ENG1005: Lecture 7 - Convergence Tests

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Contents

Basic premise

For a series $\sum_{n=1}^{\infty} a_n$ to have any chance of converging, the terms a_n must get small as n gets large.

Theorem (Tails of series)

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example

The series $\sum_{n=0}^{\infty} (-1)^n$ diverges because the limit $\lim_{n\to\infty} (-1)^n$ DNE.

Definition

A series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Remark

Absolute convergence implies convergence.

Theorem (Compare Test)

(i) If $|a_n| \leq b_n,\ n\ \in \mathbb{N}$, and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $a_n \leq b_n, n\ \in \mathbb{N}$, and $\lim_{N\to\infty} \sum_{n=1}^{N} a_n = \infty$, then $\lim_{N\to\infty} \sum_{n=1}^{N} b_n$ diverges.

Example

Determine if the series

$$\sum_{n=0}^{\infty} \frac{\sin^2(n)}{2^n}$$

converges.

Solution

Since $0 \leq \frac{\sin^2(n)}{2^n} \leq \frac{1}{2^n},\ n \in \mathbb{N}_0$, and we know that $\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges, it follows from the Comparison test that the series $\sum_{n=0}^{\infty} \frac{\sin^2(n)}{2^n}$ converges.

Theorem (Ratio Test)

Given a sequence $\{a_n\}_{n=1}^{\infty}$, let $l = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, the series $\sum_{n=1}^{\infty} a_n$

(i) converges absolutely if $l < 1$,

(ii) and diverges if $l > 1$

Remark

If $l = 1$ or $l = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ fails to exist, then the ratio test gives no information. This is NOT the same as saying that the limit DNE, just that we can't use the ratio test to determine this.

Example

Determine if the following series converge:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ (factorial series)

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ (harmonic series)

Solution

(a)

$$\begin{aligned} a_n = \frac{1}{n!} \Rightarrow l = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| \\ &= \lim_{n\to\infty}\frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty}\frac{1}{n+1} \\ &= 0 \end{aligned}$$

This shows that the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

(b)

$$\begin{aligned} a_n = \frac{2^n}{n} \Rightarrow l = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}}\right| \\ &= \lim_{n\to\infty}\frac{n}{n+1}\,\frac{2^{n+1}}{2^n} \\ &= 2\lim_{n\to\infty}\frac{n}{n+1} \\ &= 2\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} \\ &= 2\cdot\frac{1}{1} = 2 > 1 \end{aligned}$$

Thus, the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n}$ diverges by the Ratio Test.

(c)

$$\begin{aligned} a_n = \frac{1}{n} \Rightarrow l &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{n}{n+1} \\ &= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \end{aligned}$$

This shows that the Ratio Test gives no information.

Theorem (Integral Test)

Suppose $f(x)$ is continuous, positive and decreasing on $[1, \infty)$.

(i) If $|a_n| \leq f(n),\ n \in \mathbb{N}$, and $\displaystyle\int_1^\infty f(x)\,dx$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $|a_n| \leq f(n),\ n \in \mathbb{N}$, and $\displaystyle\int_1^\infty f(x)\,dx$ diverges, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges. You can tell this is true if you consider that $\displaystyle\sum_{n=1}^{N} a_n \approx \int_1^N f(x)\,dx$.

Example

Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

Solution

Let $f(x) = \frac{1}{x}$. Then $f(x)$ is continuous on $[1, \infty)$, decreasing and positive, and

$$f(n) = \frac{1}{n}$$

Since $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{\varepsilon\to\infty} \int_1^\varepsilon \frac{1}{x}\,dx = \lim_{\varepsilon\to\infty} \ln(x)|_1^\varepsilon = \lim_{\varepsilon\to\infty} \ln(\varepsilon) = \infty$, we conclude via the integral test that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.