ENG1005: Lecture 15

Lex Gallon

April 29, 2020

Contents

Video link

Click here for a recording of the lecture.

Matrix operations continued

Transposition

$$[A_{ij}]^T = [A_{ji}]$$

This means the transposition of an $m \times n$ matrix is an $n \times m$ matrix.

Example

$$\begin{bmatrix} 0 & 1 & 2 \\ -3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & 4 \\ 2 & 6 \end{bmatrix}$$

Properties of matrix operations §5.2.6

(i) $(AB)C = A(BC)$ (associative)

(ii) $(A+B)C = AC + BC$ (distributive)
This also works with scalars: $\lambda(B+C) = \lambda B + \lambda C$

Remark

In general, matrix multiplication is not commutative,

$$AB \neq BA$$

Example

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

But if you switch the matrics around, you instead get

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

So you can see that the order in which you multiply matrices does matter.

(iii) $(AB)^T = B^T A^T$

(iv) The matrix

$$\mathbb{I}_n = (\delta_{ij}),\ 1 \leq i, j \leq n$$

Note: δ_{ij} is the Kronecker delta and is defined as

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j \end{cases}$$

Example

$$\mathbb{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then

$$A\mathbb{I}_n = A$$

$$\mathbb{I}_n B = B$$

If C is an $n \times n$ matrix, then

$$\mathbb{I}_n C = C\mathbb{I}_n = C$$

Matrix formulation of linear equations §5.5

$$\begin{aligned}A_{11}x_1 + A_{12}x_2 + ... + A_{1n}x_n &= b_1\\ A_{21}x_1 + A_{22}x_2 + ... + A_{2n}x_n &= b_2\\ &\vdots\\ A_{m1}x_1 + A_{m2}x_2 + ... + A_{mn}x_n &= b_m\end{aligned}$$

We can write the above system as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & ... & A_{mn} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Terminology

Matrices of the form $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 & y_2 & ... & y_n \end{bmatrix}$ are referred to as column and row vectors, respectively, of length n.

Rank §5.6

Given an $m \times n$ matrix A, we can perform Gaussian elimination to put it into row echelon form B. Note that, although there is no unique row echelon form, the number of pivots will always be the same. Then the rank of A is defined by

$$\text{rank}(A) = \# \text{ of pivots of } A$$

Example

We previously showed using Gaussian elimination that the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 4 & 2 & 1 \\ -3 & -9 & 3 & -3 \end{bmatrix}$$

can be row-reduced to

$$\begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are 2 pivots, we get

$$\text{rank}(A) = 2$$

The solutions space for linear systems of equations

Theorem

Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is an m-column vector. Then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has

(i) no solutions if $\text{rank}(A) \neq \text{rank}([A|\mathbf{b}])$,

(ii) and at least one non-trivial solution ($\mathbf{x} \neq \mathbf{0}$, presumably also $\mathbf{b} \neq \mathbf{0}$) if $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$. Furthermore, if $\text{rank}(A) = \text{rank}([A|\mathbf{b}])$, then

(ii.a) there is a unique solution if $\text{rank}(A) = n$,

(ii.b) and an infinite number of solutions with n-$\text{rank}(A)$ free parameters if $\text{rank}(A) < n$.

LU decomposition

Gaussian elimination is an effective method for solving systems of linear equations

$$A\mathbf{x} = \mathbf{b} \quad (*)$$

- For $n \times n$ matrices A, the computational effort to solve (*) is n^3.
- $n = 10^6$ is not considered large nowadays!

Gaussian elimination acts on the augmented matrix. Often you may have a constant A, but then have many different vectors $\mathbf{b}$ so Gaussian elimination can be inefficient in this case since it requires A gets changed everytime $\mathbf{b}$ is.

Definition

A factorisation of an $n \times n$ matrix of the form

$$A = LU$$

where L is a lower triangular matrix, and U is an upper triangular matrix (both are also $n \times n$) is known as an LU-decomposition for A.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 9 & 2 \end{bmatrix} \quad \begin{bmatrix} 7 & -2 & -1 \\ 0 & 9 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

The matrix on the left is a lower triangular matrix, and the one on the right is an upper triangular matrix.

To solve

$$A\mathbf{x} = \mathbf{b}$$

we use LU decomposition to write it as follows

$$LU\mathbf{x} = \mathbf{b} \quad (**)$$

To solve (**), we do the following:

(i) Set $\mathbf{y} = U\mathbf{x}$.
Then (x) becomes

$$L\mathbf{y} = \mathbf{b} \quad (**)$$

which can solve by back-substitution to get $\mathbf{y}$(since L is in row echelon form).

(ii) We then solve $U\mathbf{x} = \mathbf{y}$
using forward substitution to get $\mathbf{x}$