ENG1005: Lecture 5

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Comparison principle: divergence

Example

Determine if

$$\int_2^\infty \frac{2+\sin(x)}{x}\,dx$$

is convergent or divergent.

Solution

$$\int_2^\infty \frac{2+\sin(x)}{x}\,dx = \lim_{\varepsilon\to\infty}\int_2^\varepsilon \frac{2+\sin(x)}{x}\,dx$$

But observe that

$$1 \le 2+\sin x,\ \ x\in\mathbb{R}$$

which gives

$$\frac{1}{x} \le \frac{2+\sin x}{x},\ \ x>0$$

So then

$$\int_2^\varepsilon \frac{1}{x}\,dx \le \int_2^\varepsilon \frac{2+\sin(x)}{x}\,dx$$

while

$$\lim_{\varepsilon\to\infty}\int_2^\varepsilon \frac{1}{x}\,dx = \lim_{\varepsilon\to\infty} \ln(x)|_2^\varepsilon = \lim_{\varepsilon\to\infty}(ln(\varepsilon)-ln(2)) = \infty$$

So we conclude by the Comparison Theorem/Test that

$$\lim_{\varepsilon \to \infty} \int_2^{\varepsilon} \frac{2 + \sin x}{x}\, dx = \infty$$

$$\text{Then the improper integral } \int_2^{\infty} \frac{2 + \sin(x)}{x}\, dx \text{ DNE.}$$

Theorem (Comparison Test)

(i) Suppose f(x), g(x) are continuous on $[a, \infty)$, $f(x) \leq g(x)$ for all $x \in [a, \infty)$, and

$$\lim_{\varepsilon \to \infty} \int_a^{\varepsilon} f(x)\, dx = \infty$$

$$\text{then } \int_a^{\infty} g(x)\, dx \text{ is divergent.}$$

(ii) Suppose f(x), g(x) are continuous on $(a, b]$, $f(x) \leq g(x)$ for all $x \in (a, b]$, and

$$\lim_{\varepsilon \searrow a} \int_{\varepsilon}^{b} f(x)\, dx = \infty$$

$$\text{then } \int_a^{b} g(x)\, dx \text{ is divergent.}$$

Proof

(i) If $f(x) \leq g(x)$ for all x $[a, \infty)$, then

$$\int_a^{\varepsilon} f(x)\, dx \leq \int_a^{\varepsilon} g(x)\, dx = \infty forall \varepsilon > a$$

So if

$$\lim_{\varepsilon \to \infty} \int_a^{\varepsilon} f(x)\, dx \leq \int_a^{\varepsilon} g(x)\, dx = \infty$$

then

$$\lim_{\varepsilon \to \infty} \int_a^{\varepsilon} g(x)\, dx \leq \int_a^{\varepsilon} g(x)\, dx = \infty$$

$$\text{This proves that } \int_a^{b} g(x)\, dx \text{ is divergent.}$$

(ii) Proof is similar to (i).

Example

For what values of $p \in \mathbb{R}$ is $\int_1^\infty (2+\sin x)x^p\,dx$ convergent?

Solution

Since

$$1 \leq 2+\sin x \leq 3, x \in \mathbb{R}$$

we get

$$x^p \leq x^p(2+\sin x) \leq 3x^p, x \geq 0$$

So then

$$\int_1^\varepsilon x^p\,dx \leq \int_1^\varepsilon (2+\sin x)x^p\,dx \leq 3\int_1^\varepsilon x^p\,dx \tag{1}$$

Since

$$\int_1^\varepsilon x^p\,dx = \begin{cases} \left.\frac{x^{p+1}}{p+1}\right|_1^\varepsilon & p \neq -1 \\ \ln(x)|_1^\varepsilon & p = -1 \end{cases}$$

$$= \begin{cases} \left.\frac{\varepsilon^{p+1}}{p+1} - \frac{1}{p+1}\right|_1^\varepsilon & p \neq -1 \\ \ln(\varepsilon)|_1^\varepsilon & p = -1 \end{cases}$$

We see

$$\lim_{\varepsilon\to\infty}\int_1^\varepsilon x^p\,dx = \begin{cases} \infty & p = -1 \\ \infty & p+1 > 0 <> p > -1 \\ -\frac{1}{p+1} & p+1 < 0 <> p < -1 \end{cases} \tag{2}$$

We can conclude from (1) and from (2) and the Comparison Test that

$$\int_1^\varepsilon (2+\sin x)x^p\,dx$$

is convergent when $p < -1$ and divergent when $p \geq -1$.

Example

Determine if the improper integral

$$\int_0^1 \frac{1}{1-x^4}\,dx$$

is convergent or divergent.

Solution

Note that

$$1 - x^4 = (1 - x^2)(1 + x^2) = (1 - x)(1 + x)(1 + x^2)$$

So then

$$\begin{aligned}
\frac{1}{(1-x)(1+x)(1+x^2)} &= \frac{1}{1-x^4}, \ 0 \le x < 1 \\
\frac{1}{1-x} \cdot \frac{1}{(1+x)(1+x^2)} &= \frac{1}{1-x^4}, \ 0 \le x < 1
\end{aligned}$$

This shows

$$\frac{1}{4} \cdot \frac{1}{1-x} \le \frac{1}{1-x^4}, \ 0 \le x < 1$$

Integrate to get

$$\begin{aligned}
\int_0^{\varepsilon} \frac{1}{4} \cdot \frac{1}{1-x}\, dx &\le \int_0^{\varepsilon} \frac{1}{1-x^4}\, dx \\
-\frac{1}{4} \ln(1-x)\Big|_0^{\varepsilon} &\le \int_0^{\varepsilon} \frac{1}{1-x^4}\, dx \\
-\frac{1}{4} \ln(1-\varepsilon) &\le \int_0^{\varepsilon} \frac{1}{1-x^4}\, dx
\end{aligned} \tag{3}$$

Now

$$\lim_{\varepsilon \nearrow 1} -\frac{1}{4} \ln(1-\varepsilon) = \infty \tag{4}$$

and we conclude via the Comparison Test and equations (4) and (3) that

$$\int_0^1 \frac{1}{1-x^4}\, dx \text{ is divergent.}$$