ENG1005: Lecture 8

Lex Gallon

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Video link

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Power Series §7.7

Definition

A series of the type $\sum_{n=0}^{\infty} a_n x^n$ is called a power series.

Key idea

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Note: For analytic/homomorphic functions.

Example

The geometric series $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n,\ |x| < 1$ is a power series.

Radius of convergence §7.7.1

From the Ratio Test, we know that the power series $\sum_{n=0}^{\infty} a_n x^n$ will converge absolutely if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| < 1$$

Notice

$$\left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| \left| \frac{a+n+1}{a_n} \right|$$

So

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$
$$\Leftrightarrow |x| < \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

But

$$\frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|} = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

and so we have that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| < 1 \Leftrightarrow |x| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Theorem

Suppose $\{a_n\}_{n=0}^{\infty}$ is a sequence for which the limit

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists (we allow for $r = \infty$). Then the power series $\sum_{n=0}^{\infty} a_n x^n$ will converge absolutely for all x satisfying $|x| < r$.

Theorem

The number $r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ is known as the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and $(-r, r)$ $(|x| < r)$ is called the interval of convergence.

Example

Show that the exponential series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad\qquad \left[e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right]$$

converges for all $x \in \mathbb{R}$.

Solution

In our case, $a_n = \frac{1}{n!}$

$$\begin{aligned} r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| &= \lim_{n\to\infty}\left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right| \\ &= \lim_{n\to\infty}(n+1) \\ &= \infty \end{aligned}$$

This shows that the exponential series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ has an infinite radius of convergence and hence it converges absolutely for all $x \in (-\infty, \infty)$.

Example

Determine the radius and interval of convergence for the power series

$$\sum_{n=0}^{\infty} n! x^n$$

Solution

Since

$$\lim_{n\to\infty}\left|\frac{a_n}{an+1}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0,$$

the radius of convergence of $\sum_{n=0}^{\infty} x! n^n$ is zero and the series converges only at $x = 0$.

General power series

Definition

A power series about $x = x$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Remark

These power series can be converted into the standard form by setting

$$y = x - x_0.$$

Because then

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n y^n$$

The radius of convergence is computed in the same way.

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

The series will then converge absolutely for

$$|y| < r \Leftrightarrow |x - x_0| < r \Leftrightarrow x \in (x_0 - r, x_0 + r)$$

Note we have the interval of convergence on the right hand side of the above.

Example

Show that the logarithmic series

$$\ln(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} (x - 1)^n$$

converges for $0 < x < 2$. (In fact it converges for $0 < x \leq 2$).

Solution

The radius of convergence is given by

$$\begin{aligned} r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{n}}{\frac{(-1)^{n+2}}{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{\frac{1}{n}}{\frac{1}{n+1}} \right| \\ &= \lim_{n\to\infty} \frac{n+1}{n} \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \\ &= 1 \end{aligned}$$

This shows that the logarithmic series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ will converge on the interval

$$|x-1| < 1 \Leftrightarrow x \in (0,2)$$

Properties of power series

Addition

$$\lim_{n\to\infty} a_n(x-x_0)^n + \lim_{n\to\infty} b_n(x-x_0)^n = \lim_{n\to\infty}(a_n+b_n)(x-x_0)^n$$

for all x satisfying $|x-x_0| < r$ where $r = \min\{r_a, r_b\}$ where r_a and r_b are the radii of convergence of the power series $\lim_{n\to\infty} a_n(x-x_0)^n$ and $\lim_{n\to\infty} b_n(x-x_0)^n$, respectively.

Differentiation

$$\frac{d}{dx}\left(\lim_{n\to\infty} a_n(x-x_0)^n\right) = \lim_{n\to\infty} a_n n(x-x_0)^{n-1}$$

for all x satisfying $|x-x_0| < r$.

Integration

$$\int \sum_{n=0}^{\infty} a_n(x-x_0)^n\, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1} + c$$

for all x satisfying $|x-x_0| < r$.