# Lecture 6 notes

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## Sequences §7.2

### Definition

A sequence is a countably infinite collection of numbers

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, ...\}$$

### Example

$$a_n = \frac{1}{n}, n \in \mathbb{N}$$

## Limits of sequences §7.5, 7.5.1

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to a limit $l$, denoted if for any $\varepsilon > 0$, there exists $N = N(\varepsilon), \varepsilon \in \mathbb{N}$ such that $|a_n - l| < \varepsilon|$ for $n > N$.

### Example

Show that

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

### Solution

Fix $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$.

Then

$$n \geq N \Rightarrow \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

and so

$$|\frac{1}{n} - 0| = \frac{1}{n} = \frac{1}{n} < \varepsilon$$

## Properties of limits §7.5.2

### Theorem

Suppose $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} b_n = m$, then

1. $$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$
2. $$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$$
3. $$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \text{ provided that } m \neq 0$$

### Theorem (Uniqueness of limit

Suppose $\lim_{n\to\infty} a_n = l$ and $\lim_{n\to\infty} a_n = l'$. Then $l = l'$. Moreover, if $\{a_{n_j}\}_{j=1}^{\infty}$ is my sub-sequence of $\{a_n\}_{n=1}^{\infty}$. Then

$$lim_{n\to\infty} \left(a_{n_j}\right) = lim_{n\to\infty} \left(a_n\right) = l$$

For example, if you took the limit of every second term in the sequence, it would still be the same as the limit of every term.

### Theorem

Suppose $\lim_{x \to x_0} f(x) = l$ and $\lim_{n \to \infty} a_n = x_0$. Then

$$\lim_{n \to \infty} f(a_n) = l$$

Conversely, if $\lim_{n \to \infty} f(a_n) = l$ for all sequences $\{a_n\}_{n=1}^{\infty}$ satisfying $\lim_{n \to \infty} a_n = x_0$, then $\lim_{x \to x_0} f(x) = l$.

## Non-existence of limits

There can be many reasons why limits of sequences do not exist. The two most common are:

1. Oscillation

   The limit $\lim_{n \to \infty} (-1)^n$ DNE.

2. Unboundedness

   The limit $\lim_{n' \to \infty} ln(n)$ DNE.

## Series §7.6

### Definition

A sum of the elements of a sequence $\{a_n\}_{n=1}^{\infty}$,
i.e.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$$

is called a series.

### Examples

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + ...$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + ...$$

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + ...$$

### Definition

Given a sequence $\{a_n\}_{n=1}^{\infty}$ and $N \in \mathbb{N}$, the Nth partial sum is defined by

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + ... + a_N$$

## Convergence and divergence of a series

### Definition

A series $\sum_{n=1}^{\infty} a_n$ is said to converge (diverge) if the limit $\lim_{N\to\infty} S_N$ of partial sums exists (does not exist).

If $\lim_{N\to\infty} S_N$ converges, then we define

$$\sum_{n=1}^{\infty} a_n := \lim_{N\to\infty} S_N$$

## Geometric Series §7.3.2

### Definition

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + ...$$

Note: this could be multiplied by a constant and still be a geometric series.

Consider the partial sums

$$S_N = \sum_{n=0}^{N} a^n = 1 + a + a^2 + ... + a^N$$

Multiply by a to get

$$aS_N = \sum_{n=0}^{N} a^n = a + a^2 + ... + a^N + a^{N+1}$$

So

$$aS_N = S_{N-1} + a^{N+1}$$

This gives

$$(a-1)S_N = -1 + a^{N+1}$$

$$S_N = \frac{1}{1-a} - \frac{a^{N+1}}{1-a}$$

Taking the limit $N \to \infty$, we find that

$$\lim_{N\to\infty} S_N = \frac{1}{1-a} - \lim_{N\to\infty} \frac{a^{N+1}}{1-a} \qquad = \begin{cases} \frac{1}{1-a} & \text{if } |a| < 1 \\ \text{DNE} & \text{if } |a| \geq 1 \end{cases}$$

Thus

$$\sum_{n=0}^{N} a^n = \frac{1}{1-a} \text{ for } |a| < 1$$

and

$$\sum_{n=0}^{N} a^n \text{ diverges for } |a| \geq 1$$