The perceptron algorithm

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Outline

Guessing gender from height

- Feature space $\mathcal{X} \subset \mathbb{R}$: e.g. height
- Label space $\mathcal{Y} = \{-1, 1\}$: e.g. gender
- Can we find some $\beta_1 \in \mathbb{R}$ and a direction $\beta_0 \in \{-1, +1\}$ so as to separate the genders?

Online learning: At time t

- We choose a separator β_0^t, β_1^t
- We observe a new datapoint x_t, y_t
- We make a mistake at time t if:

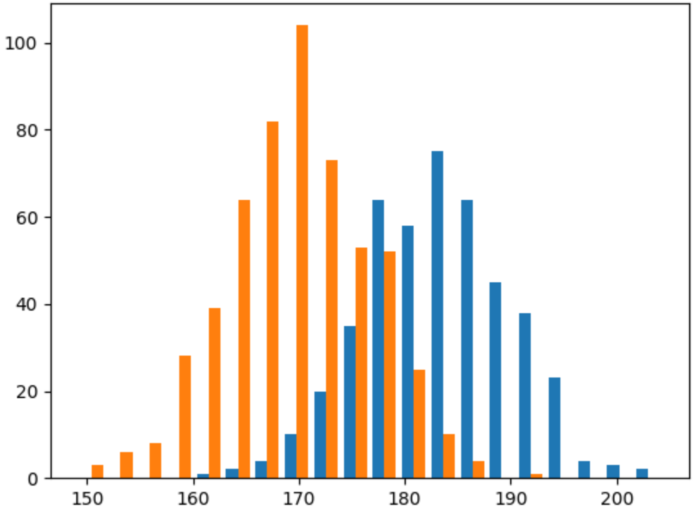
$$\beta^t x_t - \beta_0^t \leq 0.$$

- If we stop making mistakes, then we are classifying everything perfectly.

Can you find a threshold that makes a small number of mistakes?

./src/Perceptron/perceptron_simple.py

Non-separable classes



More complex example

- Feature space $\mathcal{X} \subset \mathbb{R}^n$: e.g. height and weight for $n = 2$
- Label space $\mathcal{Y} = \{-1, 1\}$: e.g. gender
- Can we find some line so as to separate the genders?

-./src/Perceptron/show_class_data_labels.py

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Linear separator

We now have parameters $\beta_0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$ defining a hyperplane $f(x) = 0$ in $\mathbb{R}^n$

$$f(x) = \beta_0 + \beta^\top x = \beta_0 + \sum_{i=1}^{n} \beta_i x_i.$$

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- The perceptron decision rule is $\pi(x) = \mathrm{sign}(f(x))$
- If $f(x) > 0$, we assign class +1
- If $f(x) < 0$, we assign class -1

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If we augment x an additional component $x_0 = 1$, we can write

$$f(x) = \beta^\top x = \sum_{i=0}^{n} \beta_i x_i.$$

The perceptron algorithm

Input

- Feature space $X \subset \mathbb{R}^n$.
- Label space $Y = \{-1, 1\}$.
- Data (x_t, y_t), $t \in [T]$, with $x_t \in X, y_t \in Y$.

Algorithm

- $\beta^0 \sim \mathrm{Normal}^n(0, I)$. % Initialise parameters
- For $t = 1, \ldots, T$
  - $a_t = \mathrm{sgn}(\beta^t \cdot x_t)$. % Classify example
  - If $a_t \neq y_t$
    - $\beta^t = \beta^{t-1} + y_t x_t$ % Move hyperplane
  - Else
    - $\beta^t = \beta^{t-1}$ % Do nothing for correct examples
  - EndIf
- Return β^T

Perceptron examples

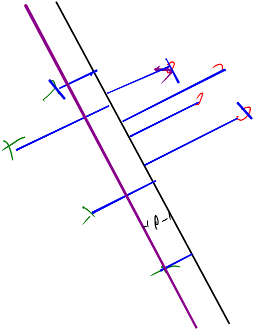
Example 1: One-dimensional data

- Done on the board
- Shows how the algorithm works.
- Demonstrates the idea of a margin

Example 2: Two-dimensional data

- See in-class programming exercise

Margins and the perceptron theorem



- The hyperplane β^* separates the examples
- The margin ρ is the minimum distance ρ between β^* and any point.

Theorem (Perceptron theorem)

The number of mistakes is bounded by ρ^{-2}, where $\|x_t\| \leq 1$, $\rho \leq y_t(x_t^\top \beta^*)$ for some margin ρ and hyperplane β^* with $\|\beta^*\| = 1$.

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- At each mistake, $\beta^\top \beta$ grows by at most 1.

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After M mistakes:

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After M mistakes:

- $\beta^\top \beta^* \geq M\rho$
- $\beta^\top \beta \leq M$

So $M\rho \leq \beta^\top \beta^* \leq \|\beta\| = \sqrt{\beta^\top \beta} \leq \sqrt{M}$.

- Thus, $M \leq \rho^{-2}$.

The gradient descent method: one dimension

- Function to minimise $f : \mathbb{R} \to \mathbb{R}$.
- Derivative $\frac{d}{d\beta} f(\beta)$

Gradient descent algorithm

- Input: initial value β^0, learning rate schedule α_t
- For $t = 1, \ldots, T$
  - $\beta^{t+1} = \beta^t - \alpha_t \frac{d}{d\beta} f(\beta^t)$
- Return β^T

Properties

- If $\sum_t \alpha_t = \infty$ and $\sum_t \alpha_t^2 < \infty$, it finds a local minimum β^T, i.e. there is $\epsilon > 0$ so that

$$f(\beta^T) < f(\beta), \forall \beta : \|\beta^T - \beta\| < \epsilon.$$

Gradient methods for expected value

Estimate the expected value

$x_t \sim P$ with $\mathbb{E}_P[x_t] = \mu$.

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Derivative

Idea: at the minimum the derivative should be zero.

$$d/d\beta \, \mathbb{E}_P[(x_t - \beta)^2] = \mathbb{E}_P[d/d\beta(x_t - \beta)^2] = \mathbb{E}_P[-(x_t - \beta)] = \mathbb{E}_P[x_t] - \beta.$$

Setting the derivative to 0, we have $\beta = \mathbb{E}_P[x_t]$. This is a simple solution.

Real-world setting

- The objective function does not result in a simple solution
- The distribution P is not known.
- We can sample $x \sim P$.

The gradient method

- Function to minimise $f : \mathbb{R}^n \to \mathbb{R}$.
- Derivative $\nabla_\beta f(\beta) = \left(\frac{\partial f(\beta)}{\partial \beta_1}, \ldots, \frac{\partial f(\beta)}{\partial \beta_n}\right)$, where $\frac{\partial f}{\partial \beta_n}$ denotes the partial derivative, i.e. varying one argument and keeping the others fixed.

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Stochastic gradient method

This is the same as the gradient method, but with added noise:

- $\beta^{t+1} = \beta^t - \alpha_t[\nabla_\beta f(\beta^t) + \omega_t]$
- $\mathbb{E}[\omega_t] = 0$ is sufficient for convergence.

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Example (When the cost is an expectation)

In machine learning, the cost is frequently an expectation of some function ℓ,

$$f(\beta) = \int_X dP(x)\ell(x, \beta)$$

This can be approximated with a sample

$$f(\beta) \approx \frac{1}{T}\sum_t \ell(x_t, \beta)$$

The same holds for the gradient:

$$\nabla_\beta f(\beta) = \int_X dP(x)\nabla_\beta \ell(x, \beta) \approx \frac{1}{T}\sum_t \nabla_\beta \ell(x_t, \beta)$$

Gradient for mean estimation

- The gradient is zero when the parameter is the expected value

$$
\begin{aligned}
\frac{d}{d\beta}\,\mathbb{E}_P[(x-\beta)^2] &= \int_{-\infty}^{\infty} dP(x)\frac{d}{d\beta}(x-\beta)^2 \\
&= \int_{-\infty}^{\infty} dP(x)2(x-\beta) \\
&= 2\,\mathbb{E}_P[x] - 2\beta.
\end{aligned}
$$

Stochastic gradient for mean estimation

Theorem (Sampling)

For any bounded random variable f,

$$\mathbb{E}_P[f] = \int_X dP(x) f(x) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} f(x_t) = \mathbb{E}_P \left[\frac{1}{T} \sum_{t=1}^{T} f(x_t) \right], \qquad x_t \sim P$$

Example (Sampling)

- If we sample x we approximate the gradient:

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- If we update β after each new sample x_t, we obtain:

$$\beta^{t+1} = \beta^t + 2\alpha_t (x_t - \beta^t)$$

Perceptron algorithm as gradient descent

Target error function

$$\mathbb{E}_{P}^{\beta}[\ell] = \int_{\mathcal{X}} dP(x) \sum_{y} P(y|x) \ell(x, y, \beta)$$

Minimises the error on the true distribution.

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Empirical error function

$$\mathbb{E}_{D}^{\beta}[\ell] = \frac{1}{T} \sum_{t=1}^{T} \ell(x_t, y_t, \beta), \qquad D = (x_t, y_t)_{t=1}^{T}, \quad x_t, y_t \sim P.$$

Minimises the error on the empirical distribution.

Cost functions and the chain rule

Perceptron cost function

The cost of each example

$$\ell(x, y, \beta) = \overbrace{\mathbb{I}\left\{y(x^\top \beta) < 0\right\}}^{\text{misclassified?}} \overbrace{[-y(x^\top \beta)]}^{\text{margin of error}} \tag{1}$$

where the indicator function $\mathbb{I}\{A\}$ is 1 when A is true and 0 otherwise. «««<
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Let $z = g(y)$, $y = f(x)$ so that $z = g(f(x))$. Then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$ «««< variant A
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- $\nabla_\beta \ell(x, y, \beta) = -\mathbb{I}\left\{y(x^\top \beta) < 0\right\} \nabla_\beta [y(x^\top \beta)]$.

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- $\frac{\partial \beta}{\partial \beta_i}[y(x_t^\top \beta)] = y x_{t,i}$

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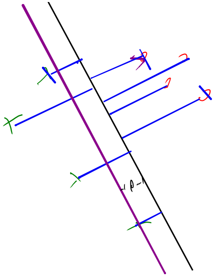
Margins and confidences

We can think of the output of the network as a measure of confidence



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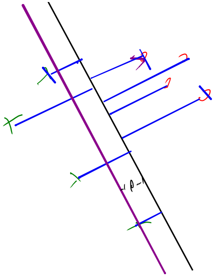

By applying the logit function, we can bound a real number x to $[0, 1]$:

$$f(x) = \frac{e^x}{1 + e^x} = \frac{1}{1 + e^{-x}}$$

Logistic regression

Output as a measure of confidence, given the parameter β

$$P_\beta(y = 1|x) = \frac{1}{1 + \exp(-x_t^\top \beta)}$$

The original output $x_t^\top \beta$ is now passed through the logit function.

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Negative Log likelihood

$\ell(x_t, y_t, \beta) = -\ln P_\beta(y_t|x_t) = \ln(1 + \exp(-y_t x_t^\top \beta))$

$$\begin{aligned}
\nabla_\beta \ell(x_t, y_t, \beta) &= \frac{1}{1 + \exp(-y x_t^\top \beta)} \nabla_\beta [1 + \exp(-y x_t^\top \beta)] \\
&= \frac{1}{1 + \exp(-y x_t^\top \beta)} \exp(-y x_t^\top \beta) [\nabla_\beta (-y_t x_t^\top \beta)] \\
&= -\frac{1}{1 + \exp(x_t^\top \beta)} (x_{t,i})_{i=1}^n e
\end{aligned}$$

- $\mathbb{E}_P(\ell) = \int_X dP(x) \sum_{y \in Y} P(y|x) P_\beta(y_t + x_t)$

The Perceptron and Gradients

`./src/Perceptron/Perceptron_gd.ipynb`

- Perceptron implemenation to fill in
- Gradient descent implementation
- Experiment on the learning rate with sklearn