ENG1005: Lecture 10

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Contents

Video link

Click here for a recording of the lecture.

Taylor series

Example

Find the Taylor series for $\ln(x)$ about $x = 1$ and determine the interval of convergence.

Solution

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k, f(x) = \ln(x)$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \frac{d^2}{dx^2}\ln(x) = \frac{-1}{x^2}, \frac{d^3}{dx^3}\ln(x) = \frac{2}{x^3}, \frac{d^4}{dx^4}\ln(x) = \frac{-6}{x^4}$$

$$\Rightarrow \frac{d^k}{dx^k}\ln(x) = (-1)^{k-1}\frac{(k-1)!}{x^k}, \ \ k = 1, 2, 3, ...$$

$$\ln(1) = 0, \ \ \frac{d^k}{dx^k}\ln(x)\bigg|_{x=1} = (-1)^{k-1}(k-1)!$$

So the Taylor series for $\ln(x)$ about $x = 1$ is

$$\sum_{k=1}^{\infty}\frac{1}{k!}(-1)^{k-1}(k-1)!(x-1)^k = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(x-1)^k$$

The radius of convergence is given by

$$r = \lim_{k\to\infty}\left|\frac{\frac{(-1)^{k-1}}{k}}{\frac{(-1)^k}{k+1}}\right| * = \lim_{k\to\infty}\frac{k+1}{k} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right) = 1$$

We conclude that the series converges uniformly for

$$|x-1| < 1 \Leftrightarrow 0 < x < 2$$

Fact

$$\ln(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1)}}{n}(x-1)^n, 0 < x \leq 2$$

What happens for x ¿ 2? Try calculating the Taylor series for $\ln(x)$ about $x = 3$.

L'Hôpital's rule §9.4.3

Suppose $f(x)$ and $g(x)$ satisfy

(i) $f''(x)$ and $g''(x)$ are continuous on $|x - x_0| < r$

(ii) $f(x_0) = g(x_0) = 0$ but $g'(x_0) \neq 0$.

Then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f(x_0) + f'(x_0)(x-x_0) + (x-x_0)^2R_2^f(x)}{g(x_0) + g'(x_0)(x-x_0) + (x-x_0)^2R_2^g(x)}$$

We know $f(x_0) = g(x_0) = 0$ and $g'(x_0) \neq 0$ so dividing both numerator and denominator by $(x - x_0)$ gives us

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x_0)(x-x_0) + (x-x_0)R_2^f(x)}{g'(x_0) + (x-x_0)R_2^g(x)}$$

And since $\lim_{x\to x_0}(x - x_0) = 0$, we get

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

Example

Compute

$$\lim_{x\to 0}\frac{\sin(x)}{x}$$

Solution

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\lim_{x\to 0}\frac{\sin'(x))}{x'}=\lim_{x\to 0}\frac{\cos(x)}{1}=1$$

Example

What is $\lim_{x\to 0}\dfrac{e^x-1}{x}$?

Solution

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}(x)}\lim_{x\to 0}\frac{e^x}{1}=1$$

Remark

L'Hôpital's rule $\lim_{x\to x_0}\dfrac{f(x)}{g(x)}=\lim_{x\to x_0}\dfrac{f'(x)}{g'(x)}$ remains valid for:

(i) one sided limits, $x\to x_0^{\pm}$

(ii) infinite limits, $x\to\pm\infty$

(iii) $\dfrac{\infty}{\infty}$ limits, i.e. $\lim_{x\to x_0}f(x)=\pm\infty$

Example

Compute

$$\lim_{x\searrow 0}x\ln(x)$$

Solution

$$\begin{aligned}\lim_{x\searrow 0}x\ln(x)&=\lim_{x\searrow 0}\frac{\ln(x)}{\frac{1}{x}} && (\frac{\infty}{\infty})\\&=\lim_{x\searrow 0}\frac{\ln'(x)}{(\frac{1}{x})'}\\&=\lim_{x\searrow 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}}\\&=\lim_{x\searrow 0}(-x)\\&=0\end{aligned}$$

Vectors §4.2.1 - 4.2.8

Vectors in $\mathbb{R}^3$

$\underline{u}=(u_1,u_2,u_3)$, $(\vec{u}=(u_1,u_2,u_3))$

Standard basis

$\vec{i}=\underline{e}_1=(1,0,0)$ $\vec{j}=\underline{e}_2=(1,0,0)$ $\vec{k}=\underline{e}_3=(1,0,0)$

n-dimensions

$$
\begin{aligned}
\underline{u} &= (u_1, u_2, u_3, ..., u_n) \\
\underline{e}_i &= (0, 0, ..., 0, 1, 0, ..., 0) \qquad\qquad (1 \text{ at } i^{\text{th}} \text{ position}) \\
\underline{u} &= \sum_{i=1}^{n} u_i \, \underline{e}_i
\end{aligned}
$$

Expansion in the standard basis

$$
\begin{aligned}
\underline{u} &= (u_1, u_2, u_3) \\
&= u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) \\
&= u_1\underline{i} + u_2\underline{j} + u_3\underline{k}
\end{aligned}
$$