Level set deformable models constrained by fuzzy spatial relations

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Abstract

Object recognition in images, especially for medical applications, is often prone to error and ambiguity due to the high variability of object features. On the other hand, spatial relations, describing the structural aspects of the image, are usually more stable, even in the presence of abnormalities. We proposed in previous works to include spatial relations into some segmentation procedures. In order to account for the intrinsic imprecision of some relations (such as "object A is to the left of object B"), we rely on fuzzy representations of these relations. In this paper, we propose an original formulation of deformable models including these fuzzy representations, in a level set framework, which provides numerical stability of the solution and flexibility in the topology of the segmented objects. The method is illustrated on synthetic images and brain MRI data.

Keywords: Spatial relations, deformable models, image segmentation, geodesics, level sets.

1 Introduction

Spatial relations play an important role in recognition of structures embedded in a complex environment and for reasoning under imprecision. When several objects or structures have to be recognized in an image, it happens in several applications that characteristics of the objects themselves may not be discriminative enough to achieve individual recognition. In such cases, spatial relations between objects become of prime importance for recognition, as a complementary information. In particular, they can solve ambiguities by providing structural information on the scene, through the description of the spatial arrangement of the objects. Examples of such descriptions include: "the train station is close to the market", "the cerebral ventricles are adjacent to the caudate nuclei". They can serve as prior information to guide the recognition of individual objects. Moreover such relations are often much less prone to variability than shape, size or grey levels.

Recently, our group has introduced a new framework for the integration of spatial constraints as a new force in a parametric deformable model [5]. The spatial constraints are modeled as fuzzy sets in the image domain.

With deformable models, segmentation is achieved through the evolution of a contour from a starting point to a final state, according to an energy minimization or an evolution equation. These models can be divided into two main types: parametric and geometric models. Level set deformable models [10, 12], also referred to as geometric deformable models, provide an elegant alternative to parametric deformable models. They rely on an implicit contour formulation which presents several advantages over the parametric formulation, including: (1) no parameterization of the contour, (2) topological flexibility, (3) good numerical stability, (4) straightforward extension of the 2D formulation to n-D.

In this paper we introduce spatial relations into implicit deformable models, merging the advantages of the implicit formulation and of the introduction of structural spatial constraints in the segmentation. We first summarize our approach for representing structural information as spatial fuzzy sets in Section 2. The background on implicit deformable models is given in Section 3. Our contribution is then detailed in Section 4 and preliminary results illustrating the interest of our approach are provided in Section 5.

2 Representation of structural information using fuzzy sets

Fuzzy sets constitute an appealing framework to represent spatial relations, since they allow representing different types of imprecision, related to the imperfections of the image, and to the intrinsic vagueness of some relations, corresponding for instance to linguistic expressions. The satisfaction of a given relation will thus be defined as a matter of degree rather than in an "all-or-nothing" manner. Given a relation with respect to a reference fuzzy object A, two types of questions can be addressed:

- compute to which degree a target object B fulfills this relation;
- find the points of the space where this relation is satisfied.

The first one has been addressed for a wide range of relations including adjacency, distances, directions and symmetries.

In this work, as in [5], we will consider the second approach, which is well suited to make the deformable model evolve towards points where the relation is satisfied. Because of lack of space, we do not detail the definitions used here (see [2] for a review and an extended list of references). We only recall that a relation is expressed as a fuzzy set in the 3D space, with membership function denoted by μ_R ($\mu_R(x)$ denotes the degree to which point x satisfies the relation). Most relations can be computed using fuzzy mathematical morphology operations. When several relations are used to describe the location of an object, the corresponding membership functions are combined using a fusion operator. In the following μ_R will be used to denote either one relation, or the fusion of several ones.

3 Level set deformable models

In their original paper [12], Osher and Sethian introduced the concept of geometric deformable models, which provides an implicit formulation of the deformable contour in a level set framework. It consists in embedding the evolving active contour $C = C(t)$, as the zero level set of a function ϕ, which is a Lipschitz continuous function that has the following properties: it is negative inside the region bounded by C, positive outside this region, and equal to 0 on C. The curve motion along the normal velocity field v_N is controlled by the following equation:

$$\frac{\partial \phi}{\partial t} + v_N |\nabla \phi| = 0 \qquad (1)$$

The level set formulation of the original snake model was first proposed by Caselles *et al.* [3] and next by Malladi *et al.* [10] as *the implicit geometric active contours model.* In their work, Caselles *et al.* proposed the following functional to segment a given image:

$$\frac{\partial \phi}{\partial t} = g(x)|\nabla \phi| \left(div \left(\frac{\nabla \phi}{|\nabla \phi|} \right) + k \right) \qquad (2)$$

where $k \geq 0$ is a constant force term, and $g : \mathbb{R}^2 \to (0, 1]$ denotes a stopping function that slows down the snake as it approaches the image edges.

Geodesic active contours were introduced simultaneously by Kichenassamy *et al.* [9] and Caselles *et al.* [4] as a segmentation framework, derived from energy-based snakes active contours, performing contour extraction via the computation of geodesics, i.e. minimal length curves in a Riemannian space derived from the image. The geodesic approach allows to connect classical "snakes" based on energy minimization and geometric active contours based on the theory of curve evolution. Geometric active contours are improved, allowing stable boundary detection when their gradients suffer from large variations, including gaps. Given an image I and a differentiable curve $C(q), q \in [0,1]$, the following energy is defined:

$$E(C) = \int_0^1 g(|\nabla(I(C(q))| \left|C'(q)\right| dq \quad (3)$$

where g is a positive decreasing function. Segmentation is achieved via minimization of this energy functional. The latter is equivalent to the computation of geodesics in a Riemannian space according to a metric that weights the Euclidian length of the curve with the term $g(|\nabla(I(C(p))|$.

In a level set framework, the geodesic energy writes in terms of ϕ as follows:

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= |\nabla \phi| div \left(g(|\nabla I| \frac{\nabla \phi}{|\nabla \phi|} \right) \\ &+ g(|\nabla I|)|\nabla \phi| div \left(\frac{\nabla \phi}{|\nabla \phi|} \right) + \nabla g(|\nabla I|).\nabla \phi \end{aligned}$$

Note that in this derivation, a convection term appears (∇g), corresponding to local variations through geodesic paths.

Recently, Gout *al.* [7] introduced a new resolution scheme under the level set formulation of the geodesic approach that we adopt in this paper. It writes as:

$$E(\phi) = \int_\Omega g(|\nabla I(x)|)|\nabla H(\phi(x))| \; dx$$

where H is the Heaviside function and Ω is the image domain. Approximating H by a C^1 or C^2 regularized function H_ε, with $\varepsilon \to 0$, and letting $\delta = H'_\varepsilon$, the energy can be written as:

$$E(\phi) = \int_\Omega g(|\nabla I(x)|)\delta(\phi)|\nabla \phi(x)| \; dx \quad (4)$$

where $\int_\Omega \delta(\phi)|\nabla \phi(x)| \; dx$ is the length of the zero level set of ϕ.

These formulations extend easily to 3D, by considering a surface instead of a curve.

4 Integration of fuzzy spatial relations into level-set deformable models

In [5], spatial relations were introduced in the evolution scheme of a deformable model, combined with edge and regularity constraints. While in the classical model the external force $\mathbf{F}_{ext}$ is only derived from edge information, we proposed to combine edge and spatial relation information, defining a new external force as:

$$\mathbf{F}_{ext} = \lambda \mathbf{F}_C + \nu \mathbf{F}_R \quad (5)$$

where $\mathbf{F}_C$ is a classical data term driving the model towards the edges, $\mathbf{F}_R$ is a force associated with spatial relations, and λ and ν are weighting coefficients. The role of $\mathbf{F}_R$ is to force the deformable model to stay in regions where the relation is fulfilled. There are different methods to compute $\mathbf{F}_R$ from a fuzzy set μ_R representing a spatial relation R or a combination of relations obtained at the first fusion level. They must satisfy the following constraints: the constructed force is directed towards high values of the membership function μ_R; it is zero inside the kernel of μ_R and non-zero elsewhere; and its modulus is proportional to $1 - \mu_R$.

In this paper, we extend this idea to other types of deformable models, in two new formulations detailed next.

4.1 Geodesic approach

Geodesic active contour models only make use of local information and are thus very sensitive to local minima. As a result the initialization remains a critical step, affecting the quality of the final segmentation.

Fuzzy landscapes, which are spatial maps of the object localization, could be integrated into the geodesic active model framework to provide a fuzzy localization of the object to extract, and compensate for the lack of accuracy in the initialization process. In addition, evolution of the geodesic curve towards a position corresponding to a global minimum of the energy is not guaranteed to correspond to the desired partition of the image due to the extreme difficulty of defining a data term energy that reflects well the real nature of the image. Including fuzzy structural information in the curve evolution process can eliminate non-desired local minima, and select the optimal “semantic” solution.

The main idea is to combine information provided by the fuzzy landscape and the energy potential derived from the image. Fuzzy sets theory provides a variety of combination operators, which may deal with heterogeneous information. A classification of these operators with respect to their behavior (conjunctive, disjunctive and compromise), the possibility of controlling this behavior, their properties and their decisiveness is given in [1].

Fuzzy sets provide a great flexibility in the choice of the fusion operator, that can be adapted to any situation at hand. Typically, fuzzy sets provide powerful operators that behave differently depending on whether the values to be combined are of the same order of magnitude or not, whether they are small or high, and operators that depend on some global knowledge.

Choosing an apropriate fusion rule highly depends on the framework that one is dealing with. In this paper, a variational framework is used to formalize our problem, finding the geodesic minimal path of a curve in the fuzzy landscape space μ_R. The derived global energy, combining fuzzy structural information and geodesic curve evolution rules, must behave as follows:

- In $Ker(\mu_R)$: only the energy potential derived from the image must be considered, which means that all solutions in the fuzzy landscape kernel are admissible.
- In $\overline{Supp}(\mu_R)$: solutions outside the support of the fuzzy landscape are not considered.
- In $Supp(\mu_R) \setminus Ker(\mu_R)$: the global energy potential corresponds to a compromise between the image information and the prior structural information expressed as a fuzzy landscape.

When dealing with different constraints (spatial information and image information) on the same dataset, a conjunctive fusion rule is appropriate. The image information is expressed as a fuzzy set, denoted by μ_I, representing the degree to which a point x belongs to a contour. The membership function μ_I must correspond to an increasing function of the gradient image. In order to respect the geodesic framework, we note $\mu_I = c\,[g(h(I))]$, where g is a positive decreasing function, $h(I(x))$ is a positive increasing function of the gradient of the image $I(x)$ ($h(I(x)) = |\nabla(I(C(q)))|$ in the original work of Caselles) and c denotes a fuzzy complementation (here we choose $c(a) = 1 - a$).

Let μ_{fusion} denote the membership function corresponding to the conjunctive fusion of the image information μ_I and spatial relation contraint μ_R. It is defined as:

$$\mu_{fusion}(x) = \top(\mu_R(x), \mu_I(x))$$

where $\top$ is a t-norm. This expression is interpreted as follows: a point x corresponds to a minimal path point (in the geodesic sense) when both $\mu_R(x)$ and $\mu_I(x)$ are satisfied, which means a high value of μ_{fusion}, which has therefore to be maximized.

Since the geodesic framework requires to minimize a potential, we can derive a global potential as:

$$\begin{aligned} P(x) &= c\,[\mu_{fusion}(x)] = c\,[\top(\mu_R(x), \mu_I(x))] \\ &= \perp(c(\mu_R(x)), c(\mu_I(x))) \\ &= \perp\big(g(h(I(x))), c(\mu_R(x))\big) \end{aligned}$$

where $\perp$ is the dual t-conorm of $\top$ with respect to c. $P(x)$ can be interpreted as the degree of non satisfaction of the constraints at a point x.

The corresponding curve evolution energy in the Riemannian space writes as: $E(C) = \int_C P(C(q))\,dq$.

Implementation of the curve evolution process in a level-set framework enables us to re-write the energy as:

$$E(\phi) = \int_\Omega P(x)\delta(\phi(x))|\nabla\phi(x)|dx \qquad (6)$$

The associated Euler-Lagrange equation derived from $E(\phi)$ is given by:

$$\frac{\partial \phi}{\partial t} = \delta(\phi)\big(P div(\frac{\nabla \phi}{|\nabla \phi|}) + \nabla P \cdot \nabla \phi\big)$$

To compute the boundary potential, the function h is defined as a gradient map of the image $I(x)$ convolved by a Gaussian kernel G_σ, and g is defined as the sigmoid function.

4.2 Choice of the t-conorm

In this section, we discuss the choice of the t-conorm among the usual ones. Let a and b be two membership degrees ($a = g(h(I(x)))$, $b = c(\mu_R(x))$) taking values in $[0, 1]$, Table 1 summarizes the different union operators ($\perp$) considered in this work.

maximum	$\max(a, b)$
algebraic sum	$a + b - ab$
Lukasiewicz t-conorm	$\min(1, a + b)$
drastic sum	$\begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$

Table 1: Some usual t-conorms

We have $\forall a, b \in [0, 1]\ \max(a, b) \leq a+b-ab \leq \min(1, a + b) \leq \perp_{drastic}(a, b)$.

More generally, the max is the smallest t-conorm and $\perp_{drastic}$ is the largest one among all t-conorms.

The drastic sum has limited interest since it is almost always equal to 1, leading to a poor discrimination power. Furthermore, it is not continuous, which is a strong drawback for its use in our formulation.

The maximum is the smallest t-conorm. It is not differentiable along the line $a = b$, which is a strong limitation when working within a variational framework.

The algebraic sum is differentiable everywhere which makes it very attractive in our application. Moreover it provides a good compromise between the involved constraints.

The Lukasiewicz sum is interesting if the values to combine are not too high; otherwise a saturation effect is produced. As a drawback, it is not differentiable along the line $a + b = 1$.

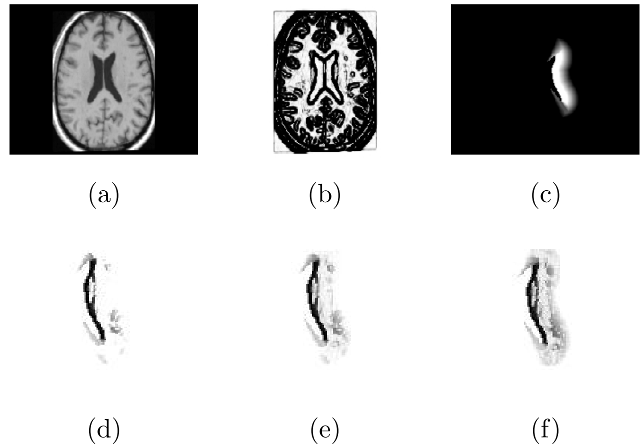


Figure 1: Representation of the potential function. The reference object is the left ventricle. (a) Original image. (b) Potential of the gradient map $g(h(I(x)))$. (c) Fuzzy landscape corresponding to the conjunctive fusion of the relations: "to the left" and "close to". Potential computed with: (d) Lukasiewicz, (e) algebraic sum, and (f) maximum t-conorms.

In addition to the theoretical properties of the t-conorms studied above, we present in Figure 1 the practical behaviors of each t-conorm. The example concerns a slice of a brain magnetic resonance image (MRI), where the potential describing "to the left of the lateral ventricle and close to it" is illustrated. One can easily observe that the maximum leaves significant minima near the desired solution. The algebraic sum and Lukasiewicz t-conorm behave better. But, as mentioned before, the Lukasiewicz t-conorm is not differentiable along the line $a+b = 1$. Therefore, we choose the **algebraic sum** as a fusion rule to combine the constraints coming from the image data and the prior structural information.

4.3 Fuzzy landscape advection-based approach

Conceptually, a fuzzy landscape represents a spatial localization rule (within a region) that is verified by the whole object, not only by its boundaries. As an alternative to the geodesic formulation, a region based approach (i.e. the energy involves not only the boundary points but all the points belonging to the object) can thus be considered in a more straightforward fashion. Furthermore, dealing with objects of the same nature (which can be seen as fuzzy sets) offers interesting fusion possibilities.

In this formulation, instead of fusing the image and prior spatial constraints μ_I and μ_R, the fuzzy region is expressed in the energy as an advection term. Note that with this formulation, the geodesic interpretation is lost.

Usually a fuzzy landscape is used to specify a region in which the object to extract should be localized. Semantically, we define the **"admissible set of points"** and the **"necessary set of points"**.

Definitions:

- An **admissible point** x is a point for which $H(\phi(x)) = 1$ and that belongs to the fuzzy set μ_R.
- A **necessary point** x is a point of the fuzzy set μ_R that ***has*** to belong to the support $H(\phi)$.

Let $\mathcal{S}$ be the set of solution points, $\mathcal{A}$ the set of admissible points and $\mathcal{N}$ the set of necessary points. We have the following inclusions: $x \in \mathcal{S} \Rightarrow x \in \mathcal{A}$ ($\mathcal{S} \subseteq \mathcal{A}$) and $x \in \mathcal{N} \Rightarrow x \in \mathcal{S}$ ($\mathcal{N} \subseteq \mathcal{S}$). Note that $x \in \mathcal{A} \nRightarrow x \in \mathcal{N}$ and $\mathcal{N} \subseteq \mathcal{S} \subseteq \mathcal{A}$.

Other relations between the fuzzy landscape and the object can be useful in practical cases. In this section we will consider only the case of **admissible** and **necessary** points.

4.3.1 Admissible points

In this case, we consider that the curve being the solution of the partial derivative equation should be a subset of the fuzzy landscape. This constraint is formulated as: $\text{Inside}(C) \subseteq Supp(\mu_R)$ and introduced in the level set energy formulation as a region term:

$$\begin{aligned} E(\phi) &= \alpha \int_\Omega g(h(I)) |\nabla H(\phi(x))| \, dx \\ &+ \beta \int_\Omega \top(c(\mu_R(x)), H(\phi(x))) \, dx \end{aligned} \tag{7}$$

The second term $\top(c(\mu_R(x)), H(\phi(x)))$ represents the conjunctive fusion between the complementary of the prior spatial region and the dynamic image region. Here we used the product for the t-norm $\top$. Thus, a penalty term is introduced to each point of the object outside the kernel of μ_R.

We then derive the Euler-Lagrange equation for this energy formulation as:

$$\begin{aligned} \frac{\partial \phi}{\partial t} = \; & \alpha\big(\delta(\phi)(g(\nabla I) div(\tfrac{\nabla\phi}{|\nabla\phi|}) \\ & +\nabla g(\nabla I) \cdot \nabla\phi\big) - \beta c(\mu_R) \end{aligned}$$

The constraint term $\beta c(\mu_R)$ acts as a pressure force outside μ_R, driving the curve inside μ_R.

4.3.2 Necessary points

In this case, the structural fuzzy information constrains the curve evolution to integrate all the points priorly defined. In the level set framework, this constraint is formulated as: $Supp(\mu_R) \subseteq \text{Inside}(C)$ and introduced in the level set formulation as a region term, penalizing each pixel outside the object but inside μ_R:

$$\begin{aligned} E(\phi) &= \alpha \int_\Omega g(\nabla I) |\nabla H(\phi(x))| \, dx \\ &+ \beta \int_\Omega \top(\mu_R(x), (1 - H(\phi(x)))) \, dx \end{aligned} \tag{8}$$

The second term $\top(\mu_R(x), (1-H(\phi(x))))$ represents the conjunctive fusion between the prior spatial region and the complementary of the dynamic image region. Again, the product rule is used.

The corresponding evolution equation is then expressed as:

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \alpha\big(\delta(\phi)(g(\nabla I) div(\frac{\nabla\phi}{|\nabla\phi|})\big) \\ &+ \nabla g(\nabla I) \cdot \nabla\phi + \beta\mu_R \end{aligned}$$

The constraint term acts here as a balloon force, forcing the curve to include μ_R.

5 Results

The implementation of the proposed approaches is based on approximations of derivatives using finite differences, as in [4].

We now present two applications on synthetic images. In Figure 2, the aim is to detect different objects using structural information.

For this purpose we assume that the triangle was previously segmented. Using the geodesic approach (Section 4.1) we extract the square and the disk separately, based on the relations they have with the triangle: the square is "below" the triangle and the disk is "on the right" of it.

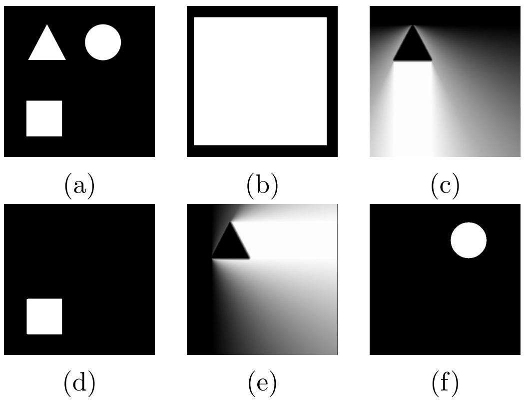


Figure 2: (a) Test image. (b) Level set initialization. (c) Modeling of the relation "the square is below the triangle". (d) Segmentation result of the square. (e) Modeling of the relation "the disk is on the right of the triangle". (f) Segmentation result of the disk.

Figure 3 illustrates the second approach. The reference objects are the square and the large disk. The aim is to extract the triangle. Two fuzzy landscapes are computed with different significations. The first one represents the **admissible set of points** (Figure 3(b)) and corresponds to a conjunctive fusion of two roughly defined relations. The second one represents an accurate definition of points belonging to the object: **neccesary set of points** that are included in the final solution, defined as an erosion of the kernel of the set of admissible points (Figure 3(c)). The result (shown in Figure 3(d)) provides the expected object. This result has been obtained by minimizing the weighted sum of the two energy functions corresponding to the constraints expressed by the admissible points and necessary points respectively (Equations 7 and 8). A higher weight is given to necessary points.

In Figure 4, we present a result of the geodesic approach on a brain magnetic resonance image, for the segmentation of the caudate nucleus using its relative spatial relation with the ventricle. The results show that our

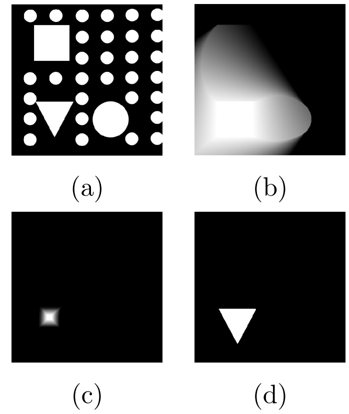


Figure 3: (a) Test image. (b) Admissible set of points. (c) Necessary set of points. (d) Segmentation result.

method detects and extracts automatically with success the caudate nucleus despite its small size and difficult localization. This approach assumes that one initial object is segmented first and that subsequent object segmentations rely on relations to the previously segmented objects. In this example, the ventricles can indeed be easily segmented in the MR volume, without using spatial relations to other objects. This approach has already proved to be very powerful [5]. Figure 5 illustrates the important improvement obtained when introducing the spatial relations into the deformable model.

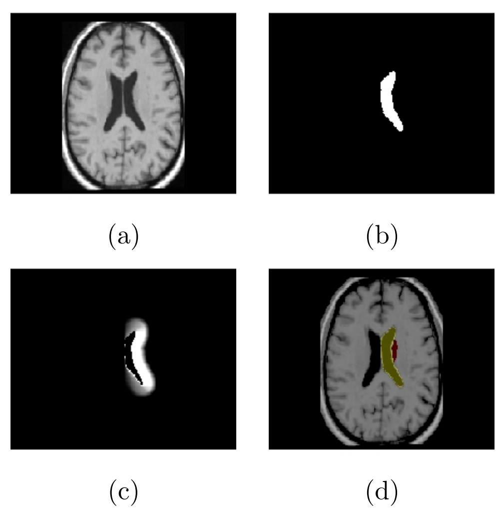
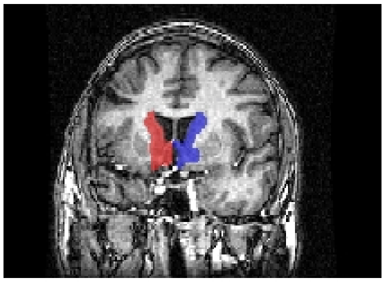


Figure 4: (a) Brain data (a slice of 3D MRI). (b) The right ventricle: reference object. (c) Fuzzy landscape representing the relation "the right caudate nucleus is close to the right ventricle". (h) Caudate nucleus segmentation result.



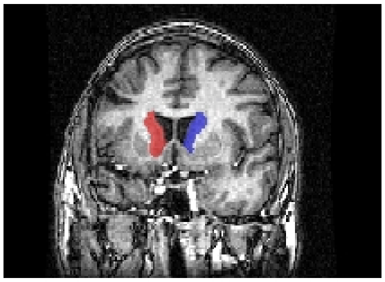


Figure 5: Segmentation of both caudate nuclei (one coronal slice of the MR volume is shown): without using spatial relations (top), using spatial relations (bottom).

6 Conclusion

We proposed in this paper an original contribution to include spatial constraints in level set deformable models. Two distinct formulations were introduced: the first one relies on finding a solution of a curve evolution as a minimal path in the fuzzy landscape, and on fusing information coming from the prior spatial relations and the image data, both types of information being expressed as fuzzy subsets. The second formulation makes use of the fuzzy landscape as an advection force in the evolution process. This formulation is more flexible as it allows modeling different semantics of the fuzzy landscapes. Furthermore, the integration of other constraints such as image region homogeneity ("active contours without edges" for instance) or texture information could be performed in a straigthforward fashion. This is left for future work.

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