ENG1005: Lecture 26

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Contents

Video link

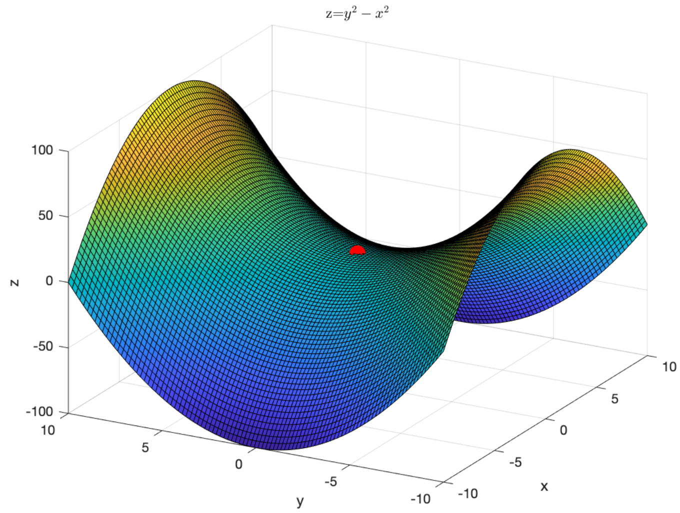
`https://echo360.org.au/lesson/G_32340f5d-ff38-43d2-be9d-d88ddb1b3611_b944cecf-8ba5-40d3-a870-02`
`2020-05-20T14:58:00.000_2020-05-20T15:53:00.000/classroom#sortDirection=desc`

Examples - continued

(iii) $f(x, y) = y^2 - x^2$.

$$\nabla f(x, y) = (-2x, 2y)$$
$$\nabla f(0, 0) = (0, 0) \Rightarrow (0, 0) \text{ is a critical point}$$

But note that it is neither a maximum or a minimum. It is a **saddle point**.

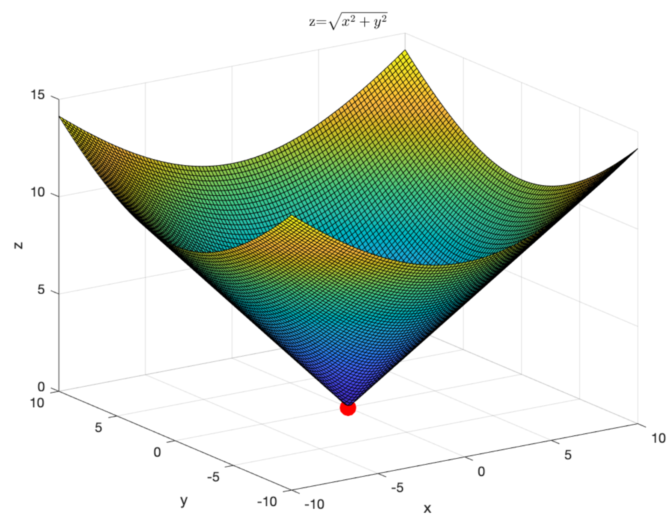


(iv) $f(x, y) = \sqrt{x^2 + y^2}$

$$\nabla f(x, y) = (\frac{2x}{\sqrt{x^2 + y^2}}, \frac{2y}{\sqrt{x^2 + y^2}})$$

$\nabla f(0, 0)$ DNE

This is still a critical point, and from the graph it is clear that this is an absolute minimum.



Example

Let $f(x, y) = 4xy - x^{-}y^4$. Find all the critical points of $f(x, y)$.

Solution

$$\begin{aligned}
\nabla f(x,y) = \mathbf{0} &\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 4y - 4x^3 = 0 \\ \frac{\partial f}{\partial x} = 4x - 4y^3 = 0 \end{cases} \\
&\Rightarrow \begin{cases} y = x^3 \\ x - y^3 = 0 \end{cases} \\
&\Rightarrow x - x^9 = 0 \\
&\Rightarrow x(x - x^8) = 0 && \Rightarrow x = 0 \text{ or } x^8 = 1 \\
&\Rightarrow x = 0, \pm 1
\end{aligned}$$

Therefore,

$$(0,0), (1,1), (-1,-1)$$

are all the critical points of $f(x,y)$.

Classifying critical points §9.7.2

Theorem

Suppose $f(x,y)$ and its derivatives up to 3rd order are continuous on $B_R((a,b))$, $\nabla f(x,y) = \mathbf{0}$ and let

$$A = \frac{\partial^2 f}{\partial x^2}(a,b),\ B = \frac{\partial^2 f}{\partial x \partial y}(a,b) \text{ and } C = \frac{\partial^2 f}{\partial y^2}(a,b)$$

then

(i) $f(x,y)$ has a local minimum at (a,b) if $AC > B^2$ and $A > 0$.

(ii) $f(x,y)$ has a local maximum at (a,b) if $AC > B^2$ and $A < 0$.

(iii) $f(x,y)$ has a saddle point at (a,b) if $AC < B^2$.

(iv) and the test gives no information if $AC = B^2$.

Example

Classify all the critical points of $f(x,y) = 4xy - x^4 - y^4$.

Solution

From before, we know that the critical points are

$$(0,0), (1,1), (-1,-1)$$

Now,

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -12x^2 && \left(A = \frac{\partial^2 f}{\partial x^2}(a,b)\right) \\
\frac{\partial^2 f}{\partial y^2} &= -12y^2 && \left(C = \frac{\partial^2 f}{\partial y^2}(a,b)\right) \\
\frac{\partial^2 f}{\partial x \partial y} &= 4 && \left(B = \frac{\partial^2 f}{\partial x \partial y}(a,b)\right)
\end{aligned}$$

and

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 144x^2y^2 - 16 \quad (AC - B^2)$$

Critical point	A	$AC - B^2$	Conclusion
$(0,0)$	0	$-16 < 0$	Saddle point
$(1,1)$	$-12 < 0$	$128 > 0$	Local max
$(-1,-1)$	$-12 < 0$	$128 > 0$	Local max

Absolute maxima and minima §9.7.4

Theorem

If $D \subset \mathbb{R}^2$ is closed and bounded and $f(x,y)$ is continuous on D then there exists points in D where $f(x,y)$ achieves an absolute maximum and an absolute minimum.

Terminology

- D is bounded if $D \subset B_R(\mathbf{0})$ for some $R > 0$.
- D is closed if it is defined by closed inequalities.
  e.g.

$$D = \{(x,y)|x^2 + y^2 \leq 1 \text{ and } x + y \geq 1\} \text{ is closed.}$$
$$D = \{(x,y)|x^2 + y^2 \leq 1 \text{ and } x > 0\} \text{ is NOT closed.}$$

  Note the $\geq$ as opposed to the $>$. I'll have to check but I believe what Todd means is that it is closed if there is an equality, not simply a $>$ or $<$.

Example

The temperature at every point in the closed unit disc

$$D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1$$

is given by

$$T(x,y) = (x+y)e^{-(x^2+y^2)}$$

Find the maximum and minimum temperature and where these are achieved on the disc.

Solution

First note that this is a nice continuous function.

Step 1: Find all the critical points of $T(x,y)$ in the interior of D.

$$\nabla T(x,y) = \mathbf{0} \Rightarrow \begin{cases} \frac{\partial T}{\partial x} = (1 - 2x(x+y))e^{-(x^2+y^2)} = 0 \\ \frac{\partial T}{\partial y} = (1 - 2y(x+y))e^{-(x^2+y^2)} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial T}{\partial x} = 1 - 2x(x+y) = 0 \\ \frac{\partial T}{\partial y} = 1 - 2y(x+y) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial T}{\partial x} = 2x(x+y) = 1 \\ \frac{\partial T}{\partial y} = 2y(x+y) = 1 \end{cases}$$

I'm gonna solve this by noting that $x = y$ as you can see that both equations are very symmetric. Therefore, from either equation we get

$$\begin{aligned} 2x(x+x) &= 1 \\ 4x^2 &= 1 \\ \Rightarrow x &= \pm\frac{1}{2} \end{aligned}$$

Therefore the critical points are

$$\left(-\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)$$

Note also that both are inside D.

Step 2: Find all the critical points of $T(x, y)$ on the boundary of D. We can parametrise the boundary of D by

$$\mathbf{r}(t) = (\cos(t), \sin(t)),\ 0 \leq t \leq 2\pi$$

Restricting $T(x, y)$ to the boundary gives

$$\begin{aligned} g(t) &= T(\cos(t), \sin(t)),\ 0 \leq t \leq 2\pi \\ &= (\cos(t) + \sin(t))e^{-1} \end{aligned}$$