ENG1005: Lecture 24

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Video link

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Taylor's theorem in higher dimensions §9.7.1

Suppose $f(x, y)$ has partial derivatives up to $(n+1)$th order that are continuous on some disc $\overline{B}_\rho((a, b))$. Then for $\sqrt{h^2 + k^2} \leq p$, the function

$$g(t) = f(a + th, b + tk) \in \overline{B}_\rho((a, b)) \text{ for } t \in [-1, 1]$$

and its derivatives up to $(n + 1)$th order are continuous on $[-1, 1]$.
Thus, by Taylor's Theorem, we know that

$$g(t) = \sum_{r=0}^{n} \frac{1}{r!} \frac{d^r g}{dt^r} t^r + \frac{t^{n+1}}{n!} \frac{d^{n+1} g}{dt^{n+1}}(\theta t), \quad 0 < \theta < 1$$

By the chain rule,

$$\begin{aligned} \frac{dg}{dt}(0) &= \frac{\partial f}{\partial x}(a, b) \frac{d}{dt}(a + th)\bigg|_{t=0} + \frac{\partial f}{\partial y}(a, b) \frac{d}{dt}(b + tk)\bigg|_{t=0} \\ &= \frac{\partial f}{\partial x}(a, b)h + \frac{\partial f}{\partial y}(a, b)k \end{aligned}$$

where $x = a + th,\ y = b + tk$.
Similar calculations show that

$$\frac{d^2 g}{dt^2}(0) = h^2 \frac{\partial^2 f}{\partial x^2}(a, b) + 2hk \frac{\partial^2 f}{\partial x \partial y}(a, b) + k^2 \frac{\partial^2}{\partial y^2}(a, b)$$

and, more generally,

$$\frac{d^r g}{dt^r}(0) = \sum_{\ell=0}^{r} \binom{r}{\ell} h^\ell k^{r-\ell} \frac{\partial^r f}{\partial x^\ell \partial y^{r-\ell}}(a,b)$$

where

$$\binom{r}{\ell} = \frac{r!}{\ell!(r-\ell)!}$$

These formulae show that at $t = 1$,

$$f((a,b)+(h,k)) = f(a,b) + h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b) + h^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2hk\frac{\partial^2 f}{\partial x \partial y}(a,b) + k^2\frac{\partial^2 f}{\partial y^2}(a,b)$$
$$+ \sum_{r=3}^{n} \frac{1}{r!} \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{\partial^r f}{\partial x^\ell \partial y^{r-\ell}}(a,b) h^\ell k^{\ell-r} + \sum_{\ell=0}^{n+1} R_\ell(h,k) h^\ell k^{n+1-\ell}$$

where the R_ℓ are continuous on $\overline{B}_\rho((0,0))$, and $h, k \in \overline{B}_\rho((0,0))$. This formula is the 2 dimensional version of Taylor's Theorem.
Note that the first line is the most important part for us.

Setting

$$h = x - a \text{ and } k = y - b$$

we can write the above result as

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(y-b)^2$$
$$+ \sum_{r=3}^{n} \frac{1}{r!} \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{\partial^2 f}{\partial x^\ell \partial y^{r-\ell}}(a,b)(x-a)^\ell (y-b)^{\ell-r} + \sum_{\ell=0}^{n+1} R_\ell(x-a, y-b)(x-a)^\ell (y-b)^{n+1-\ell}$$

Again, we're mainly going to be interested in the first and second order Taylor polynomials.

Linearisation

By Taylor's Theorem, we have that

$$f(x,y) \approx L(x,y) \text{ for } \sqrt{(x-a)^2 + (y-b)^2} \ll 1$$

where

$$L(x,y) = f(x,y) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

is known as the **linearisation** (or linear approximation) of $f(x,y)$ at the point (a,b).

Example

Find the linear approximation to $f(x,y) = xe^{xy}$ at the point $(1,0)$ and use this to approximate $f(1.2, -0.1)$.

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(xe^{xy}\right) = e^{xy} + xye^{xy} = e^{xy}(1+xy)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xe^{xy}\right) = x^2e^{xy}$$

So,

$$\begin{aligned}
\frac{\partial f}{\partial x}(1,0) &= e^0(1+0) = 1 \\
\frac{\partial f}{\partial y}(1,0) &= 1^2e^0 = 1 \\
f(1,0) &= 1e^0 = 1 \\
\Rightarrow L(x,y) &= 1 + (x-1) + y = x + y \\
\Rightarrow f(1.2,-0.1) &\approx L(1.2,-0.1) = 1.2 + (-0.1) = 1.1
\end{aligned}$$

If you check the error here, you should find it to be $|f(1.2,-0.1) - L(1.2,-0.1)| = 0.0142$, which is quite good considering we only used the first order approximation.

Note: In higher dimensions, the linearisation of $f(x_1, x_2, ..., x_n)$ at the point $\mathbf{a} = (a_1, a_2, ..., a_n)$ is

$$L(x_1, x_2, ..., x_n) = f(a_1, a_2, ..., a_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a_1, a_2, ..., a_n)(x_i - a_i)$$

Differential §9.6.9

The differential of a function $f(x,y)$ is defined by

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

or, when evaluated at a point $(x_0, y_0) \in \mathbb{R}^2$ by

$$df(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\,dx + \frac{\partial f}{\partial y}(x_0,y_0)\,dy$$

Note that the differential is related to the linearisation of $f(x,y)$ at the point (x_0, y_0) by

$$L(x,y) = f(x_0,y_0) + df(x_0,y_0)$$

where $dx = x - x_0$ and $dy = y - y_0$.
Note that you can think of df as measuring the approximate change in f due to the change $(x_0, y_0) \mapsto (x_0 + dx, y_0 + dy)$.

Example

Estimate the change in the period

$$T = 2\pi\sqrt{\frac{L}{g}}$$

of a simple pendulum due to a 2% increase in length L and a 0.6% decrease in the gravitational acceleration g.

Solution

$$dL = \frac{2}{100}L \text{ and } dg = \frac{-6}{100}g$$

Substituting this into

$$\begin{aligned}
dT &= \frac{\partial T}{\partial L}\,dL + \frac{\partial T}{\partial g}\,dg \\
&= 2\pi \frac{1}{2\sqrt{\frac{L}{g}}}\frac{1}{g}\,dL + 2\pi \frac{1}{2\sqrt{\frac{L}{g}}}\left(\frac{-L}{g^2}\right)\,dg \\
&= \frac{\pi}{\sqrt{Lg}}\,dL - \frac{\pi\sqrt{L}}{g^{\frac{3}{2}}}\,dg
\end{aligned}$$

yields

$$\begin{aligned}
dT &= \frac{\pi}{\sqrt{Lg}}\frac{1}{50}L + \frac{\pi\sqrt{L}}{g^{\frac{3}{2}}}\frac{3}{50}g \\
&= \frac{\pi}{50}\sqrt{\frac{L}{g}} + \frac{3\pi}{50}\sqrt{\frac{L}{g}} \\
&= \frac{2\pi}{25}\sqrt{\frac{L}{g}}
\end{aligned}$$