of 2 in the horizontal direction. I wonder if we could ever say anything useful about cases where we stretch the x-axis by a different amount at each point?

Devyn: Whao, that is a wild thought. That seems really hard. Since derivatives measure how much a function stretches a little piece of the domain, maybe the derivative will come into play here?

Riley: Hmmmm, but I do not see exactly how. Maybe we should ask our TA about this?

Problem 1. *Say we know that*

$$\int_{1}^{4} f(x)\, dx = 5.$$

Then, using this transformation idea, we can evaluate

$$\int_{a}^{b} f(3x+1)\, dx$$

if $a = \boxed{?}$ *and* $b = \boxed{?}$. *The value of the integral on this interval is* $\boxed{?}$.

Dig-In:

44.2 The idea of substitution

Computing antiderivatives is not as easy as computing derivatives. One issue is that the chain rule can be difficult to "undo." We have a general method called "integration by substitution" that will somewhat help with this difficulty. The idea is this, we know from the chain rule that

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

so if we conisder

$$\begin{aligned}\int_a^b f'(g(x))g'(x)\,dx &= \Big[f(g(x))\Big]_a^b \\ &= f(g(b)) - f(g(a)) \\ &= \Big[f(g)\Big]_{g(a)}^{g(b)} \\ &= \int_{g(a)}^{g(b)} f'(g)\,dg.\end{aligned}$$

This "transformation" is worth stating explicity:

Theorem 67 (Integral Substitution Formula)**.** *If g is differentiable on the interval $[a,b]$ and f is differentiable on the interval $[g(a), g(b)]$, then*

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(g)\,dg.$$

Three similar techinques

There are several different ways to think about substitution. The first is directly using the formula

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(g)\,dg.$$

Example 221. *Compute:*

$$\int_1^3 x\cos(x^2)\,dx$$

A little thought reveals that if $x\cos(x^2)$ is the derivative of some function, then it must have come from an application of the chain rule.

$$\int \underbrace{x}_{\text{derivative of inside}} \cos(\underbrace{x^2}_{\text{inside}})\,dx.$$

Set $g(x) = x^2$, so $g'(x) = 2x$, and now it must be that $f'(g) = \frac{\cos(g)}{2}$. Now we see

$$\begin{aligned}\int_1^3 x\cos(x^2)\,dx &= \int_1^9 \frac{\cos(g)}{2}\,dg \\ &= \left[\frac{\sin(g)}{2}\right]_1^9 \\ &= \frac{\sin(9)-\sin(1)}{2}.\end{aligned}$$

Notice the change of endpoints in the first equality! We obtained the new integrands by the computations

$$\begin{aligned}g(1) &= 1^2 = 1 \\ g(3) &= 3^2 = 9.\end{aligned}$$

We will usually solve these problems in a slightly different way. Let's do the same example again, this time we will think in terms of differentials.

Example 222. *Compute:*

$$\int_1^3 x\cos(x^2)\,dx$$

Here we will set $g(x) = x^2$. Then $dg = 2x\,dx$, where we are thinking in terms of differentials. So we can solve for dx to get $dx = \frac{dg}{2x}$. We then see that

$$\int_1^3 x\cos(x^2)\,dx = \int_{g(1)}^{g(3)} x\cos(g)\frac{dg}{2x}$$
$$= \int_1^9 \frac{\cos(g)}{2}\,dg.$$

At this point, we can continue as we did before and write

$$\int_1^3 x\cos(x^2)\,dx = \frac{\sin(9)-\sin(1)}{2}.$$

Finally, sometimes we simply want to deal with the antiderivative on its own, we'll repeat the example one more time demonstrating this.

Example 223. *Compute:*

$$\int_1^3 x\cos(x^2)\,dx$$

Here we start as we did before, setting $g(x) = x^2$. Now $dg = 2x\,dx$, again thinking in terms of differentials. Now we see that

$$\int x\cos(x^2)\,dx = \int x\cos(g)\frac{dg}{2x} = \int \frac{\cos(g)}{2}\,dg.$$

Hence

$$\int x\cos(x^2)\,dx = \frac{\sin(g)}{2} = \frac{\sin(x^2)}{2}.$$

So

$$\int_1^3 x\cos(x^2)\,dx = \left[\frac{\sin(x^2)}{2}\right]_1^3$$
$$= \frac{\sin(9)-\sin(1)}{2}.$$

More examples

With some experience, it is (usually) not too hard to see what to substitute as g. We will work through the following examples in the same way that we did for Example 222. Let's see another example.

Example 224. *Compute:*

$$\int x^4(x^5+1)^9\,dx$$

Here we set $g(x) = x^5+1$, so $dg = 5x^4\,dx$. Then

$$\int x^4(x^5+1)^9\,dx = \frac{1}{5}\int 5x^4(x^5+1)^9\,dx$$
$$= \frac{1}{5}\int g^9\,dg$$
$$= \frac{g^{10}}{50}.$$

Notice that this example is an indefinite integral and not a definite integral, meaning that there are no limits of integration. So we do not need to worry about changing the endpoints of the integral. However, we do need to back-substitute into our answer, so that our final answer is a function of x. Recalling that $g(x) = x^5+1$, we have our final answer

$$\int x^4(x^5+1)^9\,dx = \frac{(x^5+1)^{10}}{50} + C.$$

If substitution works to solve an integral (and that is not always the case!), a common trick to find what to substitute for is to locate the "ugly" part of the function being integrated. We then substitute for the "inside" of this ugly part. While this technique is certainly not rigorous, it can prove to be very helpful. This is especially true for students new to the technique of substitution. The next two problems are really good examples of this philosophy.

Example 225. *Compute:*

$$\int_{-1}^{0} 12x^3 e^{x^4}\, dx$$

The "ugly" part of the function being integrated is e^{x^4}. The "inside" of this term is then x^4. So a good possibility is to try

$$g(x) = x^4.$$

Then

$$dg = 4x^3\, dx \qquad \Rightarrow \qquad dx = \frac{1}{4x^3}\, dg$$

and so

$$\begin{aligned}\int_{-1}^{0} 12x^3 e^{x^4}\, dx &= \int_{g(-1)}^{g(0)} 12x^3 e^{g} \frac{1}{4x^3}\, dg \\ &= \int_{1}^{0} 3e^{g}\, dg \\ &= \Big[3e^{g}\Big]_{1}^{0} \\ &= 3(1-e).\end{aligned}$$

Example 226. *Compute:*

$$\int_{1}^{e^{\frac{\pi}{4}}} \frac{\cos(\ln x)}{x}\, dx$$

Here the "ugly" part here is $\cos(\ln x)$. So we substitute for the inside:

$$g(x) = \ln x.$$

Then

$$dg = \frac{1}{x}\, dx \qquad \Rightarrow \qquad dx = x\, dg.$$

Notice that

$$\begin{aligned}g(1) &= \ln(1) = 0 \\ g\left(e^{\frac{\pi}{4}}\right) &= \ln\left(e^{\frac{\pi}{4}}\right) = \frac{\pi}{4}.\end{aligned}$$

Then we substitute back into the original integral and solve:

$$\begin{aligned}\int_{1}^{e^{\frac{\pi}{4}}} \frac{\cos(\ln x)}{x}\, dx &= \int_{0}^{\frac{\pi}{4}} \frac{\cos(g)}{x} x\, dg \\ &= \int_{0}^{\frac{\pi}{4}} \cos(g)\, dg \\ &= \Big[\sin(g)\Big]_{0}^{\frac{\pi}{4}} \\ &= \frac{\sqrt{2}}{2} - 0 = \frac{\sqrt{2}}{2}.\end{aligned}$$

To summarize, if we suspect that a given function is the derivative of another via the chain rule, we let g denote a likely candidate for the inner function, then translate the given function so that it is written entirely in terms of g, with no x remaining in the expression. If we can integrate this new function of g, then the antiderivative of the original function is obtained by replacing g by the equivalent expression in x.

45 Working with substitution

After completing this section, students should be able to do the following.

- Determine when a function is a composition of two or more functions.
- Calculate indefinite and definite integrals requiring complicated substitutions.
- Recognize common patterns in substitutions.
- Evaluate indefinite and definite integrals through a change of variables.

Break-Ground:

45.1 Integrals are puzzles!

Check out this dialogue between two calculus students (based on a true story):

Devyn: Yo Riley, is it just me, or are integrals kind of fun?
Riley: I always feel accomplished when I finish one.
Devyn: I know! Also, even though antiderivatives are difficult, we can always check our work by taking the derivative.
Riley: So awesome!
Devyn: But something is bothering me. When we are doing substitution, we have to find f and g such that

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(g)\,dg.$$

How do we choose f and g?
Riley: Well, never ever pick $g(x) = x$, this doesn't change anything!
Devyn: And never ever pick $g(x)$ to be the entire integrand, this doesn't help either.
Riley: Somehow we must "see" one function "nested" inside of another.
Devyn: I'm not sure there's an easy path to doing, this, I think it's gonna take practice.

In the problems that follow, we will be using the substitution formula

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(g)\,dg$$

While you may use a slightly different method to compute your integrals, the skills developed by answering the problems below will help you in your quest to conquer calculus.

Problem 1. *Consider*

$$\int \sin^5(3x)\cos(3x)\,dx = \int f(g(x)) \cdot g'(x)\,dx$$

if $g(x) = 3x$*, and*

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(g)\,dg.$$

what is $f(g)$*?*

Problem 2. *Consider*

$$\int \sin^5(3x)\cos(3x)\,dx = \int f(g(x)) \cdot g'(x)\,dx$$

if $f(g) = \frac{g^5}{3}$*, and*

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(g)\,dg.$$

what is $g(x)$*?*

Problem 3. *In your own words, explain why Devyn and Riley claim we should never pick* $g(x) = x$ *or* $g(x)$ *to be the entire integrand.*

Dig-In:

45.2 Working with substitution

We begin by restating the substitution formula.

Theorem 68 (Integral Substitution Formula)**.** *If g is differentiable on the interval $[a,b]$ and f is differentiable on the interval $[g(a),g(b)]$, then*

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(g)\,dg.$$

We spend pretty much this entire section working out examples.

Example 227. *Compute:*

$$\int_2^3 \frac{1}{x\ln(x)}\,dx$$

Let

$$g = \ln(x),$$

computing dg, we find

$$dg = \frac{1}{x}\,dx$$

and solving for dx we find

$$dx = x\,dg.$$

Now

$$\begin{aligned}\int_2^3 \frac{1}{x\ln(x)}\,dx &= \int_{g(2)}^{g(3)} \frac{1}{x\cdot g} x\,dg\\ &= \int_{\ln(2)}^{\ln(3)} \frac{1}{g}\,dg\\ &= \Big[\ln(g)\Big]_{\ln(2)}^{\ln(3)}\\ &= \ln(\ln(3)) - \ln(\ln(2)).\end{aligned}$$

The next example requires a new technique.

Example 228. *Compute:*

$$\int x^3\sqrt{1-x^2}\,dx$$

Here it is not apparent that the chain rule is involved. However, if it was involved, perhaps a good guess for g would be

$$g = 1-x^2$$

and then

$$\begin{aligned}dg &= -2x\,dx,\\ dx &= \frac{-1}{2x}\,dg.\end{aligned}$$

Now we consider the integral we are trying to compute

$$\int x^3\sqrt{1-x^2}\,dx$$

and we substitute using our work above. Write with me

$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx &= \int x^3\sqrt{g}\left(\frac{-1}{2x}\right)dg\\ &= \int \frac{-x^2\sqrt{g}}{2}\,dg.\end{aligned}$$

However, we cannot continue until each x is replaced. We know that

$$\begin{aligned} & g = 1 - x^2 \\ \Rightarrow \qquad & g - 1 = -x^2 \\ \Rightarrow \qquad & 1 - g = x^2 \end{aligned}$$

so now we may replace x^2

$$\int x^3\sqrt{1-x^2}\,dx = \int -\frac{(1-g)\sqrt{g}}{2}\,dg.$$

At this point, we are close to being done. Write

$$\begin{aligned} \int -\frac{(1-g)\sqrt{g}}{2}\,dg &= \int \left(\frac{g\sqrt{g}}{2} - \frac{\sqrt{g}}{2}\right) dg \\ &= \int \frac{g^{3/2}}{2}\,dg - \int \frac{\sqrt{g}}{2}\,dg \\ &= \frac{g^{5/2}}{5} - \frac{g^{3/2}}{3}. \end{aligned}$$

Now recall that $g = 1 - x^2$. Hence our final answer is

$$\int x^3\sqrt{1-x^2}\,dx = \frac{(1-x^2)^{5/2}}{5} - \frac{(1-x^2)^{3/2}}{3} + C.$$

Sometimes it is not obvious how a fraction could have been obtained using the chain rule. A common trick though is to substitute for the *denominator* of a fraction. Like all tricks, this technique does not always work. Regardless the next two examples present how this technique can be used.

Example 229. *Compute:*

$$\int \frac{\sec(y)\tan(y) + \sec^2(y)}{\sec(y) + \tan(y)}\,dy$$

We substitute

$$g = \sec(y) + \tan(y)$$

and we immediately see that

$$\begin{aligned} dg &= (\sec(y)\tan(y) + \sec^2(y))\,dy, \\ dy &= \frac{1}{\sec(y)\tan(y) + \sec^2(y)}\,dg. \end{aligned}$$

But this cancels perfectly with the numerator! So we have that

$$\begin{aligned} \int \frac{\sec(y)\tan(y) + \sec^2(y)}{\sec(y) + \tan(y)}\,dy &= \int \frac{1}{g}\,dg \\ &= \ln|g| + C \\ &= \ln|\sec(y) + \tan(y)| + C. \end{aligned}$$

Notice that

$$\frac{\sec(x)\tan(x) + \sec^2(x)}{\sec(x) + \tan(x)} = \frac{\sec(x)(\tan(x) + \sec(x))}{\sec(x) + \tan(x)} = \sec(x)$$

when $\sec(x) \neq -\tan(x)$. So in a very contrived way, we have just proved

Theorem 69.

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C.$$

Notice the variable in this next example.

Example 230. *Compute:*

$$\int \frac{g}{1-g^2}\,dg$$

We want to substitute for $1-g^2$. But the variable "g" has already been used...OH NO! Never fear! We can substitute with whatever variable that we want. In particular, let us use "h" for this problem. So we let

$$h = 1 - g^2$$

and then

$$\begin{aligned} dh &= -2g\,dg, \\ dg &= \frac{-1}{2g}\,dh. \end{aligned}$$

Thus

$$\begin{aligned} \int \frac{g}{1-g^2}\,dg &= \int \frac{g}{h}\left(\frac{-1}{2g}\right)dh \\ &= -\frac{1}{2}\int \frac{1}{h}\,dh \\ &= -\frac{1}{2}\ln|h| + C \\ &= -\frac{1}{2}\ln|1-g^2| + C. \end{aligned}$$

Example 231. *Compute:*

$$\int \tan(x)\,dx$$

We begin by writing

$$\begin{aligned} \tan(x) &= \frac{\sin(x)}{\cos(x)} \\ &= \frac{\sin(x)}{\cos(x)}\cdot\frac{\cos(x)}{\cos(x)} \\ &= \frac{\sin(x)\cos(x)}{1-\sin^2(x)}. \end{aligned}$$

We then make the substitution

$$g = \sin(x)$$

and so

$$\begin{aligned} dg &= \cos(x)\,dx, \\ dx &= \frac{1}{\cos(x)}\,dg. \end{aligned}$$

Then

$$\begin{aligned} \int \tan(x)\,dx &= \int \frac{\sin(x)\cos(x)}{1-\sin^2(x)}\,dx \\ &= \int \frac{g\cos(x)}{1-g^2}\cdot\frac{1}{\cos(x)}\,dg \\ &= \int \frac{g}{1-g^2}\,dg. \end{aligned}$$

But this is the same problem as Example 230! And so we know that

$$\begin{aligned} \int \tan(x)\,dx &= -\frac{1}{2}\ln|1-g^2| + C \\ &= -\frac{1}{2}\ln|1-\sin^2(x)| + C \\ &= -\frac{1}{2}\ln|\cos^2(x)| + C \\ &= \ln|\cos^2(x)|^{-\frac{1}{2}} + C \\ &= \ln|\sec(x)| + C. \end{aligned}$$

We have just proved

Theorem 70.

$$\int \tan(x)\,dx = \ln|\sec(x)| + C.$$

Note that in Example 231, we could have instead made the substitution

$$g = 1 - \sin^2(x).$$

This would have gotten us to the answer quicker and without using Example 230. You are encouraged to work this out on your own right now!

We end this section with two more difficult examples.

Example 232. *Compute:*

$$\int \frac{e^{2x}}{1-e^{2x}}\,dx$$

Maybe the biggest key to solving this problem is to recall that

$$e^{2x} = (e^x)^2.$$

So we can rewrite the problem

$$\int \frac{e^{2x}}{1-e^{2x}}\,dx = \int \frac{(e^x)^2}{1-(e^x)^2}\,dx.$$

Now, if we make the substitution $g = e^x$, we have that

$$\begin{aligned} dg &= e^x\,dx, \\ dx &= \frac{1}{e^x}\,dg, \end{aligned}$$

and

$$\begin{aligned} \int \frac{e^{2x}}{1-e^{2x}}\,dx &= \int \frac{(e^x)^2}{1-g^2}\cdot\frac{1}{e^x}\,dg \\ &= \int \frac{e^x}{1-g^2}\,dg \\ &= \int \frac{g}{1-g^2}\,dg. \end{aligned}$$

But now we are back to Example 230, and so we know that

$$\begin{aligned} \int \frac{e^{2x}}{1-e^{2x}}\,dx &= -\frac{1}{2}\ln|1-g^2| + C \\ &= -\frac{1}{2}\ln|1-e^{2x}| + C. \end{aligned}$$

Again, in the previous example we could have instead made the substitution

$$g = 1-e^{2x}$$

and avoided using Example 230. In general, any time that you make two successive substitutions in a problem, you could have instead just made one substitution. This one substitution is the *composition* of the two original substitutions. But sometimes it may not be obvious to make one clever substitution, and so two substitutions makes more sense. The next example helps to demonstrate this.

Example 233. *Compute:*

$$\int_0^{16} \sqrt{4-\sqrt{x}}\,dx$$

While it is not obvious at all, let us try the substitution

$$g = \sqrt{x}.$$

Then

$$\begin{aligned} dg &= \frac{1}{2\sqrt{x}}\,dx, \\ dx &= 2\sqrt{x}\,dg = 2g\,dg, \end{aligned}$$

and so

$$\begin{aligned} \int_0^{16} \sqrt{4-\sqrt{x}}\,dx &= \int_{g(0)}^{g(16)} \sqrt{4-g}\cdot 2g\,dg \\ &= \int_0^4 2g\sqrt{4-g}\,dg. \end{aligned}$$

From here we now make the second (and more obvious) substitution

$$h = 4 - g.$$

Then $g = 4 - h$, and

$$\begin{aligned} dh &= -\,dg, \\ dg &= -\,dh. \end{aligned}$$

So

$$\begin{aligned}
\int_0^{16} \sqrt{4 - \sqrt{x}}\, dx &= \int_0^4 2g\sqrt{4 - g}\, dg \\
&= \int_{h(0)}^{h(4)} 2(4 - h)\sqrt{h}(-1)\, dh \\
&= -\int_4^0 (8h^{\frac{1}{2}} - 2h^{\frac{3}{2}})\, dh \\
&= \int_0^4 (8h^{\frac{1}{2}} - 2h^{\frac{3}{2}})\, dh \\
&= \left[8\left(\frac{2}{3}\right)h^{\frac{3}{2}} - 2\left(\frac{2}{5}\right)h^{\frac{5}{2}}\right]_0^4 \\
&= \left(\frac{16}{3}(4)^{\frac{3}{2}} - \frac{4}{5}(4)^{\frac{5}{2}}\right) - (0 - 0) \\
&= \frac{128}{3} - \frac{128}{5} \\
&= \frac{256}{15}.
\end{aligned}$$

Dig-In:

45.3 The Work-Energy Theorem

In physics, we take measurable quantities from the real world, and attempt to find meaningful relationships between them. A basic example of this would be the physical ideal of **force**. Force applied to an object changes the motion of an object. Here's the deal though, at a basic level

$$\text{force} = \text{mass} \cdot \text{acceleration}.$$

and while we can put a physical interpretation to this arithmetical definition, at the end of the day force is simply "mass times acceleration." The SI unit of force is a **newton**, which is defined to be

$$1\,\mathrm{N} = 1\,\mathrm{kg} \cdot \mathrm{m}/\,\mathrm{s}^2.$$

Question 112. *To get a feel for what a newton is, consider this: if an apple has a mass of* $0.1\,\mathrm{kg}$*, what force would an apple exert on your hand due to the acceleration due to gravity?*

In a similar way, the idea of **kinetic energy**, is "energy" objects have from motion. It is defined by the formula

$$E_k = \frac{m \cdot v^2}{2}.$$

The SI unit of energy is a **joule**, which is defined to be

$$1\,\mathrm{J} = 1\,\mathrm{kg} \cdot \mathrm{m}^2/\,\mathrm{s}^2 = 1\,\mathrm{N} \cdot \mathrm{m}.$$

To get a feel for the "size" of a joule, consider this: if an apple has a mass of $0.1\,\mathrm{kg}$ and it is dropped from a height of $1\,\mathrm{m}$, then approximately 1 joule of energy is released when it hits the ground. Let's see if we can explain why this is true.

Example 234. *If an apple has a mass of* $0.1\,\mathrm{kg}$ *and it is dropped from a height of* $1\,\mathrm{m}$*, how much energy is released when it hits the ground? Assume that the acceleration due to gravity is* $-9.8\,\mathrm{m}/\,\mathrm{s}^2$.

First we need to find the velocity at which the apple hits the ground. Let $a(t) = -9.8$ represent acceleration at time t and $v(t)$ represent velocity. Since $v(0) = 0$, we know that

$$\begin{aligned} v(t) &= \int_0^t a(x)\,dx \\ &= \int_0^t -9.8\,dx \\ &= -9.8t. \end{aligned}$$

Now we need to know how long it takes for the apple to hit the ground, after being dropped from a height of 1 meter. For this we'll need a formula for position. Here $s(0) = 1$, so we'll need to use an indefinite integral:

$$\begin{aligned} s(t) &= \int v(t)\,dt \\ &= \int -9.8t\,dt \\ &= \frac{-9.8t^2}{2} + C \end{aligned}$$

Since $s(0) = 1$, write with me

$$s(0) = \frac{-9.8 \cdot 0}{2} + C = 1,$$

hence $C = 1$ and $s(t) = \dfrac{-9.8t^2}{2} + 1$. Solving the equation

$$s(t) = 0$$

for t tells us the time the apple hits the ground. Write with me

$$\begin{aligned}
s(t) &= 0 \\
\frac{-9.8t^2}{2} + 1 &= 0 \\
-9.8t^2 &= -2 \\
t^2 &= \frac{2}{9.8} \\
t &= \sqrt{\frac{2}{9.8}}.
\end{aligned}$$

So the apple hits the ground after $\sqrt{\frac{2}{9.8}}$ seconds. Finally, the formula for kinetic energy is

$$\begin{aligned}
E_k &= \frac{m \cdot v^2}{2} \\
&= \frac{0.1 \cdot (a \cdot t)^2}{2} \\
&= \frac{0.1 \cdot \left((-9.8) \cdot \sqrt{\frac{2}{9.8}}\right)^2}{2} \\
&= \frac{0.1 \cdot (9.8)^2 \cdot \frac{2}{9.8}}{2} \\
&= \frac{0.1 \cdot 9.8 \cdot 2}{2} \\
&= 0.98.
\end{aligned}$$

Ah! So the kinetic energy released by an apple dropped from a height of 1 meter is approximately 1 joule.

Finally **work** is defined to be accumulated force over a distance. Note, there must be some force *in the direction* (or opposite direction) that the object is moving for it to be considered *work*.

Question 113. *Which of the following are examples where work of this kind is being done?*

Select All Correct Answers:

(a) *studying calculus*

(b) *a car applying breaks to come to a stop over a distance of* 100 ft

(c) *a young mathematician climbing a mountain*

(d) *a young mathematician standing still, holding a* 1000 *page calculus book for* 10 *minutes*

(e) *a young mathematician walking around with a* 1000 *page calculus book*

(f) *a young mathematician picking up a* 1000 *page calculus book*

We can write the definition of work in the language of calculus as,

$$W = \int_{s_0}^{s_1} F(s)\,ds.$$

The SI unit of work is also a **joule**. To help understand this, 1 joule is approximately how much work is done when you raise an apple one meter.

Let's again see why this is true.

Example 235. *If an apple has a mass of* 0.1 kg, *how much work is required to lift this apple* 1 *meter? Assume that the acceleration due to gravity is* $-9.8\,\mathrm{m/s^2}$.

Well, work is computed by

$$W = \int_{s_0}^{s_1} F(s)\,ds.$$

Since force is mass times acceleration,

$$\begin{aligned}
F(s) &= 0.1 \cdot (-9.8) \\
&= -0.98.
\end{aligned}$$

So, our integral becomes

$$\int_0^1 -0.98\,ds = \Big[-0.98s\Big]_0^1 = -0.98.$$

Ah! So when lifting an apple 1 meter, requires -0.98 joules of work. The sign is negative since we are lifting **against** the gravitational force.

Now we have a question:

> **Why do work and kinetic energy have the same units?**

One way to answer this is via the *Work-Energy Theorem*.

Theorem 71 (Work-Energy Theorem)**.** *Suppose that an object of mass m is moving along a straight line. If s_0 and s_1 are the the starting and ending positions, v_0 and v_1 are the the starting and ending velocities, and $F(s)$ is the force acting on the object for any given position, then*

$$W = \int_{s_0}^{s_1} F(s)\,ds = \frac{m \cdot v_1^2}{2} - \frac{m \cdot v_0^2}{2}.$$

First we need to get all of our symbolism out in the open. Let:

- $s(t)$ represent position with respect to time,
- $v(t)$ represent velocity with respect to time,
- $a(s)$ represent acceleration with respect to position,
- t_0 represent the starting time,
- t_1 represent the ending time,

then we also have that

- $s(t_0)$ represents the starting position, s_0,
- $s(t_1)$ represents the ending position, s_1,
- $v(t_0)$ represents the starting velocity, v_0,
- $v(t_1)$ represents the ending velocity, v_1.

Now write with me,

$$W = \int_{s_0}^{s_1} F(s)\,ds = \int_{s(t_0)}^{s(t_1)} F(s)\,ds$$

here we are working with functions of distance. We will use the substitution formula,

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(g)\,dg$$

transforming from right to left, to see that

$$\int_{s(t_0)}^{s(t_1)} F(s)\,ds = \int_{t_0}^{t_1} F(s(t))s'(t)\,dt$$

and we are now working with functions of time. Since $s'(t) = v(t)$, we may write

$$\int_{t_0}^{t_1} F(s(t))s'(t)\,dt = \int_{t_0}^{t_1} F(s(t))v(t)\,dt$$

and now remember that $F = m \cdot a$, so

$$\int_{t_0}^{t_1} F(s(t))v(t)\,dt = \int_{t_0}^{t_1} m \cdot a(s(t))v(t)\,dt.$$

However, $a(s(t)) = v'(t)$, so rearranging we have,

$$\int_{t_0}^{t_1} m \cdot a(s(t))v(t)\,dt = m \cdot \int_{t_0}^{t_1} v(t)v'(t)\,dt.$$

Now we apply the substitution formula again, this time we will transform left to right

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(g)\,dg$$

and so we see

$$m \cdot \int_{t_0}^{t_1} v(t)v'(t)\,dt = m \cdot \int_{v(t_0)}^{v(t_1)} v\,dv$$

and we are working with functions of velocity. At last, setting $v(t_0) = v_0$ and $v(t_1) = v_1$, we can evaluate this integral,

$$\begin{aligned} m \cdot \int_{v_0}^{v_1} v\,dv &= m \cdot \left[\frac{v^2}{2}\right]_{v_0}^{v_1} \\ &= \frac{m \cdot v_1^2}{2} - \frac{m \cdot v_0^2}{2}. \end{aligned}$$

The Work-Energy theorem says that:

$$\int_{s_0}^{s_1} F(s)\,ds = \frac{m \cdot v_1^2}{2} - \frac{m \cdot v_0^2}{2}$$

This could be interpreted as:

The accumulated force over distance is the change in kinetic energy.

Moreover, this answers our initial question of why work and kinetic energy have the same units. In essence, energy powers work.

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