ENG1005: Lecture 30

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Contents

Video link

https://echo360.org.au/lesson/G_35fe23e0-41ee-4e6f-b0f5-05f4155bb7b0_b944cecf-8ba5-40d3-a870-02
2020-05-28T15:58:00.000_2020-05-28T16:53:00.000/classroom#sortDirection=desc

Linear homogeneous 2nd order constant coefficients ODEs - continued

Case 3: $b^2 - 4ac = 0$

Then

$$\lambda = \zeta$$

is the only root of the characteristic equation which yields the solution

$$y_1(t) = e^{\zeta t}.$$

But we need 2 linearly independent solutions to form a general solutions. So, to find a second, linearly independent solution, we set

$$y_2(t) = ty_1(t) = te^{\zeta t}.$$

Now, quickly note that the first and second derivatives are given by

$$\frac{dy_2}{dt} = y_1 + t\frac{dy_1}{dt},$$
$$\frac{d^2y_2}{dt^2} = \frac{dy_1}{dt} + \left(t\frac{d^2y_1}{dt^2} + \frac{dy_1}{dt}\right) = 2\frac{dy_1}{dt} + t\frac{d^2y_1}{dt^2}.$$

Now, observe that

$$\begin{aligned}
a\frac{d^2y_2}{dt^2} + b\frac{dy_2}{dt} + cy_2 &= a\left(2\frac{dy_1}{dt} + t\frac{d^2y_1}{dt^2}\right) + b\left(y_1 + t\frac{dy_1}{dt}\right) + cty_1 \\
&= t\left(a\frac{d^2y_1}{dt^2} + b\frac{dy_1}{dt} + cy_1\right) + \left(2a\frac{dy_1}{dt} + by_1\right) \\
&= 0t + (2a\zeta + b)e^{\zeta t} \\
&= \left(2a\left(\frac{-b}{2a}\right) + b\right)e^{\zeta t} \\
&= 0
\end{aligned}$$

So we have two linearly independent solutions

$$y_1(t) = e^{\zeta t}, \quad y_2(t) = te^{\zeta t},$$

and so the general solution is given by

$$y(t) = c_1e^{\zeta t} + c_2te^{\zeta t},\ c_1, c_2 \in \mathbb{R}$$

Summary

The general solution of the

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad (a, b, c \in \mathbb{R},\ a \neq 0)$$

is

$$y(t) = \begin{cases} c_1e^{(\zeta+\omega)t} + c_2e^{(\zeta-\omega)t} & \text{if } b^2 - 4ac > 0, \\ e^{\zeta t}(c_1\cos(\omega t) + c_2\sin(\omega t)) & \text{if } b^2 - 4ac < 0, \\ e^{\zeta t}(c_1 + c_2t) & \text{if } b^2 - 4ac = 0 \end{cases}$$

where

$$\zeta = \frac{-b}{2a}, \quad \omega = \frac{\sqrt{|b^2 - 4ac|}}{2a}$$

Example

Find the general solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0.$$

Solution

The trial solution $y = e^{\lambda x}$ yields the characteristic equation

$$\lambda^2 - 2\lambda - 8\lambda = 0.$$

Since

$$(2)^2 - 4)(1)(-8) = 36 > 0,$$

we know there are two distinct real roots

$$\lambda_\pm = \frac{2 \pm \sqrt{36}}{2} = \frac{2 \pm 6}{2} = 1 \pm 3 \Rightarrow \lambda_+ = 4,\ \lambda_- = -2.$$

Thus the general solution is

$$y(x) = Ae^{4x} + Be^{-2x},\ A, B \in \mathbb{R}$$

Example

Find the general solution of
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = x.$$

Solution

First, we try to find a particular solution by guessing
$$y_p = c_0 + c_1 x.$$

Then
$$\begin{aligned}\frac{d^2y_p}{dx^2} - 2\frac{dy_p}{dx} - 8y_p &= 0 - 2c_1 - 8(c_0 + c_1 x)\\ &= -2c_1 - 8c_0 - 8c_1 x\\ &= x \Leftrightarrow \begin{cases} -2c_1 - 8c_0 = 0, \\ -8c_1 = 1 \end{cases}\\ &\Leftrightarrow \begin{cases} c_1 = -\frac{1}{8} \\ c_0 = -\frac{1}{4}c_1 = \frac{1}{32} \end{cases}\end{aligned}$$

shows that
$$y_p = \frac{1}{32} - \frac{1}{8}x$$
is a particular solution.
Since we know from the previous example that
$$y_h = Ae^{4x} + Be^{-2x},\ A, B \in \mathbb{R}$$
is the general solution to the homogeneous ODE
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0,$$
we conclude that
$$y = y_h + y_p = Ae^{4x} + Be^{-2x} + \frac{1}{32} - \frac{1}{8}x,\ A, B \in \mathbb{R}$$
is the general solution of the non-homogeneous ODE
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = x$$

Example

Solve the IVP
$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0,$$
$$y(0) = 1, \frac{dy}{dt}(0) = 2.$$

Solution

The trial solution $y = e^{\lambda t}$ yields the characteristic equation

$$\lambda^2 - 2\lambda + 5\lambda = 0$$

Since

$$(-2)^2 - 4(1)(5) = -16 < 0,$$

there are two complex roots

$$\lambda_\pm = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i \Rightarrow \lambda_+ = 1 + 2i, \lambda_- = 1 - 2i$$

Thus the general complex solution is

$$\begin{aligned} y(t) &= Ae^{(1+2i)t} + Be^{(1-2i)t} \\ &= Ae^t(\cos(2t) + i\sin(2t)) + Be^t(\cos(2t) - i\sin(2t)), \end{aligned}$$

which yields the general real solution

$$y(t) = e^t(A\cos(2t) + B\sin(2t)), \;\; A, B \in \mathbb{R}.$$

Then

$$\begin{aligned} \frac{dy}{dt} &= e^t\left[A\cos(2t) + B\sin(2t) + 2(-A\sin(2t)B\cos(2t))\right] \\ &= e^t\left[(A + 2B)\cos(2t) + (B - 2A)\sin(2t)\right]. \end{aligned}$$

The initial conditions then imply that

$$\begin{aligned} y(0) &= A = 1, \\ \frac{dy}{dt}(0) &= A + 2B = 2 \Rightarrow B = \frac{1}{2}. \end{aligned}$$

Thus,

$$y(t) = e^t(\cos(2t) + \frac{1}{2}\sin(2t))$$

solves the IVP.

Example

Find the general solution of

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

Solution

The trial solution $y = e^{\lambda x}$ yields the characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0.$$

Since

$$(-2)^2 - 4(1)(1) = 0,$$

there is only one repeated real root given by

$$\lambda = \frac{-2}{2} = -1.$$

Thus the general solution is

$$y(x) = Ae^{-x} + Bxe^{-x}, \;\; A, B \in \mathbb{R}.$$

Guessing particular solutions

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = q(x)$$

(i) If

$$q(x) = e^{kx}\sum_{j=0}^{n} b_j x^j,$$

try

$$y_p(x) = \begin{cases} e^{kx}\sum_{j=0}^{n} c_j x^j & \text{if } k \text{ is not a root of the characteristic equation,} \\ xe^{kx}\sum_{j=0}^{n} c_j x^j & \text{if } k \text{ is a non-repeated root,} \\ x^2e^{kx}\sum_{j=0}^{n} c_j x^j & \text{if } k \text{ is a repeated root} \end{cases}$$

(ii) If

$$q(x) = e^{kx}(b_1\cos(vt) + b_2\sin(vt)),$$

try

$$y_p(x) = \begin{cases} e^{kx}(c_1\cos(vt) + c_2\sin(vt)) & \text{if } k \pm iv \text{ are not roots of the characteristic equation,} \\ xe^{kx}(c_1\cos(vt) + c_2\sin(vt)) & \text{otherwise} \end{cases}$$