ENG1005: Lecture 23

Lex Gallon

May 15, 2020

Contents

Video link

Click here for a recording of the lecture.

Chain rule - continued

Example

Show that

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(s,t)\,ds = b'(t)F(b(t),t) - a'(F(a(t),t) + \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}(s,t)\,ds$$

Solution

Define

$$G(x,y,t) = \int_x^y F(s,t)\,ds$$

$$\frac{\partial G}{\partial x}(x,y,t) = \frac{\partial}{\partial x}\left(-\int_y^x F(s,t)\,ds\right)$$
$$= -F(x,t)$$

This comes from the fact that
$$\frac{d}{dx}\int_c^x f(s)\,ds = f(x)$$

Similarly,
$$\frac{\partial G}{\partial y}(x,y,t) = \frac{\partial}{\partial y}\left(\int_x^y F(s,t)\,ds\right) = F(y,t)$$

and
$$\frac{\partial G}{\partial t}(x,y,t) = \frac{\partial}{\partial t}\left(\int_x^y F(s,t)\,ds\right) = \int_x^y \frac{\partial F}{\partial t}(s,t)\,ds$$

Note that last movement of the derivative inside the integral does not always work, but does in this case so don't worry about it.

So by the chain rule, we have

$$\begin{aligned}
\frac{d}{dt}\int_{a(t)}^{b(t)} F(s,t)\,dt &= \frac{d}{dt}\left(G(a(t),b(t),t)\right)\\
&= \frac{\partial G}{\partial x}(a(t),b(t),t)a'(t) + \frac{\partial G}{\partial y}(a(t),b(t),t)b'(t) + \frac{\partial G}{\partial x}(a(t),b(t),t)\\
&= -F(a(t),t)a'(t) + F(b(t),t)b'(t) + \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}(s,t)\,ds
\end{aligned}$$

Directional derivatives and gradients

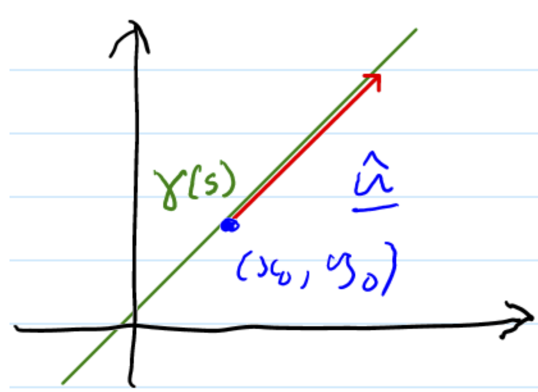
Definition

The directional derivative of a function $f(x,y)$ of two variables at a point $(x_0,y_0) \in \mathbb{R}^2$ in the direction of $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ is defined by

$$D_{\mathbf{u}} f(x_0,y_0) = \frac{d}{ds} f(\gamma(s))|_{s=0}$$

where
$$\gamma(s) = (x_0,y_0) + s\hat{\mathbf{u}} \quad \left(\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}\right)$$

I.e, $\gamma(s)$ is a line that goes in the direction of $\mathbf{u}$.



So,
if

$$\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$$

then

$$\gamma(s) = (x(s), y(s))$$

where

$$x(s) = x_0 + \hat{u}_1 s, \quad y(s) = y_0 + \hat{u}_2 s$$

Then

$$D_{\mathbf{u}} f(x_0, y_0) = \frac{d}{ds} f(x(s), y(s))|_{s=0}$$

Definition

The gradient of a differentiable function $f(x, y)$ of two variables at the point $(x_0, y_0) \in \mathbb{R}^2$ is defined as

$$\nabla f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0) \right)$$

Meaning of the directional derivative

The directional derivative $D_{\mathbf{u}} f(x_0, y_0)$ measures the rate of change of the function $f(x, y)$ at the point (x_0, y_0) in the direction of the vector $\mathbf{u} = (u_1, u_2)$.

$$|D_{\mathbf{u}} f(x_0, y_0)| \equiv \text{ rate of change of } f(x, y) \text{ at } (x_0, y_0) \text{ in the direction of } \mathbf{u} = (u_1, u_2).$$

$$D_{\mathbf{u}} f(x_0, y_0) > 0 \Rightarrow f \text{ is } \textbf{increasing} \text{ at } (x_0, y_0) \text{ in the direction of } \mathbf{u}.$$

$$D_{\mathbf{u}} f(x_0, y_0) < 0 \Rightarrow f \text{ is } \textbf{decreasing} \text{ at } (x_0, y_0) \text{ in the direction of } \mathbf{u}.$$

Relationship between the gradient and the directional derivative

Theorem

If $f(x, y)$ is differentiable at (x_0, y_0) and $\mathbf{u} = (u_1, u_2)$ is a vector in $\mathbb{R}^2$, then

$$D_{\mathbf{u}} f(x_0, y_0) = \hat{\mathbf{u}} \cdot \nabla f(x_0, y_0)$$

Proof

Let

$$\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$$

and

$$x(s) = x_0 + \hat{u}_1 s \text{ and } y(s) = y_0 + \hat{u}_2$$

Then

$$\begin{aligned}
D_{\mathbf{u}}f(x_0,y_0) &= \frac{d}{ds}f(x(s),y(s))|_{s=0} \\
&= \left(\frac{\partial f}{\partial x}(x(s),y(s))\frac{\partial x}{\partial s}(s) + \frac{\partial f}{\partial y}(x(s),y(s))\frac{\partial y}{\partial s}(s)\right)\Big|_{s=0} \\
&= \frac{\partial f}{\partial x}(x_0,y_0)\hat{u}_1 + \frac{\partial f}{\partial y}(x_0,y_0)\hat{u}_2 \\
&= \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0\right)\cdot(\hat{u}_1,\hat{u}_2) \\
&= \nabla f(x_0,y_0)\cdot\hat{\mathbf{u}}
\end{aligned}$$

Example

Find the rate of change (ROC) of $f(x,y) = x^2\sin(y)$ at $(1,\pi)$ in the direction $\mathbf{v} = (1,3)$.

Solution

(i)

$$\begin{aligned}
\nabla f(x,y) &= \left(\frac{\partial}{\partial x}\left(x^2\sin(y)\right), \frac{\partial}{\partial y}\left(x^2\sin(y)\right)\right) \\
&= \left(2x\sin(y), x^2\cos(y)\right) \\
\Rightarrow \nabla f(1,\pi) &= (2\sin(\pi),\cos(\pi)) = (0,-1)
\end{aligned}$$

(ii)

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(1,3)}{\sqrt{10}}$$

By (i) and (ii), we have

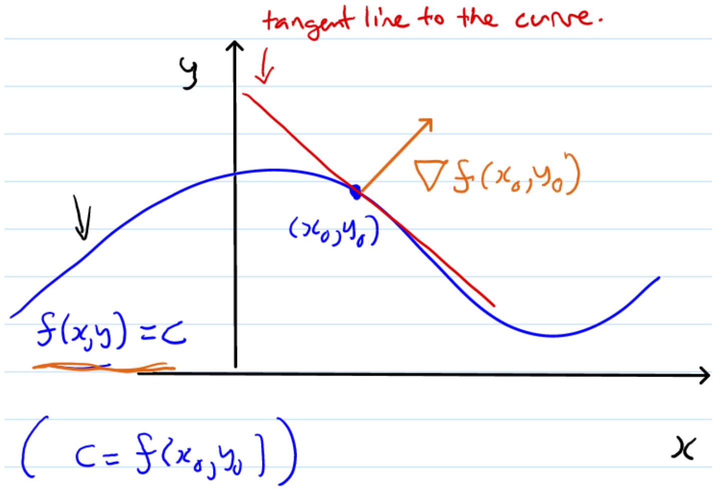
$$D_{\mathbf{v}}f(1,\pi) = \nabla f(1,\pi)\cdot\hat{\mathbf{v}} = (0,-1)\cdot\frac{\sqrt{10}}{10}(1,3) = \frac{-3\sqrt{10}}{10}$$

Therefore, the rate of change of $f(x,y)$ at the point $(1,\pi)$ in the direction of $\mathbf{v} = (1,3)$ is $\frac{3\sqrt{10}}{10}$ and f is decreasing.

Geometric significance of ∇f

Theorem

If $f(x,y)$ is differentiable at the point (x_0,y_0) and $\nabla f(x_0,y_0) \neq \mathbf{0}$, then $\nabla f(x_0,y_0)$ is orthogonal to the level curve of f that passes through the point (x_0,y_0)



Proof

$$\text{Let } \mathbf{r}(t) = (x(t), y(t))$$

$$\text{where } \mathbf{r}(0) = (x_0, y_0)$$

be a parametrisation of the level curve $f(x, y) = c = f(x_0, y_0)$.

Then

$$f(\mathbf{r}(t)) = f(x(t), y(t)) = c$$

Differentiating with respect to t,

$$\begin{aligned}
0 &= \frac{d}{dt}\left(f(x(t), y(t))\right)\Big|_{t=0} \\
&= \left(\frac{\partial f}{\partial x}(x(t), y(t))\frac{\partial x}{\partial t}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\frac{\partial y}{\partial t}(t)\right)\Bigg|_{t=0} \\
&= \frac{\partial f}{\partial x}(x_0, y_0)\frac{\partial x}{\partial t}(0) + \frac{\partial f}{\partial y}(x_0, y_0)\frac{\partial y}{\partial t}(0) \\
&= \nabla f(x_0, y_0) \cdot \frac{d\mathbf{r}}{dt}(0)
\end{aligned}$$

Since $\dfrac{d\mathbf{r}}{dt}(0)$ is tangent to the level curve at the point (x_0, y_0) and since its dot product with $\nabla f(x_0, y_0)$ is zero, we can conclude that the derivative of the function $f(x, y)$ at the point (x_0, y_0) is orthogonal to the level curve passing through the point (x_0, y_0)