ENG1005: Lecture 14

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April 29, 2020

Contents

Video link

Click here for a recording of the lecture.

Gaussian elimination §5.5.2 - continued

Example - continued

(i) Getting matrix into row echelon form.

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & -1 \\ -1 & -1 & -1 & 2 \end{array}\right]$$

$\vdots$ (see previous lecture notes)

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -7 & 8 \end{array}\right]$$

$$R_3 \to -\frac{1}{7}R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -\frac{8}{7} \end{array}\right]$$

Our augmented matrix is now in **row echelon form**. This means we have that diagonal of 1s, called **pivots**, and zeros beneath them.

(ii) Back substitution.

$$z = -\frac{8}{7} \Rightarrow y = 2z + 1 = -\frac{16}{7} + 1 = -\frac{9}{7}$$
$$\Rightarrow x = -1 - 2y + z = -1 + \frac{18}{7} - \frac{8}{7} = \frac{3}{7}$$

Row echelon form §5.6

Definition

A matrix is in row echelon form if

(i) all rows with a non-zero element are above any rows with all zero,

(ii) and the leading coefficient of each row (i.e. first non-zero number in row) is 1 (the **pivot**) and is to the left of all leading coefficients in the rows below.

Examples

(a)

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 5 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in fact in row echelon form!

(b)

$$\begin{bmatrix} 1 & 7 & 3 & 4 \\ 0 & 1 & 0 & 5 \\ 5 & 0 & 1 & 0 \end{bmatrix}$$

Note that (1, 1) is not above all zeros (because of the 5), and also that (3, 1) = 5 is a leading coefficient that isn't 1! This is NOT in row echelon form.

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 3 & 2 \\ 4 & 1 & 0 \end{bmatrix}$$

Problems include that the first row is all zeros and the last row's leading coefficient is not 1. This is NOT in row echelon form.

Remark

When using Gaussian elimination to solve a linear system of equations the row-reduction (i.e. elimination) process stops when the augmented matrix is in row echelon form. At that point, back substitution is used recursively to find the solution.

Gauss-Jordan elimination §5.5.2

$$\begin{aligned} 2x - y + z &= 1 \\ x + 2y - z &= -1 \\ -x - y - z &= 2 \end{aligned}$$

Last time, we stopped row-reduction when the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & -1 \\ -1 & -1 & -1 & 2 \end{array}\right]$$

reached the following row echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -\frac{8}{7} \end{array}\right]$$

$$R_2 \to R_2 + 2R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 0 & -\frac{9}{7} \\ 0 & 0 & 1 & -\frac{8}{7} \end{array}\right]$$

$$R_1 \to R_1 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -\frac{15}{7} \\ 0 & 1 & 0 & -\frac{9}{7} \\ 0 & 0 & 1 & -\frac{8}{7} \end{array}\right]$$

$$R_1 \to R_1 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{7} \\ 0 & 1 & 0 & -\frac{9}{7} \\ 0 & 0 & 1 & -\frac{8}{7} \end{array}\right]$$

Now, the matrix above is in **reduced row echelon form**.

$$\Rightarrow (x, y, z) = \left(\frac{3}{7}, -\frac{9}{7}, -\frac{8}{7}\right)$$

Reduced row echelon form §5.6

Definition

A matrix A is in reduced row echelon form if it is in row echelon form and any column with a pivot has zeros elsewhere.

Examples

(a)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in row echelon form, and since everything else is zeros, this is in reduced row echelon form.

(b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

This is in row echelon form but is NOT in reduced row echelon form.

(c)

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is NOT in reduced row echelon form nor row echelon form.

(d)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Similarly, this one is not even in row echelon form and so cannot be in reduced row echelon form.

Remark

The Gauss-Jordan eliminations process terminates when the augmented matrix in in reduced row echelon form.

Example

$$\begin{aligned} x + 3y - z &= 1 \\ 2x + 4y + 2z &= 1 \\ -3x - 9y + 3z &= -3 \end{aligned}$$

First, make the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 2 & 4 & 2 & 1 \\ -3 & -9 & 3 & -3 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & -2 & 4 & -1 \\ -3 & -9 & 3 & -3 \end{array}\right]$$

$$R_3 \to R_3 + 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2 \to -\frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & 1 & -2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which translates back into

$$x + 3y - z = 1$$
$$y - 2z = \frac{1}{2}$$

Note that z can be any real number, so set $z = \lambda,\ \lambda \in \mathbb{R}$. We then proceed by back-substitution.

$$z = \lambda \Rightarrow y = 2z + \frac{1}{2} = 2\lambda + \frac{1}{2}$$

$$\Rightarrow x = z - 3y + 1 = \lambda - 3\left(2\lambda + \frac{1}{2}\right) + 1 = -5\lambda - \frac{1}{2}$$

Thus

$$(x, y, z) = \left(-5\lambda - \frac{1}{2}, 2\lambda + \frac{1}{2}, \lambda\right)$$

is a 1-parameter family of solutions.

Matrix operations §5.2.2, 5.2.4

Addition

$$[A_{ij}] + [B_{ij}] = [A_{ij} + Bij]$$

Note, both A and B must be $m \times n$ matrices, and the result will be of the same size.

Example

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1+3 & -1+1 \\ 2+(-1) & 4+2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 1 & 6 \end{bmatrix}$$

Scalar multiplication

$$\lambda[A_{ij}] = [\lambda A_{ij}]$$

Example

$$3\begin{bmatrix} 1 & 2 & -1 \\ 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3\times 1 & 3\times 2 & 3\times(-1) \\ 3\times 4 & 3\times 0 & 3\times 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -3 \\ 12 & 0 & 9 \end{bmatrix}$$

Matrix multiplication

$$[A_{ij}][B_{jk}] = [C_{ik}]$$

where

$$C_{ik} = \sum_{j=1}^{n} A_{ij}B_{jk},\ 1 \leq i \leq m,\ 1 \leq k \leq p$$

Note that A is an $m \times n$ matrix, and that B must be a $n \times p$ matrix. The result will therefore be a $m \times p$ matrix.

Example

$$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times (-1) & 1 \times 2 + 2 \times (-1) & 1 \times 0 + 2 \times 3 \\ (-1) \times 1 + (-2) \times (-1) & (-1) \times 2 + (-2) \times (-1) & (-1) \times 0 + (-2) \times 3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 6 \\ 1 & 0 & - \end{bmatrix}$$