ENG1005: Lecture 29

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Video link

`https://echo360.org.au/lesson/G_32340f5d-ff38-43d2-be9d-d88ddb1b3611_b944cecf-8ba5-40d3-a870-024`
`2020-05-27T14:58:00.000_2020-05-27T15:53:00.000/classroom#sortDirection=desc`

Linear differential equations - continued

Reduction of non-homogeneous ODE to a homogeneous ODE

Suppose $y_p(x)$ is a (particular) solution of the linear non-homogeneous ODE

$$\sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k} = q(x).$$

Setting

$$y_h(x) = y(x) - y_p(x),$$

we see that

$$\sum_{k=0}^{n} a_k(x)\frac{d^k y_h}{dx^k} = \sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k} - \sum_{k=0}^{n} a_k(x)\frac{d^k y_p}{dx^k} = 0.$$

This shows that

$$y_h(x) = y(x) - y_p(x)$$

satisfies the linear, homogeneous ODE

$$\sum_{k=0}^{n} a_k(x)\frac{d^k y}{dx^k} = 0$$

So you could find a general solution for a non-homogeneous equation as follows

$$y = y_h + y_p$$

General solutions of linear ODEs §10.8.2

Definition

Functions $y_1(x), y_2(x), ..., y_p(x)$ are said to be **linearly dependent** if there exists numbers $c_1, c_2, ..., c_p$, not all zero, such that

$$\sum_{j=1}^{p} c_j y_j(x) = 0.$$

Otherwise, the functions $y_1(x), y_2(x), ..., y_p(x)$ are said to be **linearly independent**.

Example

Determine if any of the following sets of functions are linearly independent.

(i) $\{1, t, t^2, (1+t)^2\}$

(ii) $\{\sin(x), \cos(x)\}$

Solution

(i) Since

$$(1+t)^2 = t^2 + 2t + 1,$$

we see that

$$1 + 2t + t^2 - (1+t)^2 = 0.$$

This shows that $\{1, t, t^2, (1+t)^2\}$ is linearly dependent.

(ii) Suppose there exists $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \sin(x) + c_2 \cos(x) = 0.$$

Evaluating this at $x = 0$ and $x = \frac{\pi}{2}$ show that

$$c_2 = 0 \text{ and } c_1 = 0.$$

So clearly, $\{\sin(x), \cos(x)\}$ is linearly independent.

Theorem

If $y_1(x), y_2(x), ..., y_n(x)$ are linearly independent solutions of the following nth order linear homogeneous ODE,

$$\sum_{k=1}^{n} a_k(x) \frac{d^k y}{dx^k} = 0,$$

then the general solution is given by

$$y(x) = \sum_{k=1}^{n} c_k y_k(x), \;\; c_k \in \mathbb{R}$$

(note this is an n-parameter family of solutions).
Moreover, if $y_p(x)$ is any solution of the non-homogeneous linear ODE

$$\sum_{k=1}^{n} a_k(x) \frac{d^k y}{dx^k} = q(y),$$

then the general solution of the non-homogeneous linear ODE

$$y(x) = \sum_{k=1}^{n} c_k y_k(x) + y_p(x)s.$$

Linear homogeneous 2nd order constant coefficients ODEs §10.9.1, 10.10

A 2nd order linear homogeneous ODE with constant coefficients is of the form

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0,$$

where,

$$a, b, c \in \mathbb{R} \text{ and } a \neq 0.$$

To find solutions, we try

$$y(t) = e^{\lambda t} \quad (\lambda \text{ is constant}).$$

Then,

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \left(a\lambda^2 + b\lambda + c\right) e^{\lambda t}.$$

We want this to equal zero. This shows that $y(t) = e^{\lambda t}$ will solve the ODE if and only if

$$a\lambda^2 + b\lambda + c = 0.$$

This is called the **characteristic equation**.

Case 1: $b^2 - 4ac > 0$

Then

$$\lambda_{\pm} = \zeta \pm \omega,$$

where

$$\zeta = -\frac{b}{2a} \text{ and } \omega = \frac{\sqrt{|b^2 - 4ac|}}{2a}$$

are the distinct real roots of the characteristic equation that yield the linearly independent solution

$$y_{\pm}(t) = e^{\lambda_{\pm} t} = e^{(\zeta \pm \omega)t}.$$

Thus the general solution is then

$$y(t) = c_1 + e^{(\zeta+\omega)t} + c_2 - e^{(\zeta-\omega)t},\ c_1, c_2 - \in \mathbb{R}$$

Case 2: $b^2 - 4ac < 0$

Then

$$\lambda_{\pm} = \zeta \pm \omega i,$$

are distinct complex roots of the characteristic equation that yield complex solutions

$$y_{\pm}(t) = e^{\lambda_{\pm} t} = e^{\zeta t} e^{\pm \omega t i} = e^{\zeta t}(\cos(\omega t) \pm i \sin(\omega t)).$$

These complex solutions yield the real, linearly independent solution

$$y_1(t) = e^{\zeta t}\cos(\omega t) \text{ and } y_2(t) = e^{\zeta t}\sin(\omega t).$$

Thus the general solution is given by

$$y(t) = c_1 e^{\zeta t}\cos(\omega t) + c_2 e^{\zeta t}\sin(\omega t),\ c_1, c_2 \in \mathbb{R}$$

Case 3: $b^2 - 4ac = 0$

Then

$$\lambda = \zeta$$

is the only root of the characteristic equation which yields the solution

$$y_1(t) = e^{\zeta t}.$$

But we need 2 linearly independent solutions to form a general solutions. So, to find a second, linearly independent solution, we set

$$y_2(t) = t y_1(t) = t e^{\zeta t}.$$

Now, quickly note that the first and second derivatives are given by

$$\begin{aligned}
\frac{dy_2}{dt} &= y_1 + t\frac{dy_1}{dt}, \\
\frac{d^2y_2}{dt^2} &= \frac{dy_1}{dt} + \left(t\frac{d^2y_1}{dt^2} + \frac{dy_1}{dt} \right) = 2\frac{dy_1}{dt} + t\frac{d^2y_1}{dt^2}.
\end{aligned}$$

Now, observe that

$$\begin{aligned}
a\frac{d^2y_2}{dt^2} + b\frac{dy_2}{dt} + cy_2 &= a\left(2\frac{dy_1}{dt} + t\frac{d^2y_1}{dt^2} \right) + b\left(y_1 + t\frac{dy_1}{dt} \right) + cty_1 \\
&= t\left(a\frac{d^2y_1}{dt^2} + b\frac{dy_1}{dt} + cy_1 \right) + \left(2a\frac{dy_1}{dt} + by_1 \right) \\
&= 0t + (2a\zeta + b)e^{\zeta t} \\
&= \left(2a\left(\frac{-b}{2a} \right) + b \right) e^{\zeta t} \\
&= 0
\end{aligned}$$