pg. 199, Problem 2. Show that if X is regular, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.

Let our disjoint closed sets be A, B. Then since X is regular, there are disjoint open U, V with $U \supset A$, $V \supset B$. Since these are disjoint, we have that $\overline{U}$ and B are also disjoint. Then we may again apply the regularity condition to get disjoint open sets $\widetilde{U}, \widetilde{V}$, with $\widetilde{U} \supset \overline{U}$ and $\widetilde{V} \supset B$. Then, as above, we have $\overline{\widetilde{V}}$ is disjoint from $\overline{U}$. But then $\widetilde{V}$ and U are open sets containing B and A, with disjoint closures.

pg. 199, Problem 5. Let $f, g : X \to Y$ be continuous, and Y Hausdorff. Show $\{x \mid f(x) = g(x)\}$ is closed in X.

It suffices to show $\{x \mid f(x) \neq g(x)\}$ is open. Suppose x_0 is in this set. Since Y is Hausdorff, there are disjoint open sets U, V containing $f(x_0)$ and $g(x_0)$, respectively. Then $f^{-1}(U) \cap f^{-1}(V)$ is an open set containing x_0. Moreover, if there were any $x \in f^{-1}(U) \cap f^{-1}(V)$ with $f(x) = g(x)$, then we would have $f(x) \in U \cap V$, contradicting disjointness. Thus $f^{-1}(U) \cap f^{-1}(V)$ is an open set in X containing x_0 and completely inside $\{x \mid f(x) \neq g(x)\}$, hence this set is open.

pg. 218, Problem 1. Give an example showing that a Hausdorff space with a countable basis need not be metrizable.

Note that all metric spaces are regular. It therefore suffices to find such a space that is not regular. Then consider $\mathbb{R}_K$, $K = \{\frac{1}{n}, n \in \mathbb{Z}^{>0}\}$, is Hausdorff with countable basis, but K and $\{0\}$ don't have disjoint neighborhoods that contain them in $\mathbb{R}_K$. Hence $\mathbb{R}_K$ is not regular and hence not metrizable.

Problem 4 (Munkres 36.1)

Every manifold is regular, and hence Metrizable (by the Urysohn Metrization Theorem, as every manifold has a countable base by definition).

Proof. Let M be a manifold of dimension m. M is Hausdorff, so one-point sets are closed in it. Therefore by Lemma 31.1, M is regular is equivalent to the condition that for every $x \in M$ and neighborhood $U \subset M$ containing x, there is a neighborhood V of x so that $\overline{V} \subset U$. We prove this.

Let $x \in M$ and $U \subset M$ an open set containing x. As M is a manifold, by definition there exists a chart (neighborhood of x and homeomorphism into $\mathbb{R}^m$) containing x; call it (U_x, ϕ_x). Then there is a homeomorphism

$$\phi\Big|_{U \cap U_x} : U \cap U_x \longrightarrow \phi(U \cap U_i) \subset \mathbb{R}^m$$

where the image is an open subset of $\mathbb{R}^m$. As $\mathbb{R}^m$ is regular, we may find a V satisfying $x \in V \subset \overline{V} \subset \phi(U \cap U_i)$. We claim that $\phi^{-1}(V)$ satisfies the desired condition.

We have $\phi^{-1}(V) \subset U \cap U_i$. I claim that $\overline{\phi^{-1}(V)} \subset U \cap U_i$ as well. Note this is not immediate, as we do not know a priori that $\overline{\phi^{-1}(V)} = \phi^{-1}(\overline{V})$. This is only true if M is Hausdorff, so this is where we must explicitly use the Hausdorffness of the M. If the Hausdorff claim is dropped, the space need not be regular, and the final problem provides a counterexample.

Thus we claim, $\overline{\phi^{-1}(V)} = \phi^{-1}(\overline{V})$. $\phi(\overline{V}) \subset \overline{\phi(V)}$ by continutiy of f by Theorem 18.1. We now show $\phi(\overline{V})$ is closed, and it contains $\phi(V)$, so is contained in the closure, as the closure is the intersection of all closed sets containing $\phi(V)$. M is first countable (because it is second countable), so by the extra credit problem we may use the limit point definition of continuity. Suppose x is a limit point of $\phi(\overline{V})$. Then there is a sequence $x_i \to x$ with $x_i \in \phi(\overline{V})$. Then there are points $v_i \in \overline{V}$ so $\phi(v_i) = x_i$. But v_i must be Cauchy, as is a sequence convering in a neighborhood homeomorphic to Euclidean space, thus it must have a limit $y \in \overline{V}$. By limit point continuity and the uniqueness of limits in Hausdorff spaces, $\phi(y) = x$. □

Problem 5 (Munkres)

Let X be a compact Hausdorff Space that is locally Euclidean. Then there exists an embedding $\Phi : X \to \mathbb{R}^K$ for some large K.

Proof. X is locally euclidean so for every x there is a pair (U_x, ϕ_x) where $\phi_x : U_x \to V_x \subset \mathbb{R}^k$ is a homeomorphism. This is a cover of X, so by compactness there is a finite subcover, call these sets $U_1, ..., U_N$.

As X is compact and Hausdorff, it is normal by Theorem 32.3. Therefore by Theorem 36.1 there is a partition of unity $\{\psi_i\}$ subordinate to the open cover $U_1, ..., U_n$. Then we define

$$\Phi(x) = (\psi_1 \cdot \phi_1(x), \psi_1(x), ..., \psi_n \cdot \phi_n(x), \psi_n(x))$$

As a map $X \to \mathbb{R}^{n(k+1)}$, where we extend the maps $\psi \cdot \phi_i$ to be zero outside of each U_i. It is a homeomorphism by the same argument as in the proof of Theorem 36.2. □

1 Page 227 Problem 3

Let X be a Compact Hausdorff space such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. Show that X is an n-manifold.

Since X is a compact h.d. space, by 227.2, we may embed it into $\mathbb{R}^N$ for some large N. $\mathbb{R}^N$ has a countable base consisting of products of open intervals with rational endpoints, thus any subspace of $\mathbb{R}^N$ has a countable base by intersecting the basis with the subspace, so X is second countable, and thus a manifold. □

2 Page 227 Problem 5

Let X be the union of $\mathbb{R}-0$ with the two point set $\{p,q\}$. Topologize X by taking the basis to be the set of all open intervals not containing zero, along with sets of the form $(-a,0)\cup\{p\}\cup(0,a)$ or $(-a,0)\cup\{q\}\cup(0,a)$ for $a>0$.

SOLUTION BY TARA AIDA **4.36.5**

(a) - Check that this is a basis for a topology

Let $x \in X = \mathbb{R}\setminus\{0\}\cup\{p,q\}$. If $x\in\mathbb{R}\setminus\{0\}$, then it may be contained in some open interval $(x-\epsilon, x+\epsilon)$, choosing ϵ sufficiently small so that it does not contain 0. If $x=p$ or $x=q$ the basis elements of the form $(-a,0)\cup\{p\}\cup(0,a)$ and $(-a,0)\cup\{q\}\cup(0,a)$ for $a>0$ contain p,q respectively. Thus, it remains to check the intersection property of a basis. There are multiple cases.

Consider: $(a_1,b_1)\cap(a_2,b_2)$. This is either empty or equal to an another open interval (still not containing 0). Let $x\in(a_1,b_1)\cap(a_2,b_2)=(a_3,b_3)$. Then, we may find an open interval containing x and contained in (a_3,b_3).

Consider: $(a_1,b_1)\cap((-a,0)\cup\{p\}\cup(0,a))$ (this analysis also holds for the case where we replace p with q). If this is nonempty, then, the intersection is either (a_1,b_1), $(-a,b_1)$, or (a_1,a) Given a point x contained in this intersection, we can find an open interval not containing 0, but containing x, and contained in (a_1,b_1), $(-a,b_1)$, or (a_1,a)...(actually the interval itself works).

Consider: $(-a,0)\cup\{p\}\cup(0,a)$ intersected with $(-b,0)\cup\{p\}\cup(0,b)$. This is simply another basis element of the same form: $(-c,0)\cup\{p\}\cup(0,c)$, so any point in the intersection is trivially contained in another basis element (the intersection itself). This analysis holds true if we replace p with q.

Consider: $(-a,0)\cup\{p\}\cup(0,a)$ intersected with $(-b,0)\cup\{q\}\cup(0,b)$. Then, (assuming WLOG that $a<b$), this is equal to $(-a,0)\cup(0,a)$. Given any x contained in this, we may find an open interval small enough s.t. it is contained in $(-a,0)$ or $(0,a)$ and contains x.

(b) Show that each of the spaces $X-\{p\}$ and $X-\{q\}$ is homeomorphic to $\mathbb{R}$..

It suffices to show this for one (p). Claim: the function $f: X-\{p\}\to\mathbb{R}$ s.t. $f(x)=0$ if $x=q$ and $f(x)=x$ otherwise. This is certainly injective. It is surjective since $X-\{p\}=\mathbb{R}\setminus\{0\}\cup\{q\}$ (the $x\in\mathbb{R}\setminus\{0\}$ hit every nonzero element, and q hits 0). To show it is continuous, it is enough to show that $f^{-1}((a,b))$ is an open set in $X-\{p\}$ (since the open intervals form a basis for $\mathbb{R}$).

If (a,b) does not contain 0 then this is obviously true, since the inverse image will be an open interval in $\mathbb{R}$ not containing 0. If it does, then the inverse image is some $(-m,0)\cup\{q\}\cup(0,n)$, $m,n>0$,which is an open set in $X-\{p\}$, as it may be written as the union of two basis elements. Thus, f is continuous.

Now, we check f^{-1} is continuous. The image of an open interval of $\mathbb{R}$ (not containing 0) is again, an open interval in $\mathbb{R}$, which is open. The image of a set of the form $(-a,0)\cup\{q\}\cup(0,a)$ is simply $(-a,a)$ which is also open in $\mathbb{R}$.

Thus, f is a bijective continuous function with continuous inverse, so $X-\{p\}$ homeomorphic to $\mathbb{R}$. The same argument holds if we switch every p for q and vice versa, so $X-\{q\}$ is also homeomorphic to $\mathbb{R}$.

(c) - Show that X satisfies the T_1 axiom, but is not Hausdorff. X is not Hausdorff because any two neighborhoods of $\{p\}$ must be of the form $(-a,0)\cup\{p\}\cup(0,a)$, and so they must intersect each other at some open interval $(0,b)$.

Now for T_1: Let $x,y\in X$ and $x\neq y$. X is T_1 if I can find a neighborhood of x that doesn't contain y and a neighborhood of y that doesn't contain x. If $x,y\in\mathbb{R}\setminus\{0\}$ then we can find disjoint open intervals, one containing x and the other y ($\mathbb{R}$ is hausdorff).

Suppose $x=p$ and $y\in\mathbb{R}\setminus\{0\}$. Then, we may find a b small enough s.t. $(-b,0)\cup(0,b)$ does not contain y, but then clearly $(-b,0)\cup\{p\}\cup(0,b)$ is an open neighborhood of p not containing y. Alternatively, any open interval (not containing 0) containing y will not contain p. The same analysis holds if we replace p with q.

Finally, what if $x=p, y=q$. Then for $a>0$, $(-a,0)\cup\{p\}\cup(0,a)$ is an open neighborhood of p not containing q and $(-a,0)\cup\{q\}\cup(0,a)$ is an open neighborhood of q not containing p.

Thus, X satisfies the T_1 axiom!

(d) - Show that X satisfies all conditions for a 1-manifold except for the Hausdorff condition. It clearly has countable basis: simply restrict the open intervals of all the basis elements to have rational endpoints.

Given any $x\in X$ does it have a neighborhood homeomorphic to an open subset of $\mathbb{R}$? If $x\in\mathbb{R}\setminus\{0\}$ then it certainly has a neighborhood contained in $X-\{p\}$, and so by (b) it has a neighborhood homeomorphic to an open subset of $\mathbb{R}$. If $x=p$ then it has an open neighborhood contained in $(X-\{q\})$, by the fact that X is T_1, and if $x=q$ then it has an open neighborhood contained in $X-\{p\}$, again by T_1-ness, thus, in both cases, $x\in X$ contains a neighborhood homeomorphic to a open subset of $\mathbb{R}$ by the homeomorphisms defined in (b). Thus, we conclude that every $x\in X$ has a neighborhood homeomoprhic with an open subset of $\mathbb{R}$, so we are done.