ENG1005: Lecture 22

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May 14, 2020

Contents

Video link

Click here for a recording of the lecture.

Partial derivatives - continued

Example

Compute all 2nd order partial derivatives for

$$f(x, y) = e^{x^2 y}$$

Solution

$$\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial y}\left(2xye^{x^2y}\right) \\
&= 2x\frac{\partial}{\partial y}\left(ye^{x^2y}\right) \\
&= 2x\left(e^{x^2y} + yx^2e^{x^2y}\right) \\
&= 2xe^{x^2y} + 2x^3ye^{x^2y}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(2xye^{x^2y}\right) \\
&= 2y\frac{\partial}{\partial x}\left(xe^{x^2y}\right) \\
&= 2y\left(e^{x^2y} + x2xye^{x^2y}\right) \\
&= 2ye^{x^2y} + 4x^2y^2e^{x^2y}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(x^2e^{x^2y}\right) \\
&= x^2 2xye^{x^2y} + 2xe^{x^2y} \\
&= 2x^3ye^{x^2y} + 2xe^{x^2y}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\left(x^2e^{x^2y}\right) \\
&= x^2\frac{\partial}{\partial y}\left(e^{x^2y}\right) \\
&= x^2\left(x^2e^{x^2y}\right) \\
&= x^4e^{x^2y}
\end{aligned}$$

Observation

$$\frac{\partial^2}{\partial x \partial y}\left(e^{x^2y}\right) = \frac{\partial^2}{\partial y \partial x}\left(e^{x^2y}\right)$$

Theorem: Equality of mixed partials

Suppose that there exists some $R > 0$ such that $f(x,y), \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$ are continuous on the ball $B_R((a,b))$. Then

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

Challenge

Find a function $f(x,y)$ such that $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ both exist on some disc $B_R((a,b))$ in $\mathbb{R}^2$, but

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) \neq \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

The Chain Rule §9.6.5

Single variable chain rule

Start with some differentiable functions

$$f(x) \text{ and } x(t)$$

and define a new function

$$y(t) = f(x(t))$$

Then

$$\frac{dy}{dt}(t) = \frac{df}{dx}\,(x(t))\,\frac{dx}{dt}(t)$$

$$\text{Alternatively, } \frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

Two variable chain rule

Theorem

Suppose $f(x,y)$, $x(s,t)$ and $y(s,t)$ are all differentiable functions, and define

$$z(s,t) = f(x(s,t), y(s,t))$$

Then $z(s,t)$ is differentiable and

$$\frac{\partial z}{\partial s}(s,t) = \frac{\partial f}{\partial x}\,(x(s,t),y(s,t))\,\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}\,(x(s,t),y(s,t))\,\frac{\partial y}{\partial s}(s,t)$$

$$\frac{\partial z}{\partial s}(s,t) = \frac{\partial f}{\partial x}\,(x(s,t),y(s,t))\,\frac{\partial x}{\partial t}(s,t) + \frac{\partial f}{\partial y}\,(x(s,t),y(s,t))\,\frac{\partial y}{\partial t}(s,t)$$

Informal versions

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Full Chain Rule

Theorem

Suppose $f(x_1, x_2, ..., x_n)$ is differentiable and

$$x_i(t_1, t_2, ..., t_m),\ i = 1, 2, ..., n$$

are differentiable. Then

$$z(t_1, t_2, ..., t_m) = f(x_1(t_1, t_2, ..., t_m), x_2((t_1, t_2, ..., t_m), ..., x_n(t_1, t_2, ..., t_m))$$

is differentiable and

$$\frac{\partial z}{\partial t_j}(t_1, ..., t_m) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1(t_1, ..., t_m), ..., x_n(t_1 ..., t_m))\frac{\partial x_i}{\partial t_j}(t_1, ..., t_m)$$

Informal versions

$$\frac{\partial z}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t_j} \text{ for } j = 1, 2, ..., m$$

Example

Let $f(x, y) = \ln(x^2 + y^2)$, $x(t) = \cos(t)$, $y(s, t) = st$ and define

$$z(s, t) = f(x(t), y(s, t)) = \ln\left(\cos^2(t) + s^2t^2\right)$$

Solution

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}\\ &= 0 + \frac{2y}{x^2 + y^2}t\\ &= \frac{2st^2}{\cos^2(t) + s^2t^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\\ &= \frac{2x}{x^2 + y^2}(-\sin(t)) + \frac{2y}{x^2 + y^2}s\\ &= \frac{-2\cos(t)\sin(t)}{\cos^2(t) + s^2t^2} + \frac{2s^2t}{\cos^2(t) + s^2t^2} = \frac{2s^2t - 2\cos(t)\sin(t)}{\cos^2(t) + s^2t^2}\end{aligned}$$