Thus we need a little bit relaxed concept morphism of functor, i.e. natural transformation.

Exercise 1.3. i) Functor between groups?

$$\begin{tikzcd} G \arrow[r, "F"] & H \\ \bullet \arrow[d, "g"'] \arrow[r, mapsto] & \bullet \arrow[d] \\ \bullet \arrow[r, mapsto] & \bullet \end{tikzcd} \qquad g \longmapsto h$$

$F(g) = h$ s.t.

a) $F(gg') = F(g)F(g')$

b) $F(e_G) = e_H$.

$\Longrightarrow$ If F is functor, then it is gp homo.

Conversely, gp homo: satisfy a) $\Rightarrow$ b) $\therefore$ by adding $\bullet \longrightarrow \bullet$. it is functor.

1.3. ii) Functor between preorders?

$$\begin{tikzcd} (P, \leq) & (Q, \leq) \\ x \arrow[d, "x \leq y"'] \arrow[r, mapsto] & F(x) \arrow[d] \\ y \arrow[r, mapsto] & F(y) \end{tikzcd}$$

$F(x) \leq F(y)$ $\dashrightarrow$ a)

b): done.

F is order preserving function

[redacted]

1.3. iii) Dr. Hayesmeyer Notation.

C : $\underset{a}{\bullet} \longrightarrow \underset{b}{\bullet}$ $\quad$ $\underset{c}{\bullet} \longrightarrow \underset{d}{\bullet}$

$F : C \longrightarrow D$

$a \longrightarrow b \longmapsto x \longmapsto y$.

$c \longrightarrow d \longmapsto y \longmapsto z$.

$$D: \begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur] \arrow[rr] & & z \end{tikzcd} \qquad F(C): \begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur] & & z \end{tikzcd}$$

We have composable morphisms in $F(C)$

But composite doesn't exists.

Ex 1.3. iv) $f = 1_x \Rightarrow$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "1_x"'] & C(c,x) \arrow[d, "(1_x)_*"] & g \arrow[d, mapsto] \\ x \arrow[r, mapsto] & C(c,x) & (1_x g) = g \end{tikzcd}$$

identity

$f: x \to y, \quad g: y \to z$

$$\Rightarrow \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & C(c,x) \arrow[d, "f_*"] & h \arrow[d, mapsto] \\ y \arrow[r] \arrow[d, "g"'] & C(c,y) \arrow[d, "g_*"] & fh \arrow[d, mapsto] \\ z \arrow[r] & C(c,z) & gfh \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r] \arrow[d, "gf"'] \arrow[dr, phantom, "\mapsto"] & C(c,x) \arrow[d, mapsto] & h \arrow[d, mapsto] \\ z \arrow[r] & C(c,z) & gfh \end{tikzcd}$$

Ex 1.3.v) Claim: $F: C^{op} \to D \overset{\text{iff}}{\Longleftrightarrow} G: C \to D^{op}$

Let $F: C^{op} \to D$. Construct $G: C \to D^{op}$ s.t.

$G(c) := F(c)$. From $\mathrm{Hom}_D(Fb, Fa) \Leftrightarrow \mathrm{Hom}_{D^{op}}(Fa, Fb)$

So given $f \in \mathrm{Hom}_C(a, b)$

define $G(f) = (Ff)^{op}$ in $\mathrm{Hom}_{D^{op}}(Fa, Fb)$

Thus. $F: C \to C^{op} \xrightarrow{F} D \qquad G: C \xrightarrow{G} D^{op} \to D$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & x \arrow[r, mapsto] & Fx \\ y \arrow[r, mapsto] & y \arrow[u, "f^{op}"'] \arrow[r, mapsto] & Fy \arrow[u, "Ff"'] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & Fx \arrow[r, mapsto] \arrow[d, "(Ff)^{op}"] & Fx \\ y \arrow[r, mapsto] & Fy \arrow[r, mapsto] & Fy \arrow[u, "Ff"'] \end{tikzcd}$$

F, G are the same functor. Also, $F: C \to D \Leftrightarrow G: C^{op} \to D^{op}$

$$\begin{tikzcd} C \arrow[r] & D \\ x \arrow[d] & x \arrow[d, "Ff"] \\ y \arrow[r, mapsto] & y \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r] & C^{op} \arrow[r] & D^{op} \arrow[r] & D \\ x \arrow[d, "f"'] & x & Fx & Fx \arrow[d, "Ff"] \\ y & y \arrow[u, "f^{op}"'] & Fy \arrow[u, "(Ff)^{op}"'] & Fy \end{tikzcd}$$

1.3.vi) $F: D \to C$, $G: E \to C$.

$F \downarrow G$: Obj := $(d, e, f: \ )$

$d \in D$, $e \in E$, $f: Fd \to Ge \in C$

$\mathrm{Mor}((d,e,f), (d',e',f'))$

$= \{ (h, k) \in \mathrm{Mor}\, D \times \mathrm{Mor}\, E$:

$h: d \to d'$, $k: e \to e'$ s.t.

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\circlearrowright"] & Ge \arrow[d, "Gk"] \\ Fd' \arrow[r, "f'"'] & Ge' \end{tikzcd}$$

in C, i.e $f' \cdot Fh = Gk \cdot f$

It is category since,

① $\mathbb{1}_{(d,e,f)} = (\mathbb{1}_d, \mathbb{1}_k)$.

② $(d,e,f) \xrightarrow{(h_1,k_1)} (d',e',f') \xrightarrow{(h_2,k_2)} (d'',e'',f'')$

$\Rightarrow$

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "Fh_2"'] \arrow[dr, phantom, "\circlearrowright"] & Ge \arrow[d, "Fk_1"] \\ Fd' \arrow[r, "f'"] \arrow[d, "Fh_2"'] \arrow[dr, phantom, "\circlearrowright"] & Ge' \arrow[d, "Fk_2"] \\ Fd'' \arrow[r, "f''"'] & Ge'' \end{tikzcd}$$

$\Rightarrow (Fk_2)(Fk_1) \cdot f = Fk_2(f') \cdot Fh_1 = f'' Fh_2 Fh_1$

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "F(h_2h_1)"'] & Ge \arrow[d, "Fk_2k_1"] \\ Fd'' \arrow[r, "f''"'] & Ge'' \end{tikzcd}$$

$\Rightarrow f'' F(h_2h_1) = F(k_2k_1) f$

Functor: $\mathrm{dom}: F \downarrow G \longrightarrow D$, $\mathrm{cod}: F\downarrow G \longrightarrow E$

$$\begin{tikzcd} (d,e,f) \arrow[d, "{(h,k)}"'] \arrow[r, mapsto] & d \arrow[d, "h"] \\ (d',e',f') \arrow[r, mapsto] & d' \end{tikzcd} \qquad \begin{tikzcd} (d,e,f) \arrow[d, "{(h,k)}"'] \arrow[r, mapsto] & e \arrow[d, "k"] \\ (d',e',f') \arrow[r, mapsto] & e' \end{tikzcd}$$

Ex 1.3.vii) $D = (\{\cdot\}, 1_\cdot)$ $F: D \longrightarrow C$ $G = 1_C$
$E = C$ $\qquad \cdot \longmapsto c$

c/C : under

Obj $= (\cdot, x, f: c \longrightarrow x)$

Mor: $((\cdot, x, f: c\longrightarrow x), (\cdot, y, g: c \longrightarrow y))$

$= \{ (1_\cdot, h: x \longrightarrow y) \in \mathrm{Mor}\, D \times \mathrm{Mor}\, E$

s.t.

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "1_c"'] \arrow[dr, phantom, "\circlearrowright"] & x \arrow[d, "Gh = h"] \\ c \arrow[r, "g"'] & y \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\circlearrowright"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

C/c : over

$D = C$ $\qquad F = 1_C$ $\quad G = \cdot \longmapsto c$
$E = (\{\cdot\}, 1_\cdot)$

Obj : $(x, \cdot, f: x \longrightarrow c)$

Mor $((x, \cdot, f: x \rightarrow c), (y, \cdot, g: y \rightarrow c))$

$= \{ (h: x \rightarrow y, 1_\cdot) \in \mathrm{Mor}\, D \times \mathrm{Mor}\, E$

s.t.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & c \arrow[d, "1_c"] \\ y \arrow[r, "g"'] & c \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} & c & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & \circlearrowright & y \arrow[ul, "g"'] \end{tikzcd}$$

projection functor:

$$\mathrm{dom}: {}^{c}/_{C} \longrightarrow D = (\{\cdot\}, 1) \qquad \mathrm{Cod}: {}^{c}/_{C} \longrightarrow E = C$$

$$\begin{array}{ccc} (\cdot, x, f: c \to x) & \longmapsto & \cdot \\ (1, h)\downarrow & & \downarrow 1. \\ (\cdot, y, g: c \to y) & \longmapsto & \cdot \end{array} \qquad \begin{array}{ccc} (\cdot, x, f: c \to x) & \longmapsto & x \\ (1, h)\downarrow & \longmapsto & \downarrow h. \\ (\cdot, y, g: c \to y) & \longmapsto & y \end{array}$$

Thus: $\mathrm{Cod}: {}^{c}/_{C} \longrightarrow C$ $\qquad$ $\mathrm{dom}\ {}^{C}/_{c} \longrightarrow C.$

$$\begin{array}{ccc} f: c \to x & \longmapsto & x \\ h\downarrow & \longmapsto & \downarrow h. \\ g: c \to y & \longmapsto & y \end{array} \qquad \begin{array}{ccc} f: x \to c & \longmapsto & x \\ \downarrow h & \longmapsto & \downarrow h. \\ g: y \to c & \longmapsto & y \end{array}$$

Similarly., $\mathrm{dom}\ {}^{C}/_{c} \longrightarrow D = C.$ $\qquad$ $\mathrm{Cod}: {}^{C}/_{c} \longrightarrow E = (\{\cdot\}, 1)$

$$\begin{array}{ccc} (x, \cdot, f: x \to c) & \longmapsto & x \\ (h, 1)\downarrow & \longmapsto & \downarrow h. \\ (y, \cdot, g: y \to c) & \longmapsto & y \end{array} \qquad \begin{array}{ccc} (x, \cdot, f: x \to c) & \longmapsto & \cdot \\ (h, 1)\downarrow & \longmapsto & \downarrow 1. \\ (y, \cdot, g: y \to c) & \longmapsto & \cdot \end{array}$$

Ex 1.3. viii) Ex: Functor need not reflect isomorphism

Find $F: C \longrightarrow D$, $f \in \mathrm{Mor}(C)$ s.t. Ff is iso in D

f is not iso in C.

$$\underset{C}{\cdot \xrightarrow{f} \cdot} \qquad\qquad \underset{D.}{\cdot \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \cdot}$$

ex) Homology
Homotopy
Functor:
(Quasi-iso
but not homeo)

More generically, $1 = (\{\cdot\}, 1_\cdot)$

$C \longrightarrow 1$ by $c \longmapsto \cdot$ $\quad \forall f, \forall c.$

$f \longmapsto 1.$

Ex 1.3. ix)

Source \ Map	$Group_{iso.}$	$Group_{epi}$	Group
$Z(-)$	Yes. (iso)	Yes	No ①
$C(-)$	Yes	Yes	Yes 2)
$Aut(-)$	Yes	Don't know.	No 3)

1). If $G \overset{f}{\cong} H \Rightarrow Z(G) \cong Z(H)$

pf)

Claim 1: $G \overset{f}{\twoheadrightarrow} H$ surj, then $f(Z(G)) \leq Z(H)$

pf) $h f(g) = f(h'g) = f(gh') = f(g)h$. $\forall h \in H$,

$g \in Z(G)$, h' : preimage of h.

Claim 2 : $G \overset{\cong f}{\hookrightarrow\!\!\twoheadrightarrow} H$ then $f(Z(G)) = Z(H)$

pf)

If $h \in Z(H)$, $\exists\, g \in G$ s.t. $f(g) = h$

$g'g = f^{-1}(h'h) = f^{-1}(hh') = f^{-1}(h) f^{-1}(h') = gg'$.

$\Rightarrow g \in Z(G) \Rightarrow f(Z(G)) \supseteq Z(H)$. done.

Claim 3 : $Z(-)$: $Group_{iso}$, $Group_{epi}$ $\longrightarrow$ Group is functorial.

pf)
$$\begin{array}{ccc} G & \xrightarrow{Z(-)} & Z(G) \\ \downarrow 1_G & & \downarrow Z(1_G) \\ G & \longrightarrow & Z(G) \end{array}$$
, Let $G \xrightarrow{\phi} H \xrightarrow{\psi} K$ iso (or epi)

Then, from $\phi(Z(G)) \subseteq Z(H)$,

$\psi(Z(H)) \subseteq Z(K)$

we have map

$$Z(G) \xrightarrow{\phi|_{Z(G)}} Z(H) \xrightarrow{\psi|_{Z(H)}} Z(K)$$

Thus, $\psi|_{Z(H)} \circ \phi|_{Z(G)} = (\psi \circ \phi)|_{Z(G)}$

implies ϕ first functoriality axiom.

Claim 4: $Z(-)$ Group $\to$ Group is not a functor.

pf) $1_{C_2} = C_2 \longrightarrow S_3 \longrightarrow S_3/A_3 \cong C_2$

$1 \longmapsto (12) \longmapsto (12)+A_3 \mapsto 1$

Where C_2: cyclic group of order 2,

S_3: Sym. " 3.

A_3: alternating gp of S_3.

Then, If Z is a functor,

$Z(C_2) \longrightarrow Z(S_3) \longrightarrow Z(C_2) = Z(C_2) \xrightarrow{1_{Z(C_2)}} Z(C_2)$

$= C_2 \longrightarrow 0 \longrightarrow C_2$, contradiction! $C_2 \xrightarrow{1_{C_2}} C_2$

2). Claim 1: $G \xrightarrow{f} H \Rightarrow C(f(G)) = f(C(G)) \le C(H)$

pf) $C(f(G)) = \{ f(x) f(y) f(x)^{-1} f(y)^{-1} : x, y \in G \}$

$= \{ f(xyx^{-1}y^{-1}) : x, y \in G \}$

$= f(C(G)) \subseteq \{ xyx^{-1}y^{-1} : x, y \in H \}$

$= C(H)$

Claim 2: $C(-)$: $\mathrm{Group}_{Iso} \longrightarrow \mathrm{Group}$ are

$\mathrm{Group}_{epi} \longrightarrow$

$\mathrm{Group} \longrightarrow \mathrm{Group}$

functors

pf). $G \xrightarrow{f} H \Rightarrow C(H) \supseteq f(C(G))$

$\Rightarrow C(G) \xrightarrow{f|_{C(G)}} C(H)$ is gp homo.

If $f = 1_G \Rightarrow C(f) = 1_{C(G)}$

$G \xrightarrow{\phi} H \xrightarrow{\varphi} K$

$\Rightarrow C(G) \xrightarrow{\phi|_{C(G)}} C(H) \xrightarrow{\varphi|_{C(H)}} C(K) = (\varphi \circ \phi)|_{C(G)}$

3)

Claim 1. $\mathrm{Group}_{epi} \xrightarrow{Aut} \mathrm{Group}$ is not a functor

$\mathrm{Group} \longrightarrow \mathrm{Group}$

pf) $G = \mathbb{F}_{11} \rtimes \mathbb{F}_{11}^{\times}$ by multiplication

$G = \{ (a,b) : a \in \mathbb{F}_{11},\ b \in \mathbb{F}_{11}^{\times} \}$

and $(a,b)\cdot(c,d) = (a+bc, bd)$

Also, let $N = \{ (a,1) : a \in k \} \cong \mathbb{F}_{11}$

$K = \{ (0,b) : b \in k^{\times} \} \cong \mathbb{F}_{11}^{\times}$

N, K are sub gp of G.

and by construction N is normal;

$$\Big((b,c)\,(a,1)\,(b,c)^{-1} = (b+ca, c)(b, c^{-1})$$

$$= (b+ca-bc, 1) \in N \Big)$$

$\Rightarrow$ $NK = G$. (by construction)

By Schur - Zassenhaus lemma,

Any subgp of G with order $|K|$

Conjugate to each other, i.e, if K_1, K_2 has order $|K|$, then $\exists g \in G$ s.t

$g K_1 g^{-1} = K_2$

$(a,b)(c,d) = (0,1)$
$= (a+bc, bd)$.

Notes that $(0,1)$ is identity in G.

and $(a,b)^{-1} = \left(-\frac{a}{b}, b^{-1}\right)$

Hence let $\alpha \in Aut(G)$

$\Rightarrow$ $\alpha(K)$ is a subgp of G with order $|K|$

$\Rightarrow$ $\exists g \in G$ s.t.

$$g\alpha(K)g^{-1} = K$$

$\Rightarrow$ Let $\varphi_g \in Inn(G)$ s.t. $\varphi_g(g') = g g' g^{-1}$

$\Rightarrow$ $\varphi_g \circ \alpha$ fix K

Also, Notes that $\varphi_g \circ \alpha$ fix N since N is unique 11-Sylow sub gp of G.

Thus, if $\varphi_g \circ \alpha(1,1) = (a,1)$

then, $a \neq 0$ since $\varphi_g \circ \alpha^{-1}(0,1) = (0,1)$ (identity)

thus $\exists b \in \mathbb{F}_{11}^{\times}$ s.t. $ba = 1$

$\therefore$ $(0,b)(a,1)(0,b^{-1}) = (ab,b)(0,b^{-1}) = (ab,1) = (1,1)$

$\therefore$ $\varphi_b \varphi_g \circ \alpha(1,1) = (1,1)$

And since $(0,b)(0,c)(0,b^{-1}) = (0,c)$,

φ_b still fix K.

Let $\beta = \varphi_b \circ \varphi_g \circ \alpha$.

Then, $\beta(K) = K$, $\beta((1,1)) = (1,1)$.

and β is auto. ($\beta(N) = N$)

If $a \neq 0$ in $\mathbb{F}_{11}$

$\Rightarrow$ $\beta((a,1)) = \beta((0,a)(1,1)(0,a^{-1}))$

$= (\beta(0,a))(1,1)(\beta(0,a))^{-1}$

Since β fix N, $\beta(a,1) = (b,1)$ for some $b \in \mathbb{F}_{11}^{\times}$

And β fix K $\Rightarrow$ $\beta(0,a) = (0,c)$. $c \in \mathbb{F}_{11}^{\times}$

Thus, $(b,1) = (0,c)(1,1)(0,c^{-1})$

$= (c,c)(0,c^{-1}) = (c,1)$

$\Rightarrow$ $b = c$. And

$\beta(0,a) = (0,c) = (0,b)$

Thus, $\gamma: \mathbb{F}_{11} \longrightarrow N \xrightarrow{\beta} N \longrightarrow \mathbb{F}_{11}$

$a \longmapsto (a,1) \longmapsto (b,1) \longrightarrow b$.

$0 \longmapsto (0,1) \longmapsto (0,1) \longmapsto 0$.

induces a map on $\mathbb{F}_{11}$ (we didn't show it is homo)

similarly, $\mathbb{F}_{11}^{\times} \longrightarrow K \xrightarrow{\beta} K \longrightarrow \mathbb{F}_{11}^{\times}$

$a \longmapsto (0,a) \longmapsto (0,b) \longrightarrow b$.

We claim that γ is field homo.

$\sigma(a)\sigma(b) = (\sigma(a), 1)\cdot(\sigma(b), 1)$ in G

$(\;\; , \;\;) = (0, \sigma(a))(\sigma(b), 1)\,(0, \sigma(a)^{-1})$

$= \beta(0, a)\,\beta(b, 1)\,\beta(0, a^{-1})$

since $\sigma(a) = ! \; a \longmapsto (0, a) \longmapsto \beta(0, a)$

$= (0, c)$

for some c

$\longmapsto c$

thus $\beta(0, a) = (0, c) = (0, \sigma(a))$

and $b \longmapsto (b, 1) \longmapsto \beta(b, 1) = (c, 1)$

for some c

$\Rightarrow \sigma(b) = c$

$\Rightarrow \beta(b, 1) = (c, 1) = (\sigma(b), 1)$

$= \beta(ab, 1) = (c, 1)$ for some $c \in \mathbb{F}_{11}$

$\Rightarrow \sigma(a)\sigma(b) = c$

and

$ab \longmapsto (ab, 1) \longmapsto \beta(ab, 1) = (c, 1)$

$\Rightarrow \sigma(ab) = c$

Hence $\sigma(ab) = \sigma(a)\sigma(b)$. Thus β induces a field homomorphism on $\mathbb{F}_{11}$.

Thus $\mathrm{Aut}(G) \cong \mathrm{Inn}(G) \cdot \mathrm{Aut}(\mathbb{F}_{11})$

$$\cong \mathrm{Inn}(G)$$
$$\cong G.$$

Since any automorphism of G is field homomorphism times inner automorphism

and $\mathrm{Aut}(\mathbb{F}_{11}) = ①$ since there are only $\mathbb{F}_{11}^\times$ gp homomorphisms and only 1 homomorphism fix $1 \mapsto 1$.

Now Think about a map.

$\phi : \mathbb{F}_{11}^\times \longrightarrow G \longrightarrow G/N \cong \mathbb{F}_{11}^\times$! isomorphism.

$\mathrm{Aut}(-)$ induces $\mathrm{Aut}(\mathbb{F}_{11}^\times) \to \mathrm{Aut}(G) \to \mathrm{Aut}(\mathbb{F}_{11}^\times)$

But $|\mathrm{Aut}(\mathbb{F}_{11}^\times)| = (\mathbb{Z}/4\mathbb{Z}) = 4$ and $(4, 110) = 1$
$|\mathrm{Aut}(G)| = |G| = 110.$

$\Longrightarrow \mathrm{Aut}(\mathbb{F}_{11}^\times) \to \mathrm{Aut}(G)$ is zero.

But $\phi = 1_{\mathbb{F}_{11}^\times}$, thus

$\mathrm{Aut}(\phi) = 1_{\mathrm{Aut}(\mathbb{F}_{11}^\times)} \neq 0$. So Aut doesn't satisfy functoriality axiom.

1.3.X) Let G, H gp, $f: G \to H$ homo,

X_G, X_H, set of conjugacy classes of G, H.

$g: G \to X_G$, $h: H \to X_H$ class function

i.e. $g(a) = g(bab^{-1})$ $\forall b \in G$.

Let $\text{Conj}: \text{Group} \to \text{Set}$

$$\begin{tikzcd} G \arrow[r, mapsto] \arrow[d, "\phi"'] & X_G \arrow[d, "\text{Conj}(\phi):"] \\ H \arrow[r, mapsto] & X_H \end{tikzcd} \qquad \begin{matrix} \bar{g} \\ \downarrow \\ \bar{h} \end{matrix}$$

① $\text{Conj}(\phi)$ is well-defined;

if $g' \in \bar{g}$, then, $g' = bgb^{-1}$ for some $b \in G$.

Thus, $\phi(g') = \phi(b)\phi(g)\phi(b)^{-1} \Rightarrow \phi(g') \in \overline{\phi(g)}$.

② It is functorial; If $\psi: H \to J$ morphism,

$\text{Conj}(\psi) \cdot \text{Conj}(\phi) = \text{Conj}(\psi \circ \phi)$

and $\text{Conj}\left(1_G : G \to G\right) = I_{X_G}$.

Thus, if $|X_G| \neq |X_H|$, then $\text{Conj}(\phi)$ is not isomorphism, since function preserves iso, $\phi: G \to H$ is not iso.

Exercise 1.4. i.

Since α is natural iso, $\forall c \in C$,

$\exists\ \alpha_c^{-1}$ s.t. $\alpha_c \circ \alpha_c^{-1} = 1_{Gc}$

$\alpha_c^{-1} \circ \alpha_c = 1_{Fc}$

Thus, let α^{-1}: collection of α_c^{-1}. Then,

$\forall f: c \longrightarrow c'$ WTS

$$\begin{tikzcd} Gc \arrow[r, "\alpha_c^{-1}"] \arrow[d, "Gf"'] & Fc \arrow[d, "Ff"] \\ Gc' \arrow[r, "\alpha_{c'}^{-1}"'] & Fc' \end{tikzcd}$$

$Ff \circ \alpha_c^{-1} = \alpha_{c'}^{-1} \circ Gf$

Since $\alpha_{c'} \circ Ff = Gf \circ \alpha_c$,

$\Rightarrow \alpha_{c'} \circ Ff \circ \alpha_c^{-1} = Gf \circ 1_{Gc}$

$\Rightarrow 1_{Fc'} \circ Ff \circ \alpha_c^{-1} = \alpha_{c'}^{-1} \circ Gf$.

1.4. ii. If $\phi: G \longrightarrow H$ be a functor,

then, $\phi(e_G) = e_H$. $\phi(g) \cdot \phi(g') = \phi(gg')$

Thus, ϕ is any homomorphism

If $\phi, \psi : G \rightrightarrows H$, then, if $\phi \overset{\alpha}{\Rightarrow} \psi$ exists,

① $\alpha \in H$

$\forall g \in G$,

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] \arrow[d, "\phi(g)"'] & \cdot \arrow[d, "\psi(g)"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

② $\psi(g)\alpha = \alpha\phi(g)$

$\Rightarrow \alpha^{-1}\psi(g)\alpha = \phi(g)$.

Hence, $\alpha \in \mathrm{Inn}(M)$ s.t. $\alpha \circ \psi = \phi$.

Ex 1.4. iii) $(P, \leq) \overset{F}{\underset{G}{\rightrightarrows}} (Q, \leq)$,

then F, G are order preserving function.

If $F \overset{\alpha}{\Rightarrow} G$, then $\forall f: p \to p'$
$(p \leq p')$

$$\begin{tikzcd} F_p \arrow[r, "\alpha_p"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & G_p \arrow[d] \\ F_{p'} \arrow[r, "\alpha_{p'}"'] & G_{p'} \end{tikzcd}$$

i.e. $G_{p'} \geq F_{p'} \geq F_p$

iff $G_{p'} \geq G_p \geq F_p$.

$\Rightarrow G_p \geq F_p \quad \forall p$.

Thus, natural transformation of F, G order preserving function is $F \leq G$.

i.e. $\forall p \in P, \quad F_p \leq G_p$.

(Notes that $F \leq G \Leftrightarrow$ Natural transf hold.)

1.4. iv)

Ex 1.4 iv) Each $f_* \; g_*$, f^*, g^* defines n.t. by Ex 1.4.7.

If $f_* = g_*$ $\Rightarrow$ for $h : y \to x$.

$$\begin{tikzcd} C(x, c) \arrow[r, "f_*"] \arrow[d, "-\cdot h"'] \arrow[dr, phantom, "\circlearrowright"] & C(x, d) \arrow[d, "-\cdot h"] \\ C(y, c) \arrow[r, "f_*"'] & C(y, d) \end{tikzcd}$$

$\Rightarrow \forall l \in C(x, c)$. $l \longmapsto f l h$

$f l h = g l h$. Now pick $h = 1_x$

$\Rightarrow$ $f l = g l$, Pick $x = c$, $l = 1_c$.

$\Rightarrow f = g$, contradiction $\therefore f_* \neq g_*$ as natural transformation. (f^*, g^*: similar manner)

Ex 1.4 v.

$$\begin{array}{ccccc} F \downarrow G & \xrightarrow{\text{dom}} & D & \xrightarrow{F} & C \\ (d, e, f) & \longmapsto & d & \longrightarrow & Fd \\ (h,k) \downarrow & \longmapsto & \downarrow h & & \downarrow Fh \\ (d', e', f') & \longmapsto & d' & \longrightarrow & Fd' \end{array}$$

$$
\begin{tikzcd}
F \downarrow G \arrow[r, "\mathrm{cod}"] & E \arrow[r, "G"] & C \\
(d,e,f) \arrow[r, mapsto] \arrow[d, "{(h,k)}"'] & e \arrow[r, mapsto] \arrow[d, "k"] & Ge \arrow[d, "Gk"] \\
(d',e',f') \arrow[r, mapsto] & e' \arrow[r, mapsto] & Ge'
\end{tikzcd}
$$

Thus, For each $(d,e,f) \xrightarrow{(h,k)} (d',e',f')$

we need to find $\alpha_{(d,e,f)}$, $\alpha_{(d',e',f')}$

s.t.

$$
\begin{tikzcd}
Fd \arrow[r, "\alpha_{(d,e,f)}"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\circlearrowright"] & Ge \arrow[d, "Gk"] \\
Fd' \arrow[r, "\alpha_{(d',e',f')}"'] & Ge'
\end{tikzcd}
$$

Commute,

Set $\alpha_{(d,e,f)} = f$. By construction of $F \downarrow G$, it holds.

Ex 1.4 vi) If F, G has different target category

i.e.) $F : A \times B \times B^{op} \longrightarrow D$

$G : A \times C \times C^{op} \longrightarrow D'$ with $D \neq D'$.

then, we ~~don't know how to~~ cannot have a morphism $F(a,b,b) \to G(a,c,c)$ in general

$$
\begin{tikzcd}
F\downarrow G \arrow[r, "\mathrm{cod}"] & E \arrow[r, "G"] & C \\
(d,e,f) \arrow[r, mapsto] \arrow[d, "{(h,k)}"'] & e \arrow[r, mapsto] \arrow[d, "k"] & Ge \arrow[d, "Gk"] \\
(d',e',f') \arrow[r, mapsto] & e' \arrow[r, mapsto] & Ge'
\end{tikzcd}
$$

Thus, For each $(d,e,f) \xrightarrow{(h,k)} (d',e',f')$

we need to find $\alpha_{(d,e,f)}$, $\alpha_{(d',e',f')}$

s.t.

$$
\begin{tikzcd}
Fd \arrow[r, "\alpha_{(d,e,f)}"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\circlearrowright"] & Ge \arrow[d, "Gk"] \\
Fd' \arrow[r, "\alpha_{(d',e',f')}"'] & Ge'
\end{tikzcd}
$$

Commute,

Set $\alpha_{(d,e,f)} = f$. By construction of $F\downarrow G$, it holds.

Ex 1.4 vi) If F, G has different target category,

i.e.) $F: A \times B \times B^{op} \to D$ with $D \neq D'$.

$G: A \times C \times C^{op} \to D'$

then, we ~~don't know how to~~ Cannot have a morphism $F(a,b,b) \to G(a,c,c)$ in general