

# Mathematical background

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# Outline

## Probability background

- Logic and Set theory

- Probability facts

- Conditional probability and independence

- Random variables, expectation and variance

## Linear algebra

- Vectors

- Linear operators and matrices

## Calculus

- Univariate calculus

- Multivariate calculus

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## Linear algebra

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# Logic

## Statements

- ▶ A statement  $A$  may be true or false

## Unary operators

- ▶ negation:  $\neg A$  is true if  $A$  is false (and vice-versa).

## Binary operators

- ▶ or:  $A \vee B$  ( $A$  or  $B$ ) is true if either  $A$  or  $B$  are true.
- ▶ and:  $A \wedge B$  is true if both  $A$  and  $B$  are true.
- ▶ implies:  $A \Rightarrow B$ : is false if  $A$  is true and  $B$  is false.
- ▶ iff:  $A \Leftrightarrow B$ : is true if  $A, B$  have equal truth values.

## Operator precedence

$\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

# Set theory

- ▶ First, consider some universal set  $\Omega$  and the empty set  $\emptyset$
- ▶ A set  $A$  is a collection of points  $x$  in  $\Omega$ .
- ▶  $\{x \in \Omega : f(x)\}$ : the set of points in  $\Omega$  for which  $f(x)$  is true.

## Unary operators

- ▶  $\neg A = \{x \in \Omega : x \notin A\}$ .

## Binary operators

- ▶  $A \cup B$  if  $\{x \in \Omega : x \in A \vee x \in B\}$  - (c.f.  $A \vee B$ )
- ▶  $A \cap B$  if  $\{x \in \Omega : x \in A \wedge x \in B\}$  - (c.f.  $A \wedge B$ )

## Binary relations

- ▶  $A \subset B$  if  $x \in A \Rightarrow x \in B$  - (c.f.  $A \Rightarrow B$ )
- ▶  $A = B$  if  $x \in A \Leftrightarrow x \in B$  - (c.f.  $A \Leftrightarrow B$ )

## Interesting cases

- ▶ If  $A \cap B = \emptyset$ , then they are **disjoint**, or mutually exclusive.
- ▶ If  $A \cap B = A$  only if  $A \subset B$ .

# Probability fundamentals

## Probability measure $P$

- ▶ Defined on a universe  $\Omega$
- ▶  $P : \Sigma \rightarrow [0, 1]$  is a function of subsets of  $\Omega$ .
- ▶ A subset  $A \subset \Omega$  is an **event** and  $P$  measures its likelihood.

## Axioms of probability

- ▶  $P(\Omega) = 1$
- ▶ For  $A, B \subset \Omega$ , if  $A \cap B = \emptyset$  then  $P(A \cup B) = P(A) + P(B)$ .

## Partition

$\{A_i\}$  is a partition of  $\Omega$  if  $A_i \cap A_j = \emptyset \forall i \neq j$  and  $\bigcup_{i=1}^n A_i = \Omega$ . A partition of  $\Omega$  defines a **complete set of mutually exclusive alternatives**.

## Marginalisation

Let  $A_1, \dots, A_n \subset \Omega$  be a **partition** of  $\Omega$ . Then

$$P(B) = \sum_{i=1}^n P(B \cap A_i).$$

# Conditional probability

## Definition (Conditional probability)

The conditional probability of an event  $A$  given an event  $B$  is defined as

$$P(A|B) \triangleq \frac{P(A \cap B)}{P(B)}$$

The above definition requires  $P(B)$  to exist and be positive.

## Conditional probabilities as a collection of probabilities

More generally, we can define conditional probabilities as simply a collection of probability distributions:

$$\{P_{\theta}(A) \mid \theta \in \Theta\},$$

where  $\Theta$  is an arbitrary set.

# The theorem of Bayes

Theorem (Bayes's theorem)

$$P(A|B) = \frac{P(B|A)}{P(B)}$$



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## The general case

If  $A_1, \dots, A_n$  are a partition of  $\Omega$ , meaning that they are mutually exclusive events (i.e.  $A_i \cap A_j = \emptyset$  for  $i \neq j$ ) such that one of them must be true (i.e.  $\bigcup_{i=1}^n A_i = \Omega$ ), then

$$P(B) = \sum_{i=1}^n P(B|A_i)P(A_i)$$

and

$$P(A_j|B) = \frac{P(B|A_j)}{\sum_{i=1}^n P(B|A_i)P(A_i)}$$

# Independence

## Independent events

$A, B$  are independent iff  $P(A \cap B) = P(A)P(B)$ .

## Conditional independence

$A, B$  are conditionally independent given  $C$  iff  
 $P(A \cap B|C) = P(A|C)P(B|C)$ .

# Random variables

A random variable  $f : \Omega \rightarrow \mathbb{R}$  is a real-value function measurable with respect to the underlying probability measure  $P$ , and we write  $f \sim P$ .

## The distribution of $f$

The probability that  $f$  lies in some subset  $A \subset \mathbb{R}$  is

$$P_f(A) \triangleq P(\{\omega \in \Omega : f(\omega) \in A\}).$$

## Independence

Two RVs  $f, g$  are independent in the same way that events are independent:

$$P(f \in A \wedge g \in B) = P(f \in A)P(g \in B) = P_f(A)P_g(B).$$

In that sense,  $f \sim P_f$  and  $g \sim P_g$ .

## IID (Independent and Identically Distributed) random variables

A sequence  $x_t$  of r.v.s is IID if  $x_t \sim P$  ( $x_1, \dots, x_t, \dots, x_T$ )  $\sim P^T$ .

# Expectation

For any real-valued random variable  $f : \Omega \rightarrow \mathbb{R}$ , the expectation with respect to a probability measure  $P$  is

$$\mathbb{E}_P(f) = \sum_{\omega \in \Omega} f(\omega)P(\omega).$$

## Linearity of expectations

For any RVs  $x, y$ ,  $\mathbb{E}_P(x + y) = \mathbb{E}_P(x) + \mathbb{E}_P(y)$

## Correlation

If  $x, y$  are **not** correlated then  $\mathbb{E}_P(xy) = \mathbb{E}(x) \mathbb{E}(y)$ .

## Independence

If  $x, y$  are independent RVs then they are also uncorrelated (but not vice-versa)

## Conditional expectation

The conditional expectation of a random variable  $f : \Omega \rightarrow \mathbb{R}$ , with respect to a probability measure  $P$  conditioned on some event  $B$  is simply

$$\mathbb{E}_P(f|B) = \sum_{\omega \in \Omega} f(\omega)P(\omega|B).$$

# Variance

For any real-valued random variable  $f : \Omega \rightarrow \mathbb{R}$ , the variance with respect to a probability measure  $P$  is

$$\mathbb{V}_P(f) = \sum_{\omega \in \Omega} [f(\omega) - \mathbb{E}_P(f(\omega))]^2 P(\omega).$$

## Linearity of variance

If  $f, g$  are uncorrelated RVs

$$\mathbb{V}_P(f + g) = \mathbb{V}_P(f) + \mathbb{V}_P(g).$$

## Variance products

If  $f, g$  are independent RVs

$$\mathbb{V}_P(f + g) = \mathbb{E}_P(f)^2 \mathbb{V}_P(g) + \mathbb{E}_P(g)^2 \mathbb{V}_P(f) + \mathbb{V}_P(f) \mathbb{V}_P(g).$$

## Vector space $F$ axioms

Here we consider a vector space  $F$ . Typically, it is a subset of the Euclidean  $d$ -dimensional space, ie.  $F \subset \mathbb{R}^d$ .

- ▶  $(x + y) + z = x + (y + z)$ , for all  $x, y, z \in F$ .
- ▶  $x + y = y + x$ , for all  $x, y \in F$ .
- ▶ There is a zero element  $0 \in F$  such that  $x + 0 = x$  for all  $x \in F$ .
- ▶ For all  $x \in F$ , there is an element  $-x \in F$  so that  $x + (-x) = 0$ .
- ▶  $a(x + y) = ax + ay$ , For any  $a \in \mathbb{R}$ ,  $x, y \in F$ .
- ▶  $(a + b)x = ax + bx$ , For any  $a, b \in \mathbb{R}$ ,  $x \in F$ .

The real vector space  $F = \mathbb{R}^d$

For  $a \in \mathbb{R}$  and  $x, y \in F$ ,

- ▶  $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d)$
- ▶  $x + y = (x_1 + y_1, \dots, x_d + y_d).$
- ▶  $ax = (ax_1, \dots, ax_d).$
- ▶  $-x = (-1)x.$
- ▶  $0 = (0, \dots, 0)$

# Linear operators

Linear operator  $A : F \rightarrow G$

- ▶  $A(x + y) = Ax + Ay$
- ▶  $A(ax) = a(Ax)$ .

Matrices in  $\mathbb{R}^{n \times m}$ .

A matrix  $A \in \mathbb{R}^{n \times m}$  is a tabular array  $A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix}$  Matrices  
can be seen as linear operators when used to multiply vectors.



# Multiplication operators

## Matrix multiplication

For  $A \in \mathbb{R}^{n \times d}$ ,  $B \in \mathbb{R}^{d \times m}$ , the  $ij$ -th element of the result of the multiplication  $AB$  is

$$(AB)_{i,j} = \sum_{k=1}^d A_{i,k} B_{k,j}.$$

so that  $AB \in \mathbb{R}^{n \times m}$ .

## Matrix-vector multiplication

A matrix  $A \in \mathbb{R}^{n \times m}$  defines the following linear operator  $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$ .

$$Ax = \left( \sum_{j=1}^m A_{i,j} x_j : i = 1, \dots, n \right)$$

All vectors  $x \in \mathbb{R}^m$  are equivalent to matrices in  $\mathbb{R}^{m \times 1}$ .

# Matrix inverses

The identity matrix  $I \in \mathbb{R}^{n \times n}$

- ▶ For this matrix,  $I_{i,i} = 1$  and  $I_{i,j} = 0$  when  $j \neq i$ .
- ▶  $Ix = x$  and  $IA = A$ .

The inverse of a matrix  $A \in \mathbb{R}^{n \times n}$

$A^{-1}$  is called the inverse of  $A$  if

- ▶  $AA^{-1} = I$ .
- ▶ or equivalently  $A^{-1}A = I$ .

The pseudo-inverse of a matrix  $A \in \mathbb{R}^{n \times m}$

- ▶  $\tilde{A}^{-1}$  is called the **left pseudoinverse** of  $A$  if  $\tilde{A}^{-1}A = I$ .
- ▶  $\tilde{A}^{-1}$  is called the **right pseudoinverse** of  $A$  if  $A\tilde{A}^{-1} = I$ .

# Derivatives

## Derivative

The derivative of a single-argument function is defined as:

$$\frac{d}{dx}f(x) = \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon) - f(x)}{\epsilon}.$$

$f$  must be absolutely continuous at  $x$  for the derivative to exist.

## Subdifferential

For non-differential functions, we can sometimes define the set of all subderivatives:

$$\partial f(x) = \left[ \lim_{\epsilon \rightarrow 0} \frac{f(x) - f(x - \epsilon)}{\epsilon}, \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon) - f(x)}{\epsilon} \right]$$

# Integrals

## Riemann integral

The Riemann integral is obtained by taking a **horizontal** discretisation of a function to the limit:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{t=1}^n f(x_t) \frac{b-a}{n}, \quad x_t = a + (t-1) \cdot \frac{b-a}{n}$$

## Lebesgue integral

This integral is obtained by taking a **vertical** discretisation of a function to the limit. Let  $\lambda$  be the Lebesgue measure (i.e. area) of a set. Then:

$$\int_X f(x) d\lambda(x) = \lim_{n \rightarrow \infty} \sum_{t=1}^n y_t \lambda(S_t),$$

$$S_t = \{x : f(x) \in (y_{t-1}, y_t]\}, \quad y_0 = -\infty, \quad y_n = \sup_x f(x).$$

# Fundamental theorem of calculus

$$f(x) = \frac{d}{dx} \int_a^x f(t) dt$$

If  $\frac{d}{dx} F = f$  then its integral from  $a$  to  $b$  is:

$$\int_a^b f(x) dx = F(b) - F(a),$$

# Multivariate Functions

We consider functions operating in multi-dimensional Euclidean spaces.

$$f : \mathbb{R}^n \rightarrow \mathbb{R}.$$

- ▶ Any  $x \in \mathbb{R}^n$  is  $x = (x_1, \dots, x_n)$ , with  $x_i \in \mathbb{R}$ .
- ▶ We write  $f(x)$  instead of  $f(x_1, \dots, x_n)$ .

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m.$$

- ▶ If  $y = f(x)$  then  $y_i$  is the  $i$ -th component of  $y \in \mathbb{R}^m$ .
- ▶ Can be seen as  $m$  functions  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ , with  $y_i = f_i(x)$ .

# Derivatives in many dimensions

## Partial derivative

The partial derivative of  $f$  with respect to its  $i$ -th argument is:  $\frac{\partial}{\partial x_i} f(x)$ , where we see all  $x_j$  with  $j \neq i$  as fixed.

## Gradient of $f$

This is the vector of all its partial derivatives:

$$\nabla_x f(x) = \left( \frac{\partial}{\partial x_1} f(x) \cdots \frac{\partial}{\partial x_i} f(x) \cdots \frac{\partial}{\partial x_n} f(x) \right)^\top$$

## Directional derivative

$$D_\delta f(x) = \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon \delta) - f(x)}{\epsilon}.$$