# Finite volume schemes

An introduction

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# Some problems can be solved analytically…

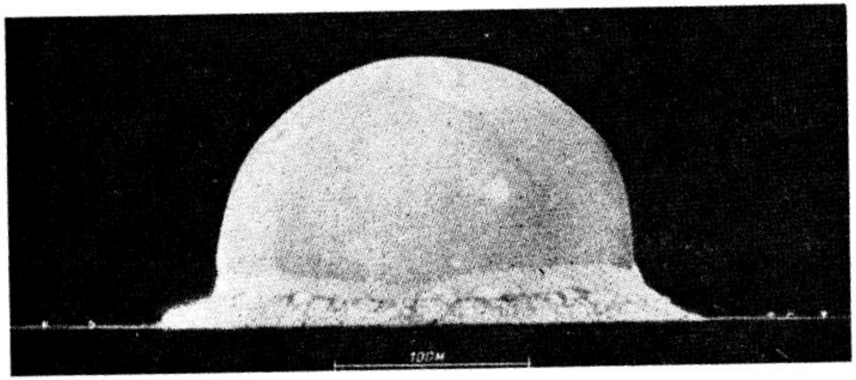


FIGURE 68. The fire ball at $t = 15 \times 10^{-3}$ s.

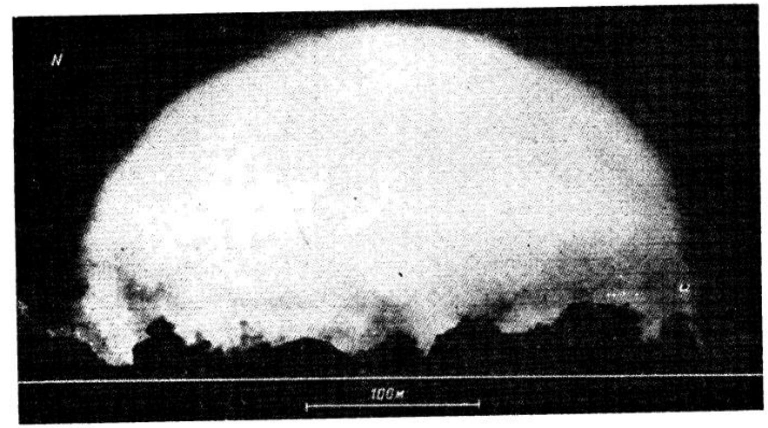


FIGURE 69. Photograph of the explosion at $t = 127 \times 10^{-3}$ s.

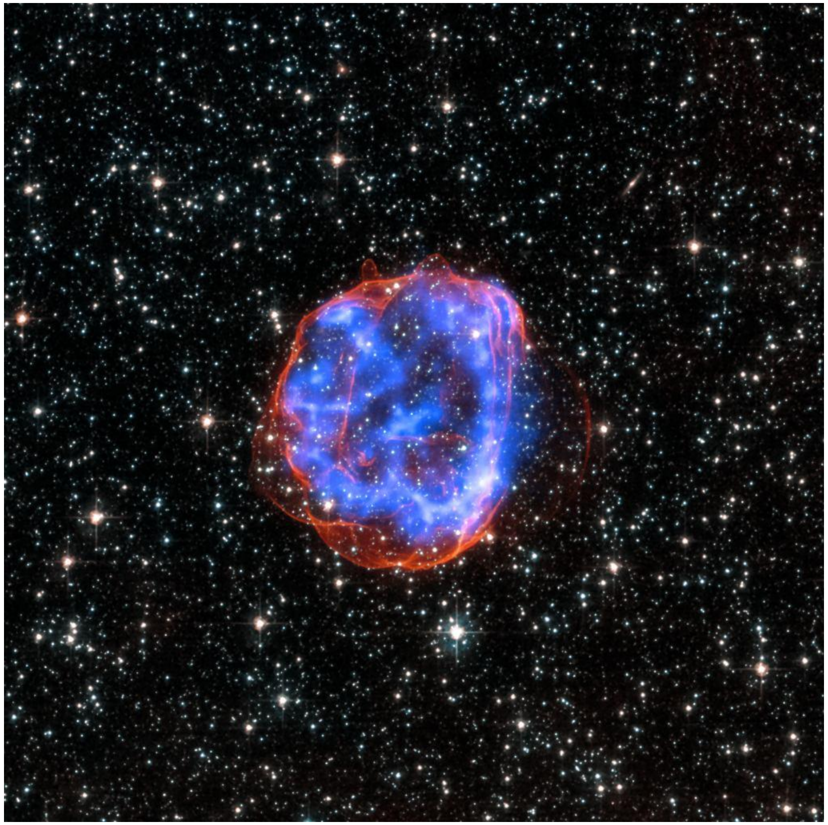
Thus, for the law of motion and for shock wave velocity in the case of spherical symmetry we obtain

$$r_2 = \left(\frac{E}{\rho_1}\right)^{1/5} t^{2/5}, \qquad c = \frac{2}{5}\left(\frac{E}{\rho_1}\right)^{1/5} t^{-3/5} = \frac{2}{5}\sqrt{\frac{E}{\rho_1}}\,\frac{1}{\sqrt{r_2^3}}; \quad (11.4)$$

*Sedov, L. I., Similarity and dimensional methods in mechanics, 10th edition (CRC Press, 1993), pp. 261-290*

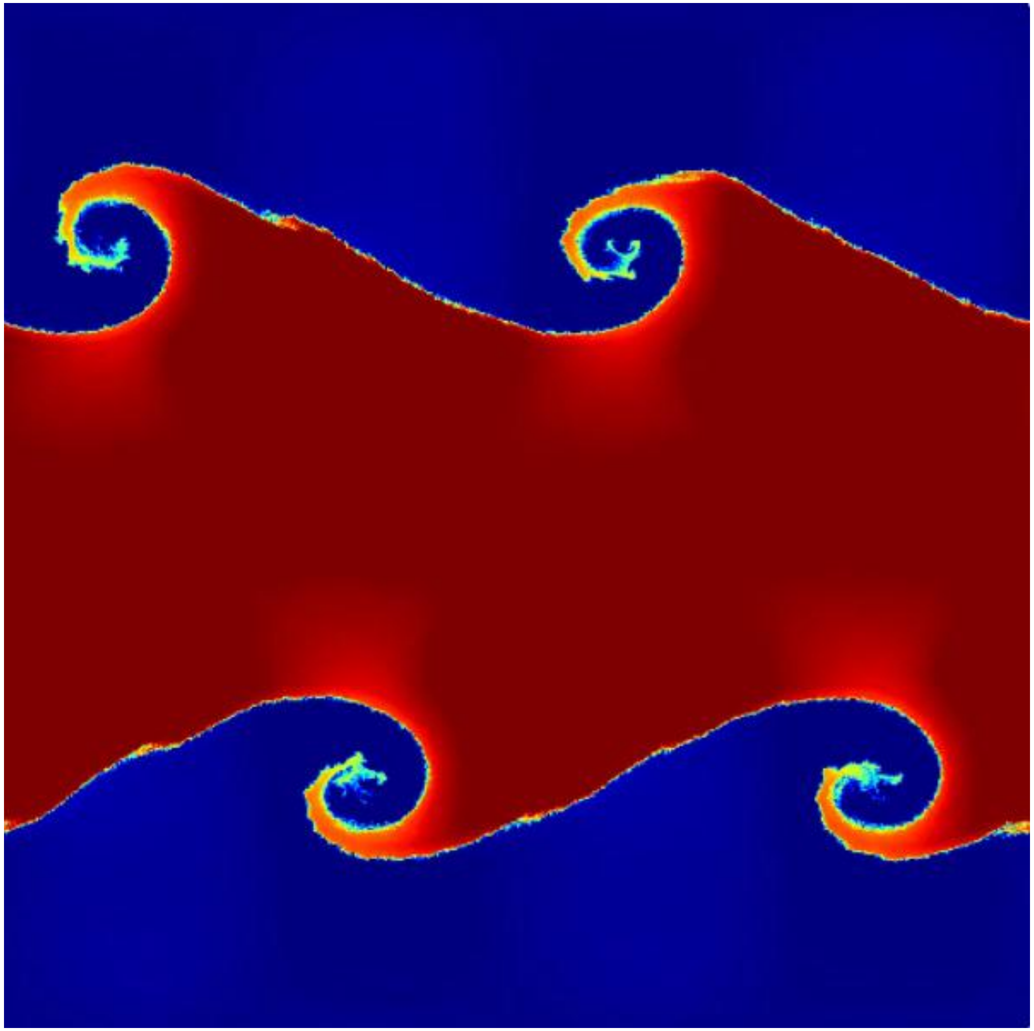
# … most cannot.

*https://www.pinterest.com/flywithcptjoe/pretty-flying-pictures/*



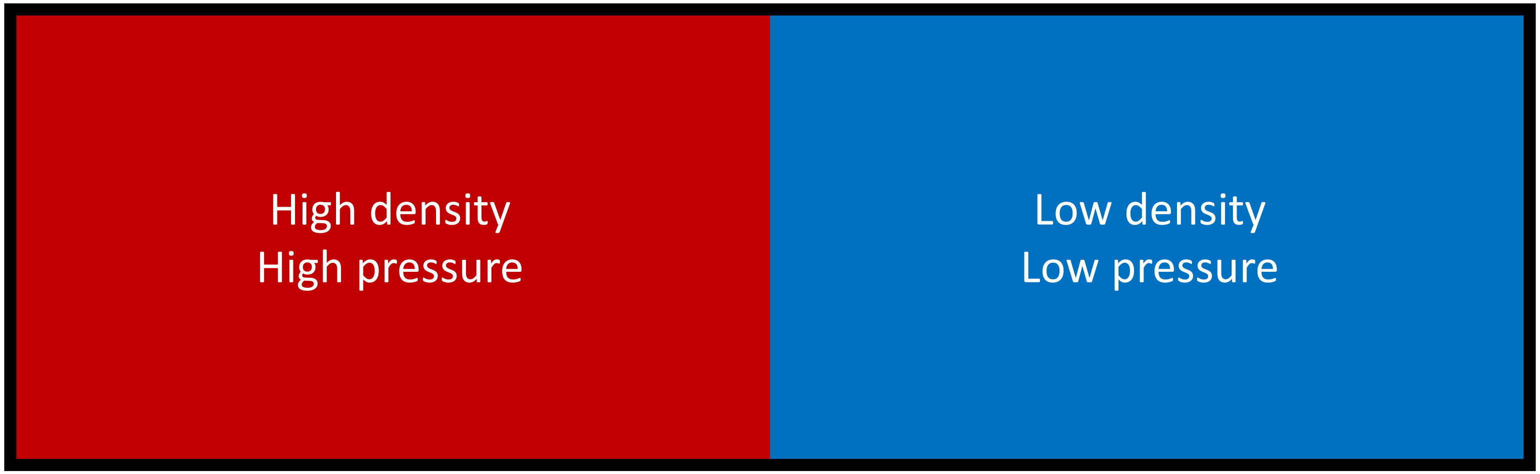
*X-ray: NASA/CXC/Rutgers/J.Hughes; Optical: NASA/STScI*

# We need numerical simulations.

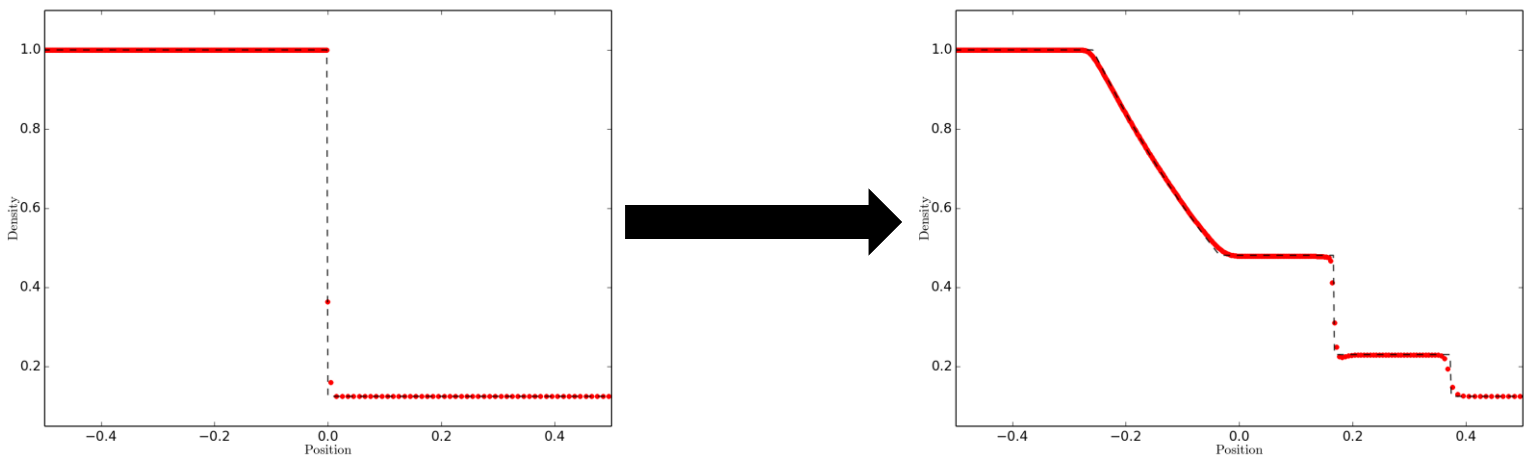


*https://gitlab.cosma.dur.ac.uk/swift/swiftsim*

# The 1D Sod shock tube

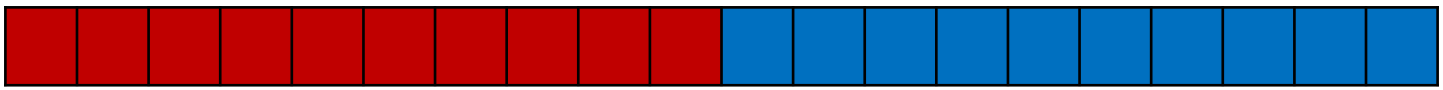



t = 0.0 t = 0.2



# Finite volume schemes: discretization in space

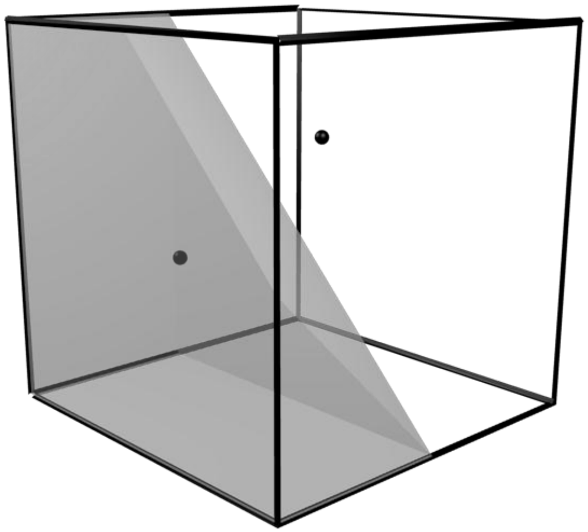
The 1D “volume” is subdivided into cells. Each cell holds a single value for the density, pressure, and fluid velocity.



We need to solve the equations of hydrodynamics for this discrete set of cells.

We will derive equations for a general, 3D, unstructured cell.

A 1D regular cell is just a special case of this more general cell shape.



# We start from the Euler equations:

Continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla}.\left(\rho \underline{u}\right) = 0}$$

Momentum equation

$$\boxed{\rho \frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}.\underline{\nabla})\underline{u} = -\underline{\nabla} p + \rho \underline{g}}$$

Energy equation

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \underline{\nabla}.\left(\frac{1}{2}\rho u^2 + \rho e\right)\underline{u} = -L - \underline{\nabla}.\left(p\underline{u}\right) - \rho \underline{u}.\underline{\nabla}\psi}$$

We will assume a *polytropic* equation of state: $p = (\gamma - 1)\rho e$

See lectures 4, 5, and 6.

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}.\left(\rho \underline{u}\right) = 0$$

We consider pure hydro, so:

- no gravity
- no energy sources or sinks

$$\rho \frac{\partial \underline{u}}{\partial t} + \boxed{\rho\left(\underline{u}.\underline{\nabla}\right)\underline{u}} = -\underline{\nabla} p + \cancel{\rho \underline{g}}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \underline{\nabla}.\left(\frac{1}{2}\rho u^2 + \rho e\right)\underline{u} = \cancel{\cdots} - \underline{\nabla}.\left(p\underline{u}\right) - \rho \underline{u}.\cancel{\underline{\nabla}\Phi}$$

$$\rho e_{tot}$$

$$\boxed{\rho\left(\underline{u}.\underline{\nabla}\right)\underline{u}} = \rho\underline{\nabla}.\left(\underline{u}\;\underline{u}\right) - \rho\underline{u}\left(\underline{\nabla}.\underline{u}\right)$$

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}.\left(\rho\underline{u}\right) = \frac{\partial\rho}{\partial t} + \rho\left(\underline{\nabla}.\underline{u}\right) + \underline{u}.\underline{\nabla}\rho = 0$$

$$\Longrightarrow \quad \underline{\nabla}.\underline{u} = -\frac{1}{\rho}\frac{\partial\rho}{\partial t} - \frac{1}{\rho}\underline{u}.\underline{\nabla}\rho$$

$$\rho\left(\underline{u}.\underline{\nabla}\right)\underline{u} = \rho\underline{\nabla}.\left(\underline{u}\;\underline{u}\right) + \underline{u}\frac{\partial\rho}{\partial t} + \underline{u}\;\underline{u}.\underline{\nabla}\rho$$

$$= \underline{\nabla}.\left(\rho\underline{u}\;\underline{u}\right) + \underline{u}\frac{\partial\rho}{\partial t}$$

$$\rho\frac{\partial \underline{u}}{\partial t}+\underline{\nabla}.\left(\rho\underline{u}\,\underline{u}\right)+\underline{u}\frac{\partial \rho}{\partial t}=-\underline{\nabla}p$$

Momentum equation'

$$\boxed{\frac{\partial \rho\underline{u}}{\partial t}+\underline{\nabla}.\left(\rho\underline{u}\,\underline{u}\right)=\boxed{-\underline{\nabla}p}}$$

$$\underline{\nabla}p=\frac{\partial p}{\partial x}\underline{i}+\frac{\partial p}{\partial y}\underline{j}+\frac{\partial p}{\partial z}\underline{k}$$

Note for a general tensor

$$\underline{\underline{A}}=A_{11}\underline{i}\,\underline{i}+A_{12}\underline{i}\,\underline{j}+A_{13}\underline{i}\,\underline{k}+\cdots$$

The divergence is given by

$$\underline{\nabla}.\underline{\underline{A}}=\left(\frac{\partial A_{11}}{\partial x}+\frac{\partial A_{21}}{\partial y}+\frac{\partial A_{31}}{\partial z}\right)\underline{i}+\cdots$$

Hence, we can also write the gradient of a scalar as a divergence of a tensor:

$$\underline{\nabla}p=\underline{\nabla}.\left(p\underline{\underline{1}}\right)=\underline{\nabla}.\begin{pmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{pmatrix}$$

Hence, we find:

$$\boxed{\frac{\partial \rho\underline{u}}{\partial t}+\underline{\nabla}.\left(\rho\underline{u}\,\underline{u}+p\underline{\underline{1}}\right)=0}$$

$$\boxed{\frac{\partial \rho \underline{u}}{\partial t} + \underline{\nabla}.\left(\rho \underline{u}\ \underline{u} + p\underline{\underline{1}}\right) = 0}$$

This is a *vector equation*, and is shorthand for:

$$\boxed{\begin{aligned}
&\frac{\partial \rho u_x}{\partial t} + \frac{\partial}{\partial x}(\rho u_x u_x) + \frac{\partial}{\partial y}(\rho u_y u_x) + \frac{\partial}{\partial z}(\rho u_z u_x) + \frac{\partial p}{\partial x} = 0 \\
&\frac{\partial \rho u_y}{\partial t} + \frac{\partial}{\partial x}(\rho u_x u_y) + \frac{\partial}{\partial y}(\rho u_y u_y) + \frac{\partial}{\partial z}(\rho u_z u_y) + \frac{\partial p}{\partial y} = 0 \\
&\frac{\partial \rho u_z}{\partial t} + \frac{\partial}{\partial x}(\rho u_x u_z) + \frac{\partial}{\partial y}(\rho u_y u_z) + \frac{\partial}{\partial z}(\rho u_z u_z) + \frac{\partial p}{\partial z} = 0
\end{aligned}}$$

# This gives us the Euler equations in *conservative form*:

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla}.\left(\rho \underline{u}\right) = 0}$$

$$\boxed{\frac{\partial \rho \underline{u}}{\partial t} + \underline{\nabla}.\left(\rho \underline{u}\,\underline{u} + p\underline{\underline{1}}\right) = 0}$$

$$\boxed{\frac{\partial \rho e_{tot}}{\partial t} + \underline{\nabla}.\left(\rho e_{tot}\underline{u} + p\underline{u}\right) = 0}$$

All equations together can be written in the more compact form

$$\frac{\partial U}{\partial t} + \underline{\nabla}.\underline{F}(U) = 0$$

with

$$U = \begin{pmatrix} \rho \\ \rho \underline{u} \\ \rho e_{tot} \end{pmatrix}, \quad \underline{F}(U) = \begin{pmatrix} \rho \underline{u} \\ \rho \underline{u}\,\underline{u} + p\underline{\underline{1}} \\ \rho e_{tot}\underline{u} + p\underline{u} \end{pmatrix}$$

## We call this a *conservation law*

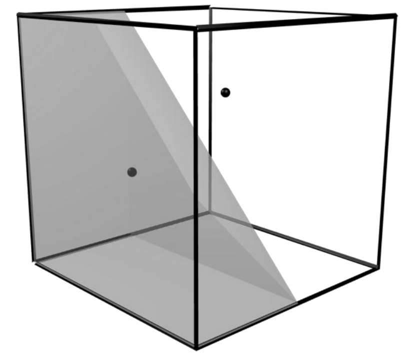
Note that the fluxes $\underline{F}(U)$ can be expressed in terms of the *primitive* variables:

- Density $\rho$

- Fluid velocity $\underline{u}$

- Pressure $p = (\gamma - 1)\rho e = (\gamma - 1)\left(\rho e_{tot} - \frac{1}{2}\rho u^2\right)$

# Conservation law?



Integrate the conservation law over an arbitrary volume $V$

$$\iiint \left( \frac{\partial U}{\partial t} + \underline{\nabla}.\underline{F}(U) \right) \mathrm{d}V = 0$$

Conserved quantities

We get if $V$ is independent of $t$

$$\iiint \frac{\partial U}{\partial t} \mathrm{d}V = \frac{\partial}{\partial t} \iiint U \mathrm{d}V = \frac{\partial}{\partial t} \iiint \begin{pmatrix} \rho \\ \rho \underline{u} \\ \rho e_{tot} \end{pmatrix} \mathrm{d}V \equiv \frac{\partial}{\partial t} \begin{pmatrix} m \\ m\underline{u} \\ E_{tot} \end{pmatrix} = \frac{\partial Q}{\partial t}$$

and if $\underline{F}(U)$ is continuous over $V$

$$\iiint \underline{\nabla}.\underline{F}(U) \mathrm{d}V = \iint \underline{F}(U).\mathrm{d}\underline{S}$$

$\underline{S}$ is the boundary surface of $V$

This means that the change of the conserved quantities $Q$ is given by the fluxes $\underline{F}(U)$ through the surface $\underline{S}$ of the volume $V$:

$$\frac{\partial Q}{\partial t} = -\iint \underline{F}(U).\mathrm{d}\underline{S}$$

Since this holds for an arbitrary volume, we can apply the same logic to a set of discrete cells $i$ with volumes $V_i$ and boundary surfaces $\underline{S_i}$:

$$\iiint \left(\frac{\partial U}{\partial t} + \underline{\nabla}.\underline{F}(U)\right) \mathrm{d}V_i = 0$$

to get:

$$\frac{\partial Q_i}{\partial t} = -\iint \underline{F}(U).\mathrm{d}\underline{S_i} \qquad\qquad Q_i = \begin{pmatrix} m_i \\ m_i\underline{u_i} \\ E_{tot,i} \end{pmatrix}$$

The change in conserved cell quantities $Q_i$ is given by the flux through the boundary surface of the cell.

Since an outflux in cell $i$ corresponds to an influx in another cell $j$, the entire integration scheme is reduced to an *exchange of fluxes* between cells.

*http://woolshed1.blogspot.co.uk/2010/02/sheep-yards-design-and-construction.html*