The perceptron algorithm

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Outline

The Perceptron

- Introduction
- The algorithm

Gradient methods

- Gradients for optimisation
- The perceptron as a gradient algorithm

Lab and Assignment

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Lab and Assignment

Guessing gender from height

- Feature space $\mathcal{X} \subset \mathbb{R}$: e.g. height
- Label space $\mathcal{Y} = \{-1, 1\}$: e.g. gender
- Can we find some $\beta_1 \in \mathbb{R}$ and a direction $\beta_0 \in \{-1, +1\}$ so as to separate the genders?

Online learning: At time t

- We choose a separator β_0^t, β_1^t
- We observe a new datapoint x_t, y_t
- We make a mistake at time t if:

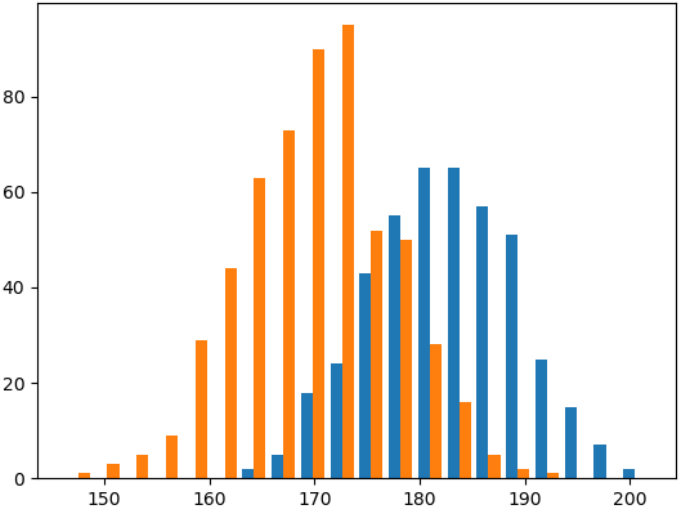
$$\beta^t x_t - \beta_0^t \leq 0.$$

- If we stop making mistakes, then we are classifying everything perfectly.

Can you find a threshold that makes a small number of mistakes?

./src/Perceptron/perceptron_simple.py

Non-separable classes



More complex example

- Feature space $\mathcal{X} \subset \mathbb{R}^n$: e.g. height and weight for $n = 2$
- Label space $\mathcal{Y} = \{-1, 1\}$: e.g. gender
- Can we find some line so as to separate the genders?

-./src/Perceptron/show_class_data_labels.py

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Linear separator

We now have parameters $\beta_0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$ defining a hyperplane $f(x) = 0$ in $\mathbb{R}^n$

$$f(x) = \beta_0 + \beta^\top x = \beta_0 + \sum_{i=1}^{n} \beta_i x_i.$$

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- If $f(x) > 0$, we assign class $+1$
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If we augment x an additional component $x_0 = 1$, we can write

$$f(x) = \beta^\top x = \sum_{i=0}^{n} \beta_i x_i.$$

The perceptron algorithm

Input

- Feature space $X \subset \mathbb{R}^n$.
- Label space $Y = \{-1, 1\}$.
- Data (x_t, y_t), $t \in [T]$, with $x_t \in X, y_t \in Y$.

Algorithm

```
▶ $\beta^0 \sim \mathrm{Normal}^n(0, I)$. % Initialise parameters
▶ For $t = 1, \ldots, T$
    ▶ $a_t = \mathrm{sgn}(\beta^t \cdot x_t)$. % Classify example
    ▶ If $a_t \neq y_t$
        ▶ $\beta^t = \beta^{t-1} + y_t x_t$ % Move hyperplane
    ▶ Else
        ▶ $\beta^t = \beta^{t-1}$ % Do nothing for correct examples
    ▶ EndIf
▶ Return $\beta^T$
```

Perceptron examples

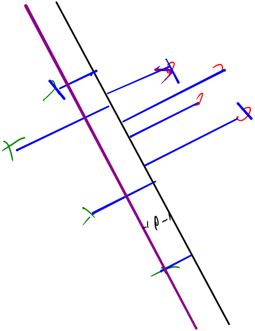
Example 1: One-dimensional data

- Done on the board
- Shows how the algorithm works.
- Demonstrates the idea of a margin

Example 2: Two-dimensional data

- See in-class programming exercise

Margins and the perceptron theorem



- The hyperplane β^* separates the examples
- The margin ρ is the minimum distance ρ between β^* and any point.

Theorem (Perceptron theorem)

The number of mistakes is bounded by ρ^{-2}, where $\|x_t\| \leq 1$, $\rho \leq y_t(x_t^\top \beta^*)$ for some margin ρ and hyperplane β^* with $\|\beta^*\| = 1$.

The Perceptron

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The algorithm

Gradient methods

Gradients for optimisation

The perceptron as a gradient algorithm

Lab and Assignment

The gradient descent method: one dimension

- Function to minimise $f : \mathbb{R} \to \mathbb{R}$.
- Derivative $\frac{d}{d\beta} f(\beta)$

Gradient descent algorithm

- Input: initial value β^0, learning rate schedule α_t
- For $t = 1, \ldots, T$
  - $\beta^{t+1} = \beta^t - \alpha_t \frac{d}{d\beta} f(\beta^t)$
- Return β^T

Properties

- If $\sum_t \alpha_t = \infty$ and $\sum_t \alpha_t^2 < \infty$, it finds a local minimum β^T, i.e. there is $\epsilon > 0$ so that

$$f(\beta^T) < f(\beta), \forall \beta : \|\beta^T - \beta\| < \epsilon.$$

Gradient methods for expected value

Estimate the expected value

$x_t \sim P$ with $\mathbb{E}_P[x_t] = \mu$.

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Here $\ell(x, \beta) = (x - \beta)^2$.

$$\min_\beta \mathbb{E}_P[(x_t - \beta)^2].$$

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Derivative

Idea: at the minimum the derivative should be zero.

$$d/d\beta \, \mathbb{E}_P[(x_t - \beta)^2] = \mathbb{E}_P[d/d\beta (x_t - \beta)^2] = \mathbb{E}_P[-(x_t - \beta)] = \mathbb{E}_P[x_t] - \beta.$$

Setting the derivative to 0, we have $\beta = \mathbb{E}_P[x_t]$. This is a simple solution.

Real-world setting

- The objective function does not result in a simple solution
- The distribution P is not known.
- We can sample $x \sim P$.

The gradient method

- Function to minimise $f : \mathbb{R}^n \to \mathbb{R}$.
- Derivative $\nabla_\beta f(\beta) = \left(\frac{\partial f(\beta)}{\partial \beta_1}, \ldots, \frac{\partial f(\beta)}{\partial \beta_n}\right)$, where $\frac{\partial f}{\partial \beta_n}$ denotes the partial derivative, i.e. varying one argument and keeping the others fixed.

Gradient descent algorithm

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Stochastic gradient method

This is the same as the gradient method, but with added noise:

- $\beta^{t+1} = \beta^t - \alpha_t [\nabla_\beta f(\beta^t) + \omega_t]$
- $\mathbb{E}[\omega_t] = 0$ is sufficient for convergence.

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Example (When the cost is an expectation)

In machine learning, the cost is frequently an expectation of some function ℓ,

$$f(\beta) = \int_X dP(x)\ell(x, \beta)$$

This can be approximated with a sample

$$f(\beta) \approx \frac{1}{T}\sum_t \ell(x_t, \beta)$$

The same holds for the gradient:

$$\nabla_\beta f(\beta) = \int_X dP(x)\nabla_\beta \ell(x, \beta) \approx \frac{1}{T}\sum_t \nabla_\beta \ell(x_t, \beta)$$

Stochastic gradient for mean estimation

- If we sample x we approximate the gradient:

$$\frac{d}{d\beta}\,\mathbb{E}_P[(x-\beta)^2] \approx \frac{1}{T}\sum_{t=1}^{T}\frac{d}{d\beta}(x_t-\beta)^2 = \frac{1}{T}\sum_{t=1}^{T}2(x_t-\beta)$$

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- If we update β after each new sample x_t, we obtain:

$$\beta^{t+1} = \beta^t + 2\alpha_t(x_t-\beta^t)$$

Perceptron algorithm as gradient descent

Target error function

$$\mathbb{E}_{P}^{\beta}[\ell] = \int_{\mathcal{X}} dP(x) \sum_{y} P(y|x)\ell(x, y, \beta)$$

Minimises the error on the true distribution.

Perceptron algorithm as gradient descent

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Empirical error function

$$\mathbb{E}_{D}^{\beta}[\ell] = \frac{1}{T}\sum_{t=1}^{T} \ell(x_t, y_t, \beta), \qquad D = (x_t, y_t)_{t=1}^{T}, \quad x_t, y_t \sim P.$$

Minimises the error on the empirical distribution.

Cost functions and the chain rule

Perceptron cost function

The cost of each example

$$\ell(x, y, \beta) = \overbrace{\mathbb{I}\left\{y(x^\top \beta) < 0\right\}}^{\text{misclassified?}} \overbrace{[-y(x^\top \beta)]}^{\text{margin of error}} \tag{1}$$

where the indicator function $\mathbb{I}\left\{A\right\}$ is 1 when A is true and 0 otherwise.

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Let $z = g(y)$, $y = f(x)$ so that $z = g(f(x))$. Then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

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Derivative: Chain rule

- $\nabla_\beta \ell(x, y, \beta) = -\mathbb{I}\left\{y(x^\top \beta) < 0\right\} \nabla_\beta [y(x^\top \beta)]$.

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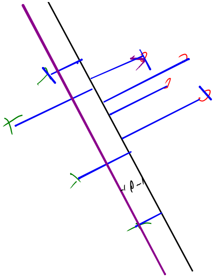
The classification error cost function is not differentiable :(

Margins and confidences

We can think of the output of the network as a measure of confidence



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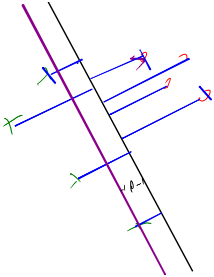

By applying the logit function, we can bound a real number x to $[0, 1]$:

$$f(x) = \frac{e^x}{1 + e^x} = \frac{1}{1 + e^{-x}}$$

Logistic regression

Output as a measure of confidence, given the parameter β

$$P_\beta(y = 1|x) = \frac{1}{1 + \exp(-x_t^\top \beta)}$$

The original output $x_t^\top \beta$ is now passed through the logit function.

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Negative Log likelihood

$\ell(x_t, y_t, \beta) = -\ln P_\beta(y_t|x_t) = \ln(1 + \exp(-y_t x_t^\top \beta))$

$$\begin{aligned}
\nabla_\beta \ell(x_t, y_t, \beta) &= \frac{1}{1 + \exp(-y x_t^\top \beta)} \nabla_\beta [1 + \exp(-y x_t^\top \beta)] \\
&= \frac{1}{1 + \exp(-y x_t^\top \beta)} \exp(-y x_t^\top \beta) [\nabla_\beta (-y_t x_t^\top \beta)] \\
&= -\frac{1}{1 + \exp(x_t^\top \beta)} (x_{t,i})_{i=1}^n e
\end{aligned}$$

- $\mathbb{E}_P(\ell) = \int_X dP(x) \sum_{y \in Y} P(y|x) P_\beta(y_t + x_t)$

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Lab demonstrations

- How to use kNN and Perceptron with sklearn (and perhaps statsmodels, time permitting)
- Use an example where there is no default 'class' label

Assignment

1. In the class data, find one categorical variable of interest that we want to predict.
2. Formulate the appropriate classification problem.
3. Perform model selection through train/validate or crossvalidation to find the best model (kNN or perceptron) and hyperparameters (k for the kNN)
4. Discuss anything of interest in the data such as: feature scaling/selection, missing data, outliers.
5. We cannot independently measure the quality of the model, as we have no test set. What can we do?