ENG1005: Lecture 21

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Contents

Video link

Click here for a recording of the lecture.

Parametric surfaces - continued

Examples

(a) $f(x, y) = \sqrt{1 - x^2 - y^2},\ x^2 + y^2 \leq 1$.
The graph of f is the (unit) hemisphere.
$S = \{(x, y, \sqrt{1 - x^2 - y^2}) \mid x^2 + y^2 \leq 1\}$.

(b) $\mathbf{r}(\theta, z) = (\cos\theta, \sin\theta, z), \quad 0 \leq \theta \leq 2\pi,\ -\infty < z < \infty$.
This is a cylinder centred on the z-axis. Is this a graph?

Level surface

Given a function of 3 variable $f(x, y,)$, we can define a surface

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = c\}$$

This type of surface is known as a level surface (note however that there can be some degenerate cases).

Examples

(a) Let $f(x, y, z) = x + 2y + 3z - 1$.
Then $f(x, y, z) = 0$ defines a plane in $\mathbb{R}^3$.

(b) Let $f(x, y, z) = x^2 + y^2 + z^2$.
Then $f(x, y, z) = 1 \Leftrightarrow x^2 + y^2 + z^2 = 1$
defines a unit sphere in $\mathbb{R}^3$.

(c) If $g(x, y)$ is a function of 2 variables, then the graph of $g(x, y)$ can be represented as a level surface by setting

$$f(x, y, z) = g(x, y) - z$$

because then

$$f(x, y, z) = 0 \Leftrightarrow z = g(x, y)$$

(d) Let $f(x, y) = x^2 + y^2 - 1$. Then

$$f(x, y) = 0 \Leftrightarrow x^2 + y^2 = 1$$

defines the unit circle in $\mathbb{R}^2$.

Limits and continuity

Definition

Given a function $f(\mathbf{x})$ of n variables ($\mathbf{x} = (x_1, x_2, ..., x_n)$) that is defined for all $\mathbf{x}$ satisfying $0 < |\mathbf{x}-\mathbf{p}| < R$, we say that the limit of $f(\mathbf{x})$ equals ℓ as $\mathbf{x}$ goes to $\mathbf{p}$, denoted by

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \ell$$

if for every $\epsilon > 0$, there exists a $\delta \in (0, R)$ such that $|f(\mathbf{x}) - \ell| < \epsilon$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Moreover,

(i) If $f(\mathbf{x})$ is defined for $\mathbf{x} = \mathbf{p}$ and $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$, then we say that $f(\mathbf{x})$ is continuous at $\mathbf{p}$.

(ii) If $D \subset \mathbb{R}^n$ and $f(\mathbf{x})$ is continuous at each $\mathbf{p} \in D$, then we say that $f(\mathbf{x})$ is continuous on D.

Our notation for balls

$$B_R(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{p}| < R\}$$
$$B_R^X(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n \mid 0 < |\mathbf{x} - \mathbf{p}| < R\} \quad \text{(doesn't include center point)}$$
$$\overline{B}_R(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{p}| \leq R\}$$

Note, B stands for 'ball', R stands for 'radius'.

Partial derivatives §9.6.2

Definition

The first partial derivatives of a two variable function $f(x, y)$ with respect to the variables x and y at the point (a, b) are defined by

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

and similarly

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Remarks

(i) If we define

$$h(x) = f(x,y),\ y\text{-fixed}, \qquad g(y) = f(x,y),\ x\text{-fixed}$$

then

$$\frac{\partial f}{\partial x}(x,y) = \frac{dh}{dx}(x) \text{ and } \frac{\partial f}{\partial y}(x,y) = \frac{dg}{dy}(y)$$

(ii) There exist many notations for partial derivatives

$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = \partial_x f(a,b) = D_1 f(a,b) = \partial_1 f(a,b)$$

Example

$$f(x,y) = \cos(x)\sin(y)$$

Compute

$$\frac{\partial f}{\partial x}(x,y)$$

Solution

Treating y as a constant, then

$$\frac{\partial}{\partial x}(\cos(x)\sin(y)) = \sin(y)\frac{d}{dx}(\cos(x)) = -\sin(y)\sin(x)$$

Therefore

$$\frac{\partial f}{\partial x}(x,y) = -\sin(y)\sin(x)$$

Example

Compute

$$\frac{\partial}{\partial x}\left(y^{yx^2}\right)$$

Solution

$$\begin{aligned}
\frac{\partial}{\partial x}\left(y^{yx^2}\right) &= \ln(y)a^{yx^2}\frac{\partial}{\partial x}\left(yx^2\right)\\
&= \ln(y)a^{yx^2}y\frac{d}{dx}\left(x^2\right)\\
&= 2\ln(y)a^{yx^2}y
\end{aligned}$$

Successive differentiation §9.6.7

Given a function $f(x, y)$, then are 4 partial derivatives of second order.

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &:= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), & \frac{\partial^2 f}{\partial x \partial y} &:= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\\
\frac{\partial^2 f}{\partial y^2} &:= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), & \frac{\partial^2 f}{\partial y \partial x} &:= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)
\end{aligned}$$