Monomial ideals in an affine semigroup rings

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Contents

Monomial ideals in an Affine Semigroup Ring

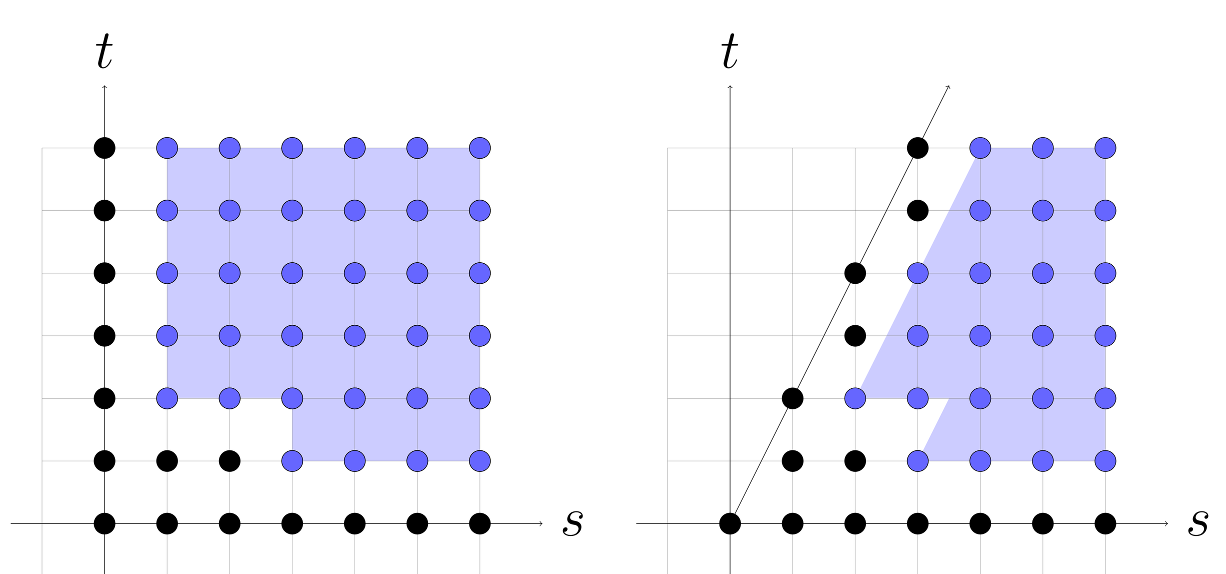
A *monomial* denotes a polynomial with one term over a field $\mathbb{K}$. A *monomial ideal* is an ideal generated by monomials. These are rich subject for combinatorial study. Thus, it is natural to generalize them to combinatorial friendly ring, called an *affine semigroup ring*.

Definition

- Set $A := \{a_1, \cdots, a_n\} \subset \mathbb{Z}^d$ and identify it as a $d \times n$ matrix; then $\mathbb{N}A := \{A \cdot u : u \in \mathbb{N}^n\}$ form a monoid, called an *affine semigroup*.
- $\mathbb{K} = [\mathbb{N}A] := \mathbb{K}[t^{a_1}, \cdots, t^{a_n}]$ is a subring of the Laurent polynomial ring $\mathbb{K}[t_1^{\pm}, \cdots, t_d^{\pm}]$ is called an *affine semigroup ring*.
- $\mathbb{K}[\mathbb{N}A]$ is $\mathbb{Z}^d$-graded by setting $\deg(t^a) = a$. A *monomial* ideal is a homogeneous ideal in $\mathbb{K}[\mathbb{N}A]$.

(a):Canonical Example: Polynomial ring
Let A be a standard basis of $\mathbb{Z}^d$. Then $\mathbb{N}A = \mathbb{N}^d$. $\mathbb{K}[\mathbb{N}A] = \mathbb{K}[x_1, \cdots, x_n]$, graded by $\deg(x_i) = e_i$.

(b):Nontrivial Example
$A = \{(1,0)^t, (1,1)^t, (1,2)^t\}$, $\mathbb{K}[\mathbb{N}A] = \mathbb{K}[s, st, st^2]$, and $I = \langle s^2t^2, s^3t\rangle$ (Figure)



(a):$\mathbb{K}[x, y]$ $I = \langle x^3y^1, xy^2\rangle$

$(b) : \mathbb{K}[s, st, st^2]$ $I = \langle s^2t^2, s^3t\rangle$

Figure 1: Affine semigroup rings and monomial ideals.

Motivation of Standard Pairs

Facts for a monomial ideal $I \subset \mathbb{K}[\mathbb{N}A]$ [2,3]

- A monomial prime ideal $\overset{1\text{-}1}{\leftrightarrow}$ A face of $\mathbb{R}_{\geq 0}A$.
- Irreducible decomposition exists.
- If $\mathbb{K}[\mathbb{N}A]$ is *normal*, $\exists$ an algorithmic irreducible decomposition and irreducible resolution.

$\mathbb{K}[\mathbb{N}A]$ is *normal* if $\mathbb{N}A = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}A$.
Standard pairs can be used to generalize the above to the nonnormal case.

Standard Pairs

Definition

- F: *face* of A if $F = A \cap H$ for a face H of $\mathbb{R}_{\geq 0}A$.
- (a, F) : *proper pair* if $(a + \mathbb{N}F) \cap I = \emptyset$.
- $(a, F) < (b, G)$ if $a + \mathbb{N}F \subseteq b + \mathbb{N}G$.
- (a, F) is *standard* if maximal w.r.t. $<$.
- (a, F) *divides* (b, G) if $\exists c \in \mathbb{N}A$ s.t. $a + c + \mathbb{N}F \subset b + \mathbb{N}G$
- (a, F) and (b, G) *overlap* if they divide each other. (Equivalence relation)

Main Theorem(Matusevich, Yu.) [1]

Given a monomial ideal I in $\mathbb{K}[\mathbb{N}A]$,

- I is *primary* iff all standard pairs of I correspond to a same face.
- I is *irreducible* iff primary + unique maximal overlap classes of the standard pairs w.r.t. divisibility.
- I has associate prime P_F iff I has a standard pair (a, F).

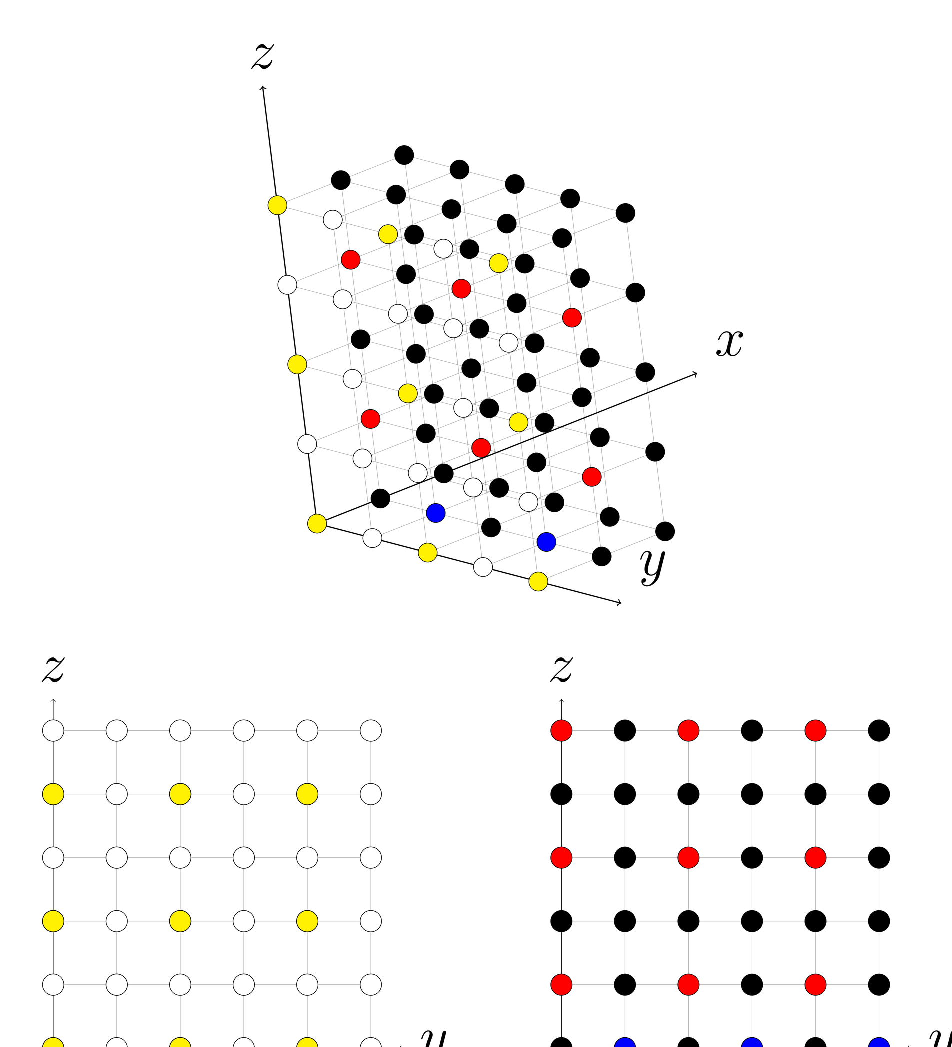
\# of maximal overlap classes of I = \# of components of an irreducible irredundant decomposition of I.

Example of a nonnormal case

$$A = \begin{bmatrix} 0&0&1&1&1&1 \\ 2&0&0&1&0&1 \\ 0&2&0&0&1&1 \end{bmatrix}, \begin{vmatrix} \mathbb{K}[\mathbb{N}A] \subset \mathbb{K}[x, y, z] \\ I = \langle x, xyz, xyz^2\rangle \end{vmatrix}, \text{ Then,}$$

$\mathbb{K}[x, y, z] \setminus \mathbb{K}[\mathbb{N}A] = x^{\{(a,b,0):a,b \in \mathbb{N}_{\text{odd}}\}}$.

Std Pairs: $(0, \{a_1, a_2\})$, $(a_5, \{a_1, a_2\})$, $(a_4, \{a_2\})$.



$x = 0$ $x = 1$

Figure 2: Standard pairs of $I = \langle x, xyz, xyz^2\rangle$

Example of an irr. decomposition

$$A = \begin{pmatrix} 1&1&1&1 \\ 0&1&2&3 \end{pmatrix}, \begin{vmatrix} \mathbb{K}[\mathbb{N}A] = \mathbb{K}[s, st, st^2, st^3] \\ I = \langle s^3, s^2t, s^2t^4\rangle \end{vmatrix}.$$

An irreducible decomposition is

$$I = \langle s, st\rangle \cap \langle s^2, s^2t^1, s^2t^2, s^2t^4, s^2t^5, s^2t^6\rangle \cap \langle st^3, s^2t, s^2\rangle \cap \langle st, st^2, st^3, s^3\rangle.$$

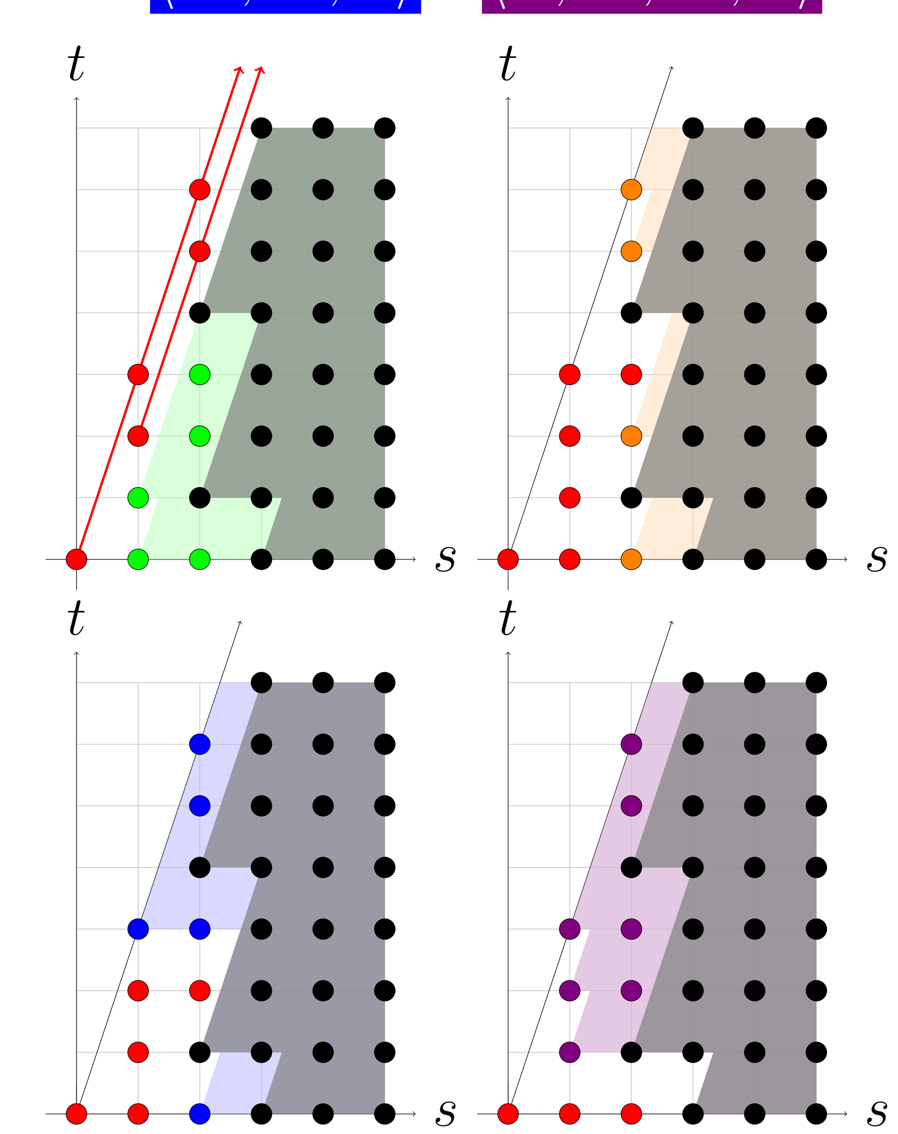


Figure 3: An irreducible decomposition

Example of the standard pairs

Given $A = \{(0,0,1)^t, (1,0,1)^t, (0,1,1)^t, (1,1,1)^t\}$, face $F = \{(0,0,1)^t, (0,1,1)^t\}$, and an ideal $I = \langle x^{(2,0,2)}, x^{(2,1,2)}, x^{(2,1,1)}\rangle$, $\exists$ three standard pairs

$(0, F)$, $(x^{(0,0,1)}, F)$, and $(x^{(0,1,1)}, F)$

Overlap happens for $(x^{(0,0,1)}, F)$ and $(x^{(0,1,1)}, F)$.

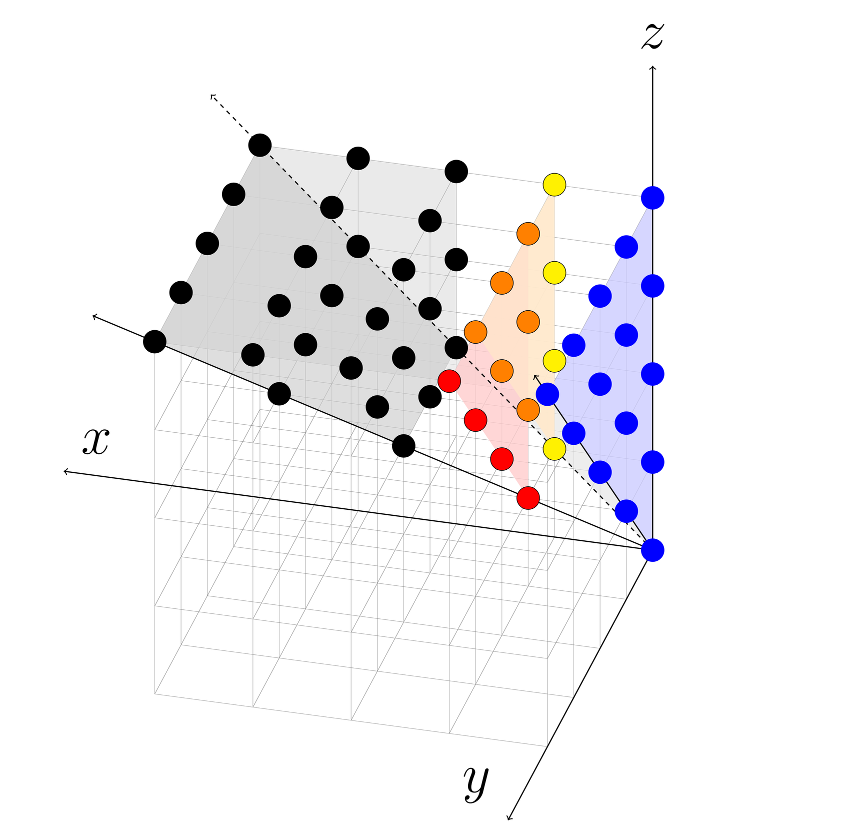


Figure 4: Standard pairs of $I = \langle x^{(2,0,2)}, x^{(2,1,2)}, x^{(2,1,1)}\rangle$

Computation of standard pairs

- Polynomial ring case is known and adopted into Macaulay 2. [4]
- Normal Case:
  - Embed into a big polynomial ring and compute.
  - Move standard pairs back by an *integer programming*. (Need many *Gröbner Bases*.)
- Nonnormal case: Use the result [5] of holes of an affine semigroup ring.

Bibliography

[1] Laura Matusevich and Byeongsu Yu, *Standard pairs for monomial ideals in semigroup rings*, preprint (2020).

[2] David Helm and Ezra Miller, *Algorithms for graded injective resolutions and local cohomology over semigroup rings*, J. Symbolic Comput. **39** (2005), no. 3-4, 373–395, DOI 10.1016/j.jsc.2004.11.009.

[3] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005.

[4] David Eisenbud, Daniel R. Grayson, Michael Stillman, and Bernd Sturmfels (eds.), *Computations in algebraic geometry with Macaulay 2*, Algorithms and Computation in Mathematics, vol. 8, Springer-Verlag, Berlin, 2002.

[5] Raymond Hemmecke, Akimichi Takemura, and Ruriko Yoshida, *Computing holes in semi-groups and its applications to transportation problems*, Contrib. Discrete Math. **4** (2009), no. 1, 81–91.