ENG1005: Lecture 12

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Contents

Lines cont.

Example

Let

$$\mathbf{r}(t) = \mathbf{u} + t\mathbf{w},\ t \in \mathbb{R}$$

parametrise a line ℓ. Show that the line m that connects the origin to the closest point on ℓ is orthogonal to ℓ.

Solution

$$\begin{aligned}
\text{Let } d(t) = |\mathbf{r}(t)|^2 &= \mathbf{r}(t) \cdot \mathbf{r}(t) \\
&= (\mathbf{u} + t\mathbf{w}) \cdot (\mathbf{u} + t\mathbf{w}) \\
&= \mathbf{u} \cdot \mathbf{u} + t\mathbf{u} \cdot \mathbf{w} + t\mathbf{w} \cdot \mathbf{u} + t^2\mathbf{w} \cdot \mathbf{w} \\
&= |\mathbf{u}|^2 + 2t\mathbf{u} \cdot \mathbf{w} + t^2|\mathbf{w}|^2
\end{aligned}$$

It's obvious that as t approaches $\pm\infty$, the distance also goes to infinity. Therefore we know there must be some minimum in between.

$d(t)$ has minimum where

$$\begin{aligned}
d'(t) = 0 &\Rightarrow 2\mathbf{u} \cdot \mathbf{w} + 2t|\mathbf{w}|^2 = 0 \\
&\Rightarrow t = \frac{-\mathbf{u} \cdot \mathbf{w}}{|\mathbf{w}|^2}
\end{aligned}$$

This shows that the point $\mathbf{p}$ on ℓ that is closest to $\mathbf{0}$ is

$$\mathbf{p} = \mathbf{r}\left(\frac{-\mathbf{u}\cdot\mathbf{w}}{|\mathbf{w}|^2}\right) = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$$

Now we want to show that $\mathbf{p}$ is orthogonal to w.

$$\begin{aligned}\mathbf{w}\cdot\mathbf{p} &= \mathbf{w}\cdot\left(\mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\right)\\ &= \mathbf{u}\cdot\mathbf{w} - \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}\cdot\mathbf{w}\\ &= \mathbf{u}\cdot\mathbf{w} - \frac{\mathbf{u}\cdot\mathbf{w}}{|\mathbf{w}|^2}|\mathbf{w}|^2\\ &= \mathbf{u}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{u} = 0\end{aligned}$$

This shows that $\ell \perp m$.

Algebraic/vector equation of a plane §4.3.3

A plane $\mathbb{P} \subset \mathbb{R}^3$ is determined by

(i) a point $\mathbf{p} \in \mathbb{P}$

(ii) and a normal vector $\mathbf{n}$ to the plane.

So $q \in \mathbb{P}$ if and only if

$$\mathbf{n}\cdot(\mathbf{q}-\mathbf{p}) = 0$$

This is known as the algebraic (vector) equation of the plane $\mathbb{P}$.

$$\mathbb{P} = \{\mathbf{q} \in \mathbb{R}^3 \mid \mathbf{n}\cdot(\mathbf{q}-\mathbf{p}) = 0\}$$

Letting

$$\mathbf{n} = (a, b, c),\ \ \mathbf{p} = (x_0, y_0, z_0),\ \ \mathbf{q} = (x, y, z)$$

then

$$\begin{aligned}\mathbf{n}\cdot(\mathbf{q}-\mathbf{p}) = 0 &\Leftrightarrow (a, b, c)\cdot(x - x_0, y - y_0, z - z_0) = 0\\ &\Leftrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0\end{aligned}$$

Note that it's easy to read the normal vector $\mathbf{n}$ from this as $\mathbf{n} = (a, b, c)$

m-dimension

The same equation

$$\mathbf{n}\cdot(\mathbf{q}-\mathbf{p}) = 0 \qquad\qquad (\mathbf{n} = (n_1, n_2, ..., n_m),\ \ \mathbf{p} = (p_1, p_2, ..., p_m),\ \ \mathbf{q} = (x_1, x_2, ..., x_m)$$

also describe a (hyper) plane in m-dimension , $m \in \mathbb{N}$.

Example

Find the algebraic equation of the plane $\mathbb{P}$ that contains the points $(1, 3, 2),\ (3, -1, 6)$ and $(5, 2, 0)$.

Solution

Remember that, to describe a plane, we need one point (we have 3!) and we need a normal vector.

$$\text{Let } \mathbf{a} = \mathbf{p_2} - \mathbf{p_1},\ \mathbf{b} = \mathbf{p_3} - \mathbf{p_1}$$

where $\mathbf{p_1} = (1,3,2)$, $\mathbf{p_2} = (3,-1,6)$, $\mathbf{p_3} = (5,2,0)$

We can find a normal vector with $\mathbf{n} = \mathbf{a} \times \mathbf{b}$.

So

$$\mathbf{a} = (3,-1,6) - (1,3,2) = (2,-4,4)$$

$$\mathbf{b} = (5,2,0) - (1,3,2) = (4,-1,-2)$$

and

$$\begin{aligned}\mathbf{n} = \mathbf{a} \times \mathbf{b} &= (-4 \cdot (-2) - 4 \cdot (-1), 4 \cdot 4 - 2 \cdot (-2), 2 \cdot (-1) - (-4) \cdot 4 \\ &= (12, 20, 14) \qquad \text{(who knows if I wrote that down correctly, I don't care)}\end{aligned}$$

This the equation of the plane is

$$\mathbf{n} \cdot (\mathbf{q} - \mathbf{p} = 0$$

$$\Leftrightarrow$$

$$\begin{aligned}&(12,20,14) \cdot ((x,y,z) - (1,3,2)) = 0 \\ \Rightarrow\ &(12,20,14) \cdot ((x-1, y-3, z-2)) = 0 \\ \Rightarrow\ &12(x-1) + 20(y-3) + 14(z-2) = 0\end{aligned}$$

Parametric equation of a plane

Suppose $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{P}$, and let

$$\mathbf{a} = \mathbf{p}_2 - \mathbf{p}_1$$

$$\mathbf{b} = \mathbf{p}_3 - \mathbf{p}_1$$

Then any point $\mathbf{q} \in \mathbb{P}$ can be written as

$$\mathbf{q} = \mathbf{p}_1 + u\mathbf{a} + v\mathbf{b} \Leftrightarrow \mathbf{q} - \mathbf{p}_1 = u\mathbf{a} + v\mathbf{b}$$

for some numbers $u, v \in \mathbb{R}$.

The equation

$$\mathbf{q}(u,v) = \mathbf{p}_1 + u\mathbf{a} + v\mathbf{b},\ u,v \in \mathbb{R}$$

is called the parametric equation of the plane $\mathbb{P}$, while

$$\mathbb{P} = \{\mathbf{q}(u,v) \mid u,v \in \mathbb{R}\}$$

Example

Find the parametric representation of the plane passing through the points

$$(1,3,2),\ (3,-1,6),\ (5,2,0)$$

Solution

Set

$$\mathbf{p}_1 = (1,3,2),\ \ \mathbf{p}_2 = (3,-1,6),\ \ \mathbf{p}_3(5,2,0)$$

We let

$$\mathbf{a} = \mathbf{p}_2 - \mathbf{p}_1 = (2,-4,4)$$

$$\mathbf{b} = \mathbf{p}_3 - \mathbf{p}_1 = (4,-1,-2)$$

From these, we see that the parametrised equation of the plane is

$$\begin{aligned} \mathbf{q}(u,v) &= \mathbf{p_1} + u\mathbf{a} + v\mathbf{b} \\ &= (1,3,2) + u(2,-4,4) + v(4,-1,-2) \\ &= (1+2u+4v, 3-4u-v, 2+4u-2v) \\ \Rightarrow \mathbf{q}(u,v) &= (1+2u+4v, 3-4u-v, 2+4u-2v),\ \ u,v \in \mathbb{R} \end{aligned}$$