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Math 201, Tutorial 3

1) Our goal here is to model the velocity of a rocket. Recall from Physics, that the momentum, p, of an object is the product of its mass m and its velocity, v:

$$p = mv$$

Newton's Second Law states that the time-rate of change of the momentum of an object is equal to the applied force, F:

$$F = \frac{d}{dt}(mv) = \frac{dm}{dt}v + m\frac{dv}{dt} \tag{1}$$

Usually the mass of an object is constant, and so we have the more familiar form of Newton's Second Law, $F = \frac{dp}{dt} = m\frac{dv}{dt} = ma$ where a is the acceleration of the object. However, the mass of a rocket is not constant; as the rocket burns fuel, its mass decreases and so we must use the more general equation 1.

Let's consider a rocket travelling a straight horizonal path in space, where there is no gravity, but let's also assume that we are in a nebula where there is "air" resistance. The rocket will experience two forces: the force generated by burning and expelling its fuel, and a drag force from air resistance. The drag force, D, increases with velocity and so we will approximate it as a force which is linear with velocity:

$$D = kv$$

where $k > 0$ is some constant. This approximation is reasonable as long as the velocities are not too great. Let's assume that the force exerted by the burning of the rocket's fuel is a constant $F_R > 0$, and that the mass changes in time as $m(t) = m_o - t$ where m_o is the initial mass of the rocket. The sum of the forces on the rocket is then $F = F_R - D$ since the drag acts in the opposite direction. Then, using equation 1, we get that

$$F_R - D = \frac{d}{dt}(mv) = \frac{dm}{dt}v + m\frac{dv}{dt}$$

which gives

$$F_R = (k-1)v + (m_o - t)\frac{dv}{dt}$$

This is a linear equation, and so we put it into standard form:

$$\frac{dv}{dt} + \frac{k-1}{m_o - t}v = \frac{F_R}{m_o - t}$$

Find a solution to the differential equation for $0 < t < m_o$ in terms of $k > 1$, m_o, F_R, assuming that our rocket starts from rest.

Suppose our rocket has initial mass, 50 kg, our rocket produces a constant force of 2000 N (a newton is 1 $kg \cdot m \cdot s^{-2}$), our drag constant $k = 3\ N/(m/s)$ and that our rocket again starts from rest at $t = 0\ s$. Find an equation for the velocity $v(t)$.

2) Solve the differential equation

$$(2xy + 3y)dx + (4y^3 + x^2 + 3x + 4)dy = 0$$