

Linear Regression

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Outline

The Linear Model

- Simple linear regression

- History

- Multiple linear regression

Optimisation algorithms

- Gradient Descent

- Least-Squares

Regression libraries in Python

- sklearn

- statsmodels

Problems

- Interpretation Problem parameters

- Exercises

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Simple linear regression

Input and output

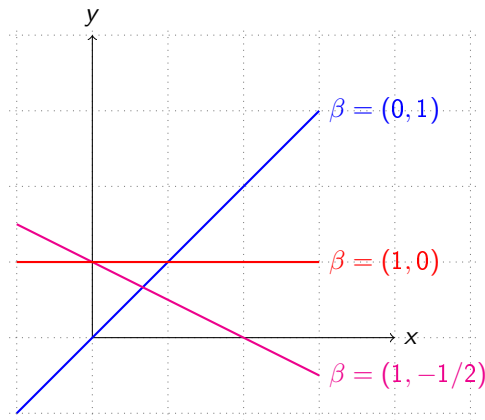
- ▶ Data pairs (x_t, y_t) , $t = 1, \dots, T$.
- ▶ Input $x_t \in \mathbb{R}$
- ▶ Output $y_t \in \mathbb{R}$.

Modelling the conditional expectation $\mathbb{E}[y_t|x_t]$

- ▶ Parameters $\beta_0, \beta_1 \in \mathbb{R}$
- ▶ Function $\pi_\beta : \mathbb{R} \rightarrow \mathbb{R}$, defined as

$$\pi_\beta(x_t) = \beta_0 + \beta_1 x_t$$

Linear models



$$\pi_{\beta}(x) = \beta_0 + \beta_1 x = [\beta_0, \beta_1] \begin{bmatrix} 1 \\ x \end{bmatrix}$$

Two views of the problem

Learning as optimisation

- ▶ Each value $\pi_\beta(x)$ is a **prediction** about the value of y
- ▶ We suffer a loss $\ell(y, \pi_\beta(x))$ for every example (x, y) that we see.
- ▶ We want to minimise the average loss over the data
- ▶ Ideally, we want to minimise the expected loss, but the distribution P is unknown.

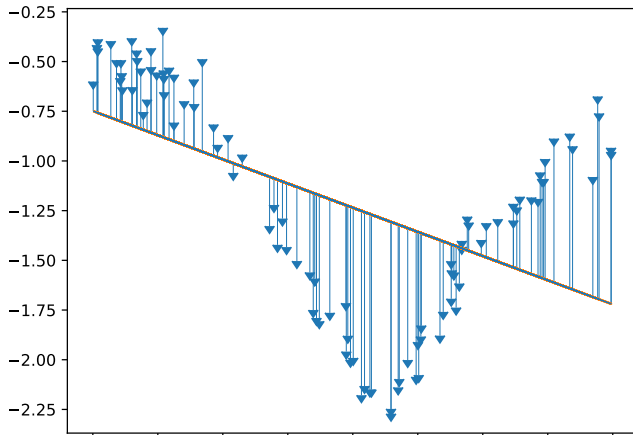
Learning as inference

- ▶ The parameters β define a **probabilistic model** $P_\beta(y|x)$ for every value of y .
- ▶ We want to find the parameters giving the highest probability on the observed data.
- ▶ Ideally, we want to find the true conditional distribution $P(y|x)$.

Learning as Optimisation

Find the parameters β minimising squared error

$$\min_{\beta} \frac{1}{T} \sum_{t=1}^T \underbrace{[y_t - \pi_{\beta}(x_t)]^2}_{\text{residual}}$$



Origins



Figure: Gauss: originator



Figure: Legendre: first publication

17 Bc

Diagram illustrating the orbits of Ceres and other celestial bodies around the Sun (Sonne).

Key features and labels:

- Sonne**: The central body, labeled "Sonne" and "Centr. Ell. prot. Ceresis".
- Erdbahn**: The Earth's orbit, labeled "Erdbahn".
- Ceres' Discovery**: "Ceres entdeckt von Piazzi Jan. 1. 1801".
- Ceres' Last Observation**: "Ceres letztes Beob. d. Ceres Febr. 11. 1801".
- Ceres' Return**: "Ceres wieder gesehen im Jahr Dec. 7. 1801".
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Maximum likelihood inference

Gaussian noise model:

$$y_t = f(x_t) + \epsilon_t, \quad \epsilon_t \sim \text{Normal}(0, \sigma)$$

With conditional density

$$p_\beta(y_t|x_t) \propto \exp(-[y_t - \pi_\beta(x_t)]^2/2\sigma^2)$$

Maximum likelihood inference

Idea: For data D , find parameters maximising $P_\beta(D)$

$$\begin{aligned} \arg \max_{\beta} P_\beta(D) &= \arg \max_{\beta} p_\beta(y_1, \dots, y_T | x_1, \dots, x_T) = \arg \max_{\beta} \ln \prod_t p_\beta(y_t | x_t) \\ &= \arg \max_{\beta} \sum \ln p_\beta(y_t | x_t) \\ &= \arg \max_{\beta} \sum_t \ln \{ \exp(-[y_t - \pi_\beta(x_t)]^2/2\sigma^2) \} \\ &= \arg \max_{\beta} \sum_t -[y_t - \pi_\beta(x_t)]^2/2\sigma^2 = \arg \min_{\beta} \sum_t |y_t - \pi_\beta(x_t)|^2 \end{aligned}$$

Coding break

- ▶ Show implementation
- ▶ Fit and residuals
- ▶ Multiple draws from the distribution
- ▶ Fit on non-linear data?

Multiple linear regression

Input and output

- ▶ Data pairs (\mathbf{x}_t, y_t) , $t = 1, \dots, T$.
- ▶ Input $\mathbf{x}_t \in \mathbb{R}^n$
- ▶ Output $y_t \in \mathbb{R}^m$.

Modelling the conditional expectation $\mathbb{E}[y_t | \mathbf{x}_t]$

- ▶ Parameters $\beta \in \mathbb{R}^{n \times m}$
- ▶ Function $\pi_\beta : \mathbb{R}^n \rightarrow \mathbb{R}^m$, defined as

$$\pi_\beta(\mathbf{x}_t) = \beta^\top \mathbf{x}_t = \sum_{i=1}^n \beta_i \mathbf{x}_{t,i}$$

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Gradient descent algorithm

Minimising a function

$$\min_{\beta} f(\beta) \leq f(\beta') \forall \beta', \quad \beta^* = \arg \min_{\beta} f(\beta) \Rightarrow f(\beta^*) = \min_{\beta} f(\beta)$$

Gradient descent algorithm

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Gradient descent for minimisation

- ▶ Input β_0
- ▶ For $n = 0, \dots, N$:
- ▶ $\beta_{n+1} = \beta_n - \eta_n \nabla_{\beta} f(\beta_n)$

Gradient descent algorithm

Minimising a function

$$\min_{\beta} f(\beta) \leq f(\beta') \forall \beta', \quad \beta^* = \arg \min_{\beta} f(\beta) \Rightarrow f(\beta^*) = \min_{\beta} f(\beta)$$

Gradient descent for minimisation

- ▶ Input β_0
- ▶ For $n = 0, \dots, N$:
- ▶ $\beta_{n+1} = \beta_n - \eta_n \nabla_{\beta} f(\beta_n)$

Step-size η_n

- ▶ η_n fixed: for online learning
- ▶ $\eta_n = c/[c + n]$ for asymptotic convergence
- ▶ $\eta_n = \arg \min_{\eta} f(\theta_n + \eta \nabla_{\beta})$: Line search.

Gradient descent for squared error

The cost function

$$L(\beta, D) = \sum_{t=1}^T (y_t - \pi_{\beta}(x_t))^2 = \sum_{t=1}^T \epsilon_t^2, \text{ with } \epsilon_t \triangleq y_t - \pi_{\beta}(x_t).$$

Gradient descent for squared error

The cost function

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Cost gradient

Using the chain rule of differentiation, $\nabla_{\beta} f(\epsilon) = \nabla_{\epsilon} f(\epsilon) \nabla_{\beta} \epsilon$.

$$\begin{aligned} \nabla_{\beta} L(\beta, D) &= \nabla_{\beta} \sum_{t=1}^T \epsilon_t^2 = \sum_{t=1}^T \nabla_{\beta} \epsilon_t^2 = \sum_{t=1}^T \nabla_{\epsilon_t} \epsilon_t^2 \nabla_{\beta} \epsilon_t \\ &= \sum_{t=1}^T 2\epsilon_t \nabla_{\beta} [y_t - \pi_{\beta}(x_t)] = \sum_{t=1}^T 2[y_t - \pi_{\beta}(x_t)] [-\nabla_{\beta} \pi_{\beta}(x_t)] \end{aligned}$$

Gradient descent for squared error

The cost function

$$L(\beta, D) = \sum_{t=1}^T (y_t - \pi_{\beta}(x_t))^2 = \sum_{t=1}^T \epsilon_t^2, \text{ with } \epsilon_t \triangleq y_t - \pi_{\beta}(x_t).$$

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Parameter gradient for linear regression

Remember $\nabla_{\beta} f = (\partial/\partial_1 f, \dots, \partial/\partial_n f)$

$$\frac{\partial}{\partial \beta_j} \pi_{\beta}(x_t) = \frac{\partial}{\partial \beta_j} \sum_{i=1}^n \beta_i x_{t,i} = \sum_{i=1}^n \frac{\partial}{\partial \beta_j} \beta_i x_{t,i} = x_{t,j}.$$

Stochastic gradient descent algorithm

When f is an expectation

$$f(\beta) = \int_{\mathcal{X}} dP(x) g(x, \beta).$$

Replacing the expectation with a sample:

$$\begin{aligned} \nabla f(\beta) &= \int_{\mathcal{X}} dP(x) \nabla g(x, \beta) \\ &\approx \frac{1}{K} \sum_{k=1}^K \nabla g(x^{(k)}, \beta), \end{aligned} \quad x^{(k)} \sim P.$$

Analytical Least-Squares Solution

We need to solve the following equations for A :

$$\begin{aligned}y_1 &= x_1^\top \beta \\ \dots & \dots \\ y_t &= x_t^\top \beta \\ \dots & \dots \\ y_T &= x_T^\top \beta\end{aligned}$$

We can rewrite it in matrix form:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_t \\ \vdots \\ y_T \end{pmatrix} = \begin{pmatrix} x_1^\top \\ \vdots \\ x_t^\top \\ \vdots \\ x_T^\top \end{pmatrix} \beta$$

Resulting in

$$\mathbf{y} = \mathbf{X}\beta.$$

How can we get β ?

Finding the β

We now have a linear equation,

$$\mathbf{y} = X\beta.$$

We want to solve for β . If X had an inverse X^{-1} , we could obtain

$$X^{-1}\mathbf{y} = X^{-1}X\beta = I\beta = \beta.$$

But X^{-1} **does not exist**.

Least-squares solution

The **left-pseudo inverse** $\tilde{X}^{-1} \triangleq (X^{\top}X)^{-1}X^{\top}$ can be used to obtain

$$\beta = \tilde{X}^{-1}\mathbf{y},$$

This follows as:

$$\begin{aligned}\mathbf{y} &= X\beta \\ \tilde{X}^{-1}\mathbf{y} &= \tilde{X}^{-1}X\beta \\ \tilde{X}^{-1}\mathbf{y} &= \underbrace{(X^{\top}X)^{-1}}_{A^{-1}} \underbrace{X^{\top}X}_A \beta.\end{aligned}$$

Some matrix algebra reminders

The identity matrix $I \in \mathbb{R}^{n \times n}$

- ▶ For this matrix, $I_{i,i} = 1$ and $I_{i,j} = 0$ when $j \neq i$.
- ▶ $Ix = x$ and $IA = A$.

The inverse of a matrix $A \in \mathbb{R}^{n \times n}$

A^{-1} is called the inverse of A if

- ▶ $AA^{-1} = I$.
- ▶ or equivalently $A^{-1}A = I$.

The pseudo-inverse of a matrix $A \in \mathbb{R}^{n \times m}$

- ▶ \tilde{A}^{-1} is called the **left pseudoinverse** of A if $\tilde{A}^{-1}A = I$.

$$\tilde{A}^{-1} = (A^T A)^{-1} A^T, \quad n > m$$

- ▶ \tilde{A}^{-1} is called the **right pseudoinverse** of A if $A\tilde{A}^{-1} = I$.

$$\tilde{A}^{-1} = A^T (AA^T)^{-1}, \quad m > n$$

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Fitting a model to data

```
from sklearn.linear_model import LinearRegression  
model = LinearRegression().fit(X, Y)
```

Getting predictions

We can get predictions for all inputs as an array

```
Z = model.predict(X)
```

Statsmodels

Fitting a model to data X, Y

```
import statsmodels.api as sm
Xa = sm.add_constant(X) # adds a constant factor to the data
model = sm.OLS(Y, Xa)
results = model.fit()
```

Getting predictions

The prediction is not just a point!

```
z = results.get_prediction(Xa[t])
z.predicted_mean # This is  $E[y|x]$ 
```

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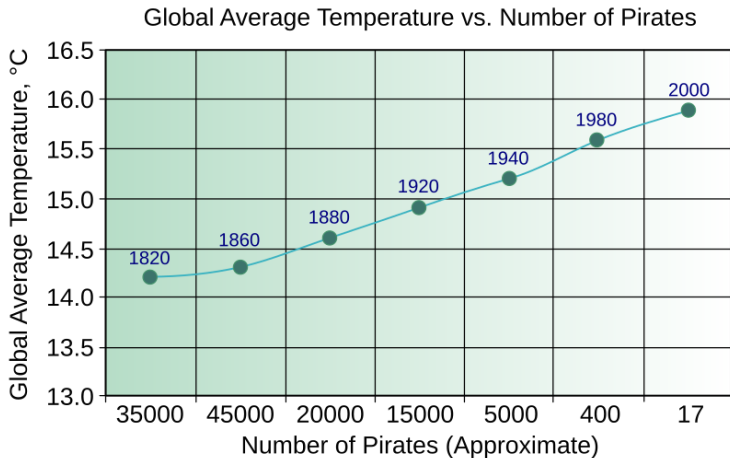
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Pitfalls

- ▶ β_i tells us how much y is correlated with $x_{t,i}$
- ▶ However, multiple correlations might be evident.
- ▶ Some features may be irrelevant
- ▶ The relationship may not be linear
- ▶ Correlation is not causation

Correlation is not causation



Linear regression exercises

- ▶ Exercises 8, 13 from ISLP
- ▶ A variant of Ex. 13 but with Y generated independently of X .