Recite 2: A Brief Review of Basic Statistics

Introduction to Econometrics,Fall 2021

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Section 1

Basic Concepts

Population(总体) and Sample(样本)

Definition

- A **population** is a collection of people, items, or events about which you want to make inferences.
  - Population always have a probability distribution.
- A **sample** is a subset of population, which draw from population in a certain way.
- The sample could also follow a probability distribution.
- To represent the population well,a sample should be **randomly** collected and adequately **large**.

Random sample(随机样本) and i.i.d(独立同分布)

Definition

- The r.v.s are called a **random sample** of size n from the population $f(x)$ if $X_1, ..., X_n$ are mutually independent and have the same p.d.f/p.m.f $f(x)$.
- Alternatively, $X_1, ..., X_n$ are called independent,and identically distributed random variable with p.d.f/p.m.f ,commonly abbreviated to **i.i.d.** r.v.s.
  - e.g.Random sample of n respondents in a survey.
- And the joint p.d.f/p.m.f of $X_1, ..., X_n$ is given by $f(x_1, ..., x_n) = f(x_1)...f(x_n) = \prod_{i=1}^{n} f(x_i)$

Statistic(统计量) and Sampling Distribution(抽样分布)

Definition

- $X_1, ..., X_n$ is a random sample of size n from the population $f(x)$.
- A **statistic**(T) is a real-valued or vector-valued function fully depended on $X_1, ..., X_n$, thus $T = T(X_1, ..., X_n)$
  - The expectation of sample is a statistic.
  - A statistic is only a function of the sample(统计量是样本的函数).
  - The probability distribution of a statistic T is called the **sampling distribution(抽样分布)** of T.

Sample Mean(样本均值) and Sample Variance(样本方差)

Definition –Two common and important estimators

- The **sample average** or **sample mean**, $\bar{X}$ of the n observations $X_1, ..., X_n$ is

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + ... + X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

- Accordingly, the **sample variance** is defined by

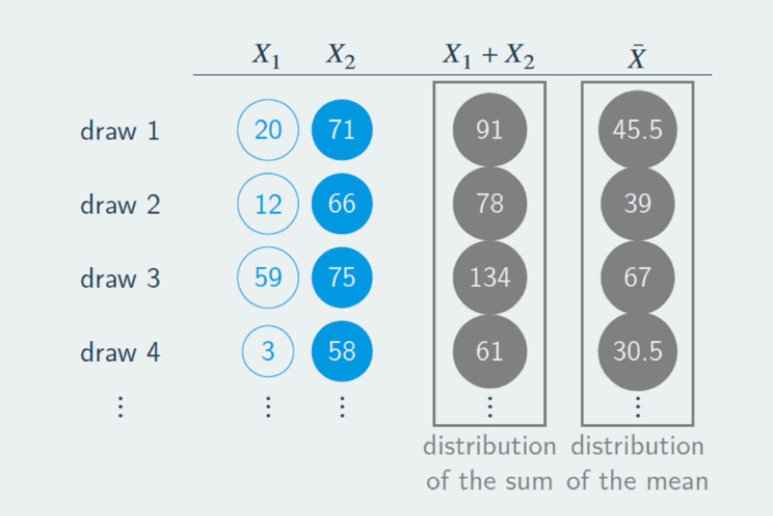
$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})$$

Deduction

- As we know that if X_i is a random variable(r.v.), then $f(X_i)$, which is a function of X_i, is also a r.v. (随机变量的函数还是随机变量)
- So if X_i is a r.v., then $\sum X_i$ is also a r.v..
  - The sample mean and the sample variance are also functions of sums, **therefore they are r.v.s, too.**
  - Assume that there are some certain probability functions which can describe distributions of the sample mean and the sample variance.
  - Then, naturally, what is the expectation, variance or p.d.f./c.d.f of these distributions?

A simple case of sample mean

- Let $X_n \in [1, 100]$, assume n $= 2$, thus only X_1 and X_2



Section 2

Large-Sample Approximations to Sampling Distributions

Sampling Distributions

1. There are two approaches to characterizing sampling distributions:
   - **exact/finite sample distribution**: The sampling distribution that exactly describes the distribution of X for any n is called the exact/finite sample distribution of X.
   - **approximate/asymptotic distribution**: When the sample size n is large, the sample distribution approximates to a certain distribution function.

Two Key Tools: L.L.N and C.L.T

2. Two key tools used to approximate sampling distributions when the sample size is large,assume that $n \to \infty$

- **The Law of Large Numbers(L.L.N.)**: when the sample size is large,X will be close to μ_Y the population mean with very high probability.
- **The Central Limit Theorem(C.L.T.)**: when the sample size is large, the sampling distribution of the standardized sample average,

$$\frac{(\bar{Y} - \mu_Y)}{\sigma_Y}$$

is approximately normal.

Convergence in probability(依概率收敛)

Definition

- Let $X_1, ..., X_n$ be an random variables or sequence, is said to converge in probability to a value b if for every $\varepsilon > 0$,

$$P(\|X_n - b\| > \varepsilon) \to 0$$

as $n \to \infty$. We denote this as $X_n \xrightarrow{p} b$ or $plim(X_n) = b$.

  - It is similar to the concept of a limitation in a probability way.

The Law of Large Numbers(大数定律)

Theorem

- Let $X_1, ..., X_n$ be an **i.i.d** draws from a distribution with mean μ and finite variance σ^2(a population) and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the **sample mean** then

$$\bar{X} \to \mu$$

  - Intuition: the distribution of $\bar{X}_n$ "collapses" on μ
  - 直观解释：抽样的样本量越大，样本平均值越接近总体平均值，即抽样分布更紧凑。

A simple case

Example

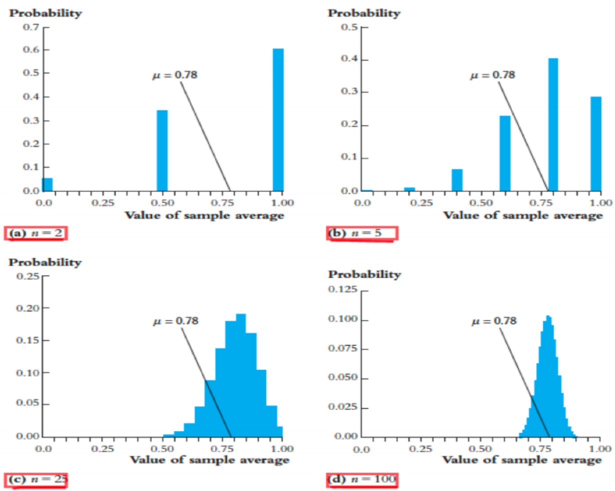
- Suppose X has a **Bernoulli** distribution if it have a binary values $X \in 0, 1$ and its probability mass function is

$$P(X = x) = \begin{cases} 0.78, & if\ x = 1 \\ 0.22, & if\ x = 0 \end{cases}$$

  - Then $E(X) = p = 0.78$ and $Var(X) = p(1-p) = 0.1716$

A simple case

FIGURE 2.8 Sampling Distribution of the Sample Average of n Bernoulli Random Variables



Convergence in Distribution(分布收敛)

Definition

Let $X_1, ..., X_n$ be a sequence of r.v.s, and for n = 1,2,... let $F_n(x)$ be the c.d.f of X_n. Then it is said that $X_1, X_2, ...$ **converges** in distribution to r.v. W with c.d.f, F_W if

$$\lim_{n \longrightarrow \infty} F_n(x) = F_W(x)$$

which we write as

$$X_n \xrightarrow{p} W$$

- Basically: when n is big, the distribution of X_n is **very similar** to the distribution of W.
- Standardize: by subtracting its expectation and dividing by its standard deviation
$$Z = \frac{X - E[X]}{Var[X]}$$

The Central Limit Theorem(中心极限定理)

Theorem

Let $X_1, ..., Xn$ be an i.i.d draws from a distribution with sample size n with mean μ and $0 < \sigma^2 < \infty$, then

$$\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \overset{d}{\sim} N(0,1)$$

- Because we don't have to make any specific assumption about the distribution of X_i, so whatever the distribution of X_i, when n is big,
  - the standardized

$$\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$$

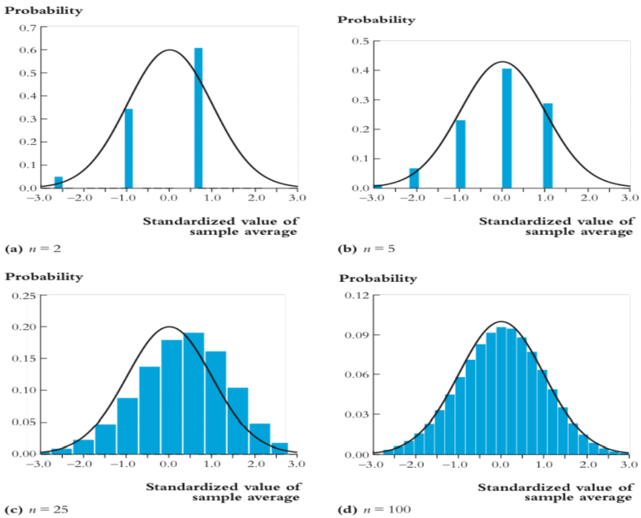
  - or

$$\bar{X}_n \sim N(0,1)$$

- 直观理解：选取的样本量越大，样本均值的分布越趋于**正态分布**

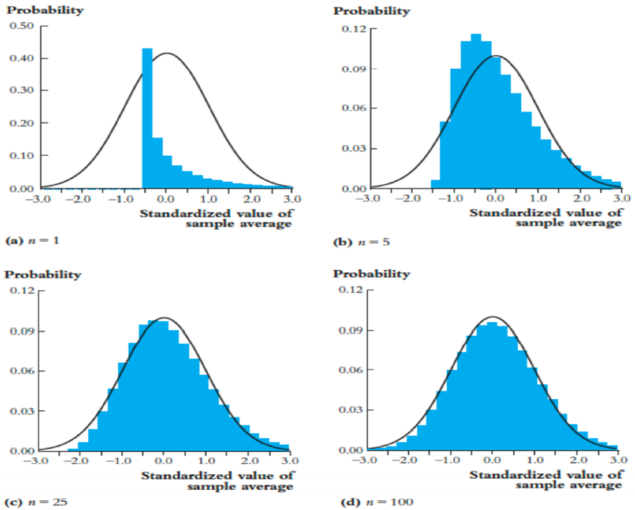
Example

FIGURE 2.9 Distribution of the Standardized Sample Average of n Bernoulli Random Variables with $p = 0.78$



Example

FIGURE 2.10 Distribution of the Standardized Sample Average of n Draws from a Skewed Distribution



How large is “large enough”?

- How large is **large enough**?
  - how large must n be for the distribution of $\bar{Y}$ to be approximately normal?
- The answer: it depends.
  - if Y_i are themselves normally distributed, then $\bar{Y}$ is exactly normally distributed for all n.
  - if Y_i themselves have a distribution that is far from normal, then this approximation can require $n = 30$ or even more.

Section 3

Statistical Inference: Estimation, Confident Intervals and Testing

Statistical Inference: From Samples to Population

Inference

- What is our best guess about some quantity of interest?
- What are a set of plausible values of the quantity of interest?
- Our focus: $\{Y_1, Y_2, ..., Y_n\}$ are **i.i.d.** draws from $f(y)$ or $F(Y)$, thus **population** distribution.
- **Statistical inference** or learning is using **samples** to infer $f(y)$.
  - Two ways: parametric and Non-parametric
  - Normally, we don't need to know everything of the population, just some measures (the moment) enough to describe the characteristics of the population.

Statistical Inference: Point estimation

Point estimation: providing a single "best guess" as to the value of some fixed, unknown quantity of interest,θ which is a feature of the population distribution, $f(y)$.

Example

1. $\mu = E[Y]$
2. $\sigma^2 = Var[Y]$
3. $\mu_y - \mu_x = E[Y] - E[X]$

Three Characteristics of an Estimator

Let $\hat{\mu}_Y$ denote some estimation value of the population moment, μ_Y and $E(\hat{\mu}_Y)$ is the mean of the sampling distribution of $\hat{\mu}_Y$

1. **Unbiasedness**: the estimator of μ_Y is unbiased if

$$E(\hat{\mu}_Y) = \hat{\mu}_Y$$

2. **Consistency**: the estimator of μ_Y is consistent if

$$E(\hat{\mu}_Y) \rightarrow \hat{\mu}_Y$$

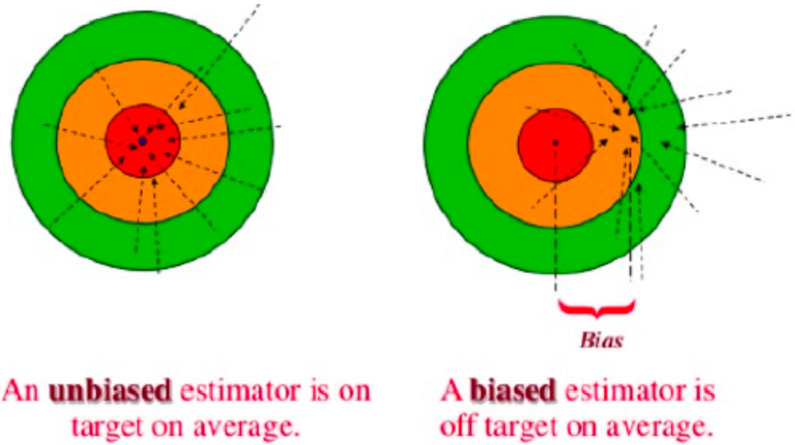
Three Characteristics of an Estimator

3. **Efficiency**: Let $\tilde{\mu}_Y$ be another estimator of μ_Y and suppose that both $\tilde{\mu}_Y$ and $\hat{\mu}_Y$ are unbiaesd.Then $\hat{\mu}_Y$ is said to be more **efficient** than $\tilde{\mu}_Y$

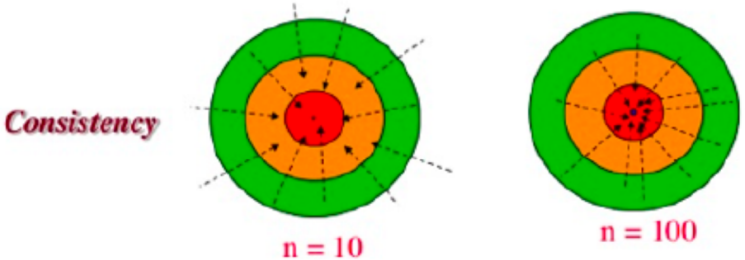
$$var(\hat{\mu}_Y) < var(\tilde{\mu}_Y)$$

- Comparing variances is difficult if we don't restrict our attention to unbiased estimators because we could always use a trivial estimator with variance zero that is biased.

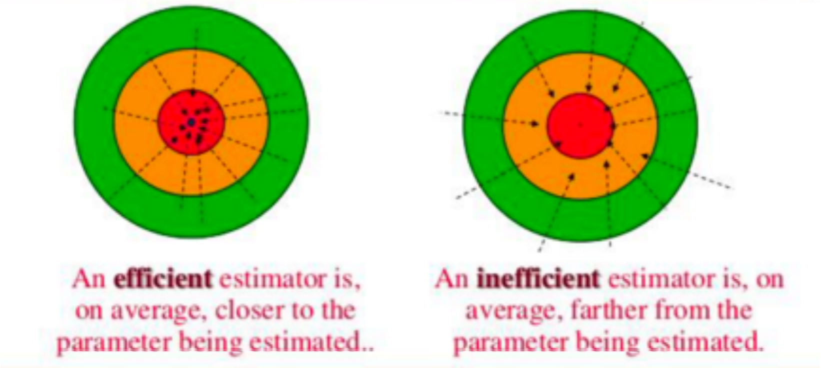
Unbiased


Bias
An unbiased estimator is on target on average.
A biased estimator is off target on average.

Consistency



Efficiency


An **efficient** estimator is, on average, closer to the parameter being estimated..
An **inefficient** estimator is, on average, farther from the parameter being estimated.

Properties of the sample mean

- Let μ_Y and σ_Y^2 denote the mean and variance of Y (总体的均值和方差)
- Let $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ of Y_i(样本均值)
- Then the expectation of the sample mean(样本均值的期望) is

$$E(\bar{Y}) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \mu_Y$$

  so Y is an **unbiased** estimator of μ_Y
- Based on the L.L.N., $\bar{Y} \to \mu_Y$ so $\bar{Y}$ is also **consistent**.

Properties of the sample mean

- The variance of sample mean (样本均值的方差)

$$Var(\bar{Y}) = var(\frac{1}{n}\sum_{i=1}^{N} Y_i) = \frac{1}{n^2}\sum_{i=1}^{N} Var(Y_i) = \frac{\sigma_Y^2}{n}$$

- Then the standard deviation of the sample mean is

$$\sigma_{\bar{Y}} = \frac{\sigma_Y}{\sqrt{n}}$$

Properties of the sample mean

- Follow the C.L.T, the $\bar{Y} \overset{d}{\sim} N(\mu_Y, \frac{\sigma_Y^2}{n})$
- And let Z be the standardized $\bar{Y}$,then $Z = \frac{\bar{Y}-\mu_Y}{\frac{\sigma}{\sqrt{n}}} \overset{d}{\sim} N(0,1)$
- Let μ_Y and σ_Y^2 denote the mean and variance of Y_i, then the **sample variance**:

$$S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

- Then it is easy to prove that
  1. $E(S_Y^2) = \sigma_Y^2$, thus S^2 is an **unbiased** estimator of σ_Y^2 which is also the reason why the average uses the divisor $n-1$ instead of n.
  2. $S_Y^2 \xrightarrow{P} \sigma_Y^2$, thus the sample variance is a **consistent estimator** of the **population variance**.

The Standard Error

- Recall: the standardized sample mean will be approximately follow a standard normal distribution when n is large.

$$Z = \frac{\bar{Y} - \mu_Y}{\frac{\sigma}{\sqrt{n}}} \overset{d}{\sim} N(0,1)$$

- But in general σ_Y, the standard deviation of population is **unknown**, so we have to use sample to estimate it.
- Let $\sigma_{\bar{Y}} = \frac{\sigma_Y}{\sqrt{n}}$, because S_Y^2 is an unbiased and consistent estimator of the σ_Y^2, then we can use $\frac{S_Y}{\sqrt{n}}$ as an estimator of the standard deviation of the sample mean.

The Standard Error

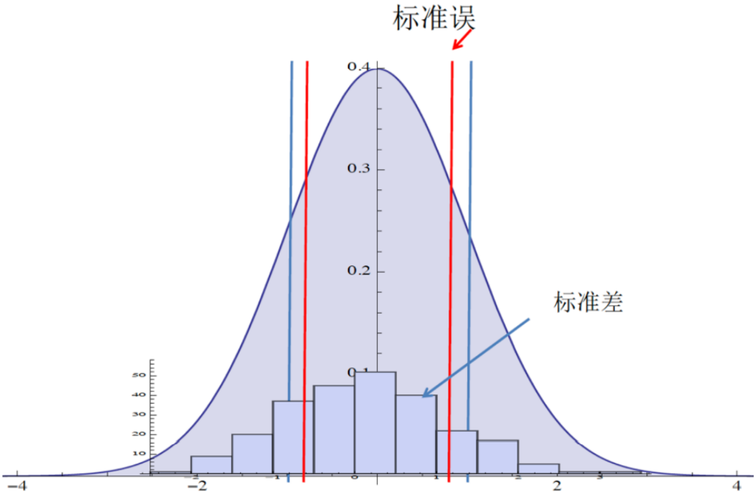
- It is called the **standard error(标准误)** of the **sample mean**

$$SE[\bar{Y}] = \hat{\sigma}_{\bar{Y}} = \frac{S_Y}{\sqrt{n}}$$

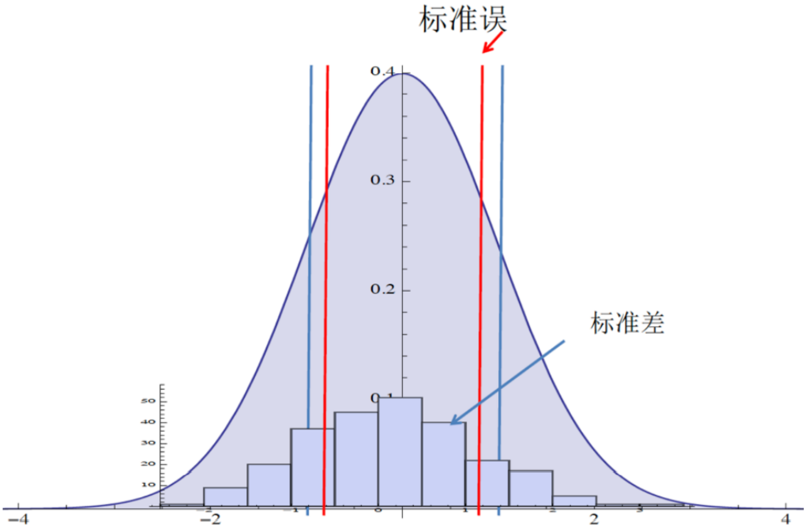
- Equivalence to the **standard deviation(标准差)** of the sample distribution which measures the **deviations** of the sample mean.

Application: Sample Size and Standard Error(Population)

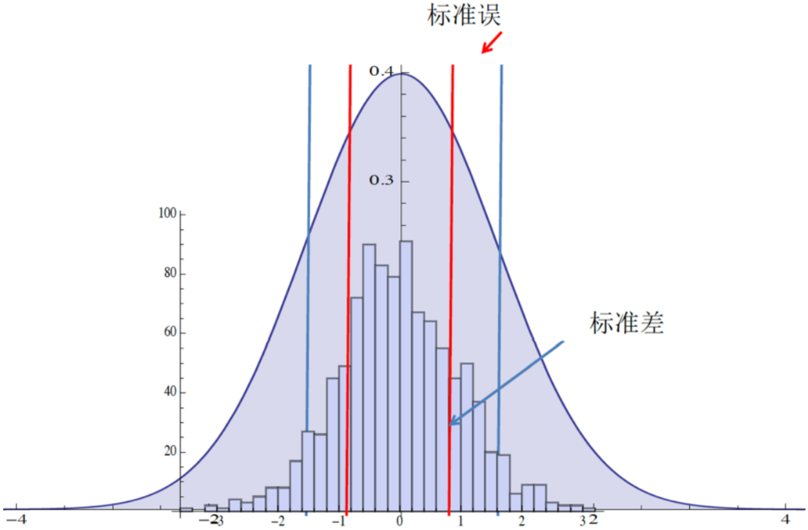
- Population $\sim N(0, SD)$



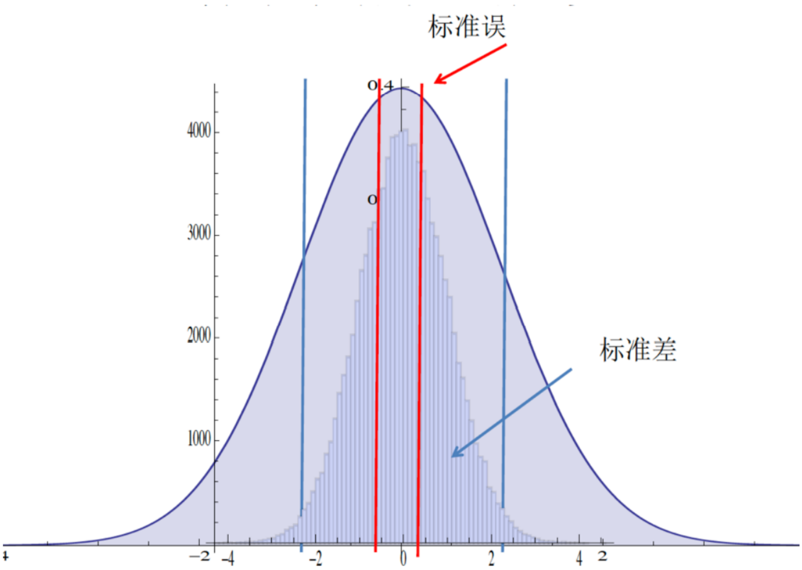
Sample Size and Standard Error(sample n=250)



Sample Size and Standard Error(n=500)



Sample Size and Standard Error(sample n=1000)



Recall: The Chi-Square Distribution

- Let $Z_i(i = 1, 2, ..., m)$ be independent random variables, each distributed as standard normal. Then a new random variable can be defined as the sum of the squares of Z_i:

$$X = \sum_{i=1}^{m} Z_i^2$$

  Then X has a **chi-squared distribution** with m degrees of freedom.

- Then, it can be prove that a variation of the sample variance will follow a **Chi-Square** distribution:

$$\frac{(n-1)S_Y^2}{\sigma_2} \sim \chi^2_{n-1}$$

The Student-t Distribution

- The Student t distribution can be obtained from a **standard normal** and a **chi-square** random variable.
- Let Z have a standard normal distribution, let X have a chi-square distribution with n degrees of freedom and assume that Z and X are independent. Then the random variable

$$T = \frac{Z}{\sqrt{\frac{X}{n}}}$$

has a **t-distribution** with n degrees of freedom, denoted as $T \sim t_n$

- Then, the $\bar{Z}$ will follow a student t distribution.

$$\bar{Z} = \frac{\bar{Y} - \mu_Y}{\frac{S_Y}{\sqrt{n}}} \overset{d}{\sim} t(n-1)$$

The Student-t Distribution

- It does not matter a lot in the large sample.
- As the degrees of freedom get large which is highly correlated with the sample size n, the **t-distribution** actually approaches the **standard normal distribution**.

Section 4

Confidence Interval and Interval Estimation

Interval Estimation

- A point estimate provides no information about how close the estimate is likely to be to the population parameter.
- We cannot know how close an estimate for a particular sample is to the population parameter because the population is never known.
- A different (complementary) approach to estimation is to produce **a range of values** that will contain the truth with some fixed probability.

What is a Confidence Interval?

Definition

- A $100(1-\alpha)\%$ confidence interval for a population parameter θ is an interval $C_n = (a, b)$, where $a = a(Y_1, Y_2, ..., Y_n)$ and $b = b(Y_1, Y_2, ..., Y_n)$ are functions of the data such that

$$P(a < \theta < b) = 1 - \alpha$$

  - In general, this **confidence level** is $1-\alpha$ (置信水平); where α is called **significance level**(显著性水平).
  - The key is how to obtain or construct the values of a and b

Interval Estimation and Condence Intervals

- Suppose the population has a normal distribution $N(\mu, \sigma^2)$ and let $Y_1, Y_2, ..., Y_n$ be a random sample from the population.
  - Then the sample mean $\bar{Y}$ has a normal distribution: $\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$
  - The standardized sample mean $\bar{Z}$ is given by:

$$\bar{Z} = \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$

- Then let $\theta = \bar{Z}$,then $P(a < \theta < b) = 1 - \alpha$ turns into

$$a < \frac{\bar{Y} - \mu}{\frac{\sigma}{\sqrt{n}}} < b$$

  then it follows that

$$P(\bar{Y} - a\frac{\sigma}{\sqrt{n}} < \mu < \bar{Y} + b\frac{\sigma}{\sqrt{n}}) = 1 - \alpha$$

- Thus the **random interval** contains the population mean with a probability $1 - \alpha$

Interval Estimation and Condence Intervals

Two cases:σ is known and unknown

- When σ is known, for example, $\sigma = 1$,thus $Y \sim N(\mu, 1)$,$\bar{Y} \sim N(\mu, \frac{\sigma^2}{n} = \frac{1}{n})$
- From this we can standardize $\bar{Y}$, and because the standardized version of $\bar{Y}$ has a standard normal distribution, and we let $\alpha = 0.05$,then we have

$$P(-1.96 < \frac{\bar{Y} - \mu}{\frac{1}{\sqrt{n}}} < 1.96) = 1 - 0.05$$

- The event in parentheses is identical to $\bar{Y} - \frac{1.96}{\sqrt{n}} < \mu < \bar{Y} + \frac{1.96}{\sqrt{n}}$
- So $P(\bar{Y} - \frac{1.96}{\sqrt{n}} < \mu < \bar{Y} + \frac{1.96}{\sqrt{n}}) = 0.95$
- The interval estimate of μ may be written as $[\bar{Y} - \frac{1.96}{\sqrt{n}}, \bar{Y} + \frac{1.96}{\sqrt{n}}]$

Interval Estimation and Condence Intervals

- When σ is unknown, we could use an estimate of σ, thus $SE[\bar{Y}] = \hat{\sigma}_{\bar{Y}} = \frac{S_Y}{\sqrt{n}}$, the sample **standard error**, replacing unknown σ thus

$$\bar{Z}_t = \frac{\bar{Y} - \mu_Y}{\frac{S_Y}{\sqrt{n}}} = \frac{\bar{Y} - \mu_Y}{SE[\bar{Y}]}$$

- We just suggested that it follows a student t distribution.

Definition

The **t-statistic** or **t-ratio**:

$$\frac{\bar{Y} - \mu}{SE(\bar{Y})} \sim t_{n-1}$$

Interval Estimation and Condence Intervals

- To construct a 95% confidence interval, let c denote the 97.5^th percentile in the t_{n-1} distribution.

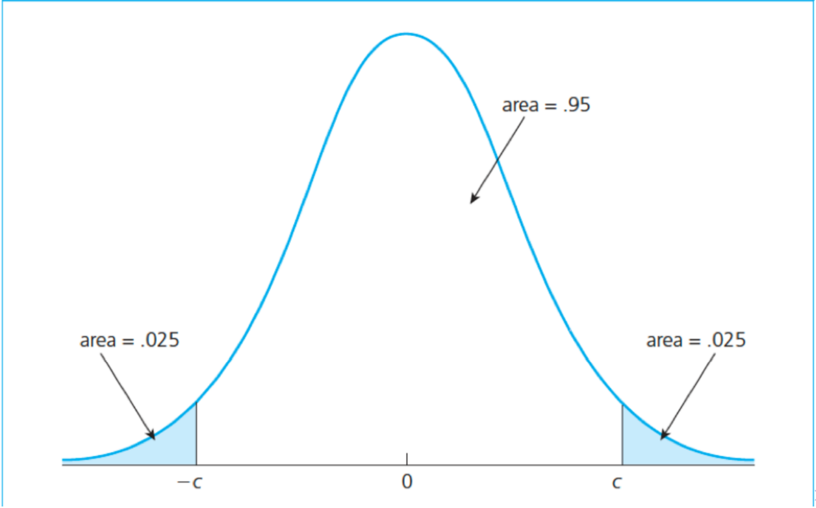
$$P(-c < t \leq c) = 0.95$$

where $c_{\frac{\alpha}{2}}$ is the critical value of the t distribution.

- The confidence interval may be written as $[Y \pm c_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}}]$

Interval Estimation and Condence Intervals

FIGURE C.4 The 97.5th percentile, c, in a t distribution.



A simple rule of thumb for a 95% confidence interval

- Because as the degrees of freedom get large which is highly correlated with the sample size n, the **t-distribution** approaches **the standard normal distribution**.
- And $\Phi(1.96) = 0.975$, so a rule of thumb for an approximate 95% confidence interval is

$$[\bar{Y} \pm 1.96 \times SE(\bar{Y})]$$

- Or

$$[\bar{Y} \pm 2 \times SE(\bar{Y})]$$

Section 5

Hypothesis Testing

Hypothesis Testing(假设检验)

Definition

A hypothesis is a **statement** about a population parameter,thus θ.Formally,we want to test whether is **significantly** different from a certain value μ_0

$$H_0 : \theta = \mu_0$$

which is called **null hypothesis**.**The alternative hypothesis** is

$$H_0 : \theta \neq \mu_0$$

- If the value μ_0 does not lie within the calculated confidence interval, then we **reject** the null hypothesis.
- If the value μ_0 lie within the calculated confidence interval,then we **fail to reject** the null hypothesis.

Hypothesis Testing(假设检验)

- In criminal law, institutions in most countries follow the rule: "**innocent until proven guilty**" (疑罪从无)
  - The prosecutor wants to prove their hypothesis that the accused person is guilty.
  - However, the burden is on the prosecutor to show guilt.
  - The jury or judge starts with the "null hypothesis" that the accused person is innocent.

Hypothesis Testing(假设检验)

- In program evaluations,instead of “presumption of innocence”, the rule is: “presumption of insignificance”.
- Policymaker’s hypothesis: the program improves learning.
- Evaluators approach experiments using the hypothesis:
  - There is zero impact of the program
  - Then we test this “null hypothesis”
- The burden of proof is on the program
  - it should show a **statistically significant** impact.

Two Type Errors

In both cases, there is a certain risk that our conclusion is wrong.

Definition

- A **Type I** error is when we reject the null hypothesis when it is in fact true.
- A **Type II** error is when we fail to reject the null hypothesis when it is false.
    - In criminal trial -**The Type I**: the judge reject the null hypothesis when the suspect is actually no guilty. “宁可错杀一千，不能放过一个.” -**The Type II**: the judge fail to reject the null hypothesis when the suspect is actually guilty. “宁可放过一千，不能错杀一个.”

The Significance level(显著性水平)

Definition

- The **significance level** or size of a test is the maximum probability for the $Type\ I\ Error$

$$P(Type\ I\ error) = P(reject\ H_0|H_0\ is\ true) = \alpha$$

- Usually, we has to carry the "burden of proof,"
- We would like to prove that the assertion of H1 is true by showing that the data rejects H_0.

Testing Procedure

- The following are the steps of the hypothesis testing:

1. Specify H_0 and H_1.
2. Choose the significance level α.
3. Define a decision rule **(critical value)**.
4. Given the data compute the test statistic and see if it falls into the critical region.

Decision Rule

- The decision rule that leads us to reject or not to reject H_0 is based on a test statistic, which is a function of the data

$$T_n = T(Y_1, Y_2, ..., Y_n)$$

- Usually, one rejects H_0 if the test statistic falls into a critical region(rejection region). A critical region is constructed by taking into account the probability of making wrong decisions,thus α. By convention, α is chosen to be a small number, for example, $\alpha = 0.01$, 0.05, or 0.10

P-value

- To provide additional information, we could ask the question: What is the largest significance level at which we could carry out the test and still fail to reject the null hypothesis?

We can consider the **p-value** of a test

- Calculate the t-statistic
- The largest significance level at which we would fail to reject H_0 is the significance level associated with using t as our critical value.

$$p - value = 1 - \Phi(t)$$

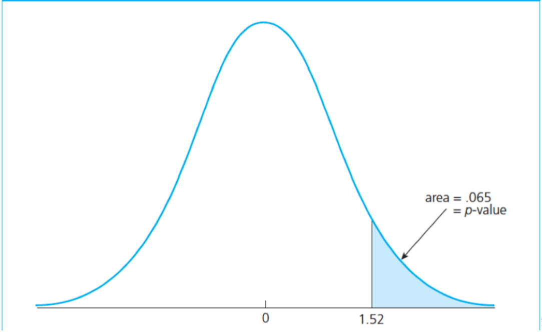
where denotes the standard normal c.d.f.(we assume that n is large enough)

P-Value

- Suppose that $t = 1.52$, then we can find the largest significance level at which we would fail to reject H_0

$$p - value = P(T > 1.52 | H_0) = 1 - \Phi(1.52) = 0.065$$

FIGURE C.7 The *p*-value when $t = 1.52$ for the one-sided alternative μ . μ_0.



Hypothesis Test of of $\bar{Y}$

- Specify H_0 and H_1

$$H_0 : E[Y] = \mu_{Y,0};\ H_1 : E[Y] \neq \mu_{Y,0}.$$

- Choose the significance level α and define a decision rule (critical region or critical value)
  - e.g. if we choose $\alpha = 0.05$, then the critical value is 1.96, then the region is $(-\infty, 1.96]$ and $[1.96, +\infty)$

Hypothesis Test of of $\bar{Y}$

FIGURE C.4 The 97.5th percentile, c, in a t distribution.