Math 201, Assignment 5 Solutions

1) Solve the following IVP:

$$y'' + 6y' + 5y = \delta(t-2), \qquad y(0) = 0, \quad y'(0) = 0$$

2) Find the Laplace Transform of the periodic triangular wave, given by:

$$f(t) = 2t, \;\; 0 \le t < 2 \text{ and } f(t+2) = f(t)$$

3) Solve the integral equation:

$$y(t) = t + \int_0^t y(x)dx + \int_0^t (t-x)y(x)dx$$

4) Use the convolution theorem to find

$$\mathscr{L}^{-1}\left\{\frac{s^2}{(s^2+1)^2}\right\}$$

5) Use a power series about the point $x = 0$ to solve the following differential equation:

$$y'' - xy' - y = 0$$

Obtain a recursion formula for the coefficients and write the first 3 nonzero terms of the power series of each of the two linearly independent solutions. Include the interval of convergence of the power series.

Solutions

1) Applying the Laplace Transform to both sides, we get

$$s^2Y(s) - sy(0) - y'(0) + 6(sY(s) - y(0)) + 5Y(s) = e^{-2s}$$
$$\implies Y(s) = \frac{e^{-2s}}{s^2 + 6s + 5}$$

Here we can either use partial fractions or complete the square. I'll show completing the square since sometimes partial fractions is not possible (when the bottom is irreducible like $x^2 + x + 1$). Also, it will use more of the Theorems that you need to be comfortable using. Completing the square we get that

$$Y(s) = \frac{e^{-2s}}{(s+3)^2 - 4}$$

In the table of Laplace Transforms in the back of the book (which you will have during your exam), we see that

$$\mathscr{L}\{sinh(kt)\} = \frac{k}{s^2 - k^2}$$

There are a few things to take care of here. We need to apply the First and Second Translation Theorems. First, by the Second Translation Theorem,

$$\mathscr{L}^{-1}\{e^{-2s}F(s)\} = f(t-2)\mathscr{U}(t-2)$$

and so we just need to calculate

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+3)^2 - 4}\right\}$$

and then apply this theorem.

We use the first Translation Theorem,

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

to calculate this inverse:

$$\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s+3)^2-4}\right\} = \frac{1}{2}e^{-3t}sinh(2t)$$

Thus,

$$\mathscr{L}^{-t}\left\{\frac{e^{-2s}}{(s+3)^2-4}\right\} = \frac{1}{2}e^{-3(t-2)}sinh(2(t-2))\mathscr{U}(t-2) \quad t \geq 0.$$

In the exam, you can leave it as above, but the solution is easier to understand as written below in piecewise form:

$$y(t) = \begin{cases} 0 & 0 \leq t < 2 \\ \frac{1}{2}e^{-3(t-2)}sinh(2(t-2)) & t \geq 2 \end{cases}$$

This looks different than the solution you would get using partial fractions, but you could recover it by using $sinh(x) = \frac{e^x - e^{-x}}{2}$.

2) Applying the theorem for the Laplace Transform of periodic functions, we get that

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 2te^{-st}dt$$

Using integration by parts, we have that

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \frac{2}{1-e^{-2s}} - \frac{t}{s}e^{-st}|_0^2 + \int_0^2 \frac{1}{s}e^{-st}dt \\ &= \frac{2}{1-e^{-2s}}\left\{-\frac{t}{s}e^{-st} - \frac{e^{-st}}{s^2}\right\}_0^2 \\ &= \frac{2}{1-e^{-2s}}\left\{-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2}\right\} \\ &= \frac{2}{1-e^{-2s}}\left\{\frac{1}{s^2}(1-e^{-2s}) - \frac{2e^{-2s}}{s}\right\} \\ &= \frac{2}{s^2} - \frac{4e^{-2s}}{s(1-e^{-2s})}\end{aligned}$$

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3) Taking the Laplace Transform of both sides we get that

$$Y(s) = \frac{1}{s^2} + \frac{1}{s}Y(s) + \frac{1}{s^2}Y(s)$$

$$\implies s^2Y(s) = 1 + sY(s) + Y(s) \implies Y(s) = \frac{1}{s^2 - s + \frac{1}{4} - \frac{5}{4}}$$

$$\implies Y(s) = \frac{1}{(s-\frac{1}{2})^2 - \frac{5}{4}} = \frac{2}{\sqrt{5}} \frac{\frac{\sqrt{5}}{2}}{(s-\frac{1}{2})^2 - \frac{5}{4}}$$

and so using the First Translation Theorem we get that

$$y(t) = \frac{2}{\sqrt{5}} e^{\frac{1}{2}t} sinh(\frac{\sqrt{5}}{2}t) \quad t > 0$$

4)

$$\mathscr{L}^{-1}\left\{\frac{s^2}{(s^2+1)^2}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{(s^2+1)}\right\} * \mathscr{L}^{-1}\left\{\frac{s}{(s^2+1)}\right\}$$
$$= cos(t) * cos(t) = \int_0^t cos(\tau)cos(t-\tau)d\tau$$

Now we'll use the identity $cos(A)cos(B) = \frac{1}{2}(cos(A+B) + cos(A-B))$ and so our integral becomes

$$\int_0^t \frac{1}{2}(cos(\tau + (t-\tau)) + cos(\tau - (t-\tau))dt$$

$$= \frac{1}{2}\int_0^t cos(t) + cos(2\tau - t)dt = \frac{1}{2}\{\tau cos(t) + \frac{1}{2}sin(2\tau - t)\}_0^t$$
$$= \frac{1}{2}tcos(t) + \frac{1}{4}sin(t) - \frac{1}{4}sin(-t) = \frac{1}{2}(tcos(t) + sin(t))$$

5) We assume a power series solution, $y = \sum_{n=0}^{\infty} c_n x^n$. Since $P(x) = -x$ is a polynomial, and $Q(x) = -1$ is constant, both of which are analytic on $\mathbb{R}$, there are no singular points. Thus, using Theorem 6.2.1 from text, we know the series converges on at least $(-\infty, \infty)$, and since this

is the largest possible interval, it must be the interval of convergence. The derivatives of $y = \sum_{n=0}^{\infty} c_n x^n$ are

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} y'' \qquad = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

and so substituting these into the D.E., we get

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

, or after multiplying the x into the second sum termwise,

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^{n} - \sum_{n=0}^{\infty} c_n x^n = 0.$$

Next, we observe that the first and last sums start with an x^0 term, while the second sum starts with an x^1 term. We need all the sums to start with the same power of x before we can combine them, so we pull off the first term from the first and last sums, giving

$$c_2(2)(1)x^0 + \sum_{n=3}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^{n} - c_0 x^0 - \sum_{n=1}^{\infty} c_n x^n = 0.$$

Now all the sums start with the same power of x, but the first one has x written as powers in $n-2$, which we need to change to k. The last two sums are already in the proper form, so we just set $n = k$ for both of them. This gives:

$$c_2(2)(1)x^0 - c_0 x^0 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_k k x^k - \sum_{k=1}^{\infty} c_k x^k = 0.$$

Combining the sums together, we get

$$2c_2 - c_0 + \sum_{k=1}^{\infty} [c_{k+2}(k+2)(k+1) - c_k k - c_k] x^k = 0.$$

Now we can use the identity property, to conclude that

$$\begin{aligned} &c_2 = \frac{c_0}{2}, \\ &c_{k+2}(k+2)(k+1) - c_k(k+1) = 0 \implies c_{k+2} = \frac{c_k}{k+2} \quad k \geq 1 \end{aligned}$$

Thus, using this recursive relation, we find

$$\begin{aligned} k = 1: &\quad c_3 = \frac{c_1}{3} \\ k = 2: &\quad c_4 = \frac{c_2}{4} = \frac{c_0}{(4)(2)} \\ k = 2: &\quad c_5 = \frac{c_3}{5} = \frac{c_1}{(5)(3)} \end{aligned}$$

Thus, collecting all the terms with c_0, and c_1 respectively, we get that

$$y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{(4)(2)}x^4 + \cdots$$

and

$$y_2 = x + \frac{1}{3}x^3 + \frac{1}{(5)(3)}x^5 + \cdots$$

with the general solution being $y = c_0 y_1 + c_1 y_2$ on $(-\infty, \infty)$.