The Riemann problem

The gory details.

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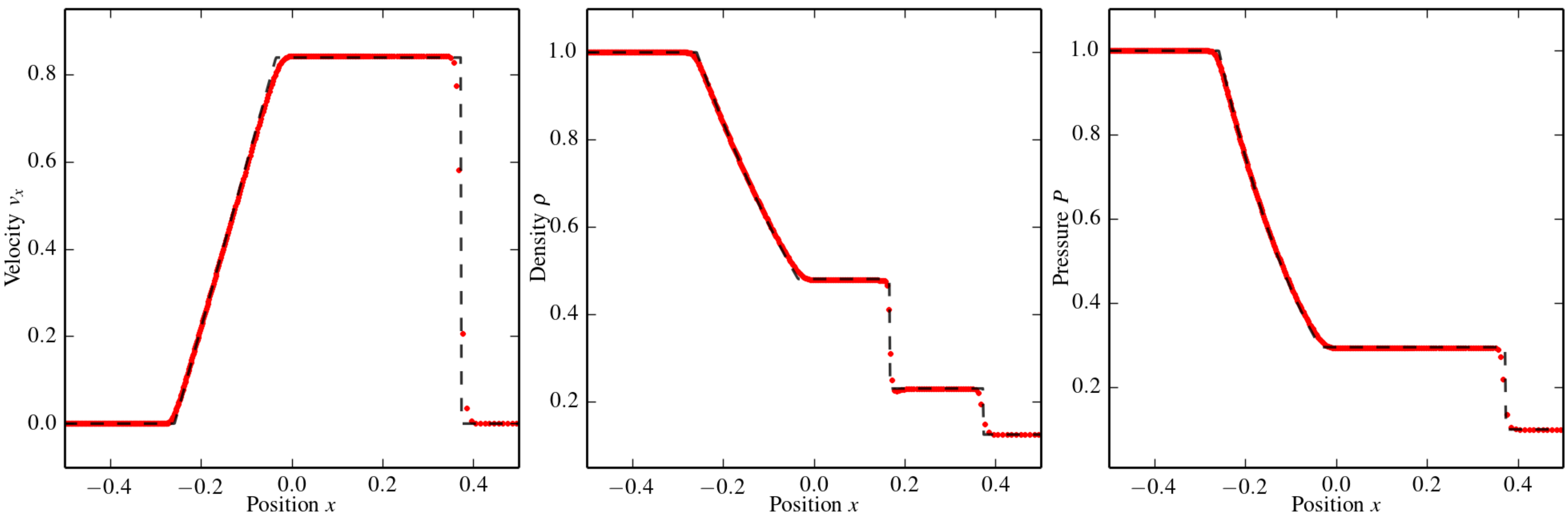
Riemann problem

L	R
$\rho_L, \underline{u}_L, p_L$	$\rho_R, \underline{u}_R, p_R$



Some questions:

What are these *waves*?

Why are there always 3 waves (in 1D)?

Why is the middle wave always a contact discontinuity?

How do we get this solution?

Remember our conservation law:

$$\frac{\partial U}{\partial t} + \underline{\nabla}.\underline{F}(U) = 0$$

In 1D:

$$\boxed{\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}F(U) = 0}$$

We will try to say something about the solutions of our conservation law by looking at

$$\frac{\partial f(x,t)}{\partial t} + a\frac{\partial f(x,t)}{\partial x} = 0$$ with a a constant and $f(x,t)$ a scalar function

and

$$\frac{\partial f(x,t)}{\partial t} + \frac{\partial g\big(f(x,t)\big)}{\partial x} = 0$$ with $g\big(f(x,t)\big)$ a function of $f(x,t)$

$$\boxed{\frac{\partial f(x,t)}{\partial t} + a\frac{\partial f(x,t)}{\partial x} = 0 \qquad \text{with } a \text{ a constant and } f(x,t) \text{ a scalar function}}$$

Introduce a new function $q(t)$, so that $\frac{\mathrm{d}q\,(t)}{\mathrm{d}t} = a$, (e.g. $q(t) = q_0 + at$)

We call this a *characteristic curve*

Along the characteristic curve, we have a function $f'(t) = f(q(t),t)$, with the property

$$\frac{\mathrm{d}f'(t)}{\mathrm{d}t} = \frac{\partial f(q(t),t)}{\partial t} + \frac{\mathrm{d}q(t)}{\mathrm{d}t}\frac{\partial f(q(t),t)}{\partial x} = \frac{\partial f(q(t),t)}{\partial t} + a\frac{\partial f(q(t),t)}{\partial x} = 0$$

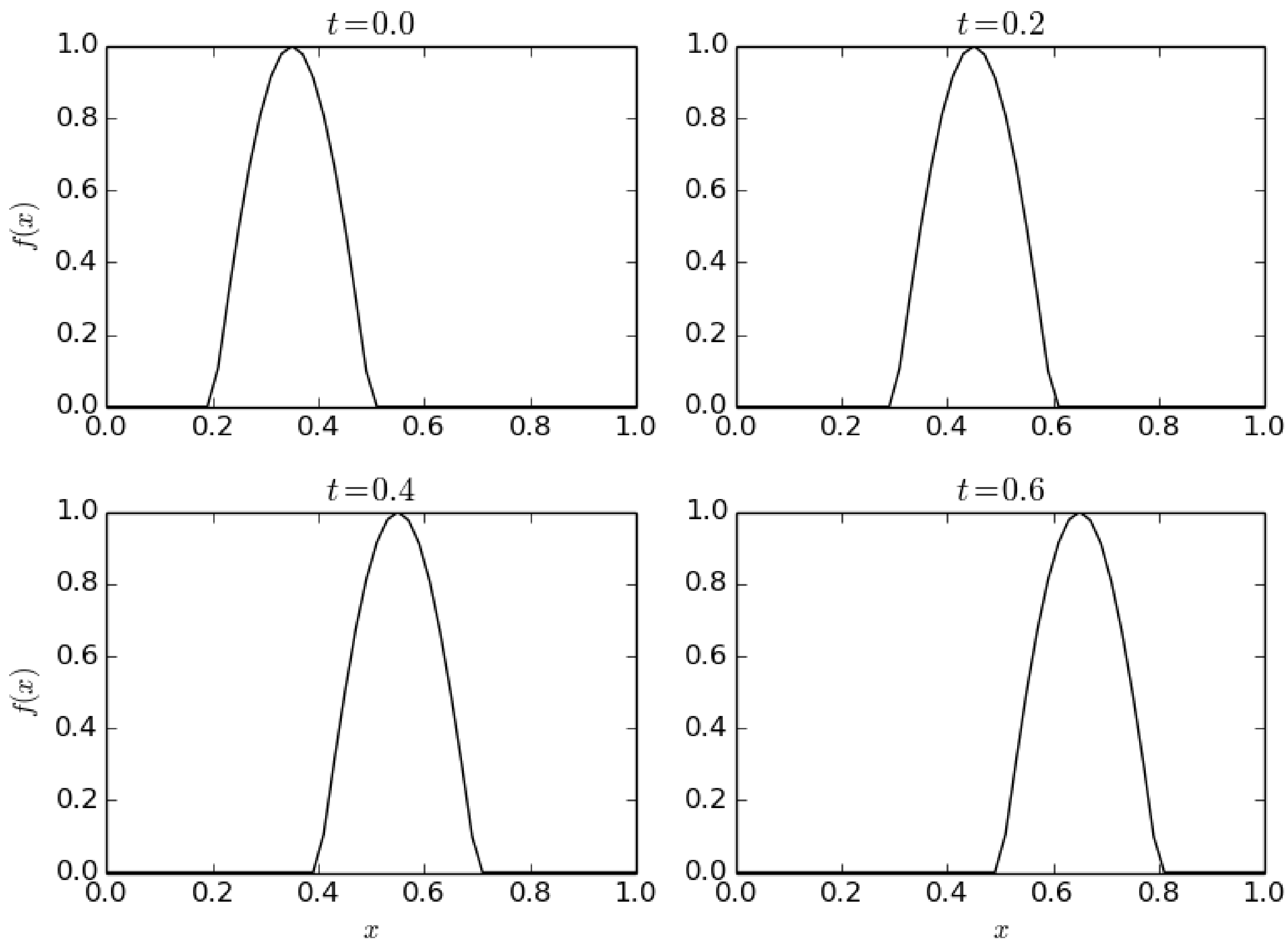
Assume $f(x,0) = f_0(x)$, then this means

$$f'(0) = f'(t) = f(q_0 + at, t) = f(q_0, 0) = f_0(q_0)$$

Hence:

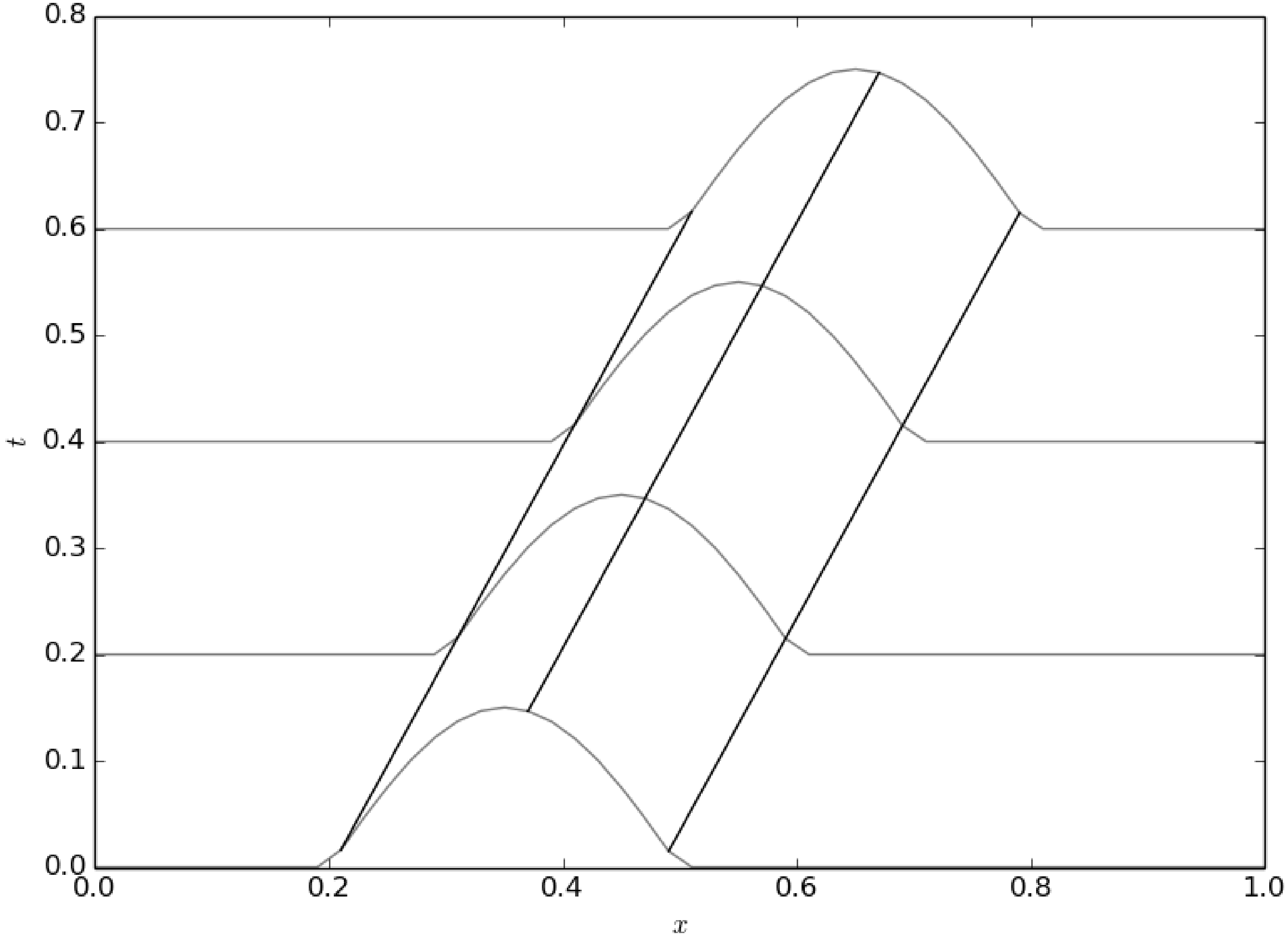
$$\boxed{f(x,t) = f_0(x - at)}$$

$$f(x,t) = f_0(x - at)$$



The solution of the differential equation is a linear translation of the initial condition with the *characteristic speed* a

$$f(x,t) = f_0(x - at)$$



All characteristic curves are just straight lines in $x - t$ space

$$\boxed{\frac{\partial f(x,t)}{\partial t} + \frac{\partial g\big(f(x,t)\big)}{\partial x} = 0 \quad \text{with } g\big(f(x,t)\big) \text{ a function of } f(x,t)}$$

$$\frac{\partial g(f(x,t))}{\partial x} = \frac{\partial g(f(x,t))}{\partial f}\frac{\partial f(x,t)}{\partial x} = J(f(x,t))\frac{\partial f(x,t)}{\partial x}$$

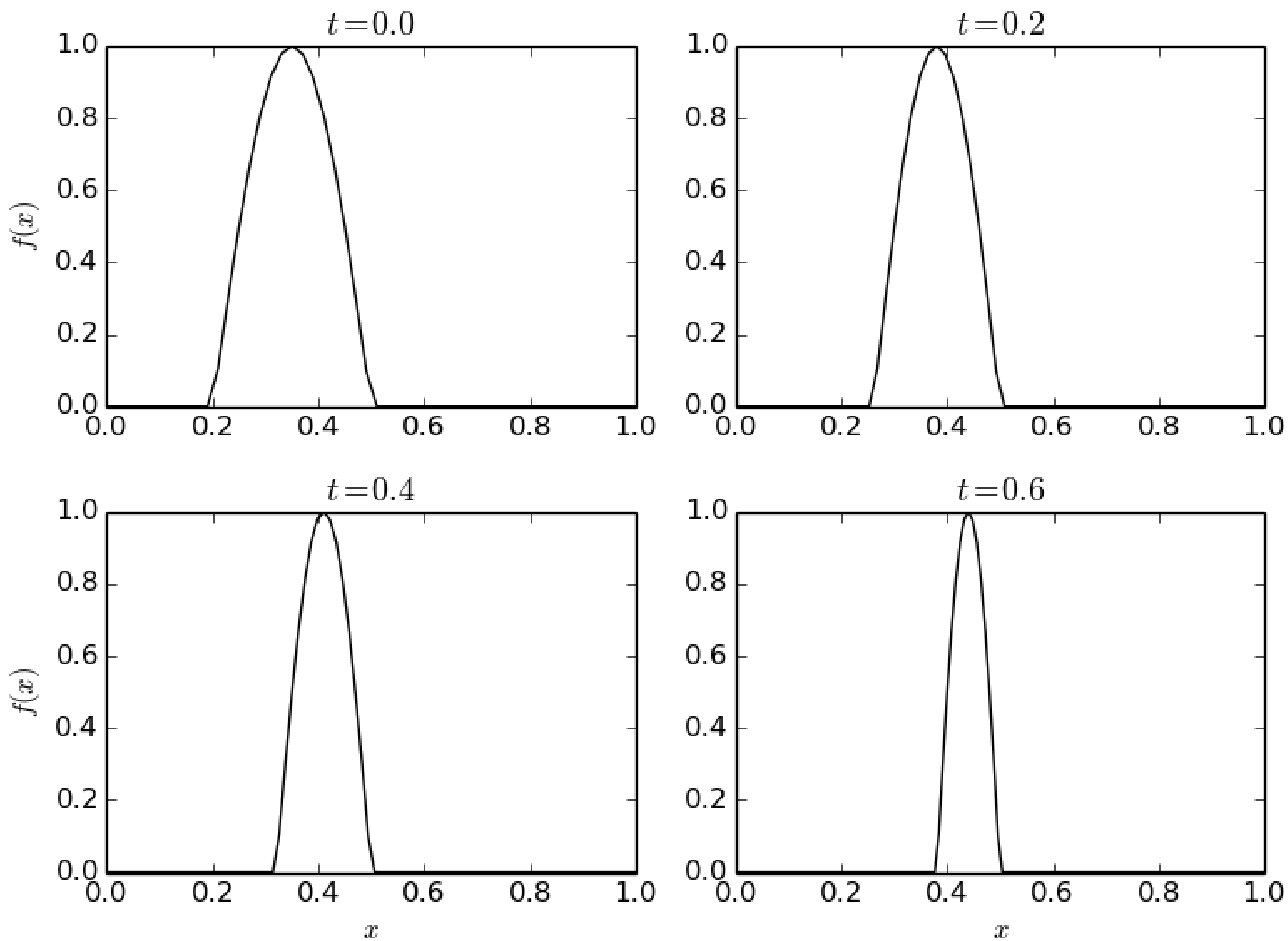
$$\Longrightarrow \frac{\partial f(x,t)}{\partial t} + J\big(f(x,t)\big)\frac{\partial f(x,t)}{\partial x} = 0$$

This is the same characteristic equation as before, but now $J(f(x,t))$ is no longer constant

Characteristic curves will run into each other (compressive region):
shock wave, discontinuous solutions

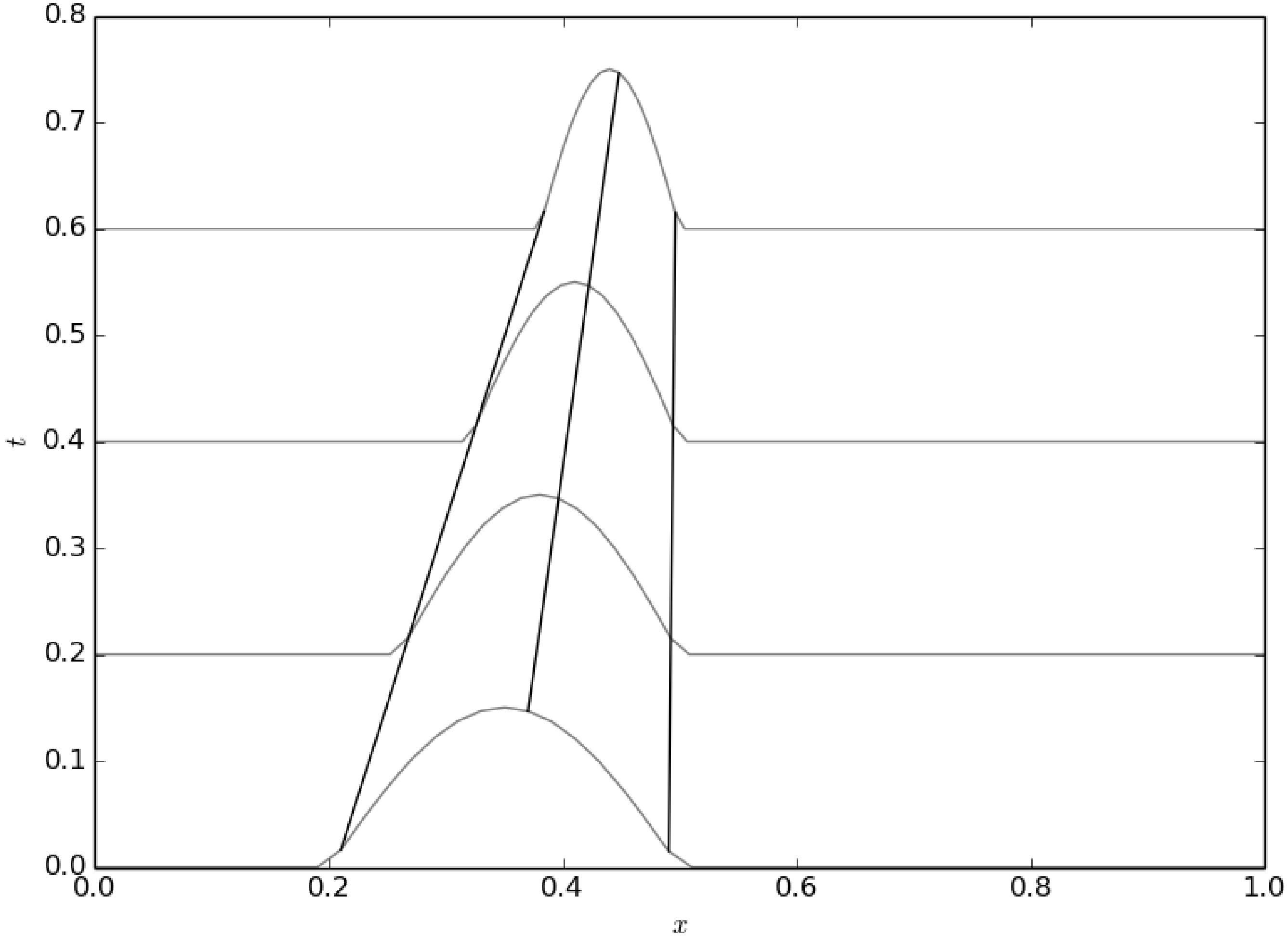
or will run away from each other (expansive region):
rarefaction wave, continuous solutions

$$f(x,t) = f_0(x - a(x)t),\, a(x) = b - x$$



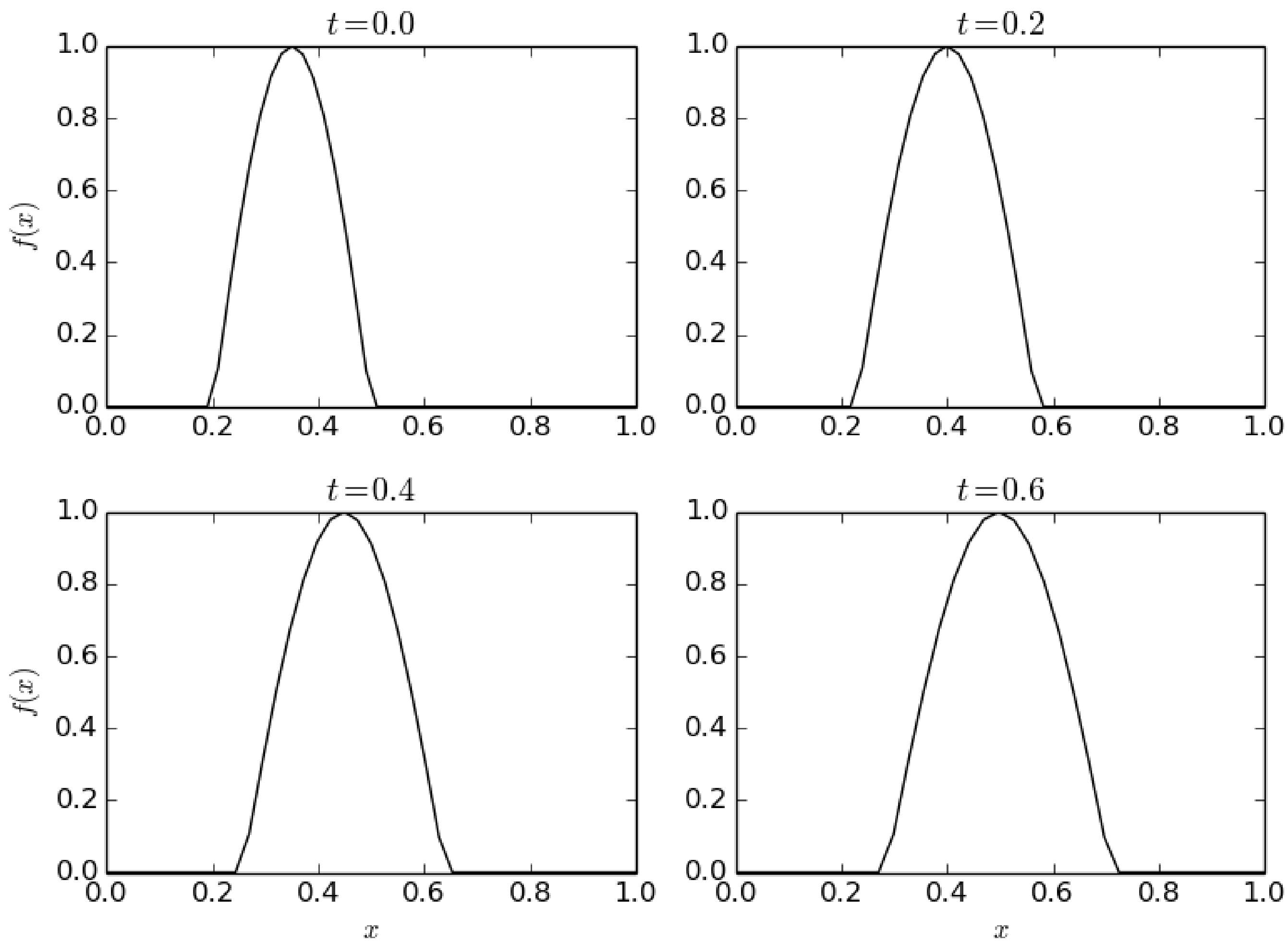
Shock wave

$$f(x,t) = f_0(x - a(x)t),\, a(x) = b - x$$



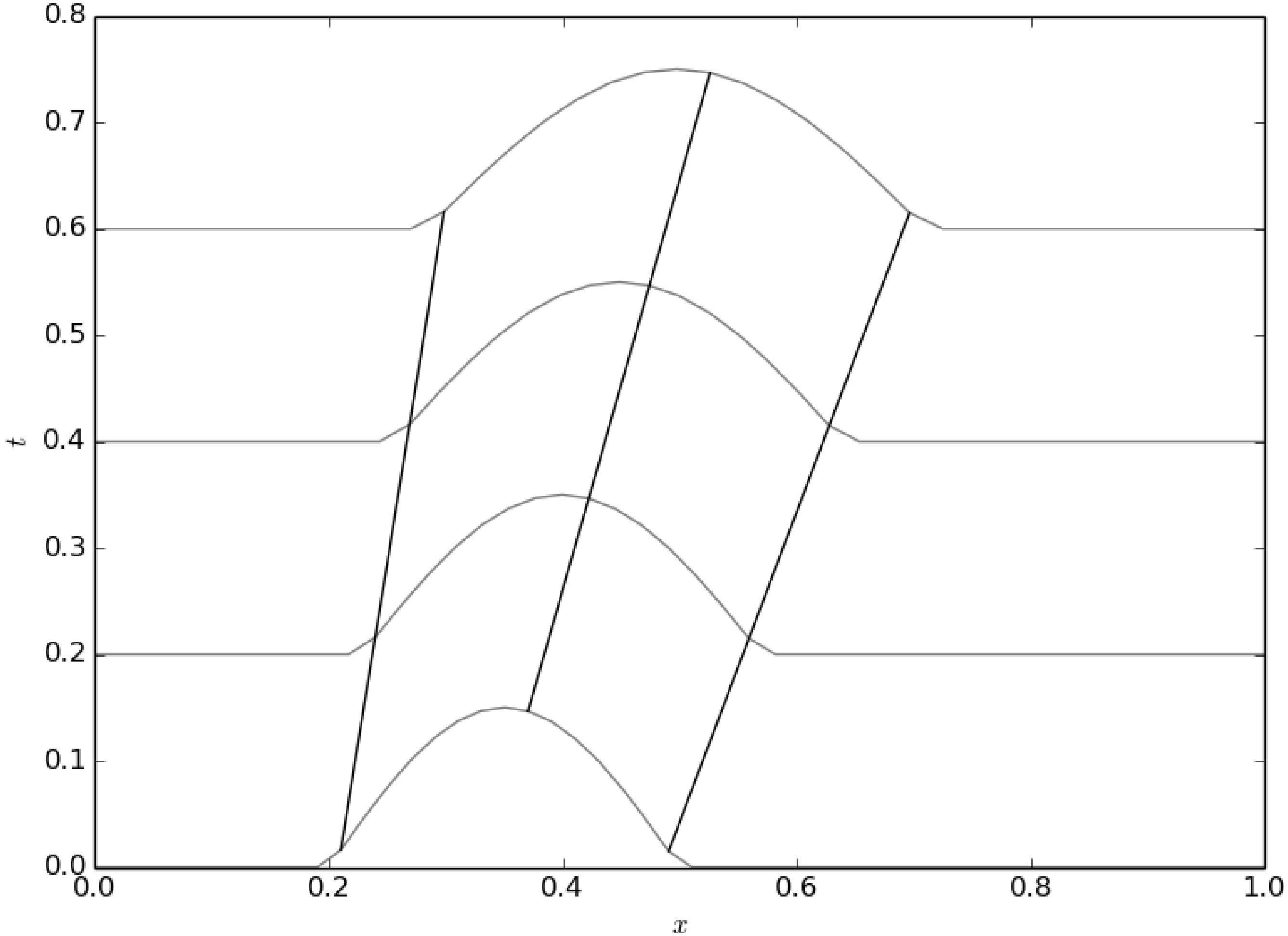
Shock wave

$$f(x,t) = f_0(x - a(x)t),\ a(x) = bx$$



Rarefaction wave

$$f(x,t) = f_0(x - a(x)t),\, a(x) = bx$$



Rarefaction wave

Important note

If the solution is discontinuous (shock wave), we can no longer use the conservation law!

However, we can still deduce a *Rankine-Hugoniot condition* across the discontinuity, by imposing the integral form of the conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_L}^{x_R} f(x,t)\mathrm{d}x = g\big(f(x_L,t)\big) - g\big(f(x_R,t)\big)$$

Suppose this law holds everywhere in an interval $[x_L, x_R]$, except on a line $s(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_L}^{s(t)} f(x,t)\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{s(t)}^{x_R} f(x,t)\mathrm{d}x = g\big(f(x_L,t)\big) - g\big(f(x_R,t)\big)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_L}^{s(t)} f(x,t)\mathrm{d}x = \int_{x_L}^{s(t)} \frac{\partial f(x,t)}{\partial t}\mathrm{d}x + \frac{\mathrm{d}s(t)}{\mathrm{d}t} f(s_L\,,t)$$

$$s_L = \lim_{x\to s(t)} s(t)\,, x < s(t)$$

$$\int_{x_L}^{s(t)} \frac{\partial f(x,t)}{\partial t} \mathrm{d}x + \int_{s(t)}^{x_R} \frac{\partial f(x,t)}{\partial t} \mathrm{d}x + (f(s_L,t) - f(s_R,t))S = g\big(f(x_L,t)\big) - g\big(f(x_R,t)\big)$$

$$S = \frac{\mathrm{d}s(t)}{\mathrm{d}t}$$

If the function $f(x,t)$ is a physical field, its value is bounded and the integrals vanish if $x_L \to s(t)$ and $x_R \to s(t)$

We are left with

$$\boxed{g\big(f(s_L,t)\big) - g\big(f(s_R,t)\big) = S\big(f(s_L,t) - f(s_R,t)\big)}$$

This links the change of the solution across a discontinuity moving with speed S to the change in fluxes across the discontinuity

Summary so far

What are these *waves*?

Waves are the solution of a 1D conservation law

They have a characteristic speed: the wave speed

The wave speed can be function of the solution, in which case the behaviour of the wave speed function sets the type of wave:

A compressive wave is called a shock wave
it has a discontinuous jump across the wave front
Rankine-Hugoniot conditions apply

An expansive wave is called a rarefaction wave
the solution is continuous
the differential form of the conservation law applies

Unfortunately, the Euler equations are not a 1D conservation law

However, we will show that it is possible to reduce the 3 1D Euler equations to a linearized system of 1D conservation laws

We start by looking at

$$\frac{\partial F(x,t)}{\partial t} + A\frac{\partial F(x,t)}{\partial x} = 0$$

with A a constant matrix and $F(x,t)$ a matrix function

$$\frac{\partial F(x,t)}{\partial t} + A\frac{\partial F(x,t)}{\partial x} = 0$$ with A a constant matrix and $F(x,t)$ a matrix function

This system is *hyperbolic* if A has real eigenvalues, in which case we can rewrite A as

$A = K\Lambda K^{-1}$, with $AK^{i} = \lambda_i K^{i}$

(λ_i are the eigenvalues of A, with corresponding eigenvectors K^{i})

If we introduce a new matrix $Z(x,t) = K^{-1}F(x,t)$, then we find

$$K\frac{\partial Z(x,t)}{\partial t} + AK\frac{\partial Z(x,t)}{\partial x} = 0$$

or

$$\frac{\partial Z(x,t)}{\partial t} + \Lambda\frac{\partial Z(x,t)}{\partial x} = 0$$

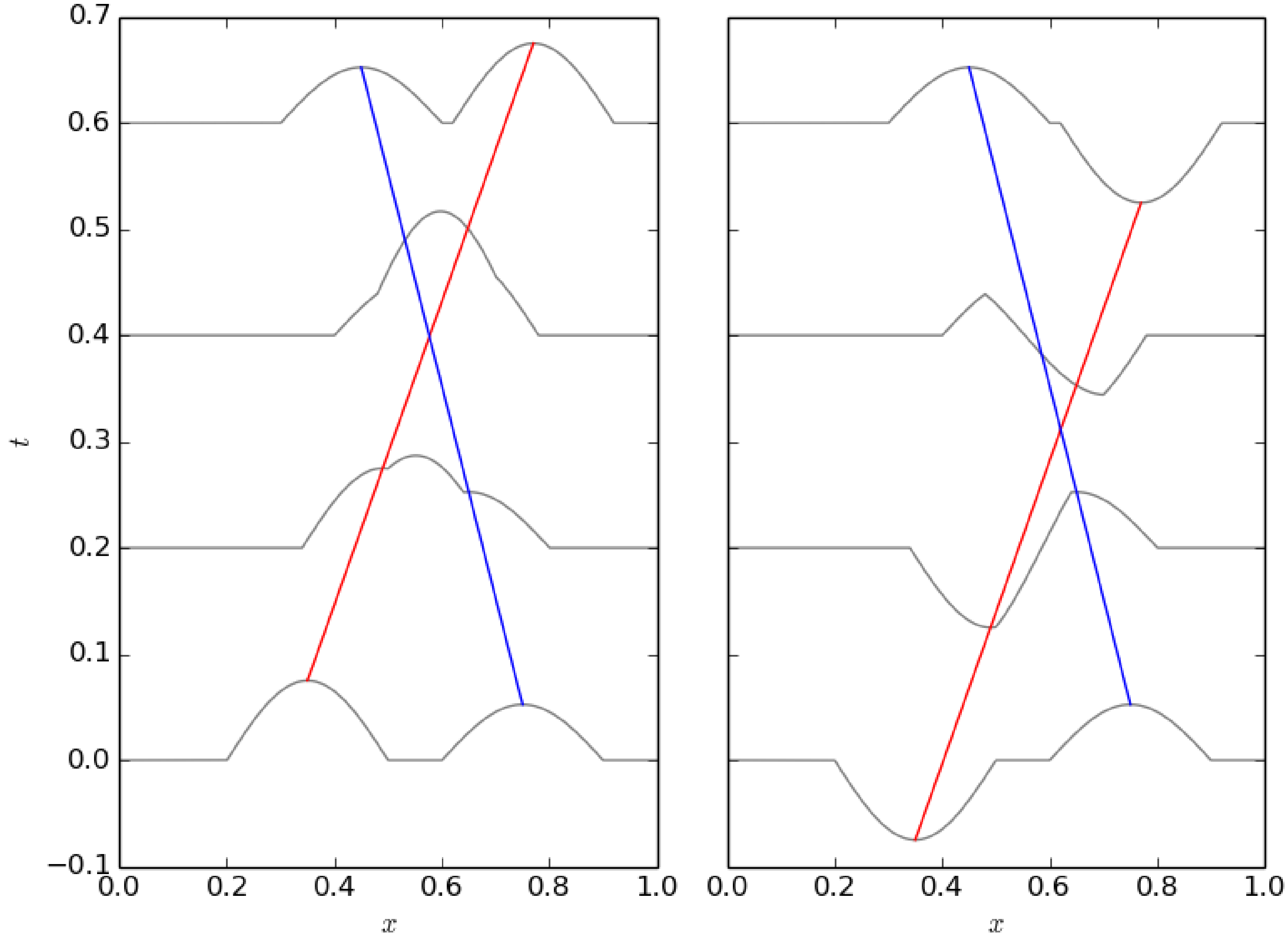
$$\frac{\partial Z(x,t)}{\partial t} + \Lambda \frac{\partial Z(x,t)}{\partial x} = 0$$

If A and Λ are $m \times m$ matrices, then this is a system of m independent characteristic equations
The solutions of this system are m characteristic waves with wave speeds λ_i

The solutions $F(x,t)$ of the original equation are linear combinations of the characteristic waves:

$$F(x,t) = K\,Z(x,t)$$

2D example



We now return to our general conservation law:

$$\boxed{\frac{\partial F\big(U(x,t)\big)}{\partial t} + \frac{\partial F\big(U(x,t)\big)}{\partial x} = 0}$$

Just as in the 1D case, we introduce the Jacobian (now *Jacobian matrix*):

$$J(x,t) = \frac{\partial F(U)}{\partial U}$$

We obtain the linearized system of equations

$$\frac{\partial F\big(U(x,t)\big)}{\partial t} + J(x,t)\frac{\partial U(x,t)}{\partial x} = 0$$

$$\frac{\partial F\big(U(x,t)\big)}{\partial t} + J(x,t)\,\frac{\partial U(x,t)}{\partial x} = 0$$

This system is hyperbolic if the Jacobian matrix has real eigenvalues

In this case, we can replace it with a system of independent 1D conservation laws, each with a characteristic wave speed which can be a function of x and t

The general solution of the conservation law will be a linear combination of these elementary characteristic waves

There is a characteristic for every equation in the system

Summary so far

What are these *waves*? Characteristic solutions of a 1D conservation law

Why are there always 3 waves (in 1D)?

A *hyperbolic* system of m conservation laws can be reduced to a system of m independent 1D conservation laws

The general solution of the system is a linear combination of the m characteristic waves defined by these 1D conservation laws

The 1D Euler equations consist of 3 coupled conservation laws, and hence there are 3 elementary waves in the solution of the 1D Riemann problem

Summary so far

$$\boxed{\frac{\partial F\big(U(x,t)\big)}{\partial t} + \frac{\partial F\big(U(x,t)\big)}{\partial x} = 0}$$

$$J(x,t) = \frac{\partial F(U)}{\partial U}$$

$$\frac{\partial F\big(U(x,t)\big)}{\partial t} + J(x,t)\,\frac{\partial U(x,t)}{\partial x} = 0$$

$$J(x,t) = K(x,t)\Lambda(x,t)K^{-1}(x,t),$$

with $A(x,t)K^i(x,t) = \lambda_i(x,t)K^i(x,t)$

Now we need to apply this to the Euler equations

The Riemann problem

The gory details.

PART 2

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Summary so far

What are these *waves*? Characteristic solutions of a 1D conservation law

Why are there always 3 waves (in 1D)? Because there are 3 1D Euler equations

Why is the middle wave always a contact discontinuity?

How do we get this solution?

Summary so far

$$\boxed{\frac{\partial F\big(U(x,t)\big)}{\partial t} + \frac{\partial F\big(U(x,t)\big)}{\partial x} = 0}$$

$$J(x,t) = \frac{\partial F(U)}{\partial U}$$

$$\frac{\partial F\big(U(x,t)\big)}{\partial t} + J(x,t)\,\frac{\partial U(x,t)}{\partial x} = 0$$

$$J(x,t) = K(x,t)\Lambda(x,t)K^{-1}(x,t),$$

with $A(x,t)K^{i}(x,t) = \lambda_i(x,t)K^{i}(x,t)$

Now we need to apply this to the Euler equations

For the 1D Euler equations

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x} F(U) = 0$$

with

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho e_{tot} \end{pmatrix}, \; F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho e_{tot} u + pu \end{pmatrix}, \text{ and } p = (\gamma - 1)\left(\rho e_{tot} - \tfrac{1}{2}\rho u^2\right)$$

we find

$$J(U) = \frac{\partial F(U)}{\partial U} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma - 3)u^2 & (3 - \gamma)u & \gamma - 1 \\ -\gamma u e_{tot} + (\gamma - 1)u^3 & \gamma e_{tot} - \frac{3}{2}(\gamma - 1)u^2 & \gamma u \end{pmatrix}$$

$$J(U) = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma - 3)u^2 & (3-\gamma)u & \gamma - 1 \\ -\gamma u e_{tot} + (\gamma - 1)u^3 & \gamma e_{tot} - \frac{3}{2}(\gamma - 1)u^2 & \gamma u \end{pmatrix}$$

For physical input values, this Jacobian can be diagonalized as $\Lambda(\mathrm{U}) = K(U)J(U)K^{-1}(U)$, with

$$\Lambda(U) = \begin{pmatrix} u - c_s & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u + c_s \end{pmatrix} \qquad K(U) = \begin{pmatrix} 1 & 1 & 1 \\ u - c_s & u & u + c_s \\ e_{tot} + \frac{p}{\rho} - uc_s & \frac{1}{2}u^2 & e_{tot} + \frac{p}{\rho} + uc_s \end{pmatrix}$$

We have a middle wave moving with the fluid velocity u, and a left and right wave moving with the sound speed $c_s = \sqrt{\frac{\gamma p}{\rho}}$ relative w.r.t. the middle wave

Note that for the 3D Euler equations (assuming a variation in x only), we find

$$J(U) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -u_x^2 + \frac{1}{2}(\gamma-1)\underline{u}^2 & (3-\gamma)u_x & -(\gamma-1)u_y & -(\gamma-1)u_z & \gamma-1 \\ -u_x u_y & u_y & u_x & 0 & 0 \\ -u_x u_z & u_z & 0 & u_x & 0 \\ -\gamma e_{tot} u_x - 2u_x^3 + (\gamma-1)u_x\underline{u}^2 & \gamma e_{tot} + (4-\gamma)u_x^2 - \frac{1}{2}(\gamma-1)\underline{u}^2 & -(\gamma-1)u_x u_y & -(\gamma-1)u_x u_z & \gamma u_x \end{pmatrix}$$

Which diagonalizes to (proof left as exercise):

$$\Lambda(U) = \begin{pmatrix} u_x - c_s & 0 & 0 & 0 & 0 \\ 0 & u_x & 0 & 0 & 0 \\ 0 & 0 & u_x & 0 & 0 \\ 0 & 0 & 0 & u_x & 0 \\ 0 & 0 & 0 & 0 & u_x + c_s \end{pmatrix}$$

This means that the 2 extra waves *coincide* with the middle wave
As long as we choose a reference frame where the spatial variation of the Riemann problem is limited to one coordinate dimension, the general solution only has 3 waves

Back to the 1D Euler equations:

$$\Lambda(U) = \begin{pmatrix} u - c_s & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u + c_s \end{pmatrix} \qquad K(U) = \begin{pmatrix} 1 & 1 & 1 \\ u - c_s & u & u + c_s \\ e_{tot} + \frac{p}{\rho} - uc_s & \frac{1}{2}u^2 & e_{tot} + \frac{p}{\rho} + uc_s \end{pmatrix}$$

Across the waves, we have (these are called *Generalised Riemann Invariants*):

Left wave
$$\frac{\mathrm{d}(\rho)}{1} = \frac{\mathrm{d}(\rho u)}{u - c_s} = \frac{\mathrm{d}(\rho e_{tot})}{e_{tot} + \frac{p}{\rho} - uc_s}$$

Middle wave
$$\frac{\mathrm{d}(\rho)}{1} = \frac{\mathrm{d}(\rho u)}{u} = \frac{\mathrm{d}(\rho e_{tot})}{\frac{1}{2}u^2}$$

Right wave
$$\frac{\mathrm{d}(\rho)}{1} = \frac{\mathrm{d}(\rho u)}{u + c_s} = \frac{\mathrm{d}(\rho e_{tot})}{e_{tot} + \frac{p}{\rho} + uc_s}$$

Middle wave $$\frac{\mathrm{d}(\rho)}{1} = \frac{\mathrm{d}(\rho u)}{u} = \frac{\mathrm{d}(\rho e_{tot})}{\frac{1}{2}u^2}$$

$$u\mathrm{d}\rho = \rho\mathrm{d}u + u\mathrm{d}\rho \Longrightarrow \mathrm{d}u = 0$$

$$\mathrm{d}e_{tot} = \mathrm{d}\left(\frac{1}{2}u^2 + \frac{p}{(\gamma-1)\rho}\right) = \frac{\mathrm{d}p}{(\gamma-1)\rho} - \frac{p\mathrm{d}\rho}{(\gamma-1)\rho^2}$$

$$\frac{1}{2}u^2\mathrm{d}\rho = \rho\mathrm{d}e_{tot} + e_{tot}\mathrm{d}\rho = \frac{\mathrm{d}p}{(\gamma-1)} - \frac{p\mathrm{d}\rho}{(\gamma-1)\rho} + \left(\frac{1}{2}u^2 + \frac{p}{(\gamma-1)\rho}\right)\mathrm{d}\rho$$

$$\Longrightarrow \mathrm{d}p = 0$$

u and p are constant across the middle wave: we have *a contact discontinuity*

Left wave

$$\frac{\mathrm{d}(\rho)}{1} = \frac{\mathrm{d}(\rho u)}{u - c_s} = \frac{\mathrm{d}(\rho e_{tot})}{e_{tot} + \frac{p}{\rho} - u c_s}$$

$$u\mathrm{d}\rho - c_s\mathrm{d}\rho = \rho\mathrm{d}u + u\mathrm{d}\rho$$

For $c_s \neq 0$, every change in ρ leads to a change in u (and p)

The left wave (and similarly the right wave) *cannot* be a contact discontinuity

Summary so far

What are these *waves*? Characteristic solutions of a 1D conservation law

Why are there always 3 waves (in 1D)? Because there are 3 1D Euler equations

Why is the middle wave always a contact discontinuity?

Because that follows from the Generalised Riemann invariants that are derived from the eigenvectors that diagonalize the linearized equations

Similarly, we can show that the left and right waves cannot be contact discontinuities and hence have to be shock waves or rarefaction waves

Interlude: non-conservative formulation

The Euler equations can also be rewritten in terms of the primitive variables:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + \gamma p\frac{\partial u}{\partial x} + u\frac{\partial p}{\partial x} = 0$$

or:

$$\frac{\partial W}{\partial t} + A(W)\frac{\partial W}{\partial x} = 0 \qquad \text{with} \qquad W = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}, A(W) = \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma p & u \end{pmatrix}$$

$$\frac{\partial W}{\partial t} + A(W)\frac{\partial W}{\partial x} = 0 \qquad \text{with} \qquad W = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}, A(W) = \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma p & u \end{pmatrix}$$

This can be diagonalized as

$$\Lambda(W) = \begin{pmatrix} u - c_s & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u + c_s \end{pmatrix} \qquad K(W) = \begin{pmatrix} 1 & 1 & 1 \\ -\frac{c_s}{\rho} & 0 & \frac{c_s}{\rho} \\ c_s^2 & 0 & c_s^2 \end{pmatrix}$$

Unsurprisingly, these equivalent equations have the same characteristics

The Generalised Riemann invariants for the middle wave now immediately show that u and p are constant across the middle wave

We can also introduce an *entropic function* $S = \frac{p}{\rho^\gamma}$, to get

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + \frac{c_s^2}{\rho}\frac{\partial \rho}{\partial x} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial S}\frac{\partial S}{\partial x} = 0$$

$$\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = 0$$

or:

$$\frac{\partial W'}{\partial t} + A(W')\frac{\partial W'}{\partial x} = 0 \qquad \text{with} \qquad W' = \begin{pmatrix} \rho \\ u \\ S \end{pmatrix}, A(W') = \begin{pmatrix} u & \rho & 0 \\ \frac{c_s^2}{\rho} & u & \frac{1}{\rho}\frac{\partial p}{\partial S} \\ 0 & 0 & u \end{pmatrix}$$

$$\frac{\partial W'}{\partial t} + A(W')\frac{\partial W'}{\partial x} = 0 \qquad \text{with} \qquad W' = \begin{pmatrix} \rho \\ u \\ S \end{pmatrix}, A(W') = \begin{pmatrix} u & \rho & 0 \\ \frac{c_s^2}{\rho} & u & \frac{1}{\rho}\frac{\partial p}{\partial S} \\ 0 & 0 & u \end{pmatrix}$$

This can be diagonalized as

$$\Lambda(W') = \begin{pmatrix} u - c_s & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u + c_s \end{pmatrix} \qquad K(W') = \begin{pmatrix} 1 & -\frac{\partial p}{\partial S} & 1 \\ -\frac{c_s}{\rho} & 0 & \frac{c_s}{\rho} \\ 0 & c_s^2 & 0 \end{pmatrix}$$

We immediately see that $\mathrm{d}S = 0$ across the left and right wave (isentropic = adiabatic flow)

However, this only applies if the primitive variables change continuously, and we have a rarefaction wave

$$\Lambda(W') = \begin{pmatrix} u - c_s & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u + c_s \end{pmatrix} \qquad K(W') = \begin{pmatrix} 1 & -\dfrac{\partial p}{\partial S} & 1 \\ -\dfrac{c_s}{\rho} & 0 & \dfrac{c_s}{\rho} \\ 0 & c_s^2 & 0 \end{pmatrix}$$

Across the left and right wave we also have

$$\mathrm{d}u \pm \frac{c_s}{\rho}\mathrm{d}\rho = 0$$

We know that $\mathrm{d}S = 0$, hence

$$\mathrm{d}\left(\frac{p}{\rho^\gamma}\right) = \frac{1}{\rho^\gamma}\mathrm{d}p - \gamma\frac{p}{\rho^{\gamma+1}}\mathrm{d}\rho = 0 \longrightarrow \mathrm{d}p = c_s^2\,\mathrm{d}\rho$$

$$\mathrm{d}u \pm \frac{c_s}{\rho}\mathrm{d}\rho = 0$$

We also know that

$$\mathrm{d}(c_s^2) = 2c_s\mathrm{d}c_s = \mathrm{d}\left(\frac{\gamma p}{\rho}\right) = \frac{\gamma}{\rho}\mathrm{d}p - \frac{\gamma p}{\rho^2}\mathrm{d}\rho = \frac{\gamma}{\rho}\mathrm{d}p - \frac{c_s^2}{\rho}\mathrm{d}\rho$$

or:

$$\mathrm{d}\rho = \frac{2\rho}{(\gamma-1)c_s}\mathrm{d}c_s$$

This leads to

$$\mathrm{d}u \pm \frac{2}{(\gamma-1)}\mathrm{d}c_s = \mathrm{d}\left(u \pm \frac{2c_s}{(\gamma-1)}\right) = 0$$

Interlude summary

If the left or right wave is a rarefaction wave, we can assume the following relations across the wave:

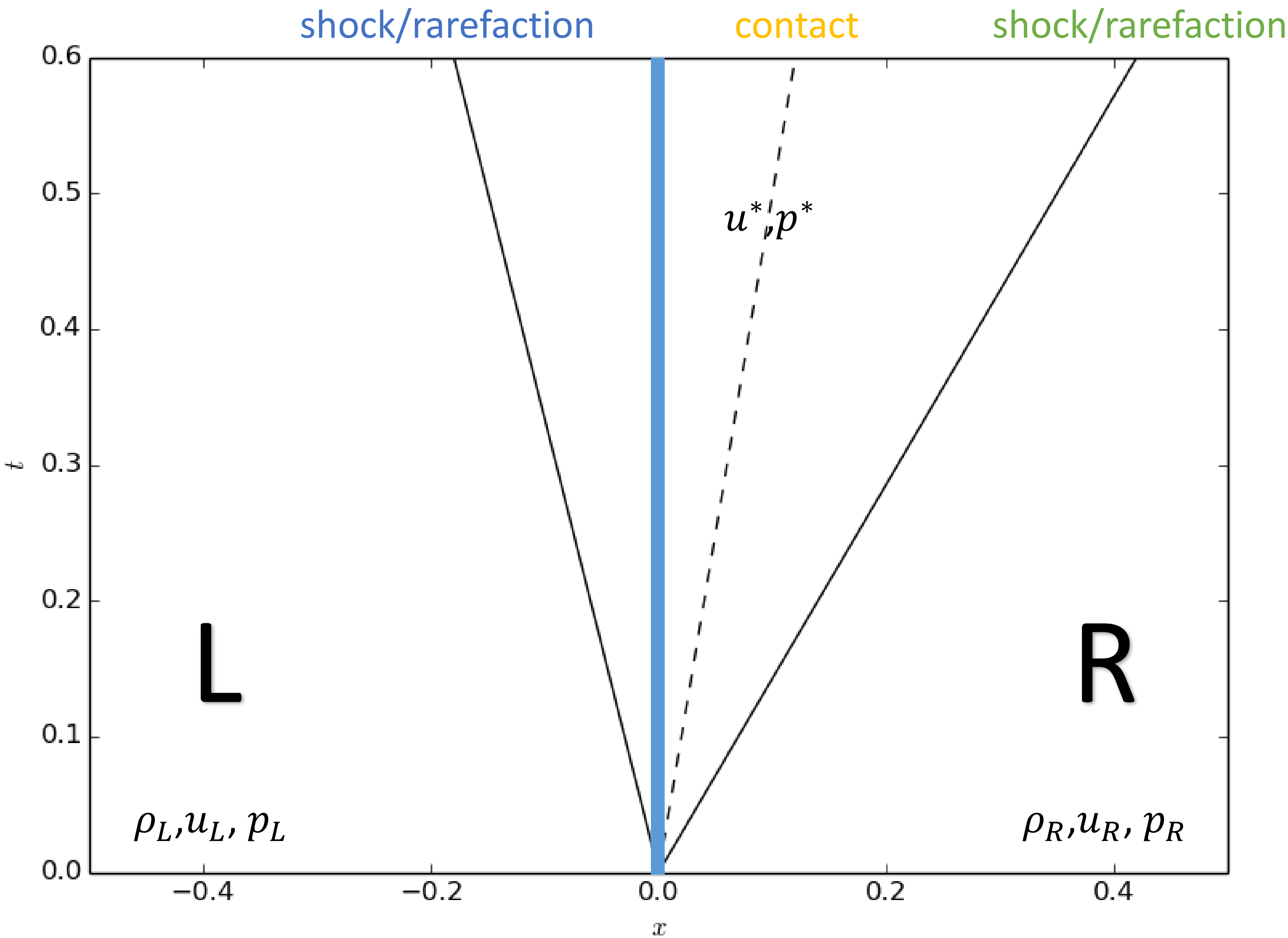
$$S = \frac{p}{\rho^\gamma} = \text{constant} \qquad \text{(isentropic law)}$$

$$u \pm \frac{2c_s}{(\gamma - 1)} = \text{constant}$$

For a left or right shock wave, we already found the Rankine-Hugoniot conditions:

$$F(U_L) - F(U_R) = S_{shock}(U_L - U_R)$$

The entire solution



The Riemann problem

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PART 3

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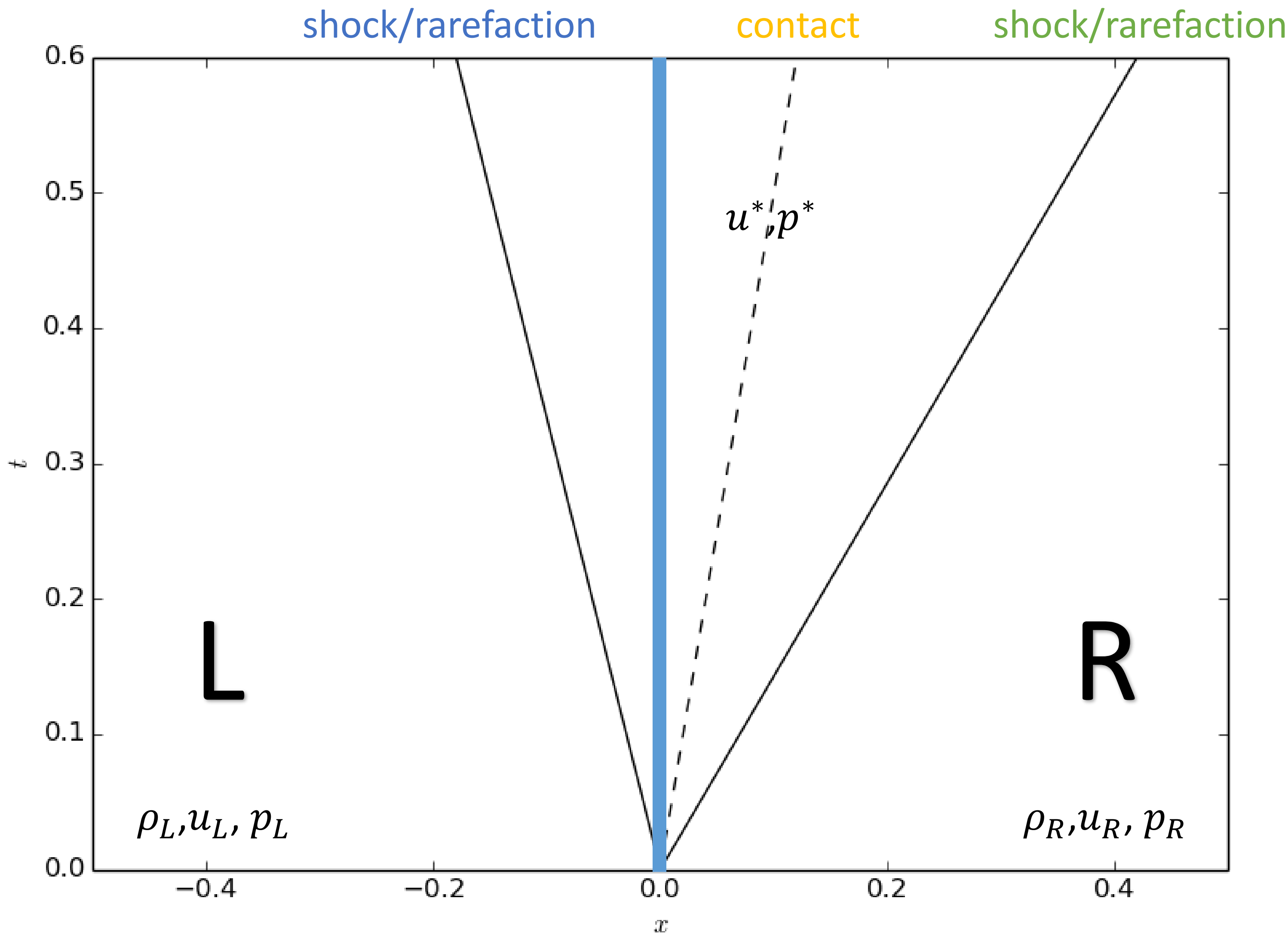
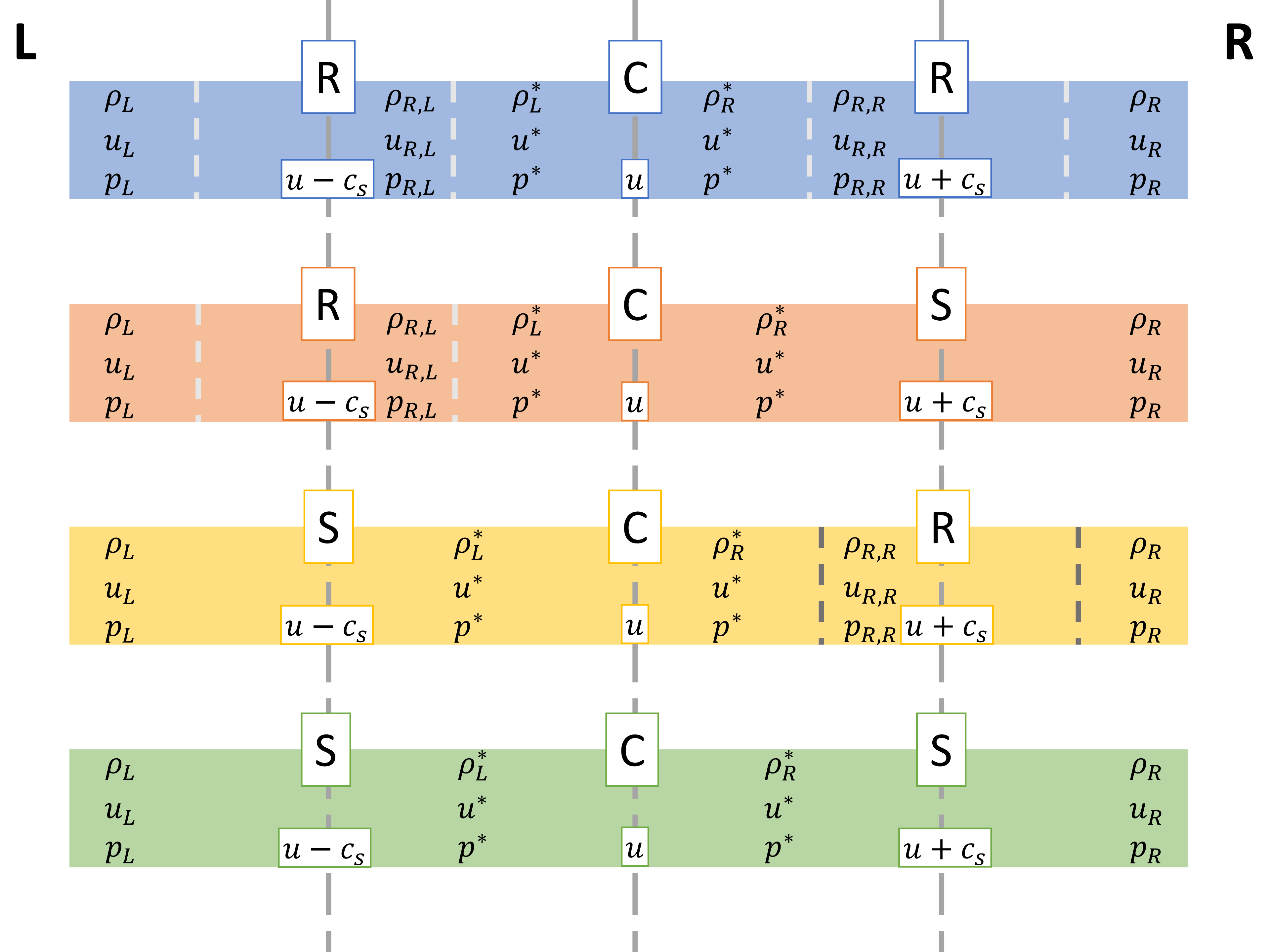
The entire solution





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ρ_L u_L p_L
$u - c_s$
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ρ_R^* u^* p^*
$\rho_{R,R}$ $u_{R,R}$ $p_{R,R}$
$u + c_s$
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C
R
ρ_L u_L p_L
$u - c_s$
ρ_L^* u^* p^*
u
ρ_R^* u^* p^*
$\rho_{R,R}$ $u_{R,R}$ $p_{R,R}$
$u + c_s$
ρ_R u_R p_R
S
C
S
ρ_L u_L p_L
$u - c_s$
ρ_L^* u^* p^*
u
ρ_R^* u^* p^*
$u + c_s$
ρ_R u_R p_R

Solution across a shock wave

Known: ρ_L, u_L, p_L

Unknown: ρ_L^*, u^*, S_L

Assume we know p^*

We use the Rankine-Hugoniot conditions

$$\boxed{F(U_L) - F(U_R) = S_{shock}(U_L - U_R)}$$

$$\rho_L \hat{u}_L = \rho_L^* \hat{u}^*$$

$$\rho_L \hat{u}_L^2 + p_L = \rho_L^* \hat{u}^{*2} + p^*$$

$$\rho_L e_{tot,L} \hat{u}_L + p_L \hat{u}_L = \rho_L^* e_{tot}^* \hat{u}^* + p^* \hat{u}^*$$

Where all velocities are expressed relative to the unknown shock speed S_L:

$$\hat{u} = u - S_L$$

Solution across a shock wave

Known: ρ_L, u_L, p_L

Unknown: ρ_L^*, u^*, S_L

Assume we know p^*

After a lot of manipulations (left as exercise), we obtain:

$$u^* = u_L - (p^* - p_L)\sqrt{\frac{2}{(\gamma+1)\rho_L\left(p^* + \frac{(\gamma-1)}{(\gamma+1)}p_L\right)}}$$

$$S_L = u_L - c_{s,L}\sqrt{\frac{(\gamma+1)}{2\gamma}\frac{p^*}{p_L} + \frac{(\gamma-1)}{2\gamma}}$$

$$\rho_L^* = \rho_L\left(\frac{\frac{p^*}{p_L} + \frac{(\gamma-1)}{(\gamma+1)}}{\frac{(\gamma-1)}{(\gamma+1)}\frac{p^*}{p_L} + 1}\right)$$

Solution across a shock wave

Known: ρ_R, u_R, p_R

Unknown: ρ_R^*, u^*, S_R

Assume we know p^*

For a right shock wave, we find

$$u^* = u_R + (p^* - p_R)\sqrt{\frac{2}{(\gamma+1)\rho_R\left(p^* + \frac{(\gamma-1)}{(\gamma+1)}p_R\right)}}$$

$$S_R = u_R + c_{s,R}\sqrt{\frac{(\gamma+1)}{2\gamma}\frac{p^*}{p_R} + \frac{(\gamma-1)}{2\gamma}}$$

$$\rho_R^* = \rho_R\left(\frac{\frac{p^*}{p_R} + \frac{(\gamma-1)}{(\gamma+1)}}{\frac{(\gamma-1)}{(\gamma+1)}\frac{p^*}{p_R} + 1}\right)$$

Solution across a rarefaction wave

Known: ρ_L, u_L, p_L Unknown: $\rho_L^*, u^*, \rho_{R,L}, u_{R,L}, p_{R,L}$ Assume we know p^*

We use the isentropic law to find the density

$$\frac{p_L}{\rho_L^\gamma} = \frac{p^*}{(\rho_L^*)^\gamma} \Leftrightarrow \rho_L^* = \rho_L \left(\frac{p^*}{p_L}\right)^{\frac{1}{\gamma}}$$

Using the other Riemann invariant, we find

$$u_L + \frac{2c_{s,L}}{(\gamma - 1)} = u^* + \frac{2c_s^*}{(\gamma - 1)}$$

Solution across a rarefaction wave

Known: ρ_L, u_L, p_L Unknown: $\rho_L^*, u^*, \rho_{R,L}, u_{R,L}, p_{R,L}$ Assume we know p^*

And after some more algebraic manipulations:

$$\rho_L^* = \rho_L \left(\frac{p^*}{p_L}\right)^{\frac{1}{\gamma}}$$

$$u^* = u_L - \frac{2c_{s,L}}{(\gamma - 1)}\left(\left(\frac{p^*}{p_L}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right)$$

Solution across a rarefaction wave

Known: ρ_R, u_R, p_R Unknown: $\rho_R^*, u^*, \rho_{R,R}, u_{R,R}, p_{R,R}$ Assume we know p^*

For the right rarefaction wave:

$$\rho_R^* = \rho_R \left(\frac{p^*}{p_R}\right)^{\frac{1}{\gamma}}$$

$$u^* = u_R + \frac{2c_{s,R}}{(\gamma - 1)}\left(\left(\frac{p^*}{p_R}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right)$$

Solution across a rarefaction wave

Known: ρ_L, u_L, p_L Unknown: $\rho_L^*, u^*, \rho_{R,L}, u_{R,L}, p_{R,L}$ Assume we know p^*

The rarefaction wave itself is the entire region in between two characteristics:

$$S_{HL} = u_L - c_{s,L}$$

$$S_{TL} = u^* - c_s^*$$

We call these the head and tail of the rarefaction wave

In between head and tail, the characteristic equation applies:

$$\frac{x}{t} = u - c_s$$

Solution across a rarefaction wave

Known: ρ_L, u_L, p_L Unknown: $\rho_L^*, u^*, \rho_{R,L}, u_{R,L}, p_{R,L}$ Assume we know p^*

Solving the characteristic equation together with the Riemann Invariant, we find

$$\rho_{R,L} = \rho_L \left(\frac{2}{(\gamma+1)} + \frac{(\gamma-1)}{(\gamma+1)c_{s,L}} \left(u_L - \frac{x}{t} \right) \right)^{\frac{2}{\gamma-1}}$$

$$u_{R,L} = \frac{2}{(\gamma+1)} \left(c_{s,L} + \frac{(\gamma-1)}{2} u_L + \frac{x}{t} \right)$$

$$p_{R,L} = p_L \left(\frac{2}{(\gamma+1)} + \frac{(\gamma-1)}{(\gamma+1)c_{s,L}} \left(u_L - \frac{x}{t} \right) \right)^{\frac{2\gamma}{\gamma-1}}$$

Solution across a rarefaction wave

Known: ρ_R, u_R, p_R Unknown: $\rho_R^*, u^*, \rho_{R,R}, u_{R,R}, p_{R,R}$ Assume we know p^*

And for the right rarefaction wave

$$\rho_{R,R} = \rho_R \left(\frac{2}{(\gamma+1)} - \frac{(\gamma-1)}{(\gamma+1)c_{s,R}} \left(u_R - \frac{x}{t} \right) \right)^{\frac{2}{\gamma-1}}$$

$$u_{R,R} = \frac{2}{(\gamma+1)} \left(-c_{s,R} + \frac{(\gamma-1)}{2} u_R + \frac{x}{t} \right)$$

$$p_{R,R} = p_R \left(\frac{2}{(\gamma+1)} - \frac{(\gamma-1)}{(\gamma+1)c_{s,R}} \left(u_R - \frac{x}{t} \right) \right)^{\frac{2\gamma}{\gamma-1}}$$

Shock wave or rarefaction wave?

We know we have a shock wave if the characteristics run into each other, i.e.

$$u_L - c_{s,L} > u^* - c_{s,L}^*$$

We also know that $u^* = u_L - C(p^* - p_L)$ for a shock wave, with C a positive constant

It can also be shown that $c_{s,L}^* = f\left(\frac{p^*}{p_L}\right) c_{s,L}$, with $f(1) = 1, f(x < 1) < 1, f(x > 1) > 1$

Together, this tells us that we can only have a left shock wave if $p_L < p^*$

Similarly, we can only have a rarefaction wave if $p_L \geq p^*$

Solution strategy

Starting from an initial guess for the pressure p^*, we can get two estimates of u^*:

Left state:

$$p^* > p_L \Longrightarrow u_L^* = u_{shock}^*$$

$$p^* \leq p_L \Longrightarrow u_L^* = u_{rarefaction}^*$$

Right state:

$$p^* > p_R \Longrightarrow u_R^* = u_{shock}^*$$

$$p^* \leq p_R \Longrightarrow u_R^* = u_{rarefaction}^*$$

We close the system of equations by requiring $u_L^* = u_R^* = u^*$

Solution strategy

We are basically trying to find the roots of the following function:

$$f(p, W_L, W_R) = f_L(p, W_L) + f_R(p, W_R) + u_R - u_L$$

With ($X = L$ or R)

$$f_X(p, W_X) = \begin{cases} (p - p_X)\sqrt{\dfrac{2}{(\gamma+1)\rho_X\left(p + \dfrac{(\gamma-1)}{(\gamma+1)}p_X\right)}}, & p > p_X \\ \dfrac{2c_{s,X}}{(\gamma-1)}\left(\left(\dfrac{p}{p_X}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right), & p \leq p_X \end{cases}$$

Solution strategy

Once p^* (and hence u^*) are known, we know what the solution looks like

We can then also obtain the speeds associated with the different waves:

Left/right shock wave:

$$S_{L/R} = u_{L/R} \mp c_{s,L/R} \sqrt{\frac{(\gamma+1)}{2\gamma}\frac{p^*}{p_{L/R}} + \frac{(\gamma-1)}{2\gamma}}$$

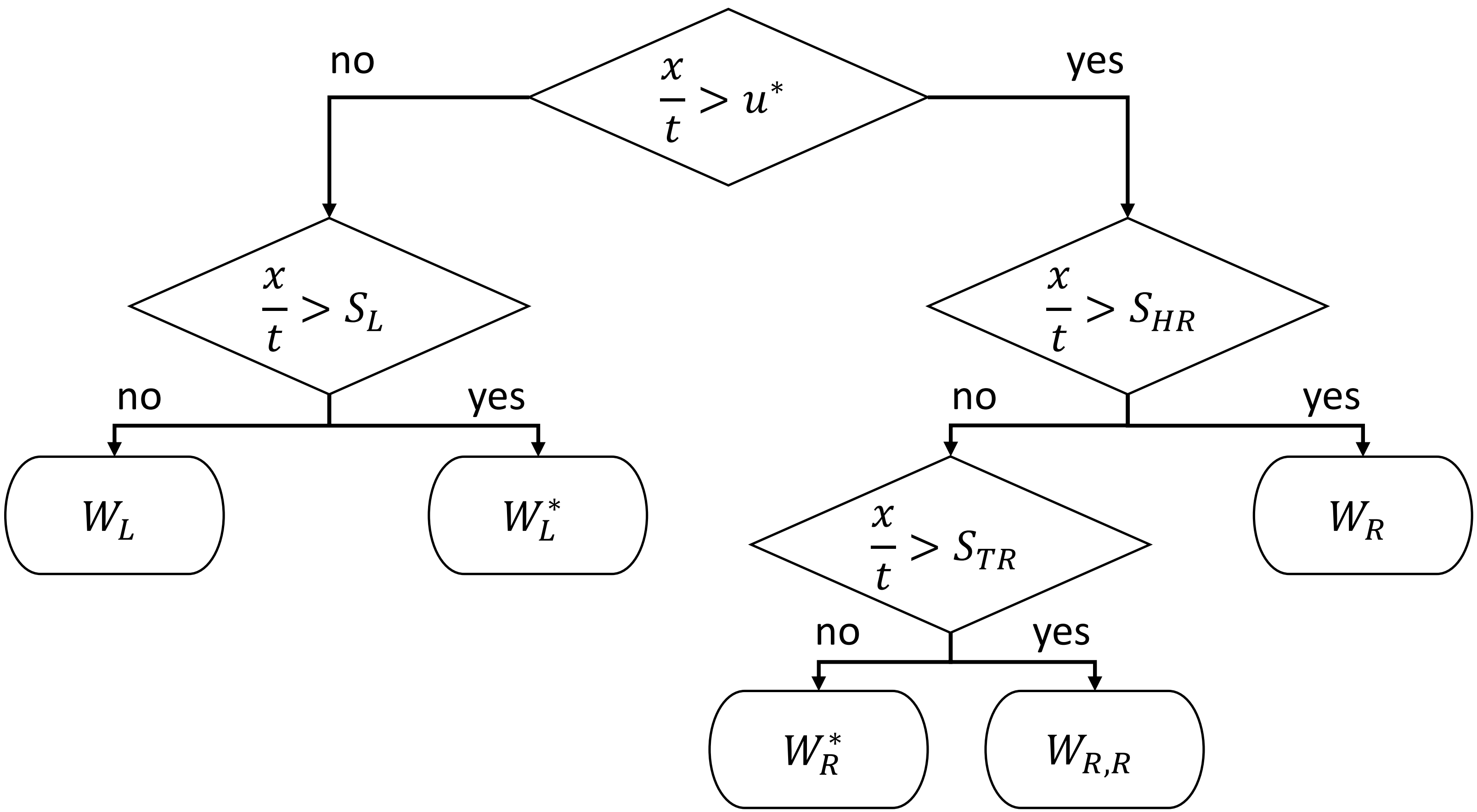
Left/right rarefaction wave:

$$S_{H,L/R} = u_{L/R} \mp c_{s,L/R}$$

$$S_{T,L/R} = u^* \mp c^*_{s,L/R}$$

Solution strategy

At a given time t, the solution at a position x will depend on where we are in characteristic space, i.e. the value of $\frac{x}{t}$



Last note: extra dimensions

We saw before that we can solve the 3D Riemann problem as if it were a 1D problem with two extra middle waves

The extra middle waves will still be contact discontinuities, across which the two extra velocity components jump from their left to their right state

The same is true for any other variable that is *advected* with the flow, i.e.

$$\frac{\partial X}{\partial t} + \underline{u}.\underline{\nabla} X = 0$$