MATH 201, SUMMER 2015, TEST 2

Name: Solutions.

Student Number:

Mark: / 30

1) [6 Marks] Find the general solution to the following differential equations.

a) [3 Marks]

$$y'' - 5y' + 6y = 0$$

Auxiliary equation:

$m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

so $m = 2, 3$

Thus $y = c_1 e^{2x} + c_2 e^{3x}$ on $(-\infty, \infty)$

b) [3 Marks]

$$y'' + 6y' + 9y = 0$$

$m^2 + 6m + 9 = (m+3)^2 = 0$

so $m = -3$ twice

Thus, $y = c_1 e^{-3x} + c_2 x e^{-3x}$ on $(-\infty, \infty)$

2) [6 Marks] Find the general solution to the following differential equation.

$$y'' - 4y = 8e^{2x}$$

Complementary:

$m^2 - 4 = 0 \implies (m+2)(m-2) = 0$ so $m = \pm 2$

$y_c = c_1 e^{2x} + c_2 e^{-2x}$

$y_p = Ae^{2x}$ but this appears as a complementary solution

so take

$y_p = Axe^{2x}$, $y_p' = Ae^{2x} + 2Axe^{2x}$

$y_p'' = 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x}$

Subbing into the D.E.:

$4Ae^{2x} + \cancel{4Axe^{2x}} - \cancel{4Axe^{2x}} = 8e^{2x}$

Thus, $A = 2$.

The general solution is then

$y = c_1 e^{2x} + c_2 e^{-2x} + 2xe^{2x}$ on $(-\infty, \infty)$

3) [6 Marks] Find the general solution to the following differential equation.

$$x^2y'' - xy' + y = 2x$$

This is Cauchy-Euler so assume $y_c = x^m$ for the complementary:

$m(m-1) - m + 1 = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$ so $m = 1$ twice

Thus, $y_c = c_1 x + c_2 x \ln|x|$ Pick an interval $\boxed{(0,\infty).}$ $(-\infty, 0)$ or

For y_p, put the D.E. into standard form:

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{2}{x}$$

$$W = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x\ln x + x - x\ln x = x$$

$$W_1 = \begin{vmatrix} 0 & x\ln x \\ \frac{2}{x} & \ln x + 1 \end{vmatrix} = -2\ln x$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & \frac{2}{x} \end{vmatrix} = 2$$

$u_1' = \frac{-2\ln x}{x}$

$u_2' = \frac{2}{x}$

$u_1 = -2\int \frac{\ln x}{x}\,dx$ $\quad u = \ln x$, $du = \frac{1}{x}dx$

$= -2\int u\,du$ don't need

$= -u^2 \boxed{+C} = -(\ln x)^2 \boxed{+C}$

$u_2 = \int \frac{2}{x}\,dx = 2\ln x + C$

Thus $y_p = -x(\ln x)^2 + 2x(\ln x)^2$

$= x(\ln x)^2$

Therefore, the general solution is $y = c_1 x + c_2 x \ln x + x(\ln x)^2$ on $(0,\infty)$

4) [6 marks] Find **a** nontrivial solution to the following non-linear differential equation. [Hint: Since you are only looking for **a** solution, you can pick your constants "conveniently"...]

$$y'' + 2y(y')^3 = 0$$

No independent variable, so set $u = y'$ with u considered a function of y. $y'' = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}u$.

The D.E. becomes

$$\frac{du}{dy}u + 2yu^3 = 0$$

$$\Rightarrow \int \frac{du}{u^2} = \int -2y\,dy \Rightarrow \frac{-1}{u} = -y^2 + c \qquad \text{set } c = 0.$$

$$\Rightarrow u = \frac{1}{y^2} \Rightarrow \frac{dy}{dx} = \frac{1}{y^2} \Rightarrow \int y^2 dy = \int dx$$

$$\Rightarrow \frac{y^3}{3} = x + c_2 \qquad \text{set } c_2 = 0.$$

$$y = \sqrt[3]{3x}.$$

No need to provide an interval in non-linear questions

5) [6 marks] Use the Laplace Transform to solve the following initial value problem. No other method of solution will receive any credit.

$$y' - y = 1 \qquad y(0) = 0$$

$$sY(s) - y(0) - Y(s) = \frac{1}{s}$$

$$Y(s)(s-1) = \frac{1}{s}$$

$$Y(s) = \frac{1}{s(s-1)} \qquad \frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1}$$

$$1 = A(s-1) + Bs$$

$$B = 1 \quad A = -1$$

Thus,

$$Y(s) = \frac{1}{s-1} - \frac{1}{s}$$ & so

$$\boxed{y(t) = e^{t} - 1}$$ on $(-\infty, \infty)$.