Math 201, Assignment 4
Due at the beginning of tutorial on July 15, 2015
Illegible or disorganized solutions will receive no credit! Please, for the sake of our marker, be neat!

1) Evaluate the following Laplace Transforms using the table in the textbook.

   a)
   $$\mathscr{L}\{t^3 - sinh(2t)\}$$

   b)
   $$\mathscr{L}\{(t-2)^2 e^{4t}\}$$

   c)
   $$\mathscr{L}\{sin^2(kt)\}$$

   d)
   $$\mathscr{L}\{x(t)\}$$

   where

   $$x(t) = \begin{cases} 1 & 0 \leq t < 2 \\ 2 & 2 \leq t < 4 \\ 0 & t \geq 4 \end{cases}$$

2) Show that all bounded functions are of exponential order.

3) Evaluate the following inverse Laplace Tranforms

   a)
   $$\mathscr{L}^{-1}\left\{\frac{s^2}{(s+1)^3}\right\}$$

   b)
   $$\mathscr{L}^{-1}\left\{\frac{1}{s^2+4s+10}\right\}$$

4) Use the Laplace Transform to solve the following initial value problem.

$$y'' + 4y = e^{-t}, \quad y(0) = 2,\ y'(0) = 1$$

Solutions:

1) a)

$$\mathscr{L}\{t^3 - sinh(2t)\} = \mathscr{L}\{t^3\} - \mathscr{L}\{sinh(2t)\} = \frac{3!}{s^4} - \frac{2}{s^2-4}$$

b)

$$\mathscr{L}\{(t-2)^2 e^{4t}\} = \mathscr{L}\{t^2e^{4t} - 4te^{4t} + 4e^{4t}\} = \frac{2}{(s-4)^3} - 4\frac{1}{(s-4)^2} + 4\frac{1}{s-4}$$

c)

$$\mathscr{L}\{sin^2(kt)\} = \mathscr{L}\{\frac{1}{2}(1-cos(2kt)\} = \frac{1}{2s} - \frac{s}{2(s^2+4k^2)} = \frac{s^2+4k^2-s^2}{2s(s^2+4k^2)} = \frac{2k^2}{s(s^2+4k^2)}$$

d) First, we write $x(t)$ using the unit step function, $\mathscr{U}(t-a)$:

$$x(t) = 1 + \mathscr{U}(t-2) - 2\mathscr{U}(t-4)$$

Then,

$$\mathscr{L}\{x(t)\} = \frac{1}{s} + \frac{e^{-2s}}{s} - 2\frac{e^{-4s}}{s}$$

2) Recall that a function $f(x)$ is bounded if there exists an B such that $|f(x)| \leq B$ for all x. Pick $c = 0$, $T = 0$ and $M = B$. Then $|f(x)| \leq B = Me^{0t}$ for all $t > 0$ and so $f(x)$ is of exponential order.

3) a) We need to use partial fractions to break up the quotient.

$$\frac{s^2}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3}$$

$$\implies s^2 = A(s+1)^2 + B(s+1) + C \implies s^2 = As^2 + (2A+B)s + A + B + C$$

Thus, $A = 1$, $B = -2$, and $C = 1$ and so

$$\mathscr{L}^{-1}\left\{\frac{s^2}{(s+1)^3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{2}{2(s+1)^3}\right\}$$
$$= e^{-t} - 2te^{-t} + \frac{1}{2}t^2e^{-t}$$

b)

$$\frac{1}{s^2+4s+10} = \frac{1}{s^2+4s+4+6} = \frac{1}{(s+2)^2+6}$$

If $F(s) = \frac{1}{s^2+6}$, then $F(s+2) = \frac{1}{(s+2)^2+6}$. We know that $\mathscr{L}^{-1}\{F(s)\} = \frac{1}{\sqrt{6}} sin(\sqrt{6}t)$ and so by the First Translation Theorem,

$$\mathscr{L}^{-1}\{F(s+2)\} = e^{-2t}\frac{1}{\sqrt{6}} sin(\sqrt{6}t)$$

4) Taking the Laplace Transform of both sides gives

$$s^2F(s) - sy(0) - y'(0) + 4F(s) = \frac{1}{s+1} \implies s^2F(s) - 2s - 1 + 4F(s) = \frac{1}{s+1}$$

$$\implies F(s) = \frac{1}{(s+1)(s^2+4)} + \frac{2s+1}{s^2+4}$$

We need to now use a partial fractions decomposition of the first quotient.

$$\frac{1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4} \implies 1 = As^2 + 4A + (Bs+C)(s+1)$$

$$\implies 1 = (A+B)s^2 + (B+C)s + 4A + C$$

and so $4A + C = 1$, $B + C = 0$ and $A + B = 0$. Thus, $A = C = \frac{1}{5}$ and $B = -\frac{1}{5}$. Thus,

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\{\frac{\frac{1}{5}}{s+1} + \frac{-\frac{1}{5}s}{s^2+4} + \frac{\frac{1}{5}}{s^2+4} + \frac{2s}{s^2+4} + \frac{1}{s^2+4}\} \\ &= \mathscr{L}^{-1}\{\frac{\frac{1}{5}}{s+1}\} + \mathscr{L}^{-1}\{\frac{\frac{9}{5}s}{s^2+4}\} + \mathscr{L}^{-1}\{\frac{\frac{6}{5}}{s^2+4}\} \\ &= \frac{1}{5}e^{-t} + \frac{9}{5}cos(2t) + \frac{3}{5}sin(2t) \qquad \text{on } (-\infty, \infty) \end{aligned}$$