0. Preliminary: Modular arithmetic

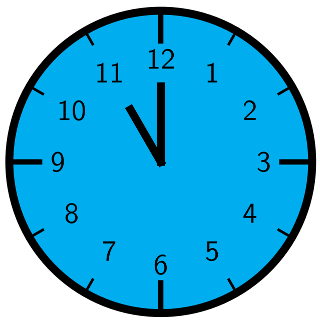
Definition 0.1. *Identification numbers* is numbers for identifying objects.

Thus it should be ________________. So each number represents a different person or thing. Some examples of identification numbers are Social Security numbers, UINs, Driver's License numbers, routing numbers, VINs, UPCs, etc.

Identification number often contain one or more ________________to help catch data entry and data transmission errors.

0.1. **Modular artihmetic.**

Example 0.2. If a class starts at 11:00 a.m. and lasts for 3 hours, what time does it end?



This is a form of modular arithmetic. With modular arithmetic, we perform calculations modulo some number m, which we usually say as "mod m." To find a number a mod m, you find the ________________when a is divided by m.

Example 0.3. Find the following values.

(1) 17 mod 5 is ________________
(2) 47 mod 11 is ________________
(3) 53 mod 7 is ________________
(4) 8 mod 13 is ________________

Definition 0.4. Two numbers a and b are *congruent* mod m if $a \mathrm{mod} m = b \mathrm{mod} m$. If a and b are congruent mod m, we write it as

$$a \equiv b \text{ mod } m$$

Example 0.5. 7 mod 5 = ________________and 12 mod 5 = ________________ $\implies$ $7 \equiv 12$ mod 5

Instead of having to do two calculations, though, another way to determine if two numbers are congruent mod m is the following:

$$a \equiv b \text{ mod } m \text{ if and only if } a - b \text{ is divisible by } m.$$

In other words, $a - b$ is a multiple of m. In other words, $a - b$ has a remainder of 0 when we divided by m.

Example 0.6. Determine if the following congruences are true or false.

(1) $28 \equiv 1$ mod 9.
(2) $29 \equiv 15$ mod 10.
(3) $131 \equiv 113$ mod 6.

1. Check Digits

Example 1.1. Common Data Entry Errors

(1) ________________: Occurs when an incorrect digit is entered. This is also called a substitution error.
   - Ex:

(2) ________________: Occurs when adjacent digits are entered in reverse order.
   - Ex:

(3) ________________: Occurs when digits that are separated by another digit are entered in reverse order.
   - Ex:

There are several examples of identification numbers to avoid the data entry errors.

(1) Sum of the digits mod 9

The first 10 digits of a USPS Money Order identify the money order. The last digit is a check digit, which should equal the sum of the first 10 digits mod 9.

(2) Number mod 7:

Airline Tickets, UPS packages, and Avis and National rental car companies reserve the last digit as a check digit. The check digit is the identification number mod 7.
   - Is the check digit for the National rental car number 5007125 valid? Note: The number in this problem includes the identification number with the check digit appended to the end.

(3) Universal Product Codes (UPC)

0 21000 65371 3

0 51000 08723

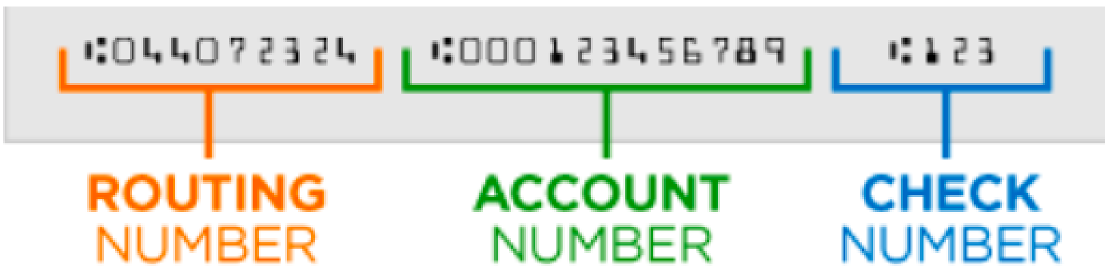
The blocks of digits indicate the category of the item, the manufacturer, the item, and the check digit, respectively. Let

$$S = 3a_1 + a_2 + 3a_3 + a_4 + 3a_5 + a_6 + 3a_7 + a_8 + 3a_9 + a_{10} + 3a_{11} + a_{12}.$$

Then, check digits, a_{12} is chosen so that S is a multiple of 10, i.e., $S \equiv 0 \bmod 10$.

- Check to see if the first code and second codes in the picture have a valid check digit.

(4) Banking



(Image from Nationwide.com)

For routing number, let

$$R = 7a_1 + 3a_2 + 9a_3 + 7a_4 + 3a_5 + 9a_6 + 7a_7 + 3a_8$$

Then, the check digit, a_9, equals the last digit of R.

- Does this sample routing number have a valid check digit?

Example 1.2. The HR representative could not read a new doctor's handwriting on her direct deposit form. The HR representative could tell the routing number was $314x77337$. Can you determine the missing digit x for the HR representative?

x	0	1	2	3	4	5	6	7	8	9
R										

(5) ISBN-10

ISBN-10: 1-4292-0900-3

The blocks of digits represent the language of the country in which the book was published, the publisher, the book, and the check digit, respectively. Let

$$V = \mathbf{10}a_1 + \mathbf{9}a_2 + \mathbf{8}a_3 + \mathbf{7}a_4 + \mathbf{6}a_5 + \mathbf{5}a_6 + \mathbf{4}a_7 + \mathbf{3}a_8 + \mathbf{2}a_9 + a_{10}$$

The check digit, a_{10}, is chosen so V is divisible by 11.
Note: A remainder of 10 is possible when dividing by 11. So if a_{10} needs to be 10, an X is used so that it only takes up one space.

- Is the check digit correct for the ISBN-10 listed above?

- Suppose a book you own has an ISBN where one of the digits is unreadable and looks like: $1 - 531x - 7002 - 6$ Determine the value of x if the check digit is correct.

x	0	1	2	3	4	5	6	7	8	9
R										

(6) Credit Cards

Credit cards have 15-digit numbers with a check digit in position 16. Let

$D =$ the sum of the digits in odd-numbered positions

$E =$ the sum of the digits in even-numbered positions, except the check digit a_{16}

$T =$ the number of digits in odd-numbered positions that are larger than 4

$C = 2D + E + T + a_{16}$.

Then the check digit a_{16} is chosen so C is a multiple of 10.

- The number 4128 0012 3456 7890 was listed on a credit card advertisement. Is the check digit correct?

Example 1.3. An ID number consists of 3 digits followed by a 4^{th} check digit, $a_1a_2a_3a_4$ where the check digit is given by:

$$a_4 = (9a_1 + 3a_2 + 8a_3) \bmod 7$$

You receive the number $57x6$. Given that the check digit is correct, what is the value of x?

x	0	1	2	3	4	5	6	7	8	9
R										

Check digits are designed to catch errors, but unfortunately, in some cases, not all errors will be caught. So how do we determine which errors will *NOT* be caught.

KEY IDEA: For an error not to be caught, the correct number and the number with an error must produce the _________________check digit.

(1) Write a general form of the correct number. Label the correct number with $a_1a_2\cdots$ which a_i represents a _________________digit. (Note: A digit is some whole number 0, 1, $\cdots$,9.)
(2) Write a general form of the incorrect number.
   (a) For transposition errors, we will rearrange the a_i according to the type of transposition.
   (b) For single digit errors, write the number using e_i to represent the error for the digit that should have been a_i.
(3) Find the check digits for the correct number and incorrect number mod m.
(4) The error will NOT be caught if the check digits are the same mod m. Use the fact that if $x \equiv y \bmod m$, then $x - y$ is a _________________of m.
(5) Determine the multiples that are integers (whole numbers) between 1 and 9.
(6) The error will not be caught when $|x - y|$ takes on these values.
(7) List the pairs of digits where this occurs.

Example 1.4. Suppose for a four-digit number, the 4th digit is the check digit and is the sum of first three digits mod 9. Determine if a single digit error in the first digit will be caught.

Proof. Let's look at an error in the first digit, a_1.

Correct code: $a_1a_2a_3a_4$
Incorrect code: $e_1a_2a_3a_4$

Check digit for Correct code: $(a_1 + a_2 + a_3) \bmod 9$
Check digit for Incorrect code: $(e_1 + a_2 + a_3) \bmod 9$

The error will NOT be caught if the check digits are the same, i.e., if

$$(a_1 + a_2 + a_3) \equiv (e_1 + a_2 + a_3) \bmod 9$$

But, remember this is the same as saying:

$$(a_1 + a_2 + a_3) - (e_1 + a_2 + a_3) \text{ is a multiple of } 9$$

This simplifies algebraically to $a_1 - e_1$ is a multiple of 9.

What are the integer multiples of 9 between 1 and 9? ________________

Why do we not care about multiples that are 0?
Why do we not care of multiples that are larger than 9?

So, if $|a_1 - e_1| = 9$, then the error will not be caught. (We use absolute values here because, depending on which is bigger, we may get a negative number, and we only really care about the absolute difference.)

When does this occur? What pair(s) of digits are separated by 9 units?
The error will not be caught if we use 0 instead of 9 or 9 instead of 0. □

Example 1.5. Let the check digit for a 4-digit number be given by

$$a_4 = (3a_1 + a_2 + 9a_3) \text{ mod } 4$$

Determine if this scheme will catch single digit errors in the second digit.

Correct code: $a_1a_2a_3a_4$
Incorrect Code:

Check digit for Correct code: $(3a_1 + a_2 + 9a_3)$ mod 4
Check digit for Incorrect Code:

The error will NOT be caught if the check digits are the same, i.e., if

________________ − ________________ is a multiple of ________________

This simplifies algebraically to ________________is a multiple of ________________

The integer multiples of 4 between 1 and 9 are ________________.

So, if $|a_2 - e_2| =$ ________________, the error will not be caught.

The pairs for which this occurs are:

Example 1.6. Let the check digit for a 4-digit number be given by

$$a_4 = (3a_1 + a_2 + 9a_3) \bmod 4.$$

Determine if this scheme will catch transposition errors of the first and second digits.

Correct code: $a_1a_2a_3a_4$
Incorrect Code: $a_2a_1a_3a_4$

Check digit for Correct code: $(3a_1 + a_2 + 9a_3) \bmod 4$
Check digit for Incorrect Code:

The error will NOT be caught if the check digits are the same, i.e., if

$$\_\_\_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_\_\_ \text{ is a multiple of } \_\_\_\_\_\_\_\_\_\_$$

After cancelling terms, this simplifies to:
$2a_1 - 2a_2$ is a multiple of 4.
We now factor out the 2 to get:
$2\,(a_1 - a_2)$ is a multiple of 4.
Dividing by 2 we get:
$a_1 - a_2$ is a multiple of 2.

The integer multiples of 2 between 1 and 9 are __________.
So, if $|a_1 - a_2| =$ __________, the error will not be caught.
The pairs for which this occurs are:

Example 1.7. Let the check digit $a_4 = (4a_1 + 7a_2 + 2a_3)$ mod 10 Will this catch all single-digit errors in the third digit?

Correct Code:
Incorrect Code:

Check digit for Correct Code:
Check digit for Incorrect Code:

The error will NOT be caught if the check digits are the same, i.e., if

________________ − ________________ is a multiple of ________________.

Simplify:

The error will not be caught if ________________is a multiple of ________________.
The integer multiples ________________of between 1 and 9 are ________________.

Conclusion:

Pairs:

Example 1.8. Let the check digit $a_4 = (4a_1 + 7a_2 + 2a_3) \bmod 10$ Will this catch all transposition errors of the 1st and 2nd digits?

Correct Code:
Incorrect Code:

Check digit for Correct Code:
Check digit for Incorrect Code:

The error will NOT be caught if the check digits are the same, i.e., if

________________ − ________________ is a multiple of ________________.

Simplify:

The error will not be caught if ________________is a multiple of ________________.
The integer multiples ________________of between 1 and 9 are ________________.

Conclusion:

Pairs:

Example 1.9. Let the check digit$a_4 = (5a_1 + 3a_2 + 8a_3)$ mod 6 Will this catch all transposition errors of the 2nd and 3rd digits?

Correct Code:
Incorrect Code:

Check digit for Correct Code:
Check digit for Incorrect Code:

The error will NOT be caught if the check digits are the same, i.e., if

________________ − ________________ is a multiple of ________________.

Simplify:

The error will not be caught if ________________is a multiple of ________________.
The integer multiples ________________of between 1 and 9 are ________________.

Conclusion:

Pairs:

Example 1.10. Let the check digit $a_4 = (4a_1 + 5a_2 + 7a_3) \bmod 6$. Will this catch the single digit errors of the 1st digit?

Correct Code:
Incorrect Code:

Check digit for Correct Code:
Check digit for Incorrect Code:

The error will NOT be caught if the check digits are the same, i.e., if

________________ – ________________ is a multiple of ________________.

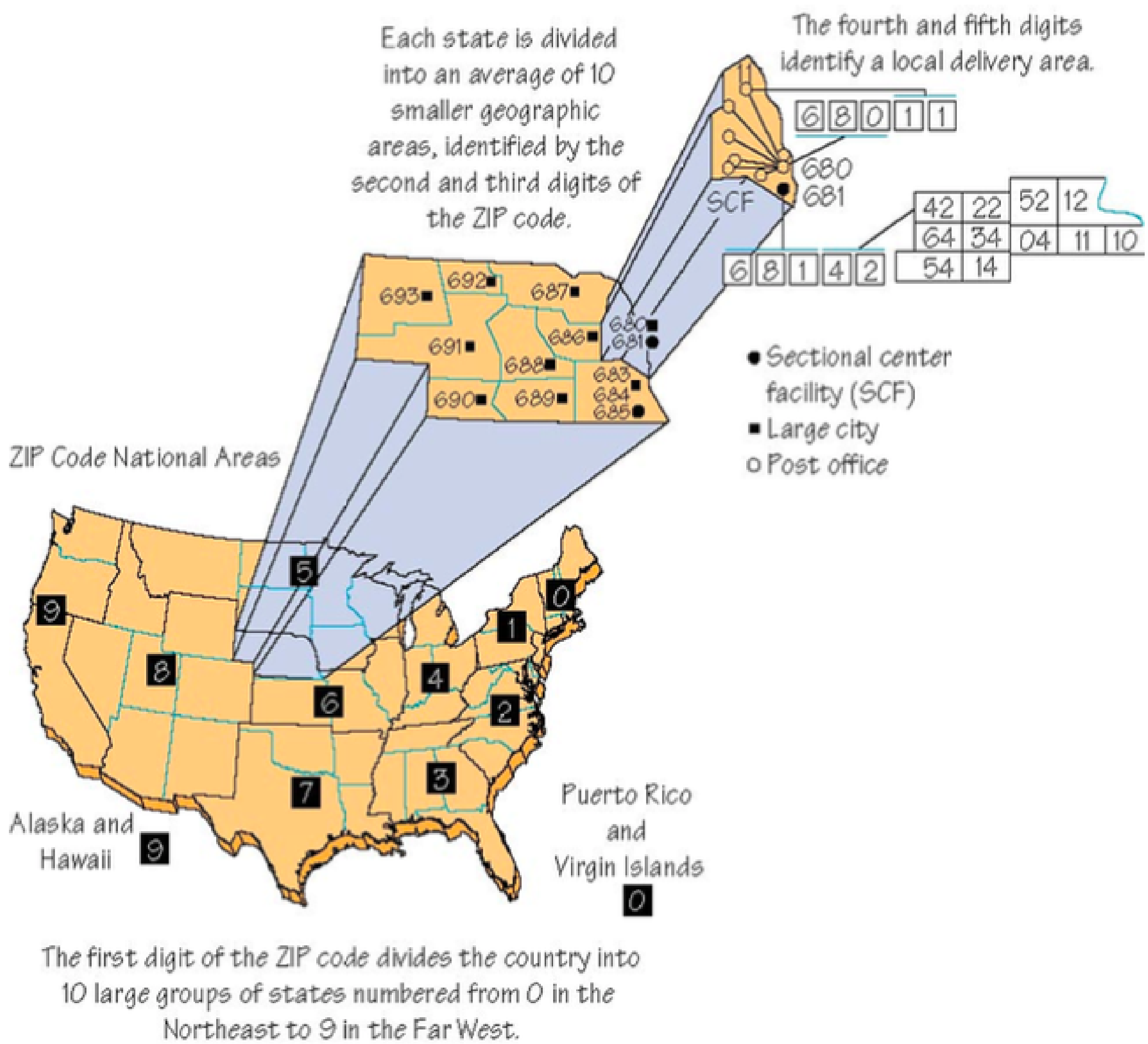
Simplify:

The error will not be caught if ________________is a multiple of ________________.
The integer multiples ________________of between 1 and 9 are ________________.

Conclusion:

Pairs:

2. Zip Codes

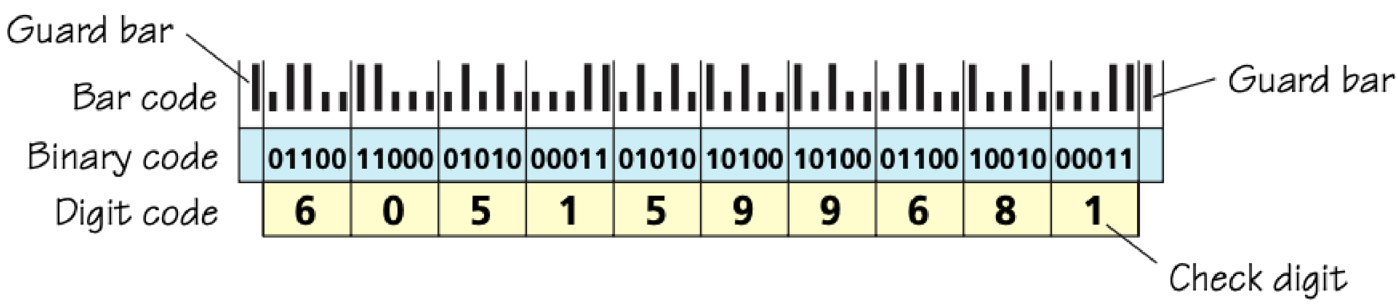


Example 2.1. What does Texas A&M's zip code of 77843-0001 tell us?

3. Barcodes

Definition 3.1. A system for representing data with only two symbols is a ________________system.
A ________________is a binary system that is made up of a series of dark bars and light spaces that represent characters.
A ________________code is used to encode ZIP+4 numbers. Each digit is represented by ________________vertical bars (________________long and ________________short). The ZIP+4 is followed by a check digit determined by adding the first 9 digits and choosing the check digit so the sum of all ten digits is a multiple of 10. The bars for these ten digits are surrounded by a tall bar on each end called guard bars to designate the beginning and end of the code.

Example 3.2. Determine the ZIP+4 displayed below and determine if the check digit is correct.



(Images from text page 584)

Example 3.3.

This special bar code is called a ________________code. These can encode much more information and are popular in print media. The Cooking Light magazine from September, 2011 had quite a few, including this one.

4. Personal Data

Example 4.1 (Social Security Number)**.** The first three digits tell the state where the application was filed.

Example 4.2 (Florida Driver's License)**.**

(image from
http://restaurantandlodging.com/a-la-carte/new-law-permits-veterans-designation-on-florida-driver-licen.html)

- Take on the form XXXX-XXX-YY-DDD-N
- XXXX-XXX is based on a coding of your name.
- YY gives the last two digits of the year of your birth
- DDD is coded from the month (m) and day (d) of your birth. These three digits are
  - Male: $40(m-1)+d$
  - Female: $40(m-1)+d+500$
  - This system uses 40 as a loose representation for the number of days in a month.
- N is an overflow digit in case multiple people have the same number.

What are the range of values of DDD for a male and for a female?

Are the digits YY-DDD correct for Joe Sample?

Determine the Florida driver's license digits YY-DDD for a female who was born on September 18, 1942.

What can you determine about a person who holds a Florida driver's license with the digits YY-DDD as 61-607?

What can you determine about a person who holds a Florida driver's license with the digits YY-DDD as 34-475?

5. Summary of Check Digits

The check digit is the last digit unless noted otherwise.

(1) Bank Routing Number
- The check digit, a_9, is the last digit of R where

$$R = 7a_1 + 3a_2 + 9a_3 + 7a_4 + 3a_5 + 9a_6 + 7a_7 + 3a_8$$

(2) Codabar Scheme - Credit Cards. Libraries, Blood Banks. South Dakota Driver's License
- The smallest nonnegative integer, a_{16}, such that C is a multiple of 10 where

$C = 2D + E + T + a_{16}$

$D =$ the sum of the digits in odd-numbered positions

$E =$ the sum of the digits in even-numbered positions (not including the check digit)

$T =$ the number of digits in odd-numbered positions that are larger than 4.

(3) I S B N-10
- The smallest nonnegative integer, a_{10}, such that V is a multiple of 11 where

$$V = 10a_1 + 9a_2 + 8a_3 + 7a_4 + 6a_5 + 5a_6 + 4a_7 + 3a_8 + 2a_9 + a_{10}$$

- The letter X is used to represent the integer 10 .

(4) ISBN-13 and EAN (European Article Number)
- The smallest nonnegative integer, a_{13}, such that S is a multiple of 10 where

$$s = a_1 + 3a_2 + a_3 + 3a_4 + a_5 + 3a_6 + a_7 + 3a_8 + a_9 + 3a_{10} + a_{11} + 3a_{12} + a_{13}$$

(5) Money Order from US Postal Service
- The check digit, a_{11}, is the sum of the first ten digits mod9

(6) Rental Cars (Avis and National) and Airline Tickets
- The number formed by the digits prior to the check digit mod 7 . Note that this does NOT use the sum of the digits.

(7) Travelers Cheques (American Express and Visa) and Euro Bank Notes
- The smallest nonnegative integer such that the sum of the digits (including the check digit) is a multiple of 9.

(8) UPC (Universal Product Code)
- The smallest nonnegative integer, a_{12}, such that S is a multiple of 10 where $S = 3a_1 + a_2 + 3a_3 + a_4 + 3a_5 + a_6 + 3a_7 + a_8 + 3a_9 + a_{10} + 3a_{11} + a_{12}$

(9) ZIP+4
- The smallest nonnegative integer such that the sum of the digits (including the check digit) is a multiple of 10.