ENG1005: Lecture 11

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Video link

Click here for a recording of the lecture.

Vector products

Scalar (or inner or dot) product

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= (u_1, u_2, u_3)\cdot(v_1, v_2, v_3)\\ &= u_1v_1 + u_2v_2 + u_3v_3 \qquad \text{(a scalar)}\end{aligned}$$

This is equivalent to

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos(\theta)$$

Where $|\mathbf{u}|$ is known as the norm, length or magnitude of $\mathbf{u}$.

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Cross product

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= (u_1,u_2,u_3)\times(v_1,v_2,v_3)\\ &= (u_2v_3-u_3v_2, u_3v_1-u_1v_3, u_1v_2-u_2v_1)\\ &= (u_2v_3-u_3v_2)\hat{\mathbf{i}}+(u_3v_1-u_1v_3)\hat{\mathbf{j}}+(u_1v_2-u_2v_1)\hat{\mathbf{k}}\end{aligned}$$

Geometric interpretation

¡MAYBE INSERT PICTURE HERE¿ We can note that the cross product of 2 vectors $\mathbf{u},\mathbf{v}$ is perpendicular to the plane on which both vectors lie.

a) $\mathbf{u}\times\mathbf{v} = |\mathbf{u}||\mathbf{v}|\sin(\theta)\mathbf{n}$, where $\mathbf{n}$ is that normal vector (and is a unit vector!).

   Note that vectors are perpendicular/orthogonal ($\mathbf{u}\perp\mathbf{v}$) if and only if their dot product is **zero**.

   Also note that, in the case of the cross product, $\mathbf{n}\perp\mathbf{u}$ and $\mathbf{n}\perp\mathbf{v}$. It then also follows that $\mathbf{n}\cdot\mathbf{u}=\mathbf{n}\cdot\mathbf{v}=0$.

b) $\mathbf{u}\times\mathbf{v} = 0 = \mathbf{u}\times\mathbf{v}\cdot\mathbf{v}$ $\qquad$ $(\mathbf{u}\times\mathbf{v}\perp\mathbf{u}\ \&\ \mathbf{v})$

c) $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\theta)$ ¡MAYBE INSERT PICTURE HERE OF PARALLELOGRAM¿

Example

Compute $\mathbf{u}\times\mathbf{v}$, where $\mathbf{u}=(1,2,3)$ and $\mathbf{v}=(4,-3,2)$

Solution

$$\begin{aligned}\mathbf{u}\times\mathbf{v} &= (u_2v_3-u_3v_2, u_3v_1-u_1v_3, u_1v_2-u_2v_1)\\ &= (2\cdot 2-3\cdot(-3), 3\cdot 4-1\cdot 2, 1\cdot(-3)-2\cdot 4)\\ &= (13,10,-11)\end{aligned}$$

You could then verify this is true by checking that:
$\mathbf{u}\times\mathbf{v}\cdot\mathbf{u}=0\ \&\ \mathbf{u}\times\mathbf{v}\cdot\mathbf{v}=0$

Lines §4.3.1

Parametric/vector equation

¡MAYBE INSERT PICTURE HERE¿ Basically, if we have two vectors representing points, we have the vector from one point to the other being $\mathbf{w}=\mathbf{v}-\mathbf{u}$.
$\mathbf{r}(t)=\mathbf{u}+t(\mathbf{v}-\mathbf{u})$
So $\mathbf{r}(0)=\mathbf{u}, r(1)=\mathbf{v}$. This gives us some function $\mathbf{r}(t)$ that lies between $\mathbf{u}$ and $\mathbf{v}$ for $0\le t\le 1$ (note that t can be any real number if you want).
$l=\{\mathbf{r}(t)|-\infty<t<\infty\}$ $\qquad$ (line)

Example

Find the parametric equation of the line passing through $(1,2,3)$ and $(-1,3,-2)$.

Solution

Let $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (-1, 3, -2)$.
We then set

$$\mathbf{w} = \mathbf{v} - \mathbf{u} = (-1, 3, -2) - (1, 2, 3) = (-2, 1, -5)$$

Thus the parametric equation of the line is

$$\mathbf{r}(t) = \mathbf{u} + t\mathbf{w} = (1, 2, 3) + t(-2, 1, -5),\ t \in \mathbb{R}$$

$$\Leftrightarrow$$

$$\mathbf{r}(t) = (1 - 2t, 2 + t, 3 - 5t),\ t \in \mathbb{R}$$

Algebraic/Cartesian equation

Let $\mathbf{r}(t) = \mathbf{u} + t\mathbf{w},\ t \in \mathbb{R}$ parametrise a line l. Then

$$\begin{aligned}
\mathbf{x} = (x_1, x_2, x_3) \in l &\Leftrightarrow \mathbf{x} = \mathbf{r}(t) \text{ for some } t \in \mathbb{R} \\
&\Leftrightarrow (x_1, x_2, x_3) = (u_1, u_2, u_3) + t(w_1, w_2, w_3) \\
&\Leftrightarrow (x_1, x_2, x_3) = (u_1 + tw_1, u_2 + tw_2, u_3 + tw_3) \\
&\Leftrightarrow (x_1, x_2, x_3) = u_i + tw_i,\ i = 1, 2, 3 \\
&\Leftrightarrow \frac{x_i - u_i}{w_i} = t,\ i = 1, 2, 3\ (w_i \neq 0) \\
&\Leftrightarrow \frac{x_1 - u_1}{w_1} = \frac{x_2 - u_2}{w_2} = \frac{x_3 - u_3}{w_3}
\end{aligned}$$

which is the algebraic equation of the line. This shows that we can determine the points that lie on a line l passing through the vectors $\mathbf{u}$ and $\mathbf{v} = \mathbf{u} + \mathbf{w}$ by solving

$$\frac{x_1 - u_1}{w_1} = \frac{x_2 - u_2}{w_2} = \frac{x_3 - u_3}{w_3}$$

Note: If $w_1 = 0$,

$$x_1 = u_1,\ \frac{x_2 - u_2}{w_2} = \frac{x_3 - u_3}{w_3}$$

Example

Find the algebraic equation of the line that passes through the points $(-1, 2, -1)$ and $(1, 1, 1)$.

Solution

Set $\mathbf{u} = (-1, 2, -1)$ and $\mathbf{v} = (1, 1, 1)$.
Then

$$\mathbf{w} = \mathbf{v} - \mathbf{u} = (1, 1, 1) - (-1, 2, -1) = (2, -1, 2)$$

So the algebraic equation of the line is

$$\frac{x + 1}{2} = \frac{y - 2}{-1} = \frac{z + 1}{2}$$