ENG1005: Lecture 33

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Video link

`https://echo360.org.au/lesson/G_35fe23e0-41ee-4e6f-b0f5-05f4155bb7b0_b944cecf-8ba5-40d3-a870-02`
`2020-06-04T15:58:00.000_2020-06-04T16:53:00.000/classroom#sortDirection=desc`

Laplace transforms of integrals §11.3.2

Let

$$g(t) = \int_0^t f(\tau)\, d\tau.$$

Then, by the Fundamental Theorem of Calculus, we have

$$\frac{dg}{dt}(t) = f(t).$$

So

$$\mathscr{L}[f(t)](s) = \mathscr{L}\left[\frac{dg}{dt}(t)\right](s) = s\mathscr{L}[g(t)](s) - g(0) = s\mathscr{L}\left[\int_0^t f(\tau)\, d\tau\right](s).$$

It follows that

$$\mathscr{L}\left[\int_0^t f(\tau)\, d\tau\right](s) = \frac{1}{s}\mathscr{L}[f(t)](s).$$

The inverse Laplace Transform §11.2.7

If $f(t)$ and $g(t)$ both satisfy

$$|f(t)| \leq Me^{\sigma t} \text{ and } |g(t)| \leq Me^{\sigma t},\ t \geq 0,$$

for some $M \geq 0$ and $\sigma \in \mathbb{R}$, and

$$\mathscr{L}[f(t)](s) = \mathscr{L}[g(t)](s),\ \operatorname{Re}(s) > \sigma.$$

Then it can be shown that

$$f(t) = g(t),\ t \geq 0.$$

This allows us to uniquely define the inverse Laplace transform by

$$\mathscr{L}^{-1}[\mathscr{L}[f(t)]] = H(t)f(t),$$

where $H(t)$ is the Heaviside step function

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{if } t < 0 \end{cases}$$

Bronwich integral (not examinable)

The inverse Laplace transform can be computed using

$$f(t) = \mathscr{L}^{-1}[F(s)](t) := \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma - iT}^{\sigma + iT} F(s)\, ds$$

where all singularities of $F(s)$ lie in the region $\operatorname{Re}(s) > \sigma$.

Example

Compute

$$\mathscr{L}^{-1}\left[\frac{s+1}{s(s^2+2)}\right](t)$$

Solution

Use partial fractions, we know that

$$\begin{aligned} \frac{s+1}{s(s^2+1)} &= \frac{1}{2}\frac{1}{s} + \frac{1 - \frac{1}{2}s}{s^2+2} \\ &= \frac{1}{2}\frac{1}{s} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{s^2+(\sqrt{2})^2} - \frac{1}{2}\frac{s}{s^2+(\sqrt{2})^2} \\ &= \mathscr{L}\left[\frac{1}{2} + \frac{1}{\sqrt{2}}\sin(\sqrt{2}t) - \frac{1}{2}\cos(\sqrt{2}t)\right](s) \end{aligned}$$

So then

$$\mathscr{L}^{-1}\left[\frac{s+1}{s(s^2+2)}\right](t) = \frac{1}{2} + \frac{1}{\sqrt{2}}\sin(\sqrt{2}t) - \frac{1}{2}\cos(\sqrt{2}t)$$

Inverse of the s-shift property §11.2.9

Theorem

Suppose $f(t)$, $t \geq 0$, has a Laplace Transform

$$F(s) = \mathscr{L}[f(t)](s),\ \mathrm{Re}(s) > \sigma,$$

and $a \in \mathbb{R}$. Then

$$\mathscr{L}^{-1}[e^{-as}F(s)](t) = H(t-a)f(t-a).$$

Proof

By definition

$$\begin{aligned}
\mathscr{L}[H(t-a)f(t-a)](s) &= \int_0^\infty H(t-a)f(t-a)\,dt \\
&= \int_{-a}^\infty e^{-s(\tau+a)}H(\tau)f(\tau)\,d\tau, \quad ,\tau = t-a \\
&= \int_0^\infty e^{-s(\tau+a)}f(\tau)\,d\tau, \quad ,\tau = t-a \\
&= \int_0^\infty e^{-s\tau}e^{-sa}f(\tau)\,d\tau, \quad ,\tau = t-a \\
&= e^{-sa}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau, \quad ,\tau = t-a \\
&= e^{-sa}\mathscr{L}[f(t)](s) \\
\Rightarrow \mathscr{L}[H(t-a)f(t-a)](s) &= e^{-sa}\mathscr{L}[f(t)](s) = e^{-sa}F(s) \\
\Rightarrow H(t-a)f(t-a) &= \mathscr{L}^{-1}[e^{-sa}F(s)](t)
\end{aligned}$$

Convolutions

The convolution of two functions $f(t)$ and $g(t)$, $t \geq 0$, denoted by $f * g(t)$, is the function defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Theorem

Suppose $f(t)$ and $g(t)$ are two functions satisfying

$$|f(t)| \leq Me^{\sigma t} \text{ and } |g(t)| \leq Me^{\sigma t},\ t \geq 0$$

for some $M \geq 0$, and $\sigma \in \mathbb{R}$. Then

$$\mathscr{L}[(f * g)(t)](s) = F(s)G(s),\ \mathrm{Re}(s) > \sigma,$$

where

$$F(s) = \mathscr{L}[f(t)](s) \text{ and } G(s) = \mathscr{L}[g(t)](s).$$

Example

Use the Laplace transform to show that

$$\int_0^t t\sin(t-\tau)\,d\tau = t - \sin(t).$$

Note you could probably do integration by parts but whatever.

Solution

$$\begin{aligned}
\mathscr{L}\left[\int_0^t t\sin(t-\tau)\,d\tau\right](s) &= \mathscr{L}[t * \sin(t)](s) \\
&= \mathscr{L}[t](s)\mathscr{L}[\sin(t)](s) \\
&= \frac{1}{s^2}\frac{1}{s^2+1} \\
&= \frac{1}{s^2} - \frac{1}{s^2+1} \\
\Rightarrow \int_0^t t\sin(t-\tau)\,d\tau &= \mathscr{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right](t) \\
&= \mathscr{L}^{-1}\left[\frac{1}{s^2}\right](t) - \mathscr{L}^{-1}\left[\frac{1}{s^2+1}\right](t) \\
&= t - \sin(t)
\end{aligned}$$

Impulse

Mathematically, a short, sharp forcing at $t = 0$ is represented by the Dirac Delta Function (although it's not really a function), denoted by $\delta(t)$, which is defined by

$$\delta(t) = 0,\ t = 0,$$

and

$$\int_a^b f(t)\delta(t)\,dt = \begin{cases} f(0) & \text{if } 0 \in (a,b) \\ 0 & \text{otherwise} \end{cases}$$

for all continuous functions $f(t)$ that are defined on $[a, b]$.

Example

Compute the Laplace transform of

$$\delta(t-a),\ a > 0.$$

Solution

$$\mathscr{L}[\delta(t-a)](s) = \int_0^\infty e^{-st}\delta(t-a)]\,dt = e^{-sa}.$$