MATH 201, SUMMER 2012, PRACTICE TEST 2

1) Find the general solution, including an interval, to the following differential equations.

   a)
$$y''' + 3y'' + 3y' + 1 = 0$$

   b)
$$y'' + 2y' + y = e^{-x} lnx$$

2) Find a solution to the following nonlinear differential equation. Don't worry about an interval.

$$y^2 y'' = y'$$

3) Show using the definition of the Laplace Transform that the Laplace Transform of 1 is $\frac{1}{s}$.

4) Use the Laplace Transform to solve the following differential equation:

$$y'' + 4y = e^t$$

   In the test (and the final exam), you will be provided with the table on the very last 2 pages of the text, so use that in order to help you solve this question.

Solutions

1) a) First note that the homogeneous equation, which we solve first, is $y''' + 3y'' + 3y' = 0$. The auxiliary equation is thus $m^3 + 3m^2 + 3m = m(m^2 + 3m + 3) = 0$. Using the quadratic formula, we see that

$$m = \frac{-3 \pm \sqrt{9 - 4(3)}}{2}$$

and so we have 3 roots,

$$m = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i,\ 0$$

Thus,

$$y_c = c_1 + e^{-3/2}(c_2 cos(\frac{\sqrt{3}}{2}x) + c_3 sin(\frac{\sqrt{3}}{2}x))$$

Now we find a particular solution, using undetermined coefficients. Our initial guess for y_p is a constant A, but this duplicates a complementary solution. Thus, we set $y_p = Ax$ which eliminates the duplication and we have

$$y_p' = A$$
$$y_p'' = 0$$
$$y_p''' = 0$$

Plugging these into the D.E., we get

$$3A = -1$$

and so the general solution is

$$y = c_1 + e^{-3/2}(c_2 cos(\frac{\sqrt{3}}{2}x) + c_3 sin(\frac{\sqrt{3}}{2}x)) - \frac{1}{3}x$$

for $x \in (-\infty, \infty)$.

b) Again we first solve the homogeneous case: The auxiliary equation is $m^2 + 2m + 1 = 0$ which gives $m = -1$ twice and so the complementary solution is

$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$

Now we'll use variation of parameters to find the particular solution. We assume a solution of the form $y_p = u_1 y_1 + u_2 y_2$ where $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ and u_1 and u_2 are the unknown functions of x.

$$u_1' = \frac{W_1}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W}$$

where $W_1 = -y_2 f(x) = -xe^{-2x} lnx$, $W_2 = y_1 f(x) = e^{-2x} lnx$ and W is the Wronskian.

$$W = Det\begin{pmatrix} e-x & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{pmatrix} = e^{-2x} - xe^{-2x} + xe^{-2x} = e^{-2x}$$

Thus, using integration by parts for both integrals,

$$u_1 = \int -xlnxdx = -(\frac{1}{2}x^2lnx - \int \frac{1}{2}xdx) = \frac{1}{4}x^2 - \frac{1}{2}x^2lnx$$

and

$$u_2 = \int lnxdx = xlnx - \int dx = xlnx - x$$

Thus, $y_p = (\frac{1}{4}x^2 - \frac{1}{2}x^2lnx)e^{-x} + (xlnx - x)xe^{-x} = x^2e^{-x}(\frac{1}{2}lnx - \frac{3}{4})$
and so the general solution is

$$y = y_c + y_p = c_1e^{-x} + c_2xe^{-x} + x^2e^{-x}(\frac{1}{2}lnx - \frac{3}{4})$$

on $(0, \infty)$ since lnx is only defined and continunous on the positive reals.

2) We'll use the substitution $u = y'$ and since the independent variable x does not appear explicitly, we will make y the independent variable and u the dependent variable. Then, $\frac{d^2y}{dx^2} = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{du}{dy}u$. Thus our differential equation becomes

$$y^2uu' = u \implies u' = y^{-2} \implies u = -\frac{1}{y} + c_1$$

Since we are only asked to find **a** solution, we'll set $c_1 = 0$ to simplify things. Then,

$$\frac{dy}{dx} = -\frac{1}{y} \implies y^2 = -2x + c \implies y = \sqrt{-2x + c}$$

Check:

$$y' = \frac{1}{2}(-2x + c)^{-\frac{1}{2}}(-2) \quad y'' = \frac{1}{2}(-\frac{1}{2})(-2x + c)^{-\frac{3}{2}}(-2)^2$$

and so $y^2y'' = (-2x + c)(-(-2x + c)^{-\frac{3}{2}}) = -(-2x + c)^{-\frac{1}{2}} = y'$

3) This exact question is done in section 7.1 in the text.

4) You will typically be given initial conditions in this problem, but if you are not, you can assign arbitrary initial conditions, $y(0) = y_0$, $y'(0) = y_1$. We begin by taking the transform of the entire equation:

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{1}{s-1}$$
$$Y(s)(s^2 + 4) - sy_0 - y_1 = \frac{1}{s-1}$$
$$Y(s) = \frac{1}{(s-1)(s^2+4)} + \frac{sy_0}{s^2+4} + \frac{y_1}{s^2+4}$$

Before we apply the inverse transform, we need to break up the first term using partial fractions.

$$\frac{1}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs + C}{s^2+4}$$
$$1 = As^2 + 4A + (Bs + C)(s-1)$$
$$1 = (A + B)s^2 + (C - B)s + (4A - C)$$

and so $A+B=0$, $C-B=0$ and $4A-C=1$, giving $-A=B=C$ and so $5A=1$. Thus, $A=\frac{1}{5}$, $B=C=-\frac{1}{5}$.

We have then that

$$Y(s)=\frac{1/5}{s-1}+\frac{-1/5s}{s^2+4}+\frac{-1/5}{s^2+4}+\frac{sy_0}{s^2+4}+\frac{y_1}{s^2+4}$$

$$Y(s)=\frac{1}{5}\frac{1}{s-1}-\frac{1}{5}\frac{s}{s^2+4}-\frac{1}{10}\frac{2}{s^2+4}+y_0\frac{s}{s^2+4}+\frac{y_1}{2}\frac{2}{s^2+4}$$

and so

$$y(t)=1/5e^t+(y_0-1/5)cos(2t)+\frac{1}{2}(y_1-1/5)sin(2t)$$