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1 Question 1

Let $\vec{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{v} = \begin{bmatrix}4\\5\\6\end{bmatrix}$, and $\vec{w} = \begin{bmatrix}7\\8\\9\end{bmatrix}$. Explain whether the set $A = \{\vec{u}, \vec{v}, \vec{w}\}$ is a basis for $\mathbb{R}^3$. Make sure to include all relevant definitions.

1.1 Solution for Question 1

A *basis* for a subspace $V \subset \mathbb{R}^3$ is a linearly independent set of vectors A such that $\operatorname{span} A = V$.

First, let's check whether a is a linearly independent set. A set is considered *linearly independent* if the only way to express it as a linear combination of vectors is trivial. (ie, the combination where all coefficients are 0).

Does

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = c_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + c_2 \begin{bmatrix}4\\5\\6\end{bmatrix} + c_3 \begin{bmatrix}7\\8\\9\end{bmatrix}$$

have a nontrivial solution? We can rewrite the equation above as:

$$\begin{aligned}0 &= c_1 + 4c_2 + 7c_3\\ 0 &= 2c_1 + 5c_2 + 8c_3\\ 0 &= 3c_1 + 6c_2 + 9c_3.\end{aligned}$$

By solving this system, we can conclude that any c's satisfying the condition $c_2 = -2c_3 = -2c_1$ will lead to a valid solution. If we arbitrarily chose the value of 1 for c_1, we can plug in and find that $c_2 = -2$ and $c_3 = 1$ give a solution to the above system. Since c_1, c_2, and $c_3 \neq 0$, $\vec{0} = \vec{u} - 2\vec{v} + \vec{w}$ is a solution.

Since $\vec{0}$ can be written as a non-trivial linear combination of vectors in A, A is a linearly dependent set, and therefore cannot be a basis for R^3.

2 Question 2

Fix $\vec{u}, \vec{v} \in \mathbb{R}^n$. Show that $\operatorname{span}(\operatorname{span}\{\vec{u}, \vec{v}\}) = \operatorname{span}\{\vec{u}, \vec{v}\}$. Make sure to include all relevant definitions.

2.1 Solution for Question 2

The *span* of a set of vectors X is defined as set of all linear combinations of vectors in X.

We want to show that for any two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $\operatorname{span}(\operatorname{span}\{\vec{u}, \vec{v}\}) = \operatorname{span}\{\vec{u}, \vec{v}\}$. To do this, we show that $\operatorname{span}(\operatorname{span}\{\vec{u}, \vec{v}\})$ and $\operatorname{span}\{\vec{u}, \vec{v}\}$ are subsets of each other.

$\Leftarrow$:
Let $\vec{u}, \vec{v}$ be some arbitrary vectors in $\mathbb{R}^n$ and $A = \operatorname{span}\{\vec{u}, \vec{v}\}$.

$A = \{\vec{x} \in \mathbb{R}^n : \vec{x} = m\vec{u} + n\vec{v} \text{ for some } m, n \in \mathbb{R}\}$.

If we also let $\vec{w}$ be some arbitrary vector in $\operatorname{span} A$, then by definition $\vec{w} = \sum_{i=1}^{n} k_i \vec{s}_i$. for some $s_i \in \vec{A}$ and some value $k \in \mathbb{R}$.

By the definition of A, we can expand the above into:

$$\vec{w} = \sum_{i=1}^{n} k_i(m_i\vec{u} + n_i\vec{v}) = (\sum_{i=1}^{n} k_i m_i)\vec{u} + (\sum_{i=1}^{n} k_i n_i)\vec{v}$$

for some $m_i, n_i, k_i \in \mathbb{R}$.

This shows that $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$, and thus $\vec{w} \in A$ and thus $\operatorname{span} A \subseteq A$.

$\Rightarrow$:
By definition, A is a subset of $\operatorname{span} A$.

$\therefore A = \operatorname{span} A$.

3 Question 3

The worksheets define $\text{proj}_{\vec{v}}\vec{u}$ as the vector in the direction $\vec{v}$ such that $\vec{u} - \text{proj}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$. Call this definition (a). Your textbook defined $\text{proj}_{\vec{v}}\vec{u}$ as the vector $\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$. Call this definition (b). Show that definitions (a) and (b) are equivalent by showing that that the vector arising from definition (b) must be the same as the vector arising from definition (a). In your answer, elaborate on definition (a) by including the definition of *vector in the direction of* $\vec{v}$ and *orthogonal*.

3.1 Solution for Question 3

We'll assume that $\vec{v} \neq 0$ because otherwise it would break definition (b). [1]

Remember that a vector $\vec{u}$ is in the direction of a vector $\vec{v}$ if $\vec{u} = k\vec{v}$ for some nonzero $k \in \mathbb{R}$. Also, two vectors $\vec{u}, \vec{v}$ are considered *orthogonal* if $\vec{u} \cdot \vec{v} = 0$.

Let's take a further look at the two definitions:

By Definition (a), $(\vec{u} - \text{proj}_{\vec{v}}\vec{u}) \cdot \vec{v} = 0$. We know that $\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$ from definition (b). Let's substitute it into the equation.

$$(\vec{u} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}) \cdot \vec{v} = 0$$

$$\vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\cdot\vec{v} = 0$$

Since for any vectors $\vec{u}, \vec{v}$, $\vec{u}\cdot\vec{v}$ is a scalar. Furthermore, for any vector $\vec{v}$, $\vec{v}\cdot\vec{v} \neq 0$ because a vector cannot be perpendicular to itself. Thus, in the equations above, $\vec{v}\cdot\vec{v}$ is a non-zero scalar, and we can happily cancel.

$$\vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{v} = 0$$

This means that $(\vec{u} - \text{proj}_{\vec{v}}\vec{u}) \cdot \vec{v}$ is orthoganal to $\vec{v}$.

4 Addendum

This is the first time one of us is submitting an assignment in LaTeX. Please be gentle in grading our assignment and excuse the obvious poor formatting. Many thanks to the people in the assistance centre, who provided invaluable help for questions 2 and 3.

[1] I'm also a computer scientist, and in my profession, dividing by zeros usually throws syntax errors.