ENG1005: Lecture 9 - Taylor Series

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Video link

Click here for a recording of the lecture.

Taylor's Theorem §9.4.1

Theorem (Taylor's Theorem)

If the derivatives $f^{(k)}(x), k = 1, 2, ..., n, n+1$ exist and are continuous on $[x_0 - r, x_0 + r], r > 0$, then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + (x - x_0)^{n+1} R_n(x), |x - x_0| \leq r$$

where $R_n(x)$ is continuous on $[x_0 - r, x_0 + r]$. We further note that for fixed $x \in [x_0 - r, x_0 + r], R_n(x)$ can be expressed as

$$R_n(x) = \frac{1}{(n+1)!} f^{n+1}(x_0 + \theta(x - x_0)), 0 < \theta < 1$$

Note that this $R_n(x)$ is a remainder of sorts.

So, informally we have that

$$f(x) \approx p_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)$$

Example

Approximate $\sin(x)$ by a 4th order Taylor polynomial near $x = \frac{\pi}{2}$.

Solution

Let

$$f(x) = \sin(x),\ x_0 = \frac{\pi}{2}$$

Then

k	$f^k(x)$	$f^k(\frac{\pi}{2}$
0	$\sin(x)$	1
1	$\cos(x)$	0
2	$-\sin(x)$	-1
3	$-\cos(x)$	0
4	$\sin(x)$	1

So the 4th order Taylor polynomial expansion of $f(x)$ about $x = \frac{\pi}{2}$ is

$$p_4(x) = \sin(\frac{\pi}{2}) + \sin^{(1)}(\frac{\pi}{2})(x - \frac{\pi}{2}) + \frac{\sin^{(2)}(\frac{\pi}{2})}{2!}(x - \frac{\pi}{2})^2 + \frac{\sin^{(3)}(\frac{\pi}{2})}{3!}(x - \frac{\pi}{2})^3 + \frac{\sin^{(4)}(\frac{\pi}{2})}{4!}(x - \frac{\pi}{2})^4$$
$$p_4(x) = 1 + 0 + \frac{-1}{2}(x - \frac{\pi}{2})^2 + 0 + \frac{1}{4!}(x - \frac{\pi}{2})^4$$
$$p_4(x) = 1 + \frac{-1}{2}(x - \frac{\pi}{2})^2 + \frac{1}{24}(x - \frac{\pi}{2})^4$$

Taylor series §9.4.2

We know from Taylor's theorem that if $f(x)$ is infinitely differentiable on some interval $|x - x_0| \leq r$, then for $n \in \mathbb{N}$, we can express $f(x)$ as

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k + (x - x_0)^{n+1} R_n(x)$$

If the remainder term satisfies

$$(x - x_0)^{n+1} R_n(x) \to 0 \text{ as } n \to \infty$$

then $f(x)$ can be represented by the power series

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$$

This series is called the Taylor series expansion of $f(x)$ about $x = x_0$.
If $x_0 = 0$, then the Taylor series

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0) x^k$$

is known as the Maclaurin series expansion of $f(x)$.

Remark

Whenever $f(x)$ is infinitely differentiable, we can compute its Taylor series expansion about a point $x = x_0$ by

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$$

It is then a separate question to determine if the series converges, and if it does converge, whether or not

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$$

is true.

Example

Compute the Taylor series for e^x about $x = 0$.

Solution

Let $f(x) = e^x$, then

$$f^{(k)}(x) = e^x \Rightarrow f^{(k)}(0) = 1$$

This shows that the Taylor series for e^x about $x = 0$ is

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0) x^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

Follow-up question

Is it true that

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k ?$$

Answer 1

Yes, if we define

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k, x \in \mathbb{R}$$

Answer 2

Still yes even if we define e^x some other way.

$$e^x = \lim_{\lambda \to \infty} \left(1 + \frac{x}{\lambda}\right)^\lambda$$

But you would need to show that the remainder terms $x^{n+1} R_n(x) \to 0$ as $n \to \infty$.

Question

Can all infinitely differentiable functions be represented by their Taylor series?

Answer

No. For example, define

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Then it can be shown that $f(x)$ is infinitely differentiable for $x \in \mathbb{R}$ but

$$f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{n!} x^n \text{ for } x \text{ near } 0$$