BLACKBOARD PROOFS

CSE202 – WEEK 2

Here are details on two more steps of the complexity analysis of Karatsuba's algorithm. They would work equally well with the naive divide-and-conquer algorithm for multiplication and will pop up in all our proofs of complexity for divide-and-conquer algorithms.

1. Iterating the inequality

The starting point of the analysis is

$$C(n) \leq 3C(\lceil n/2 \rceil) + \lambda n, \tag{1}$$

which is valid for all $n > 1$. In particular, if $\lceil n/2 \rceil > 1$, then this inequality applies to $\lceil n/2 \rceil$, leading to

$$C(\lceil n/2 \rceil) \leq 3C(\lceil \lceil n/2 \rceil /2 \rceil) + \lambda \lceil n/2 \rceil = 3C(\lceil n/2 \rceil_2) + \lambda \lceil n/2 \rceil,$$

where the last equality comes from our definition of $\lceil n/2 \rceil_k$. Finally, using this upper bound for $C(\lceil n/2 \rceil)$ in the right-hand side of Inequality (1) yields our second inequality:

$$C(n) \leq 3\left(3C(\lceil n/2 \rceil_2) + \lambda \lceil n/2 \rceil\right) + \lambda n = \lambda n + 3\lambda \lceil n/2 \rceil + 9C(\lceil n/2 \rceil_2).$$

2. Exponent in the complexity

The sequence of inequalities reaches

$$C(n) \leq (2\lambda + 1)3^{\lceil \log_2 n \rceil}.$$

The final step is as follows:

$$3^{\lceil \log_2 n \rceil} \leq 3^{\log_2 n + 1} = 3 \exp\left(\frac{\log n \log 3}{\log 2}\right) = 3n^{\log 3/\log 2} = 3n^{\log_2 3},$$

where we use nothing more that the definitions of power and logarithm

$$a^x = \exp(x \log a) \qquad \log_k a = \log a / \log k.$$