John Forbes Nash - First Edition

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CO480 - Spring 2015

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1 Introduction to Game Theory & Strategic Games

The creation of the field of Game Theory is largely attributed to John von Neumann and Oskar Morgenstern who fully introduced the concepts of cooperative games and 2-player zero-sum games in their paper *Theory of Games and Economic Behaviour* (Nashs Thesis). This exposition was published in 1944 and built on works published by the two authors dating back to 1928 (Theory of Games & Economic Behaviour). This work provided a new approach to a number of problems in economics.

Since this early work, the field of Game Theory has exploded. Game Theory has is used to study and explain phenomena not only in economics, but also in military tactics, biology, and in real-world corporate business decisions (e.g. Mergers & Acquisitions, pricing decisions, supplier negotiations).

2 Strategic Games

We first define the class of games that Nash, and subsequently this paper will study. For the purposes of this paper, strategic games are games with N players, each of which has a finite number of pure strategies. This induces the following notation and definitions.

2.1 Definitions & Notation

Definition A strategic game with N players has **player set** $\{1, ..., n\}$ denoted by N

Each player $i \in N$ has a finite number of pure strategies. The set of all of player i's strategies is denoted as S_i, and an individual pure strategy is denoted as s_i.

A strategy profile denoted S is a N-tuple, where each element i is a pure strategy of player i.

The collection of all strategy profiles S is denoted as $\mathbb{S}$. We have then

$$\mathbb{S} = S_1 \times S_2 \times \cdots \times S_n$$

where $\times$ represents the Cartesian product of sets.

We also define the following utility/payoff functions.

$$\forall i, u_i : \mathbb{S} \Rightarrow \mathbb{R}$$

Each of the u_i takes in a strategy profile and returns the utility value that the player receives under the strategy profile S. That is, if each player i plays the strategy s_i in S, the player i will receive a payoff of $u_i(S)$

Note: The specific payoff amount may have no meaning in a strategic game. The only claim is that a player prefers a higher payoff to a smaller payoff. We cannot, however, say that a player prefers a payoff of 2, twice as much as a payoff of 1.

We also introduce the following substitution notation:

Suppose $S = (s_1, ..., s_n)$. Then the strategy profile $(S_{-i}, s'_i) = (s_i, ..., s_{-1}, s'_i, s_{i+1}, ..., s_n)$. In words, (S_{-1}, s'_i) is the strategy profile obtained from S when player i changes their strategy from s_i to s'_i

2.2 How are Strategic Games Played?

In strategic games each player moves simultaneously. That is, each player selects a strategy at the same time, and each players strategy selection is independent of the strategies chosen by the other players. Each player then receives payoff $u_i(S)$ based on the actions of each player.

We make two key assumptions when playing strategic games:

1. Each player is a rational actor with the single goal of maximizing their own utility, meaning that a player will always choose the action that - given the other players strategies - will yield the highest payoff.

2. We assume that players have played these games extensively in the past. It is assumed that this has led each player to form beliefs about how their opponents will play the game. This assumption is applied to each player, and it is assumed that all such beliefs are consistent.

2.3 Examples of Strategic Games

To illustrate strategic games, two-player games are often given in matrix form. The rows correspond to each of player 1s moves, and the columns correspond to player 2s moves. The elements of the matrix are ordered pairs (x, y) where x and y are the payoffs to player 1 and 2 respectively.

Prisoner's Dilemma

Set-up: Two prisoners have been captured and are being interrogated about their involvement with a crime. Each prisoner has two options remaining quiet (Q), or confessing (C). If both prisoners remain quiet, the police can only convict them of a minor charge. If both prisoners confess, the police will convict them of a major charge, but the sentence will be reduced because of their cooperation. If one prisoner confesses, they will go free and their accomplice will be convicted of the major offence. This situation can be modelled as the following strategic game:

	Q	C
Q	(2,2)	(0,3)
C	(3,0)	(1,1)

This results in the following instances of our definitions:

- $\mathbb{S} = \{(Q,Q),(Q,C),(C,Q),(C,C)\}$
- $S_1 = S_2 = \{Q, C\}$
- $u_1((Q,Q)) = 1$
- $u_2((C,Q)) = 0$

Matching Pennies

Set-up: Two players are each holding a penny. They each choose to show either Heads (H) or Tails (T). If both players select the same side, Player 2 pays player 1 \$1. If the sides do not match, then player 1 pays player 2 \$1. This can be represented by the following matrix form:

	H	T
H	(1,-1)	(-1,1)
T	(-1,1)	(1,-1)

	Q	C
Q	(2,2)	(0,3)
C	(3,0)	(1,1)

This results in the following instances of our definitions:

- $\mathbb{S} = \{(H,H),(H,T),(T,H),(T,T)\}$
- $S_1 = S_2 = \{H, T\}$
- $u_1((H,H)) = 1$
- $u_2((H,H)) = -1$

3 Pure Nash Equilibria

Nashs key result in game theory revolved around the concept of equilibria. In leymans terms, an equilibrium point in a strategic game is a strategy profile where no player can improve their payoff by just changing their own strategy. Mathematically we write:

$$S \in \mathbb{S} \text{ is a pure Nash Equilibria if } \forall i \in N, \forall s_i'$$
$$U_i(S) \geq u_i(S_{-1}, s_i')$$

We illustrate this concept by showing the pure Nash Equilibria in *The Prisoners Dillema* Recall the matrix form of this game: =

	Q	C
Q	(2,2)	(0,3)
C	(3,0)	(1,1)

Clain: The pure strategy profile (C,C) is a pure Nash Equilibrium.

Proof. Player 1s current payoff is 2. Given that player 2 is playing pure strategy C, player 1 can only change the strategy profile to (Q,C). However, this would yield payoff of 0 for player 1 which is suboptimal. Thus, player 1 has no incentive to change their strategy. A similar argument can be made for player 2. It is a simple exercise for the reader to check that S = (C,C) is the only pure Nash Equilibrium in this game.

□

The Prisoners Dilemma is a small 2-player game, but how can we check whether a strategy profile is a pure Nash Equilibrium in a larger game. To do this we introduce the concept of best response functions.

The idea is given the moves of the other players, what strategy will maximize player is utility.

Definition Best Response Function:

$$B_i : \mathbb{S}_{-1} \Rightarrow S_i$$
$$B_i(S_{-1}) = \{s_i \in S_i | u_i(S_{-1}, s_i) \geq u_i(S_{-1}, s_i') \forall s_i' \in S_i\}$$

We can now use the best response function to prove the following theorem.

Theorem 1 $S^* = (s_1^*, ..., s_n^*)$ is a pure Nash Equilibrium iff $s_i^* \in B_i(S_{-i}^*) \forall i \in N$

Proof. S^* is a pure Nash Equilibrium iff $\forall i u_i(S^*) >= u_i(S_{-i}^*, s_i') \ \forall s_i' \in S_i$. iff $s_{-i}^* \in B_i(S_-^* i)$. $\square$

Using Theorem 2 (only defined theorem 1), we can check for pure Nash Equilibirum using best response functions. This can be done by computing the best response function value for each player for each combination of their opponents strategies. In a two player game, a Nash Equilibrium will just be the strategy profiles $S = (s_1, s_2)$ where $s_1 \in B_1(s_2)$ and $s_2 \in B_2(s_1)$. We illustrate this with an example.
Consider the following game:

	L	C	R
T	(1,2)	(2,1)	(1,0)
M	(2,1)	(0,1)	(0,0)
B	(0,1)	(0,0)	(1,2)

We compute the best response functions in each case:

- $B_1(L) = \{M\}$
- $B_1(C) = \{T\}$
- $B_1(R) = \{T, B\}$
- $B_2(T) = \{L\}$
- $B_2(M) = \{L, C\}$
- $B_2(B) = \{R\}$

It is then easy to check that $S = (M, L)$ and $S' = (B, R)$ are pure Nash Equilibria as they satisfy the requirements for theorem 1.

To this point we have only considered pure Nash Equilibria. The key question is whether or not all games have such an equilibrium point. The answer in this case is no.

Consider again the Matching Pennies game

	H	T
H	(1,-1)	(-1,1)
T	(-1,1)	(1,-1)

It is easy to see that there are no pure Nash Equilibria. If the pennies currently match, then Player 2 could change their strategy so that the pennies do not match, increasing their payoff from -1 to 1. Similarly, if the pennies do not match, player 1 can change their strategy which results in an increase of their payoff from -1 to 1.

4 Mixed Strategies

As illustrated above, not all games have pure Nash Equilibria. However, we will show that all games contain an equilibrium point when we allow mixed strategies. First, we introduce the concept of mixed strategies, mixed Nash Equilibrium, and the accompanying notation.

Definition x^i is a **mixed strategy** of player i which represents a probability distribution over S_i. We define the elements of x^i as $x^i_{s_i}$ which represents the probability assigned to pure strategy s_i. These elements are defined for all s_i in S_i.

Consistent with a probability distribution we have the following constraints on x^i:

Constraint 1: $x^i_{s_i} \geq 0 \forall s_i \in S_i$

Constraint 2: $\sum) s_i \in S_i x^i_{s_i} = 1$

A pure strategy s_i is simply the mixed strategy where:

$$X^i_{s_i} = 1 \text{and} x^i_{s'_i} = 0 | s'_i \neq s_i$$

We denote a mixed strategy profile as $X = (x^1, ..., x^n)$. Compactly we have,

Constraint 3: $x^i \in \mathbb{R}^{|s_i|}_{+}$ with $\mathbb{1}^T x^i = 1$
Our interpretation of payoff necessarily shifts to the concept of expected payoff and we overload our payoff function notation to define, in the mixed strategy framework payoff functions u_i as:

Function 1: $U_i(X) = \sum_{s \in \mathbb{S}} U_i(S) \prod_{j \in N} x^j_{\mathbb{1}_j}$

As in the pure strategy case $(x^{-i}, \bar{x}^i)$ represents the mixed strategy profile $x = (x^1, ..x^{i-1}, \bar{x}^i, x^{i+1}, ..., x^n)$ which is the strategy profile obtained from x where player i has changed their strategy from x^i to $\bar{x}^i$.

Similarly, (x^{-i}, s_i) is the strategy profile where player i has replaced their strategy x^i with pure strategy s_i.

We define FUNCTION 4 as the expected payoff of pure strategy s_i.

Function 4: $U_i(x^{-i}, s_i) = \sum_{s_{-i} \in \mathbb{S}_{-i}} U_i(S_{-i}, s_i) \prod_{j \neq i} x^j_{S_j}$

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This allows us to re-write $U_i(x)$ as FUNCTION 5.

Function 5: $\sum_{s_i \in S_i} x^i_{s_i} U_i(x^{-i}, s_i)$

We can also redefine the concepts of equilibrium points and best response function in the context of mixed strategies as follows:

Definition 1: X is a **Mixed Nash Equilibrium** if $U_i(x) \geq U_i(x^{-i}, \bar{x}^{-i})$ $\forall i \in N$, for all mixed strategies $\bar{x}^{-i}$ of player i.