ENG1005: Lecture 13

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Video link

Click here for a recording of the lecture.

Linear systems of equations §5.5, 5.5.2

Example

(a) **1 equation and 2 unknowns.**

$$-2x - 3y = 4$$

We solve this to get

$$y = \frac{-1}{3}(4 + 2x),\ x \in \mathbb{R}$$

which implies an infinite number of solutions. I.e.

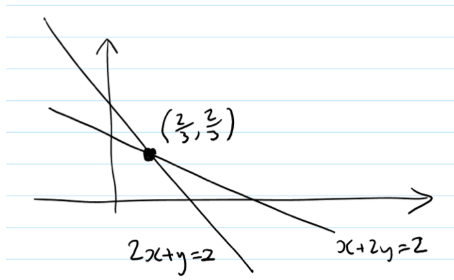
$$\{(x, y) | x \in reals, y = \frac{-1}{3}(4 + 2x)\}$$

(b) **2 equations and 2 unknowns.**

$$\begin{aligned} x + 2y &= 2 && (1) \\ 2x + y &= 2 && (2) \\ \hline x + 2y &= 2 && (1) \\ -3y &= -2 && (2) \to -2 \times (1) + (2) \\ \hline x + 2y &= 2 && (1) \\ y &= \frac{2}{3} && (2) \to \frac{-1}{3}(2) \\ \hline \end{aligned}$$

Back-substitution to solve:

$$x + \frac{4}{3} = 2 \Rightarrow x = \frac{2}{3} \text{ is the unique solution to the given equations}$$

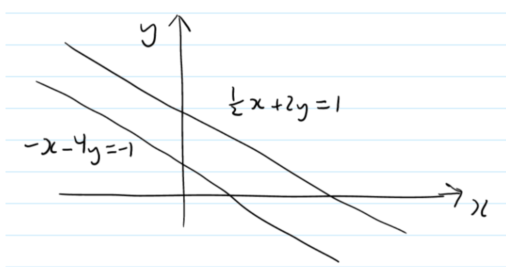


Since this system of equations has at least one solution, we call it a **consistent** system (same goes for that of part (a)).

(b.2)

$$\begin{aligned} \frac{1}{2}x + 2y &= 1 && (1) \\ -x - 4y &= 0 && (2) \\ \hline \frac{1}{2}x + 2y &= 1 && (1) \\ 0 &= 1 && (2) \to 2 \times (1) + (2) \end{aligned}$$

This contradiction implies no solution! This also means this is an **inconsistent** system.



Definition

A linear system of m-equations in n-unknowns is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m$$

This system is called **consistent** if there exists at least 1 solution, and otherwise, it is called **inconsistent**.

Possible outcomes

(i) No solution.

(ii) A unique solution.

(iii) An infinite number of solutions.

Elementary row operations §5.2.2

Given a linear system of equations, we are free to:

(1) Multiply any one equation by a non-zero scalar $\lambda \in \mathbb{R}$. E.g.

$$\begin{aligned} 3x + 2y - z &= 1 && (1) & x + \frac{2}{3}y - \frac{1}{3}z &= \frac{1}{3} && (1) \to \frac{1}{3}(1) \\ 4x - y + 2z &= -2 && (2) \Leftrightarrow & 4x - y + 2z &= -2 && (2) \\ -x + y + z &= 4 && (3) & -x + y + z &= 4 && (3) \end{aligned}$$

(2) Switch any two equations.

$$\begin{aligned} 3x + 2y - z &= 1 && (1) & -x + y + z &= 4 && (1) \to (3) \\ 4x - y + 2z &= -2 && (2) \Leftrightarrow & 4x - y + 2z &= -2 && (2) \\ -x + y + z &= 4 && (3) & 3x + 2y - z &= 1 && (3) \to (1) \end{aligned}$$

(3) Add λ times one equation to another.

$$\begin{aligned} 3x + 2y - z &= 1 && (1) & 5y + 2z &= 13 && (1) \to 3 \times (3) + (1) \\ 4x - y + 2z &= -2 && (2) \Leftrightarrow & 4x - y + 2z &= -2 && (2) \\ -x + y + z &= 4 && (3) & -x + y + z &= 4 && (3) \end{aligned}$$

Matrices §5.2.1

An $m \times n$ matrix is an m by n array of real numbers (complex numbers are allowed but whatever).

$$A = [A_{ij}]$$

All A_{ij} are the **coefficients** of the array A.

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ A_{31} & A_{32} & ... & A_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & ... & A_{mn} \end{bmatrix}$$

where $1 \leq i \leq m$ is the row index and $1 \leq j \leq n$ is the column index.

Definition

Given a linear system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n = b_m$$

then the matrix

$$[A_{ij}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & ... & a_{1n} & b1 \\ a_{21} & a_{22} & ... & a_{2n} & b2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} & b_m \end{array}\right]$$

is known as the **augmented matrix** of the linear system. (Note, the vertical line isn't require but does nicely distinguish it from a general matrix).

Gaussian elimination §5.5.2

Example

(i)

$$\begin{aligned} 2x - y + z &= 1 \\ x + 2y - z &= -1 \\ -x - y - z &= 2 \end{aligned}$$

Switch R1 with R2

$$\begin{aligned} x + 2y - z &= -1 \\ 2x - y + z &= 1 \\ -x - y - z &= 2 \end{aligned}$$

R3 becomes R1 + R3

$$\begin{aligned} x + 2y - z &= -1 \\ 2x - y + z &= 1 \\ y - 2z &= 1 \end{aligned}$$

R2 becomes R2 - 2R1

$$\begin{aligned} x + 2y - z &= -1 \\ -5y + 3z &= 3 \\ y - 2z &= 1 \end{aligned}$$

Switch R2 with R3

$$\begin{aligned} x + 2y - z &= -1 \\ y - 2z &= 1 \\ -5y + 3z &= 3 \end{aligned}$$

R3 becomes 5R2 + R3

$$\begin{aligned} x + 2y - z &= -1 \\ y - 2z &= 1 \\ -7z &= 8 \end{aligned}$$

Now, with augmented matrices

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 1 & 2 & -1 & -1 \\ -1 & -1 & -1 & 2 \end{array}\right]$$

$$R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & -1 & 1 & 1 \\ -1 & -1 & -1 & 2 \end{array}\right]$$

$$R_3 \to R_1 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & -1 & 1 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -5 & 3 & 3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & -5 & 3 & 3 \end{array}\right]$$

$$R_3 \to 5R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -7 & 8 \end{array}\right]$$

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