May 31, 2015
Math 201, Practice Midterm Questions

1) a) State sufficient conditions for the existence and uniqueness of a solution on some interval, I, including the point x_o of the IVP

$$\frac{dy}{dx} = f(x, y) \quad \text{with} \quad \mathrm{y(x_o) = y_o}$$

where x_o, $y_o \in \mathbb{R}$.

b) For what values of x_o, $y_o \in \mathbb{R}$ is the IVP

$$\frac{dy}{dx} = y^{2/3}\sqrt{x-2} \quad \text{with} \quad \mathrm{y(x_o) = y_o}$$

guaranteed to have a unique solution on some interval containing the point x_o?

2) Find an implicit family of solutions to

$$\frac{dy}{dx} = \frac{x^3}{\mathrm{cosy}}$$

3) Find the general solution to the IVP (i.e. find the 1 parameter family of solutions to the IVP)

$$\frac{dy}{dx} + 2xy = x^3$$

4) Verify that the following differential equation is exact (you will want to quote a theorem from the text, which you should know ,in order to verify this) and then find a family of implicit solutions.

$$(3x^2y + 8xy^2)dx + (x^3 + 6y^2 + 8x^2y)dy$$

5) Solve the homogeneous differential equation

$$(y^2 + xy)dx - x^2dy = 0$$

using the substitution $y = ux$.

6)
$$y^2\mathrm{dx} + (4\mathrm{xy} + 1)\mathrm{dy} = 0$$

is a nonseparable differential equation which can be made exact by multiplying through by an appropriate integrating factor. Find the integrating factor, verify that the new equation is in fact exact and then find a family of implicit solutions.

7) Identify the type of D.E. (Linear, Bernoulli, Exact, Homogeneous, Separable)

a)
$$\frac{dy}{dx} + 3x^4y = xy^3$$

b)
$$(x^3 + xy^2)dx + (y^3)dy = 0$$

c)
$$\frac{dy}{dx} + (x^3\sin(\mathrm{x}))\mathrm{y} = \mathrm{e}^{4\mathrm{x}}$$

d)
$$(e^{2y} - y)\cos(\mathrm{x})\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{e}^{\mathrm{y}}\sin(\mathrm{x})$$

e)
$$12x^3y^5dx + (15x^4y^4 + 6y)dy = 0$$

8) Consider the autonomous differential equation:

$$\frac{dy}{dx} = (y^3 + y^2 - 12y)e^y$$

Find and classify the equilibrium solutions and draw the 1 dimensional phase portrait.

9) Suppose the rate at which a student complete his or her homework is proportional to the amount of homework he or she has remaining. If $h(t)$ represents the amount of homework remaining, write a differential equation which describes the system. If the student begins with H_o amount of homework, and after 50 hours has $\frac{4H_o}{5}$ remaining, find an explicit solution for $h(t)$ for $t > 0$. How long will it take for the student to finish $\frac{9}{10}$ of the total homework?

Solutions

1) a) This is just the uniqueness and existence theorem from the textbook that we have used repeatedly. We need (x_o, y_o) to be contained in the interior of a rectangular region R in the xy-plane given by $a \leq x \leq b$, $c \leq y \leq d$ on which $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous.

   b) From above, we need to find the region R, in particular, the a and b. $f(x, y) = y^{2/3}\sqrt{x-2}$ and this function is continuous for all $y \in \mathbb{R}$ and for all $x \geq 2$. $\frac{\partial f}{\partial y} = y^{-1/3}\sqrt{x-2}$ which is continuous for $y \neq 0$ and again $x \geq 2$. Thus our points x_o, y_o need to lie in interior of any rectangular region in $\{(x, y) : x \geq 2, y \neq 0\}$. Thus the possible values for (x_o, y_o) are $\{(x, y) : x > 2, y \neq 0\}$.

2) This DE is separable and so we get that $\int \cos y \, dy = \int x^3 \, dx$ which gives

$$\sin y = \frac{1}{4}x^4 + c$$

3) This DE is linear and already in standard form, so $P(x) = 2x$. Thus our integrating factor is

$$\mu(x) = e^{\int 2x \, dx} = e^{x^2}$$

Multiplying our DE through by $\mu(x)$ gives

$$\frac{d}{dx}(e^{x^2}y) = e^{x^2}x^3 \implies e^{x^2}y = \int e^{x^2}x^3 \, dx$$

Making the substitution $u = x^2$, $du = 2xdx$ on the right integral gives

$$e^{x^2}y = \frac{1}{2}\int ue^u \, du \implies e^{x^2}y = \frac{1}{2}(ue^u - e^u) + c = \frac{1}{2}(x^2e^{x^2} - e^{x^2}) + c$$

Dividing by e^{x^2} (which is never zero, so this is valid for all x) gives

$$y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$$

Check:

$$2xy = x^3 - x + 2cxe^{-x^2} \qquad \frac{dy}{dx} = x + c(-2x)e^{-x^2}$$

and so their sum is in fact x^3.

4) An equation of the form given is exact on some interval I if and only if the first partial derivatives of $M(x,y)$ and $N(x,y)$ are continuous on I and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ where $M = 3x^2y + 8xy^2$ and $N = x^3 + 6y^2 + 8x^2y$.

$$\frac{\partial M}{\partial y} = 3x^2 + 16xy = \frac{\partial N}{\partial x}$$

and both M and N are continuous everywhere. Thus the equation is exact.

Since the DE is exact, we seek an implicit solution of the form $f(x,y) = c$ for some function f with continuous first and second partial derivatives, where $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$. Thus,

$$f = \int M\,dx + g(y) = x^3y + 4x^2y^2 + g(y)$$

Now,

$$x^3 + 6y^2 + 8x^2y = N = \frac{\partial f}{\partial y} = x^3 + 8x^2y + g'(y)$$

and so $g'(y) = 6y^2$ which gives $g(y) = 2y^3$ (no need to worry about the constant yet) and so

$$f(x,y) = x^3y + 4x^2y^2 + 2y^3$$

and so an implicit solution to our DE is

$$x^3y + 4x^2y^2 + 2y^3 = C$$

This method of solution is slightly different from the one I presented in class, but works equally well.

5) For homogeneous equations, such as this one, $(y^2 + xy)dx - x^2dy = 0$, we make the substitution, $y = ux$ or $x = vy$. It is recommended to use $y = ux$ when coefficient function of dy is simpler, and this is the given substitution. Thus, with $y = ux$, we have $dy = udx + xdu$ and our DE becomes

$$(u^2x^2+ux^2)dx-x^2(udx+xdu) = 0 \implies (u^2x^2+ux^2-x^2u)dx-x^3du = 0$$

and so we get the separable equation

$$u^2x^2dx = x^3du \implies \int \frac{dx}{x} = \int \frac{du}{u^2}$$

Thus, solving the integral and then substituting back in for $u = \frac{y}{x}$

$$ln|x| = -\frac{1}{u} + C \implies ln|x| = -\frac{x}{y} + C$$

Thus,

$$y = -\frac{x}{ln|x| - C}.$$

6) We need $\frac{M_y - N_x}{N}$ to be a function of only x, or $\frac{N_x - M_y}{M}$ to be a function of only y.

$$\frac{M_y - N_x}{N} = \frac{2y - 4y}{4xy + 1} \text{ and } \frac{N_x - M_y}{M} = \frac{4y - 2y}{y^2} = \frac{2}{y}$$

and so the second one works.

Thus we multiply the equation through by $e^{\int \frac{2}{y}\,dy} = y^2$, giving

$$y^4dx + (4xy^3 + y^2)dy = 0$$

and now M and N still both have continuous first partial derivatives and

$$\frac{\partial M}{\partial y} = 4y^3 = \frac{\partial N}{\partial x}$$

and so the equation is now exact. Thus,
$M = \frac{\partial f}{\partial x}$ and so

$$f(x, y) = \int M dx + g(y) \implies f(x, y) = xy^4 + g(y)$$

We also have that

$$(4xy^3 + y^2) = N = \frac{\partial f}{\partial y} = 4xy^3 + g'(y)$$

and so $g'(y) = y^2$ which means $g(y) = \frac{y^3}{3}$.

Thus, $f(x, y) = xy^4 + \frac{y^3}{3}$ and so $xy^4 + \frac{y^3}{3} = C$ is a family of implicit solutions to the DE.

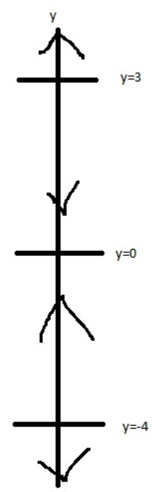
7) a) Bernoulli
   b) Homogeneous
   c) Linear
   d) Separable
   e) Exact

8) Equilibrium solutions occur when $y(y-3)(y+4)e^y = f(y) = 0$ which happens when $y = 0$, $y = 3$ or $y = -4$.

If $y < -4$, $\frac{dy}{dx} < 0$.
If $-4 < y < 0$, $\frac{dy}{dx} > 0$.
If $0 < y < 3$, $\frac{dy}{dx} < 0$.
If $3 < y$, $\frac{dy}{dx} > 0$.

Thus, $y = -4$ is unstable, $y = 0$ is stable, and $y = 3$ is unstable.

1D Phase portrait



9)

$$\frac{dh}{dt} = kh(t)$$

The solution to the DE is $h(t) = ce^{kt}$.

$h(0) = H_o = ce^0$ so $c = H_o$

$h(50) = 4/5H_o = H_oe^{50k}$ so $ln(\frac{4}{5}) = 50k$ so $k = \frac{ln(\frac{4}{5})}{50}$

$h(t) = \frac{1}{10}H_o = H_oekt$

so $ln(\frac{1}{10}) = kt$ so $t = \frac{ln(\frac{1}{10})}{k}$ where $k = \frac{ln(\frac{4}{5})}{50}$

Thus, it will take approximately 516 hours to complete $\frac{9}{10}$ of the homework.

Note that in this model, it is impossible to complete all of the homework, which one could argue supports its validity.