

Nearest Neighbour Algorithms

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Supervised learning

- ▶ Given labelled training examples $(x_1, y_1), \dots, (x_T, y_T)$ where
- ▶ $x_t \in X$ are **features**
- ▶ $y_t \in Y$ are **labels**..

Feature space

- ▶ Usually $= \mathbb{R}^n$: the n-dimensional Euclidean space
- ▶ How do we use your class data?

Classification

- ▶ $Y = \{1, \dots, m\}$ are **discrete** labels

Regression

- ▶ $Y = \mathbb{R}$ are **continuous** values

The kNN algorithm idea

- ▶ Assume an unknown example is similar to its neighbours
- ▶ Smoothness allows us to make predictions

Discriminatory analysis-nonparametric discrimination: consistency properties, Evelyn Fix and Joseph L. Hodges Jr, 1951.



Figure: Evelyn Fix



Figure: Joseph Hodges

Performance of KNN on image classification

`./fig/knn-image-performance.png`

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The Nearest Neighbour algorithm

Pseudocode

- ▶ Input: Data $(x_t, y_t)_{t=1}^T$, test point x , distance d
- ▶ $t^* = \arg\min_t d(x_t, x)$ / How do we implement this?
- ▶ Return $\hat{y}_t = y_{t^*}$

Classification

$$\hat{y}_t \in [m] \equiv \{1, \dots, m\}$$

Regression

$$\hat{y}_t \in \mathbb{R}$$

The k-Nearest Neighbour algorithm

Pseudocode

- ▶ Input: Data $(x_t, y_t)_{t=1}^T$, test point x , distance d , neighbours k
- ▶ Calculate $h_t = d(x_t, x)$ for all t .
- ▶ Get sorted indices $s = \text{argsort}(h)$ so that $d(x_{s_i}, x) \leq d(x_{s_{i+1}}, x)$ for all i . (How?)
- ▶ Return $\sum_{i=1}^k y_{s_i} / k$.

Classification

- ▶ It is not convenient to work with discrete labels.
- ▶ We use a **one-hot encoding** $(0, \dots, 0, 1, 0, \dots, 0)$.
- ▶ $y_t \in \{0, 1\}^m$ with $\|y_t\|_1 = 1$, so that the class of the t -th example is j iff $y_{t,j} = 1$.

Regression

- ▶ $y_t \in \mathbb{R}^m$, so we need do nothing

Making a decision

kNN Output

- ▶ Given features x , we get a vector
- ▶ $p_i = \hat{\Pr}(y = i|x)$.

Accuracy

- ▶ Predicted label a
- ▶ Actual label y
- ▶ $U(a, y) = a = y$

Classification decision to maximise accuracy

- ▶ $a_t = \underset{i}{\operatorname{argmax}} p_i$

The number of neighbours

$$k = 1$$

- ▶ How does it perform on the training data?
- ▶ How might it perform on unseen data?

$$k = T$$

- ▶ How does it perform on the training data?
- ▶ How might it perform on unseen data?

Distance function

For data in n , p -norm

$$d(x, y) = \|x - y\|_p$$

Scaled norms

When features having varying scales:

$$d(x, y) = \|Sx - Sy\|_p$$

Or pre-scale the data

Complex data

- ▶ Manifold distances
- ▶ Graph distance

Distances

A distance $d(\cdot, \cdot)$:

- ▶ Identity $d(x, x) = 0$.
- ▶ Positivity $d(x, y) > 0$ if $x \neq y$.
- ▶ Symmetry $d(y, x) = d(x, y)$.
- ▶ Triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$.

For data in \mathbb{R}^n , ℓ_p -norm

$$d(x, y) = \|x - y\|_p$$

Norms;

A norm $\|\cdot\|$

- ▶ Zero element $\|0\| = 0$.
- ▶ Homogeneity $\|cx\| = c\|x\|$ for any scalar a .
- ▶ Triangle inequality $\|x + y\| \leq \|x\| + \|y\|$.

ℓ_p -norm

$$\|z\|_p = \left(\sum_i z_i^p \right)^{1/p}$$

Neighbourhood calculation

If we have T datapoints

Sort and top K .

- Requires $O(T \ln T)$ time

Use the Cover-Tree or KD-Tree algorithm

- Requires $O(cK \ln T)$ time.
- c depends on the data distribution.

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KNN activity

- ▶ Implement nearest neighbours
- ▶ Introduction to scikitlearn nearest neighbours