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Robust adaptive fractional order proportional integral derivative controller design for uncertain fractional order nonlinear systems using sliding mode control

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Abstract

This article presents a robust adaptive fractional order proportional integral derivative controller for a class of uncertain fractional order nonlinear systems using fractional order sliding mode control. The goal is to achieve closed-loop control system robustness against the system uncertainty and external disturbance. The fractional order proportional integral derivative controller gains are adjustable and will be updated using the gradient method from a proper sliding surface. A supervisory controller is used to guarantee the stability of the closed-loop fractional order proportional integral derivative control system. Finally, fractional order Duffing–Holmes system is used to verify the proposed method.

Keywords

Fractional order systems, fractional order proportional integral derivative controller, gradient method, fractional order sliding mode control

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Introduction

Fractional calculus is an old field of mathematics that deals with derivatives and integrals of non-integer order and has had no applications over many years.[1] In recent years, fractional calculus has been extensively used in a variety of fields of physics and engineering applications, for example, dynamical systems,[2] viscoelastic damping,[3] signal processing,[4] diffusion wave,[5] biomedical applications,[6] stochastic systems,[7] and chaotic systems.[8–10]

Recently, many fractional order controllers are developed like fractional order proportional integral derivative (FOPID) control,[11,12] fractional order proportional integral (PI) and proportional differential (PD) control,[13,14] fractional order lead-lag control,[15,16] fractional CRONE controller,[17] fractional order model reference adaptive control,[18,19] and sliding mode control of fractional order systems.[20]

In most of the applications, the controllers are of proportional integral derivative (PID) type due to its simplicity and appropriate performance in a variety of applications.[21] By developing fractional calculus in control theory, FOPID controller—which is simply called $PI^{\lambda}D^{\mu}$—was proposed by Podlubny.[1] The main feature of designing a FOPID controller is the determination of five controller parameters: proportional gain, integral gain, derivative gain, order of integral, and order of derivative. Up to now, several methods of tuning for FOPID controller have been established,[22–24] but many real systems are mostly time varying or may have uncertainty. Hence, using adaptive control and developing methods to design adaptive $PI^{\lambda}D^{\mu}$ control must be concerned.

In recent years, some methods are proposed for tuning adaptive FOPID controller based on sliding mode control,[25] fuzzy methods,[26,27] and genetic algorithm,[28] but up to now no method has been developed for uncertain fractional order nonlinear systems and in the previous methods the stability of the closed-loop control systems is not considered.

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In this article, a robust adaptive FOPID controller is introduced to control an uncertain fractional order nonlinear system. Robustness of the closed-loop system is guaranteed using fractional order sliding mode control method. In addition, a supervisory control is used to guarantee the boundedness of the system state trajectories.

This article is organized as follows: section "Preliminaries" provides a review of the preliminary concepts of fractional calculus. In section "Problem statement," the problem statement is presented. In section "Stability analysis and supervisory controller design," the stability of the closed-loop control system is analyzed and the supervisory controller is introduced to guarantee the stability of the system. In section "Adaptation law," proper sliding surface is introduced and the gradient method is used to generate adaptation laws for FOPID controller gains. Numerical simulations are shown in section "Numerical simulations." Finally, a brief conclusion is presented in the last section.

Preliminaries

There are several definitions of fractional derivatives. The common formulations for these derivatives were generated by Grünwald–Letnikov, Riemann–Liouville, and Caputo.

Definition 1 (Riemann–Liouville fractional integral). The Riemann–Liouville fractional integral[1] of order $p \in \Re^{+}$ of function $f(\tau)$ is defined as

$$
{}_0D_t^{-p}f(t) = \frac{1}{\Gamma(p)}\int_0^t (t-\tau)^{p-1} f(\tau)d\tau \tag{1}
$$

where $\Gamma(\cdot)$ is the standard Gamma function.

Definition 2 (Riemann–Liouville fractional derivative). The Riemann–Liouville fractional derivative[1] of order $p \in \Re^{+}$ of function $f(\tau)$ is defined as

$$
{}_0^{RL}D_t^{p}f(t) = \frac{d^k}{dt^k}\left({}_0D_t^{-(k-p)}f(t)\right) \tag{2}
$$

where k is an integer such that $k-1 \leqslant p < k$.

Definition 3 (Caputo fractional derivative). The Caputo fractional derivative[1] of order $p \in \Re^{+}$ of function $f(\tau)$ is defined as

$$
{}_0^{C}D_t^{p}f(t) = \begin{cases} \frac{1}{\Gamma(k-p)}\int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{p+1-k}}d\tau & k-1 < p < k \\ \frac{d^k}{dt^k}f(t) & p = k \end{cases} \tag{3}
$$

In engineering applications, Caputo derivative has several advantages to Riemann–Liouville derivation method. The most influential one is that fractional differential equations with Caputo derivative take on the same form as for integer order differential equations. Another difference between the above-mentioned definitions is that the Caputo derivative of a constant is 0, whereas the Riemann–Liouville derivative of a constant C is not equal to 0, but

$$
{}_0^{RL}D_t^{\alpha}C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)} \tag{4}
$$

The relation between Riemann–Liouville and Caputo fractional derivatives can be expressed as[1]

$$
{}_0^{RL}D_t^{p}f(t) = {}_0^{C}D_t^{p}f(t) + \sum_{k=0}^{n-1}\Phi_{k-p+1}(t)f^{(k)}(0) \tag{5}
$$

where n is an integer such that $n-1 \leqslant p < n$ and the function $\Phi_p(t)$ is defined by

$$
\Phi_p(t) = \begin{cases} \frac{t^{p-1}}{\Gamma(p)} & t > 0 \\ 0 & t \leqslant 0 \end{cases} \tag{6}
$$

Theorem 1. If $f(t)$ and $g(t)$ along all its derivatives are continuous in $[0, t]$, then the Leibniz rule for fractional differentiation takes the form[1]

$$
{}_0^{RL}D_t^{p}(g(t)f(t)) = \sum_{k=0}^{\infty}\binom{p}{k}g^{(k)}(t){}_0^{RL}D_t^{p-k}f(t) \tag{7}
$$

For $0 < p < 1$, the Leibniz rule for Caputo fractional order derivative can be expressed as

$$
\begin{aligned} {}_0^{C}D_t^{p}(g(t)f(t)) = {} & \sum_{k=0}^{\infty}\binom{p}{k}g^{(k)}(t){}_0^{RL}D_t^{p-k}f(t) \\ & - \frac{(g(t)f(t))|_{t=0}}{t^p\Gamma(1-p)} \end{aligned} \tag{8}
$$

Definition 4. A fractional order linear system in state space form is like

$$
\begin{aligned} \begin{Bmatrix} {}_0^{C}D_t^{\alpha_1}x_1 \\ {}_0^{C}D_t^{\alpha_2}x_2 \\ \vdots \\ {}_0^{C}D_t^{\alpha_n}x_n \end{Bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \\ &= \mathbf{A}_{n\times n}\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \end{aligned} \tag{9}
$$

where $\alpha_1, \ldots, \alpha_n$ are arbitrary real numbers. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, the system is called commensurate.

Theorem 2. Consider a commensurate fractional order linear system[29]

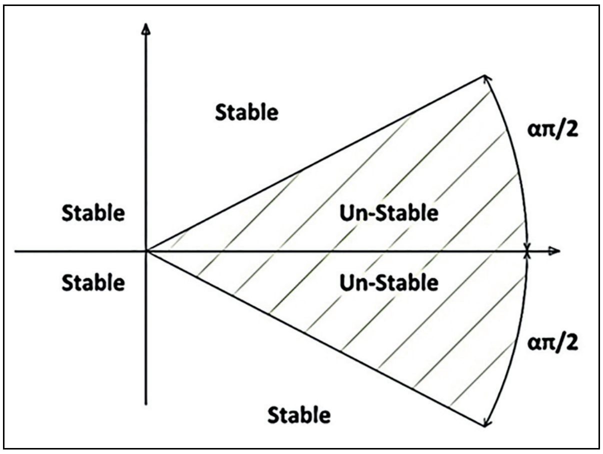

Figure 1. Stability condition in commensurate fractional order linear system.

$$\frac{d^{\alpha}}{dt^{\alpha}}\mathbf{x} = \mathbf{A}\mathbf{x} \tag{10}$$

where $\mathbf{x} \in \Re^{n}$, $\mathbf{A} \in \Re^{n\times n}$, and α is an arbitrary real number between 0 and 2. The autonomous system is asymptotically stable if the following condition is satisfied

$$|\arg(\lambda_i)| > \alpha\frac{\pi}{2} \quad \forall i \tag{11}$$

where λ_i represents the eigenvalues of matrix $\mathbf{A}$. This condition is shown in Figure 1.

Definition 5. A fractional order nonlinear system can be represented in the following state space form

$${}_0^C D_t^{\alpha_i} x_i = f_i(t, x_1, \ldots, x_n) x_i(0) = x_{i0}, i = 1, \ldots, n \tag{12}$$

where $0 < \alpha_i \leqslant 1$ for $i = 1, \ldots, n$.

Problem statement

System description

Consider the following uncertain fractional order nonlinear system

$$\begin{cases} {}_0^C D_t^{\alpha} x_i = x_{i+1}, 1 \leqslant i \leqslant n-1 \\ {}_0^C D_t^{\alpha} x_n = f(t, \mathbf{X}) + \Delta f(t, \mathbf{X}) + d(t) + u(t) \\ y(t) = x_1(t) \end{cases} \tag{13}$$

where α is an arbitrary real number and shows fractional order of the equations, $\mathbf{X} = [x_1 \quad \cdots \quad x_n]^T$ is the measurable state vector of the system, $f(\cdot)$ is a nonlinear function, $\Delta f(\cdot)$ stands for uncertainty, $d(\cdot)$ denotes the external disturbances, and $y(t)$ and $u(t)$ are the output and input of the system, respectively.

The main goal is to design an adaptive FOPID controller for the above-mentioned system such that the output tracks a desired reference signal. The controller gains will be updated through a proper sliding surface. Because of using sliding mode techniques, the controller is robust against the system uncertainties and external disturbances. In addition, a supervisory control will be developed to guarantee boundedness of the tracking error. Figure 2 shows the schematic diagram of the closed-loop control system. Let y_d be the desired output. Therefore, we define $\mathbf{r}(t)$ as a reference signal

$$\mathbf{r}(t) = [r_1(t) \quad \cdots \quad r_n(t)]^T \tag{14}$$

where $r_1 = y_d$ and $r_{i+1} = {}_0^C D_t^{\alpha} r_i$ for $i = 1, \ldots, n-1$ and it is assumed that $r_i{}'s, i = 1, \ldots, n$ are all bounded with known bounds and continuously differentiable.

Define the error vector of the system as

$$\mathbf{E} = [e_1 \quad \cdots \quad e_n]^T \tag{15}$$

where

$$\begin{aligned} e &= e_1 = r_1 - x_1 = r_1 - y \\ e_i &= r_i - x_i, 2 \leqslant i \leqslant n \end{aligned} \tag{16}$$

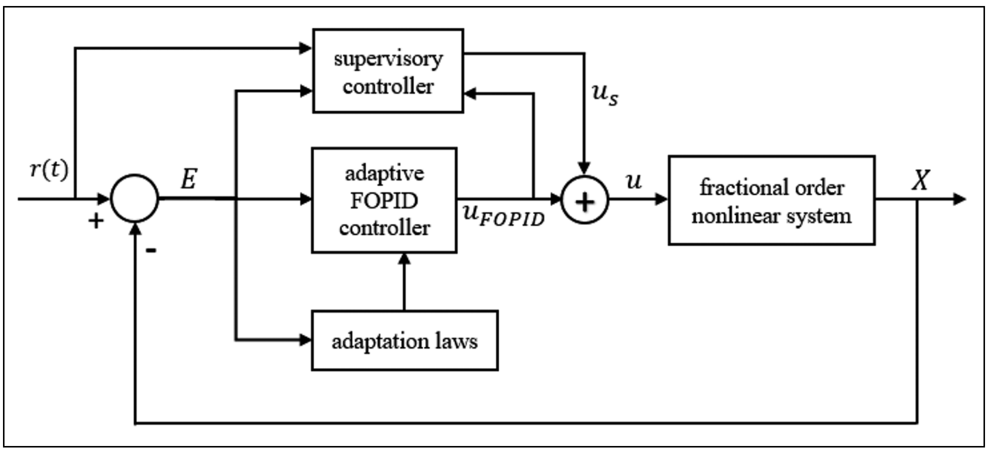

Figure 2. Schematic diagram of the closed-loop system.

From equations (13) and (16), output tracking error dynamics of the closed-loop system can be expressed as

$$\begin{cases} {}_0^C D_t^\alpha e_i = e_{i+1}, 1 \leq i \leq n-1 \\ {}_0^C D_t^\alpha e_n = {}_0^C D_t^\alpha r_n - f(t, \mathbf{X}) - \Delta f(t, \mathbf{X}) - d(t) - u(t) \end{cases} \tag{17}$$

Throughout the article, the following assumptions are made.

Assumption 1. Derivation order of equation (13), α, belongs to $(0, 1)$.

Assumption 2. All x_i's are continuously differentiable. In other words, we assume that $x_i \in C^1[0, \infty]$. This assumption is common especially when a fractional order system is aimed to be controlled.[30,31] Since all x_i's and r_i's are continuously differentiable, all e_i's are continuously differentiable too.

Assumption 3. $f(\cdot)$, $\Delta f(\cdot)$, and $d(\cdot)$ are all unknown functions. But all of them are bounded with known bounds

$$|f(\cdot)| \leq f^u \tag{18}$$

$$|\Delta f(\cdot)| \leq \Delta f^u \tag{19}$$

$$|d(\cdot)| \leq d^u \tag{20}$$

where f^u, Δf^u, and d^u are known positive functions which are the upper bounds of the nominal plant of the system, system modeling uncertainties, and external disturbances, respectively.

Control input

Control input consists of two parts: the FOPID controller and a supervisory controller. Therefore, the control input in equation (13) can be written as follows

$$u(t) = u_{FOPID}(t) + u_s(t) \tag{21}$$

where $u_{FOPID}(t)$ and $u_s(t)$ stand for the FOPID and supervisory controllers, respectively.

The FOPID controller can be expressed as

$$u_{FOPID}(t) = K_P e(t) + K_{I0}\, D_t^{-\lambda} e(t) + K_{D0}{}^C D_t^\mu e(t) \tag{22}$$

where K_P, K_I, and K_D are the proportional, integral, and derivative gains, respectively. For convenience, we define vectors $\boldsymbol{\Theta}$ and $\boldsymbol{\Phi}$ as

$$\boldsymbol{\Theta} = \begin{bmatrix} K_P & K_I & K_D \end{bmatrix}^T \tag{23}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} e & {}_0D_t^{-\lambda} e & {}_0^C D_t^\mu e \end{bmatrix}^T \tag{24}$$

So, FOPID controller can be rewritten in the following form

$$u_{FOPID}(t) = \boldsymbol{\Theta}^T \boldsymbol{\Phi} = \sum_{i=1}^{3} \theta_i \varphi_i \tag{25}$$

where θ_i and φ_i represent the elements of the vectors $\boldsymbol{\Theta}$ and $\boldsymbol{\Phi}$, respectively.

Supervisory controller will be designed to guarantee the stability of the closed-loop system. The overall procedure of the supervisory controller design is presented in the next section.

Stability analysis and supervisory controller design

In this section, we will define a bounded set Ω for tracking error trajectories and the supervisory control will be obtained in a way that Ω becomes an attracting set. For this purpose, the supervisory controller should guarantee the stability condition out of the Ω set. In other words, it should be determined such that the Lyapunov function derivative with respect to time be negative definite out of Ω.

Consider the constraint set Ω for tracking error of the system defined as

$$\Omega = \left\{ \mathbf{E} \in \Re^n : \|\mathbf{E}\|_2 \leq M \right\} \tag{26}$$

where M is a pre-specified parameter. Now, consider the following Lyapunov function candidate

$$V_e = \sum_{i=1}^{n} \left| {}_0D_t^{\alpha-1} e_i \right| \tag{27}$$

The Lyapunov function derivative with respect to time yields

$$\dot{V}_e = \sum_{i=1}^{n} \operatorname{sgn}({}_0D_t^{\alpha-1} e_i) {}_0^C D_t^\alpha e_i \tag{28}$$

Substituting equations (17) and (21) into equation (28), we have

$$\begin{aligned} \dot{V}_e &= \sum_{i=1}^{n-1} \operatorname{sgn}({}_0D_t^{\alpha-1} e_i) e_{i+1} + \operatorname{sgn}({}_0D_t^{\alpha-1} e_n) \\ &\quad \left({}_0^C D_t^\alpha r_n - f(t, \mathbf{X}) - \Delta f(t, \mathbf{X}) - d(t) - u_{FOPID} - u_s \right) \\ &\leq \sum_{i=1}^{n-1} |e_{i+1}| + \operatorname{sgn}({}_0D_t^{\alpha-1} e_n) \\ &\quad \left({}_0^C D_t^\alpha r_n - f(t, \mathbf{X}) - \Delta f(t, \mathbf{X}) - d(t) - u_{FOPID} - u_s \right) \\ &\leq (n-1)\|\mathbf{E}\|_\infty + \operatorname{sgn}({}_0D_t^{\alpha-1} e_n) \\ &\quad \left({}_0^C D_t^\alpha r_n - f(t, \mathbf{X}) - \Delta f(t, \mathbf{X}) - d(t) - u_{FOPID} - u_s \right) \\ &\leq (n-1)\|\mathbf{E}\|_2 + \operatorname{sgn}({}_0D_t^{\alpha-1} e_n) \\ &\quad \left({}_0^C D_t^\alpha r_n - f(t, \mathbf{X}) - \Delta f(t, \mathbf{X}) - d(t) - u_{FOPID} - u_s \right) \end{aligned} \tag{29}$$

Now, we define the supervisory control by

$$u_s = \begin{cases} 0 & \mathbf{E} \in \Omega \\ \operatorname{sgn}({}_0D_t^{\alpha-1}e_n)\big((k+n-1)\|\mathbf{E}\|_2 + \left|{}_0^CD_t^\alpha r_n\right| + f^u + \Delta f^u + d^u + |u_{FOPID}|\big) & \mathbf{E} \notin \Omega \end{cases} \tag{30}$$

where $k > 0$ is an arbitrary positive real number.

By substituting equation (30) into equation (29), we can easily show when the tracking errors trajectories are out of the mentioned constraint set, we have

$$\begin{aligned} \dot{V}_e \leqslant\ & (n-1)\|\mathbf{E}\|_2 + \operatorname{sgn}({}_0D_t^{\alpha-1}e_n) \\ & ({}_0^CD_t^\alpha r_n - f(t,\mathbf{X}) - \Delta f(t,\mathbf{X}) - d(t) - u_{FOPID}) \\ & - (k+n-1)\|\mathbf{E}\|_2 - \left|{}_0^CD_t^\alpha r_n\right| \\ & - |u_{FOPID}| - f^u - \Delta f^u - d^u \\ =\ & -\big(\left|{}_0^CD_t^\alpha r_n\right| - \operatorname{sgn}({}_0D_t^{\alpha-1}e_n){}_0^CD_t^\alpha r_n\big) \\ & - \big(|u_{FOPID}| + \operatorname{sgn}({}_0D_t^{\alpha-1}e_n)u_{FOPID}\big) \\ & - \big(f^u + \operatorname{sgn}({}_0D_t^{\alpha-1}e_n)f(t,\mathbf{X})\big) \\ & - \big(\Delta f^u + \operatorname{sgn}({}_0D_t^{\alpha-1}e_n)\Delta f(t,\mathbf{X})\big) \\ & - \big(d^u + \operatorname{sgn}({}_0D_t^{\alpha-1}e_n)d(t)\big) - k\|\mathbf{E}\|_2 \\ \leqslant\ & -k\|\mathbf{E}\|_2 \end{aligned} \tag{31}$$

Therefore, the supervisory control guarantees the attracting property of the Ω set for the closed-loop system. It means that the supervisory control forces the error to be attracted by the set Ω and hence be bounded.

Adaptation law

FOPID controller gains in equation (25) will be tuned using a proper sliding surface. To design a proper sliding surface, we define a signal, x_r, as follows

$${}_0^CD_t^\alpha x_r = {}_0^CD_t^\alpha r_n + \sum_{i=1}^{n} K_i e_i \tag{32}$$

Now, the sliding surface function can be defined as follows

$$S = x_n - x_r \tag{33}$$

When the sliding condition occurs

$$S = 0 \Rightarrow x_r = x_n \tag{34}$$

Substituting equation (34) into equation (32), we obtain

$${}_0^CD_t^\alpha e_n + \sum_{i=1}^{n} K_i e_i = 0 \tag{35}$$

Equation (35) can be written in state space form as

$${}_0^CD_t^\alpha \mathbf{E} = \mathbf{\Lambda}\mathbf{E} \tag{36}$$

where $\mathbf{\Lambda}$ is the gain matrix and is defined as follows

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -K_1 & -K_2 & -K_3 & \cdots & -K_{n-1} & -K_n \end{bmatrix} \tag{37}$$

where K_i's are chosen such that the eigenvalues of $\mathbf{\Lambda}$ matrix satisfy the asymptotic stability condition in fractional order linear systems. This condition can be expressed by

$$|\arg(\lambda_i)| > \alpha\frac{\pi}{2} \tag{38}$$

where λ_i for $i = 1, \ldots, n$ denotes the eigenvalues of $\mathbf{\Lambda}$ matrix. Therefore, the error asymptotically tends to zero as $t \to \infty$.

Now, consider the following Lyapunov function for sliding mode control

$$V_s = \frac{1}{2}S^2 \tag{39}$$

Differentiating the Lyapunov function with respect to time yields

$$\dot{V}_s = S\dot{S} \tag{40}$$

The sliding condition can be expressed as

$$\dot{V}_s = S\dot{S} < 0 \tag{41}$$

From equations (33) and (41), we conclude

$$\dot{V}_s = S(\dot{x}_n - \dot{x}_r) \tag{42}$$

By fractionalizing the classic derivative into a fractional type, equation (42) can be rewritten in the following form

$$\dot{V}_s = S{}_0^CD_t^{1-\alpha}({}_0^CD_t^\alpha x_n) - S\dot{x}_r \tag{43}$$

From equations (13), (21), and (43), we have

$$\dot{V}_s = S{}_0^CD_t^{1-\alpha}(f + \Delta f + d + u_{FOPID} + u_s) - S\dot{x}_r \tag{44}$$

In order to derive a proper adaptation law, we try to decrease the time derivative of Lyapunov function defined in equation (40) as much as possible. The desired goal is to negative the mentioned function. However, it may not be possible, so decreasing it as much as possible is the main purpose. To this end, the gradient descent method will be used. Using the chain rule, the gradient method can be expressed as

$$\dot{\theta}_i = -\gamma \frac{\partial \dot{V}_s}{\partial \theta_i} = -\gamma \frac{\partial \dot{V}_s}{\partial J} \frac{\partial J}{\partial \theta_i} \tag{45}$$

where J is defined as

$$J = {}_0^C D_t^{1-\alpha} u_{FOPID} \tag{46}$$

Substituting equation (25) into equation (46), we obtain

$$J = {}_0^C D_t^{1-\alpha} \sum_{i=1}^{3} (\theta_i \varphi_i) = \sum_{i=1}^{3} {}_0^C D_t^{1-\alpha} (\theta_i \varphi_i) \tag{47}$$

Using the Leibniz rule for Caputo fractional derivative, J can be rewritten in the following form

$$J = \sum_{i=1}^{3} \sum_{k=0}^{\infty} \left\{ \binom{p}{k} \theta_i^{(k)} {}_0^{RL} D_t^{1-\alpha-k} \varphi_i - \frac{1}{t^{1-\alpha}\Gamma(\alpha)} (\theta_i \varphi_i)|_{t=0} \right\} \tag{48}$$

From equations (45) and (48), one can obtain

$$\dot{\theta}_i = -\gamma S_0^{RL} D_t^{1-\alpha} \varphi_i \tag{49}$$

Adaptation laws (49) can be written as follows

$$\begin{cases} \dot{K}_P = -\gamma S_0^{RL} D_t^{1-\alpha} e \\ \dot{K}_I = -\gamma S_0^{RL} D_t^{1-\alpha} \left({}_0 D_t^{-\lambda} e \right) \\ \dot{K}_D = -\gamma S_0^{RL} D_t^{1-\alpha} \left({}_0^C D_t^{\mu} e \right) \end{cases} \tag{50}$$

In relation (50), ${}_0^{RL} D_t^{1-\alpha} ({}_0^C D_t^{\mu} e_1)$ can be expressed in terms of Caputo derivative

$$\begin{cases} \dot{K}_P = -\gamma S_0^{RL} D_t^{1-\alpha} e \\ \dot{K}_I = -\gamma S_0^{RL} D_t^{1-\alpha} \left({}_0 D_t^{-\lambda} e \right) \\ \dot{K}_D = -\gamma S \left({}_0^C D_t^{1-\alpha} \left({}_0^C D_t^{\mu} e \right) + \frac{{}_0^C D_t^{\mu} e|_{t=0}}{t^{1-\alpha}\Gamma(\alpha)} \right) \end{cases} \tag{51}$$

To have simple and more implementable adaptation laws for K_I and K_D, we choose $\lambda = 2 - \alpha$ and $\mu = \alpha$. Since we have assumed that the tracking error is continuously differentiable, using theorem 3.1 in Li and Deng,[31] we conclude that ${}_0^C D_t^{\mu} e|_{t=0} = 0$. Finally, the adaptation laws can be expressed in the following form

$$\begin{cases} \dot{K}_P = -\gamma S_0^{RL} D_t^{1-\alpha} e \\ \dot{K}_I = -\gamma S \int_0^t e(\tau) d\tau \\ \dot{K}_D = -\gamma S \dot{e} \end{cases} \tag{52}$$

Numerical simulations

In this section, output tracking of a fractional order Duffing–Holmes system is shown using the proposed adaptive FOPID controller. Numerical simulations are presented to illustrate the performance of the method.

The mathematical modeling of a fractional order Duffing–Holmes system is defined by

$$\begin{cases} {}_0^C D_t^{\alpha} x_1 = x_2 \\ {}_0^C D_t^{\alpha} x_2 = x_1(t) - 0.25 x_2(t) - x_1^3(t) + 0.3 \cos(t) \\ \quad + \Delta f(t, \mathbf{X}) + d(t) + u(t) \\ y(t) = x_1(t) \end{cases} \tag{53}$$

In this case, $\Delta f(t, \mathbf{X})$ and $d(t)$ are chosen as $0.1 \sin(t) \sqrt{x_1^2 + x_2^2}$ and $0.1 \sin(t)$, respectively. The order of the equations is considered as $\alpha = 0.8$. Therefore, the system dynamics can be described by the following equations

$$\begin{cases} {}_0^C D_t^{0.8} x_1 = x_2 \\ {}_0^C D_t^{0.8} x_2 = x_1(t) - 0.25 x_2(t) - x_1^3(t) + 0.3 \cos(t) \\ \quad + 0.1 \sin(t) \sqrt{x_1^2 + x_2^2} + 0.1 \sin(t) + u(t) \\ y(t) = x_1(t) \end{cases} \tag{54}$$

From equation (54), one can easily obtain the functions f^u, Δf^u, and d^u

$$f^u = |x_1(t)| + 0.25 |x_2(t)| + |x_1^3(t)| + 0.3 \tag{55}$$

$$\Delta f^u = 0.1 \sqrt{x_1^2 + x_2^2} \tag{56}$$

$$d^u = 0.1 \tag{57}$$

We choose $K = 4$, $K_1 = 36$, and $K_2 = 12$, so that the condition given by equation (37) is satisfied. Integral and derivative order of the FOPID controller and the tuning rate are chosen as $\lambda = 1.2$, $\mu = 0.8$, and $\gamma = 50$, respectively.

The Predict–Evaluate–Correct–Evaluate (PECE) algorithm[32] is used to solve the fractional differential equations with a time step of size 0.01. The initial conditions of the system and the initial values of the FOPID controller gains are set to $X(0) = [1.25 \quad 0.5]^T$ and $\mathbf{\Theta}(0) = [5 \quad 5 \quad 5]^T$, respectively. In the simulations, the control goal is to make output of the system follow the desired signal $y_d = 0.5 \sin(t)$ asymptotically.

Simulation results are shown in Figures 3–6. Figure 3(a) and (b) shows output response of the closed-loop control system. Figure 4 demonstrates the error of the tracking of the states. Figures 5 and 6 show the time histories of the FOPID controller gains and sliding surface S. From Figure 5, we can easily see that after less than 10 s the dynamic trajectories of K_P, K_I, and K_D are on the steady state.

Conclusion

This article has shown a robust adaptive FOPID controller for uncertain fractional order nonlinear systems. Robustness of the closed-loop control system against system uncertainty and external disturbance is considered using fractional order sliding mode control method. The adaptation mechanism is constructed from a proper sliding surface via the gradient descent method. A supervisory control is used to guarantee the

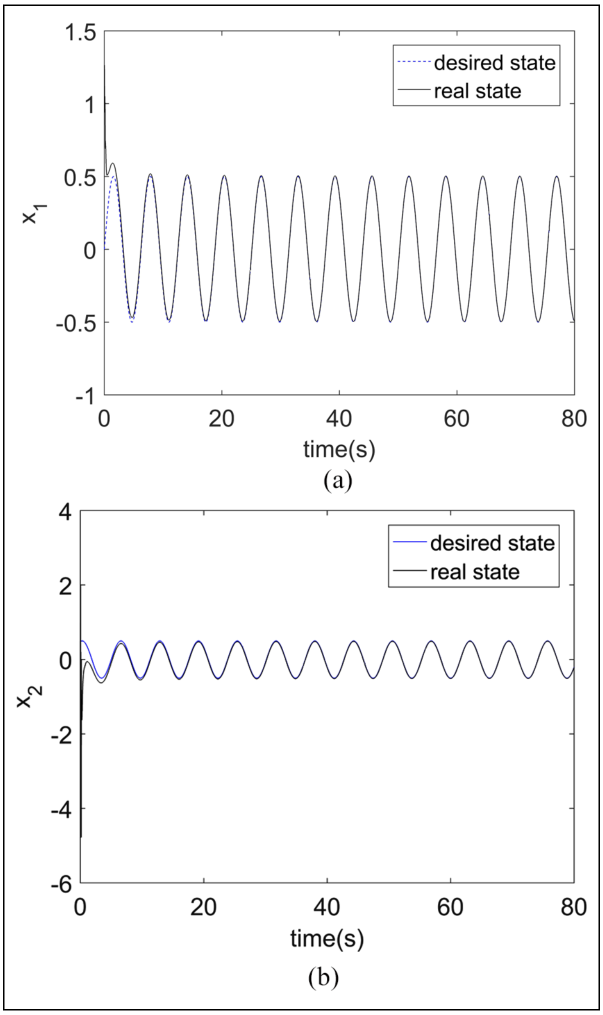

Figure 3. Output response of the closed-loop control system: (a) time history of x_1 and (b) time history of x_2.

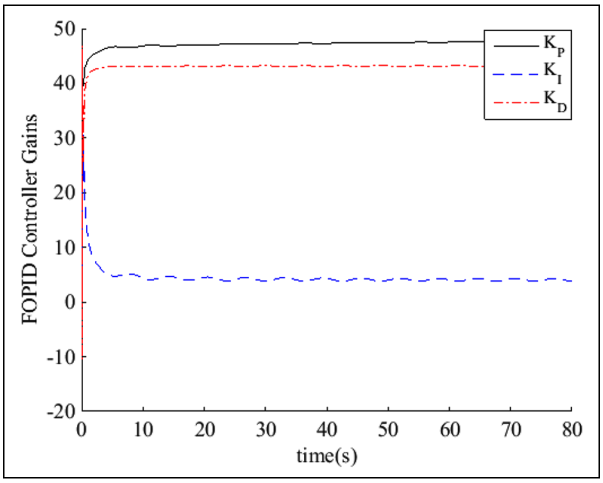

Figure 5. Time histories of the FOPID controller gains.

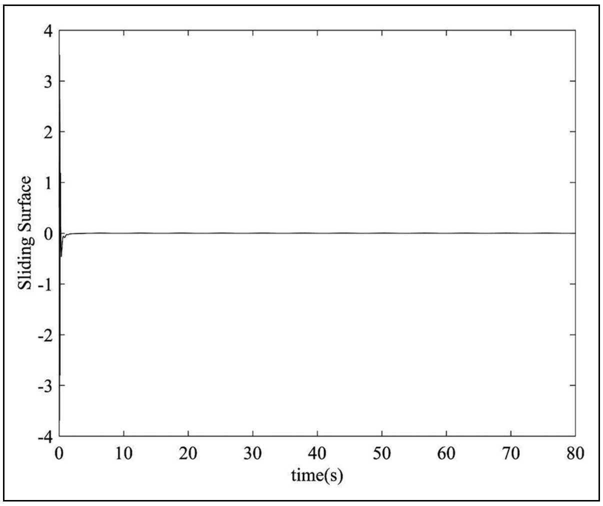

Figure 6. Time history of the sliding surface.

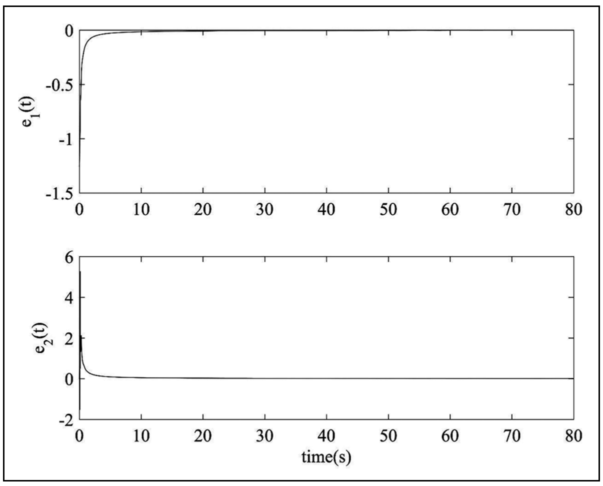

Figure 4. Time history of the tracking error.

boundedness of the system state trajectories. Finally, the proposed method is implemented to control a fractional order Duffing–Holmes system and the simulation results are included to illustrate the great performance of the proposed method.

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