Multidimensional finite volume methods

A comprehensive overview

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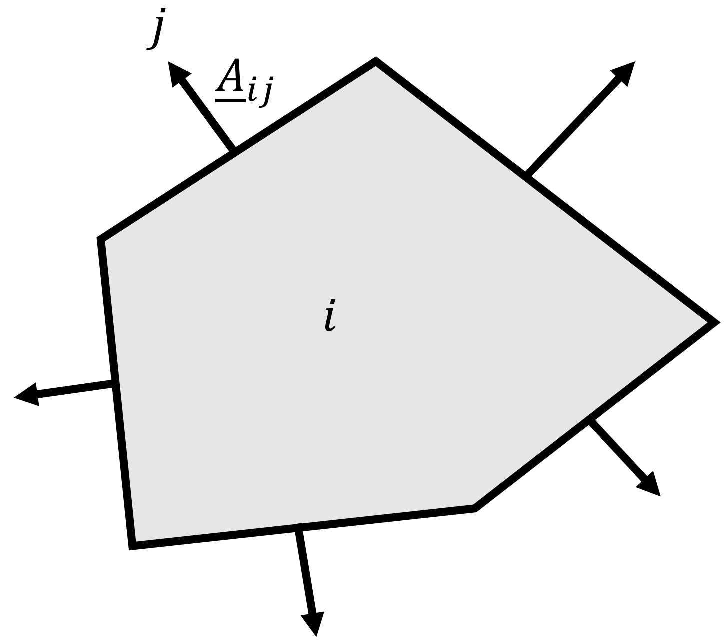
We want to solve the discretized finite volume equation:



$$\Delta Q_i = -\left(\sum_j \underline{F}_{ij} . \underline{A}_{ij}\right)\Delta t$$

With the flux vector given by

$$\underline{F}(U) = \begin{pmatrix} \rho \underline{u} \\ \rho \underline{u}\,\underline{u} + p\underline{\underline{1}} \\ \rho e_{tot}\underline{u} + p\underline{u} \end{pmatrix}$$

Remember:

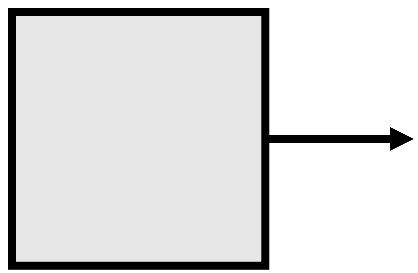
$$Q_i = \begin{pmatrix} m_i \\ m_i \underline{u_i} \\ E_{tot,i} \end{pmatrix} = V_i \begin{pmatrix} \rho_i \\ \rho_i \underline{u_i} \\ \rho_i e_{tot,i} \end{pmatrix} = V_i U_i$$

$$p_i = (\gamma - 1)\rho_i \left(e_{tot,i} - \frac{1}{2}\left|\underline{u_i}\right|^2 \right)$$

Problem: we only have a 1D Riemann solver...

$\Longrightarrow$ The split multidimensional Riemann problem

Consider a face parallel to the y-axis in 2D:



In this case, the 2D conservation law

$$\frac{\partial U}{\partial t} + \underline{\nabla}.\,\underline{F}(U) = \frac{\partial U}{\partial t} + \frac{\partial F_x(U)}{\partial x} + \frac{\partial F_y(U)}{\partial y} = 0$$

reduces to

$$\frac{\partial U}{\partial t} + \frac{\partial F_x(U)}{\partial x} = 0 \qquad \text{with} \qquad F_x(U) = \begin{pmatrix} \rho u_x \\ \rho u_x^2 + p \\ \rho u_x u_y \\ \rho e u_x + p u_x \end{pmatrix}$$

$$\frac{\partial U}{\partial t} + \frac{\partial F_x(U)}{\partial x} = 0 \qquad \text{with} \qquad F_x(U) = \begin{pmatrix} \rho u_x \\ \rho u_x^2 + p \\ \rho u_x u_y \\ \rho e u_x + p u_x \end{pmatrix}$$

These are the same equations as the 1D conservation law, with an extra equation:

$$\frac{\partial \rho u_y}{\partial t} + \frac{\partial \rho u_x u_y}{\partial x} = \rho \frac{\partial u_y}{\partial t} + u_y \frac{\partial \rho}{\partial t} + \rho u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial \rho u_x}{\partial x} = 0$$

or, using the continuity equation:

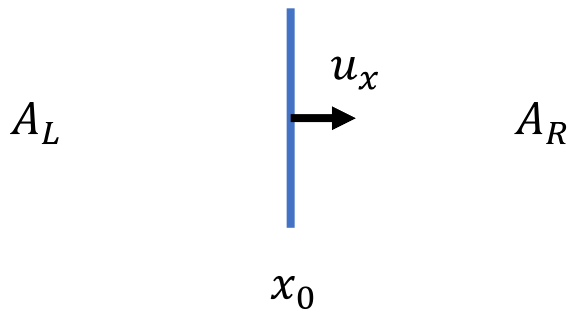
$$\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} = 0$$

This is an *advection equation*, which means that u_y is simply advected with the flow in the x direction

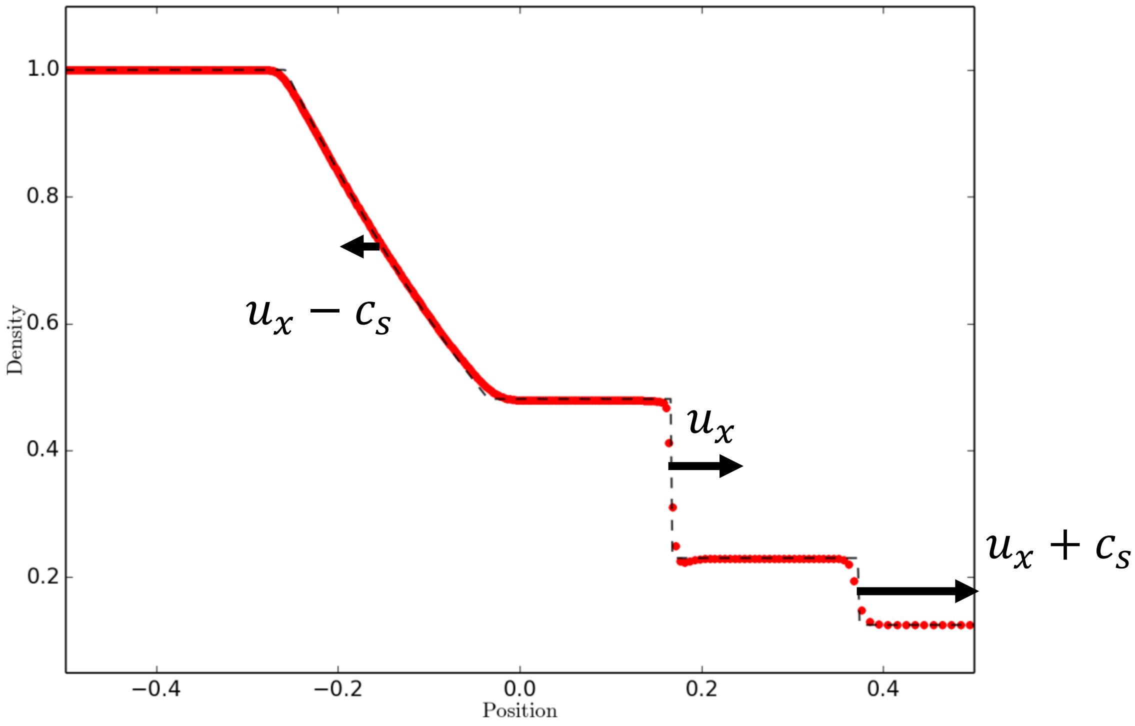
For a general advection equation

$$\frac{\partial A}{\partial t} + u_x \frac{\partial A}{\partial x} = 0$$

the Riemann problem has a trivial solution:



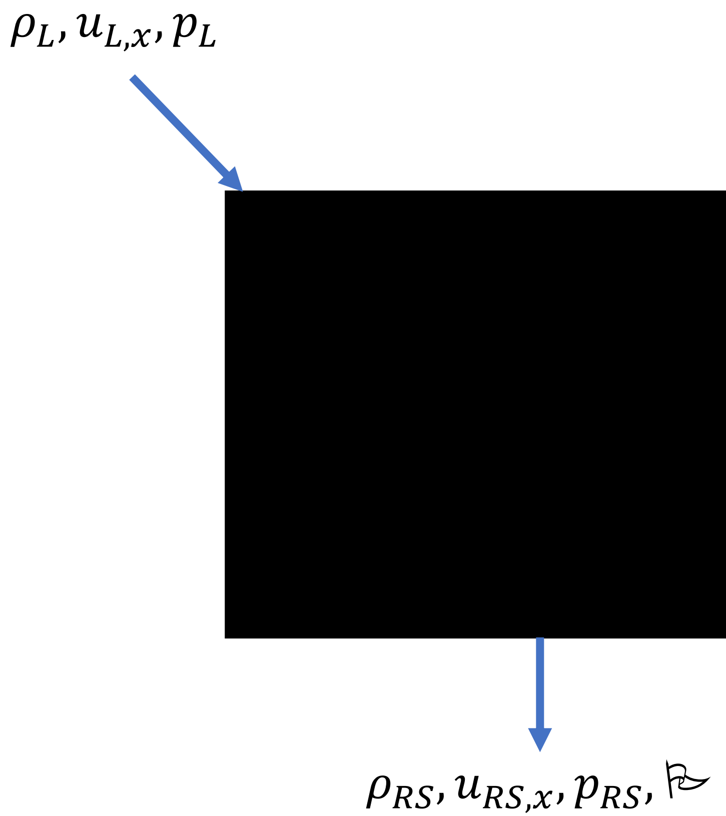
$$A(x,0) = \begin{cases} A_L, & x < x_0 \\ A_R, & x > x_0 \end{cases} \longrightarrow A(x,t) = \begin{cases} A_L, & x + u_x t < x_0 \\ A_R, & x + u_x t > x_0 \end{cases}$$



The advected quantities move at the same speed as the middle wave (contact discontinuity) in the Riemann problem solution

This means that we just need to find out on which side of the contact discontinuity we sample the Riemann solution

The Riemann solver

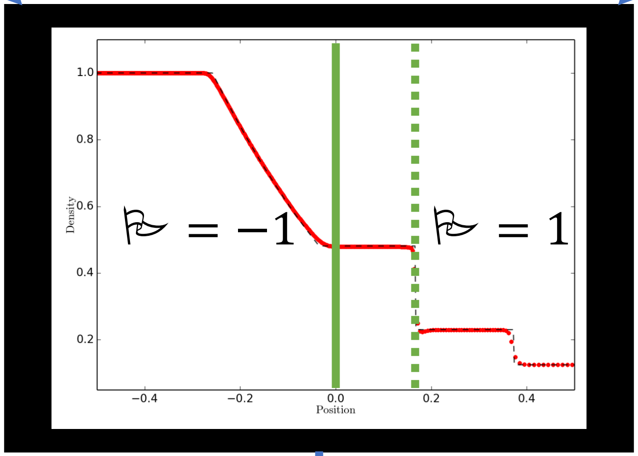


The Riemann solver gives us 3 primitive variables, we need 4...

The Riemann solver

$\rho_L, u_{L,x}, p_L$

$\rho_R, u_{R,x}, p_R$

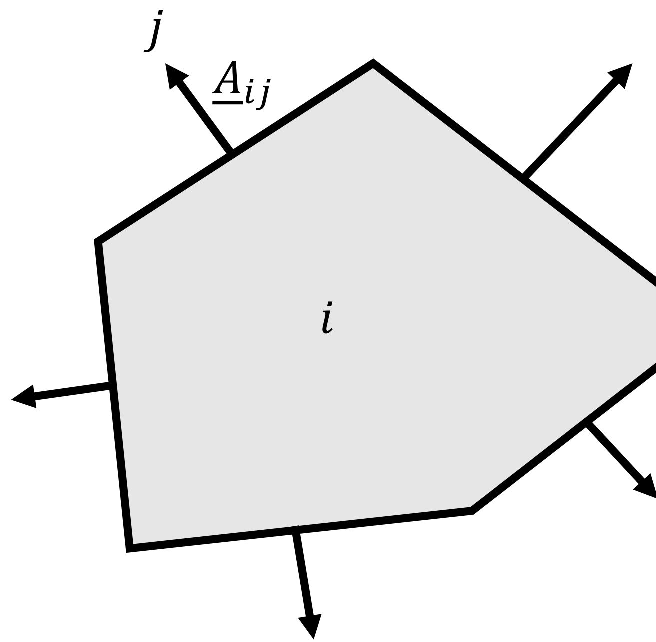


$\rho_{RS}, u_{RS,x}, p_{RS},$ ⚐

An extra flag variable tells us which side of the contact discontinuity we sample:

$$\rho_S, u_{S,x}, u_{S,y}, p_S = \begin{cases} \rho_{RS}, u_{RS,x}, u_{L,y}, p_{RS}, & \text{⚐} = -1 \\ \rho_{RS}, u_{RS,x}, u_{R,y}, p_{RS}, & \text{⚐} = +1 \end{cases}$$

For a general cell, the faces do not align with the coordinate axes



However, we can always rotate to a new reference frame where they do:

$$\rho' = \rho$$

$$u'_x = \underline{u}.\underline{n}$$

$$u'_y = \underline{u}.\underline{m}$$

$$p' = p$$

$$\underline{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix} = \frac{\underline{A}_{ij}}{\left|\underline{A}_{ij}\right|} \qquad \underline{m} = \begin{pmatrix} -n_y \\ n_x \end{pmatrix}$$

$$
\begin{aligned}
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u'_x &= \underline{u}.\underline{n} \\
u'_y &= \underline{u}.\underline{m} \\
p' &= p
\end{aligned}
\qquad
\underline{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix} = \frac{\underline{A}_{ij}}{\left|\underline{A}_{ij}\right|}
\qquad
\underline{m} = \begin{pmatrix} -n_y \\ n_x \end{pmatrix}
$$

We then solve the Riemann problem in the rotated reference frame:

$$
\rho'_S, u'_{S,x}, u'_{S,y}, p'_S = \begin{cases} \rho_{RS}, u_{RS,x}, u'_{L,y}, p_{RS}, & \text{⚐} = -1 \\ \rho_{RS}, u_{RS,x}, u'_{R,y}, p_{RS}, & \text{⚐} = +1 \end{cases}
$$

and rotate back:

$$
\begin{aligned}
\rho_S &= \rho'_S \\
u_{S,x} &= u'_{S,x} n_x + u'_{S,y} m_x = u'_{S,x} n_x - u'_{S,y} n_y \\
u_{S,y} &= u'_{S,x} n_y + u'_{S,y} m_y = u'_{S,y} n_x + u'_{S,x} n_y \\
p_S &= p'_S
\end{aligned}
$$

Note that

$$u'_{S,x} = u_{RS,x} \qquad u'_{S,y} = u'_{K,y}\,, \qquad K = L \text{ or } R$$

Hence we can rewrite

$$\begin{aligned} u_{S,x} = u'_{S,x}n_x - u'_{S,y}n_y &= u_{RS,x}n_x - \left(-u_{K,x}n_y + u_{K,y}n_x\right)n_y \\ &= \left(u_{RS,x} - u_{K,y}n_y - u_{K,x}n_x\right)n_x + u_{K,x}n_y^2 + u_{K,x}n_x^2 \\ &= \left(u_{RS,x} - \underline{u}_K.\underline{n}\right)n_x + u_{K,x} \end{aligned}$$

and similarly

$$u_{S,y} = \left(u_{RS,x} - \underline{u}_K.\underline{n}\right)n_y + u_{K,y}$$

We hence do not need to perform an actual rotation

General multidimensional Riemann problem solution

Compute the surface normal of the interface

$$\underline{n} = \frac{\underline{A}_{ij}}{\left|\underline{A}_{ij}\right|}$$

Compute the velocities along the surface normal

$$v_L = \underline{u}_L \cdot \underline{n} \qquad\qquad v_R = \underline{u}_R \cdot \underline{n}$$

Solve the 1D Riemann problem

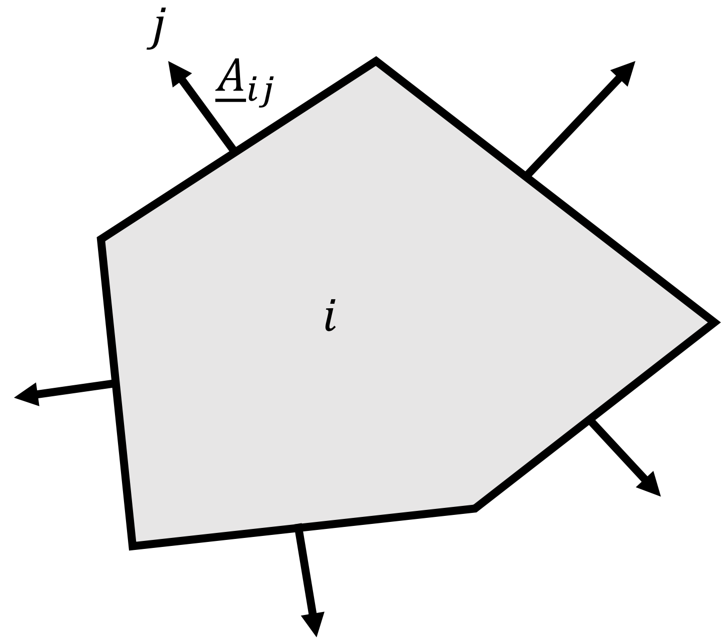
$$RS(\rho_L, v_L, p_L, \rho_R, v_R, p_R) \rightarrow \rho_S, v_S, p_S, \text{⚐}$$

Solve the multidimensional Riemann problem

$$\rho_S, \underline{u}_S, p_S = \begin{cases} \rho_S, \underline{u}_L + (v_S - v_L)\underline{n}, p_S, & \text{⚐} = -1 \\ \rho_S, \underline{u}_R + (v_S - v_R)\underline{n}, p_S, & \text{⚐} = +1 \end{cases}$$

Fluxes

$$\Delta Q_i = -\left(\sum_j \underline{F}_{ij}.\underline{A}_{ij}\right)\Delta t$$



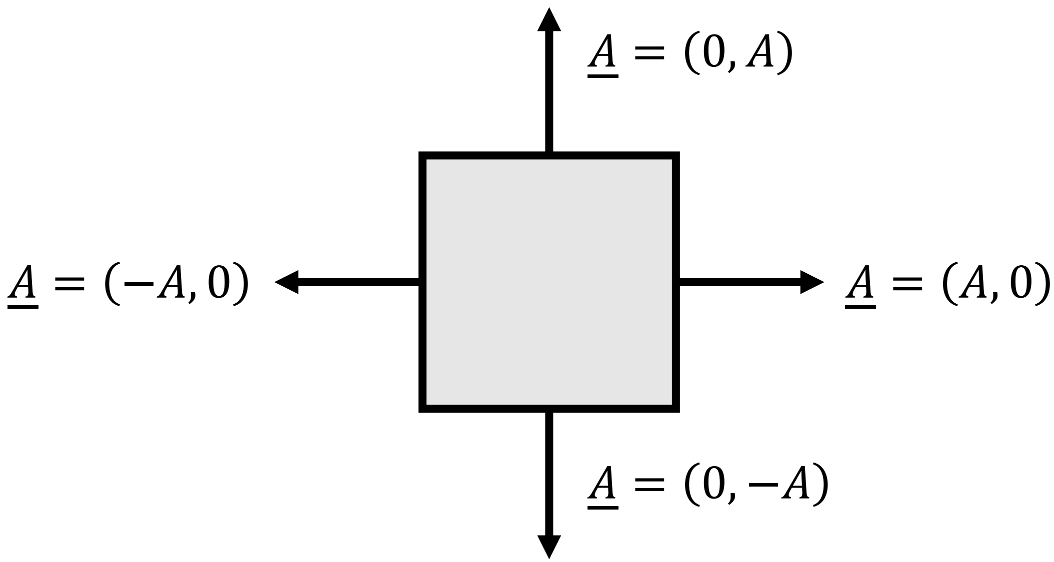
Remember that the flux is a *vector*
The actual flux contribution is the dot product of this vector with the surface normal *vector*:

$$F_{ij,x}(U) = \begin{pmatrix} \rho_S u_{S,x} \\ \rho_S u_{S,x}^2 + p_S \\ \rho_S u_{S,x} u_{S,y} \\ \rho_S e_{tot,S} u_{S,x} + p_S u_{S,x} \end{pmatrix} \qquad F_{ij,y}(U) = \begin{pmatrix} \rho_S u_{S,y} \\ \rho_S u_{S,x} u_{S,y} \\ \rho_S u_{S,y}^2 + p_S \\ \rho_S e_{tot,S} u_{S,y} + p_S u_{S,y} \end{pmatrix}$$

$$\Delta Q_i = -\left(\sum_j \left(F_{ij,x} A_{ij,x} + F_{ij,y} A_{ij,y}\right)\right)\Delta t$$

Regular Cartesian grid

For a regular Cartesian grid with square/cubic cells, the equations can be simplified



For an x face: only $F_x(U)$ contributes, Riemann problem in x with advected u_y

For a y face: only $F_y(U)$ contributes, Riemann problem in y with advected u_x

Common mistakes

Left/right confusion in the Riemann problem:

LEFT and RIGHT are defined such that the surface normal vector points from LEFT to RIGHT, they have no actual spatial meaning

Wrong fluxes:

The flux is a VECTOR, the conserved quantities are updated by taking the dot product of this VECTOR with the surface normal VECTOR
For a regular grid, the surface normal has zero components, so that some flux components drop out of the equation

Not every u^2 in 1D becomes a $|\underline{u}|^2$ in 2D and 3D (although some do!)

Wrong geometry:

A 1D volume is a line segment, a 1D surface is a point and has no surface area

A 2D volume is a surface area, a 2D surface area is a line segment

A 3D volume is - well - a volume, a 3D surface area is a surface area

Wrong geometry?

3D volume: $A \times B \times C$, surface areas: $A \times B$, $A \times C$, or $B \times C$

2D volume: $A \times B \times C$, surface areas: $A \times C$, or $B \times C$

$A \times B$ A or B

1D volume: $A \times B \times C$, surface areas: $B \times C$

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