Original Article

Institution of MECHANICAL ENGINEERS

Journal of SYSTEMS AND CONTROL ENGINEERING

Proc IMechE Part I:
J Systems and Control Engineering
1–8

DOI: 10.1177/0959651818758209
journals.sagepub.com/home/pii
SAGE

Robust adaptive fractional order proportional integral derivative controller design for uncertain fractional order nonlinear systems using sliding mode control

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Abstract

This article presents a robust adaptive fractional order proportional integral derivative controller for a class of uncertain fractional order nonlinear systems using fractional order sliding mode control. The goal is to achieve closed-loop control system robustness against the system uncertainty and external disturbance. The fractional order proportional integral derivative controller gains are adjustable and will be updated using the gradient method from a proper sliding surface. A supervisory controller is used to guarantee the stability of the closed-loop fractional order proportional integral derivative control system. Finally, fractional order Duffing–Holmes system is used to verify the proposed method.

Keywords

Fractional order systems, fractional order proportional integral derivative controller, gradient method, fractional order sliding mode control

Date received: 17 November 2017; accepted: 22 December 2017

Introduction

Fractional calculus is an old field of mathematics that deals with derivatives and integrals of non-integer order and has had no applications over many years.[1] In recent years, fractional calculus has been extensively used in a variety of fields of physics and engineering applications, for example, dynamical systems,[2] viscoelastic damping,[3] signal processing,[4] diffusion wave,[5] biomedical applications,[6] stochastic systems,[7] and chaotic systems.[8–10]

Recently, many fractional order controllers are developed like fractional order proportional integral derivative (FOPID) control,[11,12] fractional order proportional integral (PI) and proportional differential (PD) control,[13,14] fractional order lead-lag control,[15,16] fractional CRONE controller,[17] fractional order model reference adaptive control,[18,19] and sliding mode control of fractional order systems.[20]

In most of the applications, the controllers are of proportional integral derivative (PID) type due to its simplicity and appropriate performance in a variety of applications.[21] By developing fractional calculus in control theory, FOPID controller—which is simply called $PI^{\lambda}D^{\mu}$—was proposed by Podlubny.[1] The main feature of designing a FOPID controller is the determination of five controller parameters: proportional gain, integral gain, derivative gain, order of integral, and order of derivative. Up to now, several methods of tuning for FOPID controller have been established,[22–24] but many real systems are mostly time varying or may have uncertainty. Hence, using adaptive control and developing methods to design adaptive $PI^{\lambda}D^{\mu}$ control must be concerned.

In recent years, some methods are proposed for tuning adaptive FOPID controller based on sliding mode control,[25] fuzzy methods,[26,27] and genetic algorithm,[28] but up to now no method has been developed for uncertain fractional order nonlinear systems and in the previous methods the stability of the closed-loop control systems is not considered.

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In this article, a robust adaptive FOPID controller is introduced to control an uncertain fractional order nonlinear system. Robustness of the closed-loop system is guaranteed using fractional order sliding mode control method. In addition, a supervisory control is used to guarantee the boundedness of the system state trajectories.

This article is organized as follows: section "Preliminaries" provides a review of the preliminary concepts of fractional calculus. In section "Problem statement," the problem statement is presented. In section "Stability analysis and supervisory controller design," the stability of the closed-loop control system is analyzed and the supervisory controller is introduced to guarantee the stability of the system. In section "Adaptation law," proper sliding surface is introduced and the gradient method is used to generate adaptation laws for FOPID controller gains. Numerical simulations are shown in section "Numerical simulations." Finally, a brief conclusion is presented in the last section.

Preliminaries

There are several definitions of fractional derivatives. The common formulations for these derivatives were generated by Grünwald–Letnikov, Riemann–Liouville, and Caputo.

Definition 1 (Riemann–Liouville fractional integral). The Riemann–Liouville fractional integral[1] of order $p \in \Re^+$ of function $f(\tau)$ is defined as

$$ {}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau \tag{1} $$

where $\Gamma(\cdot)$ is the standard Gamma function.

Definition 2 (Riemann–Liouville fractional derivative). The Riemann–Liouville fractional derivative[1] of order $p \in \Re^+$ of function $f(\tau)$ is defined as

$$ {}_0^{RL}D_t^p f(t) = \frac{d^k}{dt^k} \left({}_0D_t^{-(k-p)} f(t) \right) \tag{2} $$

where k is an integer such that $k-1 \leqslant p < k$.

Definition 3 (Caputo fractional derivative). The Caputo fractional derivative[1] of order $p \in \Re^+$ of function $f(\tau)$ is defined as

$$ {}_0^{C}D_t^p f(t) = \begin{cases} \frac{1}{\Gamma(k-p)} \int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{p+1-k}} d\tau & k-1 < p < k \\ \frac{d^k}{dt^k} f(t) & p = k \end{cases} \tag{3} $$

In engineering applications, Caputo derivative has several advantages to Riemann–Liouville derivation method. The most influential one is that fractional differential equations with Caputo derivative take on the same form as for integer order differential equations. Another difference between the above-mentioned definitions is that the Caputo derivative of a constant is 0, whereas the Riemann–Liouville derivative of a constant C is not equal to 0, but

$$ {}_0^{RL}D_t^{\alpha} C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)} \tag{4} $$

The relation between Riemann–Liouville and Caputo fractional derivatives can be expressed as[1]

$$ {}_0^{RL}D_t^p f(t) = {}_0^{C}D_t^p f(t) + \sum_{k=0}^{n-1} \Phi_{k-p+1}(t) f^{(k)}(0) \tag{5} $$

where n is an integer such that $n-1 \leqslant p < n$ and the function $\Phi_p(t)$ is defined by

$$ \Phi_p(t) = \begin{cases} \frac{t^{p-1}}{\Gamma(p)} & t > 0 \\ 0 & t \leqslant 0 \end{cases} \tag{6} $$

Theorem 1. If $f(t)$ and $g(t)$ along all its derivatives are continuous in $[0, t]$, then the Leibniz rule for fractional differentiation takes the form[1]

$$ {}_0^{RL}D_t^p (g(t) f(t)) = \sum_{k=0}^{\infty} \binom{p}{k} g^{(k)}(t) {}_0^{RL}D_t^{p-k} f(t) \tag{7} $$

For $0 < p < 1$, the Leibniz rule for Caputo fractional order derivative can be expressed as

$$ \begin{aligned} {}_0^{C}D_t^p (g(t) f(t)) = & \sum_{k=0}^{\infty} \binom{p}{k} g^{(k)}(t) {}_0^{RL}D_t^{p-k} f(t) \\ & - \frac{(g(t) f(t))|_{t=0}}{t^p \Gamma(1-p)} \end{aligned} \tag{8} $$

Definition 4. A fractional order linear system in state space form is like

$$ \begin{aligned} \begin{Bmatrix} {}_0^{C}D_t^{\alpha_1} x_1 \\ {}_0^{C}D_t^{\alpha_2} x_2 \\ \vdots \\ {}_0^{C}D_t^{\alpha_n} x_n \end{Bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \\ &= \mathbf{A}_{n \times n} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \end{aligned} \tag{9} $$

where $\alpha_1, \ldots, \alpha_n$ are arbitrary real numbers. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, the system is called commensurate.

Theorem 2. Consider a commensurate fractional order linear system[29]

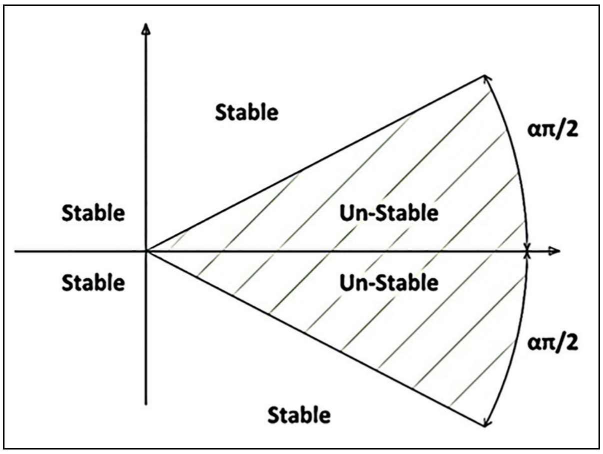

Figure 1. Stability condition in commensurate fractional order linear system.

$$\frac{d^\alpha}{dt^\alpha}\mathbf{x} = \mathbf{A}\mathbf{x} \tag{10}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{A} \in \Re^{n\times n}$, and α is an arbitrary real number between 0 and 2. The autonomous system is asymptotically stable if the following condition is satisfied

$$|\arg(\lambda_i)| > \alpha\frac{\pi}{2} \quad \forall i \tag{11}$$

where λ_i represents the eigenvalues of matrix $\mathbf{A}$. This condition is shown in Figure 1.

Definition 5. A fractional order nonlinear system can be represented in the following state space form

$${}_0^C D_t^{\alpha_i} x_i = f_i(t, x_1, \ldots, x_n) x_i(0) = x_{i0}, i = 1, \ldots, n \tag{12}$$

where $0 < \alpha_i \leqslant 1$ for $i = 1, \ldots, n$.

Problem statement

System description

Consider the following uncertain fractional order nonlinear system

$$\begin{cases} {}_0^C D_t^\alpha x_i = x_{i+1}, 1 \leqslant i \leqslant n-1 \\ {}_0^C D_t^\alpha x_n = f(t, \mathbf{X}) + \Delta f(t, \mathbf{X}) + d(t) + u(t) \\ y(t) = x_1(t) \end{cases} \tag{13}$$

where α is an arbitrary real number and shows fractional order of the equations, $\mathbf{X} = [x_1 \quad \cdots \quad x_n]^T$ is the measurable state vector of the system, $f(\cdot)$ is a nonlinear function, $\Delta f(\cdot)$ stands for uncertainty, $d(\cdot)$ denotes the external disturbances, and $y(t)$ and $u(t)$ are the output and input of the system, respectively.

The main goal is to design an adaptive FOPID controller for the above-mentioned system such that the output tracks a desired reference signal. The controller gains will be updated through a proper sliding surface. Because of using sliding mode techniques, the controller is robust against the system uncertainties and external disturbances. In addition, a supervisory control will be developed to guarantee boundedness of the tracking error. Figure 2 shows the schematic diagram of the closed-loop control system. Let y_d be the desired output. Therefore, we define $\mathbf{r}(t)$ as a reference signal

$$\mathbf{r}(t) = [r_1(t) \quad \cdots \quad r_n(t)]^T \tag{14}$$

where $r_1 = y_d$ and $r_{i+1} = {}_0^C D_t^\alpha r_i$ for $i = 1, \ldots, n-1$ and it is assumed that $r_i's, i = 1, \ldots, n$ are all bounded with known bounds and continuously differentiable.

Define the error vector of the system as

$$\mathbf{E} = [e_1 \quad \cdots \quad e_n]^T \tag{15}$$

where

$$\begin{aligned} e &= e_1 = r_1 - x_1 = r_1 - y \\ e_i &= r_i - x_i, 2 \leqslant i \leqslant n \end{aligned} \tag{16}$$

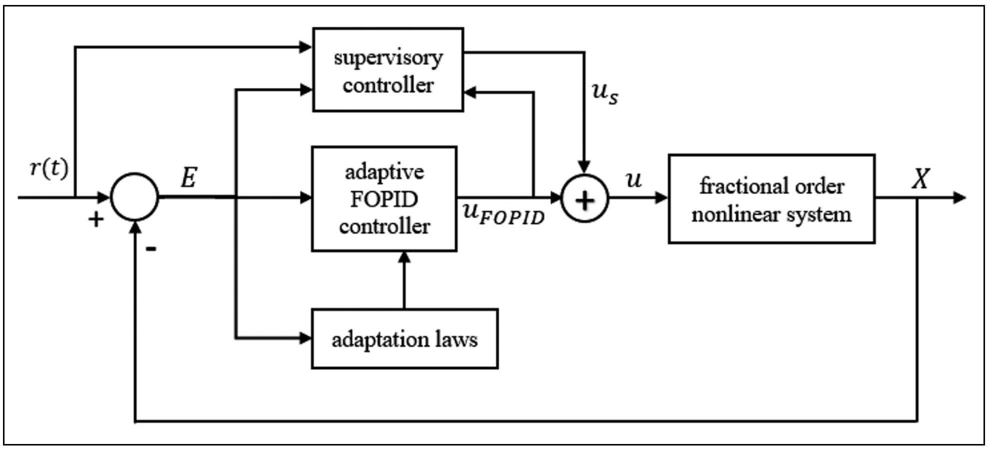

Figure 2. Schematic diagram of the closed-loop system.

From equations (13) and (16), output tracking error dynamics of the closed-loop system can be expressed as

$$\begin{cases} {}_0^C D_t^\alpha e_i = e_{i+1}, 1 \leqslant i \leqslant n-1 \\ {}_0^C D_t^\alpha e_n = {}_0^C D_t^\alpha r_n - f(t, \mathbf{X}) - \Delta f(t, \mathbf{X}) - d(t) - u(t) \end{cases} \tag{17}$$

Throughout the article, the following assumptions are made.

Assumption 1. Derivation order of equation (13), α, belongs to $(0, 1)$.

Assumption 2. All x_i's are continuously differentiable. In other words, we assume that $x_i \in C^1[0, \infty]$. This assumption is common especially when a fractional order system is aimed to be controlled.[30,31] Since all x_i's and r_i's are continuously differentiable, all e_i's are continuously differentiable too.

Assumption 3. $f(\cdot)$, $\Delta f(\cdot)$, and $d(\cdot)$ are all unknown functions. But all of them are bounded with known bounds

$$|f(\cdot)| \leqslant f^u \tag{18}$$

$$|\Delta f(\cdot)| \leqslant \Delta f^u \tag{19}$$

$$|d(\cdot)| \leqslant d^u \tag{20}$$

where f^u, Δf^u, and d^u are known positive functions which are the upper bounds of the nominal plant of the system, system modeling uncertainties, and external disturbances, respectively.

Control input

Control input consists of two parts: the FOPID controller and a supervisory controller. Therefore, the control input in equation (13) can be written as follows

$$u(t) = u_{FOPID}(t) + u_s(t) \tag{21}$$

where $u_{FOPID}(t)$ and $u_s(t)$ stand for the FOPID and supervisory controllers, respectively.

The FOPID controller can be expressed as

$$u_{FOPID}(t) = K_P e(t) + K_{I0}\, D_t^{-\lambda} e(t) + K_{D0}^{\ C} D_t^\mu e(t) \tag{22}$$

where K_P, K_I, and K_D are the proportional, integral, and derivative gains, respectively. For convenience, we define vectors $\mathbf{\Theta}$ and $\mathbf{\Phi}$ as

$$\mathbf{\Theta} = \begin{bmatrix} K_P & K_I & K_D \end{bmatrix}^T \tag{23}$$

$$\mathbf{\Phi} = \begin{bmatrix} e & {}_0D_t^{-\lambda} e & {}_0^C D_t^\mu e \end{bmatrix}^T \tag{24}$$

So, FOPID controller can be rewritten in the following form

$$u_{FOPID}(t) = \mathbf{\Theta}^T \mathbf{\Phi} = \sum_{i=1}^{3} \theta_i \varphi_i \tag{25}$$

where θ_i and φ_i represent the elements of the vectors $\mathbf{\Theta}$ and $\mathbf{\Phi}$, respectively.

Supervisory controller will be designed to guarantee the stability of the closed-loop system. The overall procedure of the supervisory controller design is presented in the next section.

Stability analysis and supervisory controller design

In this section, we will define a bounded set Ω for tracking error trajectories and the supervisory control will be obtained in a way that Ω becomes an attracting set. For this purpose, the supervisory controller should guarantee the stability condition out of the Ω set. In other words, it should be determined such that the Lyapunov function derivative with respect to time be negative definite out of Ω.

Consider the constraint set Ω for tracking error of the system defined as

$$\Omega = \left\{ \mathbf{E} \in \Re^n : \|\mathbf{E}\|_2 \leqslant M \right\} \tag{26}$$

where M is a pre-specified parameter. Now, consider the following Lyapunov function candidate

$$V_e = \sum_{i=1}^{n} \left| {}_0D_t^{\alpha-1} e_i \right| \tag{27}$$

The Lyapunov function derivative with respect to time yields

$$\dot{V}_e = \sum_{i=1}^{n} \mathrm{sgn}({}_0D_t^{\alpha-1} e_i) {}_0^C D_t^\alpha e_i \tag{28}$$

Substituting equations (17) and (21) into equation (28), we have

$$\begin{aligned} \dot{V}_e = & \sum_{i=1}^{n-1} \mathrm{sgn}({}_0D_t^{\alpha-1} e_i) e_{i+1} + \mathrm{sgn}({}_0D_t^{\alpha-1} e_n) \\ & \left({}_0^C D_t^\alpha r_n - f(t, \mathbf{X}) - \Delta f(t, \mathbf{X}) - d(t) - u_{FOPID} - u_s \right) \\ \leqslant & \sum_{i=1}^{n-1} |e_{i+1}| + \mathrm{sgn}({}_0D_t^{\alpha-1} e_n) \\ & \left({}_0^C D_t^\alpha r_n - f(t, \mathbf{X}) - \Delta f(t, \mathbf{X}) - d(t) - u_{FOPID} - u_s \right) \\ \leqslant & (n-1)\|\mathbf{E}\|_\infty + \mathrm{sgn}({}_0D_t^{\alpha-1} e_n) \\ & \left({}_0^C D_t^\alpha r_n - f(t, \mathbf{X}) - \Delta f(t, \mathbf{X}) - d(t) - u_{FOPID} - u_s \right) \\ \leqslant & (n-1)\|\mathbf{E}\|_2 + \mathrm{sgn}({}_0D_t^{\alpha-1} e_n) \\ & \left({}_0^C D_t^\alpha r_n - f(t, \mathbf{X}) - \Delta f(t, \mathbf{X}) - d(t) - u_{FOPID} - u_s \right) \end{aligned} \tag{29}$$

Now, we define the supervisory control by

$$u_s = \begin{cases} 0 & \mathbf{E} \in \Omega \\ \operatorname{sgn}({}_0D_t^{\alpha-1}e_n)\big((k+n-1)\|\mathbf{E}\|_2 + \left|{}_0^CD_t^{\alpha}r_n\right| + f^u + \Delta f^u + d^u + |u_{FOPID}|\big) & \mathbf{E} \notin \Omega \end{cases} \tag{30}$$

where $k > 0$ is an arbitrary positive real number.

By substituting equation (30) into equation (29), we can easily show when the tracking errors trajectories are out of the mentioned constraint set, we have

$$\begin{aligned} \dot{V}_e \leqslant\ & (n-1)\|\mathbf{E}\|_2 + \operatorname{sgn}({}_0D_t^{\alpha-1}e_n) \\ & ({}_0^CD_t^{\alpha}r_n - f(t,\mathbf{X}) - \Delta f(t,\mathbf{X}) - d(t) - u_{FOPID}) \\ & - (k+n-1)\|\mathbf{E}\|_2 - \left|{}_0^CD_t^{\alpha}r_n\right| \\ & - |u_{FOPID}| - f^u - \Delta f^u - d^u \\ =\ & -\big(\left|{}_0^CD_t^{\alpha}r_n\right| - \operatorname{sgn}({}_0D_t^{\alpha-1}e_n){}_0^CD_t^{\alpha}r_n\big) \\ & - \big(|u_{FOPID}| + \operatorname{sgn}({}_0D_t^{\alpha-1}e_n)u_{FOPID}\big) \\ & - \big(f^u + \operatorname{sgn}({}_0D_t^{\alpha-1}e_n)f(t,\mathbf{X})\big) \\ & - \big(\Delta f^u + \operatorname{sgn}({}_0D_t^{\alpha-1}e_n)\Delta f(t,\mathbf{X})\big) \\ & - \big(d^u + \operatorname{sgn}({}_0D_t^{\alpha-1}e_n)d(t)\big) - k\|\mathbf{E}\|_2 \\ \leqslant\ & -k\|\mathbf{E}\|_2 \end{aligned} \tag{31}$$

Therefore, the supervisory control guarantees the attracting property of the Ω set for the closed-loop system. It means that the supervisory control forces the error to be attracted by the set Ω and hence be bounded.

Adaptation law

FOPID controller gains in equation (25) will be tuned using a proper sliding surface. To design a proper sliding surface, we define a signal, x_r, as follows

$${}_0^CD_t^{\alpha}x_r = {}_0^CD_t^{\alpha}r_n + \sum_{i=1}^{n} K_i e_i \tag{32}$$

Now, the sliding surface function can be defined as follows

$$S = x_n - x_r \tag{33}$$

When the sliding condition occurs

$$S = 0 \Rightarrow x_r = x_n \tag{34}$$

Substituting equation (34) into equation (32), we obtain

$${}_0^CD_t^{\alpha}e_n + \sum_{i=1}^{n} K_i e_i = 0 \tag{35}$$

Equation (35) can be written in state space form as

$${}_0^CD_t^{\alpha}\mathbf{E} = \mathbf{\Lambda}\mathbf{E} \tag{36}$$

where $\mathbf{\Lambda}$ is the gain matrix and is defined as follows

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -K_1 & -K_2 & -K_3 & \cdots & -K_{n-1} & -K_n \end{bmatrix} \tag{37}$$

where K_i's are chosen such that the eigenvalues of $\mathbf{\Lambda}$ matrix satisfy the asymptotic stability condition in fractional order linear systems. This condition can be expressed by

$$|\arg(\lambda_i)| > \alpha\frac{\pi}{2} \tag{38}$$

where λ_i for $i = 1, \ldots, n$ denotes the eigenvalues of $\mathbf{\Lambda}$ matrix. Therefore, the error asymptotically tends to zero as $t \to \infty$.

Now, consider the following Lyapunov function for sliding mode control

$$V_s = \frac{1}{2}S^2 \tag{39}$$

Differentiating the Lyapunov function with respect to time yields

$$\dot{V}_s = S\dot{S} \tag{40}$$

The sliding condition can be expressed as

$$\dot{V}_s = S\dot{S} < 0 \tag{41}$$

From equations (33) and (41), we conclude

$$\dot{V}_s = S(\dot{x}_n - \dot{x}_r) \tag{42}$$

By fractionalizing the classic derivative into a fractional type, equation (42) can be rewritten in the following form

$$\dot{V}_s = S{}_0^CD_t^{1-\alpha}({}_0^CD_t^{\alpha}x_n) - S\dot{x}_r \tag{43}$$

From equations (13), (21), and (43), we have

$$\dot{V}_s = S{}_0^CD_t^{1-\alpha}(f + \Delta f + d + u_{FOPID} + u_s) - S\dot{x}_r \tag{44}$$

In order to derive a proper adaptation law, we try to decrease the time derivative of Lyapunov function defined in equation (40) as much as possible. The desired goal is to negative the mentioned function. However, it may not be possible, so decreasing it as much as possible is the main purpose. To this end, the gradient descent method will be used. Using the chain rule, the gradient method can be expressed as

$$\dot{\theta}_i = -\gamma \frac{\partial \dot{V}_s}{\partial \theta_i} = -\gamma \frac{\partial \dot{V}_s}{\partial J}\frac{\partial J}{\partial \theta_i} \tag{45}$$

where J is defined as

$$J = {}_0^C D_t^{1-\alpha} u_{FOPID} \tag{46}$$

Substituting equation (25) into equation (46), we obtain

$$J = {}_0^C D_t^{1-\alpha} \sum_{i=1}^{3} (\theta_i \varphi_i) = \sum_{i=1}^{3} {}_0^C D_t^{1-\alpha}(\theta_i \varphi_i) \tag{47}$$

Using the Leibniz rule for Caputo fractional derivative, J can be rewritten in the following form

$$J = \sum_{i=1}^{3}\sum_{k=0}^{\infty} \left\{ \binom{p}{k} \theta_i^{(k)}\, {}_0^{RL} D_t^{1-\alpha-k} \varphi_i - \frac{1}{t^{1-\alpha}\Gamma(\alpha)} (\theta_i \varphi_i)|_{t=0} \right\} \tag{48}$$

From equations (45) and (48), one can obtain

$$\dot{\theta}_i = -\gamma S_0^{RL} D_t^{1-\alpha} \varphi_i \tag{49}$$

Adaptation laws (49) can be written as follows

$$\begin{cases} \dot{K}_P = -\gamma S_0^{RL} D_t^{1-\alpha} e \\ \dot{K}_I = -\gamma S_0^{RL} D_t^{1-\alpha}\left({}_0 D_t^{-\lambda} e\right) \\ \dot{K}_D = -\gamma S_0^{RL} D_t^{1-\alpha}\left({}_0^C D_t^{\mu} e\right) \end{cases} \tag{50}$$

In relation (50), ${}_0^{RL} D_t^{1-\alpha}({}_0^C D_t^{\mu} e_1)$ can be expressed in terms of Caputo derivative

$$\begin{cases} \dot{K}_P = -\gamma S_0^{RL} D_t^{1-\alpha} e \\ \dot{K}_I = -\gamma S_0^{RL} D_t^{1-\alpha}\left({}_0 D_t^{-\lambda} e\right) \\ \dot{K}_D = -\gamma S\left({}_0^C D_t^{1-\alpha}\left({}_0^C D_t^{\mu} e\right) + \frac{{}_0^C D_t^{\mu} e|_{t=0}}{t^{1-\alpha}\Gamma(\alpha)} \right) \end{cases} \tag{51}$$

To have simple and more implementable adaptation laws for K_I and K_D, we choose $\lambda = 2-\alpha$ and $\mu = \alpha$. Since we have assumed that the tracking error is continuously differentiable, using theorem 3.1 in Li and Deng,[31] we conclude that ${}_0^C D_t^{\mu} e\big|_{t=0} = 0$. Finally, the adaptation laws can be expressed in the following form

$$\begin{cases} \dot{K}_P = -\gamma S_0^{RL} D_t^{1-\alpha} e \\ \dot{K}_I = -\gamma S \int_0^t e(\tau) d\tau \\ \dot{K}_D = -\gamma S \dot{e} \end{cases} \tag{52}$$

Numerical simulations

In this section, output tracking of a fractional order Duffing–Holmes system is shown using the proposed adaptive FOPID controller. Numerical simulations are presented to illustrate the performance of the method.

The mathematical modeling of a fractional order Duffing–Holmes system is defined by

$$\begin{cases} {}_0^C D_t^{\alpha} x_1 = x_2 \\ {}_0^C D_t^{\alpha} x_2 = x_1(t) - 0.25 x_2(t) - x_1^3(t) + 0.3\cos(t) \\ \quad + \Delta f(t, \mathbf{X}) + d(t) + u(t) \\ y(t) = x_1(t) \end{cases} \tag{53}$$

In this case, $\Delta f(t, \mathbf{X})$ and $d(t)$ are chosen as $0.1\sin(t)\sqrt{x_1^2 + x_2^2}$ and $0.1\sin(t)$, respectively. The order of the equations is considered as $\alpha = 0.8$. Therefore, the system dynamics can be described by the following equations

$$\begin{cases} {}_0^C D_t^{0.8} x_1 = x_2 \\ {}_0^C D_t^{0.8} x_2 = x_1(t) - 0.25 x_2(t) - x_1^3(t) + 0.3\cos(t) \\ \quad + 0.1\sin(t)\sqrt{x_1^2 + x_2^2} + 0.1\sin(t) + u(t) \\ y(t) = x_1(t) \end{cases} \tag{54}$$

From equation (54), one can easily obtain the functions f^u, Δf^u, and d^u

$$f^u = |x_1(t)| + 0.25|x_2(t)| + \left|x_1^3(t)\right| + 0.3 \tag{55}$$

$$\Delta f^u = 0.1\sqrt{x_1^2 + x_2^2} \tag{56}$$

$$d^u = 0.1 \tag{57}$$

We choose $K = 4$, $K_1 = 36$, and $K_2 = 12$, so that the condition given by equation (37) is satisfied. Integral and derivative order of the FOPID controller and the tuning rate are chosen as $\lambda = 1.2$, $\mu = 0.8$, and $\gamma = 50$, respectively.

The Predict–Evaluate–Correct–Evaluate (PECE) algorithm[32] is used to solve the fractional differential equations with a time step of size 0.01. The initial conditions of the system and the initial values of the FOPID controller gains are set to $X(0) = [1.25 \quad 0.5]^T$ and $\mathbf{\Theta}(0) = [5 \quad 5 \quad 5]^T$, respectively. In the simulations, the control goal is to make output of the system follow the desired signal $y_d = 0.5\sin(t)$ asymptotically.

Simulation results are shown in Figures 3–6. Figure 3(a) and (b) shows output response of the closed-loop control system. Figure 4 demonstrates the error of the tracking of the states. Figures 5 and 6 show the time histories of the FOPID controller gains and sliding surface S. From Figure 5, we can easily see that after less than 10 s the dynamic trajectories of K_P, K_I, and K_D are on the steady state.

Conclusion

This article has shown a robust adaptive FOPID controller for uncertain fractional order nonlinear systems. Robustness of the closed-loop control system against system uncertainty and external disturbance is considered using fractional order sliding mode control method. The adaptation mechanism is constructed from a proper sliding surface via the gradient descent method. A supervisory control is used to guarantee the

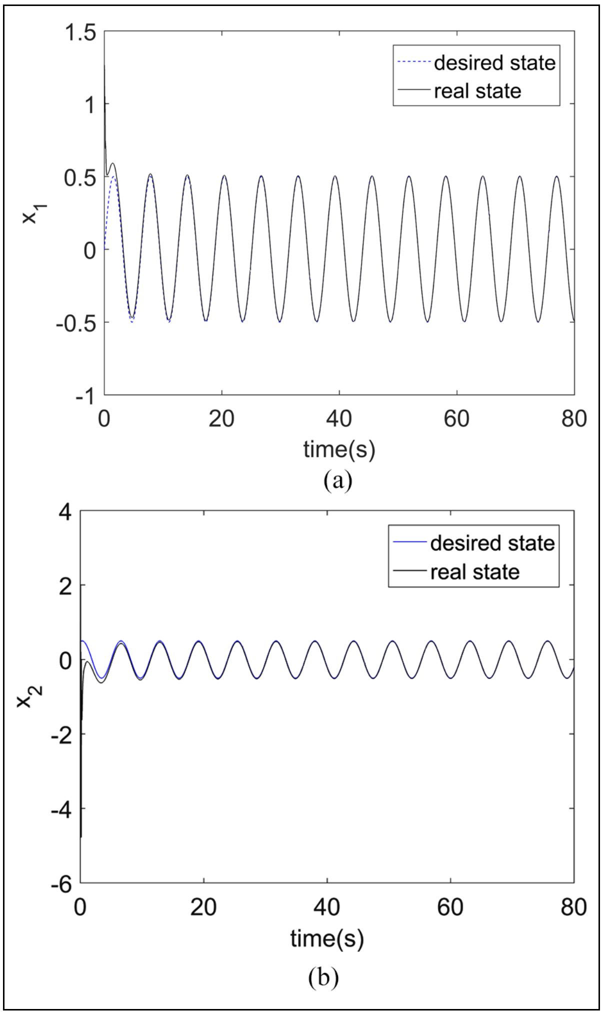

Figure 3. Output response of the closed-loop control system: (a) time history of x_1 and (b) time history of x_2.

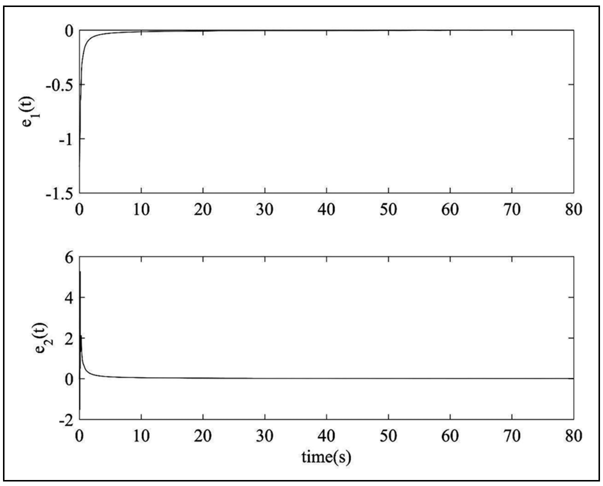

Figure 4. Time history of the tracking error.

boundedness of the system state trajectories. Finally, the proposed method is implemented to control a

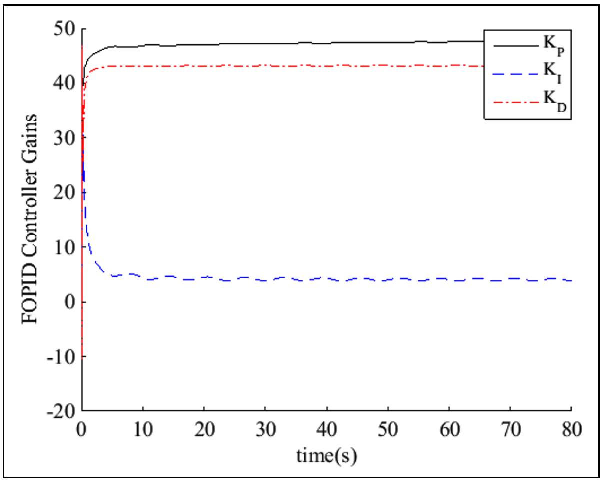

Figure 5. Time histories of the FOPID controller gains.

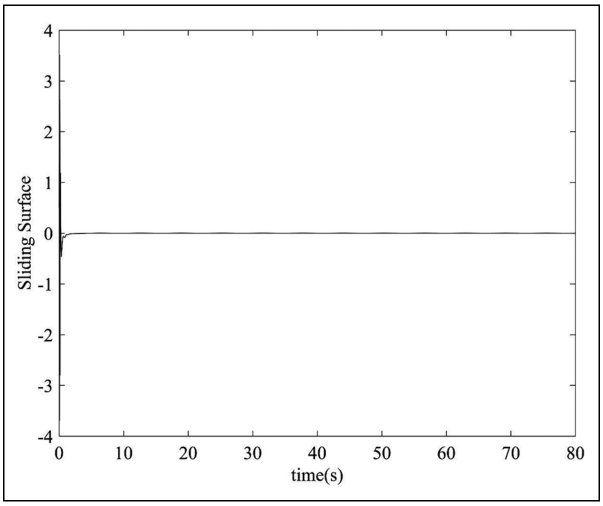

Figure 6. Time history of the sliding surface.

fractional order Duffing–Holmes system and the simulation results are included to illustrate the great performance of the proposed method.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

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Tuning Fractional Order Proportional Integral Controllers Using Dominant Pole Placement Method

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ABSTRACT

This paper presents a method of tuning for fractional order proportional integral controllers for a class of fractional order systems. Fractional order proportional integral controller inherits the advantages of the traditional integer order PI controller and has one degree of freedom more than the integer order PI controller and it has better control performance. Based on this characteristic of the FOPI controller, fractional order proportional integral (FOPI) and fractional order [proportional integral] (FO[PI]) controllers proposed and designed based on dominant pole placement method for the considered class of fractional order systems. The simulation results show that the proposed method works well for the design of fractional order PI controller.

Keywords: Integer order PI controller, Fractional order PI controller, FOPI controller, FO[PI] controller

1. INTRODUCTION

As we know, the use of Proportional-Integral-Derivative (PID) control has a long history in control engineering. Hence in many real industrial applications, the PID controller is still widely used even though lots of new control techniques have been proposed [1]. Design and tuning of PID controllers have been a large research area ever since Ziegler and Nichols presented their method in 1942 [2]. Design, applications and performance of the PID controllers have been widely treated since then [3, 4].

Fractional calculus is dealing with integration and derivation of non-integer order [5-7]. In recent years, application of fractional calculus is becoming a hot topic in control area [8-11]. The idea of fractional order PID ($PI^{\lambda}D^{\mu}$) is proposed by Podlubny I. [5]. The main advantage of using fractional order PID controllers for a linear control system is that we have more degrees of freedom in the controller design using the additional parameters of the integral and differential orders and, as a consequence, it is expected that the use of FOPID controllers can enhance the feedback control loop performance over the integer order controllers [12]. So many tuning algorithms for fractional order PID controllers were developed [12-22] and fractional order PID controllers came to real-world applications [18].

The paper is organized as follows: Section 2 is a review of the preliminary concepts of fractional calculus. In Section 3 calculation of the dominant poles is studied. In Section 4 the considered fractional order system, the FOPI and FO[PI] controllers and the tuning constraints for the robustness requirements on the loop gain variations are introduced. In Section 5 the design procedure for the controllers are presented. An application of the proposed method is given in Section 6. Finally some concluding remarks are cited in Section 7.

2. Preliminaries

There are several definitions of fractional derivatives [8]. The best definitions are Riemann-Liouville and Caputo.

Definition 1. (Riemann-Liouville fractional derivative) The Riemann-Liouville fractional derivative is defined as:

$$ {}_aD_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \int_a^t (t-\tau)^{-\alpha-1} f(\tau) d\tau & \alpha < 0 \\ f(t) & \alpha = 0 \\ \dfrac{d^n}{dt^n}\left[{}_aD_t^{\alpha-n} f(t) \right] & \alpha > 0 \end{cases} \tag{1} $$

where $\Gamma(.)$ is the standard gamma function and n is an integer number such that $n-1 \leq \alpha < n$.

Definition 2. (Caputo fractional derivative) The Caputo fractional derivative is defined as:

$$ {}_a^CD_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\mathrm{n}-\alpha)} \int_a^t \dfrac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau & n-1 < \alpha < n \\ \dfrac{d^n}{dt^n} f(t) & \alpha = n \end{cases} \tag{2} $$

Definition 3. Laplace transform of the Riemann-Liouville derivative is defined as:

$$ \mathcal{L}\left\{ {}_aD_t^{\alpha} \right\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^k \, {}_aD_t^{\alpha-k-1} f(a) \tag{3} $$

Definition 4. Laplace transform of the Caputo derivative is defined as:

$$ \mathcal{L}\left\{ {}_a^CD_t^{\alpha} \right\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(a) \tag{4} $$

The main advantage of Caputo derivative is that the initial conditions for fractional differential equations with Caputo derivative take on the same form as for integer order differential equations. Another difference between two definitions is that the Caputo derivative of a constant is 0 whereas in the cases of a finite value of the lower terminal a the Riemann-Liouville derivative of a constant C is not equal to 0, but

$$ {}_aD_t^{\alpha} C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)} \tag{5} $$

A linear fractional order system in state space form is like:

$$ \begin{Bmatrix} {}_a^CD_t^{\alpha_1} x_1 \\ {}_a^CD_t^{\alpha_2} x_2 \\ \vdots \\ {}_a^CD_t^{\alpha_n} x_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = A_{n\times n} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \tag{6} $$

3. Dominant poles

In higher order systems dominant poles are the poles which are the closest to the imaginary axis. Consider the original open-loop system is of high order, the closed loop system behaves like a second order system,

$$ G_m(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{7} $$

Consider the unit step response of a second order system. It is possible to calculate the parameters of the system as a function of peak overshoot and settling time.

$$ \zeta = -\frac{\ln(P.O./100)}{\sqrt{\pi^2 + (\ln(P.O./100)^2}} \tag{8} $$

$$\omega_n = \frac{4}{\zeta T_s} \tag{9}$$

where $P.O.$ is peak overshoot and T_s is settling time.

Dominant poles of a second order system in terms of system parameters are given as

$$s_{1,2} = re^{\pm j\theta} \tag{10}$$

Where θ is a function of damping ratio,

$$\theta = \pi - \cos^{-1}\zeta \tag{11}$$

$$r = \omega_n \tag{12}$$

4. Fractional order proportional integral controllers and the tuning constraints

The fractional order plant to be controlled has the following form of the transfer function

$$G(s) = \frac{k}{s^{2\alpha} + a_1 s^{\alpha} + a_0} \tag{13}$$

Where α is a positive real number and $\alpha \in (0,1)$.

Our work in this paper is to design a fractional order proportional integral controller for fractional order systems. We consider two types of fractional order PI controllers: fractional order proportional integral (FOPI) and fractional order [proportional integral] (FO[PI]) whose transfer functions are given as follows, respectively

$$C_1(s) = K_P + \frac{K_I}{s^{\lambda}} \tag{14}$$

$$C_2(s) = \left(K_P + \frac{K_I}{s}\right)^{\lambda} \tag{15}$$

Where K_P and K_I are proportional and integral gains and λ is a positive real number.

In this paper, the following specifications to be met by fractional controlled system are proposed:

1. **Dominant poles**

   Let the plant at the dominant poles be represented by a complex number,

$$G(s_1) = G_1 + jG_2 \tag{16}$$

   where G_1 and G_2 are given by

$$G_1 = \frac{k\,(r^{2\alpha}\cos(2\alpha\theta) + a_1 r^{\alpha}\cos(\alpha\theta) + a_0)}{(r^{2\alpha}\cos(2\alpha\theta) + a_1 r^{\alpha}\cos(\alpha\theta) + a_0)^2 + (r^{2\alpha}\sin(2\alpha\theta) + a_1 r^{\alpha}\sin(\alpha\theta))^2} \tag{17}$$

$$G_2 = \frac{-k\,(r^{2\alpha}\sin(2\alpha\theta) + a_1 r^{\alpha}\sin(\alpha\theta))}{(r^{2\alpha}\cos(2\alpha\theta) + a_1 r^{\alpha}\cos(\alpha\theta) + a_0)^2 + (r^{2\alpha}\sin(2\alpha\theta) + a_1 r^{\alpha}\sin(\alpha\theta))^2} \tag{18}$$

   The characteristic equation of the system is

$$1 + G(s)C(s) = 0 \tag{19}$$

   Dominant poles must satisfy the characteristic equation.

2. **Robustness to variations in the gain of the plant**.

   With this condition the phase is forced to be flat at gain crossover frequency and so, to be almost constant within an interval around the gain crossover frequency. It means that the system is more robust to gain changes and the overshoot of the response is almost constant within the interval.

$$\left(\frac{d\,(\arg(G(j\omega)\mathrm{C}(j\omega)))}{d\omega}\right)_{\omega=\omega_c} = 0 \tag{20}$$

   where ω_c is gain crossover frequency.

3. **Gain crossover frequency specification**

It means that the system is robust to gain change variations,

$$\left|G(j\omega_c)C(j\omega_c)\right|_{dB} = 0dB \tag{21}$$

4. Tuning controllers for the fractional order systems considered

5.1 FOPI controller design

The systematic way to design the FOPI controller for the fractional order system (13) according to the three specifications introduced in section 4, is presented below,

Let the controller at dominant pole is given by

$$C_1(s_1) = K_P + \frac{K_I}{r^\lambda}\cos(\lambda\theta) + j\frac{K_I}{r^\lambda}\sin(\lambda\theta) \tag{22}$$

Substituting (16) and (22) in characteristic equation and comparing real-real and imaginary-imaginary parts in equation, the following equations are obtained,

$$r^\lambda + G_1 K_P r^\lambda + G_1 K_I \cos(\lambda\theta) - G_2 K_I \sin(\lambda\theta) = 0 \tag{23}$$

$$G_1 K_I \sin(\lambda\theta) + G_2 K_I \cos(\lambda\theta) + G_2 K_P r^\lambda = 0 \tag{24}$$

According to the constraint (2) about the robustness to loop gain variation, one can obtain,

$$\left(\frac{d(\arg(G(j\omega)\mathrm{C}(j\omega)))}{d\omega}\right)_{\omega=\omega_c} = \frac{K_P K_I \omega_c^\lambda \sin(\lambda\pi/2)}{(K_P\omega_c^\lambda)^2 + K_I^2 + K_P K_I \omega_c^\lambda \sin(\lambda\pi)} + \frac{AB' - A'B}{A^2 + B^2} = 0 \tag{25}$$

where

$$A = \omega_c^{2\alpha}\sin\alpha\pi + a_1\omega_c^\alpha \sin(\alpha\pi/2) \tag{26}$$

$$B = \omega_c^{2\alpha}\cos\alpha\pi + a_1\omega_c^\alpha \cos(\alpha\pi/2) + a_0 \tag{27}$$

$$A' = \left.\frac{\partial A}{\partial\omega}\right|_{\omega=\omega_c} = 2\alpha\omega_c^{2\alpha-1}\sin\alpha\pi + a_1\alpha\omega_c^{\alpha-1}\sin(\alpha\pi/2) \tag{28}$$

$$B' = \left.\frac{\partial B}{\partial\omega}\right|_{\omega=\omega_c} = 2\alpha\omega_c^{2\alpha-1}\cos\alpha\pi + a_1\alpha\omega_c^{\alpha-1}\cos(\alpha\pi/2) \tag{29}$$

According to the gain crossover frequency constraint, one can obtain,

$$\left((K_P\omega_c^\lambda)^2 + K_I^2 + 2K_P K_I \omega_c^\lambda \cos(\lambda\pi/2)\right)\left(A^2 + B^2\right) = 1 \tag{30}$$

We can solve equations (23), (24), (25) and (30) to get K_P , K_I , λ and ω_c .

5.2 FO[PI] controller design

Let the controller at dominant poles is given by

$$C_2(s_1) = \left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)^{\frac{\lambda}{2}}\cos(\varphi) + j\left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)^{\frac{\lambda}{2}}\sin(\varphi) \tag{31}$$

Where

$$\varphi = \tan^{-1}\frac{K_I\sin(\theta)/r}{K_P + K_I\cos(\theta)/r} \tag{32}$$

Substituting () and () in characteristic equation and comparing real-real and imaginary-imaginary parts in equation, the following equations are obtained,

$$1 + G_1\left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)\cos(\varphi) - G_2\left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)\sin(\varphi) = 0 \tag{33}$$

$$G_1\left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)\sin(\varphi) + G_2\left(K_P^2 + \frac{K_I^2}{r^2} + 2K_P\frac{K_I}{r}\cos(\theta)\right)\cos(\varphi) = 0 \tag{34}$$

According to the constraint (2) about the robustness to loop gain variation, one can obtain,

$$\left(\frac{d\,(\arg(G\,(j\,\omega)\,C(j\,\omega)))}{d\,\omega}\right)_{\omega=\omega_c}=\frac{K_P K_I\,\sin(\theta)}{(K_P\omega_c)^2+K_I^2+K_P K_I\,\sin(2\theta)}+\frac{AB'-A'B}{A^2+B^2}=0 \tag{35}$$

According to the gain crossover frequency constraint, one can obtain,

$$\left(K_P^2+\frac{K_I^2}{\omega_c^2}+2K_P\frac{K_I}{\omega_c}\cos(\theta)\right)^{\lambda}\left(A^2+B^2\right)=1 \tag{36}$$

Clearly, again we can solve equations (33), (34), (35) and (36) to get K_P , K_I , λ and ω_c .

5. Summaries of the controllers design procedure and illustrative examples

In this section, the design procedures of the controllers are summarized. Illustrative examples are presented to verify the proposed controller designs.

6.1 Fractional order PI controller design procedure

Fractional order PI controller design procedure are summarized as below,

- Choose the peak overshoot and settling time of the model
- Find the dominant poles
- Obtain equations (23), (24), (25) and (30)
- Then we can obtain the controller parameters using fmincon function in MATLAB. The gain crossover frequency specification should be taken as the main function to minimize.

6.2 Illustrative example I

The plant to control is:

$$G(s)=\frac{2}{s^{1.2}+5s^{0.6}+2} \tag{37}$$

The design specifications required for the controlled system are the following ones:

- Peak overshoot, $P.O.\le 15\%$
- Settling time, $T_s\le 3s$

Using fmincon function the controller parameters and gain crossover frequency are,

$K_P=20.7411$

$K_I=103.3162$

$\lambda=1.4139$

$\omega_c=12.1153 rad/s$

Step response of the closed loop system and the control input are shown in Fig. 1 and Fig. 2 respectively.

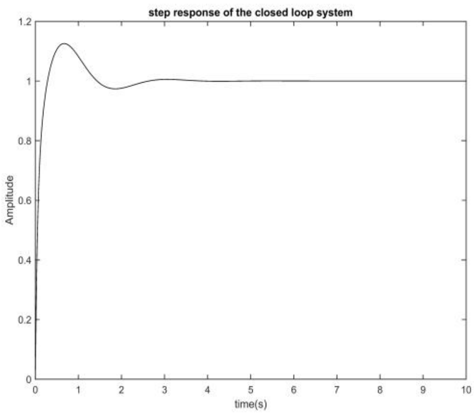

Fig. 1. Step response of the closed loop system using FOPI

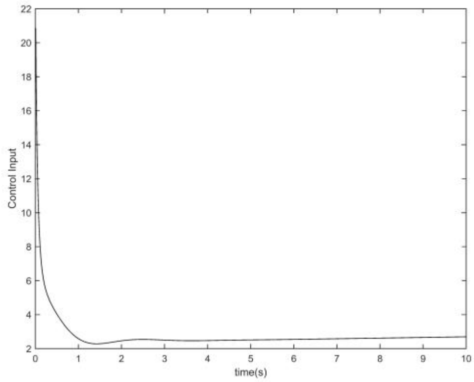

Fig. 2. Control input

6.3 Fractional order [PI] controller design procedure

Fractional order [PI] controller design procedure are summarized as below,

- Choose the peak overshoot and settling time of the model
- Find the dominant poles
- Obtain equations (33), (34), (35) and (36)
- Then we can obtain the controller parameters using fmincon function in MATLAB.

6.4 Illustrative example II

The plant to control is:

$$G(s) = \frac{2}{s^{1.2} + 5s^{0.6} + 2} \tag{38}$$

The design specifications required for the controlled system are the following ones:

- Peak overshoot, $P.O. \leq 15\%$
- Settling time, $T_S \leq 3s$

the controller parameters and gain crossover frequency are,

$K_P = 37.9223$

$K_I = 96.2488$

$\lambda = 1.2887$

$\omega_c = 25.0 rad / s$

Step response of the closed loop system and the control input are shown in Fig. 3 and Fig. 4 respectively.

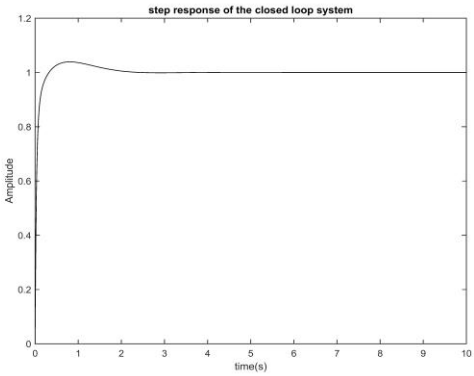

Fig. 3. Step response of the closed loop system using FO[PI]

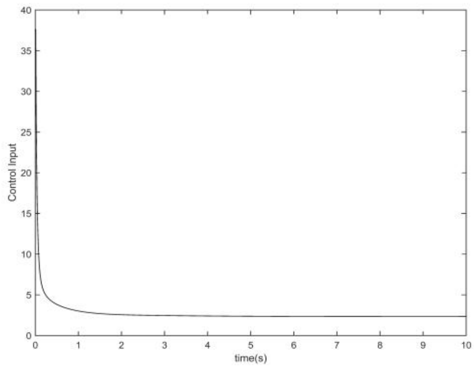

Fig. 4. Control input

6. Conclusion

In this paper a method for tuning of fractional order proportional integral and fractional order [proportional integral] controllers for a class of fractional order systems has been proposed. The presented method is based on dominant pole placement method. For fair comparison, the proposed FOPI and FO[PI] controllers are all designed following the same set of the imposed tuning constraints and design specifications. The simulation results show that the FO[PI] controller has better response than the FOPI controller.

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Adaptive Synchronization of Uncertain Fractional Order Chaotic Systems Using Sliding Mode Control Techniques

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Abstract

In this paper, an adaptive nonlinear controller is designed to synchronize two uncertain fractional order chaotic systems using fractional order (FO) sliding mode control. The controller structure and adaptation laws are chosen such that asymptotic stability of the closed loop control system is guaranteed. The adaptation laws are being calculated from a proper sliding surface by using Lyapunov approach. This method guarantees the closed loop control system robustness against the system uncertainties and external disturbances. Eventually, the presented method is used to synchronize two fractional order gyro and Duffing systems, and the numerical simulation results demonstrate the effectiveness of this method.

Keywords: Fractional order chaotic systems; Synchronization; Fractional order sliding mode control; Adaptive control.

1. Introduction

Chaos is a phenomenon that appears in nonlinear dynamical systems and has been observed in many fields of science such as economics [1], chemistry [2], biology [3], engineering [4] and so on. In recent years, synchronization of chaotic systems has attracted interest of many scientists in variety of fields [5]. In 1990, Pecora and Carroll proposed the idea of chaos synchronization [6]. In the last years, several methods have been introduced for synchronization of chaotic systems [7]. In many applications, parameters of the slave system are unknown or system has some uncertainties; therefore, using adaptive control and developing some adaptive synchronization methods have been presented [8-10].

Fractional calculus is an old field of mathematics from the 17th century that studies derivatives and integrals of non-integer order [11]. For many years, fractional calculus was a pure mathematics topic; and has no applications in real world. But recently, fractional calculus is introduced as a powerful tool for modeling many systems in various fields of physics and engineering e.g. viscoelasticity [12], dynamical systems [13], biomedical applications [14], signal processing [15], electrical networks [16], cyber-physical systems [17], diffusion wave [18-20], electromagnetism [21], stochastic systems [22], control theory [23], chaotic systems [24-26] and so on. Also, many fractional order controllers are developed like, fractional PID controller [27, 28], fractional PI controller [29], fractional PD controller [30], fractional lead-lag controller [31, 32], fractional CRONE controller [33], adaptive fractional order PID controller [34], fractional model reference adaptive control [35, 36], and fractional sliding mode control [37-39].

In recent years, many fractional order (FO) dynamic systems with chaotic behavior are introduced such as FO Duffing system [40], FO Chen system [41], FO Van der Pol dynamics [42], FO Rössler equations [43], FO Chua system [44] and so on. In the last few years, several techniques have been proposed for synchronization of fractional order chaotic systems [45-49].

In this paper, an adaptive nonlinear controller is designed to synchronize two FO chaotic systems. In the proposed method stability of the closed loop control system is guaranteed by using Lyapunov approach. In addition, robustness of the closed loop system is guaranteed by using sliding mode control methods.

The rest of this paper is organized as follows: In Section 2, the most applicable definitions of fractional order integral and derivative and stability of the fractional order systems are presented. In Section 3 the problem statement is presented. In Section 4, stability of the closed loop control system is guaranteed by introducing a proper sliding mode controller. In Section

5, numerical simulation results are shown. Finally, a concise conclusion is presented in Section 6.

2. Preliminary Concepts

Since the time Leibniz introduced non-integer order derivatives, several definitions have been generated by several mathematicians [11]. In this section, some basic definitions and stability theorems in fractional calculus are given.

Definition 1 [11]**.** The Riemann–Liouville fractional integral of order p is defined as follows:

$$ {}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau \tag{1} $$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2 [11]**.** The Riemann–Liouville fractional derivative of order p is defined as follows:

$$ {}_0D_t^{p} f(t) = \frac{d^k}{dt^k} \left({}_0^{RL}D_t^{-(k-p)} f(t) \right) \tag{2} $$

where $k-1 \leq p < k$ and k is an integer number.

Definition 3 [11]**.** The Caputo fractional integral of order p is defined as follows:

$$ {}_0^{C}D_t^{p} f(t) = \begin{cases} \dfrac{1}{\Gamma(k-p)} \displaystyle\int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{p+1-k}} d\tau & k-1 < p < k \\ \dfrac{d^k}{dt^k} f(t) & p = k \end{cases} \tag{3} $$

Definition 4. A fractional order linear system can be represented by the following state space form:

$$ \begin{Bmatrix} {}_0^{C}D_t^{\alpha_1} x_1 \\ {}_0^{C}D_t^{\alpha_2} x_2 \\ \vdots \\ {}_0^{C}D_t^{\alpha_n} x_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \tag{4} $$

where $\alpha_1, \ldots, \alpha_n$ are fractional order of derivatives. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, the system is called commensurate.

Theorem 1. Consider the following commensurate fractional order linear system:

$$\frac{d^{\alpha}}{dt^{\alpha}}\mathbf{x} = \mathbf{A}\mathbf{x} \tag{5}$$

where $\mathbf{x} \in \Re^{n}$, $\mathbf{A} \in \Re^{n\times n}$ and $\alpha \in (0,1)$. The stability of the system can be studied by using the following condition:

$$\left|\arg(eig(A))\right| > \alpha\frac{\pi}{2} \quad \forall i \tag{6}$$

where $eig(A)$ show the eigenvalues of matrix $\mathbf{A}$.

Definition 5. Using the Caputo derivative, a fractional order nonlinear system can be represtented by:

$${}_{0}^{C}D_{t}^{\alpha}x = f(t,x) \tag{7}$$

where α is the order of fractional derivatives of the system.

Theorem 2 (Fractional-order extension of Lyapunov direct method [50]). Let $x = 0$ be an equilibrium point for the autonomous fractional order system (7). Let $V(t,x(t))$ be a continuously differentiable function as a Lyapunov function candidate, and $\gamma_i (i = 1,2,3)$ be class-K functions such that

$$\gamma_1\left(\|x\|\right) \le V(t,x(t)) \le \gamma_2\left(\|x\|\right) \tag{8}$$

$${}_{0}^{C}D_{t}^{\beta}V(t,x(t)) \le -\gamma_3\left(\|x\|\right) \tag{9}$$

where $\beta \in (0,1)$. Then the system (7) is asymptotically stable.

Lemma 1. [50] Let $x(t) \in \Re$ be a continuously differentiable function. Then, for all $t \ge t_0$

$$\frac{1}{2}\,{}_{t_0}^{C}D_{t}^{\alpha}x^2(t) \le x(t)\,{}_{t_0}^{C}D_{t}^{\alpha}x(t), \qquad \forall\alpha \in (0,1) \tag{10}$$

Remark 1. [50] In the case when $x(t) \in \Re^{n}$, Lemma 1 is still valid. That is, $\forall t \ge t_0$

$$\frac{1}{2}\,{}_{t_0}^{C}D_{t}^{\alpha}x^{T}(t)x(t) \le x^{T}(t)\,{}_{t_0}^{C}D_{t}^{\alpha}x(t), \qquad \forall\alpha \in (0,1) \tag{11}$$

In [51], Barbalat's Lemma is developed for fractional order nonlinear systems.

Theorem 3. [51] Let $\phi:\Re\to\Re$ be a uniformly continuous function on $[t_0,\infty)$. Assume that there exist two positive constants p and M such that ${}_0D_t^{-\alpha}|\phi|^p \le M$ for all $t>t_0>0$ with $\alpha\in(0,1)$. Then

$$\lim_{t\to\infty}\phi(t)=0 \tag{12}$$

3. Problem Statement

3.1 System Description

In synchronization of two chaotic systems, there are two systems: a particular dynamic system as a master and another different dynamic as a slave. From the viewpoint of control, the task is to design a controller such that the slave system imitates behavior of the master system.

Consider following commensurate fractional order chaotic as a slave system:

$$\begin{cases} {}_0^C D_t^\alpha y_i = y_{i+1} \quad , \quad 1\le i\le n-1 \\ {}_0^C D_t^\alpha y_n = f(t,\mathbf{Y})+\mathbf{F}^T(t,\mathbf{Y})\mathbf{\Theta}+d(t)+g(t,\mathbf{Y})u \end{cases} \tag{13}$$

where α represents the order of fractional derivatives of the system and belongs to interval $(0,1)$; $\mathbf{Y}=\begin{bmatrix} y_1 & \cdots & y_n\end{bmatrix}^T$ is the system state vector; $f(\cdot)$ and $g(\cdot)$ are nonlinear functions, and belong to $\mathrm{C}^1\left(\Re^n\times\Re\ \to\Re\right)$; $\mathbf{F}(\cdot)$ is a known vector with $1\times m$ dimension; $\boldsymbol{\theta}\in\Re^m$ stands for the uncertain parameter vector of the system; $d(\cdot)$ denotes the external disturbance; and $u(t)$ is the input of the system. It is assumed that $g(t,\mathbf{Y})\ne 0$ and $|d(t)|\le k$ for all $t>0$, and k is a positive and unknown real number.

Let us define the following commensurate fractional order chaotic system as a master system:

$$\begin{cases} {}_0^C D_t^\alpha x_i = x_{i+1} \quad , \quad 1\le i\le n-1 \\ {}_0^C D_t^\alpha x_n = h(t,\mathbf{X}) \end{cases} \tag{14}$$

where $\mathbf{X} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ is the master system state vector.

The main objective is to design a robust control law to synchronize behavior of the master and slave systems. To achieve this goal, we will design an adaptive sliding mode control such that the closed loop control system satisfy the stability condition.

In this paper, several assumptions are used.

Assumption 1: Both of $\mathbf{\Theta}$ and $d(\cdot)$ are bounded but their exact values are unknown.

4. Controller Design

The synchronization error is defined as difference between the states of the master and the slave systems:

$$e_i = y_i - x_i \quad , \quad 1 \le i \le n \tag{15}$$

From (13), (14) and (15) the error dynamics can be written in the following form:

$$\begin{cases} {}_0^C D_t^\alpha e_i = e_{i+1} \quad , \quad 1 \le i \le n-1 \\ {}_0^C D_t^\alpha e_n = f(t, \mathbf{Y}) + \mathbf{F}^T(t, \mathbf{Y}) \mathbf{\Theta} + d(t) + g(t, \mathbf{Y}) u - h(t, \mathbf{X}) \end{cases} \tag{16}$$

The main goal in this section is to develop the control input in system (16) such that the closed loop control system be asymptotically stable and robust against the system uncertainty and external disturbance. To this purpose, adaptive sliding mode control techniques will be used. We consider the following function as a sliding surface function:

$$S = e_n + \sum_{i=1}^{n} \alpha_i \, {}_0 I_t^\alpha e_i \tag{17}$$

where $\alpha_i > 0's$ are set to obtain an exponentially stable dynamics for sliding mode, S = 0.

Assumption 2: A simplifying condition which is very common in controlling fractional order systems is assumed; that all of the system state variables as well as the sliding surface are continuously differentiable and can be measured. This assumption is common especially when a fractional order system is aimed to be controlled [52, 53].

The control action u must certify the reaching condition. It means that all error trajectories must intersect the sliding surface in a finite time. To reach this goal, the following theorem is proposed.

Theorem 4. The error dynamics system (16) is asymptotically stable at zero point under the following controller and adaptation laws:

$$u = -\frac{1}{g(t,\mathbf{Y})}\left(f(t,\mathbf{Y}) + \mathbf{F}^T(t,\mathbf{Y})\hat{\mathbf{\Theta}} + \hat{k}\,\mathrm{sgn}(S) + \eta S + \sum_{i=1}^{n}\alpha_i e_i - h(t,\mathbf{X}) \right) \tag{18}$$

$${}_0^C D_t^{\alpha}\hat{\theta}_i = \gamma_i S F_i(t,\mathbf{Y}) \tag{19}$$

$${}_0^C D_t^{\alpha}\hat{k} = \gamma_k \left| S \right| \tag{20}$$

where $\gamma_i \in \Re\ (1 \le i \le m)$ and $\gamma_k \in \Re$ are adaptation coefficients.

Proof. Let us consider the following Lyapunov function candidate:

$$V(t) = \frac{1}{2}S^2 + \frac{1}{2}\left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}\right)^T\left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}\right) + \frac{1}{2}\left(k - \hat{k}\right)^2 \tag{21}$$

Using Lemma 1, one can obtain:

$$\begin{aligned}
{}_0^C D_t^{\alpha}V(t) &= \frac{1}{2}\,{}_0^C D_t^{\alpha}S^2 + \frac{1}{2}\,{}_0^C D_t^{\alpha}\left(\left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}\right)^T\left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}\right)\right) + \frac{1}{2}\,{}_0^C D_t^{\alpha}\left(k - \hat{k}\right)^2 \\
&\le S\,{}_0^C D_t^{\alpha}S + \left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}\right)^T {}_0^C D_t^{\alpha}\left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}\right) + \left(k - \hat{k}\right){}_0^C D_t^{\alpha}\left(k - \hat{k}\right) \\
&= S\left({}_0^C D_t^{\alpha}e_n + \sum_{i=1}^{n}\alpha_i e_i \right) - \left(\mathbf{\Theta} - \hat{\mathbf{\Theta}}\right)^T {}_0^C D_t^{\alpha}\hat{\mathbf{\Theta}} - \left(k - \hat{k}\right){}_0^C D_t^{\alpha}\hat{k} \\
&= S\left({}_0^C D_t^{\alpha}e_n + \sum_{i=1}^{n}\alpha_i e_i \right) - \sum_{i=1}^{m}\left(\theta_i - \hat{\theta}_i\right){}_0^C D_t^{\alpha}\hat{\theta}_i - \left(k - \hat{k}\right){}_0^C D_t^{\alpha}\hat{k}
\end{aligned} \tag{22}$$

Substituting Eq. (16) into Eq. (22) results in:

$$\begin{aligned}
{}_0^C D_t^{\alpha}V(t) &\le S\left(f(t,\mathbf{Y}) + \sum_{i=1}^{m}F_i(t,\mathbf{Y})\,\theta_i + d(t) + g(t,\mathbf{Y})u - h(t,\mathbf{X}) + \sum_{i=1}^{n}\alpha_i e_i \right) \\
&- \sum_{i=1}^{m}\left(\theta_i - \hat{\theta}_i\right){}_0^C D_t^{\alpha}\hat{\theta}_i - \left(k - \hat{k}\right){}_0^C D_t^{\alpha}\hat{k}
\end{aligned} \tag{23}$$

Along with Eq. (18) and Inequality (23), we obtain:

$$\begin{aligned}
{}_0^C D_t^\alpha V(t) &\le S\left(f(t,\mathbf{Y}) + \sum_{i=1}^{m} F_i(t,\mathbf{Y})\,\theta_i + d(t) - h(t,\mathbf{X}) + \sum_{i=1}^{n} \alpha_i e_i \right) \\
&- S\left(f(t,\mathbf{Y}) + \sum_{i=1}^{m} F_i(t,\mathbf{Y})\,\hat{\theta}_i + \hat{k}\,\mathrm{sgn}(S) + \eta S + \sum_{i=1}^{n} \alpha_i e_i - h(t,\mathbf{X}) \right) \\
&- \sum_{i=1}^{m} \left(\theta_i - \hat{\theta}_i\right)^T {}_0^C D_t^\alpha \hat{\theta}_i - \left(k - \hat{k}\right) {}_0^C D_t^\alpha \hat{k} \\
&\le S \sum_{i=1}^{m} F_i(t,\mathbf{Y})\,\theta_i + |S| k - S \sum_{i=1}^{m} F_i(t,\mathbf{Y})\,\hat{\theta}_i + |S| \hat{k} \\
&- \sum_{i=1}^{m} \left(\theta_i - \hat{\theta}_i\right) {}_0^C D_t^\alpha \hat{\theta}_i - \left(k - \hat{k}\right) {}_0^C D_t^\alpha \hat{k} - \eta S^2 \\
&= \sum_{i=1}^{m} \left(\theta_i - \hat{\theta}_i\right)\left(S F_i(t,\mathbf{Y}) - {}_0^C D_t^\alpha \hat{\theta}_i\right) + \left(k - \hat{k}\right)\left(|S| - {}_0^C D_t^\alpha \hat{k}\right) - \eta S^2
\end{aligned} \tag{24}$$

By substituting adaptation laws into (24), we have

$${}_0^C D_t^\alpha V(t) \le -\eta S^2 \tag{25}$$

Integrating both sides of Eq. (25), we have

$$\begin{aligned}
&{}_0 D_t^{-\alpha}\, {}_0^C D_t^\alpha V = V(t) - V(0) \le -{}_0 D_t^{-\alpha} \left(\eta S^2\right) = -\eta\, {}_0 D_t^{-\alpha} \left(|S|^2\right) \Rightarrow \\
&V(t) + \eta\, {}_0 D_t^{-\alpha} \left(|S|^2\right) \le V(0)
\end{aligned} \tag{26}$$

One can obtain the following equation from (26)

$$\eta\, {}_0 D_t^{-\alpha} \left(|S|^2\right) \le V(0) \Rightarrow {}_0 D_t^{-\alpha} \left(|S|^2\right) \le \frac{V(0)}{\eta} \tag{27}$$

As we assume that the sliding surface is continuously differentiable, it is a uniformly continuous function. Accordingly, Using Theorem 3, Inequality (27) demonstrates that the sliding surface becomes zero as time approaches to infinity. Also, due to asymptotic stability of the origin in the sliding surface, the error trajectory converges to zero and the error system (16) is asymptotically stable.

5. Simulation Results

In this section, simulation results are presented to show performance of the method. The fractional order gyro system which is used as a master system in this simulation is considered as follows:

$$\begin{cases} {}_0^C D_t^{\alpha} x_1 = x_2 \\ {}_0^C D_t^{\alpha} x_2 = -c_1{}^2 \dfrac{\left(1-\cos x_1\right)^2}{\sin^3 x_1} - c_3 x_2 - c_4 x_2^3 + \left(c_2 + p\sin \omega t\right)\sin x_1 \end{cases} \tag{28}$$

Dynamics of this gyro system exhibits chaotic behavior for parameter values of $\alpha = 0.97$, $c_1 = 10$, $c_2 = 1$, $c_3 = 0.5$, $c_4 = 0.05$, $p = 35.5$, and $\omega = 2$ [54].

Slave system is the well-known fractional order Duffing system with fractional order of the equations $\alpha = 0.97$ [55]. Hence, the dynamics of the slave system in state space is considered as follows:

$$\begin{cases} {}_0^C D_t^{0.97} y_1 = y_2 \\ {}_0^C D_t^{0.97} y_2 = -\theta_1 y_1 - \theta_2 y_1^3 - \theta_3 y_2 + \theta_4 \cos\left(\omega t\right) + d\left(t\right) + g\left(t, \mathbf{Y}\right) u \end{cases} \tag{29}$$

Since the parameters of the slave system is assumed to be unknown, Eq. (29) can be restated as:

$$\begin{cases} {}_0^C D_t^{0.97} y_1 = y_2 \\ {}_0^C D_t^{0.97} y_2 = f\left(t, \mathbf{Y}\right) + \mathbf{F}^T\left(t, \mathbf{Y}\right)\mathbf{\Theta} + d\left(t\right) + g\left(t, \mathbf{Y}\right) u \end{cases} \tag{30}$$

where $f(t, \mathbf{Y}) = 0$ and $g\left(t, \mathbf{Y}\right) = 1 + y_1^2$. Furthermore, $\mathbf{F}\left(t, \mathbf{Y}\right)$ and $\mathbf{\Theta}$ are defined as:

$$\mathbf{F}\left(t, \mathbf{Y}\right) = \begin{bmatrix} -y_1 & -y_1^3 & -y_2 & \cos\left(\omega t\right) \end{bmatrix}^T \tag{31}$$

$$\mathbf{\Theta} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix}^{\mathrm{T}} \tag{32}$$

The disturbance term is taken as $d\left(t\right) = 0.2\cos\left(2t\right)$, which is bounded by $\left|d\left(t\right)\right| \le k = 0.2$. The parameter k is assumed to be unknown and should be updated by adaptation law. The initial conditions are considered as $\mathbf{X}(0)=\begin{bmatrix}0.2 & 0.2\end{bmatrix}^T$, $\mathbf{Y}(0)=\begin{bmatrix}-0.2 & -0.2\end{bmatrix}^T$, $\hat{\mathbf{\Theta}}(0)=\begin{bmatrix}-0.5 & -0.5 & 0.5 & 0.5\end{bmatrix}^{\mathrm{T}}$, and $\hat{k}(0)=0.1$. The adaptation coefficients are set to $\gamma_k=1$ and $\gamma=\begin{bmatrix}5 & 5 & 5 & 5\end{bmatrix}^T$. In order to solve fractional differential equations, the Predict–Evaluate–Correct–Evaluate algorithm [56] is used with time step size of 0.01.

Numerical simulation results are shown in Figures 1-3. Figures 1 and 2 show time history of the systems state variables and synchronization error, respectively; and Figure 3 demonstrates time history of the control input u and the sliding surface S.

6. Conclusion

This paper has shown a method for synchronization of two uncertain and chaotic fractional order systems. The proposed method is based on an adaptive sliding mode controller. The adaptation laws are derived from a sliding surface using Lyapunov approach. The most influential advantage of the presented method is the robustness of the closed loop control system against system uncertainties and external disturbance. The other one is simplicity and suitable performance of the proposed controller. Finally, the proposed method is used to control a synchronization problem of two fractional order Duffing and gyro systems. From the simulation results, it is clear to demonstrate that a satisfactory control performance can be achieved by using proposed scheme.

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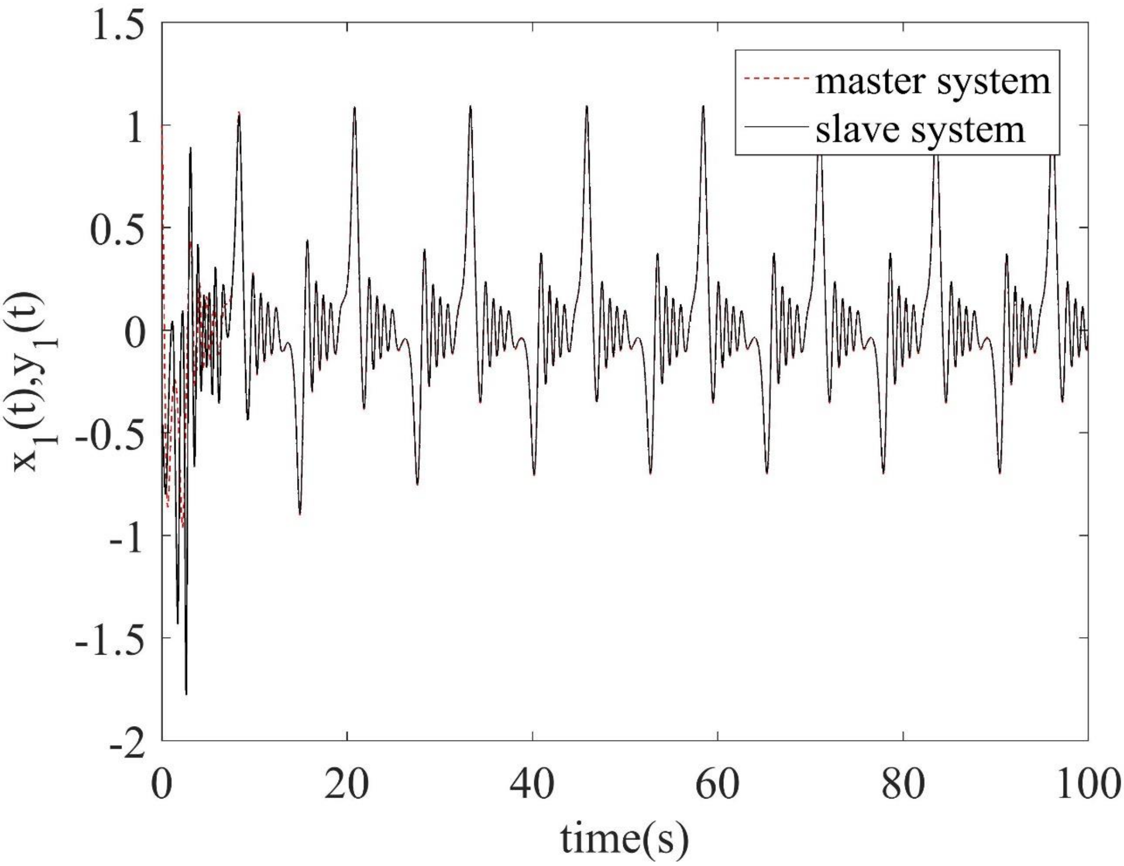
Captions:

Figure 1: Time history of the master and slave state variables; (a) first state variable, (b) second state variable.

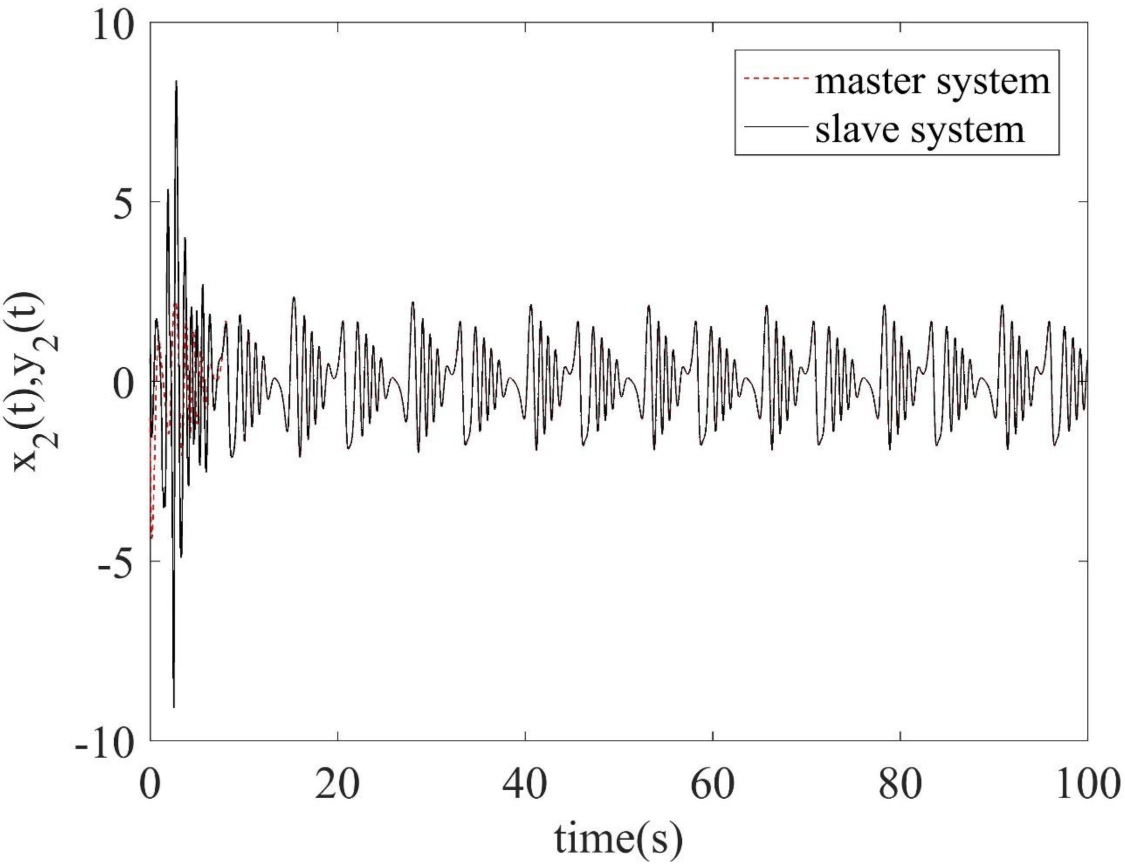
Figure 2: Time history of the synchronization error.

Figure 3: (a) Time history of the control action, (b) Time history of the sliding surface.



1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
20
40
60
80
100
$x_1(t),y_1(t)$
time(s)
master system
slave system

(a)



(b)

Figure 1: Time history of the master and slave state variables; (a) first state variable, (b) second state variable

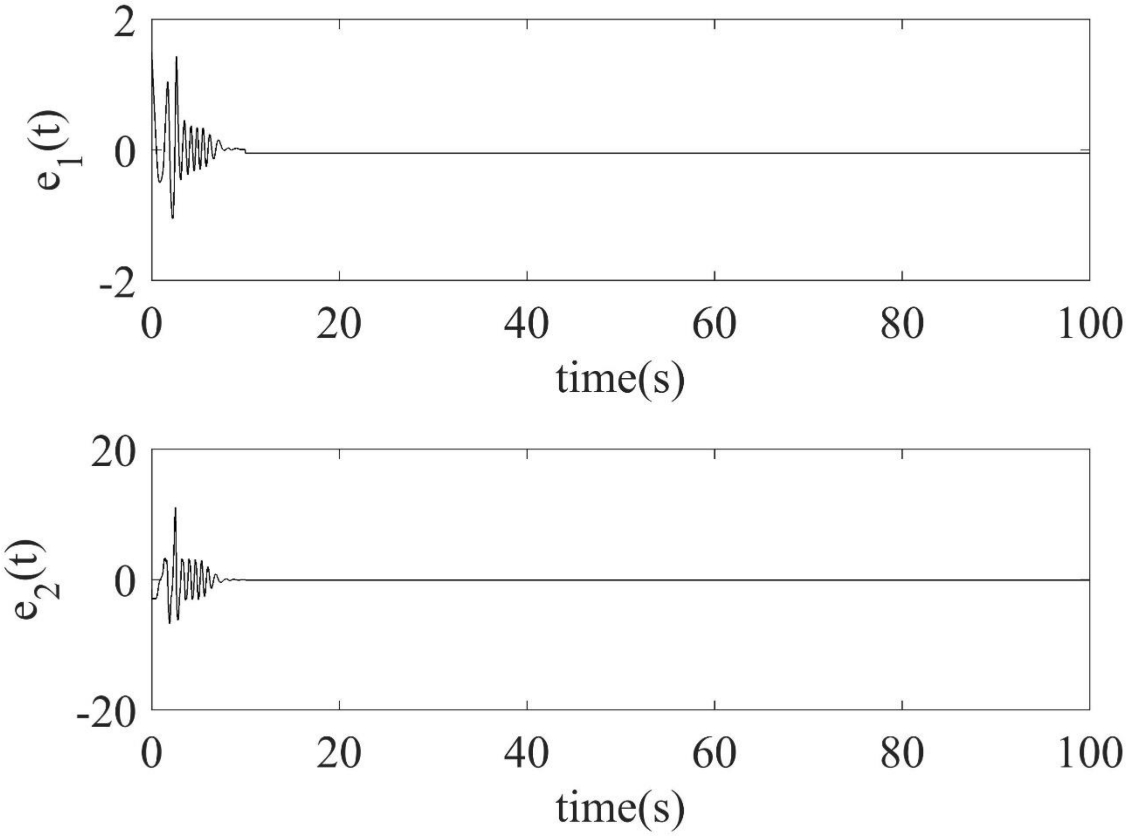


Figure 2: Time history of the synchronization error

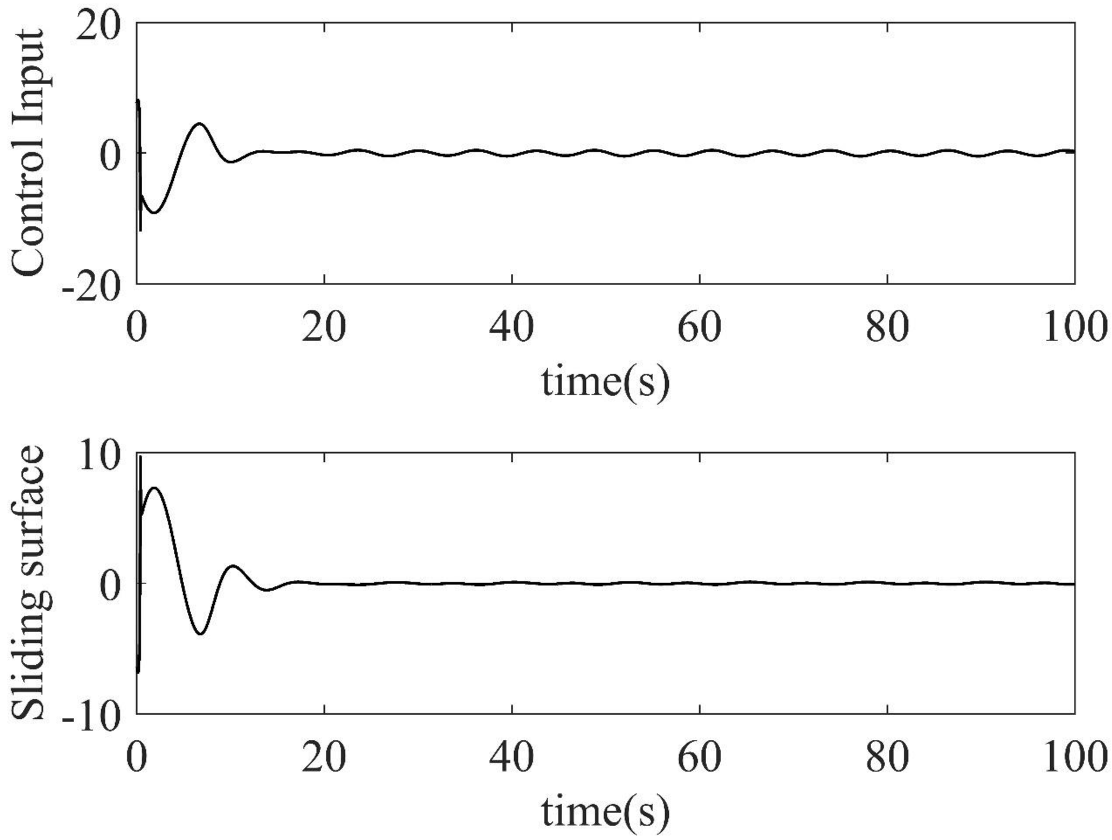


Figure 3: (a) Time history of the control action, (b) Time history of the sliding surface

Adaptive Delayed Feedback Control of Uncertain Fractional Order Chaotic Systems Using Sliding Mode Control

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Abstract

This paper presents an adaptive delayed feedback control method for stabilizing unstable periodic orbits (UPOs) of uncertain fractional order chaotic systems using sliding mode control. The main goal of this research is to develop an adaptive control method based on Lyapunov approach and sliding mode techniques such that the closed loop control system be asymptotically stable on a periodic trajectory which can be sufficiently close to the UPO of the fractional chaotic system. Robustness of the closed loop control system against the system uncertainty and external disturbances is guaranteed. Finally, the proposed method is used to stabilize the UPO of fractional order Duffing system and the numerical simulations show the effectiveness of this method.

Keywords: Fractional order systems; Chaos; Nonlinear delayed feedback; Sliding mode Control; Adaptive control.

1. Introduction

Chaos is a phenomenon that appears in nonlinear dynamical systems and has been observed in many fields of science such as economics [1], chemistry [2], biology [3], engineering [4], and so on. In recent years, control of chaotic systems have attracted many scientists in variety of fields. One of the best known methods for chaos control is stabilizing an unstable periodic orbit (UPO) [5, 6]. The first method for stabilizing the UPO of chaotic systems was introduced in 1990 is named OGY [7]. After that time several extensions of the above-mentioned method were introduced [8, 9]. Linear delayed feedback method proposed for control of chaotic systems by Pyragas. The best-known advantage of this method is that it does not need any information about the periodic solutions other than its period [10]. The delayed feedback control scheme is used for control of both discrete time systems [11, 12] and continuous time systems [13, 14]. Up to this time, several methods have been presented for the control of chaotic systems such as: feedback linearization [15, 16], sliding mode control [17, 18], fuzzy control [19, 20], variable structure control [21], backstepping [22], minimum entropy control [23], and fuzzy minimum entropy control [24].

Fractional calculus is an old field of mathematics that has had no real applications over many years, but recently, it has been extensively used in variety of fields of science and engineering. It has been used to model many systems in several fields e.g. electromagnetics problems [25, 26], viscoelastic damping [27], signal processing [28], vibration [29], diffusion wave [30-32], cyber-physical systems [33], control theory [34, 35] and chaos control [36-38]. In addition, by developing fractional calculus in control theory, many fractional order controllers are introduced like, fractional PID controller [39], fractional PI controller [40], fractional PD controller [41], fractional lead-lag controller [42, 43], fractional CRONE controller [44] and so on.

In recent years, different techniques have been developed to achieve the fractional order chaotic systems control. For example, Lyapunov-based control [45], linear feedback control [46], sliding mode control [17, 47], fuzzy control [48], and intelligent control [49]. In most of the mentioned methods, the exact model of the systems is needed to derive controllers, but in many applications, because of the system uncertainty and external disturbances, mathematical model of the system is not known.

After development of fractional calculus in dynamical systems, it has been shown that chaotic systems in fractional order form can behave like integer order ones. Afterwards, it is demonstrated that UPO can be found in these systems as it can be shown in integer order ones

[50, 51]. This paper presents an adaptive nonlinear delayed feedback control via sliding mode control for stabilizing a periodic orbit with known period, which is near to the UPO of the fractional chaotic system.

The main contribution of the paper can be itemized as:

a) The proposed method can stabilize the UPO of a fractional order chaotic system even when the trajectory of UPO is not known. In this case the period of the UPO should be known.

b) The Pyragas method is one the most well-known approaches which can be applied to stabilize the UPO when only the period of UPO is known. This method is also called delayed feedback control; however, this method has not a systematic procedure especially when a continuous-time chaotic system is aimed to be controlled, so its feedback gains should be determined by trial and error. In this paper a systematic algorithm for delayed feedback control of fractional order continuous-time chaotic systems is proposed. The proposed method is nonlinear and, it is applicable to many nonlinear fractional chaotic systems.

c) The proposed method is a fractional order version of [52]. In this work the uncertainties and disturbances are assumed to have unknown bounds, and so an adaptive method is suggested.

This paper is organized as follows: Section 2 is a review of the preliminary concepts of fractional calculus. In Section 3 the problem statement is presented. In Section 4, an adaptive sliding mode control method is introduced. The parameters are updated using an adaptation mechanism. In Section 5, the proposed scheme is utilized for stabilizing chaotic Duffing system. Simulation results confirm the effectiveness of the presented method. Finally, a brief conclusion is presented in the latest section.

2. Preliminary Concepts

There are several definitions of fractional derivatives. The best known definitions are Riemann-Liouville, Grünwald-Letnikov and Caputo.

Definition 1 [53]**.** The pth order fractional integral of function $f(t)$ is defined by

$$ {}_0I_t^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau \tag{1} $$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 [53]**.** The Riemann-Liouville fractional derivative of order p of any function $f(t)$ is defined as follows

$$ {}_0^{RL}D_t^p f(t) = \frac{d^k}{dt^k} \left({}_0I_t^{(k-p)} f(t) \right) \tag{2} $$

where k is an integer number such that $k-1 \le p < k$.

Definition 3 [53]**.** The Grünwald-Letnikov fractional derivative of order p of any function $f(t)$ is defined as follows

$$ {}_0^{GL}D_t^p f(t) = \lim_{h \to 0} \frac{(\Delta_h^p f)(t)}{h^p} = \lim_{\substack{h \to 0 \\ Nh=t}} \frac{1}{h^p} \sum_{i=0}^{N} (-1)^i \binom{\alpha}{i} f\left(t - ih\right) \tag{3} $$

where

$$ \binom{\alpha}{i} = \frac{\Gamma(p+1)}{i\,!\,\Gamma(p+1-i)} \tag{4} $$

This definition of fractional derivative leads to Riemann–Liouville definition when we perform limit operation.

Definition 4 [53]**.** The Caputo fractional derivative of order p of any function $f(t)$ is defined as follows

$$ {}_0^{C}D_t^p f(t) = \begin{cases} \dfrac{1}{\Gamma(k-p)} \displaystyle\int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{p+1-k}} d\tau & k-1 < p < k \\ \dfrac{d^k}{dt^k} f(t) & p = k \end{cases} \tag{5} $$

Definition 5. A fractional order linear system in state space form is like:

$$ \begin{Bmatrix} {}_0^{C}D_t^{\alpha_1} x_1 \\ {}_0^{C}D_t^{\alpha_2} x_2 \\ \vdots \\ {}_0^{C}D_t^{\alpha_n} x_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \mathbf{A}_{n \times n} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \tag{6} $$

where α_1, …, α_n are arbitrary real numbers. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, the system is called commensurate.

Theorem 1 [54]**.** Consider a commensurate fractional order linear system

$$\frac{d^\alpha}{dt^\alpha}\mathbf{X} = \mathbf{AX} \tag{7}$$

where $\mathbf{X} \in \Re^n$, $\mathbf{A} \in \Re^{n\times n}$ and α is an arbitrary real number between 0 and 2. The autonomous system is asymptotically stable if the following condition is satisfied:

$$\left|\arg(\lambda_i)\right| > \alpha\frac{\pi}{2} \quad \forall i \tag{8}$$

where λ_i represents the eigenvalues of matrix $\mathbf{A}$.

Definition 5. By using the Caputo derivative, a fractional order nonlinear system is defined as:

$${}_0^C D_t^\alpha x = f(t,x) \tag{9}$$

where α is the order of fractional derivatives of the system.

By using the Lyapunov direct method, the asymptotic stability of nonlinear fractional order systems can be studied. The Lyapunov direct method for the fractional-order systems is defined in the following theorem [55].

Theorem 2 [55]**.** Let $x=0$ be an equilibrium point for the autonomous fractional order system 0(9). Let $V(t,x(t))$ be a continuously differentiable function as a Lyapunov function candidate, and $\gamma_i (i=1,2,3)$ be class-K functions such that

$$\gamma_1\left(\|x\|\right) \le V(t,x(t)) \le \gamma_2\left(\|x\|\right) \tag{10}$$

$${}_0^C D_t^\beta V(t,x(t)) \le -\gamma_3\left(\|x\|\right) \tag{11}$$

where $\beta \in (0,1)$. Then the system (9) is asymptotically stable.

Lemma 1. [55] Let $x(t) \in \Re$ be a continuously differentiable function. Then, for all $t \ge t_0$

$$\frac{1}{2}\,{}_{t_0}^{C}D_t^{\alpha}x^2(t) \le x(t)\,{}_{t_0}^{C}D_t^{\alpha}x(t), \qquad \forall\alpha\in(0,1) \tag{12}$$

Remark 1. [55] In the case when $x(t)\in\Re^n$, Lemma 1 is still valid. That is, $\forall t\ge t_0$

$$\frac{1}{2}\,{}_{t_0}^{C}D_t^{\alpha}x^T(t)x(t) \le x^T(t)\,{}_{t_0}^{C}D_t^{\alpha}x(t), \qquad \forall\alpha\in(0,1) \tag{13}$$

In [56], Barbalat's Lemma is developed for fractional order nonlinear systems as follows:

Theorem 3 [56]**.** Let $\phi:\Re\to\Re$ be a uniformly continuous function on $[t_0,\infty)$. Assume that there exist two positive constants p and M such that ${}_0I_t^{\alpha}|\phi|^p \le M$ for all $t>t_0>0$ with $\alpha\in(0,1)$. Then

$$\lim_{t\to\infty}\phi(t)=0 \tag{14}$$

3. Problem Statement

Consider the following uncertain fractional order chaotic system

$$\begin{cases} {}_0^{C}D_t^{\alpha}x_i = x_{i+1} & 1\le i\le n-1 \\ {}_0^{C}D_t^{\alpha}x_n = f(t,\mathbf{X})+\mathbf{F}^T(t,\mathbf{X})\mathbf{\Theta}+g(t,\mathbf{X})u+d(t) \end{cases} \tag{15}$$

where $0<\alpha<1$ demonstrates the fractional order of the equations, $\mathbf{X}=\begin{bmatrix}x_1 & \cdots & x_n\end{bmatrix}^T$ is measurable state vector of the system, $f(\cdot)$ and $g(\cdot)$ are nonlinear functions and belong to $\mathrm{C}^1\left(\Re^n\times\Re \to\Re\right)$, and it is assumed that $g(\cdot)\ne 0$, and $\mathbf{\Theta}=\begin{bmatrix}\theta_1 & \cdots & \theta_m\end{bmatrix}^T\in\Re^m$ stands for the uncertain parameter vector of the system. $\mathbf{F}(\cdot)$ is an $m\times 1$ vector, $d(\cdot)$ is an external disturbance, and it is assumed that $|d(\cdot)|\le k<\infty$ for all $t>0$ and the value of $k>0$ is unknown. $u(t)\in\Re$ is the control input and the system shows chaotic response for $u=0$. All functions are assumed to be sufficiently smooth and Lipchitz. $f(t,\mathbf{X})+\mathbf{F}^T(t,\mathbf{X})\mathbf{\Theta}$ function is t-periodic with period T, and also the system (15) without control and disturbance has an unstable periodic solution with period T. The main goal is to design a control law u, such that the chaotic behavior of the system (15) changes to a periodic one which is near to an unstable periodic orbit of the system.

4. Controller Design

In the mentioned Pyragas method, usually we use a linear delayed feedback control to stabilize the UPO of the system. Here again the concept of the Pyragas technique is used, and it is applied to our problem.

Defining a delayed state as $\tilde{\mathbf{X}} = \mathbf{X}(t-T)$, and substituting it for $\mathbf{X}$ in Eq. (15), results in:

$$\begin{cases} {}_0^C D_t^\alpha \tilde{x}_i = \tilde{x}_{i+1} \qquad 1 \le i \le n-1 \\ {}_0^C D_t^\alpha \tilde{x}_n = f(t-T, \tilde{\mathbf{X}}) + \mathbf{F}^T (t-T, \tilde{\mathbf{X}}) \tilde{\boldsymbol{\Theta}} + \tilde{d} + g(t-T, \tilde{\mathbf{X}}) \tilde{u} \end{cases} \tag{16}$$

where $\tilde{u} = u(t-T)$, and $\tilde{\boldsymbol{\Theta}} = \boldsymbol{\Theta}(t-T)$. We subtract Eq. (16) from Eq. (15) to obtain following equations:

$$\begin{cases} {}_0^C D_t^\alpha x_i - {}_0^C D_t^\alpha \tilde{x}_i = x_{i+1} - \tilde{x}_{i+1} \qquad 1 \le i \le n-1 \\ {}_0^C D_t^\alpha x_n - {}_0^C D_t^\alpha \tilde{x}_n = f(t, \mathbf{X}) - f(t-T, \tilde{\mathbf{X}}) + \mathbf{F}^T (t, \mathbf{X}) \boldsymbol{\Theta} - \mathbf{F}^T (t-T, \tilde{\mathbf{X}}) \tilde{\boldsymbol{\Theta}} \\ \qquad\qquad + d - \tilde{d} + g(t, \mathbf{X}) u - g(t-T, \tilde{\mathbf{X}}) \tilde{u} \end{cases} \tag{17}$$

By defining $\mathbf{E} = \mathbf{X} - \tilde{\mathbf{X}}$, Eq. (17) can be re-written as (error dynamics):

$$\begin{cases} {}_0^C D_t^\alpha e_i = e_{i+1} \qquad 1 \le i \le n-1 \\ {}_0^C D_t^\alpha e_n = f(t, \mathbf{X}) - f(t-T, \tilde{\mathbf{X}}) + \mathbf{F}^T (t, \mathbf{X}) \boldsymbol{\Theta} - \mathbf{F}^T (t-T, \tilde{\mathbf{X}}) \tilde{\boldsymbol{\Theta}} \\ \qquad + d - \tilde{d} + g(t, \mathbf{X}) u - g(t-T, \tilde{\mathbf{X}}) \tilde{u} \end{cases} \tag{18}$$

To stabilize a periodic orbit of the system, we should obtain the control law u such that the following conditions are satisfied:

$$\lim_{t\to\infty} \|\mathbf{E}\| = 0 \;\equiv\; \lim_{t\to\infty} \left\|\mathbf{X}(t) - \tilde{\mathbf{X}}(t)\right\| = 0 \;\equiv\; \lim_{t\to\infty} \|\mathbf{X}(t) - \mathbf{X}(t-T)\| = 0 \tag{19}$$

Theorem 4. Assume $\tilde{\mathbf{X}} = \mathbf{X}(t-T)$ and $\tilde{u}(t) = u\left(t-T\right)$, where T is the period of the unstable periodic orbit of the chaotic system. If the following control and adaptation laws are applied to system defined by (15), the chaotic behavior of the system is substituted by a regular periodic one.

$$u = u_{\text{eq}} + u_{\text{ad}} + u_s \tag{20}$$

$$u_{\text{eq}} = -\frac{\sum_{i=1}^{n} \alpha_i e_i + f(t,\mathbf{X}) - f(t-T,\tilde{\mathbf{X}}) - g(t-T,\tilde{\mathbf{X}})\tilde{u}}{g(t,\mathbf{X})} \tag{21}$$

$$u_{\text{ad}} = -\frac{\left(\mathbf{F}^T(t,\mathbf{X})\hat{\mathbf{\Theta}}\left(t\right) - \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\,\hat{\mathbf{\Theta}}\left(t-T\right)\right) + 2\hat{k}\,\text{sign}\left(S\right)}{g(t,\mathbf{X})} \tag{22}$$

$$u_s = \frac{-\left(M+\eta\right)\text{sign}\left(S\right)}{g\left(t,\mathbf{X}\right)} \tag{23}$$

$${}_0^C D_t^{\alpha} \hat{\theta}_i = \gamma_i S \left(F_i(t,\mathbf{X}) - F_i(t-T,\tilde{\mathbf{X}})\right) \tag{24}$$

$${}_0^C D_t^{\alpha} \hat{k} = 2\gamma_k \left|S\right| \tag{25}$$

where $\hat{\theta}_i$ and $\hat{k}$ are estimates of θ_i and k, η is an arbitrary positive constant, γ_i and γ_{k} are positive constants and M is a sufficiently large positive constant. In the above equations, S is a sliding surface defined by:

$$S = e_n + \sum_{i=1}^{n} \alpha_i \, {}_0I_t^{\alpha} e_i \tag{26}$$

in which $\alpha_i > 0$'s are constants and selected such that an asymptotically stable dynamics for the sliding surface, $S = 0$ is provided.

A simplifying condition, which is very common in controlling fractional order systems, is assumed that all of the system state variables as well as the sliding surface are continuously differentiable and can be measured. This assumption is common especially when a fractional order system is aimed to be controlled [57, 58].

Remark 2. For the existence of sliding mode, it is necessary that the following manifold be asymptotically stable.

$$S = e_n + \sum_{i=1}^{n} \alpha_i \, {}_0I_t^{\alpha} e_i \tag{27}$$

By applying the time derivative of α 's order to Eq. (27), we have:

$${}_0^C D_t^{\alpha} S = {}_0^C D_t^{\alpha} e_n + \sum_{i=1}^{n} \alpha_i e_i \tag{28}$$

Therefore, the sliding mode dynamics can be obtained as follows

$${}_0^C D_t^{\alpha} e_n = -\sum_{i=1}^{n} \alpha_i e_i \tag{29}$$

where $\alpha_1,...,\alpha_n$ are chosen such that the roots of equation $s^{n\alpha} + \sum_{i=1}^{n} \alpha_i s^{(i-1)\alpha} = 0$ satisfy the asymptotic stability condition in fractional order linear systems. Using Eq. (8), this condition can be expressed by

$$\left|\arg(s_i)\right| > \alpha \frac{\pi}{2} \tag{30}$$

where s_i for $i = 1,...,n$ denotes the roots of the above-mentioned equation.

Proof of Theorem 4. Consider the following Lyapunov function candidate

$$V = \frac{1}{2} S^2 + \frac{1}{2} \sum_{i=1}^{m} \frac{1}{\gamma_i} (\theta_i - \hat{\theta}_i)^2 + \frac{1}{2} \frac{1}{\gamma_k} (k - \hat{k})^2 \tag{31}$$

Applying the time derivative of α 's order to the Lyapunov function (31) along with using Lemma 1 results in:

$$
\begin{aligned}
{}_0^C D_t^\alpha V &= \frac{1}{2}\,{}_0^C D_t^\alpha S^2 + \frac{1}{2}\sum_{i=1}^{m}\frac{1}{\gamma_i}\,{}_0^C D_t^\alpha (\theta_i-\hat{\theta}_i)^2 + \frac{1}{2}\frac{1}{\gamma_k}\,{}_0^C D_t^\alpha (k-\hat{k})^2 \\
&\le S\,{}_0^C D_t^\alpha S + \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i-\hat{\theta}_i)\,{}_0^C D_t^\alpha (\theta_i-\hat{\theta}_i) + \frac{1}{\gamma_k}(k-\hat{k})\,{}_0^C D_t^\alpha (k-\hat{k}) \\
&= S\,{}_0^C D_t^\alpha S - \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i-\hat{\theta}_i)\,{}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}(k-\hat{k})\,{}_0^C D_t^\alpha \hat{k}
\end{aligned} \tag{32}
$$

Substituting for ${}_0^C D_t^\alpha S$ by Eq. (28) gives:

$$
{}_0^C D_t^\alpha V \le S\left({}_0^C D_t^\alpha e_n + \sum_{i=1}^{n}\alpha_i e_i\right) - \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i-\hat{\theta}_i)\,{}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}(k-\hat{k})\,{}_0^C D_t^\alpha \hat{k} \tag{33}
$$

Substituting Eq. (18) into Eq. (33) yields:

$$
\begin{aligned}
{}_0^C D_t^\alpha V \le S &\begin{pmatrix} \sum_{i=1}^{n}\alpha_i e_i + f(t,\mathbf{X}) - f(t-T,\tilde{\mathbf{X}}) + \mathbf{F}^T(t,\mathbf{X})\mathbf{\Theta} - \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\tilde{\mathbf{\Theta}} \\ +d-\tilde{d}+g(t,\mathbf{X})u - g(t-T,\tilde{\mathbf{X}})\tilde{u} \end{pmatrix} \\
&- \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i-\hat{\theta}_i)\,{}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}(k-\hat{k})\,{}_0^C D_t^\alpha \hat{k}
\end{aligned} \tag{34}
$$

It was assumed that $\left|d\left(t\right)\right| \le k$, therefore $\left|d-\tilde{d}\right| \le 2\mathrm{k}$ and consequently:

$$
\begin{aligned}
{}_0^C D_t^\alpha V \le S &\begin{pmatrix} \sum_{i=1}^{n}\alpha_i e_i + f(t,\mathbf{X}) - f(t-T,\tilde{\mathbf{X}}) + \mathbf{F}^T(t,\mathbf{X})\mathbf{\Theta} - \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\tilde{\mathbf{\Theta}} \\ +g(t,\mathbf{X})u - g(t-T,\tilde{\mathbf{X}})\tilde{u} \end{pmatrix} \\
&+2k\left|S\right| - \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i-\hat{\theta}_i)\,{}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}(k-\hat{k})\,{}_0^C D_t^\alpha \hat{k}
\end{aligned} \tag{35}
$$

Using Eqs. (20)-(22) in inequality (35) yields:

$$
\begin{aligned}
{}_0^C D_t^\alpha V \le S &\begin{pmatrix} \mathbf{F}^T(t,\mathbf{X})\mathbf{\Theta} - \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\tilde{\mathbf{\Theta}} - \mathbf{F}^T(t,\mathbf{X})\hat{\mathbf{\Theta}}(t) \\ +\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\hat{\mathbf{\Theta}}\left(t\right) + g(t,\mathbf{X})u_s \end{pmatrix} \\
&+2k\left|S\right| - 2\hat{k}\left|S\right| - \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i-\hat{\theta}_i)\,{}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}(k-\hat{k})\,{}_0^C D_t^\alpha \hat{k}
\end{aligned} \tag{36}
$$

Now, we add the term $S\left(-\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\mathbf{\Theta} + \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\mathbf{\Theta}\right)$ to the right side of inequality (36).

$$
{}_{0}^{C}D_t^{\alpha}V \le S\begin{pmatrix} \mathbf{F}^T(t,\mathbf{X})\mathbf{\Theta}-\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\mathbf{\Theta}+\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\mathbf{\Theta}-\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\tilde{\mathbf{\Theta}} \\ -\mathbf{F}^T(t,\mathbf{X})\hat{\mathbf{\Theta}}(t)+\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\hat{\mathbf{\Theta}}\left(t\right)+g(t,\mathbf{X})u_s \end{pmatrix}
$$
$$
+2\left|S\right|\left(k-\hat{k}\right)-\sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i-\hat{\theta}_i)\,{}_{0}^{C}D_t^{\alpha}\hat{\theta}_i-\frac{1}{\gamma_k}(k-\hat{k})\,{}_{0}^{C}D_t^{\alpha}\hat{k} \tag{37}
$$

After some mathematical manipulations inequality (37) can be written as:

$$
{}_{0}^{C}D_t^{\alpha}V \le S\left(\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\left(\mathbf{\Theta}-\tilde{\mathbf{\Theta}}\right)+g(t,\mathbf{X})u_s\right)+(k-\hat{k})\left[2\left|S\right|-\frac{1}{\gamma_k}\,{}_{0}^{C}D_t^{\alpha}\hat{k}\right]
$$
$$
+\sum_{i=1}^{m}\left((\theta_i-\hat{\theta}_i)\left[S\left(F_i(t,\mathbf{X})-F_i(t-T,\tilde{\mathbf{X}})\right)-\frac{1}{\gamma_i}\,{}_{0}^{C}D_t^{\alpha}\hat{\theta}_i\right]\right) \tag{38}
$$

Using Eqs. (24) and (25) in inequality (38) yields:

$$
{}_{0}^{C}D_t^{\alpha}V \le S\left(\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\left(\mathbf{\Theta}(t)-\mathbf{\Theta}(t-T)\right)+g(t,\mathbf{X})u_s\right) \tag{39}
$$

Since $\mathbf{\Theta}(\cdot)$ is bounded, one can conclude that $\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\left(\mathbf{\Theta}(t)-\mathbf{\Theta}(t-T)\right)$ is also bounded. Using Eq. (23) in inequality (39) results in:

$$
{}_{0}^{C}D_t^{\alpha}V \le -\eta\left|S\right| \tag{40}
$$

Integrating both sides of Eq. (40), we have

$$
\begin{aligned}
&{}_{0}I_t^{\alpha}\,{}_{0}^{C}D_t^{\alpha}V = V(t)-V(0) \le -{}_{0}I_t^{\alpha}\left(\eta\left|S\right|\right) = -\eta\,{}_{0}I_t^{\alpha}\left(\left|S\right|\right) \Rightarrow \\
&V(t)+\eta\,{}_{0}I_t^{\alpha}\left(\left|S\right|\right) \le V(0)
\end{aligned} \tag{41}
$$

One can obtain the following equation from (41)

$$
\eta\,{}_{0}I_t^{\alpha}\left(\left|S\right|\right) \le V(0) \Rightarrow {}_{0}I_t^{\alpha}\left(\left|S\right|\right) \le \frac{V(0)}{\eta} \tag{42}
$$

Since we assume that the sliding surface is continuously differentiable, it is a uniformly continuous function. Therefore, Using Theorem 3, Inequality (42) indicates that the sliding surface becomes zero as time approaches to infinity. Also, due to asymptotic stability of the origin, the error trajectory converges to zero. Thus, the condition of Eq. (19) is satisfied and the control objective is achieved. So the Theorem 4 has been proved completely.

Remark 3. As a special case, if $\mathbf{\Theta}(t)=\mathbf{\Theta}(t-T)$ or $\mathbf{\Theta}$ is a constant, then u_s can be selected as:

$$u_s = \frac{-\eta \operatorname{sign}(S)}{g(t,\mathbf{X})} \tag{43}$$

Proof. If $\mathbf{\Theta}(t)=\mathbf{\Theta}(t-T)$, then $\mathbf{\Theta}(t)-\mathbf{\Theta}(t-T)=0$, so Eq. (39) will be simplified as follows:

$${}_0^C D_t^{\alpha} V \le S g(t,\mathbf{X}) u_s \tag{44}$$

By substituting Eq. (43) into Eq. (44), one can obtain Eq. (40).

Remark 4. Due to use of the sign function in equations of u_{ad} and u_s, chattering in implementation of control law can occur. To avoid this problem, the sign function can be replaced by saturation as follows [59]:

$$\operatorname{sat}\left(\frac{S}{\phi}\right) = \begin{cases} \dfrac{S}{\phi} & \left|\dfrac{S}{\phi}\right| < 1 \\ \operatorname{sign}\left(\dfrac{S}{\phi}\right) & \left|\dfrac{S}{\phi}\right| \ge 1 \end{cases} \tag{45}$$

where ϕ is a small positive number.

5. Simulation Results

The fractional order chaotic Duffing system with the following equation is used for simulation:

$$\begin{cases} {}_0^C D_t^{0.98} x_1 = x_2 \\ {}_0^C D_t^{0.98} x_2 = -\theta_1 x_1 - \theta_2 x_1^3 - \theta_3 x_2 + \theta_4 \cos(\omega t) + d(t) + \left(1 + x_1^2\right) u \end{cases} \tag{46}$$

where $d(t)$ is the external disturbance and u is the control action. For $u=0$ and $d(t)=0$, the fractional order Duffing equation is obtained. By setting $\theta_1=-1$, $\theta_2=1$, $\theta_3=0.15$, $\omega=1$, and $\theta_4 = 0.3$, the fractional order Duffing equation shows chaotic behavior. Moreover, existence of unstable periodic orbit in fractional order Duffing system is proved [50]. One of the UPOs of the above-mentioned system is shown in Figure 1.

We consider $f(t,\mathbf{X}) = 0$ and $g(t,\mathbf{X}) = (1 + x_1^2)$. Furthermore, $\mathbf{F}(t,\mathbf{X})$ and $\mathbf{\Theta}$ are defined as:

$$\mathbf{F}(t,\mathbf{X}) = \begin{bmatrix} -x_1 & -x_1^3 & -x_2 & \cos(\omega t) \end{bmatrix}^T \tag{47}$$

$$\mathbf{\Theta} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix}^{\mathrm{T}} \tag{48}$$

Therefore, Eq. (46) can be rewritten as follows

$$\begin{cases} {}_0^C D_t^{0.98} x_1 = x_2 \\ {}_0^C D_t^{0.98} x_2 = f(t,\mathbf{X}) + \mathbf{F}^T(t,\mathbf{X})\mathbf{\Theta} + d(t) + g(t,\mathbf{X})\,u \end{cases} \tag{49}$$

The disturbance term is assumed to be $d(t) = 0.2\cos(2t)$ which is bounded by $|d(t)| \le k = 0.2$. The parameter k is assumed to be unknown that should be updated by adaptation law (25). The initial conditions are set to $\hat{\mathbf{\Theta}}(0) = \begin{bmatrix} -1.5 & 1.5 & 0.2 & 0.5 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{X}(0) = \begin{bmatrix} 0.15 & 0.1 \end{bmatrix}^T$, $\hat{k}(0) = 0.1$, and parameters γ_k and γ are selected as $\gamma_k = 1$ and $\gamma = \begin{bmatrix} 5 & 5 & 5 & 5 \end{bmatrix}^T$.

The periodic solution with period $T = 2\pi$ is considered for stabilization, and the simulation results are shown in Figures 2-4. Figure 2 shows time history of the state variables, Figure 3 represents phase plane of the closed loop control system, and Figure 4 demonstrates time history of the control input and sliding surface. The dashed lines in Figure 2 show the main UPO of the system. As it is observed, the chaotic behavior is substituted by to a periodic orbit close to the UPO of the main system. Note that the controller is applied at $t = 4T$. Due to the system uncertainties, external disturbances, and unknown parameters, the main UPO may not be stabilized and the controller action has not converged completely to zero, and the system trajectories converge to a close vicinity of the main UPO. The closer we get to the UPO of the system, the lower control signal will be required because when the trajectory of a chaotic system reaches to the UPO of the system exactly, then the trajectory will remain on the UPO with zero control signal.

6. Conclusion

This paper has shown a robust adaptive nonlinear delayed feedback control for a class of uncertain fractional order nonlinear systems. Robustness of the closed loop control system against the system uncertainty and external disturbances is guaranteed by using fractional order sliding mode control method. The control input and adaptation mechanism is constructed from a proper sliding surface via Lyapunov method. The most influential advantage of the proposed method is that it just needs period T to derive the controller structure. Because of the system uncertainty and external disturbances, the stabilized orbit is not exactly the UPO of the system. However, it has been tried to stabilize an orbit that is very close to the UPO of the system. Finally, the proposed method is implemented to control a fractional order Duffing system and simulation results are included to illustrate the great performance of the proposed method.

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Captions:

Figure 1. Unstable periodic orbit of fractional order Duffing system.

Figure 2. Stabilizing the 2π-periodic orbit of the fractional order Duffing system: a) first state variable, b) second state variable.

Figure 3. Phase plane of the close loop control system.

Figure 4. (a) Time history of the control action, (b) Time history of the sliding surface.

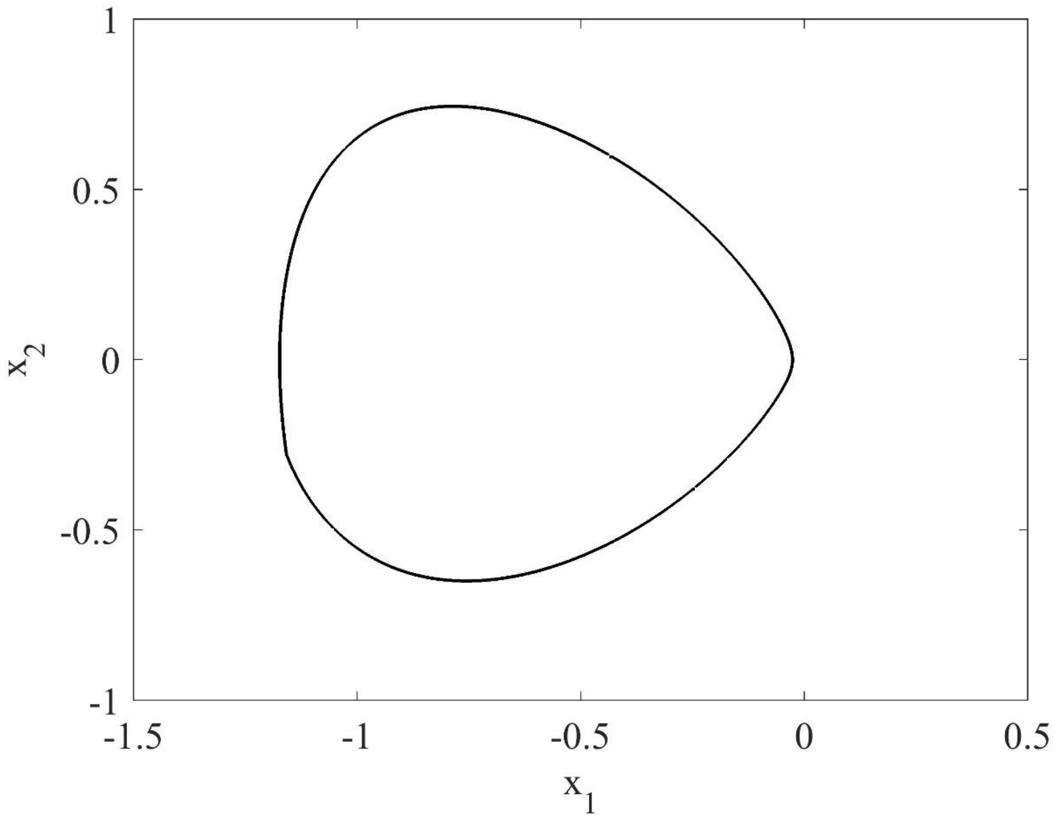

Figure 1. Unstable periodic orbit of fractional order Duffing system

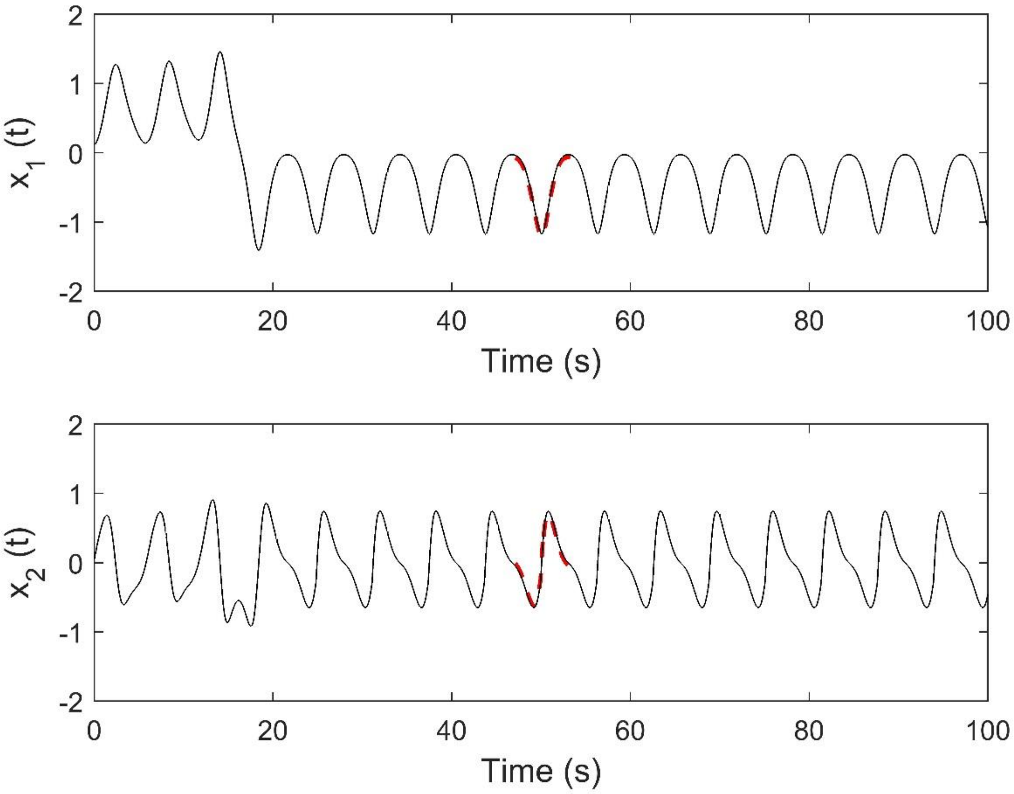

Figure 2. Stabilizing the 2π-periodic orbit of the fractional order Duffing system: a) first state variable, b) second state variable

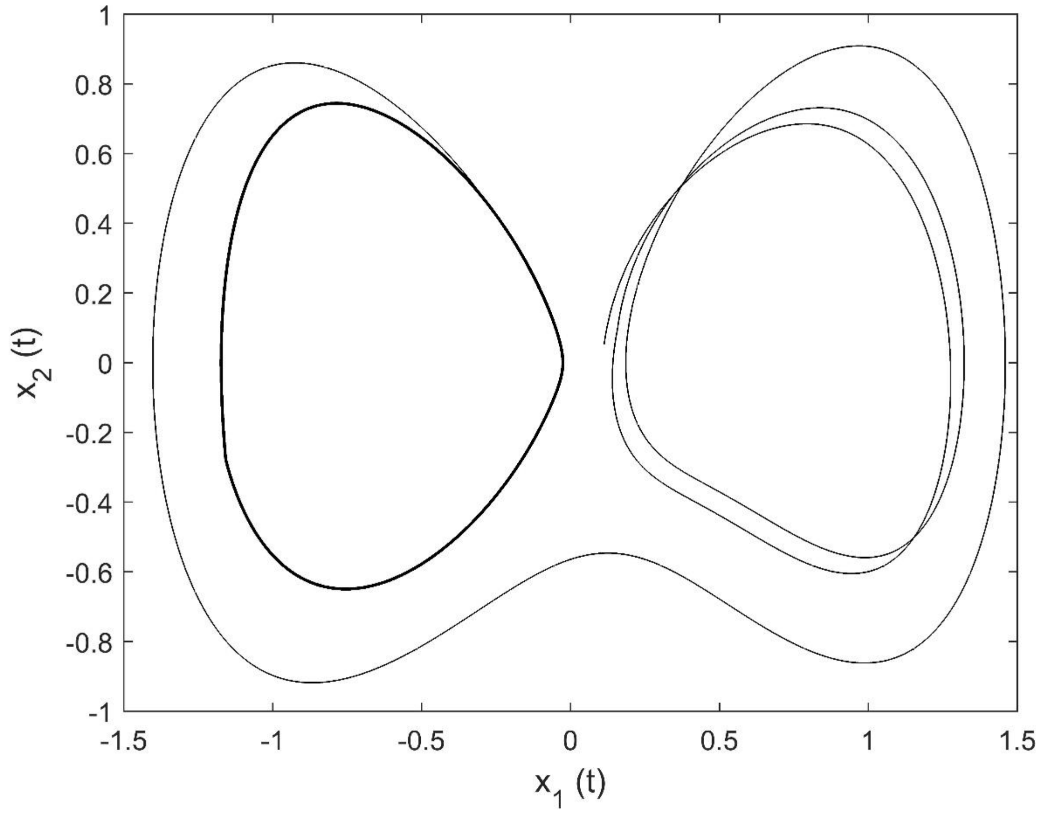

Figure 3. Phase plane of the close loop control system

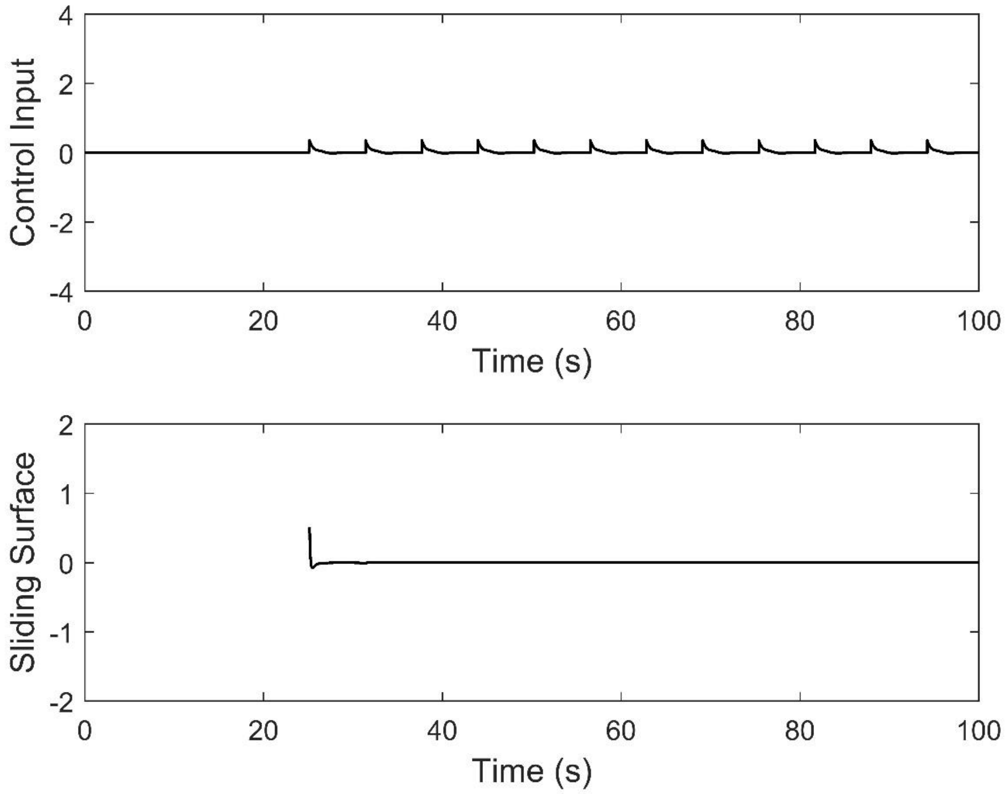

Figure 4. (a) Time history of the control action, (b) Time history of the sliding surface