Adaptive Delayed Feedback Control of Uncertain Fractional Order Chaotic Systems Using Sliding Mode Control

Bahram Yaghooti, Kaveh Safavi, Reza Hajiloo, Hassan Salarieh *

Department of Mechanical Engineering, Sharif University of Technology, Tehran, Iran, P.O. Box: 11155-9567, Tel: (+98) 21 66165538, Fax: (+98) 21 66000021

* salarieh@sharif.edu

Abstract

This paper presents an adaptive delayed feedback control method for stabilizing unstable periodic orbits (UPOs) of uncertain fractional order chaotic systems using sliding mode control. The main goal of this research is to develop an adaptive control method based on Lyapunov approach and sliding mode techniques such that the closed loop control system be asymptotically stable on a periodic trajectory which can be sufficiently close to the UPO of the fractional chaotic system. Robustness of the closed loop control system against the system uncertainty and external disturbances is guaranteed. Finally, the proposed method is used to stabilize the UPO of fractional order Duffing system and the numerical simulations show the effectiveness of this method.

Keywords: Fractional order systems; Chaos; Nonlinear delayed feedback; Sliding mode Control; Adaptive control.

1. Introduction

Chaos is a phenomenon that appears in nonlinear dynamical systems and has been observed in many fields of science such as economics [1], chemistry [2], biology [3], engineering [4], and so on. In recent years, control of chaotic systems have attracted many scientists in variety of fields. One of the best known methods for chaos control is stabilizing an unstable periodic orbit (UPO) [5, 6]. The first method for stabilizing the UPO of chaotic systems was introduced in 1990 is named OGY [7]. After that time several extensions of the above-mentioned method were introduced [8, 9]. Linear delayed feedback method proposed for control of chaotic systems by Pyragas. The best-known advantage of this method is that it does not need any information about the periodic solutions other than its period [10]. The delayed feedback control scheme is used for control of both discrete time systems [11, 12] and continuous time systems [13, 14]. Up to this time, several methods have been presented for the control of chaotic systems such as: feedback linearization [15, 16], sliding mode control [17, 18], fuzzy control [19, 20], variable structure control [21], backstepping [22], minimum entropy control [23], and fuzzy minimum entropy control [24].

Fractional calculus is an old field of mathematics that has had no real applications over many years, but recently, it has been extensively used in variety of fields of science and engineering. It has been used to model many systems in several fields e.g. electromagnetics problems [25, 26], viscoelastic damping [27], signal processing [28], vibration [29], diffusion wave [30-32], cyber-physical systems [33], control theory [34, 35] and chaos control [36-38]. In addition, by developing fractional calculus in control theory, many fractional order controllers are introduced like, fractional PID controller [39], fractional PI controller [40], fractional PD controller [41], fractional lead-lag controller [42, 43], fractional CRONE controller [44] and so on.

In recent years, different techniques have been developed to achieve the fractional order chaotic systems control. For example, Lyapunov-based control [45], linear feedback control [46], sliding mode control [17, 47], fuzzy control [48], and intelligent control [49]. In most of the mentioned methods, the exact model of the systems is needed to derive controllers, but in many applications, because of the system uncertainty and external disturbances, mathematical model of the system is not known.

After development of fractional calculus in dynamical systems, it has been shown that chaotic systems in fractional order form can behave like integer order ones. Afterwards, it is demonstrated that UPO can be found in these systems as it can be shown in integer order ones

[50, 51]. This paper presents an adaptive nonlinear delayed feedback control via sliding mode control for stabilizing a periodic orbit with known period, which is near to the UPO of the fractional chaotic system.

The main contribution of the paper can be itemized as:

a) The proposed method can stabilize the UPO of a fractional order chaotic system even when the trajectory of UPO is not known. In this case the period of the UPO should be known.

b) The Pyragas method is one the most well-known approaches which can be applied to stabilize the UPO when only the period of UPO is known. This method is also called delayed feedback control; however, this method has not a systematic procedure especially when a continuous-time chaotic system is aimed to be controlled, so its feedback gains should be determined by trial and error. In this paper a systematic algorithm for delayed feedback control of fractional order continuous-time chaotic systems is proposed. The proposed method is nonlinear and, it is applicable to many nonlinear fractional chaotic systems.

c) The proposed method is a fractional order version of [52]. In this work the uncertainties and disturbances are assumed to have unknown bounds, and so an adaptive method is suggested.

This paper is organized as follows: Section 2 is a review of the preliminary concepts of fractional calculus. In Section 3 the problem statement is presented. In Section 4, an adaptive sliding mode control method is introduced. The parameters are updated using an adaptation mechanism. In Section 5, the proposed scheme is utilized for stabilizing chaotic Duffing system. Simulation results confirm the effectiveness of the presented method. Finally, a brief conclusion is presented in the latest section.

2. Preliminary Concepts

There are several definitions of fractional derivatives. The best known definitions are Riemann-Liouville, Grünwald-Letnikov and Caputo.

Definition 1 [53]**.** The pth order fractional integral of function $f(t)$ is defined by

$$ {}_0I_t^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau \tag{1} $$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 [53]**.** The Riemann-Liouville fractional derivative of order p of any function $f(t)$ is defined as follows

$$ {}_0^{RL}D_t^p f(t) = \frac{d^k}{dt^k} \left({}_0I_t^{(k-p)} f(t) \right) \tag{2} $$

where k is an integer number such that $k-1 \le p < k$.

Definition 3 [53]**.** The Grünwald-Letnikov fractional derivative of order p of any function $f(t)$ is defined as follows

$$ {}_0^{GL}D_t^p f(t) = \lim_{h \to 0} \frac{(\Delta_h^p f)(t)}{h^p} = \lim_{\substack{h \to 0 \\ Nh=t}} \frac{1}{h^p} \sum_{i=0}^{N} (-1)^i \binom{\alpha}{i} f\left(t - ih\right) \tag{3} $$

where

$$ \binom{\alpha}{i} = \frac{\Gamma(p+1)}{i\,!\Gamma(p+1-i)} \tag{4} $$

This definition of fractional derivative leads to Riemann–Liouville definition when we perform limit operation.

Definition 4 [53]**.** The Caputo fractional derivative of order p of any function $f(t)$ is defined as follows

$$ {}_0^{C}D_t^p f(t) = \begin{cases} \dfrac{1}{\Gamma(k-p)} \displaystyle\int_0^t \frac{f^{(k)}(\tau)}{(t-\tau)^{p+1-k}} d\tau & k-1 < p < k \\ \dfrac{d^k}{dt^k} f(t) & p = k \end{cases} \tag{5} $$

Definition 5. A fractional order linear system in state space form is like:

$$ \begin{Bmatrix} {}_0^{C}D_t^{\alpha_1} x_1 \\ {}_0^{C}D_t^{\alpha_2} x_2 \\ \vdots \\ {}_0^{C}D_t^{\alpha_n} x_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \mathbf{A}_{n \times n} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \tag{6} $$

where α_1, …, α_n are arbitrary real numbers. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, the system is called commensurate.

Theorem 1 [54]**.** Consider a commensurate fractional order linear system

$$\frac{d^{\alpha}}{dt^{\alpha}}\mathbf{X} = \mathbf{A}\mathbf{X} \tag{7}$$

where $\mathbf{X} \in \Re^{n}$, $\mathbf{A} \in \Re^{n\times n}$ and α is an arbitrary real number between 0 and 2. The autonomous system is asymptotically stable if the following condition is satisfied:

$$\left|\arg(\lambda_i)\right| > \alpha\frac{\pi}{2} \quad \forall i \tag{8}$$

where λ_i represents the eigenvalues of matrix $\mathbf{A}$.

Definition 5. By using the Caputo derivative, a fractional order nonlinear system is defined as:

$${}_{0}^{C}D_{t}^{\alpha}x = f(t,x) \tag{9}$$

where α is the order of fractional derivatives of the system.

By using the Lyapunov direct method, the asymptotic stability of nonlinear fractional order systems can be studied. The Lyapunov direct method for the fractional-order systems is defined in the following theorem [55].

Theorem 2 [55]**.** Let $x = 0$ be an equilibrium point for the autonomous fractional order system 0(9). Let $V(t, x(t))$ be a continuously differentiable function as a Lyapunov function candidate, and $\gamma_i (i = 1, 2, 3)$ be class-K functions such that

$$\gamma_1\left(\|x\|\right) \le V(t, x(t)) \le \gamma_2\left(\|x\|\right) \tag{10}$$

$${}_{0}^{C}D_{t}^{\beta}V(t, x(t)) \le -\gamma_3\left(\|x\|\right) \tag{11}$$

where $\beta \in (0,1)$. Then the system (9) is asymptotically stable.

Lemma 1. [55] Let $x(t) \in \Re$ be a continuously differentiable function. Then, for all $t \ge t_0$

$$\frac{1}{2}{}_{t_0}^{C}D_t^{\alpha}x^2(t) \le x(t)\,{}_{t_0}^{C}D_t^{\alpha}x(t), \qquad \forall \alpha \in (0,1) \tag{12}$$

Remark 1. [55] In the case when $x(t) \in \Re^n$, Lemma 1 is still valid. That is, $\forall t \ge t_0$

$$\frac{1}{2}{}_{t_0}^{C}D_t^{\alpha}x^T(t)x(t) \le x^T(t)\,{}_{t_0}^{C}D_t^{\alpha}x(t), \qquad \forall \alpha \in (0,1) \tag{13}$$

In [56], Barbalat's Lemma is developed for fractional order nonlinear systems as follows:

Theorem 3 [56]**.** Let $\phi : \Re \to \Re$ be a uniformly continuous function on $[t_0, \infty)$. Assume that there exist two positive constants p and M such that ${}_0I_t^{\alpha}|\phi|^p \le M$ for all $t > t_0 > 0$ with $\alpha \in (0,1)$. Then

$$\lim_{t\to\infty}\phi(t) = 0 \tag{14}$$

3. Problem Statement

Consider the following uncertain fractional order chaotic system

$$\begin{cases} {}_0^{C}D_t^{\alpha}x_i = x_{i+1} \qquad 1 \le i \le n-1 \\ {}_0^{C}D_t^{\alpha}x_n = f(t,\mathbf{X}) + \mathbf{F}^T(t,\mathbf{X})\mathbf{\Theta} + g(t,\mathbf{X})u + d(t) \end{cases} \tag{15}$$

where $0 < \alpha < 1$ demonstrates the fractional order of the equations, $\mathbf{X} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ is measurable state vector of the system, $f(\cdot)$ and $g(\cdot)$ are nonlinear functions and belong to $\mathrm{C}^1\left(\Re^n \times \Re \to \Re\right)$, and it is assumed that $g(\cdot) \ne 0$, and $\mathbf{\Theta} = \begin{bmatrix} \theta_1 & \cdots & \theta_m \end{bmatrix}^T \in \Re^m$ stands for the uncertain parameter vector of the system. $\mathbf{F}(\cdot)$ is an $m \times 1$ vector, $d(\cdot)$ is an external disturbance, and it is assumed that $|d(\cdot)| \le k < \infty$ for all $t > 0$ and the value of $k > 0$ is unknown. $u(t) \in \Re$ is the control input and the system shows chaotic response for $u = 0$. All functions are assumed to be sufficiently smooth and Lipchitz. $f\left(t,\mathbf{X}\right) + \mathbf{F}^T\left(t,\mathbf{X}\right)\mathbf{\Theta}$ function is t-periodic with period *T*, and also the system (15) without control and disturbance has an unstable periodic solution with period T. The main goal is to design a control law u, such that the chaotic behavior of the system (15) changes to a periodic one which is near to an unstable periodic orbit of the system.

4. Controller Design

In the mentioned Pyragas method, usually we use a linear delayed feedback control to stabilize the UPO of the system. Here again the concept of the Pyragas technique is used, and it is applied to our problem.

Defining a delayed state as $\tilde{\mathbf{X}} = \mathbf{X}(t-T)$, and substituting it for $\mathbf{X}$ in Eq. (15), results in:

$$\begin{cases} {}_0^C D_t^\alpha \tilde{x}_i = \tilde{x}_{i+1} \qquad 1 \le i \le n-1 \\ {}_0^C D_t^\alpha \tilde{x}_n = f(t-T, \tilde{\mathbf{X}}) + \mathbf{F}^T(t-T, \tilde{\mathbf{X}})\tilde{\boldsymbol{\Theta}} + \tilde{d} + g(t-T, \tilde{\mathbf{X}})\tilde{u} \end{cases} \tag{16}$$

where $\tilde{u} = u\left(t-T\right)$, and $\tilde{\boldsymbol{\Theta}} = \boldsymbol{\Theta}(t-T)$. We subtract Eq. (16) from Eq. (15) to obtain following equations:

$$\begin{cases} {}_0^C D_t^\alpha x_i - {}_0^C D_t^\alpha \tilde{x}_i = x_{i+1} - \tilde{x}_{i+1} \qquad 1 \le i \le n-1 \\ {}_0^C D_t^\alpha x_n - {}_0^C D_t^\alpha \tilde{x}_n = f(t, \mathbf{X}) - f(t-T, \tilde{\mathbf{X}}) + \mathbf{F}^T(t, \mathbf{X})\boldsymbol{\Theta} - \mathbf{F}^T(t-T, \tilde{\mathbf{X}})\tilde{\boldsymbol{\Theta}} \\ \qquad\qquad + d - \tilde{d} + g(t, \mathbf{X})u - g(t-T, \tilde{\mathbf{X}})\tilde{u} \end{cases} \tag{17}$$

By defining $\mathbf{E} = \mathbf{X} - \tilde{\mathbf{X}}$, Eq. (17) can be re-written as (error dynamics):

$$\begin{cases} {}_0^C D_t^\alpha e_i = e_{i+1} \qquad 1 \le i \le n-1 \\ {}_0^C D_t^\alpha e_n = f(t, \mathbf{X}) - f(t-T, \tilde{\mathbf{X}}) + \mathbf{F}^T(t, \mathbf{X})\boldsymbol{\Theta} - \mathbf{F}^T(t-T, \tilde{\mathbf{X}})\tilde{\boldsymbol{\Theta}} \\ \qquad + d - \tilde{d} + g(t, \mathbf{X})u - g(t-T, \tilde{\mathbf{X}})\tilde{u} \end{cases} \tag{18}$$

To stabilize a periodic orbit of the system, we should obtain the control law u such that the following conditions are satisfied:

$$\lim_{t\to\infty} \|\mathbf{E}\| = 0 \;\equiv\; \lim_{t\to\infty} \left\|\mathbf{X}(t) - \tilde{\mathbf{X}}(t)\right\| = 0 \;\equiv\; \lim_{t\to\infty} \|\mathbf{X}(t) - \mathbf{X}(t-T)\| = 0 \tag{19}$$

Theorem 4. Assume $\tilde{\mathbf{X}}=\mathbf{X}(t-T)$ and $\tilde{u}(t)=u\left(t-T\right)$, where T is the period of the unstable periodic orbit of the chaotic system. If the following control and adaptation laws are applied to system defined by (15), the chaotic behavior of the system is substituted by a regular periodic one.

$$u=u_{\mathrm{eq}}+u_{\mathrm{ad}}+u_{s} \tag{20}$$

$$u_{\mathrm{eq}}=-\frac{\sum_{i=1}^{n}\alpha_i\, e_i+f(t,\mathbf{X})-f(t-T,\tilde{\mathbf{X}})-g(t-T,\tilde{\mathbf{X}})\tilde{u}}{g(t,\mathbf{X})} \tag{21}$$

$$u_{\mathrm{ad}}=-\frac{\left(\mathbf{F}^{T}(t,\mathbf{X})\hat{\mathbf{\Theta}}\left(t\right)-\mathbf{F}^{T}(t-T,\tilde{\mathbf{X}})\,\hat{\mathbf{\Theta}}\left(t-T\right)\right)+2\hat{k}\,\mathrm{sign}\left(S\right)}{g(t,\mathbf{X})} \tag{22}$$

$$u_{s}=\frac{-\left(M+\eta\right)\mathrm{sign}\left(S\right)}{g\left(t,\mathbf{X}\right)} \tag{23}$$

$${}_{0}^{C}D_{t}^{\alpha}\hat{\theta}_{i}=\gamma_{i}S\left(F_{i}(t,\mathbf{X})-F_{i}(t-T,\tilde{\mathbf{X}})\right) \tag{24}$$

$${}_{0}^{C}D_{t}^{\alpha}\hat{k}=2\gamma_{k}\left|S\right| \tag{25}$$

where $\hat{\theta}_i$ and $\hat{k}$ are estimates of θ_i and k, η is an arbitrary positive constant, γ_i and γ_{k} are positive constants and M is a sufficiently large positive constant. In the above equations, S is a sliding surface defined by:

$$S=e_{n}+\sum_{i=1}^{n}\alpha_{i}\,{}_{0}I_{t}^{\alpha}e_{i} \tag{26}$$

in which $\alpha_i>0$'s are constants and selected such that an asymptotically stable dynamics for the sliding surface, $S=0$ is provided.

A simplifying condition, which is very common in controlling fractional order systems, is assumed that all of the system state variables as well as the sliding surface are continuously differentiable and can be measured. This assumption is common especially when a fractional order system is aimed to be controlled [57, 58].

Remark 2. For the existence of sliding mode, it is necessary that the following manifold be asymptotically stable.

$$S = e_n + \sum_{i=1}^{n} \alpha_i \, {}_0I_t^{\alpha} e_i \tag{27}$$

By applying the time derivative of α 's order to Eq. (27), we have:

$${}_0^C D_t^{\alpha} S = {}_0^C D_t^{\alpha} e_n + \sum_{i=1}^{n} \alpha_i e_i \tag{28}$$

Therefore, the sliding mode dynamics can be obtained as follows

$${}_0^C D_t^{\alpha} e_n = -\sum_{i=1}^{n} \alpha_i e_i \tag{29}$$

where $\alpha_1, ..., \alpha_n$ are chosen such that the roots of equation $s^{n\alpha} + \sum_{i=1}^{n} \alpha_i s^{(i-1)\alpha} = 0$ satisfy the asymptotic stability condition in fractional order linear systems. Using Eq. (8), this condition can be expressed by

$$\left|\arg(s_i)\right| > \alpha \frac{\pi}{2} \tag{30}$$

where s_i for $i = 1, ..., n$ denotes the roots of the above-mentioned equation.

Proof of Theorem 4. Consider the following Lyapunov function candidate

$$V = \frac{1}{2} S^2 + \frac{1}{2} \sum_{i=1}^{m} \frac{1}{\gamma_i} (\theta_i - \hat{\theta}_i)^2 + \frac{1}{2} \frac{1}{\gamma_k} (k - \hat{k})^2 \tag{31}$$

Applying the time derivative of α 's order to the Lyapunov function (31) along with using Lemma 1 results in:

$$
\begin{aligned}
{}_0^C D_t^\alpha V &= \frac{1}{2}\,{}_0^C D_t^\alpha S^2 + \frac{1}{2}\sum_{i=1}^{m}\frac{1}{\gamma_i}\,{}_0^C D_t^\alpha (\theta_i - \hat{\theta}_i)^2 + \frac{1}{2}\frac{1}{\gamma_k}\,{}_0^C D_t^\alpha (k - \hat{k})^2 \\
&\le S\,{}_0^C D_t^\alpha S + \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i - \hat{\theta}_i)\,{}_0^C D_t^\alpha (\theta_i - \hat{\theta}_i) + \frac{1}{\gamma_k}(k - \hat{k})\,{}_0^C D_t^\alpha (k - \hat{k}) \\
&= S\,{}_0^C D_t^\alpha S - \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i - \hat{\theta}_i)\,{}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}(k - \hat{k})\,{}_0^C D_t^\alpha \hat{k}
\end{aligned} \tag{32}
$$

Substituting for ${}_0^C D_t^\alpha S$ by Eq. (28) gives:

$$
{}_0^C D_t^\alpha V \le S\left({}_0^C D_t^\alpha e_n + \sum_{i=1}^{n}\alpha_i e_i\right) - \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i - \hat{\theta}_i)\,{}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}(k - \hat{k})\,{}_0^C D_t^\alpha \hat{k} \tag{33}
$$

Substituting Eq. (18) into Eq. (33) yields:

$$
\begin{aligned}
{}_0^C D_t^\alpha V \le S &\left(\begin{aligned} &\sum_{i=1}^{n}\alpha_i e_i + f(t,\mathbf{X}) - f(t-T,\tilde{\mathbf{X}}) + \mathbf{F}^T(t,\mathbf{X})\boldsymbol{\Theta} - \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\tilde{\boldsymbol{\Theta}} \\ &+ d - \tilde{d} + g(t,\mathbf{X})u - g(t-T,\tilde{\mathbf{X}})\tilde{u} \end{aligned}\right) \\
&- \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i - \hat{\theta}_i)\,{}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}(k - \hat{k})\,{}_0^C D_t^\alpha \hat{k}
\end{aligned} \tag{34}
$$

It was assumed that $\left|d\left(t\right)\right| \le k$, therefore $\left|d - \tilde{d}\right| \le 2\mathrm{k}$ and consequently:

$$
\begin{aligned}
{}_0^C D_t^\alpha V \le S &\left(\begin{aligned} &\sum_{i=1}^{n}\alpha_i e_i + f(t,\mathbf{X}) - f(t-T,\tilde{\mathbf{X}}) + \mathbf{F}^T(t,\mathbf{X})\boldsymbol{\Theta} - \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\tilde{\boldsymbol{\Theta}} \\ &+ g(t,\mathbf{X})u - g(t-T,\tilde{\mathbf{X}})\tilde{u} \end{aligned}\right) \\
&+ 2k\left|S\right| - \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i - \hat{\theta}_i)\,{}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}(k - \hat{k})\,{}_0^C D_t^\alpha \hat{k}
\end{aligned} \tag{35}
$$

Using Eqs. (20)-(22) in inequality (35) yields:

$$
\begin{aligned}
{}_0^C D_t^\alpha V \le S &\left(\begin{aligned} &\mathbf{F}^T(t,\mathbf{X})\boldsymbol{\Theta} - \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\tilde{\boldsymbol{\Theta}} - \mathbf{F}^T(t,\mathbf{X})\hat{\boldsymbol{\Theta}}(t) \\ &+ \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\hat{\boldsymbol{\Theta}}\left(t\right) + g(t,\mathbf{X})u_s \end{aligned}\right) \\
&+ 2k\left|S\right| - 2\hat{k}\left|S\right| - \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i - \hat{\theta}_i)\,{}_0^C D_t^\alpha \hat{\theta}_i - \frac{1}{\gamma_k}(k - \hat{k})\,{}_0^C D_t^\alpha \hat{k}
\end{aligned} \tag{36}
$$

Now, we add the term $S\left(-\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\boldsymbol{\Theta} + \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\boldsymbol{\Theta}\right)$ to the right side of inequality (36).

$$
{}_{0}^{C}D_t^{\alpha}V \le S\begin{pmatrix} \mathbf{F}^T(t,\mathbf{X})\mathbf{\Theta} - \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\mathbf{\Theta} + \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\mathbf{\Theta} - \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\tilde{\mathbf{\Theta}} \\ -\mathbf{F}^T(t,\mathbf{X})\hat{\mathbf{\Theta}}(t) + \mathbf{F}^T(t-T,\tilde{\mathbf{X}})\hat{\mathbf{\Theta}}(t) + g(t,\mathbf{X})u_s \end{pmatrix}
$$

$$
+2|S|\left(k-\hat{k}\right) - \sum_{i=1}^{m}\frac{1}{\gamma_i}(\theta_i-\hat{\theta}_i)\,{}_{0}^{C}D_t^{\alpha}\hat{\theta}_i - \frac{1}{\gamma_k}(k-\hat{k})\,{}_{0}^{C}D_t^{\alpha}\hat{k} \tag{37}
$$

After some mathematical manipulations inequality (37) can be written as:

$$
{}_{0}^{C}D_t^{\alpha}V \le S\left(\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\left(\mathbf{\Theta}-\tilde{\mathbf{\Theta}}\right) + g(t,\mathbf{X})u_s\right) + (k-\hat{k})\left[2|S| - \frac{1}{\gamma_k}\,{}_{0}^{C}D_t^{\alpha}\hat{k}\right]
$$

$$
+\sum_{i=1}^{m}\left((\theta_i-\hat{\theta}_i)\left[S\left(F_i(t,\mathbf{X}) - F_i(t-T,\tilde{\mathbf{X}})\right) - \frac{1}{\gamma_i}\,{}_{0}^{C}D_t^{\alpha}\hat{\theta}_i\right]\right) \tag{38}
$$

Using Eqs. (24) and (25) in inequality (38) yields:

$$
{}_{0}^{C}D_t^{\alpha}V \le S\left(\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\left(\mathbf{\Theta}(t)-\mathbf{\Theta}(t-T)\right) + g(t,\mathbf{X})u_s\right) \tag{39}
$$

Since $\mathbf{\Theta}(\cdot)$ is bounded, one can conclude that $\mathbf{F}^T(t-T,\tilde{\mathbf{X}})\left(\mathbf{\Theta}(t)-\mathbf{\Theta}(t-T)\right)$ is also bounded. Using Eq. (23) in inequality (39) results in:

$$
{}_{0}^{C}D_t^{\alpha}V \le -\eta|S| \tag{40}
$$

Integrating both sides of Eq. (40), we have

$$
\begin{aligned}
&{}_{0}I_t^{\alpha}\,{}_{0}^{C}D_t^{\alpha}V = V(t) - V(0) \le -{}_{0}I_t^{\alpha}\left(\eta|S|\right) = -\eta\,{}_{0}I_t^{\alpha}\left(|S|\right) \Rightarrow \\
&V(t) + \eta\,{}_{0}I_t^{\alpha}\left(|S|\right) \le V(0)
\end{aligned} \tag{41}
$$

One can obtain the following equation from (41)

$$
\eta\,{}_{0}I_t^{\alpha}\left(|S|\right) \le V(0) \Rightarrow {}_{0}I_t^{\alpha}\left(|S|\right) \le \frac{V(0)}{\eta} \tag{42}
$$

Since we assume that the sliding surface is continuously differentiable, it is a uniformly continuous function. Therefore, Using Theorem 3, Inequality (42) indicates that the sliding surface becomes zero as time approaches to infinity. Also, due to asymptotic stability of the origin, the error trajectory converges to zero. Thus, the condition of Eq. (19) is satisfied and the control objective is achieved. So the Theorem 4 has been proved completely.

Remark 3. As a special case, if $\mathbf{\Theta}(t)=\mathbf{\Theta}(t-T)$ or $\mathbf{\Theta}$ is a constant, then u_s can be selected as:

$$u_s = \frac{-\eta \operatorname{sign}(S)}{g(t,\mathbf{X})} \tag{43}$$

Proof. If $\mathbf{\Theta}(t)=\mathbf{\Theta}(t-T)$, then $\mathbf{\Theta}(t)-\mathbf{\Theta}(t-T)=0$, so Eq. (39) will be simplified as follows:

$${}_0^C D_t^{\alpha} V \le S g(t,\mathbf{X}) u_s \tag{44}$$

By substituting Eq. (43) into Eq. (44), one can obtain Eq. (40).

Remark 4. Due to use of the sign function in equations of u_{ad} and u_s, chattering in implementation of control law can occur. To avoid this problem, the sign function can be replaced by saturation as follows [59]:

$$\operatorname{sat}\left(\frac{S}{\phi}\right) = \begin{cases} \dfrac{S}{\phi} & \left|\dfrac{S}{\phi}\right| < 1 \\ \operatorname{sign}\left(\dfrac{S}{\phi}\right) & \left|\dfrac{S}{\phi}\right| \ge 1 \end{cases} \tag{45}$$

where ϕ is a small positive number.

5. Simulation Results

The fractional order chaotic Duffing system with the following equation is used for simulation:

$$\begin{cases} {}_0^C D_t^{0.98} x_1 = x_2 \\ {}_0^C D_t^{0.98} x_2 = -\theta_1 x_1 - \theta_2 x_1^3 - \theta_3 x_2 + \theta_4 \cos(\omega t) + d(t) + \left(1 + x_1^2\right) u \end{cases} \tag{46}$$

where $d(t)$ is the external disturbance and u is the control action. For $u=0$ and $d(t)=0$, the fractional order Duffing equation is obtained. By setting $\theta_1=-1$, $\theta_2=1$, $\theta_3=0.15$, $\omega=1$, and $\theta_4 = 0.3$, the fractional order Duffing equation shows chaotic behavior. Moreover, existence of unstable periodic orbit in fractional order Duffing system is proved [50]. One of the UPOs of the above-mentioned system is shown in Figure 1.

We consider $f(t,\mathbf{X}) = 0$ and $g(t,\mathbf{X}) = (1 + x_1^2)$. Furthermore, $\mathbf{F}(t,\mathbf{X})$ and $\mathbf{\Theta}$ are defined as:

$$\mathbf{F}(t,\mathbf{X}) = \begin{bmatrix} -x_1 & -x_1^3 & -x_2 & \cos(\omega t) \end{bmatrix}^T \tag{47}$$

$$\mathbf{\Theta} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \end{bmatrix}^{\mathrm{T}} \tag{48}$$

Therefore, Eq. (46) can be rewritten as follows

$$\begin{cases} {}_0^C D_t^{0.98} x_1 = x_2 \\ {}_0^C D_t^{0.98} x_2 = f(t,\mathbf{X}) + \mathbf{F}^T(t,\mathbf{X})\mathbf{\Theta} + d(t) + g(t,\mathbf{X})\, u \end{cases} \tag{49}$$

The disturbance term is assumed to be $d(t) = 0.2\cos(2t)$ which is bounded by $|d(t)| \le k = 0.2$. The parameter k is assumed to be unknown that should be updated by adaptation law (25). The initial conditions are set to $\hat{\mathbf{\Theta}}(0) = \begin{bmatrix} -1.5 & 1.5 & 0.2 & 0.5 \end{bmatrix}^{\mathrm{T}}$, $\mathbf{X}(0) = \begin{bmatrix} 0.15 & 0.1 \end{bmatrix}^T$, $\hat{k}(0) = 0.1$, and parameters γ_k and γ are selected as $\gamma_k = 1$ and $\gamma = \begin{bmatrix} 5 & 5 & 5 & 5 \end{bmatrix}^T$.

The periodic solution with period $T = 2\pi$ is considered for stabilization, and the simulation results are shown in Figures 2-4. Figure 2 shows time history of the state variables, Figure 3 represents phase plane of the closed loop control system, and Figure 4 demonstrates time history of the control input and sliding surface. The dashed lines in Figure 2 show the main UPO of the system. As it is observed, the chaotic behavior is substituted by to a periodic orbit close to the UPO of the main system. Note that the controller is applied at $t = 4T$. Due to the system uncertainties, external disturbances, and unknown parameters, the main UPO may not be stabilized and the controller action has not converged completely to zero, and the system trajectories converge to a close vicinity of the main UPO. The closer we get to the UPO of the system, the lower control signal will be required because when the trajectory of a chaotic system reaches to the UPO of the system exactly, then the trajectory will remain on the UPO with zero control signal.

6. Conclusion

This paper has shown a robust adaptive nonlinear delayed feedback control for a class of uncertain fractional order nonlinear systems. Robustness of the closed loop control system against the system uncertainty and external disturbances is guaranteed by using fractional order sliding mode control method. The control input and adaptation mechanism is constructed from a proper sliding surface via Lyapunov method. The most influential advantage of the proposed method is that it just needs period T to derive the controller structure. Because of the system uncertainty and external disturbances, the stabilized orbit is not exactly the UPO of the system. However, it has been tried to stabilize an orbit that is very close to the UPO of the system. Finally, the proposed method is implemented to control a fractional order Duffing system and simulation results are included to illustrate the great performance of the proposed method.

References

[1] Peters EE. Fractal market analysis: applying chaos theory to investment and economics: John Wiley & Sons; 1994.

[2] Rössler OE. Chaos and chemistry. Nonlinear Phenomena in Chemical Dynamics: Springer; 1981. p. 79-87.

[3] Rapp PE. Chaos in Biology: Chaos in the neurosciences: cautionary tales from the frontier. BIOLOGIST-INSTITUTE OF BIOLOGY. 1993;40:89-.

[4] Sun H, Zhang Y, Baleanu D, Chen W, Chen Y. A new collection of real world applications of fractional calculus in science and engineering. Communications in Nonlinear Science and Numerical Simulation. 2018.

[5] Kawai Y, Tsubone T. Chaos control based on stability transformation method for unstable periodic orbits. Nonlinear Theory and Its Applications, IEICE. 2012;3:246-56.

[6] Bielawski S, Derozier D, Glorieux P. Controlling unstable periodic orbits by a delayed continuous feedback. Physical Review E. 1994;49:R971.

[7] OTT E, GREBOGI C, YORKE J. Controlling Chaos. Physical review letters. 1990;64:1196-9.

[8] Mehta N, Henderson R. Controlling chaos to generate aperiodic orbits. Physical review A. 1991;44:4861-5.

[9] Rega G, Lenci S, Thompson J. Controlling chaos: the OGY method, its use in mechanics, and an alternative unified framework for control of non-regular dynamics. Nonlinear Dynamics and Chaos: Advances and Perspectives: Springer; 2010. p. 211-69.

[10] PYRAGAS K. Continuous control of chaos by self-controlling feedback. Physics letters A. 1992;170:421-8.

[11] MORGüL Ö. On the stability of delayed feedback controllers for discrete time systems. Physics letters A. 2005;335:31-42.

[12] Liu D, Yan G. Stabilization of discrete-time chaotic systems via improved periodic delayed feedback control based on polynomial matrix right coprime factorization. Nonlinear Dynamics. 2013;74:1243-52.

[13] Nakajima H. On analytical properties of delayed feedback control of chaos. Physics Letters A. 1997;232:207-10.

[14] El-Dessoky M, Yassen M, Aly E. Bifurcation analysis and chaos control in Shimizu–Morioka chaotic system with delayed feedback. Applied Mathematics and Computation. 2014;243:283-97.

[15] Alasty A, Salarieh H. Nonlinear feedback control of chaotic pendulum in presence of saturation effect. Chaos, Solitons and Fractals. 2007;31:292-304.

[16] Fuh C-C, Tsai H-H, Yao W-H. Combining a feedback linearization controller with a disturbance observer to control a chaotic system under external excitation. Communications in Nonlinear Science and Numerical Simulation. 2012;17:1423-9.

[17] Yin C, Dadras S, Zhong S-m, Chen Y. Control of a novel class of fractional-order chaotic systems via adaptive sliding mode control approach. Applied Mathematical Modelling. 2013;37:2469-83.

[18] Li J, Li W, Li Q. Sliding mode control for uncertain chaotic systems with input nonlinearity. Communications in Nonlinear Science and Numerical Simulation. 2012;17:341-8.
[19] Alasty A, Salarieh H. Controlling the chaos using fuzzy estimation of OGY and Pyragas controllers. Chaos, Solitons & Fractals. 2005;26:379-92.
[20] Wu Z-G, Shi P, Su H, Chu J. Sampled-Data Fuzzy Control of Chaotic Systems Based on a T–S Fuzzy Model. Fuzzy Systems, IEEE Transactions on. 2014;22:153-63.
[21] Lin C-J, Yang S-K, Yau H-T. Chaos suppression control of a coronary artery system with uncertainties by using variable structure control. Computers & Mathematics with Applications. 2012;64:988-95.
[22] Li S-Y, Yang C-H, Lin C-T, Ko L-W, Chiu T-T. Adaptive synchronization of chaotic systems with unknown parameters via new backstepping strategy. Nonlinear Dynamics. 2012;70:2129-43.
[23] Salarieh H, Alasty A. Stabilizing unstable fixed points of chaotic maps via minimum entropy control. Chaos, Solitons & Fractals. 2008;37:763-9.
[24] Sadeghian H, Merat K, Salarieh H, Alasty A. On the fuzzy minimum entropy control to stabilize the unstable fixed points of chaotic maps. Applied Mathematical Modelling. 2011;35:1016-23.
[25] Engheia N. On the role of fractional calculus in electromagnetic theory. Antennas and Propagation Magazine, IEEE. 1997;39:35-46.
[26] Engheta N. On fractional calculus and fractional multipoles in electromagnetism. Antennas and Propagation, IEEE Transactions on. 1996;44:554-66.
[27] Bagley RL, Calico R. Fractional order state equations for the control of viscoelasticallydamped structures. Journal of Guidance, Control, and Dynamics. 1991;14:304-11.
[28] Cruz–Duarte JM, Rosales–Garcia J, Correa–Cely CR, Garcia–Perez A, Avina–Cervantes JG. A closed form expression for the Gaussian–based Caputo–Fabrizio fractional derivative for signal processing applications. Communications in Nonlinear Science and Numerical Simulation. 2018;61:138-48.
[29] Oustaloup M, Moreau X, Nouillant M. From fractal robustness to non-integer approach in vibration insulation: the CRONE suspension. Decision and Control, 1997, Proceedings of the 36th IEEE Conference on: IEEE; 1997. p. 4979-84.
[30] Agrawal OP. Solution for a fractional diffusion-wave equation defined in a bounded domain. Nonlinear Dynamics. 2002;29:145-55.

[31] El-Sayed AM. Fractional-order diffusion-wave equation. International Journal of Theoretical Physics. 1996;35:311-22.

[32] Mainardi F. Fractional relaxation-oscillation and fractional diffusion-wave phenomena. Chaos, Solitons & Fractals. 1996;7:1461-77.

[33] Bogdan P, Jain S, Goyal K, Marculescu R. Implantable pacemakers control and optimization via fractional calculus approaches: A cyber-physical systems perspective. Cyber-Physical Systems (ICCPS), 2012 IEEE/ACM Third International Conference on: IEEE; 2012. p. 23-32.

[34] Sadeghian H, Salarieh H, Alasty A, Meghdari A. Controllability of linear fractional stochastic systems. Scientia Iranica Transaction B, Mechanical Engineering. 2015;22:264.

[35] Abedini M, Nojoumian MA, Salarieh H, Meghdari A. Model reference adaptive control in fractional order systems using discrete-time approximation methods. Communications in Nonlinear Science and Numerical Simulation. 2015;25:27-40.

[36] Radwan A, Moaddy K, Salama K, Momani S, Hashim I. Control and switching synchronization of fractional order chaotic systems using active control technique. Journal of advanced research. 2014;5:125-32.

[37] Mathiyalagan K, Park JH, Sakthivel R. Exponential synchronization for fractional-order chaotic systems with mixed uncertainties. Complexity. 2014.

[38] Sadeghian H, Salarieh H, Alasty A, Meghdari A. On the control of chaos via fractional delayed feedback method. Computers & Mathematics with Applications. 2011;62:1482-91.

[39] Podlubny I. Fractional-order systems and PI/sup/spl lambda//D/sup/spl mu//-controllers. Automatic Control, IEEE Transactions on. 1999;44:208-14.

[40] Luo Y, Chen YQ, Wang CY, Pi YG. Tuning fractional order proportional integral controllers for fractional order systems. Journal of Process Control. 2010;20:823-31.

[41] Luo Y, Chen Y. Fractional order [proportional derivative] controller for a class of fractional order systems. Automatica. 2009;45:2446-50.

[42] Monje CA, Calderón AJ, Vinagre BM, Feliu V. The fractional order lead compensator. Computational Cybernetics, 2004 ICCC 2004 Second IEEE International Conference on: IEEE; 2004. p. 347-52.

[43] Monje CA, Vinagre BM, Calderon AJ, Feliu V, Chen Y. Auto-tuning of fractional lead-lag compensators. World Congress2005. p. 452-.

[44] Oustaloup A, Sabatier J, Lanusse P. From fractal robustness to CRONE control. Fract Calc Appl Anal. 1999;2:1-30.

[45] Aghababa MP. A Lyapunov-based control scheme for robust stabilization of fractional chaotic systems. Nonlinear Dynamics. 2014;78:2129-40.

[46] Hao J, Xiong X, Bin H, Sun N. Controlling chaos for fractional order loss type of coupled dynamos systems via feedback. International Journal of Bifurcation and Chaos. 2015;25:1550111.
[47] Chen D-y, Liu Y-x, Ma X-y, Zhang R-f. Control of a class of fractional-order chaotic systems via sliding mode. Nonlinear Dynamics. 2012;67:893-901.
[48] Wang B, Xue J, Chen D. Takagi-Sugeno fuzzy control for a wide class of fractional-order chaotic systems with uncertain parameters via linear matrix inequality. Journal of Vibration and Control. 2016;22:2356-69.
[49] Lin T-C, Balas VE, Kuo C-H. Fractional order chaotic system tracking design based on adaptive hybrid intelligent control. Fuzzy Systems (FUZZ), 2011 IEEE International Conference on: IEEE; 2011. p. 2890-6.
[50] Rahimi MA, Salarieh H, Alasty A. Stabilizing periodic orbits of fractional order chaotic systems via linear feedback theory. Applied Mathematical Modelling. 2012;36:863-77.
[51] Yazdani M, Salarieh H. On the existence of periodic solutions in time-invariant fractional order systems. Automatica. 2011;47:1834-7.
[52] Salarieh H, Alasty A. Chaos control in uncertain dynamical systems using nonlinear delayed feedback. Chaos, Solitons & Fractals. 2009;41:67-71.
[53] Podlubny I. Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications: Academic Press, San Diego - New York - London; 1999.
[54] Matignon D. Stability results for fractional differential equations with applications to control processing. Computational engineering in systems applications: IMACS, IEEE-SMC Lille, France; 1996. p. 963-8.
[55] Li Y, Chen Y, Podlubny I. Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability. Computers & Mathematics with Applications. 2010;59:1810-21.
[56] Zhang R, Liu Y. A new Barbalat's lemma and Lyapunov stability theorem for fractional order systems. Control And Decision Conference (CCDC), 2017 29th Chinese: IEEE; 2017. p. 3676-81.
[57] Diethelm K. The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type: Springer Science & Business Media; 2010.
[58] Li C, Deng W. Remarks on fractional derivatives. Applied Mathematics and Computation. 2007;187:777-84.
[59] Slotine J-JE. Applied nonlinear control.

Captions:

Figure 1. Unstable periodic orbit of fractional order Duffing system.

Figure 2. Stabilizing the 2π-periodic orbit of the fractional order Duffing system: a) first state variable, b) second state variable.

Figure 3. Phase plane of the close loop control system.

Figure 4. (a) Time history of the control action, (b) Time history of the sliding surface.

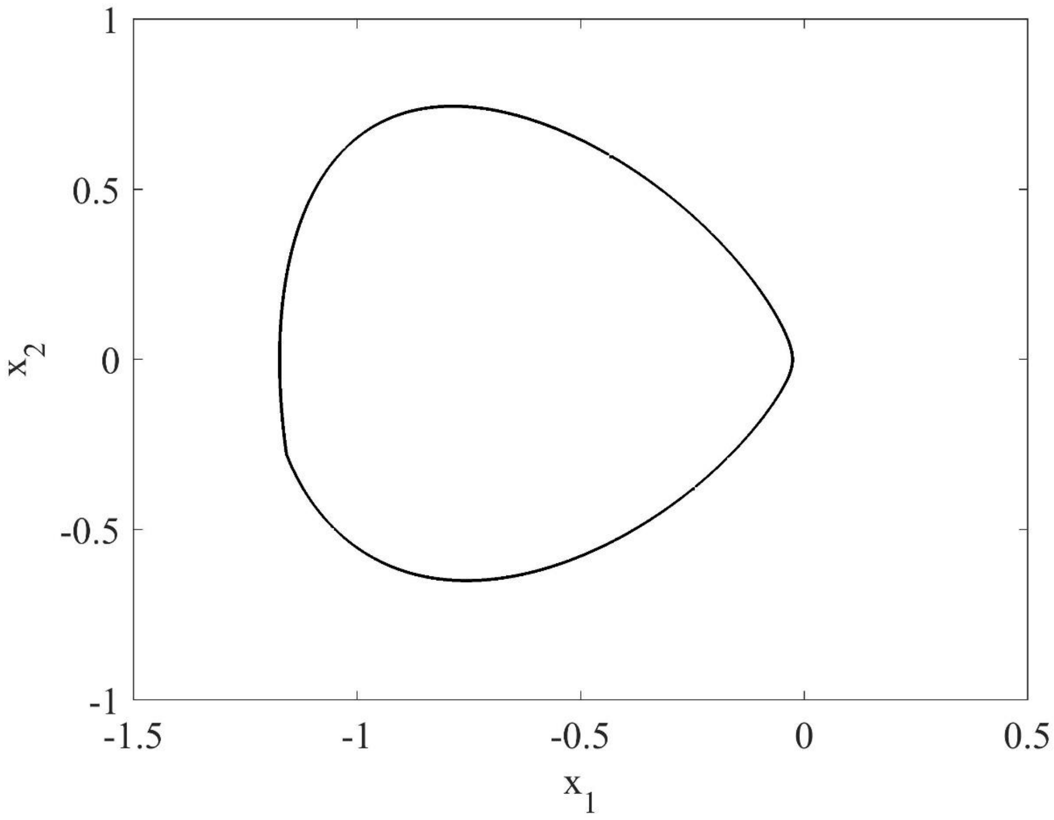

Figure 1. Unstable periodic orbit of fractional order Duffing system

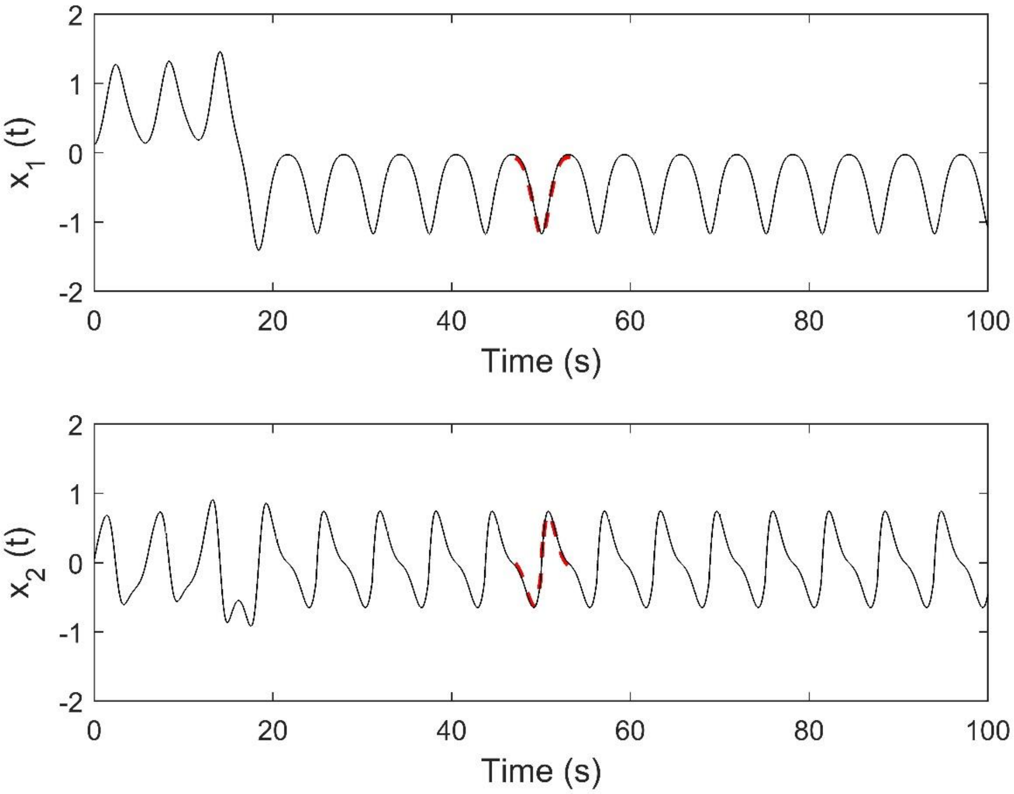

Figure 2. Stabilizing the 2π-periodic orbit of the fractional order Duffing system: a) first state variable, b) second state variable

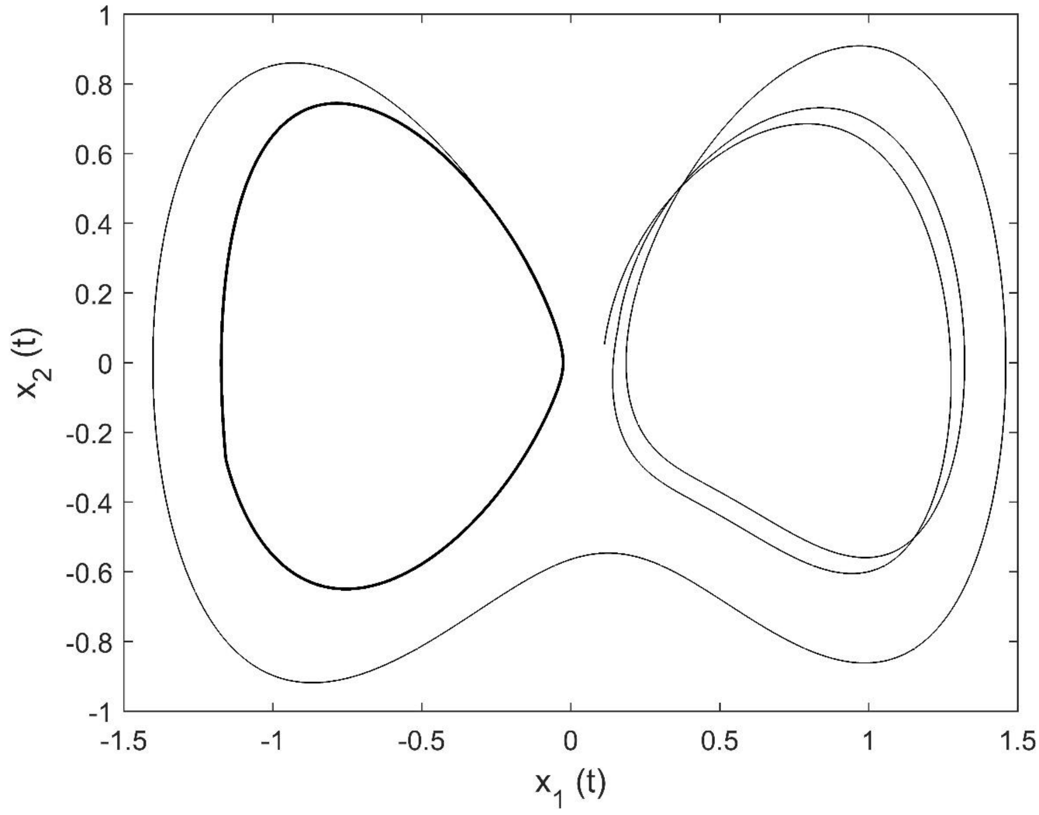

Figure 3. Phase plane of the close loop control system

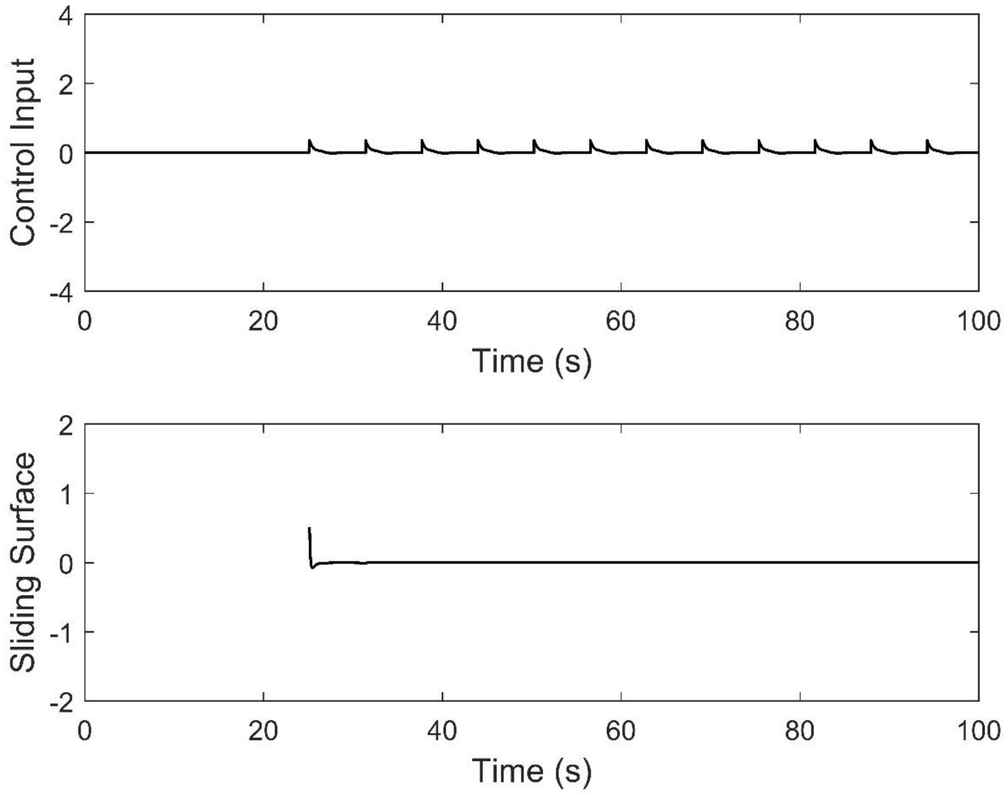

Figure 4. (a) Time history of the control action, (b) Time history of the sliding surface