MATH 201, SUMMER 2015, TEST 1

Name: Solutions

Student Number:

Mark: / 40

1) a) [4 marks] Find an explicit solution to the following initial value problem. You do not need to provide an interval.

$$\frac{dy}{dx} = -\frac{x}{y} \qquad y(0) = 3$$

b) [2 mark] Is your solution to part a) unique? Justify your answer.

c) [2 mark] Suppose we changed the initial value problem in part a) to

$$\frac{dy}{dx} = -\frac{x}{y} \qquad y(3) = 0.$$

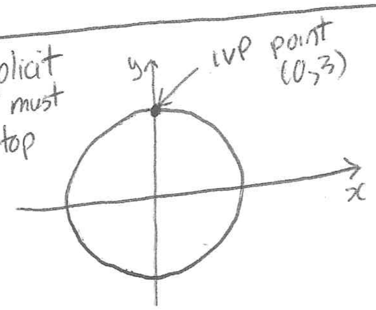
Can we apply our existence and uniqueness theorem? Justify your answer.

a) Separable: $y\,dy = -x\,dx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c$

$\Rightarrow y^2 + x^2 = c_1$

$y(0)=3 \Rightarrow 3^2 + 0^2 = c_1 \Rightarrow c_1 = 9.$

The explicit solution must be the top half of the circle in order to include the point $(0,3)$.



The implicit solution is $x^2 + y^2 = 9$.

The explicit solution is $y = \sqrt{9-x^2}$, not $y = -\sqrt{9-x^2}$

since notice for the first that $y(0) = \sqrt{9} = 3$ so it satisfies the IVP, but for the second $y(0) = -\sqrt{9} = -3$.

b) $f(x,y) = -\frac{x}{y}$ is continuous for all x & all $y \neq 0$

$\frac{\partial f}{\partial y} = \frac{x}{y^2}$ is also continuous for all x & all $y \neq 0$.

Thus, $y(x_0) = y_0$ will have a unique solution as long as $y_0 \neq 0$. Since in this case $y_0 = 3$, we are guaranteed a unique solution on some interval containing $x=0$.

c) In this case, $y_0 = 0$, & so we cannot apply the theorem, as outlined above

2) a) [2 marks] Verify that the following equation is exact.

$$(8x^3y^3 + y + 8x^7)\,dx + (6x^4y^2 + x + 4y^3)\,dy = 0$$

b) [6 marks] Find a family of implicit solutions.

a) $M(x,y) = 8x^3y^3 + y + 8x^7$ $\quad N(x,y) = 6x^4y^2 + x + 4y^3$

Both M & N are polynomials in x & y & so they both are continuous with continuous first partial derivatives, and

$$\frac{\partial M}{\partial y} = 24x^3y^2 + 1 = \frac{\partial N}{\partial x}.$$

$\therefore$ The differential is exact.

b) We seek a solution $f(x,y) = C$ such that $M = \frac{\partial f}{\partial x}$ & $N = \frac{\partial f}{\partial y}$.

$$f(x,y) = \int 8x^3y^3 + y + 8x^7\,dx + h(y)$$

$$= 2x^4y^3 + yx + x^8 + h(y)$$

$$f(x,y) = \int 6x^4y^2 + x + 4y^3\,dy + g(x)$$

$$= 2x^4y^3 + xy + y^4 + g(x).$$

Thus, $h(y) = y^4$ & $g(x) = x^8$, so

$f(x,y) = 2x^4y^3 + xy + y^4 + x^8$ & so a family of implicit solutions is $2x^4y^3 + xy + y^4 + x^8 = C$, where C is an arbitrary constant.

3) a) [8 marks] Solve the following differential equation by using the substitution $x = vy$. You may leave your answer in implicit form, and do not need to provide an interval.

$$xy\,dx - (2x^2 + y^2)\,dy = 0.$$

$x = vy$ so $dx = v\,dy + y\,dv$. Thus, the D.E. becomes

$$vy^2(v\,dy + y\,dv) - (2v^2y^2 + y^2)\,dy = 0$$

$$\Rightarrow (v^2y^2 - 2v^2y^2 - y^2)\,dy + vy^3\,dv = 0$$ Cancelling y^2 gives

$$(-v^2 - 1)\,dy + vy\,dv = 0$$

$$\Rightarrow vy\,dv = (v^2+1)\,dy$$

$$\Rightarrow \frac{v}{v^2+1}\,dv = \frac{dy}{y} \Rightarrow \int \frac{v}{v^2+1}\,dv = \int \frac{dy}{y}$$

The left side integral is solved with a substitution, $w = v^2+1$
$dw = 2v\,dv$

$$\Rightarrow \frac{1}{2}\int \frac{1}{w}\,dw = \int \frac{dy}{y} \Rightarrow \frac{1}{2}\ln|w| = \ln|y| + C$$

$$\Rightarrow \frac{1}{2}\ln|v^2+1| = \ln|y| + C.$$

$$\Rightarrow \frac{1}{2}\ln\left|\frac{x^2}{y^2} + 1\right| = \ln|y| + C$$

4) a) [8 marks] Find the solution to the following initial value problem, including the largest interval of solution.

$$x\frac{dy}{dx} + y = x^2 + x, \qquad y(-1) = \frac{1}{3}$$

Standard form: $\frac{dy}{dx} + \frac{y}{x} = x+1$ so $P(x) = \frac{1}{x}$

Thus, $\mu(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|$.

Setting $\mu(x) = x$ will suffice. Thus

$$x\frac{dy}{dx} + y = x^2+x \Rightarrow \frac{d}{dx}(xy) = x^2+x$$

$$\Rightarrow xy = \int x^2+x\,dx \Rightarrow xy = \frac{x^3}{3} + \frac{x^2}{2} + C$$

$$\Rightarrow y = \frac{x^2}{3} + \frac{x}{2} + \frac{C}{x}$$

$$y(-1) = \frac{1}{3} \Rightarrow \frac{1}{3} = \frac{1}{3} - \frac{1}{2} - C \quad \text{so} \quad C = -\frac{1}{2}$$

Thus, the solution is $y = \frac{x^2}{3} + \frac{x}{2} - \frac{1}{2x}$.

For the interval, notice $P(x) = \frac{1}{x}$ is not continuous at $x=0$ & $f(x) = x+1$ is continuous everywhere. Thus, our choice for the interval are $(-\infty, 0)$ or $(0, \infty)$ & we must choose the one which contains $x_0 = -1$. Thus, the interval of solution is $(-\infty, 0)$.

5) Consider the differential equation

$$\frac{dP}{dt} = P(9 - P^2)$$

a) [5 marks] Find and classify the equilibrium solutions as stable, unstable, or semistable.
b) [3 marks] Sketch the 1 dimensional phase portrait.

Equilibria occur when $\frac{dP}{dt} = 0 = P(9-P^2) = P(3+P)(3-P)$.

Thus, they occur when $P = 0, 3, -3$

We need to check the sign of $\frac{dP}{dt}$ in 4 intervals:

$(-\infty, -3)$, $(-3, 0)$, $(0, 3)$ & $(3, \infty)$.

It suffices to plug in values in each of these intervals:

$P = -4 \Rightarrow \frac{dP}{dt} = -4(9-(-4)^2) > 0$

$P = -1 \Rightarrow \frac{dP}{dt} = -1(9-(-1)^2) < 0$

$P = 1 \Rightarrow \frac{dP}{dt} = 1(9-1^2) > 0$

$P = 4 \Rightarrow \frac{dP}{dt} = 4(9-4^2) < 0$.

Thus, $P = -3$ is stable.
$P = 0$ is unstable.
$P = 3$ is stable.

b)