Math 201, Assignment 1
Due at the beginning of tutorial on May 20, 2015
Illegible or disorganized solutions will receive no credit! Please, for the sake of our markers, be neat!

1) Determine the values of m, for which $y = e^{mx}$ is a solution of the differential equation
$$y'' - 5y' + 6y = 0.$$
If y_1 and y_2 are solutions to the differential equation above and c_1 and c_2 are constants, is $y = c_1 y_1 + c_2 y_2$ also a solution? Why or why not?

2) Find the 1 parameter family of solutions to
$$y' = (x+5)(x-3)^{-1}(x+1)^{-1}.$$
Show which values of $x_o \in \mathbb{R}$ guarantee existence and uniqueness of the solution to the IVP $y(x_o) = y_o$ by invoking an appropriate theorem from the text.

3) When an object at room temperature is placed in an oven whose temperature is constant at T_f, the temperature of the object will increase with time, approaching the temperature of the oven. The temperature T of the object is related to time by through the differential equation
$$T' = k(T - T_f)$$
where k is a real constant.
Given that $T(0) = T_i$, use separation of variables to solve this IVP for T in terms of the independent variable, t, and the constants, k, T_f and T_i.

4) Solve by separating variables, the initial value problem
$$y' = xy^2 e^x, \quad y(0) = 2$$
and comment on uniqueness of the solution.

5) Find a 1 parameter family of solutions to the following first order linear differential equation,
$$x^3 \frac{dy}{dx} + x^2 y = x.$$

Solutions

1)

$$y''-5y'+6y=0 \implies m^2e^{mx}-5me^{mx}+6e^{mx}=0 \implies (m^2-5m+6)e^{mx}=0$$

Since $e^{mx}>0$ for all $x\in\mathbb{R}$, $m^2-5m+6=(m-3)(m-2)=0$ and so $m=2$ or $m=3$.

Thus e^{3x} and e^{2x} are solutions to the differential equation. Since the differential equation is linear and homogeneous, if y_1 and y_2 are solutions to the differential equation, then $c_1y_1+c_2y_2$ is also a solution:

$$\begin{aligned}&\frac{d^2}{dx^2}(c_1y_1+c_2y_2)-5\frac{d}{dx}(c_1y_1+c_2y_2)+6(c_1y_1+c_2y_2)\\ &=c_1(\frac{d^2}{dx^2}y_1-5\frac{d}{dx}y_1+6y_1)+c_2(\frac{d^2}{dx^2}y_2-5\frac{d}{dx}y_2+6y_2)\\ &=c_1(0)+c_2(0)=0\end{aligned}$$

2)

$$\frac{dy}{dx}=\frac{x+5}{(x-3)(x+1)} \implies y=\int\frac{x+5}{(x-3)(x+1)}dx+C_1$$

We decompose the integral into a sum of two partial fractions:

$$\frac{x+5}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1} \implies x+5=A(x+1)+B(x-3)$$

which gives $A=2$ and $B=-1$, and so our integral becomes

$$\int[\frac{2}{x-3}-\frac{1}{x+1}]dx=2ln|x-3|-ln|x+1|+lnC$$

and so

$$y=\frac{C(x-3)^2}{|x+1|}$$

Our theorem of existence and uniqueness requires $f(x,y)$ and $\frac{\partial f}{\partial y}$ to be continuous in some region R which contains (x_o, y_o). $\frac{\partial f}{\partial y} = 0$ and so it's continuous everywhere. $f(x,y) = \frac{x+5}{(x-3)(x+1)}$ and is discontinuous at $x = 3$ and $x = -1$, and so any value of x_o other than 3 and -1 will guarantee existence and uniqueness of the solution to the IVP $y(x_o) = y_o$.

3)

$$\frac{dT}{dt} = k(T - T_f) \implies \frac{dT}{T - T_f} = kdt$$

and so

$$ln|T - T_f| = kt + c_1 \implies |T - T_f| = ce^{kt}$$

where $c > 0$.

We are assuming that the tempurature $T(t) < T_f$ and so

$$|T - T_f| = T_f - T = ce^{kt} \implies T(t) = T_f - ce^{kt}$$

$T_i = T(0) = T_f - ce^0 = T_f - c$ and so $c = T_f - T_i$. Thus,

$$T(t) = T_f - (T_f - T_i)e^{kt}$$

4)

$$\frac{dy}{dx} = y^2 xe^x, \quad y(0) = 2$$

$$\implies \int \frac{dy}{y^2} = \int xe^x dx$$

The left hand side is easy and for the right hand side we use integration by parts, setting $u = x$ and $dv = e^x dx$ and so $du = dx$ and $v = e^x$. Thus,

$$\int y^{-2} dy = -y^{-1} + c_1$$

$$\int xe^x dx = xe^x - \int x^x dx = xe^x - e^x + c_2$$

and so we get

$$y^{-1} = xe^x - e^x + c \implies y(x) = \frac{1}{xe^x - e^x + c}$$

Since $y(0) = 2$, we have that $\frac{1}{1+c} = y(0) = 2$ and so $c = -\frac{1}{2}$. Thus,

$$y(x) = \frac{1}{xe^x - e^x - \frac{1}{2}}$$

$f(x, y) = xy^2e^x$ is continuous on all of $\mathbb{R}$ and $\frac{\partial f}{\partial y} = 2xye^x$ is continuous on $\mathbb{R}$ and so the solution to the IVP is unique on some interval about $x = 0$.

5)

$$x^3y' + x^2y = x$$

First, rewrite the equation in standard form:

$$y' + \frac{y}{x} = \frac{1}{x^2}$$

which means $P(x) = \frac{1}{x}$ and so our integrating factor is

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{ln|x|} = |x|$$

We are searching for a solution on an interval I on which $P(x) = \frac{1}{x}$ and $\frac{1}{x^2}$ are continuous, and so this interval cannot contain 0. Thus, we will take our interval to be $(0, \infty)$. As a result, $|x| = x$. Multiplying our standard form equation by the integrating factor gives

$$xy' + y = \frac{1}{x} \implies \frac{d}{dx}(xy) = \frac{1}{x} \implies xy = \ln|x| + C$$

but again we are on the interval $(0, \infty)$, so $\ln|x| = \ln x$. Thus,

$$y = \frac{\ln x}{x} + \frac{c}{x} \quad \text{on} \quad (0, \infty)$$

is a 1 parameter family of solutions to the differential equation. It is in fact the general solution on $(0, \infty)$. We could have also taken our interval to be $(-\infty, 0)$ and carried out the solution method in a similar way, but with $|x| = -x$.