Math 201, Assignment 3
Due at the beginning of tutorial on June 24, 2015
Illegible or disorganized solutions will receive no credit! Please, for the sake of our marker, be neat!

1) Solve the following homogeneous equations

   a)
   $$y'' + 3y' + 2y = 0$$

   b)
   $$y''' = 0$$

   c)
   $$y^{(4)} + 8y'' + 16y = 0$$

2) Solve the following IVP

$$y'' + 2y' + 4y = 0 \quad y(0) = 1, \quad y'(0) = -1 + 2\sqrt{3}$$

3) When a spring is stretched or compressed, its restoring force is directly proportional to its change in length. If x represents the displacement of the weight from its equilibrium position, then by Hooke's Law

$$F = -kx \quad k > 0$$

where the minus sign indicates that the restoring force F is always opposite in direction to the displacement. Combining this with Newton's Second Law gives,

$$m\frac{d^2x}{dt^2} = -kx$$

or

$$m\frac{d^2x}{dt^2} + kx = 0$$

   a) If the spring is initially displaced to $x(0) = 5$ and it's initial velocity is zero (i.e. $x'(0) = 0$), find $x(t)$ for $t > 0$.

b) Sketch your solution in a).

c) This approximation is decent, but doesn't really reflect what happens in real life; in reality, the amplitude of the oscillations decreases until eventually the mass is at rest. Thus, we need to include another term which acts as air resistance. At slow speeds, the force of air resistance on an object is well approximated by $F_a = -c\frac{dx}{dt}$, where $c > 0$ and $\frac{dx}{dt}$ is the speed of the object. Thus our differential equation becomes

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Assume $c^2 < 4mk$ and find $x(t)$ in terms of c, m and k assuming the same initial conditions as in a).

d) Sketch your solution in c).

4) Find the general solution of

$$y''' + 4y' = x^2 + sinx$$

Solutions

1) a) The auxiliary equation is $m^2 + 3m + 2 = 0$ and we can see that it factors as $(m + 2)(m + 1)$ and so the roots are $m = -1$ and $m = -2$. Thus, the solution is

$$y = c_1e^{-x} + c_2e^{-2x} \quad \text{on } (-\infty, \infty)$$

b) The auxiliary equation is $m^3 = 0$ and so $m = 0$ is a triple root. Thus, the solution is

$$y = c_1 + c_2x + c_3x^2 \quad \text{on } (-\infty, \infty)$$

c) The auxiliary equation is $m^4 + 8m^2 + 16 = 0$ which factors as $(m^2 + 4)^2 = 0$. Thus, $m = \pm 2i$ are both double roots. Thus, the solution is

$$y = c_1cos(2x)+c_2sin(2x)+c_3xcos(2x)+c_4xsin(2x) \quad \text{on } (-\infty, \infty)$$

---

2) The characteristic equation is $m^2 + 2m + 4 = 0$ and so using the quadratic formula we get that

$$m = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i$$

and so the general solution is

$$y = e^{-x}(c_1 cos(\sqrt{3}x) + c_2 sin(\sqrt{3}x) \quad \text{on } (-\infty, \infty)$$

We use the initial conditions to get that

$$1 = y(0) = c_1 \quad \text{and} \quad -1 + 2\sqrt{3} = y'(0)$$

We find

$$\begin{aligned} y'(x) &= -e^{-x}(c_1 cos(\sqrt{3}x) + c_2 sin(\sqrt{3}x)) + e^{-x}(-c_1\sqrt{3}sin(\sqrt{3}x) + c_2\sqrt{3}cos(\sqrt{3}x)) \\ &= e^{-x}((-c_1 + c_2\sqrt{3})cos(\sqrt{3}x + (-c_2 - c_1\sqrt{3})sin(\sqrt{3}x)) \end{aligned}$$

and so $c_2 = 2$.

Thus, the solution is

$$y = e^{-x}(cos(\sqrt{3}x) + 2sin(\sqrt{3}x)) \quad \text{on } (-\infty, \infty)$$

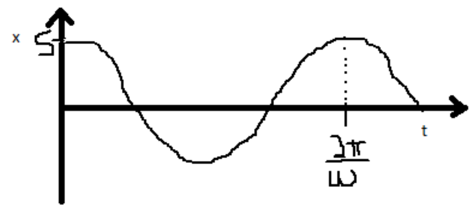
3) a) We are solving $x'' + \frac{k}{m}x = 0$ and so the auxiliary equation is $t^2 + \omega^2 = 0$ where $\omega^2 = \frac{k}{m}$. Thus, the general solution is

$$x = c_1 cos(\omega t) + c_2 sin(\omega t)$$

Since $x(0) = 5$, we get that $c_1 = 5$. $x'(0) = 0$ gives that $c_2 = 0$. Thus, our solution is

$$x(t) = 5cos(\omega t) \quad \text{for } t > 0$$

b) $x(t) = 5cos(\omega t)$



c) The auxiliary equation is now

$$s^2 + \frac{c}{m}s + \frac{k}{m} = 0$$

and so the quadratic formula gives us that

$$s = \frac{-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}}}{2}$$

Since $c^2 < 4mk$, we have that $\frac{c^2}{m^2} < 4\frac{k}{m}$ and so the expression under the square root is negative. Thus, the roots are

$$s = \frac{-\frac{c}{m} \pm i\sqrt{4\frac{k}{m} - \frac{c^2}{m^2}}}{2}$$

and so the general solution is

$$x = e^{-\frac{c}{2m}t}(c_1cos((\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}})t) + c_2sin((\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}})t))$$

Let $-\frac{c}{2m} = \alpha$ and $\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} = \beta$ so our general solution becomes

$$x = e^{-\alpha t}(c_1cos(\beta t) + c_2sin(\beta t))$$

The initial conditions give us that $c_1 = 5$. To find c_2, we need to calculate $x'(t)$ in order to use $x'(0) = 0$.

$$x' = -\alpha e^{-\alpha t}(5cos(\beta t) + c_2 sin(\beta t)) + e^{-\alpha t}(-5\beta sin(\beta t) + c_2\beta cos(\beta t))$$
$$= e^{-\alpha t}((-5\alpha + c_2\beta)cos(\beta t) + (-c_2\alpha - 5\beta)sin(\beta t))$$

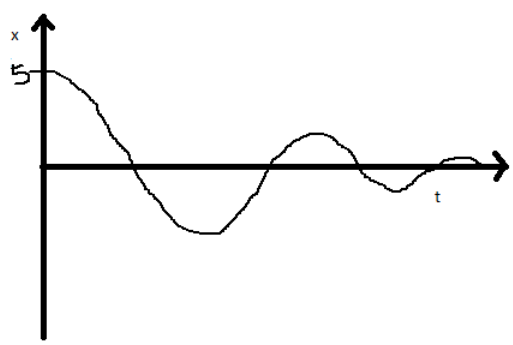
$$0 = x'(0) = -5\alpha + c_2\beta \implies c_2 = 5\frac{\alpha}{\beta}$$

Thus the solution is

$$x = e^{-\alpha t}(5cos(\beta t) + 5\frac{\alpha}{\beta}(sin(\beta t)) \quad \text{for } t > 0$$

d)

$$x = e^{-\alpha t}(5cos(\beta t) + 5\frac{\alpha}{\beta}(sin(\beta t)) \quad \text{for } t > 0$$



4) We solve the homogeneous case first and the auxiliary equation is $m^3 + 4m = m(m^2 + 4) = 0$. Thus, $m = 0$ or $m = \pm 2i$ and so the complementary solution is

$$y_c = c_1 + c_2cos(2x) + c_3sin(3x)$$

We'll use undetermined coefficients to find a particular solution, y_p. From the table, we assume a solution of the form

$$y_p = Asinx + Bcosx + Cx^2 + Dx + E$$

However, this form duplicates the constant solution of the complementary solution and so we need to take

$$\begin{aligned} y_p &= Asinx + Bcosx + Cx^3 + Dx^2 + Ex \\ y_p' &= Acosx - Bsinx + 3Cx^2 + 2Dx + E \\ y_p'' &= -Asinx - Bcosx + 6Cx + 2D \\ y_p''' &= -Acosx + Bsinx + 6C \end{aligned}$$

Substituting these back into the D.E. gives

$$-Acosx + Bsinx + 6C + 4(Acosx - Bsinx + 3Cx^2 + 2Dx + E) = x^2 + sinx$$

Grouping terms and equating coefficients on both sides, we have that

$$(-A + 4A) = 0, \quad (B - 4B) = 1, \quad 6C + 4E = 0, \quad 12C = 1, \quad 8D = 0$$

And so, $A = 0$, $B = -\frac{1}{3}$, $C = \frac{1}{12}$, $D = 0$, $E = -\frac{1}{8}$ and so the particular solution is

$$y_p = -\frac{1}{3}cosx + \frac{1}{12}x^3 + -\frac{1}{8}x$$

Thus, the general solution is

$$y = y_c + y_p = c_1 + c_2cos(2x) + c_3sin(3x) - \frac{1}{3}cosx + \frac{1}{12}x^3 - \frac{1}{8}x$$