Yoneda Lemma 2.2

Q: Given a functor $F: \cancel{Se}\, C \longrightarrow Set$,
what data is needed to define $C(c,-) \cong F$?
or $C(c,-) \Rightarrow F$?

Ex 2.2.1 $F: \omega \longrightarrow Set$ $\qquad$ ω: ordinal Category ~~of ordinal~~

$\Leftrightarrow (F_n)_{n\in\omega}$, $f_{n,n+1}: F_n \to F_{n+1}$.

Then $\omega(k,-): \omega \longrightarrow Set$

$$\Rightarrow \omega(k,m) = \begin{cases} \emptyset & m < k \\ \{*\} & m \geq k \end{cases} \quad \left(\begin{array}{l} m=k \\ \Rightarrow \omega(k,k) \\ = \{1_k\} \end{array} \right)$$

If $\alpha: \omega(k,-) \Rightarrow F$, then

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "\alpha_0"'] & \emptyset \arrow[r] \arrow[d, "\alpha_1"] & \cdots \arrow[r] & \emptyset \arrow[r] \arrow[d, "\alpha_{k-1}"] & * \arrow[r] \arrow[d, "\alpha_k"] & * \arrow[r] \arrow[d, "\alpha_{k+1}"] & \cdots \\
F_0 \arrow[r, "f_{0,1}"'] & F_1 \arrow[r] & \cdots \arrow[r] & F_{k-1} \arrow[r, "f_{k-1,k}"'] & F_k \arrow[r, "f_{k,k+1}"'] & F_{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

commutes. Notes that $m<k$, α_m is empty as a set of tuple. (So commutes vacuously.)

If $m \geq k$, α_m is identified as an element.

$\Rightarrow \alpha_{n+1} = f_{n,n+1}(\alpha_n) \Rightarrow \alpha$ is determined by choice of $\alpha_k \in F_k$

Ex 2.2.2 Let G : gp. BG : category of one element with $\mathrm{Mor}\,BG = G$

$\Rightarrow G : BG \longrightarrow \mathrm{Set}$.

$G : BG^{op} \longrightarrow \mathrm{Set}$ are unique as a representable covariant functor since $BG(\cdot,\cdot) \cong G$.
(contra ")

(as a set)

BG has only one element, i.e.,

$$\begin{tikzcd} BG(\cdot,\cdot) \arrow[r, "1_G"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[d, "\phi"] \\ BG(\cdot,\cdot) \arrow[r, "1_G"'] & G \end{tikzcd}$$

Satisfying (left) Group action.

$X : BG \longrightarrow \mathrm{Set}$

$$\begin{tikzcd} \cdot \arrow[r, mapsto] \arrow[d, "g"'] & X \arrow[d, "g_*"] \\ \cdot \arrow[r, mapsto] & X \end{tikzcd}$$

g_* : iso of X

i.e, $G \leq \mathrm{Aut}(X)$.

If $\alpha : BG(\cdot, -) \Rightarrow X$

$$\begin{tikzcd} G \arrow[r, "\alpha_\cdot"] \arrow[d, "g\cdot"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "g_*"] \\ G \arrow[r, "\alpha_\cdot"'] & X \end{tikzcd}$$

$h \mapsto$ $\alpha_\cdot(gh)$ $g\cdot\alpha_\cdot(h)$

$\Rightarrow \alpha_\cdot(g\cdot h) = g\cdot\alpha(h)$

Especially if $h = e$,

$\alpha_\cdot(g) = g\cdot\phi(e)$

So, choice of $\phi(e) \in X$ forces us to define $\phi(g)$.

And $\phi(e)$ can be any element, since left action of G on G is free. (i.e. every stabilizer gp is trivial.)

Prop 2.2.3: G-equivariant maps $G \to X$ ($f(gx) = gf(x)$)
Corresponds bijectively to elements of X
Identified as image of identity $e \in G$.

In these two example, natural transformation whose domain is a representable functor are determined by the choice of single element which lives in the set def by evaluating codomain functor at the representing object.
Moreover choice is permitted.
I.e Let $F: C \to Set$, $C(c, -)$ be a representable functor. Then, $\alpha: C(c,-) \Rightarrow F$ is determined by choice of elements in Fc.
I.e. $\mathrm{Hom}(C(c,-), F) \cong F_c$ as a set.

Thm 2.2.4 (Yoneda Lemma)
For any functor $F: C \to Set$, C: locally small
$\forall c \in \mathrm{Obj}\, C$. Then, $\exists$ bijection

$$\mathrm{Hom}(C(c,-), F) \cong F_c$$

that associates natural transf: $\alpha: C(c,-) \Rightarrow F$ to the element $\alpha_c(1_c) \in F_c$. This bijection is natural in both c and F.

This bijection is natural in both c and F.

Rmk: Since $\mathcal{C}$ is not small, $\mathrm{Hom}(\mathcal{C}(c,-),F)$ might be large. However, Yoneda Lemma shows that $\mathrm{Hom}(\mathcal{C}(c,-),F)$ is a set.

pf 1: Bijection) construct

$$\Phi : \mathrm{Hom}(\mathcal{C}(c,-),F) \longrightarrow Fc$$

$$\alpha : \mathcal{C}(c,-) \Rightarrow F \longmapsto \alpha_c(1_c)$$

WTS Φ is bijection. It suffices to show that

$\Psi : Fc \longrightarrow \mathrm{Hom}(\mathcal{C}(c,-),F)$ has inverse.

To do this, for each $x \in Fc$, we need to define $\Psi(x)$ as a natural transf.

$\Rightarrow$ Need to define $\Psi(x)_d : \mathcal{C}(c,d) \to Fd$ for any $d \in \mathrm{obj}\,\mathcal{C}$ s.t. $f : c \to d$.

$$\begin{tikzcd}
\mathcal{C}(c,c) \arrow[r, "\Psi(x)_c"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "Ff"] \\
\mathcal{C}(c,d) \arrow[r, "\Psi(x)_d"'] & Fd
\end{tikzcd}$$

Then,

$$\begin{tikzcd}
1_c \arrow[r, mapsto] \arrow[d, mapsto] & \Psi(x)_c(1_c) \arrow[dr] & \\
f \arrow[r, mapsto] & \Psi(x)_d(f) & Ff(\Psi(x)_c(1_c))
\end{tikzcd}$$

Thus, WTS $\Psi(x)_d(f) = Ff(\Psi(x)_c(1_c))$

Since Ψ is intended as a inverse of Φ.

$\Phi(\Psi(x)) := x$ naturally.

Thus, $\Phi(\Psi(x)) \underset{\uparrow}{=} \Psi(x)_c(1_c) = x$.

from def of Φ

Therefore, naturality forces to define

$\Psi : Fc \to \mathrm{Hom}(\mathcal{C}(c,-), F)$

$\Psi(x)_d(f) := Ff(x)$

It determines $\Psi(x)_d$ as a map $\mathcal{C}(c,d) \to Fd$.

To see $\Psi(x)$ is natural transformation.

let $g : d \to e$. WTS

$$\begin{tikzcd}
\mathcal{C}(c,d) \arrow[r, "\Psi(x)_d"] \arrow[d, "g_*"'] \arrow[dr, phantom, "\circlearrowright"] & Fd \arrow[d, "Fg"] \\
\mathcal{C}(c,e) \arrow[r, "\Psi(x)_e"'] & Fe
\end{tikzcd}$$

Commutes. Let $f \in \mathcal{C}(c,d)$

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \Psi(x)_d(f) \arrow[d, mapsto] \\
gf \arrow[r, mapsto] & \Psi(x)_e(gf)
\end{tikzcd}$$

$Fg(\Psi(x)_d(f))$

$\Psi(x)_e(gf)$

$= F(gf)(x)$

$= Fg.(Ff(x))$

By functoriality of F, $F(gf).x$
$= (Fg).(Ff)(x)$.

So $\Psi : Fc \longrightarrow \mathrm{Hom}(C(c,-), F)$ is well-def function.

By construction, $\Phi\Psi(x) = \Psi(x)_c(1_c) = x$

NTS $\Psi\Phi(\alpha) = \alpha$ for any $\alpha : C(c,-) \Rightarrow F$

$$\begin{array}{c} \Psi\Phi(\alpha) \\ \parallel \\ \Psi(\alpha_c(1_c)) \end{array}$$

$\Rightarrow$ It suffices to show that for any $f : c \to d$,

$$\Psi(\alpha_c(1_c))_d(f) = Ff(\alpha_c(1_c))$$

By naturality of α

$$\begin{tikzcd} C(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "Ff"] \\ C(c,d) \arrow[r, "\alpha_d"'] & Fd \end{tikzcd}$$

Thus, $Ff(\alpha_c(1_c))$
$= \alpha_d(f)$

$\Rightarrow \Psi(\alpha_c(1_c))_d(f) = \alpha_d(f)$

$\Rightarrow \Psi(\alpha_c(1_c))_d = \alpha_d$

$\Rightarrow \Psi(\Phi(\alpha))_d = \alpha_d$

$\Rightarrow \Psi\Phi(\alpha) = \alpha$

So, Ψ and Φ are inverse to eachother.

Hence $\mathrm{Hom}(C(c,-), F) \cong Fc$

Proof of Naturality) ① Naturality in the functor

WTS, given $\beta : F \Rightarrow G$, if x represents

$\beta\alpha : C(c,-) \Rightarrow F \Rightarrow G$, i.e. $\Phi_G(\beta\alpha) = x$,

then $x = \beta_c(y)$ s.t. $y \in F_c$ represents

$\alpha : C(c,-) \Rightarrow F$.

i.e. $x = \Phi_F(\beta_c(y))$

In other words,

$$\begin{tikzcd}
\mathrm{Hom}(C(c,-),F) \arrow[r, "\Phi_F", "\cong"'] \arrow[d, "\beta_*"'] \arrow[dr, phantom, "\circlearrowright"] & F_c \arrow[d, "\beta_c"] \\
\mathrm{Hom}(C(c,-),G) \arrow[r, "\Phi_G"', "\cong"] & G_c
\end{tikzcd}$$

commutes.

pf) $\Phi_G(\beta\cdot\alpha) = (\beta\cdot\alpha)_c(1_c) = \beta_c(\alpha_c(1_c))$

$= \beta_c(\Phi_F(\alpha))$.

② Naturality in the object

Given $f : c \to d$ in C, if $x \in Fd$ represent

$\alpha f^* : C(d,-) \Rightarrow C(c,-) \Rightarrow F$, i.e. $\Phi_d(\alpha f^*) = x$,

then $x = Ff(y)$ where y represents α, i.e.

$x = Ff(\Phi_c(\alpha))$.

In other words,

$$\begin{tikzcd}
\mathrm{Hom}(C(c,-),F) \arrow[r, "\Phi_c", "\cong"'] \arrow[d, "(f^*)^*"'] & Fc \arrow[d, "Ff"] \\
\mathrm{Hom}(C(d,-),F) \arrow[r, "\cong", "\Phi_d"'] & Fd
\end{tikzcd}$$

$$\alpha \longmapsto Ff(\alpha_c(1_c))$$

$$\alpha \longmapsto (\alpha \cdot f^*)_d(1_d)$$

Pf) Notes that $(\alpha f^*)_d$ is $\alpha_d \circ f^*$; Hence,

$$C(d,d) \xrightarrow{f^*} C(c,d) \xrightarrow{\alpha_d} Fd$$

Thus $\quad 1_d \longmapsto f \longmapsto \alpha_d(f)$

And in the proof of bijection,

$$\begin{tikzcd}
C(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f_*"'] & Fc \arrow[d, "Ff"] \\
C(c,d) \arrow[r, "\alpha_d"'] & Fd
\end{tikzcd} \Rightarrow \quad \alpha_d(f) = Ff(\alpha_c)$$

$$\Rightarrow (\alpha \cdot f^*)_d(1_d) = \alpha_d(f) = Ff(\alpha_c(1_c)) \qquad \blacksquare$$

Rmk 2.2.17. If we donot consider size issue, Yoneda lemma can be viewed as natural isomorphism between functors.

Let $(c, F) \in \mathrm{Obj}\ \mathcal{C} \times \mathrm{Set}^{\mathcal{C}}$.

def: $\mathrm{ev} : \mathcal{C} \times \mathrm{Set}^{\mathcal{C}} \longrightarrow \mathrm{Set}$

$$(c, F) \longmapsto Fc = \text{codomain of } \Phi$$

Also, let

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "Y"] & \mathrm{Set}^{\mathcal{C}} \\ c \arrow[d, "f"'] \arrow[r, mapsto] & \mathcal{C}(c,-) \\ d \arrow[r, mapsto] & \mathcal{C}(d,-) \arrow[u, "f^*"'] \end{tikzcd}$$

Then, $\mathrm{Hom}(Y(-), -) := \mathcal{C} \times \mathrm{Set}^{\mathcal{C}} \xrightarrow{Y \times 1_{\mathrm{Set}^{\mathcal{C}}}} (\mathrm{Set}^{\mathcal{C}})^{op} \times \mathrm{Set}^{\mathcal{C}} \longrightarrow \mathrm{SET}$.

$$(c, F) \longmapsto (\mathcal{C}(c,-), F) \longmapsto \mathrm{Hom}(\mathcal{C}(c,-), F)$$

$\mathrm{Hom}(\mathcal{C}(c,-), F) = $ domain of Φ.

If $\mathcal{C}$ is small, No problem

$\mathcal{C}$ is locally small, $\mathrm{Set}^{\mathcal{C}}$ need not be locally small.

Then, $\mathrm{Hom}(Y(-), -) : \mathcal{C} \times \mathrm{Set}^{\mathcal{C}} \longrightarrow \mathrm{Set}$

$$(c, F) \longmapsto \mathrm{Hom}(\mathcal{C}(c,-), F)$$

Then, by naturality proof, we see that

$$\begin{tikzcd} C \times Set^C \arrow[r, bend left, "Hom(Y(-),-)"] \arrow[r, bend right, "ev"'] \arrow[r, phantom, "\Downarrow \Phi \cong"] & Set \end{tikzcd}$$

Corollary 2.2.8 (Yoneda embedding)

$$\begin{tikzcd} C \arrow[r, hook, "Y"] & \mathrm{Set}^{C^{op}} \\ c \arrow[d, "f"] \arrow[r, mapsto] & C(-,c) \arrow[d, "f_*"] \\ d \arrow[r, mapsto] & C(-,d) \end{tikzcd} \qquad \begin{tikzcd} C^{op} \arrow[r, hook, "Y"] & \mathrm{Set}^{C} \\ c \arrow[d, "f"] \arrow[r, mapsto] & C(c,-) \\ d \arrow[r, mapsto] & C(d,-) \arrow[u, "f^*"] \end{tikzcd}$$

define full and faithful embeddings

Rmk: Co- and contravariant Yoneda embedding are two different incarnation of a common bifunctor.

$$\begin{tikzcd} C(-,-) : C^{op} \times C \arrow[r] & \mathrm{Set} \\ (c,d) \arrow[d, "(f^{op}, g)"] \arrow[r, mapsto] & C(c,d) \arrow[d, "g_* f^*"] \\ (e,f) \arrow[r, mapsto] & C(e,f) \end{tikzcd}$$

f: pre
g: post

pf) WTS for any $f : c \to d$,

$C(c,d) \cong \mathrm{Hom}(C(-,c), C(-,d))$

and $C(c,d) \cong \mathrm{Hom}(C(d,-), C(c,-))$.

① By exercise 1.4 iv. (distinct parallel morphism define distinct natural transf (pre, post composition)

It is injective.

Also, Yoneda lemma says that

$$\alpha : C(d,-) \Rightarrow C(c,-) \quad \text{corresponds to}$$

$$\Phi(\alpha) = \alpha_d(1_d) \in C(c,d).$$

If we denote $\alpha_d(1_d) =: f$, then

$f^* : C(d,-) \Rightarrow C(c,-)$ sends $1_d \longmapsto f$.

$\Rightarrow$ $\boxed{\alpha = f^*}$ by Yoneda lemma. ∎

Cor 2.2.8 implies that natural transformation between represented functor corresponds to morphisms between the representing object.

There are three example, but introduce two first

1). Every row operation on matrix with n rows def by left mult of $n \times n$ Matrix

2). Cayley's theorem

3). In Vect_k, $V \otimes_k W \cong W \otimes_k V$.

Cor 2.2.9. Matrix mult. (R: unital ring)

pf) Mat_R : obj := $\mathbb{N}$.

$\mathrm{Mat}(m, n) \cong \mathrm{Mat}_{n\times m}(R)$.

Row operation define natural endomorphism of $\mathrm{Hom}(-, n)$, i.e. If α is a row op. $f: m \to k$ then

$k\times m$ matrix

$$\begin{tikzcd} x \arrow[ddd, mapsto] & \mathrm{Hom}(k, n) \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(k, n) \arrow[d, "f"] \\ & \mathrm{Hom}(m, n) \arrow[r, "\alpha"'] & \mathrm{Hom}(m, n) \\ x\cdot f & & \end{tikzcd}$$

By Cor 2.2.8, α is rep by element in $\mathrm{Hom}(n,n)$ $n\times n$ mat.

Moreover, Thm 2.2.4 identify what it is

i.e $\alpha_n(1_n)$ = row op on $n\times n$ identity matrix

Cor 2.2.10: Any gp is isomorphic to subgp of permutation gp.

pf) Let BG, category of 1 elem with $\mathrm{Mor}_{BG} \cong G$ for some gp G. Ex 2.2.2 gives $BG \hookrightarrow \mathrm{Set}^{BG^{op}}$

as right G-set G.

Corollary 2.2.8 says that

G-equivariant endomorphism of right G-set G are those maps defined by left multiplication, i.e. element of $\underline{BG(\cdot, \cdot) \cong G}$.

$$\text{Thus,}\quad G \cong \mathrm{Aut}_{\substack{\text{right}\\ G\text{-set}}}(G) \leq \mathrm{Sym}(G)$$

Since $\mathrm{Set}^{BG^{op}} \longrightarrow \mathrm{Set}$ is faithful functor.