ENG1005: Lecture 16

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Contents

Video link

Click here for a recording of the lecture.

LU decomposition - continued

$$LU\mathbf{x} = \mathbf{b} \Leftrightarrow LU\mathbf{x} = \mathbf{b} \Leftrightarrow \left.\begin{cases} L\mathbf{y} = \mathbf{b} \\ U\mathbf{x} = \mathbf{y} \end{cases}\right\} (*)$$

Determining the LU decomposition

Algorithm to perform LU decomposition:

(i) Perform elementary row operations

$$R_i \to R_i - \alpha_{ij} R_j$$

on A until A is in upper triangular form, which gives U.

(ii) Replace the zeros in the (i, j) position of the identity matrix with the constants α_{ij} to obtain the lower triangular matrix L.

This algorithm then guarantees that

$$A = LU$$

provided no row swaps are needed to find U.

Remark

For $n \times n$ matrices A that require row swaps to be reduced into upper triangular form there always exists an $n \times n$ permutation matrix P such that PA has an LU decomposition, that is

$$PA = LU$$

So in this case, rather than solving

$$A\mathbf{x} = \mathbf{b},$$

we solve

$$PA\mathbf{x} = P\mathbf{b}$$

using an LU decomposition.

Example

Find an LU decomposition for

$$\begin{bmatrix} 3 & 1 & 5 \\ 3 & 4 & 8 \\ 6 & 11 & 3 \end{bmatrix}$$

(i)

$$R_2 \to R_2 - R_1,\ \alpha_{21} = 1$$

$$\begin{bmatrix} 3 & 1 & 5 \\ 0 & 3 & 3 \\ 6 & 11 & 3 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1,\ \alpha_{31} = 2$$

$$\begin{bmatrix} 3 & 1 & 5 \\ 0 & 3 & 3 \\ 0 & 9 & -7 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2,\ \alpha_{32} = 3$$

$$\begin{bmatrix} 3 & 1 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & -16 \end{bmatrix}$$

So

$$U = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & -16 \end{bmatrix}$$

(ii) So start with the identity matrix, then fill out appropriate cells with α values from above (compare indices of αs and you should be able to see what we've done).

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 5 \\ 3 & 4 & 8 \\ 6 & 11 & 3 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & -16 \end{bmatrix}$$

Example

Solve

$$
\begin{aligned}
3x_1 + x_2 + 5x_3 &= 10, \\
3x_1 + 4x_2 + 8x_3 &= 19 \\
6x_1 + 11x_2 + 3x_3 &= 31
\end{aligned}
$$

using LU decomposition.

Solution

Write this as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} 3 & 1 & 5 \\ 3 & 4 & 8 \\ 6 & 11 & 3 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 10 \\ 19 \\ 31 \end{bmatrix}$$

Then by the previous example, we have

$$LU\mathbf{x} = \mathbf{b}$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & -16 \end{bmatrix}$$

Setting $\mathbf{y} = U\mathbf{x}$, and we solve

$$L\mathbf{y} = \mathbf{b} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 19 \\ 31 \end{bmatrix}$$

It follows that

$$
\begin{aligned}
y_1 &= 10 \\
y_1 + y_2 &= 19 \\
2y_1 + 3y_2 + y_3 &= 31 \\
\Rightarrow y_1 &= 10 \\
y_2 &= 19 - y_1 = 19 - 10 = 9 \\
y_3 &= 31 - 2y_1 - 3y_2 = 31 - 2 \times 10 - 3 \times 9 = -16 \\
\Rightarrow \mathbf{y} &= \begin{bmatrix} 10 \\ 9 \\ -16 \end{bmatrix}
\end{aligned}
$$

We next solve

$$U\mathbf{x} = \mathbf{y} \Rightarrow \begin{bmatrix} 3 & 1 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & -16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \\ -16 \end{bmatrix}$$

$$
\begin{aligned}
3x_1 + x_2 + 5x_3 &= 10 \\
3x_2 + 3x_3 &= 9 \\
-16x_3 &= -16 \\
\Rightarrow x_3 &= 1 \\
x_2 &= \frac{1}{3}(9 - 3x_3) = \frac{1}{3}(9-3) = 2 \\
x_1 &= \frac{1}{3}(10 - x_2 - 5x_3) = \frac{1}{3}(10 - 2 - 5 \times 1) = 1
\end{aligned}
$$

So the solution to the system of linear equations is

$$(x_1, x_2, x_3) = (1, 2, 1)$$

Matrix inversion §5.4

Consider now a linear system of equations

$$A\mathbf{x} = \mathbf{b} \quad (*)$$

where A is an $n \times n$ matrix. Suppose we can find an $n \times n$ matrix B such that

$$BA = \mathbb{I}_n$$

Then multiplying (*) on the left by B would give

$$
\begin{aligned}
BA\mathbf{x} &= B\mathbf{b} \\
\mathbb{I}_n\mathbf{x} &= B\mathbf{b} \\
\mathbf{x} &= B\mathbf{b}
\end{aligned}
$$

This shows that (*) would have a unique solution given by

$$\mathbf{x} = B\mathbf{b}$$

Here, B is the **inverse matrix** of A.

Definition

Given an $n \times n$ matrix A, we say that A is invertible or **non-singular** if there exists an $n \times n$ matrix B such that

$$AB = BA = \mathbb{I}_n$$

The matrix B is the **inverse matrix** of A and it is denoted by A^{-1}. If no such B exists, then A is **singular** (or not invertible).
I.e.

$$AA^{-1} = A^{-1}A = \mathbb{I}_n$$