ENG1005: Lecture 31

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Video link

https://echo360.org.au/lesson/G_8402119b-734b-4e1e-a3b4-7e907e86ddba_b944cecf-8ba5-40d3-a870-024
2020-06-02T15:58:00.000_2020-06-02T16:53:00.000/classroom#sortDirection=desc

Laplace Transform §11.2, 11.2.1

Given a function $f(t)$ defined for $t \geq 0$, the Laplace Transform of f is defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad s \in D \subset \mathbb{C},$$

where D is some sort of domain. Note you could consider F as a function of two variables, namely the real and imaginary components of s, but I guess we'll see defining it as above is preferable.

The function $F(s)$ is often written

$$F(s) = \mathscr{L}[f](s).$$

It can be used to transform something from a time domain ($f(t)$)into a complex frequency domain ($\mathscr{L}[f](s)$).

Linearity §11.2.4

Given $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
\mathscr{L}[\lambda f_1(t) + f_2(t)](s) &= \int_0^\infty e^{-ts}(\lambda f_1(t) + f_2(t))\,dt \\
&= \lambda \int_0^\infty e^{-ts} f_1(t)\,dt + \int_0^\infty e^{-st} f_2(t)\,dt \\
&= \lambda \mathscr{L}[f_1(t)](s) + \mathscr{L}[f_2(t)](s) \\
\Rightarrow \mathscr{L}[\lambda f_1(t) + f_2(t)](s) &= \lambda \mathscr{L}[f_1(t)](s) + \mathscr{L}[f_2(t)](s)
\end{aligned}$$

for all $\lambda \in \mathbb{C}$ and $f_1(t),\ f_2(t)$ defined for $t \geq 0$.

Laplace Transforms of simple functions §11.2.2

Example

Let

$$f(t) = 1,\ t \geq 0.$$

Then

$$\begin{aligned}
\mathscr{L}[1](s) &= \int_0^\infty e^{-st}\,dt \\
&= \lim_{T\to\infty} \int_0^T e^{-st}\,dt \\
&= \lim_{T\to\infty} \left[-1\frac{1}{s}e^{-st}\Big|_0^T \right] \\
&= \lim_{T\to\infty} \left[-\frac{1}{s}e^{-sT} + \frac{1}{s} \right] \\
&= \lim_{T\to\infty} -\frac{1}{s}e^{-sT} + \frac{1}{s} \\
&= -\frac{1}{s} \lim_{T\to\infty} e^{-\theta T - i\omega T} + \frac{1}{s} \\
&= -\frac{1}{s} \lim_{T\to\infty} e^{-\sigma T} e^{-i\omega T} + \frac{1}{s} \\
&= -\frac{1}{s} \lim_{T\to\infty} e^{-\sigma T}(\cos(\omega T) - i\sin(\omega T)) + \frac{1}{s}.
\end{aligned}$$

Note the magnitude

$$\left|e^{-i\omega T}\right| = |\cos(\omega T) - i\sin(\omega T)| = 1.$$

So we now have

$$\mathscr{L}[1](s) = \frac{1}{s} \quad \text{IF } \sigma = \mathrm{Re}(s) > 0.$$

We got this by noting that, for that integral to converge, that limit had to reach zero, meaning

$$\lim_{T\to\infty} e^{-\sigma T} = 0,$$

which will only happen for positive σ.

Example

Let

$$f(t) = t,\ t \geq 0.$$

Then

$$\begin{aligned}
\mathscr{L}[t](s) &= \int_0^\infty e^{-st} t\, dt \\
&= \lim_{T\to\infty} \int_0^T e^{-st} t\, dt \\
&= \lim_{T\to\infty} \left[-\frac{e^{-st}}{s} t \Big|_0^T + \frac{1}{s} \int_0^T e^{-st}\, dt \right] \\
&= \lim_{T\to\infty} -\frac{e^{-sT}}{s} T + \frac{1}{s} \lim_{T\to\infty} \int_0^T e^{-st}\, dt \\
&= \lim_{T\to\infty} -\frac{e^{-sT}}{s} T + \frac{1}{s} \mathscr{L}[1](s),\ \mathrm{Re}(s) > 0 \\
&= \lim_{T\to\infty} -\frac{e^{-sT}}{s} T + \frac{1}{s^2} \\
&= 0 + \frac{1}{s^2} \\
&= \frac{1}{s^2}.
\end{aligned}$$

Thus,

$$\mathscr{L}[t](s) = \frac{1}{s^2},\ \mathrm{Re}(s) > 0.$$

Example

Let

$$f(t) = e^{kt},\ k \in \mathbb{C},\ t \geq 0.$$

Then

$$\begin{aligned}
\mathscr{L}[e^{kt}](s) &= \int_0^\infty e^{-st} e^{kt}\, dt \\
&= \int_0^\infty e^{t(k-s)}\, dt \\
&= \lim_{T\to\infty} \int_0^T e^{t(k-s)}\, dt \\
&= \lim_{T\to\infty} \left[\frac{1}{k-s} e^{t(k-s)} \Big|_0^T \right] \\
&= \lim_{T\to\infty} \left[\frac{1}{k-s} e^{T(k-s)} - \frac{1}{k-s} \right] \\
&= \frac{1}{k-s} \lim_{T\to\infty} e^{-(s-k)T} + \frac{1}{s-k}.
\end{aligned}$$

We now want

$$\mathrm{Re}(s-k) > 0 \Rightarrow \mathrm{Re}(s) > \mathrm{Re}(k),$$

so the limit reaches zero, so we then get

$$\mathscr{L}[e^{kt}](s) = \frac{1}{s-k}, \ \mathrm{Re}(s) > \mathrm{Re}(k)$$

Existence of the Laplace Transform §11.2.3

Theorem

If $f(t)$ is a piecewise continuous function for $t \geq 0$ and there exists constants $M \geq 0, \lambda \in \mathbb{R}$ such that

$$|f(t)| \leq M e^{\lambda t}, \ t \geq 0,$$

then the Laplace Transform

$$\mathscr{L}[f](s) = \int_0^\infty e^{-st} f(t)\, dt$$

exists and is well-defined for $\mathrm{Re}(s) > \lambda$.

To see why this is the case, we note that

$$\begin{aligned} \left| \int_0^T e^{-st} f(t)\, dt \right| &\leq \int_0^T \left| e^{-st} f(t) \right| \, dt \\ &\leq \int_0^T |e^{-st}||f(t)|\, dt. \end{aligned}$$

Letting

$$s = \sigma + i\omega,$$

we see that

$$e^{-st} = e^{-\sigma t + i\omega t} = e^{-\sigma t} e^{i\omega t}$$

from which we conclude that

$$e^{-st} = |e^{-\sigma t}||e^{i\omega t}| = |e^{-\sigma t}|.$$

Substituting this into the inequality gives

$$\left| \int_0^T e^{-st} f(t)\, dt \right| \leq \int_0^T e^{-\sigma t} |f(t)|\, dt.$$

Now, if

$$|f(t)| \leq M e^{\lambda t},$$

then

$$\begin{aligned} \left| \int_0^T e^{-st} f(t)\, dt \right| &\leq M \int_0^T e^{-\sigma t} e^{\lambda t}\, dt \\ &\leq M \int_0^T e^{-(\sigma - \lambda)t}\, dt \\ &=\leq M \left[\frac{-1}{\sigma - \lambda} e^{-(\sigma-\lambda)t} \Big|_0^T \right] \\ &\leq M \left[\frac{-1}{\sigma - \lambda} e^{-(\sigma - \lambda)T} + \frac{1}{\sigma - \lambda} \right] \end{aligned}$$

Assuming that

$$\sigma - \lambda > 0 \Leftrightarrow \mathrm{Re}(s) > \lambda,$$

we see from letting $T \to \infty$ in the above inequality that

$$\left| \int_0^T e^{-st} f(t)\, dt \right| \leq \frac{M}{\sigma - \lambda},$$

and

$$\int_0^\infty e^{-st} f(t)\, dt \text{ exists.}$$