ENG1005: Lecture 25

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Video link

`https://echo360.org.au/lesson/G_8402119b-734b-4e1e-a3b4-7e907e86ddba_b944cecf-8ba5-40d3-a870-02`
`2020-05-19T15:58:00.000_2020-05-19T16:53:00.000/classroom#sortDirection=desc`

Example

Estimate the change in the period

$$T = 2\pi\sqrt{\frac{L}{g}}$$

of a simple pendulum due to a 2% increase in length L and a 0.6% decrease in the gravitational acceleration g.

Solution

$$dL = \frac{2}{100}L \text{ and } dg = \frac{-6}{1000}g$$

Substituting this into

$$\begin{aligned}
dT &= \frac{\partial T}{\partial L}\,dL + \frac{\partial T}{\partial g}\,dg \\
&= 2\pi \frac{1}{2\sqrt{\frac{L}{g}}}\frac{1}{g}\,dL + 2\pi \frac{1}{2\sqrt{\frac{L}{g}}}\left(\frac{-L}{g^2}\right)dg \\
&= \frac{\pi}{\sqrt{Lg}}\,dL - \frac{\pi\sqrt{L}}{g^{\frac{3}{2}}}\,dg
\end{aligned}$$

yields

$$\begin{aligned}
dT &= \frac{\pi}{\sqrt{Lg}}\frac{2}{100}L + \frac{\pi\sqrt{L}}{g^{\frac{3}{2}}}\frac{6}{1000}g \\
&= \frac{2\pi}{100}\sqrt{\frac{L}{g}} + \frac{6\pi}{1000}\sqrt{\frac{L}{g}} \\
&= \frac{26\pi}{1000}\sqrt{\frac{L}{g}} \\
&= \frac{13}{1000}T
\end{aligned}$$

So there is a 1.3% increase in the period of a pendulum.

Tangent plane

Given a function $f(x,y)$, then the curves

$$\mathbf{r}(s) = (a+s, b, f(a+s,b)); \quad \mathbf{r}(0) = (a,b,f(a,b))$$

$$\tilde{\mathbf{r}}(t) = (a, b+t, f(a,b+t)); \quad \tilde{\mathbf{r}}(0) = (a,b,f(a,b))$$

lie on the graph

$$S = \{(x,y,f(x,y)) | (x,y) \in D\} \subset \mathbb{R}^3$$

and pass through the point $p = (a,b,f(a,b))$ at $s=0$ and $t=0$, respectively.

$$\begin{aligned}
\frac{d\mathbf{r}}{dS}(0) &= \frac{d}{ds}((a+s,b,f(a+s,b))\Big|_{s=0} \\
&= \left(1, 0, \frac{\partial f}{\partial x}(a+s,b)\frac{d}{ds}(a+s)\right)\Big|_{s=0} \\
&= \left(1, 0, \frac{\partial f}{\partial x}(a,b)\right)
\end{aligned}$$

$$\begin{aligned}
\frac{d\tilde{\mathbf{r}}}{0} &= \frac{d}{dt}(a, b+t, f(a,b+t))\Big|_{t=0} \\
&= \left(0, 1, \frac{\partial f}{\partial y}(a,b)\right)
\end{aligned}$$

Tangent plane parametrisation

$$\begin{aligned}\ell &= \mathbf{p} + h\frac{d\mathbf{r}}{ds}(0) + k\frac{d\tilde{\mathbf{r}}}{dt}(0)\\ &= (a,b,f(a,b)) + h\left(1,0,\frac{\partial f}{\partial x}(a,b)\right) + k\left(0,1,\frac{\partial f}{\partial y}(a,b)\right)\\ \Rightarrow \ell(h,k) &= \left(a+h, b+k, f(a,b) + \frac{\partial f}{\partial x}(a,b)h + \frac{\partial f}{\partial y}(a,b)k\right)\end{aligned}$$

Setting

$$h = x - a \text{ and } k = y - b$$

gives

$$\ell(x,y) = \left(x, y, f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)\right)$$

So this has now given us a **parametrisation of the tangent plane** to the graph of S of $f(x,y)$ at the point $(a,b,f(a,b))$.

But note that the tangent plane is the graph of the linearisation of $f(x,y)$! This makes sense really since, when we're linearising, we take the first order terms around a given point, which seems analogous to taking the first order derivative of a line in $\mathbb{R}^2$ (which gives the tangent line.

Maximum and minimum values §9.7.2, 9.7.3

Definition

A function $f(x,y)$ has a

(i) local maximum at (a,b) if there exists some $R > 0$ such that $f(x,y) \leq f(a,b)$ for all $(x,y) \in B_R((a,b))$.

(ii) local minimum at (a,b) if there exists some $R > 0$ such that $f(x,y) \geq f(a,b)$ for all $(x,y) \in B_R((a,b))$.

(iii) absolute maximum at (a,b) if $f(x,y) \leq f(a,b)$ for all $(x,y) \in D$ (where D is the domain of f).

(iv) absolute minimum at (a,b) if $f(x,y) \geq f(a,b)$ for all $(x,y) \in D$ (where D is the domain of f).

Definition

A point $(a,b) \in \mathbb{R}^2$ is called a critical (stationary) point of $f(x,y)$ if $\nabla f(x,y) = \mathbf{0}$ or $\nabla f(a,b)$ is undefined. (is that meant to be x,y or a,b, not sure?)

Theorem

If the gradient $\nabla f(x,y)$ exists at (a,b) and (a,b) is either a local maximum or a local minimum of $f(x,y)$ then $\nabla f(a,b) = \mathbf{0}$.

Definition

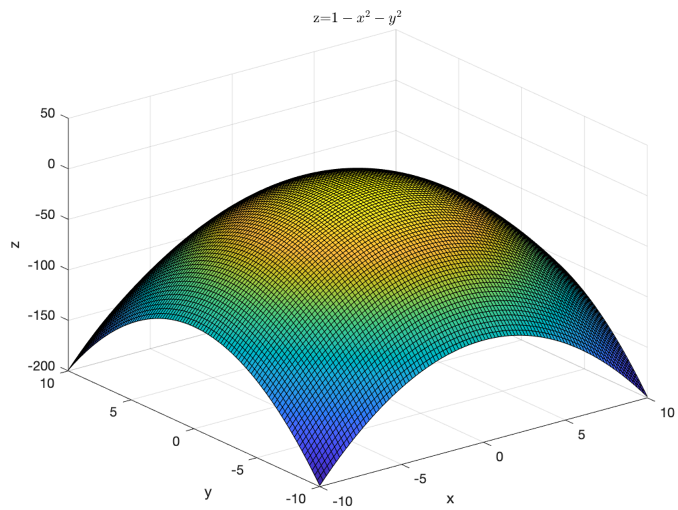
If (a,b) is a point such that $\nabla f(a,b) = \mathbf{0}$ but (a,b) is neither a local minimum or a local maximum, then (a,b) is called a saddle point.

Examples

(i) $f(x,y) = 1 - x^2 - y^2$.

$$\nabla f(x,y) = (-2x, -2y)$$
$$\Rightarrow \nabla f(0,0) = (0,0) \Rightarrow (0,0) \text{ is a critical point}$$

and if you look at the graph of $f(x,y) = 1 - x^2 - y^2$, we can see that $f(x,y)$ has an absolute maximum at $(0,0)$.



(ii) $f(x,y) = x^2 + y^2$

$$\nabla f(x,y) = (2x, 2y)$$
$$\nabla f(0,0) = (0,0) \Rightarrow (0,0) \text{ is a critical point}$$

and we can similarly look at this graph to see that $f(x,y) = x^2 + y^2$ has an absolute minimum at $(0,0)$.