ENG1005: Lecture 4

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March 20, 2020

Contents

Improper integrals: infinite integrals §9.2.2

Define integrals like

$$\int_0^\infty \frac{1}{1+x^2}\,dx$$

We first regularize the integral by

$$\int_0^\varepsilon \frac{1}{1+x^2}\,dx,\ \ \varepsilon > 0$$

Then we define the integral as a limit

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{\varepsilon\to\infty}\left(\int_0^\varepsilon \frac{1}{1+x^2}\,dx\right) = \lim_{\lambda\searrow 0}\left(\int_0^{\frac{1}{\lambda}} \frac{1}{1+x^2}\,dx\right)$$

So

$$\begin{aligned}
\int_0^\infty \frac{1}{1+x^2}\,dx &= \lim_{\varepsilon\to\infty}\left(\int_0^\varepsilon \frac{1}{1+x^2}\,dx\right) \\
&= \lim_{\varepsilon\to\infty}\left(\arctan(\varepsilon) - \arctan(0)\right) \\
&= \lim_{\varepsilon\to\infty}\left(\arctan(\varepsilon)\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

This shows that $\int_0^\infty \frac{1}{1+x^2}\,dx$ is a convergent improper integral.
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Summary

1. If $f(x)$ is continuous on $[a,\infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{\varepsilon\to\infty}\int_a^\varepsilon f(x)\,dx$$

2. If $f(x)$ is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\varepsilon\to-\infty}\int_\varepsilon^c f(x)\,dx + \lim_{\lambda\to\infty}\int_c^\lambda f(x)\,dx$$

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Comparison principle: convergence

Definition

$$\int_1^\infty e^{-x^2}\,dx = \lim_{\varepsilon\to\infty}\int_1^\varepsilon e^{-x^2}\,dx$$

but we can't compute $\int_1^\varepsilon e^{-x^2}\,dx$.

Theorem (Comparison Test)

Suppose $f(x)$ and $g(x)$ (not magnitude of g(x)?) are continuous on $[a,\infty)$, $|f(x)| \le g(x)$ for all $x \in [a,\infty)$, and $\int_a^\infty g(x)\,dx$ converges.

$$\text{Then } \int_a^\infty f(x)\,dx \text{ also converges}$$

Proof

If $|f(x)| \le g(x)$, then

$$\left|\int_a^\varepsilon f(x)\,dx\right| \le \int_a^\varepsilon |f(x)|\,dx \le \int_a^\varepsilon g(x)\,dx$$

(assuming g(x) converges)

Example

Since

$$\left|e^{-x^2}\right| = e^{-x^2} \leq e^{-x}$$

(couldn't we just say it's less than or equal to 1? No, because when we take the limit to infinity, it wouldn't converge!).

and

$$\begin{aligned}\int_1^\infty e^{-x}\,dx &= \lim_{\varepsilon\to\infty}\int_1^\varepsilon e^{-x}\,dx \\ &= \lim_{\varepsilon\to\infty}\left(-e^{-\varepsilon}+e^{-1}\right)\end{aligned}$$

So by the Comparison Test, we conclude that

$$\int_1^\infty e^{-x}\,dx \text{ is a convergent improper integral}$$

Example

Determine if the improper integral

$$\int_1^\infty \frac{1}{1+xe^x}\,dx$$

converges or diverges.

Solution

$$\begin{aligned}1 &\leq x \\ e^x &\leq xe^x \\ e^x &\leq 1+xe^x \\ \frac{1}{1+xe^x} &\leq \frac{1}{e^x} \\ \left|\frac{1}{1+xe^x}\right| &\leq e^{-x}\end{aligned}$$

Since

Since $\displaystyle\int_1^\infty e^{-x}\,dx$ is convergent, we deduce from the Comparison test that $\displaystyle\int_1^\infty \frac{1}{1+xe^x}\,dx$ is also con

Theorem (Comparsion Test)

Suppose $f(x)$ and $g(x)$ are continuous on $[a,b)$, $|f(x)| \leq g(x)$ for all $x \in [a,b)$, and $\int_a^b g(x)\,dx$ converges.

Then
$$\int_a^b f(x)\,dx \text{ converges.}$$

Example

$$\int_0^{\frac{\pi}{4}} \frac{\cos(x)}{\sqrt{x}}\,dx \text{ convergent or not?}$$

Solution

$$\left|\frac{\cos(x)}{\sqrt(x)}\right| = \frac{1}{\sqrt{x}}|\cos(x)| \leq \frac{1}{\sqrt{x}}$$

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{\cos(x)}{\sqrt{x}}\,dx &= \lim_{\varepsilon \searrow 0} \int_\varepsilon^{\frac{\pi}{4}} \frac{1}{\sqrt{x}}\,dx \\ &= \lim_{\varepsilon \searrow 0} 2\sqrt{\frac{\pi}{4}} - 2\sqrt{\varepsilon} \\ &= \sqrt{\pi}\end{aligned}$$