ENG1005: Lecture 28

Lex Gallon

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Contents

Video link

`https://echo360.org.au/lesson/G_8402119b-734b-4e1e-a3b4-7e907e86ddba_b944cecf-8ba5-40d3-a870-024`
`2020-05-26T15:58:00.000_2020-05-26T16:53:00.000/classroom#sortDirection=desc`

Initial value problems (IVP) - continued

Example

Newton's law of cooling states that the temperature of a homogeneous object satisfies

$$\frac{dT}{dt} = -K(T - T_a)$$

where T_a is the ambient temperature, $T(t)$ is temperature of body at time t, $K > 0$ is some decay constant. The initial condition is

$$T(t_0) = T_0$$

Find the temperature of the body at time $t > t_0$ assuming that $K > 0$ and $T_0 > T_a$.

Solution

$$\begin{aligned}\frac{dT}{dt} = K(T - T_a) \Rightarrow \frac{dT}{T - T_a} &= -k\,dt \\ &\Rightarrow \int_{T_0}^{T} \frac{d\tilde{T}}{T - T_a} = -K \int_{t_0}^{t} d\tilde{t} \\ &\Rightarrow \ln(\tilde{T} - T_a)\Big|_{T_0}^{T} = -Kt|_{t_0}^{t} \\ &\Rightarrow \ln\left(\frac{T - T_a}{T_0 - T_a}\right) = -K(t - t_0) \\ &\Rightarrow \frac{T - T_a}{T_0 - T_a} = e^{-K(t-t_0)} \\ &\Rightarrow T = (T_0 - T_a)e^{-K(t-t_0)} + T_a\end{aligned}$$

Note that

$$\lim_{t\to\infty} T(t) = T_a$$

Remark

The IVP for an nth order ODE

$$\frac{d^n y}{dt^n} = F\left(x, y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, ..., \frac{d^{n-1}y}{dt^{n-1}}\right)$$

need n initial conditions

$$y(t_0) = y_0, \frac{dy}{dt}(t_0) = y_1, ... \frac{d^{n-1}y}{dt^{n-1}}(t_0) = y_{n-1}$$

to be **uniquely solvable**.

Separable ODEs §10.5.3

A 1st order ODE

$$\frac{dy}{dy} = F(t, y)$$

is called separable if $F(t, y)$ can be written as

$$F(t, y) = \frac{G(t)}{H(y)}$$

To solve these equations, we write

$$\frac{dy}{dt} = \frac{G(t)}{H(y)}$$

as

$$H(y)\frac{dy}{dt} = G(t)$$

Then we integrate both sides with respect to t to get

$$\int H(y)\frac{dy}{dt}\,dt = \int G(t)\,dt$$

We then use change of variables

$$y = y(t), \quad dy = \frac{dy}{dt}\,dt$$

to get

$$\int H(y)\,dy = \int G(t)\,dt$$

In general, this yields an implicit solution to the above ODE.

For IVP with initial condition

$$y(t_0) = y_0,$$

then we can write the solution as

$$\int_{y_0}^{y} H(\tilde{y})\,d\tilde{t} = \int_{t_0}^{t} G(\tilde{t})\,d\tilde{t}$$

Informal version

$$\begin{aligned}\frac{dy}{dt} = \frac{G(t)}{h(y)} &\Rightarrow H(y)\,dy = G(t)\,dt \\ &\Rightarrow \int H(y)\,dy = \int G(t)\,dt\end{aligned}$$

Example

$$\frac{dV}{dt} = \sqrt{V}$$

Solution

This is separable since we can trivially write the RHS as

$$\sqrt{V} = \frac{1}{\frac{1}{\sqrt{V}}} = \frac{G(t)}{H(V)}$$

$$\begin{aligned}\frac{dV}{dt} = \sqrt{V} &\Rightarrow \frac{1}{\sqrt{V}}\,dV = dt \\ &\Rightarrow \int \frac{1}{\sqrt{V}}\,dV = \int dt \\ &\Rightarrow 2\sqrt{V} = t + 2c \quad \text{(only do } 2c \text{ to simplify next step)} \\ &\Rightarrow \sqrt{V} = \frac{t}{2} + c \\ &\Rightarrow V = \left(\frac{t}{2} + c\right)^2\end{aligned}$$

Linear 1st order ODEs §10.5.9

The most general 1st order ODE is

$$\frac{dy}{dt} + p(t)y = q(t)$$

If $q(t) = 0$, then the ODE is said to be **homogeneous**, and otherwise it is called inhomogeneous or nonhomogeneous.
To solve this equation, we multiply it by an integrating factor $I(t)$ to get

$$I\frac{dy}{dt} + Ip(t)y = Iq(t)$$

or equivalently

$$\frac{d}{dt}(Iy) - \frac{dI}{dt}y + Ipy = Iq$$

We can write the above ODE as

$$\frac{d}{dt}(Iy) + \left(Ip - \frac{dI}{dt}\right)y = Iq$$

Now, we could simplify this is we choose I such that those terms in the right set of brackets become zero. So we choose I to satisfy

$$\frac{dI}{dt} = Ip$$

But this is a separable ODE! So,

$$\begin{aligned}\frac{dI}{dt} = Ip \Rightarrow \frac{dI}{I} &= p\,dt \\ \Rightarrow \int \frac{dI}{I} &= \int p\,dt \\ \Rightarrow \ln(I) &= \int p(t)\,dt \\ \Rightarrow I(t) &= e^{\int p(t)\,dt}\end{aligned}$$

With this choice of $I(t)$, we get that

$$\frac{d}{dt}(Iy) = Iq.$$

Integrate both sides to get

$$\begin{aligned}I(t)y(t) &= \int I(t)q(t)\,dt + c \\ \Rightarrow y(t) &= \frac{1}{I(t)}\left[\int I(t)q(t)\,dt + c\right],\end{aligned}$$

where

$$I(t) = e^{\int p(t)\,dt}.$$

Example

Solve the IVP

$$\begin{aligned}\frac{dy}{dt} + ty &= t \\ y(0) &= 2.\end{aligned}$$

Solution

In our case,

$$p(t) = t,\ q(t) = t$$

So we can find the integrating factor

$$I(t) = e^{\int t\,dt} = e^{\frac{1}{2}t^2} \quad \text{(note no need to worry about integration constant).}$$

So the general solution is given by

$$\begin{aligned} y(t) &= \frac{1}{I(t)}\left[\int I(t)q(t)\,dt + c\right] \\ &= e^{-\frac{1}{2}t^2}\left[\int e^{\frac{1}{2}t^2} t\,dt + c\right] \\ &= e^{-\frac{1}{2}t^2}\left[e^{\frac{1}{2}t^2} + c\right] \\ &= 1 + ce^{-\frac{1}{2}t^2}. \end{aligned}$$

And now, for the initial condition,

$$\begin{aligned} y(0) = 2 &\Rightarrow y(0)1 + ce^{-\frac{1}{2}t^2} = 1 + c = 2 \\ &\Rightarrow c = 1. \end{aligned}$$

So we can conclude that

$$y(t) = 1 + e^{-\frac{1}{2}t^2}$$

solves the IVP.

Linear differential equations §10.8

Superposition principle

Suppose $y_1(x)$ and $y_2(x)$ both solve the following linear homogeneous ODE

$$\sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k} = 0$$

Let

$$\tilde{y}(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x),\ \lambda_1, \lambda_2 \in \mathbb{R}$$

Then

$$\begin{aligned} \sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k} &= \sum_{k=0}^{n} a_k(x)\left(\lambda_1\frac{d^k y_1}{dx^k} + \lambda_2\frac{d^k y_2}{dx^k}\right) \\ &= \lambda_1 \sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k} + \lambda_2 \sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k} \\ &= 0 \end{aligned}$$

shows that

$$\tilde{y}(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x),\ \lambda_1, \lambda_2 \in \mathbb{R}$$

solves the linear, homogeneous ODE.