Tuning Fractional Order Proportional Integral Controllers Using Dominant Pole Placement Method

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ABSTRACT

This paper presents a method of tuning for fractional order proportional integral controllers for a class of fractional order systems. Fractional order proportional integral controller inherits the advantages of the traditional integer order PI controller and has one degree of freedom more than the integer order PI controller and it has better control performance. Based on this characteristic of the FOPI controller, fractional order proportional integral (FOPI) and fractional order [proportional integral] (FO[PI]) controllers proposed and designed based on dominant pole placement method for the considered class of fractional order systems. The simulation results show that the proposed method works well for the design of fractional order PI controller.

Keywords: Integer order PI controller, Fractional order PI controller, FOPI controller, FO[PI] controller

1. INTRODUCTION

As we know, the use of Proportional-Integral-Derivative (PID) control has a long history in control engineering. Hence in many real industrial applications, the PID controller is still widely used even though lots of new control techniques have been proposed [1]. Design and tuning of PID controllers have been a large research area ever since Ziegler and Nichols presented their method in 1942 [2]. Design, applications and performance of the PID controllers have been widely treated since then [3, 4].

Fractional calculus is dealing with integration and derivation of non-integer order [5-7]. In recent years, application of fractional calculus is becoming a hot topic in control area [8-11]. The idea of fractional order PID ($PI^{\lambda}D^{\mu}$) is proposed by Podlubny I. [5]. The main advantage of using fractional order PID controllers for a linear control system is that we have more degrees of freedom in the controller design using the additional parameters of the integral and differential orders and, as a consequence, it is expected that the use of FOPID controllers can enhance the feedback control loop performance over the integer order controllers [12]. So many tuning algorithms for fractional order PID controllers were developed [12-22] and fractional order PID controllers came to real-world applications [18].

The paper is organized as follows: Section 2 is a review of the preliminary concepts of fractional calculus. In Section 3 calculation of the dominant poles is studied. In Section 4 the considered fractional order system, the FOPI and FO[PI] controllers and the tuning constraints for the robustness requirements on the loop gain variations are introduced. In Section 5 the design procedure for the controllers are presented. An application of the proposed method is given in Section 6. Finally some concluding remarks are cited in Section 7.

2. Preliminaries

There are several definitions of fractional derivatives [8]. The best definitions are Riemann-Liouville and Caputo.

Definition 1. (Riemann-Liouville fractional derivative) The Riemann-Liouville fractional derivative is defined as:

$$ {}_aD_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \int_a^t (t-\tau)^{-\alpha-1} f(\tau) d\tau & \alpha < 0 \\ f(t) & \alpha = 0 \\ \dfrac{d^n}{dt^n}\left[{}_aD_t^{\alpha-n} f(t) \right] & \alpha > 0 \end{cases} \tag{1} $$

where $\Gamma(.)$ is the standard gamma function and n is an integer number such that $n-1 \le \alpha < n$.

Definition 2. (Caputo fractional derivative) The Caputo fractional derivative is defined as:

$$ {}_a^C D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau & n-1 < \alpha < n \\ \dfrac{d^n}{dt^n} f(t) & \alpha = n \end{cases} \tag{2} $$

Definition 3. Laplace transform of the Riemann-Liouville derivative is defined as:

$$ \mathcal{L}\left\{ {}_aD_t^{\alpha} \right\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k \, {}_aD_t^{\alpha-k-1} f(a) \tag{3} $$

Definition 4. Laplace transform of the Caputo derivative is defined as:

$$ \mathcal{L}\left\{ {}_a^C D_t^{\alpha} \right\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(a) \tag{4} $$

The main advantage of Caputo derivative is that the initial conditions for fractional differential equations with Caputo derivative take on the same form as for integer order differential equations. Another difference between two definitions is that the Caputo derivative of a constant is 0 whereas in the cases of a finite value of the lower terminal a the Riemann-Liouville derivative of a constant C is not equal to 0, but

$$ {}_aD_t^{\alpha} C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)} \tag{5} $$

A linear fractional order system in state space form is like:

$$ \begin{Bmatrix} {}_a^C D_t^{\alpha_1} x_1 \\ {}_a^C D_t^{\alpha_2} x_2 \\ \vdots \\ {}_a^C D_t^{\alpha_n} x_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = A_{n\times n} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \tag{6} $$

3. Dominant poles

In higher order systems dominant poles are the poles which are the closest to the imaginary axis. Consider the original open-loop system is of high order, the closed loop system behaves like a second order system,

$$ G_m(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{7} $$

Consider the unit step response of a second order system. It is possible to calculate the parameters of the system as a function of peak overshoot and settling time.

$$ \zeta = -\frac{\ln(P.O./100)}{\sqrt{\pi^2 + (\ln(P.O./100)^2}} \tag{8} $$

$$\omega_n = \frac{4}{\zeta T_s} \tag{9}$$

where $P.O.$ is peak overshoot and T_s is settling time.

Dominant poles of a second order system in terms of system parameters are given as

$$s_{1,2} = re^{\pm j\theta} \tag{10}$$

Where θ is a function of damping ratio,

$$\theta = \pi - \cos^{-1}\zeta \tag{11}$$

$$r = \omega_n \tag{12}$$

4. Fractional order proportional integral controllers and the tuning constraints

The fractional order plant to be controlled has the following form of the transfer function

$$G(s) = \frac{k}{s^{2\alpha} + a_1 s^{\alpha} + a_0} \tag{13}$$

Where α is a positive real number and $\alpha \in (0,1)$.

Our work in this paper is to design a fractional order proportional integral controller for fractional order systems. We consider two types of fractional order PI controllers: fractional order proportional integral (FOPI) and fractional order [proportional integral] (FO[PI]) whose transfer functions are given as follows, respectively

$$C_1(s) = K_P + \frac{K_I}{s^{\lambda}} \tag{14}$$

$$C_2(s) = \left(K_P + \frac{K_I}{s}\right)^{\lambda} \tag{15}$$

Where K_P and K_I are proportional and integral gains and λ is a positive real number.

In this paper, the following specifications to be met by fractional controlled system are proposed:

1. **Dominant poles**

Let the plant at the dominant poles be represented by a complex number,

$$G(s_1) = G_1 + jG_2 \tag{16}$$

where G_1 and G_2 are given by

$$G_1 = \frac{k\,(r^{2\alpha}\cos(2\alpha\theta) + a_1 r^{\alpha}\cos(\alpha\theta) + a_0)}{(r^{2\alpha}\cos(2\alpha\theta) + a_1 r^{\alpha}\cos(\alpha\theta) + a_0)^2 + (r^{2\alpha}\sin(2\alpha\theta) + a_1 r^{\alpha}\sin(\alpha\theta))^2} \tag{17}$$

$$G_2 = \frac{-k\,(r^{2\alpha}\sin(2\alpha\theta) + a_1 r^{\alpha}\sin(\alpha\theta))}{(r^{2\alpha}\cos(2\alpha\theta) + a_1 r^{\alpha}\cos(\alpha\theta) + a_0)^2 + (r^{2\alpha}\sin(2\alpha\theta) + a_1 r^{\alpha}\sin(\alpha\theta))^2} \tag{18}$$

The characteristic equation of the system is

$$1 + G(s)C(s) = 0 \tag{19}$$

Dominant poles must satisfy the characteristic equation.

2. **Robustness to variations in the gain of the plant**.

With this condition the phase is forced to be flat at gain crossover frequency and so, to be almost constant within an interval around the gain crossover frequency. It means that the system is more robust to gain changes and the overshoot of the response is almost constant within the interval.

$$\left(\frac{d\,(\arg(G(j\omega)\mathrm{C}(j\omega)))}{d\omega}\right)_{\omega=\omega_c} = 0 \tag{20}$$

where ω_c is gain crossover frequency.

3. **Gain crossover frequency specification**

It means that the system is robust to gain change variations,

$$\left|G(j\omega_c)C(j\omega_c)\right|_{dB} = 0dB \tag{21}$$

4. Tuning controllers for the fractional order systems considered

5.1 FOPI controller design

The systematic way to design the FOPI controller for the fractional order system (13) according to the three specifications introduced in section 4, is presented below,

Let the controller at dominant pole is given by

$$C_1(s_1) = K_P + \frac{K_I}{r^{\lambda}}\cos(\lambda\theta) + j\,\frac{K_I}{r^{\lambda}}\sin(\lambda\theta) \tag{22}$$

Substituting (16) and (22) in characteristic equation and comparing real-real and imaginary-imaginary parts in equation, the following equations are obtained,

$$r^{\lambda} + G_1 K_P r^{\lambda} + G_1 K_I \cos(\lambda\theta) - G_2 K_I \sin(\lambda\theta) = 0 \tag{23}$$

$$G_1 K_I \sin(\lambda\theta) + G_2 K_I \cos(\lambda\theta) + G_2 K_P r^{\lambda} = 0 \tag{24}$$

According to the constraint (2) about the robustness to loop gain variation, one can obtain,

$$\left(\frac{d\,(\arg(G(j\omega)\mathrm{C}(j\omega)))}{d\,\omega}\right)_{\omega=\omega_c} = \frac{K_P K_I \omega_c^{\lambda}\sin(\lambda\pi/2)}{(K_P\omega_c^{\lambda})^2 + K_I^2 + K_P K_I \omega_c^{\lambda}\sin(\lambda\pi)} + \frac{AB' - A'B}{A^2 + B^2} = 0 \tag{25}$$

where

$$A = \omega_c^{2\alpha}\sin\alpha\pi + a_1\omega_c^{\alpha}\sin(\alpha\pi/2) \tag{26}$$

$$B = \omega_c^{2\alpha}\cos\alpha\pi + a_1\omega_c^{\alpha}\cos(\alpha\pi/2) + a_0 \tag{27}$$

$$A' = \left.\frac{\partial A}{\partial\omega}\right|_{\omega=\omega_c} = 2\alpha\omega_c^{2\alpha-1}\sin\alpha\pi + a_1\alpha\omega_c^{\alpha-1}\sin(\alpha\pi/2) \tag{28}$$

$$B' = \left.\frac{\partial B}{\partial\omega}\right|_{\omega=\omega_c} = 2\alpha\omega_c^{2\alpha-1}\cos\alpha\pi + a_1\alpha\omega_c^{\alpha-1}\cos(\alpha\pi/2) \tag{29}$$

According to the gain crossover frequency constraint, one can obtain,

$$\left((K_P\omega_c^{\lambda})^2 + K_I^2 + 2K_P K_I \omega_c^{\lambda}\cos(\lambda\pi/2)\right)\left(A^2 + B^2\right) = 1 \tag{30}$$

We can solve equations (23), (24), (25) and (30) to get K_P , K_I , λ and ω_c .

5.2 FO[PI] controller design

Let the controller at dominant poles is given by

$$C_2(s_1) = \left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)^{\frac{\lambda}{2}}\cos(\varphi) + j\left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)^{\frac{\lambda}{2}}\sin(\varphi) \tag{31}$$

Where

$$\varphi = \tan^{-1}\frac{K_I\sin(\theta)/r}{K_P + K_I\cos(\theta)/r} \tag{32}$$

Substituting () and () in characteristic equation and comparing real-real and imaginary-imaginary parts in equation, the following equations are obtained,

$$1 + G_1\left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)\cos(\varphi) - G_2\left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)\sin(\varphi) = 0 \tag{33}$$

$$G_1\left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)\sin(\varphi) + G_2\left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)\cos(\varphi) = 0 \tag{34}$$

According to the constraint (2) about the robustness to loop gain variation, one can obtain,

$$\left(\frac{d\,(\arg(G\,(j\,\omega)\,\mathrm{C}(j\,\omega)))}{d\,\omega}\right)_{\omega=\omega_c}=\frac{K_P K_I\,\sin(\theta)}{(K_P\omega_c)^2+K_I^2+K_P K_I\,\sin(2\theta)}+\frac{AB'-A'B}{A^2+B^2}=0 \tag{35}$$

According to the gain crossover frequency constraint, one can obtain,

$$\left(K_P^2+\frac{K_I^2}{\omega_c^2}+2K_P\,\frac{K_I}{\omega_c}\cos(\theta)\right)^{\lambda}\left(A^2+B^2\right)=1 \tag{36}$$

Clearly, again we can solve equations (33), (34), (35) and (36) to get K_P , K_I , λ and ω_c .

5. Summaries of the controllers design procedure and illustrative examples

In this section, the design procedures of the controllers are summarized. Illustrative examples are presented to verify the proposed controller designs.

6.1 Fractional order PI controller design procedure

Fractional order PI controller design procedure are summarized as below,

- Choose the peak overshoot and settling time of the model
- Find the dominant poles
- Obtain equations (23), (24), (25) and (30)
- Then we can obtain the controller parameters using fmincon function in MATLAB. The gain crossover frequency specification should be taken as the main function to minimize.

6.2 Illustrative example I

The plant to control is:

$$G(s)=\frac{2}{s^{1.2}+5s^{0.6}+2} \tag{37}$$

The design specifications required for the controlled system are the following ones:

- Peak overshoot, $P.O.\le 15\%$
- Settling time, $T_S\ \le 3s$

Using fmincon function the controller parameters and gain crossover frequency are,

$K_P\ = 20.7411$

$K_I\ = 103.3162$

$\lambda = 1.4139$

$\omega_c\ = 12.1153 rad\ /\ s$

Step response of the closed loop system and the control input are shown in Fig. 1 and Fig. 2 respectively.

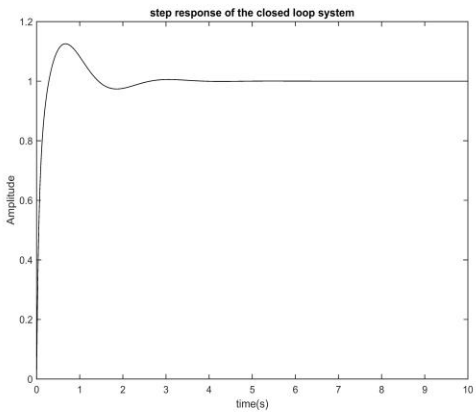

Fig. 1. Step response of the closed loop system using FOPI

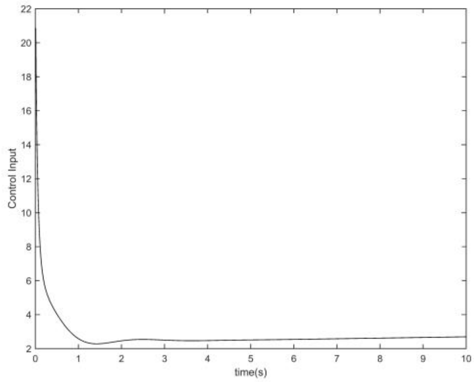

Fig. 2. Control input

6.3 Fractional order [PI] controller design procedure

Fractional order [PI] controller design procedure are summarized as below,

- Choose the peak overshoot and settling time of the model
- Find the dominant poles
- Obtain equations (33), (34), (35) and (36)
- Then we can obtain the controller parameters using fmincon function in MATLAB.

6.4 Illustrative example II

The plant to control is:

$$G(s) = \frac{2}{s^{1.2} + 5s^{0.6} + 2} \tag{38}$$

The design specifications required for the controlled system are the following ones:

- Peak overshoot, $P.O. \leq 15\%$
- Settling time, $T_S \leq 3s$

the controller parameters and gain crossover frequency are,

$K_P = 37.9223$

$K_I = 96.2488$

$\lambda = 1.2887$

$\omega_c = 25.0 rad / s$

Step response of the closed loop system and the control input are shown in Fig. 3 and Fig. 4 respectively.

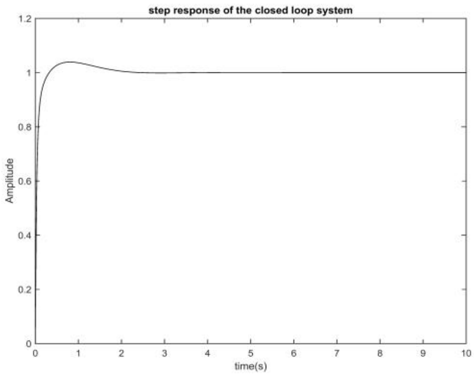

Fig. 3. Step response of the closed loop system using FO[PI]

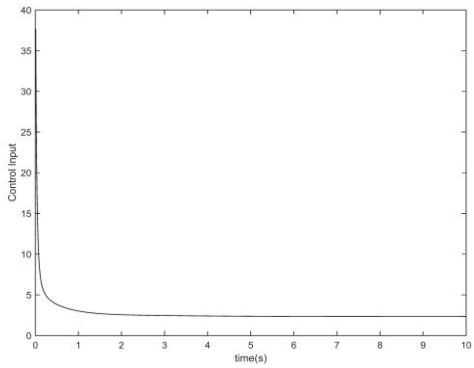

Fig. 4. Control input

6. Conclusion

In this paper a method for tuning of fractional order proportional integral and fractional order [proportional integral] controllers for a class of fractional order systems has been proposed. The presented method is based on dominant pole placement method. For fair comparison, the proposed FOPI and FO[PI] controllers are all designed following the same set of the imposed tuning constraints and design specifications. The simulation results show that the FO[PI] controller has better response than the FOPI controller.

REFERENCES

[1] Åström, K.J. and T. Hägglund, *Advanced PID control*. 2006: ISA-The Instrumentation, Systems, and Automation Society; Research Triangle Park, NC 27709.

[2] Ziegler, J.G. and N.B. Nichols, *Optimum settings for automatic controllers.* trans. ASME, 1942. **64**(11).

[3] Chen, Y., C. Hu, and K.L. Moore. *Relay feedback tuning of robust PID controllers with iso-damping property*. in *Decision and Control, 2003. Proceedings. 42nd IEEE Conference on*. 2003. IEEE.

[4] Åström, K.J. and T. Hägglund, *The future of PID control.* Control engineering practice, 2001. **9**(11): p. 1163-1175.

[5] Podlubny, I., *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications*. 1999: Academic Press, San Diego - New York - London.

[6] Monje, C.A., et al., *Fractional-order systems and controls: fundamentals and applications*. 2010: Springer Science & Business Media.

[7] Valerio, D. and J.S. Da Costa, *An introduction to fractional control*. Vol. 91. 2012: IET.

[8] Podlubny, I., *Fractional-order systems and PI/sup/spl lambda//D/sup/spl mu//-controllers.* IEEE Transactions on Automatic Control, 1999. **44**(1): p. 208-214.

[9] Monje, C.A., et al., *Tuning and auto-tuning of fractional order controllers for industry applications.* Control Engineering Practice, 2008. **16**(7): p. 798-812.

[10] Oustaloup, A., J. Sabatier, and P. Lanusse, *From fractal robustness to CRONE control.* Fract. Calc. Appl. Anal, 1999. **2**(1): p. 1-30.

[11] Lurie, B.J., *Three-parameter tunable tilt-integral-derivative (TID) controller.* 1994.

[12] Narang, A., S.L. Shah, and T. Chen. *Tuning of fractional PI controllers for fractional order system models with and without time delays*. in *American Control Conference (ACC), 2010*. 2010. IEEE.

[13] Badri, V. and M.S. Tavazoei, *On tuning fractional order [proportional–derivative] controllers for a class of fractional order systems.* Automatica, 2013. **49**(7): p. 2297-2301.

[14] Chunyang, W., F. Weicheng, and S. Yaowu. *Tuning fractional order proportional integral differentiation controller for fractional order system*. in *Control Conference (CCC), 2013 32nd Chinese*. 2013. IEEE.

[15] Li, H., Y. Luo, and Y.Q. Chen, *A fractional order proportional and derivative (FOPD) motion controller: tuning rule and experiments.* Control Systems Technology, IEEE Transactions on, 2010. **18**(2): p. 516-520.

[16] Lino, P. and G. Maione. *Fractional order PI tuning for integrating plants with time delay*. in *IFAC Conf. on Advances in PID Control (PID'12), Brescia, Italy*. 2012.

[17] Luo, Y. and Y. Chen, *Fractional order [proportional derivative] controller for a class of fractional order systems.* Automatica, 2009. **45**(10): p. 2446-2450.

[18] Luo, Y., et al., *Tuning fractional order proportional integral controllers for fractional order systems.* Journal of Process Control, 2010. **20**(7): p. 823-831.

[19] Shekher, V., P. Rai, and O. Prakash, *Tuning and Analysis of Fractional Order PID Controller.* International Journal of Electronic and Electrical Engineering, 2012. **5**(1): p. 11-21.

[20] Zamani, M., et al., *Design of a fractional order PID controller for an AVR using particle swarm optimization.* Control Engineering Practice, 2009. **17**(12): p. 1380-1387.

[21] Zhao, C., D. Xue, and Y.Q. Chen. *A fractional order PID tuning algorithm for a class of fractional order plants*. in *Mechatronics and Automation, 2005 IEEE International Conference*. 2005. IEEE.

[22] Zhe, G. and L. Xiaozhong. *Design of fractional order PID controller by hybrid adaptive particle swarm optimization based on the average velocity*. in *Control Conference (CCC), 2011 30th Chinese*. 2011. IEEE.