ENG1005: Lecture 20

Lex Gallon

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Contents

Repeated eigenvalues §5.7.4 **1**
Remarks 1

Parametric curves **2**
Examples 2

Tangent vectors **2**
Example 3

Functions of several variables §9.6, 9.6.1 **3**
Definition 3
Examples 3

Parametric surfaces **3**
Definition 3

Video link

Click here for a recording of the lecture.

Repeated eigenvalues §5.7.4

If A is an $n \times n$ matrix, then the characteristic polynomial of A is

$$\mathrm{c}(\lambda) = |A - \lambda \mathbb{I}_n| = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} + ... + (\lambda - \lambda_p)^{m_p}$$

where

(i) the λ_i, $1 \leq i \leq p \leq n$, are the distinct roots ($\lambda_i \neq \lambda_j$ for $i \neq j$) of $\mathrm{c}(\lambda)$ (in general, they are complex).

(ii) and the m_i, $1 \leq i \leq p$, is the algebraic multiplicity of the root λ_i.

Remarks

(a) If $p < n$, then A may not possess a complete set of eigenvectors.

(b) If $m_i > 1$, then the number n_i of linearly independent eigenvectors with associated eigenvalue λ_i satisfies

$$1 \leq n_i \leq m_i$$

Parametric curves

A parametric curve in $\mathbb{R}^3$ (can do this in $\mathbb{R}^n$ also) is a set of the form

$$\mathcal{C} = \{\mathbf{r}(t) | t \in (a, b)\} \quad \text{(note whether interval is open or closed doesn't matter)}$$

where

$$\mathbf{r}(t) = (x(t), y(t))$$

We call $\mathbf{r}(t)$ the **parametrisation** of the curve $\mathcal{C}$ (i.e. $\mathbf{r}(t)$ is NOT the curve).
The parametric curve $\mathcal{C}$ is said to be continuous (differentiable) if the parametrisation $\mathbf{r}(t)$ is continuous (differentiable).
If $\mathbf{r}(t)$ is differentiable, then we define

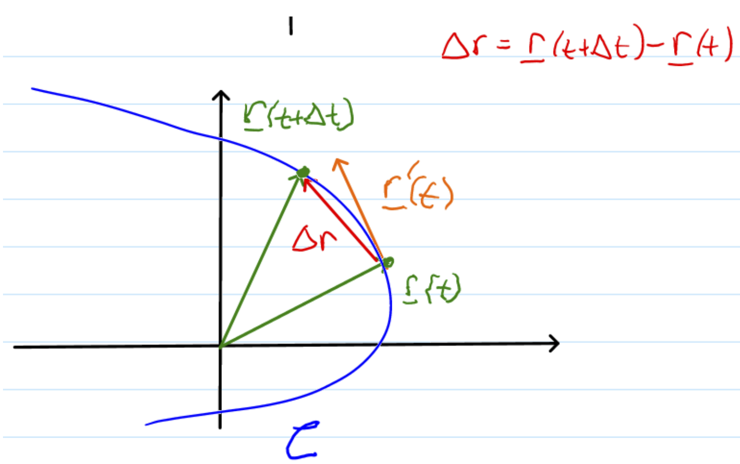
$$\mathbf{r}'(t) = (x'(t), y'(t))$$

Examples

1. $\mathbf{r}(t) = (\cos(t), \sin(t)),\ 0 \leq t \leq 2\pi$.
   Let $x = \cos(t),\ y = \sin(t)$, and then note that
   $$x^2 + y^2 = 1$$
   So this parametrisation clearly traces out a circle.

2. $\mathbf{r}(t) = (t, t^2),\ t \in \mathbb{R}$
   Let $x = t$ and $y = t^2$ and then note that
   $$y = x^2$$
   So this parametrisation traces a parabola.

Tangent vectors



The approximate tangent vector is given by

$$\frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

So clearly, the tangent is given by

$$\lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \mathbf{r}'(t)$$

So $\mathbf{r}'(t)$ gives a tangent vector to the parametric curve $\mathcal{C}$ at the point $\mathbf{r}(t)$.

Example

$$\mathbf{r}(t) = (\cos(t), \sin(t)),\ 0 \leq t \leq 2\pi$$

Note: I already know that the tangent vector of a circle is at 90 degrees to the radius vector.

$$\mathbf{r}'(t) = (-\sin(t), \cos(t))$$

If this is indeed perpendicular to the radius vector then we would expect the dot product to be zero:

$$\begin{aligned}\Rightarrow \mathbf{r}(t) \cdot \mathbf{r}'(t) &= (\cos(t), \sin(t)) \cdot (-\sin(t), \cos(t)) \\ &= -\cos(t)\sin(t) + \sin(t)\cos(t) \\ &= 0\end{aligned}$$

Functions of several variables §9.6, 9.6.1

Definition

A function of n variables, denoted $f(\mathbf{x})$, $(\mathbf{x} = (x_1, x_2, ..., x_n))$ is a rule that associates to each point $\mathbf{x} \in D \subset \mathbb{R}^n$, D is called the domain, a number $f(\mathbf{x}) \in \mathbb{R}$; symbolically, we write

$$f : D \subset \mathbb{R}^n \to \mathbb{R} : \mathbf{x} \mapsto f(\mathbf{x})$$

Note that $f(\mathbf{x})$ is *usually* defined by a formula.

Examples

(a) $f(x, y) = \ln(x^2 + y^2)$. We can assume $D = \mathbb{R}^2 \backslash \{(0, 0)\}$

(b) $f(x, y, z) = e^{xyz},\ D = \mathbb{R}^3$

Parametric surfaces

Definition

A parametric surface inf $\mathbb{R}^3$ is a set S that is defined by

$$S = \{\mathbf{r}(u, v)\}|(u, v) \subset D \subset \mathbb{R}^2$$

where $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$.
We call $\mathbf{r}(u, v)$ the **parametrisation** (or parametric equation) of S.

An important special case of a parametric surface is the graph of a function $f(x, y)$ that is defined by

$$S = \{(x, y, f(x, y))|(x, y) \in D\}$$