Exercise: 1.1 i) morphism can have at most 1 inverse

pf) If $g, h : x \to y$ are inverse of $f : y \to x$

then, $g = g 1_x = gfh = 1_y h = h$.

1.1. ii). Maximal groupoid $\to$ done.

1.1. iii) Slice Category.

Let C : category $c \in \mathrm{obj}\, C$.

$$c/C : \mathrm{obj} := \bigcup_{x \in \mathrm{obj}\, C} \mathrm{Hom}(c, x)$$

under

$\mathrm{Mor} :=$

$\mathrm{Mor}(f : c \to x,\ g : c \to y)$

$$= \left\{ h : x \to y : \begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\circlearrowleft"] & \\ x \arrow[rr, "h"'] & {} & y \end{tikzcd} \text{ commutes} \right\}$$

i.e. $g = hf$.

$$C/c : \mathrm{obj} := \bigcup_{x \in \mathrm{obj}\, C} \mathrm{Hom}(x, c)$$

over $\mathrm{Mor}(f : x \to c,\ g : y \to c)$

$$= \left\{ h : x \to y : \begin{tikzcd} & c & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} \arrow[u, phantom, "\circlearrowleft"] & y \arrow[ul, "g"'] \end{tikzcd} \text{ commutes, i.e. } f = gh \right\}$$

Duality

Def 1.2.1. From C, C^{op} is opposite category

- $Obj(C^{op}) = Obj(C)$
- $f^{op} \in Mor(C^{op})$ for each $f \in C$ with $f^{op}: cod(f) \longrightarrow dom(f)$

Then, from structure of C,

1) 1_X^{op} is id in C^{op}

2) $f^{op}: X \longrightarrow Y$, $g^{op}: Y \longrightarrow Z$

$$\Rightarrow g^{op} f^{op}: X \longrightarrow Z \quad \longleftrightarrow \quad fg: Z \to X.$$

Thus C^{op} is category iff C is category

Ex 1.2.2. Mat_R^{op} : f^{op} is transpose of f.

$(P, \leq)^{op}$: $x \longrightarrow y \iff y \leq x$.

ω^{op} : $\cdots 3 \longrightarrow 2 \longrightarrow 1 \longrightarrow 0$.

$(BG)^{op} \cong B(G^{op})$ where G^{op} : opposite gp.

gp with right multiplication

i.e. $f \cdot g = gf$ in G.

Thus, any statement of proof about C also applies to opposite: "dual theorem"

Ex).

Lemma 1.2.3 For any C, TFAE.

(1) $f : X \to Y$ is iso in C

(2) $f_* : C(c, X) \longrightarrow C(c, Y)$ is bijection $\forall c \in \mathrm{obj}\, C$
$$g \longmapsto fg$$

(3) $f^* : C(Y, c) \longrightarrow C(X, c)$ "
$$g \longmapsto gf$$

f_* : post-composition, f^* : pre-composition

pf)

(i) $\Rightarrow$ (ii) f : iso, $\Rightarrow \exists g : Y \longrightarrow X$ inverse of f.

Thus, $g_* f_*$ and $f_* g_*$ are identity $\forall c \in \mathrm{obj}\, C$

(ii) $\Rightarrow$ (i) Take $c = Y$. $\exists g \in C(Y, X)$ s.t.

$f_*(g) = 1_Y \Rightarrow fg = 1_Y$

Also,

$f_*(gf) = fgf = 1_Y f = f$

$f_*(1_X) = f 1_X = f$

By bijectivity, $1_X = gf$

(i) $\Leftrightarrow$ (iii) Apply (i) $\Leftrightarrow$ (ii) on C^{op} to get

$f^{op} : Y \longrightarrow X$ is iso $\Leftrightarrow f^{op}_* : C^{op}(c, Y) \longrightarrow C^{op}(c, X)$
$$g^{op} \longmapsto f^{op} g^{op}$$
is bijection $\forall c \in C$.

Now, $C^{op}(c, Y) \leftrightarrow C(Y, c)$ $C^{op}(c, X) \leftrightarrow C(X, c)$

$g^{op} \leftrightarrow g$, $f^{op} g^{op} \longleftrightarrow gf$

Thus f^{op}_* sends g to gf $\Rightarrow f^{op}_* = f^*$

$\Rightarrow \quad f^{op}: Y \to X$ is iso $\iff f^*: C(Y,c) \to C(X,c)$ bij. $\forall c \in C$.

$\Updownarrow$

$f: X \to Y$ is iso.

done.

Def: 1.2.7. $f: X \to Y$ morphism is

(1) monomorphism if $\forall h,k: W \rightrightarrows X$ for some W,
mono (noun)
monic (adj)
(left cancellable)

$fh = fk \Rightarrow h = k$.

(2) epimorphism if $\forall h,k: Y \rightrightarrows W$ for any W,
epi
epic
(right cancellable)

$hf = kf \Rightarrow h = k$

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f is mono. $\iff f_*: C(c,X) \to C(c,Y)$ $\forall c \in C$
injective

" epi $\iff$ "
" surjective

Ex 1.2.8. $f: X \to Y$ mono in **Set**. take $x \in X$.

Let $\{x\} \underset{h}{\overset{g}{\rightrightarrows}} X$, suppose $fg = fh$. $\Rightarrow g = h$.

Thus f is injective.

Ex 1.2.9. (split epi / split mono)

s: section (right inverse)

r: retraction (retract) (left inverse)

$X \xrightarrow{s} Y \xrightarrow{r} X$ s.t. $rs = 1_X$. Then

(s is mono
r is epic.)

In this case,

s : split mono

r : split epi.