ENG1005: Lecture 27

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Contents

Video link

`https://echo360.org.au/lesson/G_35fe23e0-41ee-4e6f-b0f5-05f4155bb7b0_b944cecf-8ba5-40d3-a870-024`
`2020-05-21T15:58:00.000_2020-05-21T16:53:00.000/classroom#sortDirection=desc`

Example

The temperature at every point in the closed unit disc

$$D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1$$

is given by

$$T(x, y) = (x + y)e^{-(x^2+y^2)}$$

Find the maximum and minimum temperature and where these are achieved on the disc.

Solution

First note that this is a nice continuous function.

Step 1: Find all the critical points of $T(x,y)$ in the interior of D.

$$\nabla T(x,y) = \mathbf{0} \Rightarrow \begin{cases} \frac{\partial T}{\partial x} = (1-2x(x+y))e^{-(x^2+y^2)} = 0 \\ \frac{\partial T}{\partial y} = (1-2y(x+y))e^{-(x^2+y^2)} = 0 \end{cases}$$
$$\Rightarrow \begin{cases} \frac{\partial T}{\partial x} = 1-2x(x+y) = 0 \\ \frac{\partial T}{\partial y} = 1-2y(x+y) = 0 \end{cases}$$
$$\Rightarrow \begin{cases} \frac{\partial T}{\partial x} = 2x(x+y) = 1 \\ \frac{\partial T}{\partial y} = 2y(x+y) = 1 \end{cases}$$

I'm gonna solve this by noting that $x = y$ as you can see that both equations are very symmetric. Therefore, from either equation we get

$$\begin{aligned} 2x(x+x) &= 1 \\ 4x^2 &= 1 \\ \Rightarrow x &= \pm\frac{1}{2} \end{aligned}$$

Therefore the critical points are

$$\left(-\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)$$

Note also that both are inside D.

Step 2: Find all the critical points of $T(x,y)$ on the boundary of D. We can parametrise the boundary of D by

$$\mathbf{r}(t) = (\cos(t), \sin(t)),\ 0 \le t \le 2\pi$$

Restricting $T(x,y)$ to the boundary gives

$$\begin{aligned} g(t) &= T(\cos(t), \sin(t)),\ 0 \le t \le 2\pi \\ &= (\cos(t) + \sin(t))e^{-1},\ 0 \le t \le 2\pi \\ g'(t) &= \frac{1}{e}(-\sin(t) + \cos(t)) = 0 \\ &\Rightarrow \tan(t) = 1 \\ &\Rightarrow t = \frac{\pi}{4}, \frac{5\pi}{4} \end{aligned}$$

The critical points of $T(x,y)$ restricted to the boundary of D are

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Step 3: Evaluate $T(x,y)$ at all the critical points.

Critical points	$T(x,y)$	Conclusion
$(\frac{1}{2}, \frac{1}{2})$	$\frac{1}{\sqrt{e}}$	Max temperature
$(-\frac{1}{2}, -\frac{1}{2})$	$-\frac{1}{\sqrt{e}}$	Min temperature
$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$\frac{\sqrt{2}}{e}$	
$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$-\frac{\sqrt{2}}{e}$	

Ordinary differential equations §10.1, 10.2, 10.3

An ordinary differential equation (ODE) is an equation that involves a function $y(x)$ and one or more of its derivatives.

The order of an ODE is the highest numbe of derivatives that appear in the equation.

An nth order ODE is called linear if is of the form

$$\sum_{k=0}^{n} a_k(x) \frac{d^k y(x)}{dx^k} = q(x)$$

If an ODE is not linear, then it called non-linear (shock!).

Examples

(i) $m\dfrac{d^2x}{dt^2} = -kx$ governs the motion of a particle of mass m that is attached to a spring. It's a 2nd order linear ODE.

(ii) $\frac{d^2h}{dt^2} + k\left(\dfrac{dh}{dt}\right)^2 - g = 0$. This roughly describes the motion of a parachute.

$$\begin{aligned} m =&\ \text{mass of box and parachute} \\ h(t) =&\ \text{height above the ground at time} \\ \rho =&\ \text{density of air, } C_d \equiv \text{ dry coefficient} \\ g =&\ \text{acceleration due to gravity} \\ k =&\ \frac{\pi \rho C_d D^2}{8m} \end{aligned}$$

Note this a 2nd order non-linear ODE¿

Solutions of ODES §10.4

A solution of an ODE is any function $y(x)$ that satisfies the equation.

Example

Verify that

$$y(x) = \sqrt{1-x^2}$$

solves the 1st order non-linear ODE

$$\frac{dy}{dx} + \frac{x}{y} = 0$$

Solution

$$\begin{aligned} \frac{dy}{dx} + \frac{x}{y} &= \frac{-2x}{2\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \\ &= \frac{-x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \\ &= 0 \end{aligned}$$

General and particular solutions §10.4.2

We say that a function $y = y(x, \alpha_1, \alpha_2, ..., \alpha_n)$ that on parameters $\alpha_1, \alpha_2, ..., \alpha_n$ is a **general solution** of an ODE if every solution of the ODE given by $y(x, \alpha_1, \alpha_2, ..., \alpha_n)$ for some choice of the parameters $\alpha_1, \alpha_2, ..., \alpha_n$.
A **particular solution** is any solution of the ODE.

Example

The general solution of the ODE

$$\frac{d^2x}{dt^2} + x = 0$$

is

$$x(t) = A\sin(t) + B\cos(t),\ \ A, B \in \mathbb{R}$$

So then setting $A = B = 1$, we get the particular solution

$$x(t) = \sin(t) + \cos(t)$$

Remark

One generally expects that the general solution of an nth order ODE will depened on n parameters.

Boundary value problems (BVP) §10.4.3

Example

Solve the BVP:

$$\frac{dy^2}{dx^2} + y = 0,\ \ 0 < x < \frac{\pi}{2}$$
$$y(0) = -1,\ \ y\left(\frac{\pi}{2}\right) = 2$$

Solution

The general solution to

$$\frac{d^2y}{dx^2} + y = 0$$

is

$$x(t) = A\sin(x) + B\cos(x),\ \ A, B \in \mathbb{R}$$

The boundary conditions (BCs) then imply

$$y(0) = B = -1 \text{ and } y\left(\frac{\pi}{2}\right) = A = 2$$

so our solution is

$$y(x) = 2\sin(x) - \cos(x)$$

Initial value problems (IVP) §10.4.3

Example

Newton's law of cooling states that the temperature of a homogeneous object satisfies

$$\frac{dT}{dt} = -K(T - T_a)$$

where T_a is the ambient temperature, $T(t)$ is temperature of body at time t, $K > 0$ is some decay constant. The initial condition is

$$T(t_0) = T_0$$

Find the temperature of the body at time $t > t_0$ assuming that $K > 0$ and $T_0 > T_a 2 \times 3$.