ENG1005: Lecture 18

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Contents

Video link

Click here for a recording of the lecture.

Determinants of $n \times n$ matrices - continued

Example

Compute the determinant of

$$\begin{bmatrix} -1 & 2 & -2 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution

Note: try finding a row with lots of zeroes to make your life easier. Here, we chose row 2.

$$\begin{vmatrix} -1 & 2 & -2 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -2\begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}$$
$$= -2(2 \times 2 - 1 \times (-2)) + 0 - 1((-1) \times 1 - 2 \times 3)$$
$$= -12 + 7 = -5$$

Properties of the determinant §5.3

(a) A is invertible $\Leftrightarrow \det(A) \neq 0$.

(b) $\det(AB) = \det(A)\det(B)$

(c) $\det(\mathbb{I}_n) = 1$

(d) $\det(A^T) = \det(A)$

Inverting matrices using determinants §5.3, 5.4

Given an $n \times n$ matrix $A = [A_{ij}]$, let

$$C_{ij}, \quad 1 \leq i,\ j \leq n \quad (C_{ij} = (-1)^{i+j}|\tilde{M}_{ij}|)$$

denote all the cofactors of A. Then the **adjugate matrix** of A is defined by

$$\text{adj}A = [C_{ij}]^T$$

If A is non-singular (i.e. $\det(A) \neq 0$) then

$$A^{-1} = \frac{1}{\det(A)}\text{adj}A$$

Example

Compute the inverse of

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution

$$M_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1, M_{12} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1, M_{13} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2$$

$$M_{21} = \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} = 6, M_{22} = \begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix} = 4, M_{23} = \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = -7$$

$$M_{31} = \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} = 2, M_{32} = \begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} = 3, M_{33} = \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix} = -4$$

$$C = \begin{bmatrix} +M_{11} & -M12 & +M_{13} \\ -M_{21} & +M22 & -M_{23} \\ +M_{31} & -M32 & +M_{33} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 2 \\ -6 & 4 & 7 \\ 2 & -3 & -4 \end{bmatrix}$$

So then

$$A^{-1} = \frac{1}{\det(A)} \text{adj} A = \frac{1}{-5} C^T = -\frac{1}{5} \begin{bmatrix} -1 & -6 & 2 \\ -1 & 4 & -3 \\ 2 & 7 & -4 \end{bmatrix}$$

Cross product

$$\begin{aligned} \mathbf{u} \times \mathbf{v} := & \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ = & \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\mathbf{\imath}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\mathbf{\jmath}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\mathbf{k}} \\ = & (u_2 v_3 - v_2 u_3) \hat{\mathbf{\imath}} - (u_1 v_3 - v_1 u_3) \hat{\mathbf{\jmath}} + (u_1 v_2 - v_1 u_2) \hat{\mathbf{k}} \\ = & (u_2 v_3 - v_2 u_3, u_1 v_3 - v_1 u_3, u_1 v_2 - v_1 u_2) \end{aligned}$$

Solution space for $n \times n$ linear systems of equations

Theorem

Suppose A is an $n \times n$ matrix and $\mathbf{b}$ is an n-column vector.

(a) If $\det(A) \neq 0$, then the linear system of equations

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution given by

$$\mathbf{x} = A^{-1}\mathbf{b}$$

(b) If $\mathbf{b} = \mathbf{0}$, then

$$A\mathbf{x} = \mathbf{0}$$

has a non-trivial solution if and only if $\det(A) = 0$.

(c) If $\mathbf{b} \neq 0$ and $\det(A) = 0$, then

$$A\mathbf{x} = \mathbf{b}$$

can either have no solution or an infinite number of solutions. (You would have to compare the ranks of A and $[A|\mathbf{b}]$ to determine which is true).

Eigenvalues and eigenvectors §5.7.2

Definition

Given an $n \times n$ matrix A, a non-zero n-column vector $\mathbf{v}$ that satisfies

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some $\lambda \in \mathbb{R}$ is called an **eigenvector**. The number λ is called the real **eigenvalue** associated to $\mathbf{v}$.

Even though we will only consider real matrices $A = [A_{ij}],\ A_{ij} \in \mathbb{R}$, it will be useful to allow for complex eigenvalues and eigenvectors. That is,

$$A\mathbf{v} = \lambda\mathbf{v}$$

where $\lambda \in \mathbb{C}$ and $\mathbf{v} = [v_j],\ v_j \in \mathbb{C},\ 1 \leq j \leq n$.

Example

Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then

$$A\mathbf{v} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow A\mathbf{v} = 4\mathbf{v}$$

Thus $\mathbf{v}$ is a (real) eigenvector of A with (real) eigenvalue $\lambda = 4$.