BLACKBOARD PROOFS

CSE202 – WEEK 5

This week, the proofs were mostly on the slides. Here are two more detailed explanations.

1. If f is increasing and $q \leq f(pn)/f(n)$ for all large enough n, then $n^{\log_p q} = O(f(n))$

Let n_0 be such that for all $n \geq n_0$, the inequality holds. If $n > pn_0$, there are two (not necessarily distinct) powers of p, n_1 and n_2 such that

$$n_1 p > n \geq n_1 \geq n_2 \geq n_0 > n_2/p.$$

Then using the inequality repeatedly,

$$f(n) \geq f(n_1) \geq q^{\log_p(n_1/n_2)} f(n_2) \geq q^{\log_p(n_1/n_2)} f(n_0),$$

with $n_1/n_2 > n/(pn_2) > n/(p^2 n_0)$. Thus $\log_p(n_1/n_2) > \log_p n - \log_p(p^2 n_0)$ and finally

$$f(n) > q^{\log_p n} \frac{f(n_0)}{q^{\log_p(p^2 n_0)}} = n^{\log_p q} \frac{f(n_0)}{q^{\log_p(p^2 n_0)}}.$$

Thus we have obtained that for $n > pn_0$

$$n^{\log_p q} < C f(n),$$

with $C = q^{\log_p(p^2 n_0)}/f(n_0)$, which shows that

$$n^{\log_p q} = O(f(n)).$$

2. Not larger

In the proof of the Master Theorem it is stated that $O(N^{\log_p m})$ is *not larger* than

$$A(N) := f(N) \times \begin{cases} O(1), & \text{if } q > m, \\ \log_p N, & \text{if } q = m, , \\ O(N^{\log_p(m/q)}), & \text{if } q < m. \end{cases}$$

which means that $N^{\log_p m} = O(A(N))$, so that $O(N^{\log_p m}) + O(A(N)) = O(A(N))$. This is a consequence of the previous assertion, namely $N^{\log_p q} = O(f(N))$. Indeed, when $q \geq m$, we have $\log_p m \leq \log_p q$ and the conclusion hold. Otherwise, writing $N^{\log_p m} = N^{\log_p(q)} N^{\log_p(m/q)}$ gives the result.