

Efficient Softmax Reformulation for Homomorphic Encryption via Moment Generating Function

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Abstract

Homomorphic encryption (HE) is a prominent framework for privacy-preserving machine learning, enabling inference directly on encrypted data. However, evaluating softmax, a core component of transformer architectures, remains particularly challenging in HE due to its multivariate structure, the large dynamic range induced by exponential functions, and the need for accurate division during normalization. In this paper, we propose MGF-softmax, a novel softmax reformulation based on the moment generating function (MGF) that replaces the softmax denominator with its moment-based counterpart. This reformulation substantially reduces multiplicative depth while preserving key properties of softmax and asymptotically converging to the exact softmax as the number of input tokens increases. Extensive experiments on Vision Transformers and large language models show that MGF-softmax provides an efficient and accurate approximation of softmax in encrypted inference. In particular, it achieves inference accuracy close to that of high-depth exact methods, while requiring substantially lower computational cost through reduced multiplicative depth.

1. Introduction

The rapid advancement of artificial intelligence (AI) has accelerated the adoption of machine learning as a service (MLaaS), where users increasingly rely on remotely hosted models for inference across a wide range of applications.

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However, this paradigm inherently raises privacy concerns, as it typically requires exposing sensitive user data to remote infrastructure. To address these risks, privacy-preserving machine learning (PPML) has emerged as a critical research area aimed at enabling secure inference without compromising the confidentiality of user data (Al-Rubaie & Chang, 2019; Riazi et al., 2019).

One promising direction within PPML is to employ homomorphic encryption (HE) (Rivest et al., 1978; Gentry, 2009), which enables computation directly over encrypted data. A broad line of prior work has leveraged HE to realize HE-based PPML inference pipelines that operate entirely in the encrypted domain. In practice, HE-based PPML protocols can be realized in either interactive or strictly non-interactive settings. Interactive protocols—often instantiated via secure multi-party computation (MPC) (Evans et al., 2018)—require client participation during inference and therefore incur substantial communication overhead. In contrast, strictly non-interactive HE allows the server to execute inference autonomously once ciphertexts are received, eliminating communication and client-side burden at the cost of increased server-side computation. In this work, we focus on the strictly non-interactive setting, where avoiding interaction during inference is essential for fully outsourced inference.

A primary challenge of HE-based PPML lies in the implementation of non-polynomial operations. Most HE schemes natively support only addition and multiplication, rendering the direct evaluation of non-polynomial functions infeasible. As a result, non-polynomial activation functions are typically approximated using polynomials, which can be evaluated homomorphically. However, achieving high approximation accuracy generally requires high-degree polynomials, leading to excessive computational latency and noise growth. This leads to increased multiplicative depth, which consumes the limited level budget and necessitates costly bootstrapping operations. To reduce this overhead, prior work has often resorted to simplified surrogate functions or low-degree polynomial approximations, at the cost of degraded model accuracy (Lou et al., 2020; Lou & Jiang, 2021; Chen et al., 2022; Rho et al., 2025).

This limitation is particularly pronounced in transformer-based architectures (Vaswani et al., 2017; Brown et al., 2020; Dosovitskiy et al., 2021; Dubey et al., 2024), where the softmax function plays a central role in the self-attention mechanism. Unlike univariate activation functions, softmax has a multivariate structure where each output depends on the entire input vector through a shared normalization term. Evaluating softmax in a strictly non-interactive HE setting therefore constitutes a major computational bottleneck, as its standard formulation involves the exponential function and division, which are inherently expensive to approximate homomorphically (Hong et al., 2022; Cho et al., 2024). Moreover, numerical stabilization techniques commonly employed in plaintext inference, such as subtracting the maximum input value, require comparison operations that are fundamentally incompatible with HE.

In response to this challenge, existing approaches to homomorphic softmax evaluation largely fall into two categories. Methods commonly referred to as *softmax approximation* aim to evaluate the softmax function with high precision under HE, but are typically computationally expensive due to high-degree polynomial approximations for the exponential function and division (Hong et al., 2022; Cho et al., 2024; Castro et al., 2025; Lim et al., 2025). To reduce this computational cost, an alternative strategy denoted as *softmax replacement* replaces the original softmax function with simpler surrogate functions (Zimerman et al., 2024b;a; Park et al., 2025). While such replacements can be effective for relatively simple natural language processing (NLP) tasks, our experiments show that they suffer from substantial accuracy degradation in more demanding settings, including large-scale image classification and complicated NLP tasks. These limitations highlight the need for a softmax evaluation method that significantly reduces computational cost while preserving high accuracy on complex tasks.

In this paper, we propose **MGF-softmax**, a novel method that *reformulates* the softmax function by leveraging the moment generating function (MGF). MGF-softmax addresses the aforementioned challenges by eliminating the need for homomorphic division as well as explicit maximum subtraction commonly used for numerical stabilization. By removing these computationally expensive operations, particularly the comparison-heavy max operation, our method substantially reduces the multiplicative depth compared to existing softmax approximation approaches. This depth reduction is critical as it minimizes the need for bootstrapping, directly translating to significant savings in computational cost. We further derive key theoretical properties of MGF-softmax and provide an asymptotic bound on its approximation error relative to the exact softmax function. Experimental results demonstrate that MGF-softmax achieves a significant reduction in the level consumption, outperforming the softmax approximation baseline (Cho et al., 2024). Moreover, MGF-

softmax consistently maintains high accuracy across diverse tasks, achieving less than a 1% accuracy drop for Vision Transformers (ViT/DeiT) (Dosovitskiy et al., 2021; Touvron et al., 2021) on ImageNet-1k (Deng et al., 2009) as well as large language model (LLaMA-3.2-1B (Meta, 2024)) on ClinC150 (Larson et al., 2019), Banking77 (Casanueva et al., 2020) and SST-2 (Wang et al., 2018). These benchmarks are particularly challenging in homomorphic inference, where existing softmax replacement methods exhibit significant degradation.

Our contributions are summarized as follows:

- **MGF-softmax Formulation:** We propose MGF-softmax, a novel reformulation of the softmax function that eliminates the need for homomorphic division and maximum subtraction, thereby substantially reducing computational cost.
- **Theoretical Analysis:** We provide a theoretical analysis establishing key properties of MGF-softmax and an asymptotic bound on its approximation error.
- **Experimental Validation:** We demonstrate through extensive experiments on Vision Transformers and a large language model that MGF-softmax achieves inference accuracy close to exact softmax while significantly reducing inference cost under homomorphic encryption.

2. Background and Related Works

2.1. Fully Homomorphic Encryption

Fully homomorphic encryption (FHE) is a cryptographic primitive that enables computations to be performed directly on encrypted data without decryption. Among existing schemes, we utilize the CKKS scheme (Cheon et al., 2017), which supports approximate arithmetic operations over encrypted real numbers.

Specifically, CKKS employs a packing technique that encrypts a vector of values into a single ciphertext. We denote the number of these values, i.e., the number of slots in a ciphertext, as s . Supported homomorphic operations include element-wise addition (Add), cyclic rotation (Rot), and multiplication. Regarding multiplication, we distinguish between *Ciphertext–Plaintext Multiplication* (PMult), which involves a plaintext constant, and the computationally expensive *Ciphertext–Ciphertext Multiplication* (CMult).

As a leveled FHE scheme, CKKS assigns a finite budget of multiplicative levels L to each ciphertext. Each CMult (and non-integer PMult) consumes one level, and the cumulative consumption determines the circuit *depth*. Once the level budget is exhausted, *bootstrapping* (Boot) is required to refresh the ciphertext, enabling further computation at the

cost of significant overhead. Consequently, we evaluate HE algorithmic efficiency based on the multiplicative depth and the counts of dominant operations, specifically CMult and Rot.

2.2. Homomorphic Softmax Evaluation

We first recall the standard softmax function in the plaintext setting. For an input vector $\mathbf{x} = (x_1, \dots, x_n)$, the i -th component of the softmax output is defined as:

$$\text{softmax}(\mathbf{x})_i = \frac{\exp(x_i)}{\sum_{j=1}^n \exp(x_j)}, \quad (1)$$

for $i \in \{1, \dots, n\}$. In standard floating-point implementations, the softmax function is typically computed in a numerically stable form to prevent overflow from the exponential function:

$$\text{softmax}(\mathbf{x})_i = \frac{\exp(x_i - x_{\max})}{\sum_{j=1}^n \exp(x_j - x_{\max})}, \quad (2)$$

where $x_{\max} = \max_{i \in \{1, \dots, n\}} x_i$. Evaluating softmax directly in the homomorphic setting presents two major challenges:

1. **Overflow and Maximum Subtraction:** The rapid growth of the exponential function renders softmax vulnerable to numerical overflow when processing large input values. In the homomorphic setting, larger ranges necessitate polynomial approximations over wider intervals, further increasing computational cost and exacerbating approximation error (Cheon et al., 2022; Cho et al., 2024). To ensure numerical stability, the input vector to softmax is typically shifted using the maximum operation, as shown in Eq. (2). However, the maximum operation is not natively supported by HE schemes and must be approximated, which incurs substantial multiplicative depth.
2. **Division Operation:** Softmax inherently involves a division operation for normalization. Since HE schemes do not natively support division, prior work (Hong et al., 2022; Lim et al., 2025; Castro et al., 2025) typically addresses this challenge by employing polynomial approximations of the reciprocal function $1/x$, such as Chebyshev polynomial approximation (Cheney, 1966) or Goldschmidt's iterative algorithm (Goldschmidt, 1964), leading to increased multiplicative depth.

2.3. Prior Work

In the strictly non-interactive setting, prior work can be broadly categorized into two classes: softmax approximation and softmax replacement. While the former directly

approximates the softmax function using polynomials, the latter replaces softmax with simpler surrogate functions that are also realized through polynomial approximations.

Softmax Approximation. Prior work on *softmax approximation* aims to approximate the original softmax function using polynomial functions. Following an early work on homomorphic softmax approximation (Hong et al., 2022), subsequent works focus on mitigating exponential overflow.

The first strategy stabilizes computation by subtracting the maximum value x_{\max} . HETAL (Lee et al., 2023) computes the exact maximum; however, this incurs high computational cost due to comparison operations, which should be approximated by polynomials. Subsequent frameworks therefore adopt heuristic or statistical estimates to approximate x_{\max} more efficiently. EncryptedLLM (Castro et al., 2025) uses a fixed empirical maximum derived from training data. Recognizing the instability of fixed values, Tricycle (Lim et al., 2025) and ARION (Yang et al., 2025) estimate an upper bound on the *expected* maximum based on statistical assumptions.

The second strategy addresses overflow by compressing the input interval. Cho et al. (2024) introduce the *normalize-and-square* algorithm, which evaluates softmax on scaled-down inputs to ensure numerical stability and subsequently recovers the original outputs via the iterative identity:

$$\text{softmax}(\mathbf{x}/2^{k-1})_i = \frac{\text{softmax}(\mathbf{x}/2^k)_i^2}{\sum_{j=1}^n \text{softmax}(\mathbf{x}/2^k)_j^2}. \quad (3)$$

THOR (Moon et al., 2025) adopts this framework, differing primarily in the choice of approximation algorithms for the exponential function and division.

Crucially, regardless of the overflow mitigation strategy, all aforementioned methods rely on polynomial approximations for both the exponential function and the computationally expensive homomorphic division, resulting in increased multiplicative depth.

Softmax Replacement. Distinct from softmax approximation, *softmax replacement* methods substitute the computationally expensive softmax operation with simpler surrogate functions. Several works replace softmax with element-wise activation functions, including standard activations (e.g., ReLU and GELU), their polynomial variants (e.g., squared ReLU), or activations augmented with auxiliary neural networks (Zimerman et al., 2024b; Chen et al., 2022). Rho et al. (2025) pursue an alternative direction by adopting Gaussian kernels in place of softmax. Another line of work replaces only the exponential term $\exp(x_i)$ with power functions or their shifted variants (Li et al., 2023; Luo et al., 2024; Zimerman et al., 2024a), while retaining the division operation for normalization. Powerformer (Park et al., 2025)

Table 1. Comparison with prior softmax approximation methods. MGF-softmax eliminates both max and division operations while remaining adaptive to input statistics. (R: Required, NR: Not Required)

Operations	(Lee et al., 2023)	(Castro et al., 2025)	(Lim et al., 2025) (Yang et al., 2025)	(Cho et al., 2024) (Moon et al., 2025)	MGF-softmax
Max	Comparison-Based Approx.	Fixed (Empirical)	Expected Max Upper Bound	Fixed (Domain Scaling)	NR
Division	R	R	R	R	NR
Data-Adaptivity	Yes	No	Yes	No	Yes

further eliminates the division operation by replacing the denominator with a fixed empirical value.

However, these softmax replacement methods face inherent limitations: they either suffer from accuracy degradation (Zimerman et al., 2024b; Rho et al., 2025) or provide limited speedup due to the retained division operation (Li et al., 2023; Luo et al., 2024; Zimerman et al., 2024a). Furthermore, even recent replacement methods (Park et al., 2025; Luo et al., 2024; Rho et al., 2025) are primarily evaluated on small-scale models such as BERT (Devlin et al., 2019) and benchmarks with small label spaces (e.g., binary or ternary classification). The effectiveness of such methods on larger models and more challenging benchmarks remains underexplored.

3. Softmax Reformulation via Moment Generating Function

In this section, we introduce a novel reformulation of the softmax function based on the MGF that enables efficient evaluation under HE.

3.1. MGF-softmax

MGF-softmax. We introduce MGF-softmax to address the challenges of softmax in HE described in Section 2.2. Given an input vector $\mathbf{x} = (x_1, \dots, x_n)$, we interpret the softmax denominator as a scaled sample mean of exponential terms. Let X denote a random variable representing the distribution of input components, and let $M_X(t) \triangleq \mathbb{E}[\exp(tX)]$ denote its MGF.

Under this interpretation, the softmax denominator $\sum_{i=1}^n \exp(x_i)$ is replaced by $nM_X(1)$ by substituting the sample mean with the ensemble mean. Accordingly, the i -th component of MGF-softmax is given by

$$\text{softmax}_{\text{MGF}}(\mathbf{x})_i = \frac{\exp(x_i)}{nM_X(1)}. \quad (4)$$

To make the structure explicit, we introduce the cumulant generating function (CGF) of X (Casella & Berger, 2024), defined as

$$K_X(t) \triangleq \ln M_X(t). \quad (5)$$

Using the CGF, Eq. (4) can be expressed in an explicit

single-exponential form as

$$\text{softmax}_{\text{MGF}}(\mathbf{x})_i = \exp(x_i - K_X(1) - \ln n). \quad (6)$$

Moreover, $K_X(1)$ can be expanded as

$$K_X(1) = \sum_{j=1}^{\infty} \frac{\kappa_j}{j!}, \quad (7)$$

where κ_j denotes the j -th *cumulant* of X .

For the Gaussian case $X \sim \mathcal{N}(\mu, \sigma^2)$, only the first two cumulants are nonzero, with $\kappa_1 = \mu$ and $\kappa_2 = \sigma^2$, resulting in $K_X(1) = \mu + \frac{\sigma^2}{2}$. Hence, MGF-softmax reduces to

$$\text{softmax}_{\text{MGF}}(\mathbf{x})_i = \exp\left(x_i - \mu - \frac{\sigma^2}{2} - \ln n\right). \quad (8)$$

MGF-softmax can be adaptively instantiated under different input distributions, allowing tractable expressions based on the empirical statistics of the input vector (see Appendix A).

Advantages of MGF-softmax. By reformulating softmax via MGF, MGF-softmax achieves both computational efficiency and improved approximation stability via a data-adaptive shift. This reformulation leads to the following key advantages:

1. MGF-softmax eliminates the need for a max operation. All input components are shifted by $K_X(1) + \ln n$ prior to exponential evaluation, which preserves numerical stability.
2. MGF-softmax removes the division operation and correspondingly reduces the multiplicative depth, as the normalization term is absorbed into the exponent of the single exponential.
3. MGF-softmax is adaptive to the input statistics through cumulants of the underlying distribution. For example, in the Gaussian case, the first two cumulants $\kappa_1 = \mu$ and $\kappa_2 = \sigma^2$ determine the shift, and these statistics can be efficiently estimated using rotation and addition operations. This data-adaptive shift leads to more stable approximation behavior across varying input distributions.

As a result, MGF-softmax directly addresses the challenges of softmax evaluation in HE outlined in Section 2.2. Table 1 compares MGF-softmax with prior softmax approximation methods and summarizes the advantages of the proposed approach. In particular, it shows that MGF-softmax uniquely eliminates both max and division operations while remaining adaptive to input statistics.

3.2. Properties of MGF-softmax

In this subsection, we present key properties of MGF-softmax that reveal its advantages under HE.

Shift Invariance. The standard softmax is *shift-invariant*, i.e., $\text{softmax}(\mathbf{x}) = \text{softmax}(\mathbf{x} - c)$ for any constant c . This property is commonly used to improve numerical stability (see Eq. (2)). MGF-softmax also preserves this shift-invariance property.

Proposition 3.1. MGF-softmax is shift-invariant. For any constant $c \in \mathbb{R}$,

$$\text{softmax}_{\text{MGF}}(\mathbf{x}) = \text{softmax}_{\text{MGF}}(\mathbf{x} - c). \quad (9)$$

Proof. For any constant c ,

$$\begin{aligned} \text{softmax}_{\text{MGF}}(\mathbf{x} - c)_i &= \frac{\exp(x_i - c)}{n\mathbb{E}[\exp(X - c)]} = \frac{\exp(x_i)}{n\mathbb{E}[\exp(X)]} \\ &= \text{softmax}_{\text{MGF}}(\mathbf{x})_i, \end{aligned}$$

which holds for all $i \in \{1, \dots, n\}$. \square

In contrast, softmax replacement methods (Luo et al., 2024; Zimerman et al., 2024a; Park et al., 2025) are generally not shift-invariant.

Domain Scaling. Efficient evaluation of the softmax function under HE is challenging when the input values span a wide range, as polynomial approximations of the exponential function rapidly deteriorate outside a limited interval (Cheon et al., 2022). To address this issue, it is crucial to control the domain of the exponential function during evaluation.

A representative approach is the normalize-and-square strategy proposed by (Cho et al., 2024), which scales the input domain to a small interval and subsequently restores the original scale via repeated squaring (see Eq. (3)). While effective, applying this strategy to the standard softmax requires repeated division operations, which are particularly costly under HE.

In contrast, MGF-softmax naturally enables a much simpler *domain scaling* mechanism.

Proposition 3.2. The domain scaling of MGF-softmax by a factor of 2^k ($k \in \mathbb{N}$) is expressed as:

$$\text{softmax}_{\text{MGF}}(\mathbf{x})_i = \exp\left(\frac{x_i - K_X(1) - \ln n}{2^k}\right)^{2^k}. \quad (10)$$

The exponent is scaled by a factor of $1/2^k$ to ensure that the exponential function is evaluated over a significantly smaller domain. The original value is then recovered by squaring the result k times. Crucially, this domain scaling reduces the effective approximation interval of the exponential function by a factor of 2^k , while incurring only an additional multiplicative depth of $k + 1$ (CMult: k , PMult: 1). Unlike the standard softmax, this procedure does not involve explicit normalization or division, enabling a substantially simpler and more efficient realization under HE.

3.3. Error Analysis

To evaluate the approximation accuracy of MGF-softmax, we analyze the *relative error* as in (Cho et al., 2024).

Proposition 3.3. The relative error between MGF-softmax and softmax is given by

$$\begin{aligned} \eta &= \frac{\|\text{softmax}(\mathbf{x}) - \text{softmax}_{\text{MGF}}(\mathbf{x})\|_{\infty}}{\|\text{softmax}(\mathbf{x})\|_{\infty}} \\ &= \left|1 - \frac{\frac{1}{n} \sum_{i=1}^n \exp(x_i)}{\mathbb{E}[\exp(X)]}\right|. \end{aligned} \quad (11)$$

Proof. The proof is given in Appendix B. \square

Importantly, the relative error η depends only on the ratio between the sample mean and the ensemble mean of $\exp(X)$, and is independent of the individual component index i .

Proposition 3.4. Let $Y = \exp(X)$ be a random variable with mean μ_Y and standard deviation σ_Y . The probability $P_{\eta} \triangleq P(\eta \geq \delta)$ is approximated by

$$P_{\eta} \approx 2 \left(1 - \Phi\left(\frac{\delta \cdot \mu_Y \sqrt{n}}{\sigma_Y}\right)\right), \quad (12)$$

where Φ denotes the cumulative distribution function (CDF) of the standard normal distribution.

Proof. The proof is given in Appendix C. \square

As n increases, $P_{\eta} \rightarrow 0$, implying that the relative error asymptotically vanishes.

4. Experimental Results

We implemented our HE algorithms using `desilofhe`, a Python-based HE library developed by (DESILO Inc.,

Table 2. Comparison with prior softmax approximation methods. MGF-softmax achieves significantly lower multiplicative depth compared to existing baselines even for larger-scale models (e.g., LLaMA-3.2-1B) and more complex benchmarks.

Method	Model	Model Size	Dataset	# Classes	Mult. Depth
EncryptedLLM (Castro et al., 2025)	GPT-2 Small	124M	SST-2, WiC, PIQA, MNLI, ANLI, Social IQA, HellaSwag, ARC (Easy)	2, 3, 4	22
	GPT-2 Medium	355M			26
	GPT-2 Large	774M			30
THOR (Moon et al., 2025) Tricycle (Lim et al., 2025) ARION (Yang et al., 2025)	BERT-Base	110M	SST-2, RTE, MRPC	2	30
	BERT-Tiny	4.4M	SST-2	2	17
	BERT-Tiny/Base	4.4M / 110M	SST-2, RTE, QNLI	2	14
Proposed	LLaMA-3.2-1B	1B	SST-2, Banking77, ClinC150	2, 77, 150	8–9
	ViT/DeiT (Tiny/Base)	5M / 86M	ImageNet-1k	1,000	7–10

2025), which supports the CKKS scheme. We set the slot size as $s = 2^{15}$, and the available multiplicative level after bootstrapping as $L = 10$. All experiments were conducted on dual AMD EPYC 7763 64-Core Processors (totaling 128 physical cores) and 1 TB of RAM, operating on Ubuntu 20.04 LTS.

In this evaluation, we instantiate MGF-softmax with a Gaussian distribution as a practical and effective choice. This design is motivated by the observation that high-dimensional representations often exhibit approximately Gaussian behavior due to aggregation effects. Importantly, the formulation of MGF-softmax does not rely on this distributional specification; nevertheless, this instantiation yields an accurate and efficient approximation in practice.

4.1. Softmax Evaluation

4.1.1. COMPLEXITY ANALYSIS

Table 2 presents a comparative analysis of the multiplicative depth required by MGF-softmax and prior softmax approximation methods (Castro et al., 2025; Lim et al., 2025; Yang et al., 2025). We note that prior approaches are optimized for fixed experimental settings tailored to specific model architectures and datasets. As a result, a direct comparison under identical input intervals is inherently challenging due to differences in the evaluated models and datasets. Despite these discrepancies, the comparison in Table 2 highlights the *scalability* and *efficiency* of MGF-softmax. In particular, MGF-softmax achieves the lowest multiplicative depth, ranging from 7 to 10, even when applied to significantly larger-scale models (e.g., LLaMA-3.2-1B) and high-complexity benchmarks such as ImageNet-1k with 1,000 classes. In contrast, prior methods require substantially higher multiplicative depths while being evaluated on smaller models and tasks with fewer classes.

Next, distinct from the softmax approximation methods discussed above, we evaluate the algorithmic complexity of MGF-softmax against (Cho et al., 2024). We adopt it as our primary softmax approximation baseline because, like our proposed method, it employs a *domain scaling* strategy that

Table 3. Algorithmic complexity comparison between MGF-softmax and (Cho et al., 2024), both based on domain scaling by $1/2^k$. Here, n denotes the input dimension of softmax, and L represents the number of available multiplicative levels after bootstrapping.

Method	Depth	# CMult	# Rot	# Boot
Cho et al.	$\geq 8k + 9$	$\geq 12k + 58$	$2k \log_2 n$	$\geq \lfloor (8k + 9)/L \rfloor$
Proposed	$k + 6$	$k + 10$	$2 \log_2 n$	$\lfloor (k + 6)/L \rfloor$

Table 4. Average execution time for basic homomorphic operations using 8 CPU threads. Bootstrapping (Boot) is the most computationally expensive operation, followed by ciphertext multiplication (CMult) and rotation (Rot).

Operation	Add	PMult	CMult	Rot	Boot
Time	1.4 ms	2.1 ms	89 ms	59 ms	14 s

enables a generalizable and scalable framework capable of handling varying input intervals, rather than relying on fixed bounds or dataset-specific tuning.

Table 3 presents a detailed comparison of algorithmic complexity. To handle large input intervals $[-M, 0]$, Cho et al. (2024) mitigates numerical instability by scaling inputs by a factor of $1/2^k$. However, recovering the original scale requires the *normalize-and-square* algorithm (see Eq. (3)), which involves an iterative process that is computationally expensive. In practice, this typically requires $k \approx \lceil \log_2 M - \log_2(\ln n) \rceil$ iterations, where n denotes the input dimension of softmax. In contrast, MGF-softmax streamlines this process by directly exploiting Eq. (10). This enables efficient handling of large input intervals while significantly reducing computational overhead. These advantages are reflected in the reduced multiplicative depth reported in Table 3. Implementation details of MGF-softmax are provided in Appendix D.

4.1.2. RUNTIME ANALYSIS

We measure the execution time of softmax evaluation under HE. For this experiment, we set the input size to a 256×256 matrix with an input interval of $[-128, 0]$, and all timings

Table 5. Breakdown of average CPU runtime (in seconds) for a single softmax evaluation on a 256×256 input matrix. The reduction in total latency is primarily attributed to the elimination of bootstrapping (Boot) and the reduced number of ciphertext multiplications (CMult).

Method	Add	PMult	CMult	Rot	Boot	Total
Cho et al.	2.37	1.15	11.49	1.46	89.24	105.74
Proposed	0.31	0.11	1.83	0.78	0.00	3.06

are measured using 8 CPU threads. Table 4 reports the unit execution costs of basic homomorphic operations. As shown in Table 5, the observed latency reduction is primarily attributed to the reduced multiplicative depth, which reduces or eliminates the need for expensive bootstrapping, as well as the reduced number of polynomial approximation steps, which significantly lowers the number of CMult operations.

4.2. Model Accuracy Evaluation

4.2.1. EXPERIMENTAL SETUP

Evaluation Scope & Objectives. To validate the scalability of MGF-softmax to large-scale architectures and its ability to handle complex tasks with high-dimensional output spaces, we consider the following evaluation settings: (1) LLaMA-3.2-1B, sourced from the Hugging Face transformers library (Wolf et al., 2020), evaluated on Clinc150 (Larson et al., 2019), Banking77 (Casanueva et al., 2020), and SST-2 (Wang et al., 2018), which involve 150, 77, and 2 classes, respectively; and (2) Vision Transformers (ViT/DeiT-Base/Tiny) implemented using the `timm` library (Wightman, 2019), evaluated on ImageNet-1k (Dosovitskiy et al., 2021) (1,000 classes).

Baselines. We select representative baselines from both the softmax approximation and replacement categories. For *softmax approximation*, we adopt (Cho et al., 2024) as the primary baseline. Unlike prior approaches that are optimized for specific experimental setting, this method provides a generalized framework capable of handling arbitrary input intervals, making it suitable for evaluation across diverse benchmarks. For *softmax replacement*, we employ Batch Power-Max (BPMax) from Powerformer (Park et al., 2025). We select this method as a baseline because it achieves low circuit depth by eliminating the division operation and was reported to have negligible inference accuracy degradation on BERT models.

Training Methodology. To recover the inference accuracy potentially affected by homomorphic softmax evaluation, prior methods typically require an additional retraining phase. Specifically, the replacement baseline (BPMax) employs a knowledge distillation (KD) framework, where the original pre-trained model serves as the *teacher* and the

model with the replaced function acts as the *student*. We adopt the same KD framework for MGF-softmax to facilitate effective accuracy recovery. Details of the training hyperparameters and configurations are provided in Appendix F.

Notably, MGF-softmax exhibits rapid convergence during retraining, leading to a substantially reduced training cost compared to prior replacement methods. For LLaMA-3.2-1B, MGF-softmax requires only 5 epochs to reach its best performance, whereas BPMax requires 20 epochs. Furthermore, for Vision Transformers, we employ attention-only tuning to efficiently recover accuracy while minimizing the computational overhead of training.

MGF-softmax Configuration. We evaluate two approximation methods for exponential function: Chebyshev polynomial approximation and the limit-based approximation $\exp(x) \approx (1 + x/2^k)^{2^k}$ (Hong et al., 2022). In the reported results, we select the method that yields better performance for each model and dataset.

We further introduce a *low-degree variant* of MGF-softmax, designed for scenarios that prioritize maximal efficiency with minimal multiplicative depth. Unlike the standard setting, this variant modifies the approximation stage by replacing the exact exponential function with a low-degree polynomial during training, rather than approximating a pre-trained model at inference time. Details of the exponential approximation schemes are provided in Appendix E.

4.2.2. INFERENCE ACCURACY

We analyze the inference accuracy results presented in Figures 1 and 2. In these figures, *exact softmax* denotes the accuracy of the original pre-trained model using the standard softmax function in the plaintext domain. The objective of HE-based softmax methods is to preserve the plaintext accuracy as closely as possible, while minimizing the required multiplicative depth.

Inference Accuracy on LLaMA-3.2-1B. Figure 1 presents the accuracy results on NLP benchmarks. The softmax replacement baseline (BPMax) achieves a very low multiplicative depth of 2, but exhibits noticeable accuracy degradation as model scale increases. In particular, BPMax shows significant drops in accuracy on NLP tasks with a large number of classes, such as Clinc150 and Banking77, and also incurs a non-negligible accuracy loss on the binary classification task SST-2 (from 93.11 % to 87.15 %). These results indicate that, while simplified replacement functions may preserve accuracy for smaller models such as BERT (Park et al., 2025), they do not consistently maintain inference accuracy when applied to larger-scale models like LLaMA-3.2-1B.

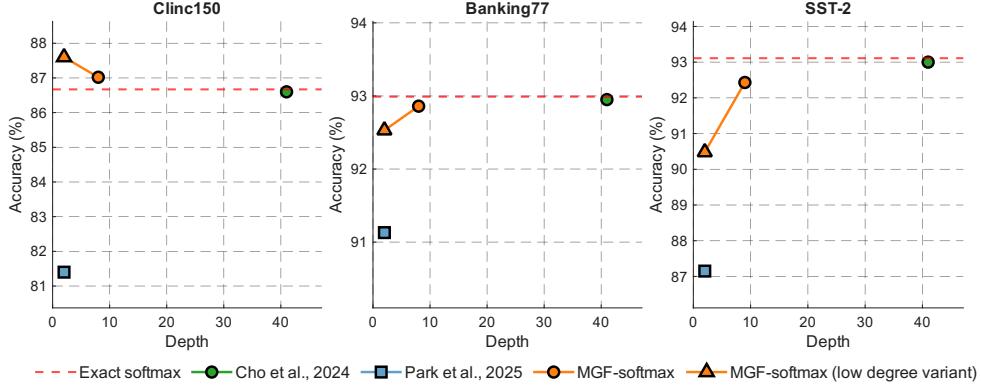


Figure 1. Top-1 accuracy of LLaMA-3.2-1B across NLP benchmarks (Clinc150, Banking77, and SST-2).

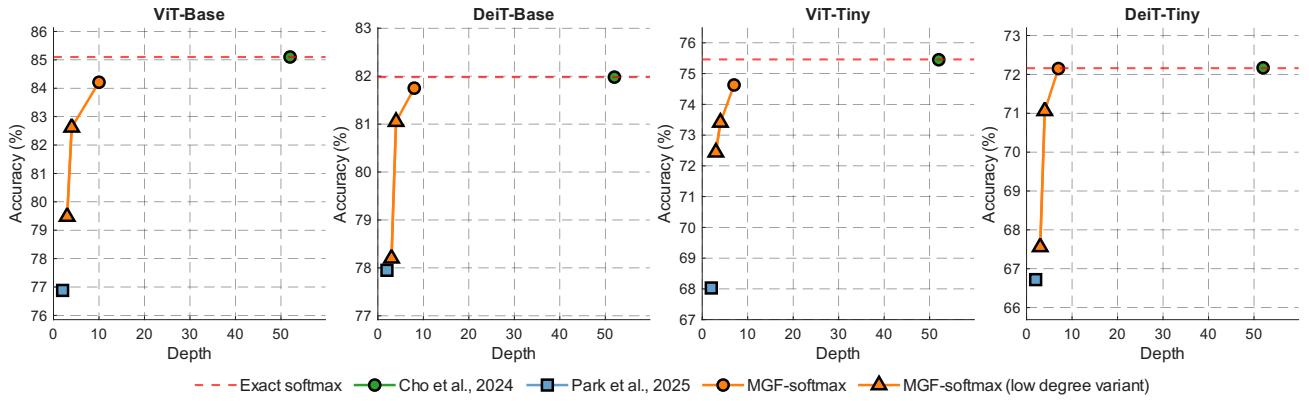


Figure 2. Top-1 accuracy of ViT and DeiT models on the ImageNet-1k dataset.

In contrast, MGF-softmax attains high inference accuracy with efficient HE circuits. Notably, the *low-degree variant* of MGF-softmax, which uses the same multiplicative depth of 2 as BPMax, consistently outperforms the replacement baseline. For example, on Clinc150, it achieves 87.60 %, exceeding that of BPMax (81.40 %) and even surpassing the plaintext baseline (86.67 %). Moreover, with a moderate multiplicative depth of 8–9, MGF-softmax recovers the plaintext accuracy with degradation bounded within 1 % across all evaluated tasks. In comparison, the softmax approximation baseline (Cho et al., 2024) requires substantially larger multiplicative depths (up to 41) to approximate the exact softmax.

Inference Accuracy on Vision Transformers. Figure 2 presents the evaluation results on ImageNet-1k, a challenging large-scale classification task with 1,000 output classes. The limitations of the softmax replacement baseline (BPMax) are more evident in this setting. BPMax incurs a notable accuracy drop of 8.2 % on ViT-Base (85.10 % to 76.88 %), indicating that softmax replacement methods are less effective at preserving accuracy on tasks with a large number of classes.

In contrast, MGF-softmax maintains inference accuracy

within 1 % of the plaintext baseline across all evaluated models, while requiring only 7–10 multiplicative depths. By comparison, the softmax approximation baseline (Cho et al., 2024) requires up to 52 multiplicative depths to approach the accuracy of exact softmax.

5. Conclusion

In this paper, we introduce MGF-softmax, a novel reformulation of the softmax function under HE. By leveraging the MGF, MGF-softmax reformulates the softmax denominator to eliminate computationally expensive operations such as maximum subtraction and homomorphic division, while remaining intrinsically adaptive to input statistics. We provide a theoretical analysis establishing the key properties of MGF-softmax, including an asymptotic bound on its approximation error. Extensive experiments on large-scale architectures, including LLaMA-3.2-1B and Vision Transformers, demonstrate that MGF-softmax preserves inference accuracy within 1 % of the plaintext baseline and consistently outperforms existing prior approaches on complex benchmarks. By reconciling high accuracy with substantially reduced multiplicative depth, MGF-softmax offers a scalable and efficient solution for secure inference.

Impact Statement

This paper presents work whose goal is to advance the field of Machine Learning, specifically within the domain of privacy-preserving machine learning (PPML). The proposed MGF-softmax facilitates the deployment of large language models and Vision Transformers, enabling users to leverage powerful AI capabilities without compromising data privacy. This advancement has significant positive societal implications for sensitive sectors such as healthcare, finance, and personal computing, where data confidentiality is paramount. Furthermore, by substantially reducing the computational complexity and multiplicative depth of homomorphic operations, our work directly addresses the environmental concern of high energy consumption typically associated with secure computing protocols.

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A. MGF-softmax of Alternative Probability Distributions

MGF-softmax is not limited to the Gaussian assumption and can be derived for a wide range of probability distributions. In this section, we demonstrate this versatility by instantiating MGF-softmax with the uniform and Laplace distributions.

A.1. Uniform Distribution

The MGF of a random variable X with a continuous uniform distribution $\mathcal{U}(a, b)$ is $M_X(t) = \frac{\exp(bt) - \exp(at)}{t(b-a)}$. Thus, $M_X(1)$ is $\frac{\exp(b) - \exp(a)}{b-a} \approx \frac{\exp(b)}{b-a}$ if $b \gg a$. Consequently, MGF-softmax instantiated with the uniform distribution is given by

$$\text{softmax}_{\text{MGF}, \mathcal{U}}(\mathbf{x})_i = \frac{\exp(x_i)}{n \cdot \frac{\exp(b)}{b-a}} = \exp\left(x_i - b - \ln \frac{n}{b-a}\right), \quad (13)$$

where $i \in \{1, \dots, n\}$.

A.2. Laplace Distribution

The MGF of a random variable X with a Laplace distribution $\mathcal{L}(\mu, b)$ is $M_X(t) = \frac{\exp(\mu t)}{1-b^2 t^2}$ for $|t| < \frac{1}{b}$. Here, μ denotes the location parameter and $b > 0$ is the scale parameter, which relates to the variance by $\sigma^2 = 2b^2$. Thus, $M_X(1)$ is $\frac{\exp(\mu)}{1-b^2}$, provided that the scale parameter satisfies $b < 1$. Consequently, MGF-softmax instantiated with the Laplace distribution is given by

$$\text{softmax}_{\text{MGF}, \mathcal{L}}(\mathbf{x})_i = \frac{\exp(x_i)}{n \cdot \frac{\exp(\mu)}{1-b^2}} = \exp\left(x_i - \mu - \ln \frac{n}{1-b^2}\right), \quad (14)$$

where $i \in \{1, \dots, n\}$ and $0 < b < 1$.

B. Relative Error

The relative error η is given by

$$\eta = \frac{\|\text{softmax}(\mathbf{x}) - \text{softmax}_{\text{MGF}}(\mathbf{x})\|_\infty}{\|\text{softmax}(\mathbf{x})\|_\infty} \quad (15)$$

$$= \frac{\left| \frac{1}{\sum_{i=1}^n \exp(x_i)} - \frac{1}{n M_X(1)} \right| \cdot \|\exp(\mathbf{x})\|_\infty}{\left| \frac{1}{\sum_{i=1}^n \exp(x_i)} \right| \cdot \|\exp(\mathbf{x})\|_\infty} \quad (16)$$

$$= \left| 1 - \frac{\frac{1}{n} \sum_{i=1}^n \exp(x_i)}{\mathbb{E}[\exp(X)]} \right|. \quad (17)$$

Note that the relative error depends only on the ratio between the sample mean and the ensemble mean.

C. Error Analysis

Let $\bar{Y}_n = \frac{1}{n} \sum_i \exp(x_i)$ be the sample mean and let $\mu_Y = \mathbb{E}[Y]$ be the true ensemble mean. We seek the probability

$$P_\eta = P(\eta \geq \delta) = P\left(\left| 1 - \frac{\bar{Y}_n}{\mu_Y} \right| \geq \delta\right). \quad (18)$$

Rearranging the inequality, this is equivalent to $|\bar{Y}_n - \mu_Y| \geq \delta \mu_Y$. Using the standard normal approximation $\bar{Y}_n \sim \mathcal{N}\left(\mu_Y, \frac{\sigma_Y^2}{n}\right)$, the probability of this deviation is given by

$$P_\eta = P(|\bar{Y}_n - \mu_Y| \geq \delta \cdot \mu_Y) \approx 2 \left(1 - \Phi\left(\frac{\delta \cdot \mu_Y \sqrt{n}}{\sigma_Y}\right) \right), \quad (19)$$

where Φ is the CDF of the standard normal distribution.

For MGF-softmax instantiated with Gaussian distribution, the random variable Y follows a log-normal distribution. Substituting μ_Y and σ_Y of the log-normal distribution into Eq. (19) yields

$$P_\eta \approx 2 \left(1 - \Phi \left(\frac{\delta\sqrt{n}}{\sqrt{\exp(\sigma^2) - 1}} \right) \right). \quad (20)$$

D. Implementation Details

D.1. Packing Strategy

In Transformer architectures, the softmax function is predominantly applied within the self-attention mechanism, necessitating row-wise execution across the input matrix. Accordingly, we target the row-wise application of MGF-softmax on an input matrix $A \in \mathbb{R}^{N_1 \times N_2}$. We assume N_1 and N_2 are powers of two; for arbitrary dimensions, we apply zero-padding to extend them to the nearest power of two. The input data is encrypted into $t = \lceil (N_1 \cdot N_2)/s \rceil$ ciphertexts. To facilitate efficient column-wise aggregation, we employ a structure similar to the column-major packing variant used by Cho et al. (Cho et al., 2024). Specifically, elements within the same row are spaced by a fixed gap g , defined as $g = (t \cdot s)/N_2$.

Let $ct_i[j]$ denote the value stored in the j -th slot of the i -th ciphertext. The element $A[n_1, n_2]$ located at row n_1 ($0 \leq n_1 < N_1$) and column n_2 ($0 \leq n_2 < N_2$) is mapped to the ciphertext as follows:

$$A[n_1, n_2] = ct_i[j], \quad \text{where } \begin{cases} i = \lfloor (n_2 \cdot g)/s \rfloor, \\ j = n_1 + (n_2 \cdot g \bmod s). \end{cases} \quad (21)$$

Any slots in the ciphertexts that are not populated by matrix elements according to this mapping are initialized to zero.

D.2. Algorithm

Based on the packing strategy described in Appendix D.1, we present the outline of homomorphic MGF-softmax in Algorithm 1. The input consists of a set of ciphertexts $\mathcal{V} = \{ct_0, \dots, ct_{t-1}\}$ representing the encrypted matrix. The algorithm is designed to evaluate MGF-softmax in a row-wise manner. Notably, the entire pre-processing stage (Steps 1 and 2) consumes 1 multiplicative level. This is because the squaring operation necessitates a single ciphertext–ciphertext multiplication. Furthermore, to optimize the circuit depth, consecutive scalar multiplications are pre-computed into a single scalar and applied to the ciphertext once; this is a common optimization technique in HE to prevent unnecessary level consumption. For the exponential function, we employ a polynomial approximation denoted as AExp, the details of which are outlined in Appendix E.

Algorithm 1 Homomorphic MGF-softmax Outline

Input: A set of ciphertexts $\mathcal{V} = \{ct_0, \dots, ct_{t-1}\}$ encrypting the matrix.
Output: A set of ciphertexts $\mathcal{W} = \{ct'_0, \dots, ct'_{t-1}\}$ encrypting the result of applying MGF-softmax row-wise to the matrix.

{// Step 1: Compute Mean (μ)}

$ct_{\text{sum}} \leftarrow$ Sum all ciphertexts in \mathcal{V} using addition.

$\mu \leftarrow$ Perform row-wise summation on ct_{sum} by repeating addition and rotation.

{// Step 2: Compute Variance Term ($\frac{1}{2}\sigma^2$)}

$\mathcal{V}_{\text{centered}} \leftarrow$ Subtract μ from each ciphertext in \mathcal{V} .

$\mathcal{V}_{\text{sq}} \leftarrow$ Square each ciphertext in $\mathcal{V}_{\text{centered}}$.

$ct_{\text{sum}} \leftarrow$ Sum all ciphertexts in \mathcal{V}_{sq} .

$\text{var_term} \leftarrow$ Perform row-wise summation on ct_{sum} .

{// Step 3: Final Approximation}

$\mathcal{V}_{\text{norm}} \leftarrow$ Subtract var_term and $\ln n$ from each ciphertext in $\mathcal{V}_{\text{centered}}$.

$\mathcal{W} \leftarrow$ Apply AExp to each ciphertext in $\mathcal{V}_{\text{norm}}$.

return \mathcal{W}

E. Approximation of the Exponential Function

In this section, we detail the implementation of the polynomial approximation function $\text{AExp}(x) \approx \exp(x)$, which constitutes the sole polynomial approximation step within the MGF-softmax framework. The configuration of MGF-softmax—specifically, whether it operates as the *standard configuration* or the efficient *Low-degree variant*—is determined by the timing of approximation relative to the training phase.

E.1. Standard Configuration (Post-training).

The standard MGF-softmax configuration applies exponential approximations in *post-training* to prioritize maximum accuracy recovery. In this setting, the model is trained using the exact MGF-softmax formulation, and the polynomial approximation is introduced only during the inference phase. We utilize two distinct polynomial approximation methods for $\text{AExp}(x)$.

Chebyshev Polynomial Approximation. To accommodate the wide dynamic range of input values while maintaining high precision, we leverage the exponential identity $\exp(x) = (\exp(x/2^k))^{2^k}$. This formulation allows us to map the input to a reduced interval (e.g., $[-8, 0]$ following (Cho et al., 2024)) by scaling the domain by a factor of $1/2^k$. Within this reduced interval, we employ a Chebyshev polynomial of degree $d = 15$. To evaluate this efficiently, we adopt the Paterson-Stockmeyer algorithm (Paterson & Stockmeyer, 1973), which reduces the multiplicative complexity of polynomial evaluation. While this method approximates the exponential function with high precision over a wide interval, it requires a relatively high multiplicative depth, calculated as $\lceil \log_2(d+1) \rceil + k + 1$, due to the combination of polynomial evaluation and the subsequent k squaring steps required to recover the original scale.

Limit Approximation. We implement this as $\text{AExp}(x) = (1 + x/2^k)^{2^k}$. This approach is computationally efficient, involving only a single scalar multiplication followed by k repeated squarings. Consequently, it consumes less multiplicative depth, requiring only $k + 1$. However, compared to the Chebyshev method, the Limit approximation is less robust, as the approximation error tends to increase rapidly when the input values fall outside the target domain.

The specific values of the parameter k utilized in our experiments are detailed in Table 6.

Table 6. Specific values of k employed for the MGF-softmax standard configuration across different models and datasets. All reported values achieve inference accuracy within 1 % of the pre-trained exact softmax baseline.

Model	LLaMA-3.2-1B			ViT-Base	DeiT-Base	ViT-Tiny	DeiT-Tiny
Dataset	Clinc150	Banking77	SST-2			ImageNet-1k	
Chebyshev approx.	3	3	4	6	3	1	1
Limit approx.	6	6	7	8	6	8	7

E.2. Low-degree Variant (In-training).

The low-degree variant is designed for scenarios requiring maximum efficiency with minimal multiplicative depth. In this approach, the approximation is applied *in-training*; the exact exponential function is replaced by a simplified polynomial approximation *before* the training phase begins. We utilize a Taylor expansion of degree d centered at x_0 , formulated as:

$$\text{AExp}(x) = \sum_{i=0}^d \frac{e^{x_0}}{i!} (x - x_0)^i \quad (22)$$

A key advantage of this method is that it eliminates the need for domain scaling, thereby avoiding the additional overhead associated with squaring operations. Total multiplicative depth required for this operation is $\lceil \log_2(d+1) \rceil$.

F. Training Details

Search Space and Configurations. To ensure a rigorous evaluation, we conducted an extensive hyperparameter sweep for both LLaMA-3.2-1B and ViT/DeiT. Table 7 details the specific search grids. For BPMax, which replace softmax with

simplified polynomial alternatives $(x + c)^p$, we significantly expanded the search space beyond the original study (Park et al., 2025)—which limited the search to shift variable $c \in \{1, 3, 5, 7\}$ and degrees $p \in \{1, 3, 5, 7\}$ —to guarantee optimal baselines.

Following the original settings (Park et al., 2025), we set the maximum training budget for BPMax to 20 epochs while applying early stopping to all experiments (including MGF-softmax). Under these conditions, MGF-softmax demonstrated significantly faster training speeds than the baseline: LLaMA-3.2-1B required only 5 epochs. For ViT/DeiT models, we employed *Attention-only Tuning*—a strategy where we freeze all model parameters and exclusively update the weights associated with the attention mechanism. This approach provided the dual benefit of minimizing the training process and facilitating effective accuracy recovery.

Fixed Configurations. Regarding input specifications, we set the sequence length to $n = 197$ for ViT/DeiT models and the maximum sequence length to 128 for LLaMA-3.2-1B. For the KD objective, we utilized a combination of Cross-Entropy loss and KL-Divergence loss, setting the temperature to $\tau = 2.0$ and the distillation weight to $\alpha = 0.5$. We fixed the random seed to 42 across all experimental runs.

Table 7. Configuration search space and fixed settings for large language models and Vision Transformers. In the Teacher Model row, ‘Self’ refers to the original backbone prior to replacing the softmax, while ‘ViT-Large (85.84%)’ denotes the pre-trained model sourced from the `timm` library.

Configuration	LLaMA-3.2-1B	Vision Transformers (ViT/DeiT)
<i>MGF-softmax (low degree variant)</i>		
Taylor Degree (d)	{1}	{3, 5}
Expansion Point (x_0)	{−3}	{−10}
<i>BPMax</i>		
Degree (p)	{1, 3, 5, 7}	{1, 3, 5, 7, 9}
Scaling (c)	{1, 3, 5, 7, 20}	{1, 3, 5, 7, 9, 20, 40, 60}
<i>Optimization</i>		
Epochs (MGF-softmax)	{5}	{15}
Epochs (BPMax)	{20}	{20}
Batch Size	{32, 64}	{64, 128}
Learning Rate	{9e-6, 1e-5, 3e-5, 5e-5}	{1e-5, 5e-5, 1e-4}
<i>Regularization & Strategy</i>		
Drop Path Rate	N/A	{0.0, 0.1, 0.2}
Attention-only Tuning	{Off}	{On, Off}
<i>Knowledge Distillation</i>		
Teacher Model	{Self}	{Self, ViT-Large (85.84%)}