ENG1005: Lecture 32

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Video link

`https://echo360.org.au/lesson/G_32340f5d-ff38-43d2-be9d-d88ddb1b3611_b944cecf-8ba5-40d3-a870-024`
`2020-06-03T14:58:00.000_2020-06-03T15:53:00.000/classroom#sortDirection=desc`

Simple Laplace Transforms - continued

$$\mathscr{L}[1](s) = \frac{1}{s},\ \mathrm{Re}(s) > 0$$
$$\mathscr{L}[t](s) = \frac{1}{s^2},\ \mathrm{Re}(s) > 0$$
$$\mathscr{L}[e^{kt}](s) = \frac{1}{s-k},\ \mathrm{Re}(s) > \mathrm{Re}(k)$$

Applying the Laplace transform to

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2},\ \omega \in \mathbb{R}, t \geq 0,$$

we see that

$$
\begin{aligned}
\mathscr{L}[\cos(\omega t)](s) &= \frac{1}{2}\left(\mathscr{L}[e^{i\omega t}](s) + \mathscr{L}[e^{-i\omega t}](s)\right) \\
&= \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right),\ \mathrm{Re}(s) > 0,
\end{aligned}
$$

or equivalently

$$\mathscr{L}[\cos(\omega t)](s) = \frac{s}{s^2+\omega^2},\ \mathrm{Re}(s) > 0.$$

Similar calculations starting from

$$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2}$$

shows that

$$\mathscr{L}[\sin(\omega t)](s) = \frac{\omega}{s^2+\omega^2},\ \mathrm{Re}(s) > 0.$$

The s-shifting property §11.2.4

Theorem

Suppose $f(t),\ t \geq 0$, is a function whose Laplace transform

$$F(s) = \mathscr{L}[f(t)](s)$$

is defined for $\mathrm{Re}(s) > \sigma$, and $a \in \mathbb{C}$. Then

$$\mathscr{L}[e^{at}f(t)](s) = F(s-a),\ \mathrm{Re}(s) > \sigma + \mathrm{Re}(a).$$

Proof

From

$$
\begin{aligned}
\mathscr{L}[e^{at}f(t)](s) &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\
&= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\
&= F(s-a) \text{where } F(s) \qquad\qquad = \int_0^\infty e^{-st}f(t)\,dt,\ \mathrm{Re}(s) > 0.
\end{aligned}
$$

But

$$\mathrm{Re}(s-a) > \sigma \Leftrightarrow \mathrm{Re}(s) > \sigma + \mathrm{Re}(a),$$

so we conclude that

$$\mathscr{L}[e^{at}f(t)](s) = F(s-a),\ \mathrm{Re}(s) > \sigma + \mathrm{Re}(a).$$

Example

Compute

$$\mathscr{L}[te^{2t}](s).$$

Solution

Let

$$F(s) = \mathscr{L}[t] = \frac{1}{s^2}, \mathrm{Re}(s) > 0.$$

So by the s-shifting property, we see that

$$\mathscr{L}[e^{2t}t](s) = F(s-2) = \frac{1}{(s-2)^2}, \ \mathrm{Re}(s) > 2.$$

Derivatives of Laplace Transforms §11.2.4

Theorem

Suppose $f(t)$, $t \geq 0$, is a function whose Laplace Transform

$$F(s) = \mathscr{L}[f(t)](s)$$

is defined for $\mathrm{Re}(s) > \sigma$.
Then

$$\mathscr{L}[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}, \ \mathrm{Re}(s) > 0, \ n \in \mathbb{N}.$$

Proof

Differentiating

$$F(s) = \mathscr{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)\, dt, \ \mathrm{Re}(s) > \sigma,$$

with respect to s gives

$$\begin{aligned}
\frac{d^n F(s)}{ds^n} &= \frac{d}{ds^n} \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^\infty \frac{d}{ds^n} \left(e^{-st} f(t)\right)\, dt \\
&= \int_0^\infty \frac{d}{ds^n} \left(e^{-st}\right) f(t)\, dt \\
&= \int_0^\infty (-1)^n t^n e^{-st} f(t)\, dt \\
&= (-1)^n \int_0^\infty e^{-st} (-1)^n f(t)\, dt \\
&= (-1)^n \mathscr{L}[t^n f(t)](s) \\
\Rightarrow \mathscr{L}[t^n f(t)](s) &= (-1)^n \frac{d^n F(s)}{ds^n}, \ \mathrm{Re}(s) > \sigma.
\end{aligned}$$

Q.E.D. (note we don't know how to prove that the domain remains the same but know that it is).

Example

Compute

$$\mathscr{L}[t^n](s).$$

Proof

Let

$$F(s) = \mathscr{L}[1](s) = \frac{1}{s},\ \mathrm{Re}(s) > 0.$$

Then

$$\mathscr{L}[t^n](s) = \mathscr{L}[t^n \cdot 1](s) = (-1)^n \frac{d^n F(s)}{ds^n} = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right).$$

It's easy to see that

$$\mathscr{L}[t^n](s) = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s}\right) = \frac{n!}{s^{n+1}}.$$

So in summary

$$\mathscr{L}[t^n](s) = \frac{n!}{s^{n+1}},\ \mathrm{Re}(s) > 0.$$

The Laplace Transforms of derivatives §11.3.1

Theorem

Suppose that $f(t),\ t \geq 0$ satisfies

$$\left|\frac{d^j f}{dt^j}(t)\right| \leq M e^{\sigma t},\ h = 1, 2, ..., n$$

for some $M > 0$, and $\sigma \in \mathbb{R}$. Then,

$$\mathscr{L}\left[\frac{d^j f}{dt^j}(t)\right](s) = s^n F(s) - \sum_{j=1}^{n} s^{n-j} \frac{d^{j-1} f}{dt^{j-1}}(0),\ j = 1, 2, ..., n$$

where

$$F(s) = \mathscr{L}[f(t)](s).$$

Some examples:

$$\mathscr{L}\left[\frac{df}{dt}(t)\right](s) = s\mathscr{L}[f(t)](s) - f(0)$$
$$\mathscr{L}\left[\frac{d^2 f}{dt^2}(t)\right](s) = s^2\mathscr{L}[f(t)](s) - sf(0) - \frac{df}{dt}(0)$$

Proof

From the definition

$$\mathscr{L}\left[\frac{df}{dt}(t)\right](s) = \int_0^\infty e^{-st} \frac{df}{dt}(t)\, dt,$$

we see that

$$
\begin{aligned}
\mathscr{L}\left[\frac{df}{dt}(t)\right](s) &= \int_0^\infty e^{-st}\frac{df}{dt}(t)\,dt \\
&= \lim_{T\to\infty}\int_0^T e^{-st}\frac{df}{dt}(t)\,dt \\
&= \lim_{T\to\infty}\left[e^{-st}f(t)\big|_0^T - \int_0^T \frac{d}{dt}\left(e^{-st}\right)f(t)\,dt\right] \\
&= \lim_{T\to\infty}\left[e^{-sT}f(T) - f(0) + s\int_0^T e^{-st}f(t)\,dt\right]
\end{aligned}
$$

But

$$
\begin{aligned}
\left|\lim_{T\to\infty} e^{-sT}f(T)\right| &= \lim_{T\to\infty}\left|e^{-sT}f(T)\right| \\
&= \lim_{T\to\infty}\left|e^{-\mathrm{Re}(s)T}e^{-\mathrm{Im}(s)iT}f(T)\right| \\
&= \lim_{T\to\infty}\left|e^{-\mathrm{Re}(s)T}\right|\left|e^{-\mathrm{Im}(s)iT}\right||f(T)| \\
&= \lim_{T\to\infty} e^{-\mathrm{Re}(s)T}\,|f(T)| \\
&= \lim_{T\to\infty} e^{-\mathrm{Re}(s)T}Me^{\sigma T} \\
&= M\lim_{T\to\infty} e^{-(\mathrm{Re}(s)-\sigma)T} = 0,\ \mathrm{Re}(s) > \sigma.
\end{aligned}
$$

Thus,

$$
\mathscr{L}\left[\frac{df}{dt}(t)\right](s) = -f(0) + s\mathscr{L}[f(t)](s)
$$