Adaptive Synchronization of Uncertain Fractional Order Chaotic Systems Using Sliding Mode Control Techniques

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Abstract

In this paper, an adaptive nonlinear controller is designed to synchronize two uncertain fractional order chaotic systems using fractional order (FO) sliding mode control. The controller structure and adaptation laws are chosen such that asymptotic stability of the closed loop control system is guaranteed. The adaptation laws are being calculated from a proper sliding surface by using Lyapunov approach. This method guarantees robustness of the closed loop control system against the system uncertainty and external disturbance. Finally, the proposed method is used to synchronize two fractional order Duffing and gyro systems, and the numerical simulations show the effectiveness of this method.

Keywords: Fractional order chaotic systems; Synchronization; FOPID controller; Fractional order sliding mode control; Gradient descent method.

1. Introduction

Chaos is a phenomenon that appears in nonlinear dynamical systems and has been observed in many fields of science such as economics [1], chemistry [2], biology [3], engineering [4] and so on. In recent years, synchronization of chaotic systems has attracted interest of many scientists in variety of fields [5]. In 1990, Pecora and Carroll proposed the idea of chaos synchronization [6]. In the last years, several methods have been introduced for synchronization of chaotic systems [7]. In many applications, parameters of the slave system are unknown or system has some uncertainties; therefore, using adaptive control and developing some adaptive synchronization methods have been presented [8-10].

Fractional calculus is an old field of mathematics from the 17th century that deals with derivatives and integrals of non-integer order [11]. For many years, fractional calculus was a pure mathematics topic; and has no applications in real world. But recently, fractional calculus is introduced as a powerful tool for modeling many systems in various fields of physics and engineering e.g. viscoelasticity [12], dynamical systems [13], biomedical applications [14], signal processing [15], electrical networks [16], cyber-physical systems [17], diffusion wave [18-20], electromagnetism [21], stochastic systems [22], control theory [23], chaotic systems [24-26] and so on. Also, many fractional order controllers are developed like, fractional PID controller [27, 28], fractional PI controller [29], fractional PD controller [30], fractional lead-lag controller [31, 32], fractional CRONE controller [33], adaptive fractional order PID controller [34], fractional model reference adaptive control [35, 36], and fractional sliding mode control [37-39].

By developing fractional calculus in chaos theory, many fractional order (FO) dynamic systems with chaotic behavior are introduced such as FO Duffing system [40], FO Chen system [41], FO Van der Pol dynamics [42], FO Rössler equations [43], FO Chua system [44] and so on. In the last few years, several techniques have been proposed for synchronization of fractional order chaotic systems [45-49].

In this paper, an adaptive nonlinear controller is designed to synchronize two FO chaotic systems. In the proposed method stability of the closed loop control system is guaranteed by using Lyapunov approach. In addition, robustness of the closed loop system is guaranteed by using sliding mode control methods.

The rest of this paper is organized as follows: In Section 2, the most applicable definitions of fractional order integral and derivative and stability of the fractional order systems are presented. In Section 3 the problem statement is presented. In Section 4, stability of the closed loop control system is guaranteed by introducing a proper sliding mode controller. In Section

5, numerical simulation results are shown. Finally, a brief conclusion is presented in the latest section.

2. Preliminaries

Since the time Leibniz introduced non-integer order derivatives, several definitions have been generated by several mathematicians [11]. The most common formulations for these derivatives are generated by Grünweld-Letnikov, Riemann-Liouville and Caputo.

Definition 1 (Riemann-Liouville Fractional Integral [11]). The Riemann-Liouville fractional integral of function $f(\tau)$ is defined as:

$$ {}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau \tag{1} $$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2 (Riemann-Liouville Fractional Derivative [11]). The Riemann-Liouville fractional derivative of function $f(\tau)$ is defined as:

$$ {}_0D_t^{p} f(t) = \frac{d^k}{dt^k} \left({}_0^{RL}D_t^{-(k-p)} f(t) \right) \tag{2} $$

where k is an integer number such that $k-1 \le p < k$.

Definition 3 (Caputo Fractional Derivative [11]). The Caputo fractional derivative of function $f(\tau)$ is defined as:

$$ {}_0^{C}D_t^{p} f(t) = \begin{cases} \dfrac{1}{\Gamma(k-p)} \displaystyle\int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{p+1-k}} d\tau & k-1 < p < k \\ \dfrac{d^k}{dt^k} f(t) & p = k \end{cases} \tag{3} $$

The relation between Riemann-Liouville and Caputo fractional derivatives can be expressed as [11]:

$$ {}_0^{RL}D_t^{p} f(t) = {}_0^{C}D_t^{p} f(t) + \sum_{k=0}^{n-1} \Phi_{k-p+1}(t) f^{(k)}(0) \tag{4} $$

where n is an integer number such that $n-1 \le p < n$ and the function $\Phi_p(t)$ is defined as:

$$\Phi_p(t) = \begin{cases} \dfrac{t^{p-1}}{\Gamma(p)} & t > 0 \\ 0 & t \le 0 \end{cases} \tag{5}$$

Definition 4. A fractional order linear system can be represented by the following state space form:

$$\begin{Bmatrix} {}_0^C D_t^{\alpha_1} x_1 \\ {}_0^C D_t^{\alpha_2} x_2 \\ \vdots \\ {}_0^C D_t^{\alpha_n} x_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \tag{6}$$

where α_1, …, α_n are fractional order of derivatives. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, the system is called commensurate.

Theorem 1. Consider the following commensurate fractional order linear system:

$$\frac{d^\alpha}{dt^\alpha}\mathbf{x} = \mathbf{Ax} \tag{7}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{A} \in \Re^{n\times n}$ and α is a real number between 0 and 2. The autonomous system is asymptotically stable if the following condition is satisfied:

$$\left|\arg(\lambda_i)\right| > \alpha\frac{\pi}{2} \quad \forall i \tag{8}$$

where λ_i represents the eigenvalues of matrix $\mathbf{A}$.

Definition 5. Using the Caputo derivative, a fractional order nonlinear system can be representented by:

$${}_0^C D_t^\alpha x = f(t,x) \tag{9}$$

where α is the order of fractional derivatives of the system.

Theorem 2 (Fractional-order extension of Lyapunov direct method [50]). Let $x = 0$ be an equilibrium point for the autonomous fractional order system (9). Let $V(t, x(t))$ be a continuously differentiable function as a Lyapunov function candidate, and $\gamma_i (i = 1,2,3)$ be class-K functions such that

$$\gamma_1\left(\|x\|\right) \le V(t, x(t)) \le \gamma_2\left(\|x\|\right) \tag{10}$$

$$ {}_0^C D_t^\beta V(t, x(t)) \le -\gamma_3\left(\|x\|\right) \tag{11}$$

where $\beta \in (0,1)$. Then the system (9) is asymptotically stable.

Lemma 1. [50] Let $x(t) \in \Re$ be a continuously differentiable function. Then, for all $t \ge t_0$

$$\frac{1}{2}\, {}_{t_0}^C D_t^\alpha x^2(t) \le x(t)\, {}_{t_0}^C D_t^\alpha x(t), \qquad \forall \alpha \in (0,1) \tag{12}$$

Remark 1. [50] In the case when $x(t) \in \Re^n$, Lemma 1 is still valid. That is, $\forall t \ge t_0$

$$\frac{1}{2}\, {}_{t_0}^C D_t^\alpha x^T(t) x(t) \le x^T(t)\, {}_{t_0}^C D_t^\alpha x(t), \qquad \forall \alpha \in (0,1) \tag{13}$$

In [51], Barbalat's Lemma is developed for fractional order nonlinear systems.

Theorem 3. [51] Let $\phi : \Re \to \Re$ be a uniformly continuous function on $[t_0, \infty)$. Assume that there exist two positive constants p and M such that ${}_0D_t^{-\alpha} |\phi|^p \le M$ for all $t > t_0 > 0$ with $\alpha \in (0,1)$. Then

$$\lim_{t\to\infty} \phi(t) = 0 \tag{14}$$

3. Problem Statement

3.1 System Description

In synchronization of two chaotic systems, there are two systems: a particular dynamic system as a master and another different dynamic as a slave. From the viewpoint of control, the task is to design a controller such that the slave system imitates behavior of the master system.

Consider following commensurate fractional order chaotic as a slave system:

$$\begin{cases} {}_0^C D_t^\alpha y_i = y_{i+1} \quad , \quad 1 \le i \le n-1 \\ {}_0^C D_t^\alpha y_n = f(t, \mathbf{Y}) + \mathbf{F}^T(t, \mathbf{Y})\,\mathbf{\Theta} + d(t) + g(t, \mathbf{Y}) u \end{cases} \tag{15}$$

where α shows fractional order of the equations and belongs to interval $(0,1)$; $\mathbf{Y}=\begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}^T$ is measurable state vector of the system; $f(\cdot)$ and $g(\cdot)$ are nonlinear functions, and belong to $C^1\left(\Re^n \times \Re \rightarrow \Re\right)$; $\mathbf{F}(\cdot)$ is a known vector with $1\times m$ dimension; $\boldsymbol{\theta}\in\Re^m$ stands for the uncertain parameter vector of the system; $d(\cdot)$ denotes the external disturbance; and $u(t)$ is the input of the system. It is assumed that $g(t,\mathbf{Y})\neq 0$ and $|d(t)|\leq k$ for all $t>0$, and k is a positive and unknown real number.

Let us define the following commensurate fractional order chaotic system as a master system:

$$\begin{cases} {}_0^C D_t^{\alpha} x_i = x_{i+1} \quad , \quad 1\leq i\leq n-1 \\ {}_0^C D_t^{\alpha} x_n = h(t,\mathbf{X}) \end{cases} \tag{16}$$

where $\mathbf{X}=\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ is measurable state vector of the master system.

The main objective is to design a robust control law to synchronize behavior of the master and slave systems. To achieve this goal, we will design an adaptive sliding mode control such that the closed loop control system satisfy the stability condition.

Throughout the paper, the following assumptions are used:

Assumption 1: The exact values of $\boldsymbol{\Theta}$ and $d(\cdot)$ are unknown, but both of them are bounded.

4. Stability Analysis and Controller Design

The synchronization error is defined as difference between the states of the master and the slave systems:

$$e_i = y_i - x_i \quad , \quad 1\leq i\leq n \tag{17}$$

From (15), (16) and (17) the error dynamics can be written in the following form:

$$\begin{cases} {}_0^C D_t^{\alpha} e_i = e_{i+1} \quad , \quad 1 \le i \le n-1 \\ {}_0^C D_t^{\alpha} e_n = f(t,\mathbf{Y}) + \mathbf{F}^T(t,\mathbf{Y})\mathbf{\Theta} + d(t) + g(t,\mathbf{Y})u - h(t,\mathbf{X}) \end{cases} \tag{18}$$

The main goal in this section is to develop the control input in system (18) such that the closed loop control system be asymptotically stable and robust against the system uncertainty and external disturbance. To this purpose, adaptive sliding mode control techniques will be used. We consider the following function as a sliding surface function:

$$S = e_n + \sum_{i=1}^{n} \alpha_i \, {}_0I_t^{\alpha} e_i \tag{19}$$

where $\alpha_i > 0's$ are set to obtain an exponentially stable dynamics for sliding mode, S = 0.

Assumption 2: A simplifying condition which is very common in controlling fractional order systems is assumed; that all of the system state variables as well as the sliding surface are continuously differentiable and can be measured. This assumption is common especially when a fractional order system is aimed to be controlled [52, 53].

The control action u must certify the reaching condition. It means that all error trajectories must intersect the sliding surface in a finite time. To reach this goal, the following theorem is proposed.

Theorem 4. The error dynamics system (18) is asymptotically stable at zero point under the following controller and adaptation laws:

$$u = -\frac{1}{g(t,\mathbf{Y})}\left(f(t,\mathbf{Y}) + \mathbf{F}^T(t,\mathbf{Y})\hat{\mathbf{\Theta}} + \hat{k}\,\mathrm{sgn}(S) + \eta S + \sum_{i=1}^{n} \alpha_i e_i - h(t,\mathbf{X}) \right) \tag{20}$$

$${}_0^C D_t^{\alpha} \hat{\theta}_i = \gamma_i S F_i(t,\mathbf{Y}) \tag{21}$$

$${}_0^C D_t^{\alpha} \hat{k} = \gamma_k |S| \tag{22}$$

where $\gamma_i \in \Re \ (1 \le i \le m)$ and $\gamma_k \in \Re$ are adaptation coefficients.

Proof. Let us consider the following Lyapunov function candidate:

$$V(t)=\frac{1}{2}S^2+\frac{1}{2}\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}\right)^T\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}\right)+\frac{1}{2}\left(k-\hat{k}\right)^2 \tag{23}$$

Using Lemma 1, one can obtain:

$$\begin{aligned}
{}_0^C D_t^\alpha V(t) &= \frac{1}{2}\,{}_0^C D_t^\alpha S^2+\frac{1}{2}\,{}_0^C D_t^\alpha\left(\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}\right)^T\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}\right)\right)+\frac{1}{2}\,{}_0^C D_t^\alpha\left(k-\hat{k}\right)^2\\
&\le S\,{}_0^C D_t^\alpha S+\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}\right)^T\,{}_0^C D_t^\alpha\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}\right)+\left(k-\hat{k}\right){}_0^C D_t^\alpha\left(k-\hat{k}\right)\\
&= S\left({}_0^C D_t^\alpha e_n+\sum_{i=1}^{n}\alpha_i e_i\right)-\left(\mathbf{\Theta}-\hat{\mathbf{\Theta}}\right)^T\,{}_0^C D_t^\alpha\hat{\mathbf{\Theta}}-\left(k-\hat{k}\right){}_0^C D_t^\alpha\hat{k}\\
&= S\left({}_0^C D_t^\alpha e_n+\sum_{i=1}^{n}\alpha_i e_i\right)-\sum_{i=1}^{m}\left(\theta_i-\hat{\theta}_i\right){}_0^C D_t^\alpha\hat{\theta}_i-\left(k-\hat{k}\right){}_0^C D_t^\alpha\hat{k}
\end{aligned} \tag{24}$$

Substituting Eq. (18) into Eq. (24) results in:

$$\begin{aligned}
{}_0^C D_t^\alpha V(t) &\le S\left(f(t,\mathbf{Y})+\sum_{i=1}^{m}F_i(t,\mathbf{Y})\,\theta_i+d(t)+g(t,\mathbf{Y})u-h(t,\mathbf{X})+\sum_{i=1}^{n}\alpha_i e_i\right)\\
&-\sum_{i=1}^{m}\left(\theta_i-\hat{\theta}_i\right){}_0^C D_t^\alpha\hat{\theta}_i-\left(k-\hat{k}\right){}_0^C D_t^\alpha\hat{k}
\end{aligned} \tag{25}$$

Along with Eq. (20) and Inequality (25), we obtain:

$$\begin{aligned}
{}_0^C D_t^\alpha V(t) &\le S\left(f(t,\mathbf{Y})+\sum_{i=1}^{m}F_i(t,\mathbf{Y})\,\theta_i+d(t)-h(t,\mathbf{X})+\sum_{i=1}^{n}\alpha_i e_i\right)\\
&-S\left(f(t,\mathbf{Y})+\sum_{i=1}^{m}F_i(t,\mathbf{Y})\,\hat{\theta}_i+\hat{k}\,\mathrm{sgn}(S)+\eta S+\sum_{i=1}^{n}\alpha_i e_i-h(t,\mathbf{X})\right)\\
&-\sum_{i=1}^{m}\left(\theta_i-\hat{\theta}_i\right)^T\,{}_0^C D_t^\alpha\hat{\theta}_i-\left(k-\hat{k}\right){}_0^C D_t^\alpha\hat{k}\\
&\le S\sum_{i=1}^{m}F_i(t,\mathbf{Y})\,\theta_i+|S|k-S\sum_{i=1}^{m}F_i(t,\mathbf{Y})\,\hat{\theta}_i+|S|\hat{k}\\
&-\sum_{i=1}^{m}\left(\theta_i-\hat{\theta}_i\right){}_0^C D_t^\alpha\hat{\theta}_i-\left(k-\hat{k}\right){}_0^C D_t^\alpha\hat{k}-\eta S^2\\
&=\sum_{i=1}^{m}\left(\theta_i-\hat{\theta}_i\right)\left(SF_i(t,\mathbf{Y})-{}_0^C D_t^\alpha\hat{\theta}_i\right)+\left(k-\hat{k}\right)\left(|S|-{}_0^C D_t^\alpha\hat{k}\right)-\eta S^2
\end{aligned} \tag{26}$$

By substituting adaptation laws into (26), we have

$${}_0^C D_t^\alpha V(t)\le-\eta S^2 \tag{27}$$

Integrating both sides of Eq. (27), we have

$$
\begin{aligned}
&{}_0D_t^{-\alpha}\,{}_0^CD_t^{\alpha}V = V(t)-V(0) \le -{}_0D_t^{-\alpha}\left(\eta S^2\right) = -\eta\,{}_0D_t^{-\alpha}\left(|S|^2\right) \Rightarrow \\
&V(t)+\eta\,{}_0D_t^{-\alpha}\left(|S|^2\right) \le V(0)
\end{aligned}
\tag{28}
$$

One can obtain the following equation from (28)

$$
\eta\,{}_0D_t^{-\alpha}\left(|S|^2\right) \le V(0) \Rightarrow {}_0D_t^{-\alpha}\left(|S|^2\right) \le \frac{V(0)}{\eta} \tag{29}
$$

As we assume that the sliding surface is continuously differentiable, it is a uniformly continuous function. Accordingly, Using Theorem 3, Inequality (29) demonstrates that the sliding surface becomes zero as time approaches to infinity. Also, due to asymptotic stability of the origin in the sliding surface, the error trajectory converges to zero and the error system (18) is asymptotically stable.

5. Simulation Results

In this section, simulation results are presented to show performance of the method. The fractional order gyro system which is used as a master system in this simulation is considered as follows:

$$
\begin{cases}
{}_0^CD_t^{\alpha}x_1 = x_2 \\
{}_0^CD_t^{\alpha}x_2 = -c_1^{\,2}\dfrac{\left(1-\cos x_1\right)^2}{\sin^3 x_1} - c_3x_2 - c_4x_2^3 + \left(c_2 + p\sin\omega t\right)\sin x_1
\end{cases}
\tag{30}
$$

Dynamics of this gyro system exhibits chaotic behavior for parameter values of $\alpha = 0.97$, $c_1 = 10$, $c_2 = 1$, $c_3 = 0.5$, $c_4 = 0.05$, $p = 35.5$, and $\omega = 2$ [54].

Slave system is the well-known fractional order Duffing system with fractional order of the equations $\alpha = 0.97$ [55]. Hence, the dynamics of the slave system in state space is considered as follows:

$$\begin{cases} {}_0^C D_t^{0.97} y_1 = y_2 \\ {}_0^C D_t^{0.97} y_2 = -\theta_1 y_1 - \theta_2 y_1^3 - \theta_3 y_2 + \theta_4 \cos(\omega t) + d(t) + g(t,\mathbf{Y})\, u \end{cases} \tag{31}$$

Since the parameters of the slave system is assumed to be unknown, Eq. (31) can be restated as:

$$\begin{cases} {}_0^C D_t^{0.97} y_1 = y_2 \\ {}_0^C D_t^{0.97} y_2 = f(t,\mathbf{Y}) + \mathbf{F}^T(t,\mathbf{Y})\boldsymbol{\Theta} + d(t) + g(t,\mathbf{Y})\, u \end{cases} \tag{32}$$

where $f(t,\mathbf{Y}) = 0$ and $g(t,\mathbf{Y}) = 1 + y_1^2$. Furthermore, $\mathbf{F}(t,\mathbf{Y})$ and $\boldsymbol{\Theta}$ are defined as:

$$\mathbf{F}(t,\mathbf{Y}) = \begin{bmatrix} -y_1 & -y_1^3 & -y_2 & \cos(\omega t) \end{bmatrix}^T \tag{33}$$

$$\boldsymbol{\Theta} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix}^{\mathrm{T}} \tag{34}$$

The disturbance term is taken as $d(t) = 0.2\cos(2t)$, which is bounded by $|d(t)| \le k = 0.2$. The parameter k is assumed to be unknown and should be updated by adaptation law. The initial conditions are considered as $\mathbf{X}(0) = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}^T$, $\mathbf{Y}(0) = \begin{bmatrix} -0.2 & -0.2 \end{bmatrix}^T$, $\hat{\boldsymbol{\Theta}}(0) = \begin{bmatrix} -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}^{\mathrm{T}}$, and $\hat{k}(0) = 0.1$. The adaptation coefficients are set to $\gamma_k = 1$ and $\gamma = \begin{bmatrix} 5 & 5 & 5 & 5 \end{bmatrix}^T$. The PECE algorithm [56] is used to solve fractional differential equations, and time step sizes are considered 0.01.

Numerical simulation results are shown in Figures 1-3. Figures 1 and 2 show time history of the systems state variables and synchronization error, respectively; and Figure 3 demonstrates time history of the control input u and the sliding surface S.

6. Conclusion

This paper has shown a method for synchronization of two uncertain and chaotic fractional order systems. The proposed method is based on an adaptive sliding mode controller. The adaptation laws are derived from a sliding surface using Lyapunov approach. The most influential advantage of the presented method is the robustness of the closed loop control system against system uncertainties and external disturbance. The other one is simplicity and suitable performance of the proposed controller. Finally, this method is implemented to synchronize two fractional order Duffing and gyro systems and numerical simulation results are included to demonstrate the great performance of the proposed method.

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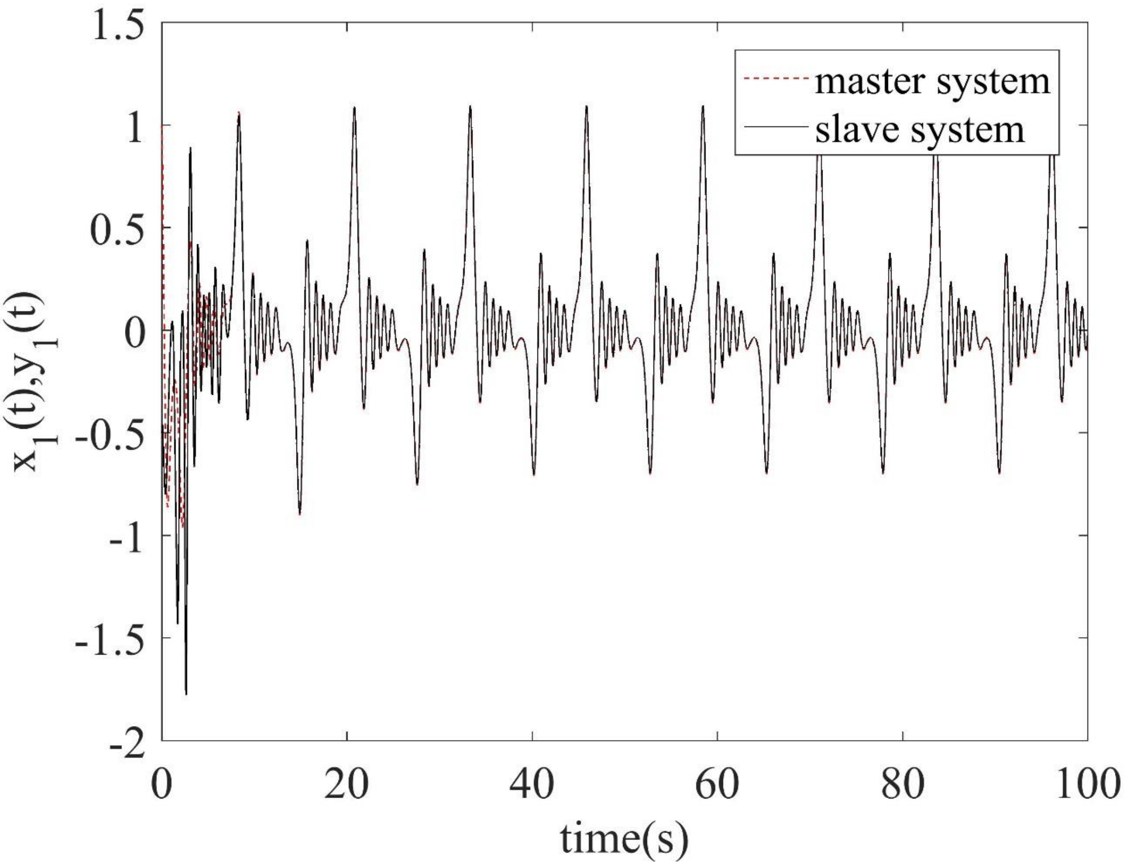
Captions:

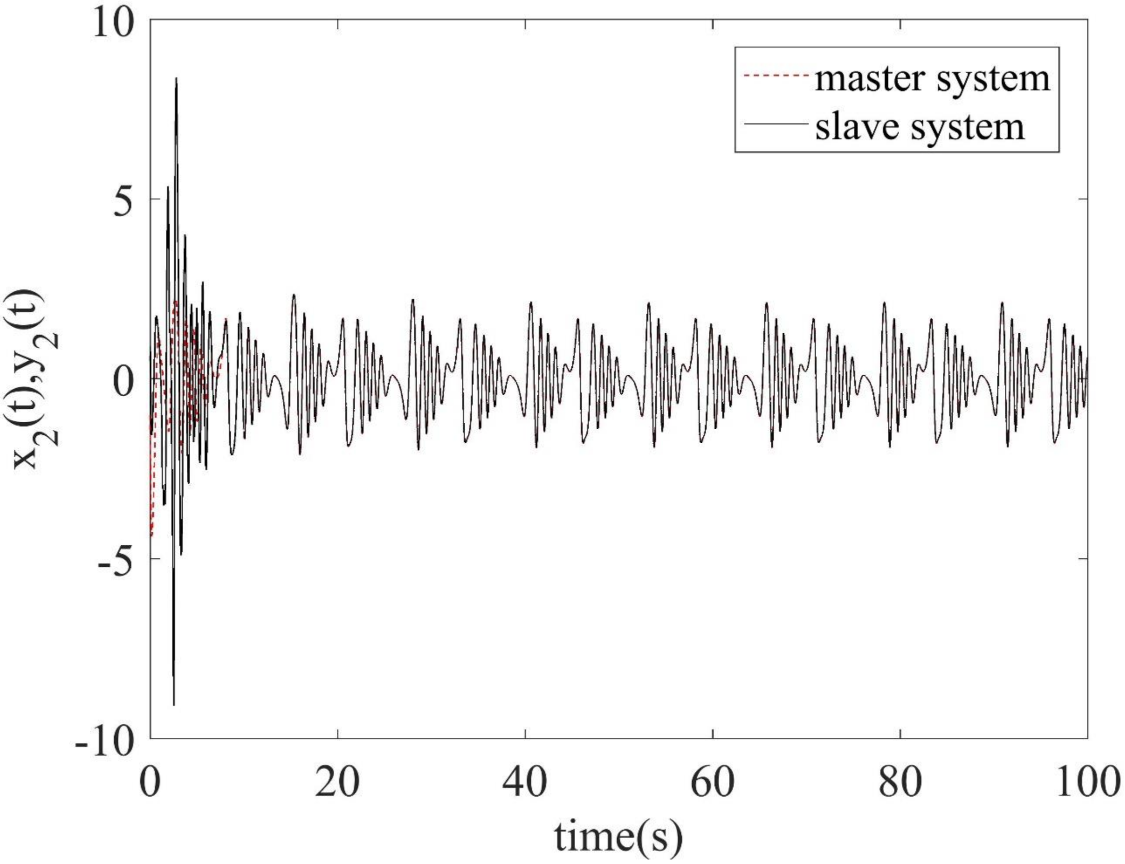
Figure 1: Time history of the master and slave state variables; (a) first state variable, (b) second state variable.

Figure 2: Time history of the synchronization error.

Figure 3: (a) Time history of the control action, (b) Time history of the sliding surface.


master system
slave system
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
20
40
60
80
100
time(s)
x_1(t),y_1(t)

(a)



(b)

Figure 1: Time history of the master and slave state variables; (a) first state variable, (b) second state variable

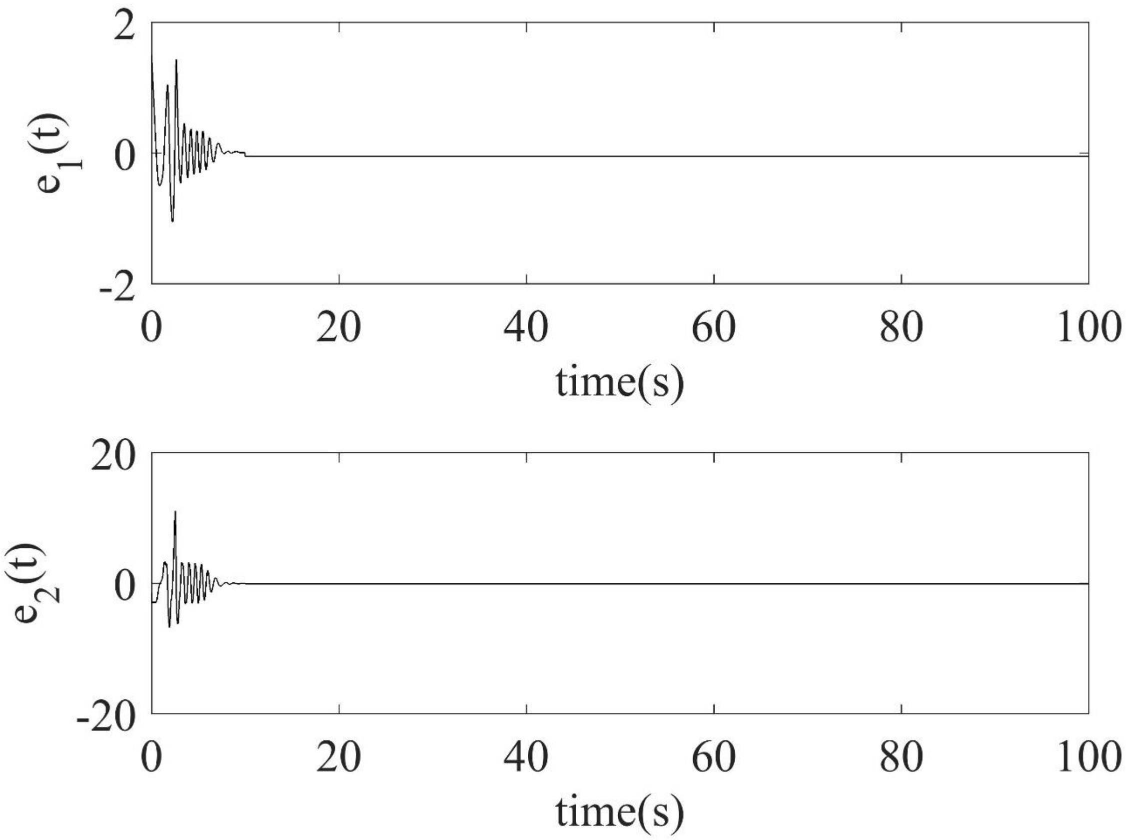


Figure 2: Time history of the synchronization error

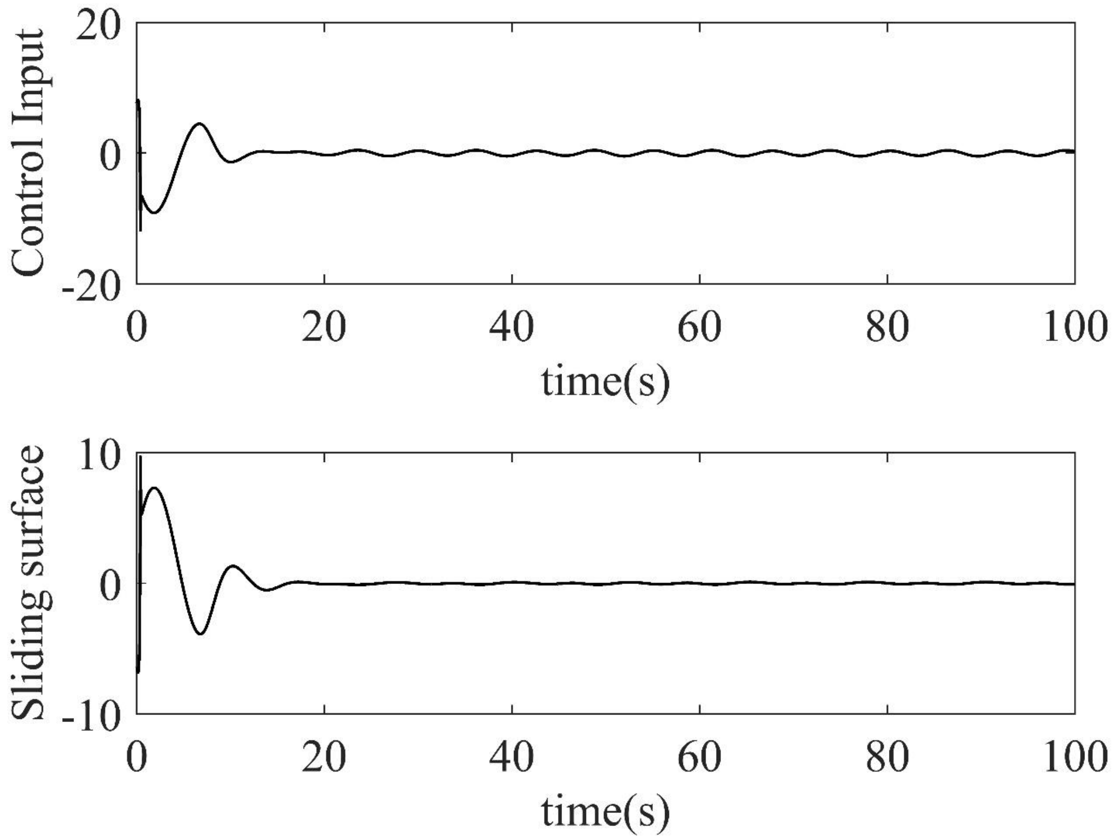


Figure 3: (a) Time history of the control action, (b) Time history of the sliding surface