1. Euler Circuit

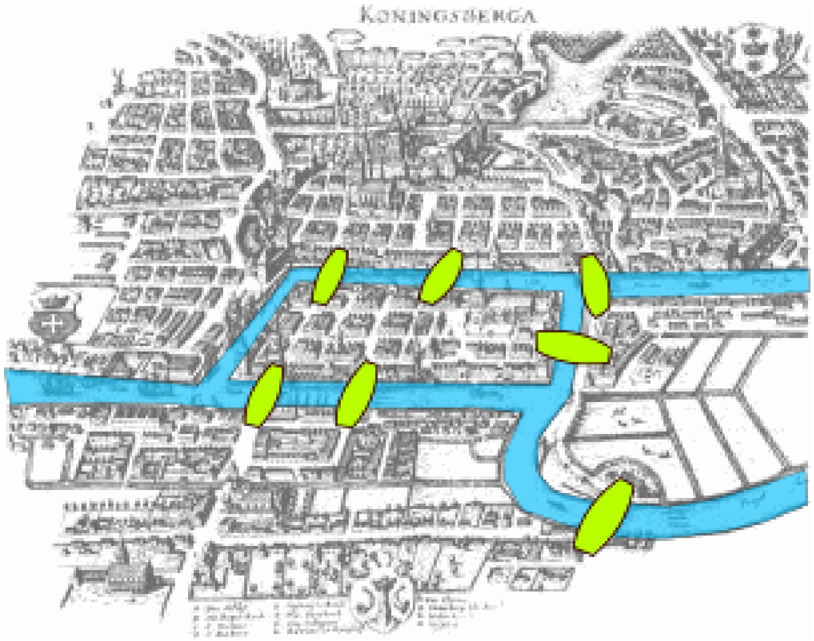


Figure 1. A map of Königsberg in Euler's time, Wikipedia, Bogdan Giusca

Can you cross each of those bridges once and only once?

Definition 1.1. A ________________ is is a collection of one or more points (called ________________) and the connections between them (called ________________)

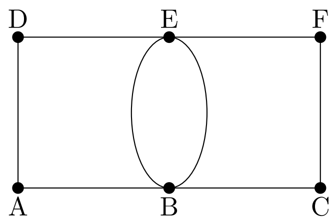
Some examples are:

Example 1.2. Draw a graph of the interstate highway connections between Oklahoma City, Dallas, Shreveport, Austin, San Antonio, Houston and Baton Rouge.

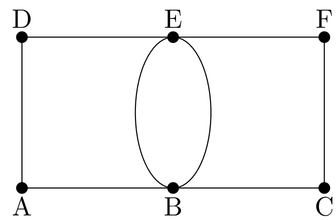
Two vertices are ________________ if they are connected by an edge. What are the pairs of adjacent vertices in the graph above?

Definition 1.3. A _________________ is a route that passes from a vertex to an adjacent vertex with each vertex used being adjacent to the next vertex.

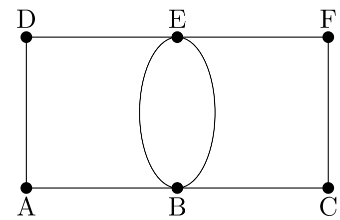
What are some paths that go from A to F?



Path



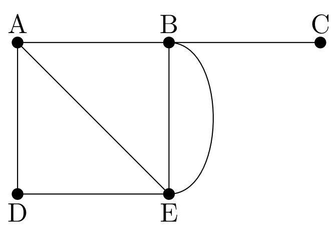
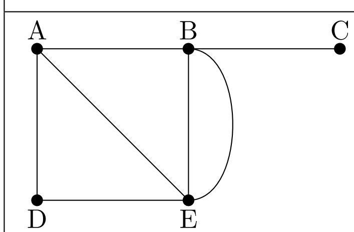
Path

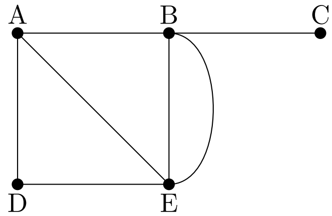
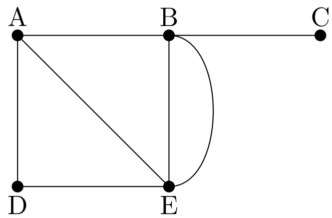
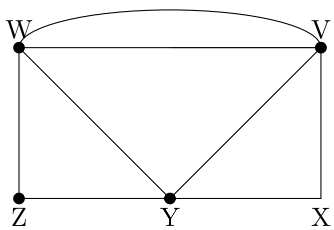
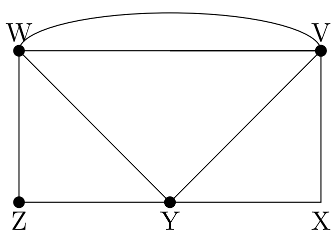
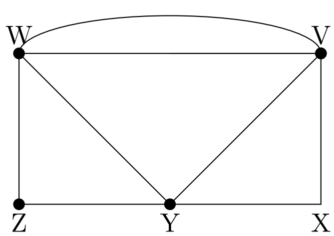


Path

Definition 1.4.
Using an edge more than once is called _________________
A path that uses every **edge** exactly once is an _________________
A path that ends at the same vertex it started from is a _________________
A circuit that uses every **edge** exactly once is an_________________

Example 1.5. Classify the following sequences of vertices for the graphs below.

Graph and sequence	Path?	Circuit?	Euler Path?	Euler Circuit?	Deadheaded Edges?
 Path:AEBC	Y/N	Y/N	Y/N	Y/N	Y/N
 Path:EDAEBE	Y/N	Y/N	Y/N	Y/N	Y/N

Graph and sequence	Path?	Circuit?	Euler Path?	Euler Circuit?	Deadheaded Edges?
 Path:ADEABEBC	Y/N	Y/N	Y/N	Y/N	Y/N
 Path:DEABEA	Y/N	Y/N	Y/N	Y/N	Y/N
 Path:WVYZ	Y/N	Y/N	Y/N	Y/N	Y/N
 Path:ZYWX	Y/N	Y/N	Y/N	Y/N	Y/N
 Path:YWVXYZWVY	Y/N	Y/N	Y/N	Y/N	Y/N

2. Finding Euler Circuits

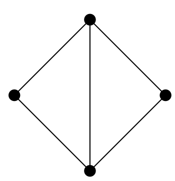
Definition 2.1.
The ________________ (or ________________) of a vertex is the number of edges at that vertex.
If this number is odd, the vertex is called ________________.
If this number is even for all vertices, then the graph is called ________________.
A ________________ is an edge that connects a vertex to itself. A loop counts twice towards the degree of a vertex.
A ________________ graph contains no loops.

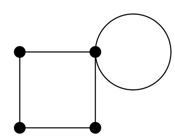
If d is the sum of the degrees of all vertices in a graph and e is the number of edges in the graph, then

$$d = 2e.$$

Notes: A loop counts twice toward the degree of a vertex.

Example 2.2. Find the valence of each vertex in each graph, and show that d = 2e.

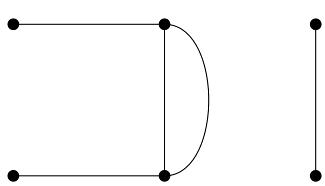


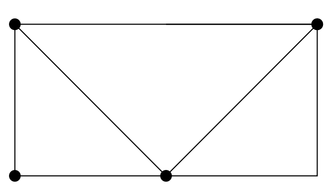


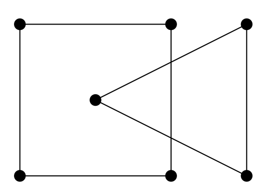
Definition 2.3.
A graph is ________________ if for every pair of vertices there is a path that connects them.
If a graph is not connected, its parts are called ________________

Example 2.4. Check the connectivity of below graphs.

Graphs:







Connected? Y/N Y/N Y/N

Theorem 2.5 (Euler's Theorem for a connected graph)**.**

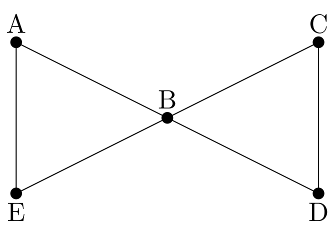
(1) If the graph has no vertices of odd degree, then it has at least one Euler ________________ and if a graph has an Euler ________________, then it has no vertices of odd degree.

(2) If a graph has exactly 2 vertices of odd degree, then there is at least one Euler ________________, but no Euler ________________. Any Euler ________________ in such a graph must start at a vertex with an odd degree and end at the other vertex of odd degree.

(3) If the graph has more than two vertices of odd degree, then it does not have an Euler ________________.

Example 2.6. Determine whether the following have an Euler circuit, an Euler path but not an Euler circuit, or neither an Euler path nor Euler circuit. Show the Euler path or Euler circuit if it exists.

Some advice for finding an Euler circuit: Never use an edge that is the only link between two parts of a graph that still need to be covered.

<table>
<tr><th>Graph and sequence</th><th>Euler Circuit?</th><th>Euler Path (but not Euler circuit)?</th></tr>
<tr><td>
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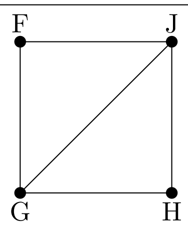
<tr><td>
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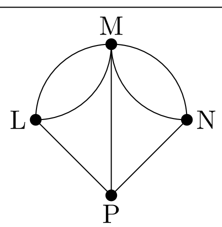
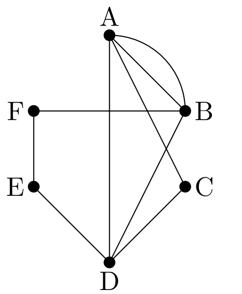
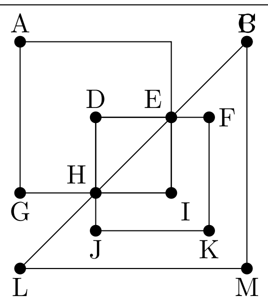
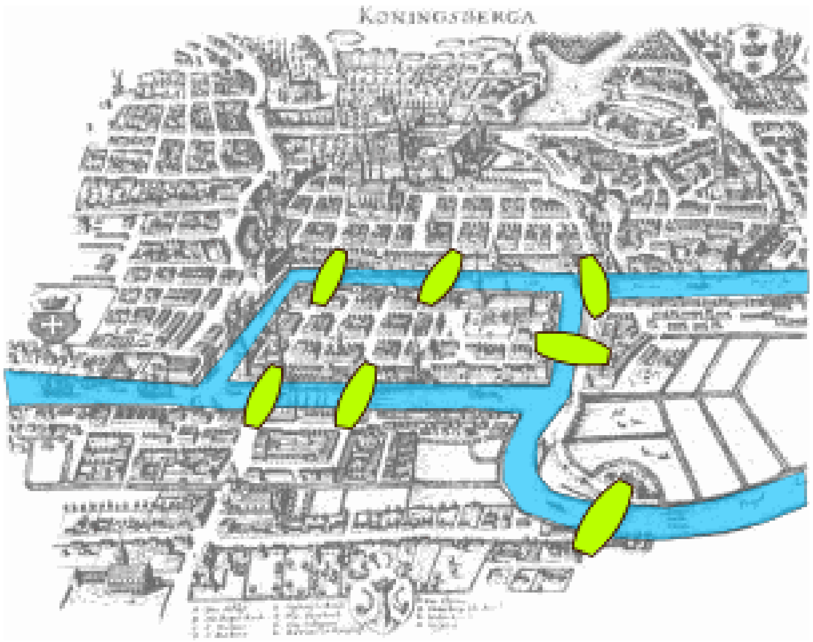
<tr><td>
</td><td></td><td></td></tr>
</table>

Example 2.7. Find an Euler circuit, if one exists, for the graphs below.

Graph and sequence	Euler Circuit?	Euler Path (but not Euler circuit)?
		
		

Example 2.8. Revisit our Königsberg bridges.



Can you make a graph from the above picture? You may think land as a vertex, and bridge as an edge. Draw the graph from the picture.

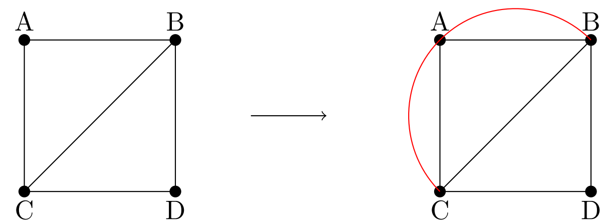
Can you cross each bridge exactly once and return to your starting spot? Explain your answer.

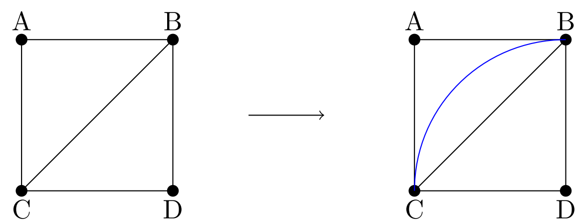
Can you cross each bridge exactly once if you do not have to return to your starting spot? Explain your answer.

3. Beyond Euler Circuits

Definition 3.1. The ____________________________ Problem works to answer how we can cover all edges with a minimum length circuit. If the graph has an Euler circuit, then that is the minimum length circuit. What is the minimum length circuit if there is not an Euler circuit?

If the graph has vertices with odd valence, **repeat edges** in such a way that there are no odd-valent vertices. This **Eulerizes** the graph. (Note that you cannot connect two vertices that were not directly connected before.)

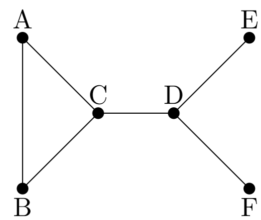




After finding the Euler circuit on the new graph you may squeeze the new graph onto the old graph by indicating where an edge is used more than once. Or you can simply dot in the extra edges to indicate where an edge is to be used more than once

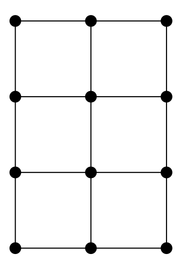
Example 3.2. In the above two Eulerizations, which one gives the Euler circuit with shorter length?

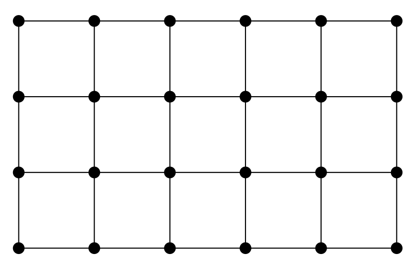
Example 3.3. Eulerize this graph.



Definition 3.4. A network is _________________ if the network consists of a series of rectangular blocks that form a larger rectangle. A rectangular network can be Eulerized by using an "_________________" to walk around the outer boundary of the large rectangle and dot in an edge to each odd-valent vertex that connects to the next vertex so that path can be used more than once.

Example 3.5. Eulerize the following rectangular graphs using an edge walker.





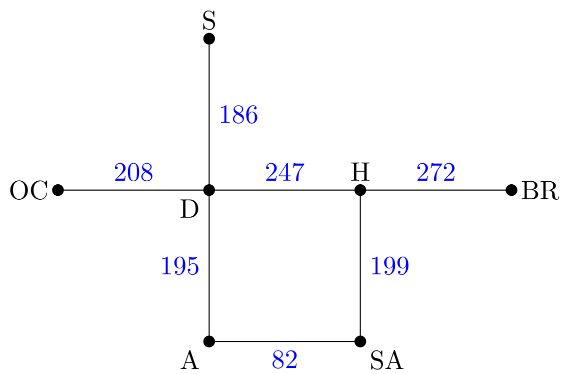
4. Urban Graph Traversal Problems

Practical applications of Euler circuits and eulerizing graphs arise when one is trying to make a job more efficient that requires checking every segment in a situation. For example, collecting garbage, checking water lines, inspecting transportation systems, et cetera.

Sometimes this efficiency requires looking at how many passes over a street is required. For example, delivering mail might require going down both sides of a street (even if it is a one-way street), but a police patrol just needs to go down a street once while looking at both sides.

Different edges can also have different "**costs**" which will affect efficiency.

Example 4.1. How can you get from Baton Rouge to Austin? Which way is more efficient with respect to mileage?



Example 4.2. Eulerize the graph below at a minimum cost (costs are in minutes).