The perceptron algorithm

Christos Dimitrakakis

October 2, 2024

Outline

Guessing gender from height

- Feature space $\mathcal{X} \subset \mathbb{R}$: e.g. height
- Label space $\mathcal{Y} = \{-1, 1\}$: e.g. gender
- Can we find some $\beta_1 \in \mathbb{R}$ and a direction $\beta_0 \in \{-1, +1\}$ so as to separate the genders?

Online learning: At time t

- We choose a separator β_0^t, β_1^t
- We observe a new datapoint x_t, y_t
- We make a mistake at time t if:

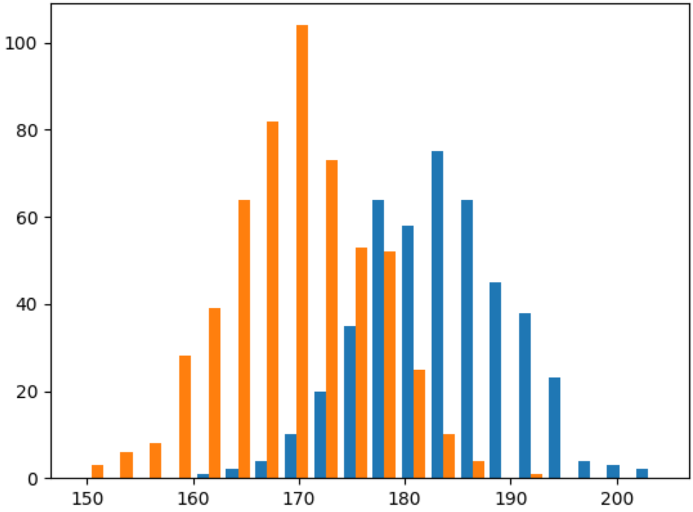
$$\beta^t x_t - \beta_0^t \leq 0.$$

- If we stop making mistakes, then we are classifying everything perfectly.

Can you find a threshold that makes a small number of mistakes?

`./src/Perceptron/perceptron_simple.py`

Non-separable classes



More complex example

- Feature space $\mathcal{X} \subset \mathbb{R}^n$: e.g. height and weight for $n = 2$
- Label space $\mathcal{Y} = \{-1, 1\}$: e.g. gender
- Can we find some line so as to separate the genders?

-./src/Perceptron/show_class_data_labels.py

More complex example

- Feature space $\mathcal{X} \subset \mathbb{R}^n$: e.g. height and weight for $n = 2$
- Label space $\mathcal{Y} = \{-1, 1\}$: e.g. gender
- Can we find some line so as to separate the genders?

`-./src/Perceptron/show_class_data_labels.py`

Linear separator

We now have parameters $\beta_0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$ defining a hyperplane $f(x) = 0$ in $\mathbb{R}^n$

$$f(x) = \beta_0 + \beta^\top x = \beta_0 + \sum_{i=1}^{n} \beta_i x_i.$$

More complex example

- Feature space $\mathcal{X} \subset \mathbb{R}^n$: e.g. height and weight for $n = 2$
- Label space $\mathcal{Y} = \{-1, 1\}$: e.g. gender
- Can we find some line so as to separate the genders?

```
-./src/Perceptron/show_class_data_labels.py
```

Linear separator

We now have parameters $\beta_0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$ defining a hyperplane $f(x) = 0$ in $\mathbb{R}^n$

$$f(x) = \beta_0 + \beta^\top x = \beta_0 + \sum_{i=1}^{n} \beta_i x_i.$$

- The perceptron decision rule is $\pi(x) = \mathrm{sign}(f(x))$
- If $f(x) > 0$, we assign class $+1$
- If $f(x) < 0$, we assign class -1

More complex example

- Feature space $\mathcal{X} \subset \mathbb{R}^n$: e.g. height and weight for $n = 2$
- Label space $\mathcal{Y} = \{-1, 1\}$: e.g. gender
- Can we find some line so as to separate the genders?

`-./src/Perceptron/show_class_data_labels.py`

Linear separator

We now have parameters $\beta_0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^n$ defining a hyperplane $f(x) = 0$ in $\mathbb{R}^n$

$$f(x) = \beta_0 + \beta^\top x = \beta_0 + \sum_{i=1}^{n} \beta_i x_i.$$

- The perceptron decision rule is $\pi(x) = \mathrm{sign}(f(x))$
- If $f(x) > 0$, we assign class +1
- If $f(x) < 0$, we assign class -1

If we augment x an additional component $x_0 = 1$, we can write

$$f(x) = \beta^\top x = \sum_{i=0}^{n} \beta_i x_i.$$

The perceptron algorithm

Input

- Feature space $X \subset \mathbb{R}^n$.
- Label space $Y = \{-1, 1\}$.
- Data (x_t, y_t), $t \in [T]$, with $x_t \in X, y_t \in Y$.

Algorithm

- $\beta^0 \sim \mathrm{Normal}^n(0, I)$. % Initialise parameters
- For $t = 1, \ldots, T$
  - $a_t = \mathrm{sgn}(\beta^t \cdot x_t)$. % Classify example
  - If $a_t \neq y_t$
    - $\beta^t = \beta^{t-1} + y_t x_t$ % Move hyperplane
  - Else
    - $\beta^t = \beta^{t-1}$ % Do nothing for correct examples
  - EndIf
- Return β^T

Perceptron examples

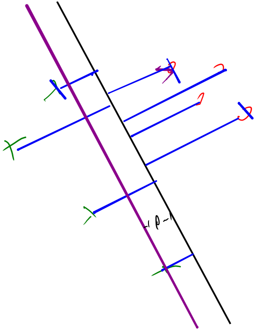
Example 1: One-dimensional data

- Done on the board
- Shows how the algorithm works.
- Demonstrates the idea of a margin

Example 2: Two-dimensional data

- See in-class programming exercise

Margins and the perceptron theorem



- The hyperplane β^* separates the examples
- The margin ρ is the minimum distance ρ between β^* and any point.

Theorem (Perceptron theorem)

The number of mistakes is bounded by ρ^{-2}, where $\|x_t\| \leq 1$, $\rho \leq y_t(x_t^\top \beta^*)$ for some margin ρ and hyperplane β^* with $\|\beta^*\| = 1$.

The gradient descent method: one dimension

- Function to minimise $f : \mathbb{R} \to \mathbb{R}$.
- Derivative $\frac{d}{d\beta} f(\beta)$

Gradient descent algorithm

- Input: initial value β^0, learning rate schedule α_t
- For $t = 1, \ldots, T$
  - $\beta^{t+1} = \beta^t - \alpha_t \frac{d}{d\beta} f(\beta^t)$
- Return β^T

Properties

- If $\sum_t \alpha_t = \infty$ and $\sum_t \alpha_t^2 < \infty$, it finds a local minimum β^T, i.e. there is $\epsilon > 0$ so that

$$f(\beta^T) < f(\beta), \forall \beta : \|\beta^T - \beta\| < \epsilon.$$

Gradient methods for expected value

Estimate the expected value

$x_t \sim P$ with $\mathbb{E}_P[x_t] = \mu$.

Gradient methods for expected value

Estimate the expected value

$x_t \sim P$ with $\mathbb{E}_P[x_t] = \mu$.

Objective: mean squared error

Here $\ell(x, \beta) = (x - \beta)^2$.

$$\min_\beta \mathbb{E}_P[(x_t - \beta)^2].$$

Gradient methods for expected value

Estimate the expected value

$x_t \sim P$ with $\mathbb{E}_P[x_t] = \mu$.

Objective: mean squared error

Here $\ell(x, \beta) = (x - \beta)^2$.

$$\min_\beta \mathbb{E}_P[(x_t - \beta)^2].$$

Derivative

Idea: at the minimum the derivative should be zero.

$$d/d\beta \, \mathbb{E}_P[(x_t - \beta)^2] = \mathbb{E}_P[d/d\beta (x_t - \beta)^2] = \mathbb{E}_P[-(x_t - \beta)] = \mathbb{E}_P[x_t] - \beta.$$

Setting the derivative to 0, we have $\beta = \mathbb{E}_P[x_t]$. This is a simple solution.

Real-world setting

- The objective function does not result in a simple solution
- The distribution P is not known.
- We can sample $x \sim P$.

The gradient method

- Function to minimise $f : \mathbb{R}^n \to \mathbb{R}$.
- Derivative $\nabla_\beta f(\beta) = \left(\frac{\partial f(\beta)}{\partial \beta_1}, \ldots, \frac{\partial f(\beta)}{\partial \beta_n}\right)$, where $\frac{\partial f}{\partial \beta_n}$ denotes the partial derivative, i.e. varying one argument and keeping the others fixed.

Gradient descent algorithm

- Input: initial value β^0, learning rate schedule α_t
- For $t = 1, \ldots, T$
  - $\beta^{t+1} = \beta^t - \alpha_t \nabla_\beta f(\beta^t)$
- Return β^T

Properties

- If $\sum_t \alpha_t = \infty$ and $\sum_t \alpha_t^2 < \infty$, it finds a local minimum β^T, i.e. there is $\epsilon > 0$ so that

$$f(\beta^T) < f(\beta), \forall \beta : \|\beta^T - \beta\| < \epsilon.$$

Stochastic gradient method

This is the same as the gradient method, but with added noise:

- $\beta^{t+1} = \beta^t - \alpha_t[\nabla_\beta f(\beta^t) + \omega_t]$
- $\mathbb{E}[\omega_t] = 0$ is sufficient for convergence.

Stochastic gradient method

This is the same as the gradient method, but with added noise:

- $\beta^{t+1} = \beta^t - \alpha_t[\nabla_\beta f(\beta^t) + \omega_t]$
- $\mathbb{E}[\omega_t] = 0$ is sufficient for convergence.

Example (When the cost is an expectation)

In machine learning, the cost is frequently an expectation of some function ℓ,

$$f(\beta) = \int_X dP(x)\ell(x, \beta)$$

This can be approximated with a sample

$$f(\beta) \approx \frac{1}{T} \sum_t \ell(x_t, \beta)$$

The same holds for the gradient:

$$\nabla_\beta f(\beta) = \int_X dP(x)\nabla_\beta \ell(x, \beta) \approx \frac{1}{T} \sum_t \nabla_\beta \ell(x_t, \beta)$$

Stochastic gradient for mean estimation

- If we sample x we approximate the gradient:

$$\frac{d}{d\beta} \mathbb{E}_P[(x-\beta)^2] \approx \frac{1}{T} \sum_{t=1}^{T} \frac{d}{d\beta} (x_t - \beta)^2 = \frac{1}{T} \sum_{t=1}^{T} 2(x_t - \beta)$$

Stochastic gradient for mean estimation

- If we sample x we approximate the gradient:

$$\frac{d}{d\beta} \mathbb{E}_P[(x-\beta)^2] \approx \frac{1}{T} \sum_{t=1}^{T} \frac{d}{d\beta}(x_t - \beta)^2 = \frac{1}{T} \sum_{t=1}^{T} 2(x_t - \beta)$$

- If we update β after each new sample x_t, we obtain:

$$\beta^{t+1} = \beta^t + 2\alpha_t(x_t - \beta^t)$$

Perceptron algorithm as gradient descent

Target error function

$$\mathbb{E}_{P}^{\beta}[\ell] = \int_{\mathcal{X}} dP(x) \sum_{y} P(y|x)\ell(x, y, \beta)$$

Minimises the error on the true distribution.

Perceptron algorithm as gradient descent

Target error function

$$\mathbb{E}_P^\beta[\ell] = \int_{\mathcal{X}} dP(x) \sum_y P(y|x)\ell(x, y, \beta)$$

Minimises the error on the true distribution.

Empirical error function

$$\mathbb{E}_D^\beta[\ell] = \frac{1}{T}\sum_{t=1}^{T} \ell(x_t, y_t, \beta), \qquad D = (x_t, y_t)_{t=1}^T, \quad x_t, y_t \sim P.$$

Minimises the error on the empirical distribution.

Cost functions and the chain rule

Perceptron cost function

The cost of each example

$$\ell(x, y, \beta) = \overbrace{\mathbb{I}\left\{y(x^\top \beta) < 0\right\}}^{\text{misclassified?}} \overbrace{[-y(x^\top \beta)]}^{\text{margin of error}} \tag{1}$$

where the indicator function $\mathbb{I}\{A\}$ is 1 when A is true and 0 otherwise.

Cost functions and the chain rule

Perceptron cost function

The cost of each example

$$\ell(x, y, \beta) = \overbrace{\mathbb{I}\left\{y(x^\top \beta) < 0\right\}}^{\text{misclassified?}} \overbrace{[-y(x^\top \beta)]}^{\text{margin of error}} \tag{1}$$

where the indicator function $\mathbb{I}\{A\}$ is 1 when A is true and 0 otherwise.

Reminder: The chain rule

Let $z = g(y)$, $y = f(x)$ so that $z = g(f(x))$. Then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

Cost functions and the chain rule

Perceptron cost function

The cost of each example

$$\ell(x, y, \beta) = \overbrace{\mathbb{I}\left\{y(x^\top \beta) < 0\right\}}^{\text{misclassified?}} \overbrace{[-y(x^\top \beta)]}^{\text{margin of error}} \tag{1}$$

where the indicator function $\mathbb{I}\{A\}$ is 1 when A is true and 0 otherwise.

Reminder: The chain rule

Let $z = g(y)$, $y = f(x)$ so that $z = g(f(x))$. Then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

Derivative: Chain rule

- $\nabla_\beta \ell(x, y, \beta) = -\mathbb{I}\left\{y(x^\top \beta) < 0\right\} \nabla_\beta [y(x^\top \beta)]$.

Cost functions and the chain rule

Perceptron cost function

The cost of each example

$$\ell(x, y, \beta) = \overbrace{\mathbb{I}\left\{y(x^\top \beta) < 0\right\}}^{\text{misclassified?}} \overbrace{[-y(x^\top \beta)]}^{\text{margin of error}} \tag{1}$$

where the indicator function $\mathbb{I}\{A\}$ is 1 when A is true and 0 otherwise.

Reminder: The chain rule

Let $z = g(y)$, $y = f(x)$ so that $z = g(f(x))$. Then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

Derivative: Chain rule

- $\nabla_\beta \ell(x, y, \beta) = -\mathbb{I}\left\{y(x^\top \beta) < 0\right\} \nabla_\beta [y(x^\top \beta)]$.
- $\frac{\partial \beta}{\partial \beta_i}[y(x_t^\top \beta)] = y x_{t,i}$ (gradient of Perceptron's output)

Cost functions and the chain rule

Perceptron cost function

The cost of each example

$$\ell(x, y, \beta) = \overbrace{\mathbb{I}\left\{y(x^\top \beta) < 0\right\}}^{\text{misclassified?}} \overbrace{[-y(x^\top \beta)]}^{\text{margin of error}} \tag{1}$$

where the indicator function $\mathbb{I}\{A\}$ is 1 when A is true and 0 otherwise.

Reminder: The chain rule

Let $z = g(y)$, $y = f(x)$ so that $z = g(f(x))$. Then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

Derivative: Chain rule

- $\nabla_\beta \ell(x, y, \beta) = -\mathbb{I}\left\{y(x^\top \beta) < 0\right\} \nabla_\beta [y(x^\top \beta)]$.
- $\frac{\partial \beta}{\partial \beta_i}[y(x_t^\top \beta)] = y x_{t,i}$ (gradient of Perceptron's output)
- Gradient update: $\beta^{t+1} = \beta^t - \nabla_\beta \ell(x, y, \beta) = \beta^t + y x_t$

Cost functions and the chain rule

Perceptron cost function

The cost of each example

$$\ell(x, y, \beta) = \overbrace{\mathbb{I}\left\{y(x^\top \beta) < 0\right\}}^{\text{misclassified?}} \overbrace{[-y(x^\top \beta)]}^{\text{margin of error}} \tag{1}$$

where the indicator function $\mathbb{I}\{A\}$ is 1 when A is true and 0 otherwise.

Reminder: The chain rule

Let $z = g(y)$, $y = f(x)$ so that $z = g(f(x))$. Then $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

Derivative: Chain rule

- $\nabla_\beta \ell(x, y, \beta) = -\mathbb{I}\left\{y(x^\top \beta) < 0\right\} \nabla_\beta [y(x^\top \beta)]$.
- $\frac{\partial \beta}{\partial \beta_i}[y(x_t^\top \beta)] = y x_{t,i}$ (gradient of Perceptron's output)
- Gradient update: $\beta^{t+1} = \beta^t - \nabla_\beta \ell(x, y, \beta) = \beta^t + y x_t$

Cost functions and the chain rule

Perceptron cost function

The cost of each example

$$\ell(x, y, \beta) = \overbrace{\mathbb{I}\left\{y(x^\top \beta) < 0\right\}}^{\text{misclassified?}} \overbrace{[-y(x^\top \beta)]}^{\text{margin of error}} \tag{1}$$

where the indicator function $\mathbb{I}\{A\}$ is 1 when A is true and 0 otherwise.

Reminder: The chain rule

Let $z = g(y)$, $y = f(x)$ so that $z = g(f(x))$. Then $\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}$

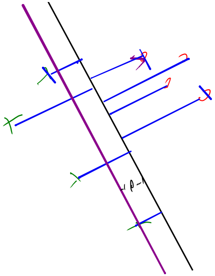
Derivative: Chain rule

- $\nabla_\beta \ell(x, y, \beta) = -\mathbb{I}\left\{y(x^\top \beta) < 0\right\} \nabla_\beta [y(x^\top \beta)]$.
- $\frac{\partial \beta}{\partial \beta_i}[y(x_t^\top \beta)] = y x_{t,i}$ (gradient of Perceptron's output)
- Gradient update: $\beta^{t+1} = \beta^t - \nabla_\beta \ell(x, y, \beta) = \beta^t + y x_t$

The classification error cost function is not differentiable :(

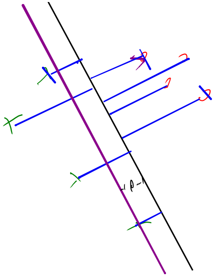
Margins and confidences

We can think of the output of the network as a measure of confidence



Margins and confidences

We can think of the output of the network as a measure of confidence



By applying the logit function, we can bound a real number x to $[0, 1]$:

$$f(x) = \frac{e^x}{1 + e^x} = \frac{1}{1 + e^{-x}}$$

Logistic regression

Output as a measure of confidence, given the parameter β

$$P_\beta(y = 1|x) = \frac{1}{1 + \exp(-x_t^\top \beta)}$$

The original output $x_t^\top \beta$ is now passed through the logit function.

Logistic regression

Output as a measure of confidence, given the parameter β

$$P_\beta(y = 1|x) = \frac{1}{1 + \exp(-x_t^\top \beta)}$$

The original output $x_t^\top \beta$ is now passed through the logit function.

Negative Log likelihood

$\ell(x_t, y_t, \beta) = -\ln P_\beta(y_t|x_t) = \ln(1 + \exp(-y_t x_t^\top \beta))$

$$\begin{aligned}
\nabla_\beta \ell(x_t, y_t, \beta) &= \frac{1}{1 + \exp(-y x_t^\top \beta)} \nabla_\beta [1 + \exp(-y x_t^\top \beta)] \\
&= \frac{1}{1 + \exp(-y x_t^\top \beta)} \exp(-y x_t^\top \beta) [\nabla_\beta (-y_t x_t^\top \beta)] \\
&= -\frac{1}{1 + \exp(x_t^\top \beta)} (x_{t,i})_{i=1}^n e
\end{aligned}$$

- $\mathbb{E}_P(\ell) = \int_X dP(x) \sum_{y \in Y} P(y|x) P_\beta(y_t + x_t)$

Example code

The Perceptron and Gradients

`./src/Perceptron/Perceptron_gd.ipynb`

- Perceptron implementation to fill in
- Gradient descent implementation
- Experiment on the learning rate with sklearn

Assignment

1. In the class data, find one categorical variable of interest that we want to predict.
2. Formulate the appropriate classification problem.
3. Perform model selection through train/validate or crossvalidation to find the best model (kNN or perceptron) and hyperparameters (k for the kNN)
4. Discuss anything of interest in the data such as: feature scaling/selection, missing data, outliers.
5. We cannot independently measure the quality of the model, as we have no test set. What can we do?