Math 201, Assignment 2
Due at the beginning of tutorial on June 10, 2015
Illegible or disorganized solutions will receive no credit! Please, for the sake of our marker, be neat!

1) Use a substitution to solve the following differential equation.

$$\frac{dy}{dx} = 3x + y + 9x^2 + 6xy + y^2 - 15$$

2) Use a substitution to solve the following differential equation.

$$\frac{dx}{dt} + \frac{x}{t} = 3x^3$$

3) Use a substitution to solve the following differential equation.

$$xy' = xe^{-y/x} + y \quad x > 0$$

4) Solve the following "almost exact" differential equation by first finding an integrating factor.

$$(2xy^3 + y^4)dx + (xy^3 - 2)dy$$

Solutions

1) The right hand side of this DE is in the form f(Ax+By+C) where A=3, B=1 and C=0, and so we make the substitution u=3x+y. Then, $\frac{du}{dx} = 3 + \frac{dy}{dx}$ and so we get the separable equation

$$\frac{du}{dx} = u + u^2 - 12 \implies \int \frac{du}{(u+4)(u-3)} = \int dx$$

We use partial fractions in order to compute the left integral:

$$\frac{1}{(u+4)(u-3)} = \frac{A}{u+4} + \frac{B}{u-3} \implies A + B = 0 \text{ and } -3A + 4B = 1$$

Thus, $A = -\frac{1}{7}$ and $B = \frac{1}{7}$ and our equation becomes

$$-\frac{1}{7}\int \frac{du}{u+4} + \frac{1}{7}\int \frac{du}{u-3} = x + c_1 \implies ln(\frac{|u-3|}{|u+4|}) = 7x + c_2$$

$$\implies \frac{|u-3|}{|u+4|} = c_3 e^{7x} \text{ where } c_3 > 0$$

(You'll notice that $u = 3$ and $u = -4$ are equilibrium solutions to this D.E. and so we know that any one solution will be confined to one of the three regions, $u < -4$, $-4 < u < 3$ and $3 < u$. In each of these regions, $\frac{u-3}{u+4}$ is either strictly positive or strictly negative and so we are justified in letting c_4 be +/- c_3.)

$$\frac{u-3}{u+4} = c_3 e^{7x} \implies u - 3 = (u+4)c_3 e^{7x}$$

$$\implies u(1 - c_3 e^{7x}) = 3 + 4c_3 e^{7x} \implies u = \frac{3 + 4c_3 e^{7x}}{1 - c_3 e^{7x}}$$

Substituting back in for $u = 3x + y$ gives

$$y = \frac{3 + 4c_3e^{7x}}{1 - c_3e^{7x}} + 3x$$

I don't expect you to provide an interval for such D.E.'s.

2) This is a Bernoulli Equation with $n = 3$ and so we make the substitution $u = x^{1-3} = x^{-2}$, or $x = u^{-1/2}$. Then

$$\frac{dx}{dt} = -\frac{1}{2}u^{-3/2}\frac{du}{dt}$$

and our DE becomes

$$-\frac{1}{2}u^{-3/2}\frac{du}{dt} + \frac{u^{-1/2}}{t} = 3u^{-3/2}$$

$$\implies u' - \frac{2u}{t} = -6$$

which is the standard form of a linear equation in u. An integrating factor for this equation is

$$e^{-\int(2/t)dt} = t^{-2}$$

where we have omitted the absolute value since t is squared. Our DE becomes

$$\frac{d}{dt}(t^{-2}v) = -6t^{-2}$$

and so

$$t^{-2}v = 6t^{-1} + c$$

and since $v = x^{-2}$ we have

$$x^2 = \frac{1}{6t + ct^2}$$

This is an implicit solution, and yields two possible explicit solutions,

$$x = \sqrt{\frac{1}{6t + ct^2}} \quad \text{and} \quad x = -\sqrt{\frac{1}{6t + ct^2}}$$

I don't expect you to provide an interval here.

3) This equation is homogenous of order 0 since

$$y' = e^{-ty/tx} + \frac{ty}{tx} = e^{-y/x} + \frac{y}{x}$$

and so we make the substitution $y = vx$ and so $y' = v + xv'$ and the given equation becomes

$$v + xv' = e^{-v} + v \implies xv' = e^{-v}$$

Separating variables gives

$$e^v dv = dx/x$$

Integrating, we get

$$e^v = ln|x| + c \implies e^{y/x} = lnx + c$$

where we drop the absolute value sign since it is given that $x > 0$. An implicit solution here is fine, and again no expectation of providing an interval here.

4)
$$\frac{N_x - M_y}{M} = \frac{y^3 - 6xy^2 - 4y^3}{2xy^3 + y^4} = \frac{-3y^3 - 6xy^2}{2xy^3 + y^4} = \frac{-3}{y}$$

and so our integrating factor is $e^{\int \frac{-3}{y} dy} = y^{-3}$ (it is ok to drop absolute values when finding integrating factors).

Our DE then becomes

$$(2x + y)dx + (x - 2y^{-3})dy$$

which is easily verified to be exact.

Thus,

$$f(x, y) = \int 2x + y dx + g(y) = x^2 + xy + g(y)$$

and

$$x - 2y^{-3} = N(x, y) = \frac{\partial f}{\partial y} = x + g'(y)$$

and so $g'(y) = -2y^{-3}$ and $g(y) = y^{-2}$.
Thus $f(x, y) = x^2 + xy + y^{-2}$
and so the implicit solution is $x^2 + xy + y^{-2} = c$