ENG1005: Lecture 19

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Contents

Video link

Click here for a recording of the lecture.

Eigenvalues - continued

Example

$$\text{Let } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

Then,

$$A\mathbf{v} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ -1 \end{bmatrix} = \begin{bmatrix} -i \\ i^2 \end{bmatrix} = i \begin{bmatrix} -1 \\ i \end{bmatrix}$$

$$\Rightarrow A\mathbf{v} = i\mathbf{v}$$

So $\mathbf{v}$ is a complex eigenvector of A with eigenvalue $\lambda = i$.

Characteristic equation

If $\mathbf{v}$ is an eigenvector of matrix A, then

$$A\mathbf{v} = \lambda\mathbf{c}$$

$$(A - \lambda\mathbb{I})\mathbf{v} = \mathbf{0}$$

This equation can have a non-trivial solution if and only if $\det(A - \lambda\mathbb{I}) = 0$. So, to find eigenvalues, we want to solve the equation

$$\det(A - \lambda\mathbb{I}) = 0$$

for λ. This is the **characteristic equation** of A.

Remarks

(a) If λ satisfies the characteristic equation $\det(A - \lambda\mathbb{I}) = 0$, then there exists at least 1 eigenvector $\mathbf{v}$ of A satisfying

$$A\mathbf{v} = \lambda\mathbf{v}$$

(b) If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then λ satisfies the characteristic equation $\det(A - \lambda\mathbb{I}) = 0$.

(c) When written out explicitly, we have

$$\mathrm{c}(\lambda) := \det(A - \lambda\mathbb{I}_n) = c_0 + c_1\lambda + ... + c_n\lambda^n$$

where A is an $n \times n$ matrix. This polynomial is known as the **characteristic polynomial** of A. It can have, at most, n distinct (complex) roots.

(d) If $\mathbf{v}$ is an eigenvector of A, then so is $\alpha\mathbf{v}$ for any $\alpha \neq 0$.

$$A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha\lambda\mathbf{v} = \lambda(\alpha\mathbf{v})$$

$$\Rightarrow A(\alpha\mathbf{v}) = \lambda(\alpha\mathbf{v})$$

(e) If $\mathbf{v}$ is a complex eigenvector of a real matrix A with a complex eigenvalue λ, then the complex conjugate $\overline{\mathbf{v}}$ is also a complex eigenvector of A with a complex eigenvalue $\overline{\lambda}$.

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \Rightarrow \overline{\mathbf{v}} = \begin{bmatrix} \overline{v_1} \\ \overline{v_2} \\ \vdots \\ \overline{v_n} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} -1 \\ i \end{bmatrix} \Rightarrow A\mathbf{v} = \lambda\mathbf{v}, \ \lambda = i$$

$$\Rightarrow A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}, \ \overline{\lambda} = -i$$

Example

Determine all the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

Solution

First, we compute the characteristic polynomial:

$$c(\lambda) = \det(A - \lambda \mathbb{I}) = \det(\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \det(\begin{bmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{bmatrix}) = (2-\lambda)^2 - 4$$

$$\Rightarrow c(\lambda) = \lambda^2 - 4\lambda = \lambda(\lambda - 4)$$

Thus, $c(\lambda) = 0 \Leftrightarrow \lambda = 0$ or $\lambda = 4$. So we now have the eigenvalues of A.

The eigenvector associated with $\lambda = 4$:

$$(A - 4\mathbb{I})\mathbf{v} = \mathbf{0} \Rightarrow \left(\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} &\Rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 4 & -2 & 0 \end{array}\right] \\ &\Rightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] R_2 \to R_2 + 2R_1 \\ &\Rightarrow -2v_1 + v_2 = 0 \end{aligned}$$

Setting $v_1 = 1$, shows that

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is an eigenvector of A with eigenvalue $\lambda_1 = 4$.

Now, the eigenvector associated with $\lambda = 0$:

$$\begin{aligned} (A - 0\mathbb{I})\mathbf{v} - \mathbf{0} &\Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 4 & 2 & 0 \end{array}\right] \\ &\Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] R_2 \to R_2 - 2R_1 \end{aligned}$$

$$\Rightarrow 2v_1 + v_2 = 0$$

Setting $v_1 = 1$, shows that

$$\mathbf{2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

is an eigenvector of A with eigenvalue $\lambda_2 = 0$.

Complete set of eigenvectors §5.7.2, 5.7.7

Definition

An $n \times n$ matrix A has a complete set of eigenvectors if there exists n linearly independent eigenvectors $v_1, v_2, ..., v_n$. That is,

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + ... + \alpha_n \mathbf{v}_n = \mathbf{0} \Leftrightarrow \alpha_1 = \alpha_2 = ... = \alpha_n = 0$$

Theorem

Suppose A is an $n \times n$ matrix.

(a) If A has n distinct eigenvalues, then A has a complete set of eigenvectors.

(b) If A is a **symmetric** matrix, ($A^T = A$), then A has a complete set of **orthonormal** eigenvectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ with associated **real** eigenvalues.

Note that orthonormal means

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} \quad , 1 \leq i, \ j \leq n$$

Note that $[\delta ij] = \mathbb{I}$.

Remark

If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ is a complete set of eigenvectors for an $n \times n$ matrix A, then any n-column vector $\mathbf{w}$ can be expressed as

$$\mathbf{w} = \sum_{j=1}^{n} c_j \mathbf{v}_j$$

Using this, we see

$$A\mathbf{w} = \sum_{j=1}^{n} c_j A\mathbf{v}_j = \sum_{j=1}^{n} c_j \lambda \mathbf{v}_j$$