ENG1005: Lecture 17

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Video link

Click here for a recording of the lecture.

Matrix inverses - continued

A is an invertible matrix (non-singular) if there exists a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = \mathbb{I}_n$$

2x2 matrix inversion

Given a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided

$$ad - bc \neq 0 \text{ (i.e. } A \text{ is invertible)}$$

Example

Solve

$$
\begin{aligned}
x + 2y &= 1, \\
x + 4y &= -1
\end{aligned}
$$

Solution

This can be repesented as

$$
\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
$$

So,

$$
\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \frac{1}{1 \times 4 - 1 \times 2} \begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 6 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -1 \end{bmatrix}
\end{aligned}
$$

So $x = 3$, $y = -1$.

Matrix inversion via Gauss-Jordan elimination

Matrix inversion algorithm

(i) Given an $n \times n$ matrix A, form the augmented matrix

$$
[A|\mathbb{I}_n]
$$

E.g.
If

$$
A = \begin{bmatrix} 4 & -7 \\ 3 & 2 \end{bmatrix}, \text{ then } [A|\mathbb{I}_n] = \left[\begin{array}{cc|cc} 4 & -7 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right]
$$

(ii) Perform elementary row operations on $[A|\mathbb{I}_n]$ until one of the following happens:

(ii.a) $[A|\mathbb{I}_n]$ is transformed into $[\mathbb{I}_n|B]$. In this case, A is non-singular with inverse $A^{-1} = B$.

(ii.b) $[A|\mathbb{I}_n]$ is transformed into $[\tilde{A}|\tilde{B}]$, where $\tilde{A}$ has a row of zeros. In this case, A is singular.

Example

Determine if the following matrices are invertible:

(a)

$$
\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 0 & 2 & 4 \end{bmatrix}
$$

(b)

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

Solution

(a)

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 0 & 2 & 4 \end{bmatrix} &\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 & 0 & 1 \end{array}\right] R_2 \to R_2 + R_1 \\
&\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 1 \end{array}\right] R_3 \to R_3 - 2R_2
\end{aligned}$$

But notice that we now have a row of zeros on the left half of the augmented matrix. This shows that the given matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

is singular/not invertible.

(b)

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \\ 0 & 2 & 3 \end{bmatrix} &\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \\
&\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] R_2 \to R_2 + R_1 \\
&\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & -2 & -2 & 1 \end{array}\right] R_3 \to R_3 - 2R_2 \\
&\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] R_3 \to \tfrac{1}{3}R_3 \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] R_1 \to R_1 - 2R_2 \\
&\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{array}\right] R_1 \to R_1 - R_3
\end{aligned}$$

Now we have the identity matrix on the left half of the augmented matrix. This means that

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & -1 \\ 0 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} \\ 1 & 1 & 0 \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Determinants §5.3

2x2 matrices

The determinant of a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined by

$$\det(A) = ad - bc (= |A|)$$

Recalling that

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \det(A) = ad - bc$$

we see that A is invertible if and only if $\det(A) \neq 0$.

nxn matrices

If $A = [A_{ij}]$ is an $n \times n$ matrix, let $\tilde{M}_{ij}$ be the $(n-1) \times (n-1)$ matrix defined by deleting the ith row and jth column from the matrix A. E.g.
If $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$, then $\tilde{M}_{12} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$ The numbers

$$M_{ij} = \det(\tilde{M}_{ij})$$

are known as minors.

The determinant of the $n \times n$ matrix $A = [A_{ij}]$ is then defined recursively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} M_{ij}$$

where i is any fixed number on $\{1, 2, ..., n\}$. I.e. you get to choose i (same result no matter what)!

The numbers

$$C_{ij} = (-1)^{i+j} M_{ij}$$

are called cofactors. So if you like, you can rewrite the determinant formula as

$$\det(A) = \sum_{j=1}^{n} A_{ij} C_{ij}, \quad i \in \{1, 2, ..., n\}$$

This formula is known as the Laplace or cofactor expansion for the determinant of A.

Note: The number $(-1)^{i+j}$ can be remembered by

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ \vdots & & & \end{bmatrix}$$