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Adaptive synchronization of uncertain fractional-order chaotic systems using sliding mode control techniques

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Abstract

In this article, an adaptive nonlinear controller is designed to synchronize two uncertain fractional-order chaotic systems using fractional-order sliding mode control. The controller structure and adaptation laws are chosen such that asymptotic stability of the closed-loop control system is guaranteed. The adaptation laws are being calculated from a proper sliding surface using the Lyapunov stability theory. This method guarantees the closed-loop control system robustness against the system uncertainties and external disturbances. Eventually, the presented method is used to synchronize two fractional-order gyro and Duffing systems, and the numerical simulation results demonstrate the effectiveness of this method.

Keywords

Fractional-order chaotic systems, synchronization, fractional-order sliding mode control, adaptive control

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Introduction

Chaos is a phenomenon that appears in nonlinear dynamical systems and has been observed in many fields of science such as economics,[1] chemistry,[2] biology,[3] and engineering.[4] In recent years, synchronization of chaotic systems has attracted interest of many scientists in variety of fields.[5] In 1990, Pecora and Carroll[6] proposed the idea of chaos synchronization. In the last years, several methods have been introduced for synchronization of chaotic systems.[7,8] In many applications, parameters of the slave system are unknown or the system has some uncertainties; therefore, using adaptive control and developing some adaptive synchronization methods have been presented.[9–11]

Fractional calculus is an old field of mathematics from the 17th century that studies derivatives and integrals of non-integer order.[12] For many years, fractional calculus was a pure mathematics topic and has no applications in real world. But recently, fractional calculus is introduced as a powerful tool for modeling many systems in various fields of physics and engineering, for example, viscoelasticity,[13] dynamical systems,[14] biomedical applications,[15] signal processing,[16] electrical networks,[17] cyber-physical systems,[18] diffusion wave,[19–21] electromagnetism,[22] stochastic systems,[23] control theory,[24] and chaotic systems.[25–27] Also, many fractional-order (FO) controllers are developed, such as fractional proportional–integral–derivative (PID) controller,[28,29] fractional PI controller,[30] fractional PD controller,[31] fractional lead-lag controller,[32,33] fractional CRONE controller,[34] adaptive FO PID controller,[35] fractional model reference adaptive control,[36,37] and fractional sliding mode control.[38–40]

In recent years, many FO dynamic systems with chaotic behavior are introduced, such as FO Duffing system,[41] FO Chen system,[42] FO Van der Pol dynamics,[43] FO Rössler equations,[44] and FO Chua system.[45] In the last few years, several techniques have been proposed for synchronization of FO chaotic systems.[46–50]

In this article, an adaptive nonlinear controller is designed to synchronize two FO chaotic systems. In the

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proposed method, stability of the closed-loop control system is guaranteed by using Lyapunov approach. In addition, robustness of the closed-loop system is guaranteed by using the sliding mode control method.

The rest of this article is organized as follows: In section "Preliminary concepts," the most applicable definitions of FO integral and derivative and stability of the FO systems are presented. In section "Problem statement," the problem statement is presented. In section "Controller design," stability of the closed-loop control system is guaranteed by introducing a proper sliding mode controller. In section "Simulation results," numerical simulation results are shown. Finally, a concise conclusion is presented in section "Conclusion."

Preliminary concepts

Since the time Leibniz introduced non-integer order derivatives, several definitions have been generated by several mathematicians.[12] In this section, some basic definitions and stability theorems in fractional calculus are given.

Definition 1. The Riemann–Liouville fractional integral of order p is defined as follows[12]

$$ {}_0D_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1}f(\tau)d\tau \tag{1}$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2. The Caputo fractional integral of order p is defined as follows[12]

$$ {}_0^CD_t^pf(t) = \begin{cases} \dfrac{1}{\Gamma(k-p)}\displaystyle\int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{p+1-k}}d\tau & k-1<p<k \\ \dfrac{d^k}{dt^k}f(t) & p=k \end{cases} \tag{2}$$

Theorem 1 (FO extension of Lyapunov direct method). Consider the following FO nonlinear system[51]

$$ {}_0^CD_t^\alpha x = f(t,x) \tag{3}$$

Let $x=0$ be an equilibrium point for system (3), $V(t,x(t))$ be a continuously differentiable function as a Lyapunov function candidate, and $\gamma_i (i=1,2,3)$ be class-K functions such that

$$ \gamma_1(\|x\|) \leqslant V(t,x(t)) \leqslant \gamma_2(\|x\|) \tag{4}$$

$$ {}_0^CD_t^\beta V(t,x(t)) \leqslant -\gamma_3(\|x\|) \tag{5}$$

where $\beta \in (0,1)$. Then, system (3) is asymptotically stable.

Lemma 1. Let $x(t) \in \Re$ be a continuously differentiable function. Then, for all $t \geqslant t_0$[51]

$$ \frac{1}{2}{}_{t_0}^CD_t^\alpha x^2(t) \leqslant x(t){}_{t_0}^CD_t^\alpha x(t), \quad \forall\alpha \in (0,1) \tag{6}$$

Remark 1. In the case when $x(t) \in \Re^n$, Lemma 1 is still valid. That is, $\forall t \geqslant t_0$[51]

$$ \frac{1}{2}{}_{t_0}^CD_t^\alpha x^T(t)x(t) \leqslant x^T(t){}_{t_0}^CD_t^\alpha x(t), \quad \forall\alpha \in (0,1) \tag{7}$$

In recent years, Barbalat's Lemma is developed for FO nonlinear systems.[52]

Theorem 2. Let $\phi : \Re \to \Re$ be a uniformly continuous function on $[t_0, \infty)$.[52] Assume that there exist two positive constants p and M such that ${}_0D_t^{-\alpha}|\phi|^p \leqslant M$ for all $t > t_0 > 0$ with $\alpha \in (0,1)$. Then

$$ \lim_{t\to\infty}\phi(t) = 0 \tag{8}$$

Problem statement

System description

In synchronization of two chaotic systems, there are two systems: a particular dynamic system as a master and another different dynamic as a slave. From the viewpoint of control, the task is to design a controller such that the slave system imitates behavior of the master system.

Consider the following commensurate FO chaotic system as the slave

$$ \begin{cases} {}_0^CD_t^\alpha y_i = y_{i+1}, 1 \leqslant i \leqslant n-1 \\ {}_0^CD_t^\alpha y_n = f(t,\mathbf{Y}) + F^T(t,\mathbf{Y})\mathbf{\Theta} + d(t) + g(t,\mathbf{Y})u \end{cases} \tag{9}$$

where α represents the order of fractional derivatives of the system and belongs to interval $(0,1)$; $\mathbf{Y} = [y_1 \ \cdots \ y_n]^T$ is the system state vector; $f(\cdot)$ and $g(\cdot)$ are nonlinear functions and belong to $C^1(\Re^n \times \Re \to \Re)$; $\mathbf{F}(\cdot)$ is a known vector with $1 \times m$ dimension; $\boldsymbol{\theta} \in \Re^m$ stands for the uncertain parameter vector of the system; $d(\cdot)$ denotes the external disturbance; and $u(t)$ is the input of the system. It is assumed that $g(t,\mathbf{Y}) \neq 0$ and $|d(t)| \leqslant k$ for all $t > 0$, and k is a positive and unknown real number.

Let us define the following commensurate FO chaotic system as a master system

$$ \begin{cases} {}_0^CD_t^\alpha x_i = x_{i+1}, 1 \leqslant i \leqslant n-1 \\ {}_0^CD_t^\alpha x_n = h(t,\mathbf{X}) \end{cases} \tag{10}$$

where $\mathbf{X} = [x_1 \ \cdots \ x_n]^T$ is the master system state vector. Again, it should be mentioned that we assume the systems given by equations (9) and (10) have chaotic behavior; the master–slave synchronization of them is our main objective. To achieve this goal, we will design an adaptive sliding mode control such that the closed-loop control system defined later satisfies the stability condition. In this article, two main assumptions are used; the first one is given here, and the second one will be presented in the next section.

Assumption 1. Both $\mathbf{\Theta}$ and $d(\cdot)$ are bounded, but their exact values are unknown.

Controller design

The synchronization error is defined as difference between the states of the master and the slave systems

$$e_i = y_i - x_i, 1 \leqslant i \leqslant n \tag{11}$$

From equations (9), (10), and (11), the error dynamics can be written in the following form

$$\begin{cases} {}_0^C D_t^\alpha e_i = e_{i+1}, 1 \leqslant i \leqslant n-1 \\ {}_0^C D_t^\alpha e_n = f(t,\mathbf{Y}) + \mathbf{F}^T(t,\mathbf{Y})\mathbf{\Theta} + d(t) + g(t,\mathbf{Y})u - h(t,\mathbf{X}) \end{cases} \tag{12}$$

The main goal in this section is to develop the control input in system (12) such that the closed loop control system of the error dynamics is asymptotically stable and robust against the system uncertainty and external disturbance. To this purpose, adaptive sliding mode control techniques will be used. We consider the following function as a sliding surface function

$$S = e_n + \sum_{i=1}^{n} \alpha_{i0} D_t^{-\alpha} e_i \tag{13}$$

where $\alpha_i > 0s$ are set to obtain an exponentially stable dynamics for sliding mode, $S = 0$.

Assumption 2. A simplifying condition which is very common in controlling FO systems is assumed, that is, all of the system state variables as well as the sliding surface are continuously differentiable and can be measured. This assumption is common especially when a FO system is aimed to be controlled.[53,54]

The control action u must certify the reaching condition. It means that all error trajectories must intersect the sliding surface in a finite time. To reach this goal, the following theorem is proposed.

Theorem 3. The error dynamics system (12) is asymptotically stable at zero point under the following controller and adaptation laws

$$u = -\frac{1}{g(t,\mathbf{Y})}\left(f(t,\mathbf{Y}) + \mathbf{F}^T(t,\mathbf{Y})\hat{\mathbf{\Theta}} + \hat{k}\operatorname{sgn}(S) + \eta S + \sum_{i=1}^{n}\alpha_i e_i - h(t,\mathbf{X})\right) \tag{14}$$

$${}_0^C D_t^\alpha \hat{\theta}_i = \gamma_i S F_i(t,\mathbf{Y}) \tag{15}$$

$${}_0^C D_t^\alpha \hat{k} = \gamma_k |S| \tag{16}$$

where $\gamma_i \in \Re (1 \leqslant i \leqslant m)$ and $\gamma_k \in \Re$ are adaptation coefficients, F_is are the elements of $\mathbf{F}$, $\operatorname{sgn}(\cdot)$ is the standard sign function, η is an arbitrary positive real number, and $\hat{\theta}_i$s are the elements of $\hat{\mathbf{\Theta}}$ as an estimation of $\mathbf{\Theta}$.

The initial conditions of equations (15) and (16) do not affect the stability proof, and any initial conditions work. However, the initial conditions can affect the transient response of the system before convergence, which has not been studied here.

Proof. Let us consider the following Lyapunov function candidate

$$V(t) = \frac{1}{2}S^2 + \sum_{i=1}^{m}\frac{1}{2\gamma_i}\left(\theta_i - \hat{\theta}_i\right)^2 + \frac{1}{2\gamma_k}\left(k - \hat{k}\right)^2 \tag{17}$$

Using Lemma 1, and regarding the unknown parameters of the system are constant, one can obtain

$$\begin{aligned} {}_0^C D_t^\alpha V(t) &= \frac{1}{2}{}_0^C D_t^\alpha S^2 + \sum_{i=1}^{m}\left(\frac{1}{2\gamma_i}{}_0^C D_t^\alpha\left(\theta_i - \hat{\theta}_i\right)^2\right) \\ &+ \frac{1}{2\gamma_k}{}_0^C D_t^\alpha\left(k - \hat{k}\right)^2 \\ &\leqslant S{}_0^C D_t^\alpha S + \sum_{i=1}^{m}\left(\frac{1}{\gamma_i}\left(\theta_i - \hat{\theta}_i\right){}_0^C D_t^\alpha\left(\theta_i - \hat{\theta}_i\right)\right) \\ &+ \frac{1}{\gamma_k}\left(k - \hat{k}\right){}_0^C D_t^\alpha\left(k - \hat{k}\right) \\ &= S\left({}_0^C D_t^\alpha e_n + \sum_{i=1}^{n}\alpha_i e_i\right) \\ &- \left(\sum_{i=1}^{m}\frac{1}{\gamma_i}\left(\theta_i - \hat{\theta}_i\right){}_0^C D_t^\alpha\hat{\theta}_i\right) - \frac{1}{\gamma_k}\left(k - \hat{k}\right){}_0^C D_t^\alpha\hat{k} \end{aligned} \tag{18}$$

Substituting equation (12) into equation (18) results in

$$\begin{aligned} {}_0^C D_t^\alpha V(t) \leqslant S\left(f(t,\mathbf{Y}) + \sum_{i=1}^{m}F_i(t,\mathbf{Y})\theta_i + d(t) + g(t,\mathbf{Y})u - h(t,\mathbf{X}) + \sum_{i=1}^{n}\alpha_i e_i\right) \\ - \sum_{i=1}^{m}\frac{1}{\gamma_i}\left(\theta_i - \hat{\theta}_i\right){}_0^C D_t^\alpha\hat{\theta}_i - \frac{1}{\gamma_k}\left(k - \hat{k}\right){}_0^C D_t^\alpha\hat{k} \end{aligned} \tag{19}$$

Along with equation (14) and inequality (19), we obtain

$$
\begin{aligned}
{}_0^C D_t^\alpha V(t) \leqslant & S\left(f(t,\mathbf{Y}) + \sum_{i=1}^{m} F_i(t,\mathbf{Y})\theta_i + d(t) - h(t,\mathbf{X}) + \sum_{i=1}^{n} \alpha_i e_i\right) \\
& - S\left(f(t,\mathbf{Y}) + \sum_{i=1}^{m} F_i(t,\mathbf{Y})\hat{\theta}_i + \hat{k}\operatorname{sgn}(S) + \eta S + \sum_{i=1}^{n} \alpha_i e_i - h(t,\mathbf{X})\right) \\
& - \sum_{i=1}^{m} \frac{1}{\gamma_i}\left(\theta_i - \hat{\theta}_i\right){}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}\left(k - \hat{k}\right){}_0^C D_t^\alpha \hat{k} \\
\leqslant & S\sum_{i=1}^{m} F_i(t,\mathbf{Y})\theta_i + |S|k - S\sum_{i=1}^{m} F_i(t,\mathbf{Y})\hat{\theta}_i + |S|\hat{k} \\
& - \sum_{i=1}^{m} \frac{1}{\gamma_i}\left(\theta_i - \hat{\theta}_i\right){}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}\left(k - \hat{k}\right){}_0^C D_t^\alpha \hat{k} - \eta S^2 \\
= & \sum_{i=1}^{m} \left(\theta_i - \hat{\theta}_i\right)\left(SF_i(t,\mathbf{Y}) - \frac{1}{\gamma_i}{}_0^C D_t^\alpha \hat{\theta}_i\right) + \left(k - \hat{k}\right)\left(|S| - \frac{1}{\gamma_k}{}_0^C D_t^\alpha \hat{k}\right) - \eta S^2
\end{aligned} \tag{20}
$$

By substituting adaptation laws into equation (20), we have

$$
\begin{aligned}
{}_0^C D_t^\alpha V(t) \leqslant & \sum_{i=1}^{m} \left(\theta_i - \hat{\theta}_i\right)\left(SF_i(t,\mathbf{Y}) - \frac{1}{\gamma_i}\left(\gamma_i S F_i(t,\mathbf{Y})\right)\right) + \left(k - \hat{k}\right)\left(|S| - \frac{1}{\gamma_k}\left(\gamma_k |S|\right)\right) - \eta S^2 \\
= & \sum_{i=1}^{m} \left(\theta_i - \hat{\theta}_i\right)\left(SF_i(t,\mathbf{Y}) - SF_i(t,\mathbf{Y})\right) + \left(k - \hat{k}\right)\left(|S| - |S|\right) - \eta S^2
\end{aligned} \tag{21}
$$

It is obvious that two terms of the right side of current inequality is suppressed by substituting the adaptation laws. Therefore, inequality (21) can be simplified as follows

$$
{}_0^C D_t^\alpha V(t) \leqslant -\eta S^2 \tag{22}
$$

Integrating both sides of equation (22), we have

$$
\begin{aligned}
& {}_0 D_t^{-\alpha}\, {}_0^C D_t^\alpha V = V(t) - V(0) \\
& \leqslant -{}_0 D_t^{-\alpha}\left(\eta S^2\right) = -\eta\, {}_0 D_t^{-\alpha}\left(|S|^2\right) \Rightarrow \\
& V(t) + \eta\, {}_0 D_t^{-\alpha}\left(|S|^2\right) \leqslant V(0)
\end{aligned} \tag{23}
$$

Since $V(t)$ is a positive definite function, one can obtain the following equation from equation (23)

$$
\eta\, {}_0 D_t^{-\alpha}\left(|S|^2\right) \leqslant V(0) \Rightarrow {}_0 D_t^{-\alpha}\left(|S|^2\right) \leqslant \frac{V(0)}{\eta} \tag{24}
$$

As we assumed that the sliding surface is continuously differentiable, it is a uniformly continuous function. Accordingly, using Theorem 3, Inequality (24) demonstrates that the sliding surface becomes zero as time tends to infinity. Also, due to asymptotic stability of the origin in the sliding surface, the error trajectory converges to zero and the error system (12) is asymptotically stable.

Simulation results

In this section, simulation results are presented to show the performance of the proposed method. The FO gyro system which is used as a master system in this simulation is considered as follows

$$
\begin{cases}
{}_0^C D_t^\alpha x_1 = x_2 \\
{}_0^C D_t^\alpha x_2 = -c_1{}^2 \frac{(1-\cos x_1)^2}{\sin^3 x_1} - c_3 x_2 - c_4 x_2^3 + (c_2 + p\sin\omega t)\sin x_1
\end{cases} \tag{25}
$$

Dynamics of this gyro system shows chaotic behavior for parameter values of $\alpha = 0.97$, $c_1 = 10$, $c_2 = 1$, $c_3 = 0.5$, $c_4 = 0.05$, $p = 35.5$, and $\omega = 2$.[55]

The slave system is the well-known FO Duffing system with fractional order of the equations $\alpha = 0.97$.[56] Hence, the dynamics of the slave system in state space is considered as follows

$$
\begin{cases}
{}_0^C D_t^{0.97} y_1 = y_2 \\
{}_0^C D_t^{0.97} y_2 = -\theta_1 y_1 - \theta_2 y_1^3 - \theta_3 y_2 + \theta_4\cos(\omega t) + d(t) + g(t,\mathbf{Y})u
\end{cases} \tag{26}
$$

Since the parameters of the slave system is assumed to be unknown, equation (26) can be restated as

$$
\begin{cases}
{}_0^C D_t^{0.97} y_1 = y_2 \\
{}_0^C D_t^{0.97} y_2 = f(t,\mathbf{Y}) + \mathbf{F}^T(t,\mathbf{Y})\boldsymbol{\Theta} + d(t) + g(t,\mathbf{Y})u
\end{cases} \tag{27}
$$

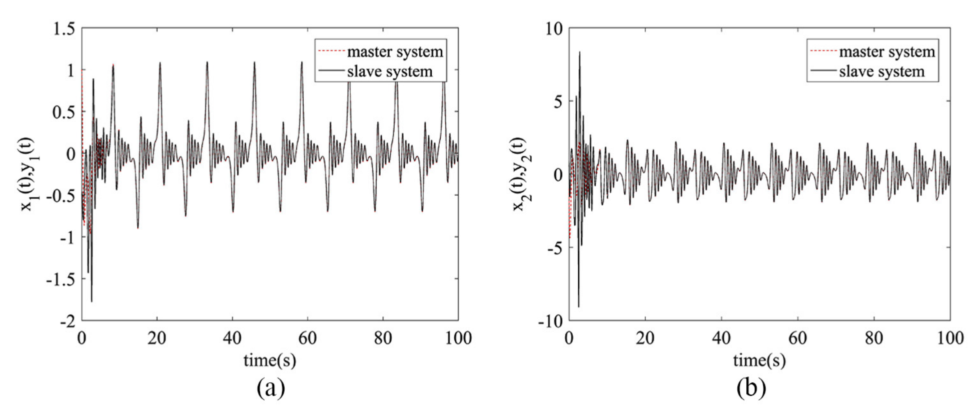

Figure 1. Time history of the master and slave state variables: (a) first state variable and (b) second state variable.

where $f(t, \mathbf{Y}) = 0$ and $g(t, \mathbf{Y}) = 1 + y_1^2$. Furthermore, $\mathbf{F}(t, \mathbf{Y})$ and $\mathbf{\Theta}$ are defined as

$$\mathbf{F}(t, \mathbf{Y}) = \begin{bmatrix} -y_1 & -y_1^3 & -y_2 & \cos(\omega t) \end{bmatrix}^T \tag{28}$$

$$\mathbf{\Theta} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix}^T \tag{29}$$

The disturbance term is taken as $d(t) = 0.2\cos(2t)$, which is bounded by $|d(t)| \leqslant k = 0.2$. The parameter k is assumed to be unknown and should be updated by adaptation law. The initial conditions are considered as $\mathbf{X}(0) = [0.2 \quad 0.2]^T$, $\mathbf{Y}(0) = [-0.2 \quad -0.2]^T$, $\hat{\mathbf{\Theta}}(0) = [-0.5 \quad -0.5 \quad 0.5 \quad 0.5]^T$, and $\hat{k}(0) = 0.1$. The adaptation coefficients are set to $\gamma_k = 1$ and $\gamma = [5 \quad 5 \quad 5 \quad 5]^T$. In order to solve fractional differential equations, the Predict–Evaluate–Correct–Evaluate algorithm[57] is used with time step size of 0.01.

Numerical simulation results are shown in Figures 1–3. Figures 1 and 2 show time histories of the system state variables and synchronization error, respectively; from these Figures 1 and 2, one can easily see that after less than 10 units of time, the synchronization of two chaotic systems is completely achieved, and the state trajectories of the slave system follow those of the master system after this short period of time, and it is seen that the synchronization error has been suppressed. Figure 3 demonstrates time history of the control input and the sliding surface. As it is observed, they converge toward a bounded region around zero. This phenomenon is reasonable, since after the synchronization, the closed-loop control system is stabilized, and the system has reached the sliding surface.

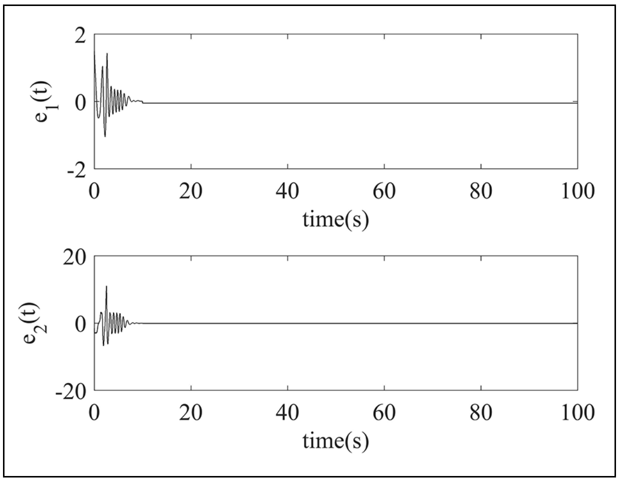

Figure 2. Time history of the synchronization error.

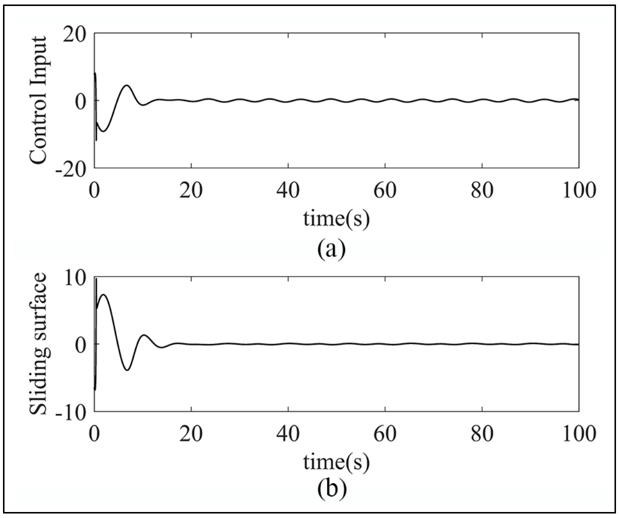

Figure 3. (a) Time history of the control action and (b) time history of the sliding surface.

Conclusion

This article has shown a method for synchronization of two uncertain and chaotic FO systems. The proposed method is based on an adaptive sliding mode controller. The adaptation laws are derived from a sliding surface using the Lyapunov stability theory. The most influential advantage of the presented method is the robustness of the closed-loop control system against system uncertainties and external disturbance. The other one is simplicity and suitable performance of the proposed controller. Finally, the proposed method is used to control a synchronization problem of two FO Duffing and gyro systems. From the simulation results, it is seen that a satisfactory control performance is achieved by using the proposed scheme. Future research can focus on exploiting the proposed control method in Li et al.,[58,59] to extend our presented controller to be utilized for multisystem synchronization. Further studies should be carried out about this matter.

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