$\forall f: m_+ \longrightarrow n_+$, def $M^f: M^m \longrightarrow M^n$

$$(a_1, \dots, a_m) \longmapsto (b_1 \cdots b_n)$$

where $b_i = \begin{cases} \prod_{j \in f^{-1}(i)} a_j & \text{if } f^{-1}(i) \neq \emptyset \\ 1 & \text{if empty} \end{cases}$

Then, M^f preserves unit. So,

$$\mathrm{Fin}_* \xrightarrow{M^-} \mathrm{Set}$$

$$\begin{array}{ccc} m_+ \to n_+ & n_+ & \mathrm{Fin}_* \\ \downarrow & \downarrow & \downarrow \\ M^m \to M^n & (M^n, (1, \dots, 1)) & \mathrm{Set}_* \rightleftarrows M^n \\ (a_i)_1^m \mapsto (b_i)_1^n & & \end{array}$$

$b_i = \left(\prod_{j \in f^{-1}(i)} a_j \right)$.

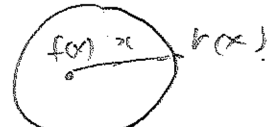
Segal 74: Cohomology on some suitable category

(Alg. K-theory from Quillen).

1.3.3 (Brouwer fixed pt Theorem)

Any cts endo $f: D^2 \longrightarrow D^2$ has a fixed pt.

pf) Let $r: D^2 \longrightarrow S^1$ by



r is cts and $i: S^1 \longrightarrow D^2$ is inclusion

$\Longrightarrow r i = 1_{S^1}$ r: split epi (retract) i: split mono (section)

Since π_1 is functor, $\mathrm{Top}_* \longrightarrow \mathrm{Group}$,

$$\pi_1(S_1, x) \xrightarrow{\pi_1(i)} \pi_1(D^2, x) \xrightarrow{\pi_1(r)} \pi_1(S^1, x)$$

$\pi_1(r)\cdot\pi_1(i) = \pi_1(r\cdot i) = \pi_1(I_{S^1}) = 1_{\pi_1(S^1)}$

ut $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(D^2) = 0$.

$\Rightarrow$ $\mathbb{Z} \to 0 \to \mathbb{Z}$ is identity, contradiction

ef 1.3.5 (Contravariant functor) $F: C^{op} \to D$

① $Fc \in D$, $\forall c \in \text{obj } C$

② $Ff : Fc' \to Fc$ $\quad \forall f: c \to c' \in \text{Hom } C$

s.t. 1) $f: x \to y$ $g: y \to z$, $\Rightarrow$ $Ff \cdot Fg = F(gf)$

$(Fy \to Fx,\ Fz \to Fy)$

2) $F1_c = 1_{Fc}$

Thus

$$\begin{tikzcd}
C^{op} \arrow[r, "F"] & D \\
c \arrow[r, mapsto] \arrow[d, "f"'] & Fc \\
{} \arrow[r, mapsto] & {} \arrow[u, "Ff"'] \\
c' \arrow[r, mapsto] & Fc
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "F"] & D \\
c \arrow[r, mapsto] \arrow[d, "f"'] & Fc \arrow[d, "Ff"] \\
{} \arrow[r, mapsto] & {} \\
c' \arrow[r, mapsto] & Fc
\end{tikzcd}$$

(Contravariant Functor $\qquad$ (Covariant) Functor

Ex 1.3.7

(1) $P: \text{Set}^{op} \to \text{Set}$

$$\begin{tikzcd}
A \arrow[r, mapsto] \arrow[d, "f"'] & PA & f^{-1}(B') \subset A \\
{} \arrow[r, mapsto] & {} \arrow[u] & {} \arrow[u] \\
B \arrow[r, mapsto] & PB & B' \subset B
\end{tikzcd}$$

) $(-)^* : \mathrm{Vect}_k^{op} \longrightarrow \mathrm{Vect}_k$

$$\begin{tikzcd} V \arrow[r, mapsto] \arrow[d, "\phi"'] & V^* & w\circ\phi \\ {} \arrow[r, mapsto] & {} & {} \\ W \arrow[r, mapsto] & W^* \arrow[uu, "\phi^*"'] & w \arrow[uu, mapsto] \end{tikzcd}$$

) $\mathcal{O} : \mathrm{Top}^{op} \longrightarrow \mathrm{Poset}$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{O}(X) & f^{-1}(U_Y) \\ {} \arrow[r, mapsto] & {} & {} \\ Y \arrow[r, mapsto] & \mathcal{O}(Y) \arrow[uu, "f^{-1}"'] & U_Y \arrow[uu, mapsto] \end{tikzcd}$$

$\mathcal{O}(X)$: topology as poset by inclusion

$C : \mathrm{Top}^{op} \longrightarrow \mathrm{Poset}$ by collection of closed sets

4) $\mathrm{Spec} : \mathrm{CRing}^{op} \longrightarrow \mathrm{Top}$

$$\begin{tikzcd} R \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Spec}(R) & f^{-1}(P) \\ {} \arrow[r, mapsto] & {} & {} \\ S \arrow[r, mapsto] & \mathrm{Spec}(S) \arrow[uu, mapsto, "f_*"'] & P \arrow[uu, mapsto] \end{tikzcd}$$

$\mathrm{Spec}(R)$ with Zariski top.

(5) (set-valued) Presheaf : any functor $C \longrightarrow \mathrm{Set}$ from small category C.

ex) $\mathcal{O}(X)^{op} \longrightarrow \mathrm{Set}$

$$\begin{tikzcd} U \arrow[r, mapsto] \arrow[d, "\mathrm{incl}"'] & C(U) = \{f : U \to \mathbb{R} \text{ cts}\} & f|_U \\ {} \arrow[r, mapsto] & {} & {} \\ V \arrow[r, mapsto] & C(U) = \{f : V \to \mathbb{R} \text{ cts}\} \arrow[uu, "\mathrm{res}(V,U)"'] & f \arrow[uu, mapsto] \end{tikzcd}$$

6) Presheaves on Δ : Simplex Category

Δ : Obj : $[n] = \{0, 1, \cdots n\}$, $n \in \mathbb{N}$ (finite nonempty ordinal)

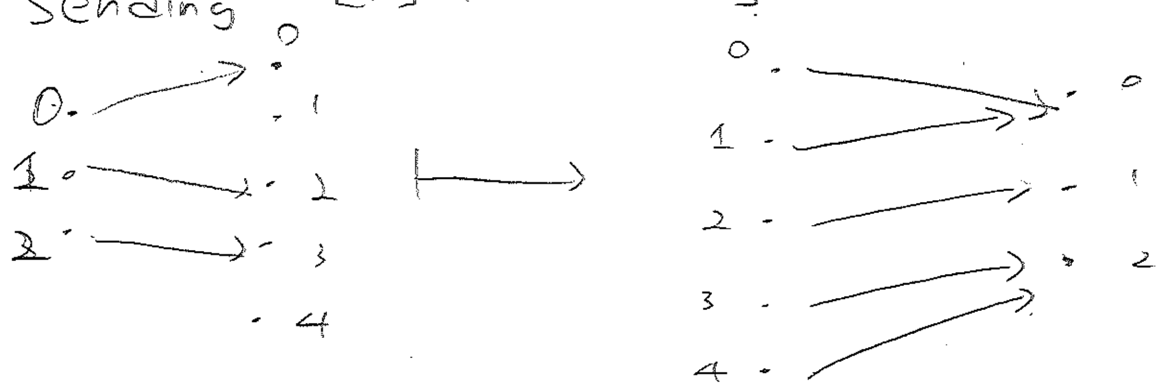
Mor : $f : [n] \longrightarrow [m]$ order preserving function.

$$(i \leq j \longmapsto f(i) \leq f(j))$$

A simplicial set : any presheaf on Δ, i.e.

$$X : \Delta^{op} \longrightarrow Set$$

ex) X : sending $[n] \longmapsto [n]$



Lem 1.3.8 Functor preserves iso.

pf) $F : C \longrightarrow D \Rightarrow$ $F(g)F(f) = F(gf) = F(1_x) = 1_{Fx}$

$\begin{pmatrix} f : x \longrightarrow y \\ g : y \longrightarrow x \end{pmatrix}$ iso. $F(f)F(g) = F(fg) = F(1_y) = 1_{Fy}$.

So does contravariant functor

Ex 1.3.9. G : gp. BG : group as a category.

Let $X : BG \longrightarrow C$: any category

$\bullet \longmapsto X$ some obj C.

$$g \longmapsto g_* : X \longrightarrow X$$

By functoriality axiom, ① $h_* g_* = (hg)_*$ $\forall g, h \in G$.

② $e_* = 1_X$ e : id in G.

$$\mathcal{C}(c, -) \qquad\qquad \mathcal{C}(-, c)$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, mapsto] & \mathrm{Set} \\
x \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{C}(c, x) \arrow[d, "f_*"] \\
y \arrow[r, mapsto] & \mathcal{C}(c, y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{op} \arrow[r] & \mathrm{Set} \\
x \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{C}(x, c) \\
y \arrow[r, mapsto] & \mathcal{C}(y, c) \arrow[u, "f^*"']
\end{tikzcd}$$

post composition $\qquad\qquad$ pre composition

"post" composition = Covariant action = left action

pre " = Contravariant " = right action

Bifunctor: ~~$\mathcal{C}(-,-)$~~ or functor of two variable

Def 1.3.12 $\mathcal{C}, \mathcal{D}$: Category $\quad \mathcal{C} \times \mathcal{D}$ is category

Obj $(\mathcal{C} \times \mathcal{D})$: (c, d), $c \in \mathrm{Obj}\,\mathcal{C}$, $d \in \mathrm{Obj}\,\mathcal{D}$

Mor: $(f, g): (c, d) \longrightarrow (c', d')$

for $f: c \longrightarrow c' \in \mathrm{Mor}\,\mathcal{C}$
$\quad g: d \longrightarrow d' \in \mathrm{Mor}\,\mathcal{D}$.

Def 1.3.13, Two sided represented functor

$$\mathcal{C}(-, -): \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Set}$$

($\mathcal{C}$: locally small)

$$\begin{tikzcd}
(x, y) \arrow[r] \arrow[d, "f^{op}"', "h"] & \mathcal{C}(x, y) \arrow[d, "(f^*, h_*)"] & g \arrow[d, mapsto] \\
(w, z) \arrow[r] & \mathcal{C}(w, z) & hgf
\end{tikzcd}$$

(So it is covariant functor.)

Thus $X: BG \to C$ defines an left action of the gp G on the object $X \in C$.

ex) If $C = \begin{pmatrix} \text{Set} \\ \text{Vect}_k \\ \text{Top} \end{pmatrix} \Rightarrow X: BG \to C$ is $\begin{pmatrix} G\text{ set} \\ G\text{-rep.} \\ G\text{-space} \end{pmatrix}$.

And Contravariant functor $BG^{op} \to C$ defines right action similar way.

Since Functor preserves iso, and all morphisms in BG are iso, so g_* is automorphism in that category.

Corollary 1.3.10 When G acts functorially on an object X in a category C (i.e. $\exists X: BG \to C$ exists)

g must act by automorphism $g_*: X \to X$

and $(g_*)^{-1} = (g^{-1})_*$.

Remark: Functor may not preserve mono or epic but preserves split mono / split epic.

pf) Almost same as 1.3.8.

Def 1.3.11 C: locally small. $c \in \text{Obj}\, C$.
Functor represented by c: $C(c, -)$, $C(-, c)$

s.t.

Cat : Category of small categories.
Mor : functors ~~between~~ them

$\Rightarrow$ Locally Small (Functors are set map between small categories)

but not small (contains Set, Poset, Monoid, Group, Groupoid as proper subcategory.)

CAT : Category of locally small categories
(avoid Russell's paradox)

$Cat \xhookrightarrow[inj.]{} CAT$ obvious.

Def (Isomorphism of Category)

$F: C \longrightarrow D$, $G: D \longrightarrow C$ are iso.

if $GF = 1_C$, $FG = 1_D$ where $1_C, 1_D$ are Identity functor.

F : Iso of Category induces bijection between Obj, Mor.

Ex 1.3.14

(i) $(-)^{op}: CAT \longrightarrow CAT$ nontrivial automorphism

$$\begin{array}{ccc} C & \longrightarrow & C^{op} \\ F \downarrow & & \downarrow F' \\ D & \longrightarrow & D^{op} \end{array}$$

(Notes: F defines $C^{op} \longrightarrow D^{op}$ as functor)

(ii) $(\ )^{-1}: BG \longrightarrow BG^{op}$ iso.

$g \longmapsto g^{-1}$.

$\Rightarrow$ Any right action can be converted into a left action by precompose $(\ )^{-1}$.

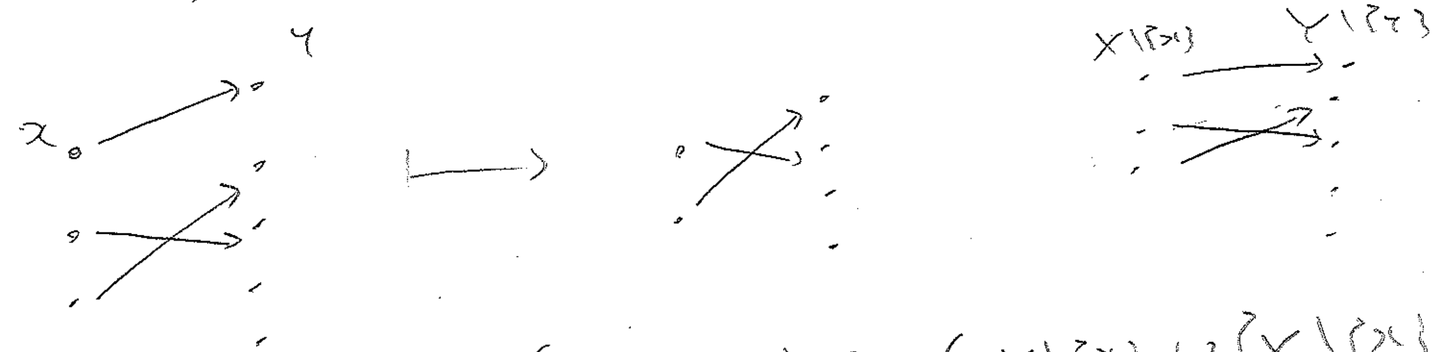
$U : \mathrm{Set}_* \longrightarrow \mathrm{Set}^{\circ}$

$$\begin{tikzcd}
(X, \{x\}) \arrow[r, mapsto] \arrow[d, "f"'] & X \setminus \{x\} \arrow[d, "f|_{X \setminus \{x\}} \text{ as a partial function}"] \\
(Y, \{y\}) \arrow[r, mapsto] & Y \setminus \{y\}
\end{tikzcd}$$

hen $U \circ (-)_+$: Identity endofunctor.

wever $(-)_+ \circ U$: $\mathrm{Set}_* \longrightarrow \mathrm{Set}_*$

$$\begin{tikzcd}
(X, \{x\}) \arrow[r, mapsto, "U"] \arrow[d, "f"'] & X \setminus \{x\} \arrow[r, "(-)_+"] \arrow[d, "f|_{X \setminus \{x\}}"] & ((X \setminus \{x\}) \cup \{X \setminus \{x\}\}) \arrow[d, "(f|_{X \setminus \{x\}})_+"] \\
(Y, \{y\}) \arrow[r, mapsto] & Y \setminus \{y\} \arrow[r] & (Y \setminus \{y\} \cup \{Y \setminus \{y\}\})
\end{tikzcd}$$



o as a set, $(X, \{x\}) \cong (X \setminus \{x\} \cup \{X \setminus \{x\}\})$

$$\begin{tikzcd}
(X, \{x\}) \arrow[r, phantom, "\cong"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & (X \setminus \{x\} \cup \{X \setminus \{x\}\}) \arrow[d, "(f|_{X \setminus \{x\}})_+"] \\
(Y, \{y\}) \arrow[r, phantom, "\cong"] & (Y \setminus \{y\}, \{Y \setminus \{y\}\})
\end{tikzcd}$$

ut not identical.

So it is not isomorphism of category, even if they act very similar.