Monomial ideals in affine semigroup rings

Byeongsu Yu (joint w. Dr. Laura Matusevich)

Texas A&M University

October 24, 2020

Goal and monomial ideal

- Goal: Understand the monomial ideal in the affine semigroup ring in a combinatorial way.
- Motivation: Both monomial ideals and affine semigroup rings are rich subject for combinatorial study.

Definition (Monomial and its ideal.)

A *monomial* denotes a polynomial with one term over a field $\mathbb{K}$. A *monomial ideal* is an ideal generated by monomials.

Example

$x^2yz^3 \in \mathbb{K}[x, y, z] \to x^{(2,1,3)} \in \mathbb{K}[x_1, x_2, x_3]$

It is natural to depict a monomial as a lattice point in $\mathbb{Z}^d$ ($\mathbb{Z}^d$-graded).

Affine semigroup ring

- *Affine semigroup*: $\mathbb{N}A := \{A \cdot u : u \in \mathbb{N}^n\}$ where $A := \{a_1, \cdots, a_n\} \subset \mathbb{Z}^d$ as a $d \times n$ matrix;
- *Affine semigroup ring*: $\mathbb{K}[\mathbb{N}A] := \mathbb{K}[t^{a_1}, \cdots, t^{a_n}]$ as a subring of the Laurent polynomial ring $\mathbb{K}[t_1^{\pm}, \cdots, t_d^{\pm}]$ ($\mathbb{Z}^d$-graded)
- *Monomial ideal*: a homogeneous ideal in $\mathbb{K}[\mathbb{N}A]$.
- $\mathbb{K}[\mathbb{N}A]$ is *normal* if $\mathbb{N}A = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}A$.

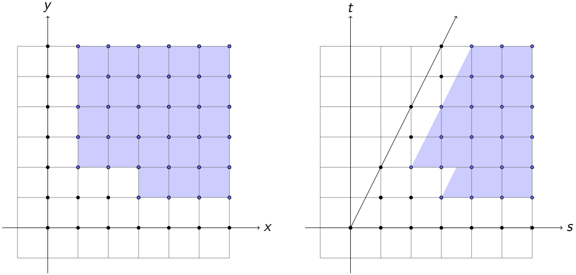
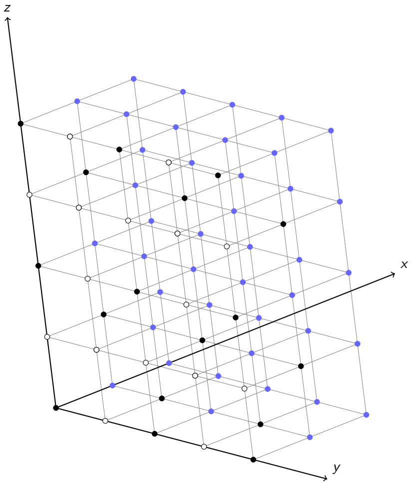


Figure: (L)$\begin{cases} \mathbb{K}[x, y] \\ I = \langle x^3y^1, xy^2 \rangle \end{cases}$ (R) $\begin{cases} \mathbb{K}\left[s, st, st^2\right] \\ I = \langle s^2t^2, s^3t \rangle \end{cases}$

Non-normal example

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \end{pmatrix},\ \mathbb{K}[\mathbb{N}A] \subset \mathbb{K}[x, y, z],\ I = \langle x, xyz, xyz^2 \rangle:$$



Facts for a monomial ideal $I \subset \mathbb{K}[\mathbb{N}A]$ (Helm and Miller, 2005; Miller and Sturmfels, 2005)

- A monomial prime ideal $\overbrace{\leftrightarrow}^{\text{1-1}}$ A face of $\mathbb{R}_{\geq 0}A$.
- Irreducible decomposition exists.
- If $\mathbb{K}[\mathbb{N}A]$ is *normal*, $\exists$ an algorithmic irreducible decomposition and irreducible resolution.

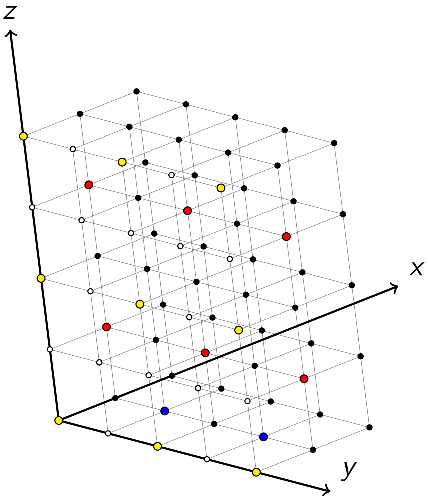
Standard pairs can be used to generalizes the above to the nonnormal case.

Standard Pairs

- F: *face* of A if $F = A \cap H$ for a face H of $\mathbb{R}_{\geq 0}A$.
- (a, F) : *proper pair* if $(a + \mathbb{N}F) \cap I = \emptyset$.
- $(a, F) < (b, G)$ if $a + \mathbb{N}F \subseteq b + \mathbb{N}G$.
- (a, F) is *standard* if maximal w.r.t. $<$.
- (a, F) *divides* (b, G) if $\exists c \in \mathbb{N}A$ s.t. $a + c + \mathbb{N}F \subset b + \mathbb{N}G$
- (a, F) and (b, G) *overlap* if they divide each other. (Equivalence relation)

Example of Standard Pairs

$$A = \left(\begin{smallmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \end{smallmatrix}\right),\ \mathbb{K}[\mathbb{N}A] \subset \mathbb{K}[x, y, z],\ I = \langle x, xyz, xyz^2 \rangle:$$



Example of Standard Pairs

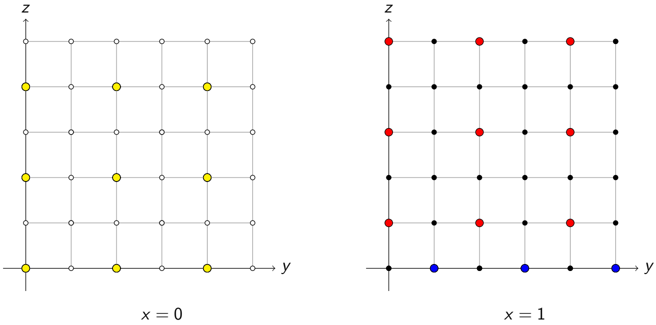


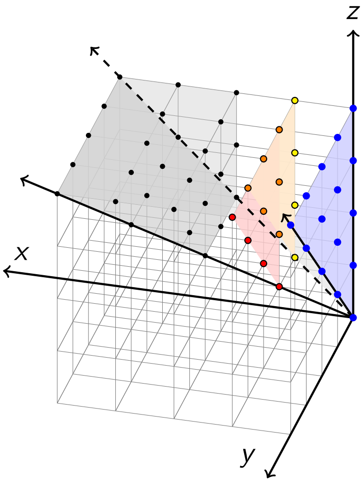
Figure: Standard pairs of $I = \langle x, xyz, xyz^2 \rangle$ in $\mathbb{K}[\mathbb{N}A]$

Example of Standard Pairs: Overlap class

$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $I = \langle x^{(2,0,2)}, x^{(2,1,2)}, x^{(2,1,1)} \rangle$, $F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

Standard Pairs: $(0, F)$, $(x^{(0,0,1)}, F)$, and $(x^{(0,1,1)}, F)$

Overlap happens between $(x^{(0,0,1)}, F)$ and $(x^{(0,1,1)}, F)$.



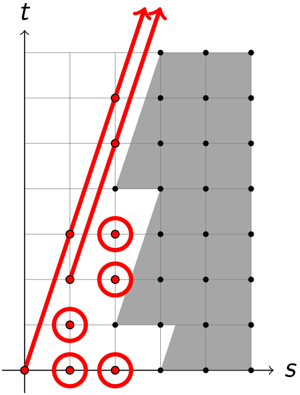
Main Result of (Matusevich and Yu, 2020)

Given a monomial ideal I in $\mathbb{K}[\mathbb{N}A]$,

- I is *primary* iff all standard pairs of I correspond to a same face.
- I is *irreducible* iff I is primary and has the unique maximal overlap classes of the standard pairs w.r.t. divisibility.
- I has associated prime P_F iff I has a standard pair (a, F).
- The *multiplicity* of $P_F = \#$ of overlap classes of I whose face belongs to F.
- $\#$ of maximal (w.r.t divisibility) overlap classes of I $= \#$ of components of an irreducible irredundant decomposition of I.

Example of Irreducible Decomposition

$A = \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{smallmatrix}\right)$, $\mathbb{K}[\mathbb{N}A] = \mathbb{K}[s, st, st^2, st^3], I = \langle s^3, s^2t, s^2t^4 \rangle$.
Standard Pairs: two red lines and four red circles.



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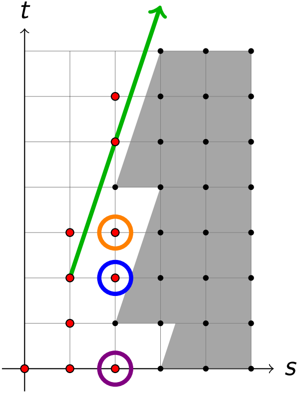
Maximal Overlap Classes



Example of Irreducible Decomposition

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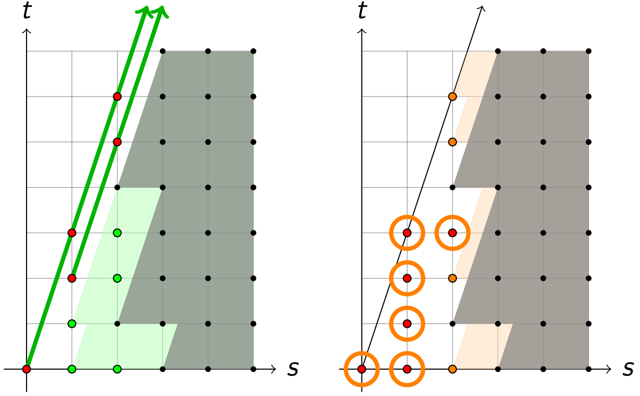
$$I = \langle s, st \rangle \cap \langle s^2, s^2t^1, s^2t^2, s^2t^4, s^2t^5, s^2t^6 \rangle$$
$$\cap \langle st^3, s^2t, s^2 \rangle \cap \langle st, st^2, st^3, s^3 \rangle.$$



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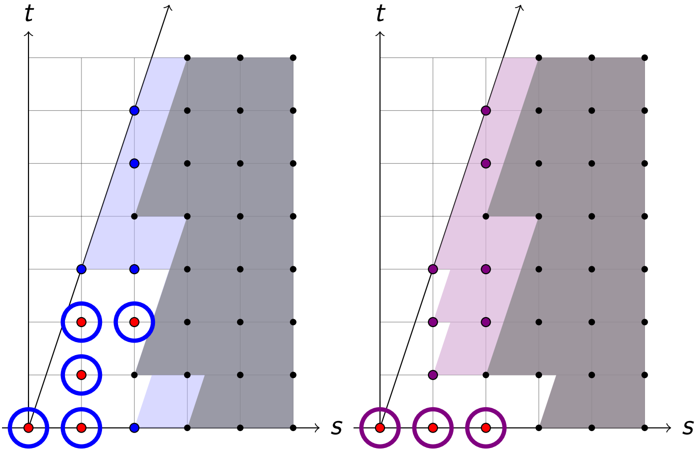
$$\cap \langle st^3, s^2t, s^2 \rangle \cap \langle st, st^2, st^3, s^3 \rangle.$$



Computation of monomial ideals

- Polynomial ring case is known and adopted into Macaulay 2. (Eisenbud et al., 2002)
- General Affine semigroup: `stdPairs.spyx` (Yu, 2020)
  - Library in a `SageMath`.
  - Compatible with Macaulay2.
  - One can save and load his/her computation on the monomial ideal.

```
byeongsuyu — IPython: Users/byeongsuyu — python3.7 ◂ sudo — 80×26
Last login: Fri Oct 23 21:33:02 on ttys000
byeongsuyu@yubyeongsuui-MacBook-Air ~ % sudo /Applications/SageMath-9.1.app/sage
Password:

  SageMath version 9.1, Release Date: 2020-05-20
  Using Python 3.7.3. Type "help()" for help.

sage: load("~/stdPairs.spyx")
Compiling /Users/byeongsuyu/stdPairs.spyx...
sage: A = matrix(ZZ,[[1,2],[0,2]])
sage: Q = affineMonoid(A)
sage: Q
An affine semigroup whose generating set is
[[1 2]
 [0 2]]
sage: M = matrix(ZZ,[[4,6],[4,6]])
sage: I = monomialIdeal(Q,M)
sage: I.standardCover()
{(0,): [(([[0], [0]]^T,[[1], [0]]), ([[2], [2]]^T,[[1], [0]]))]}
sage: I.associatedPrimes()
{(0,): [An ideal whose generating set is
 [[2]
  [2]]]}
sage: I.save("~/2D_ideal")
1
```

Thank you for listening!

References

Matusevich, Laura and Byeongsu Yu. 2020. *Standard pairs for monomial ideals in semigroup rings*, available at `arXiv:2005.10968`.

Helm, David and Ezra Miller. 2005. *Algorithms for graded injective resolutions and local cohomology over semigroup rings*, J. Symbolic Comput. **39**, no. 3-4, 373–395, DOI 10.1016/j.jsc.2004.11.009.

Miller, Ezra and Bernd Sturmfels. 2005. *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York.

Eisenbud, David, Daniel R. Grayson, Michael Stillman, and Bernd Sturmfels (eds.) 2002. *Computations in algebraic geometry with Macaulay 2*, Algorithms and Computation in Mathematics, vol. 8, Springer-Verlag, Berlin.

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