3.1. Limit and Colimit as a universal cone

Recall: diagram (or shape) J in a category C

$\Rightarrow$ $F : J \to C$ a functor.

Objective: Introduce $\lim$ and colim of diagram as universal cones over and under the diagram

Def 3.1.1 [Cone]

$c \in C$, J : category.

$c : J \to C$ is called constant functor.

$$\begin{array}{ccc} x & \longmapsto & c \\ f \downarrow & \longmapsto & \downarrow 1_c \\ y & \longmapsto & c \end{array}$$

$$\Rightarrow \quad \begin{array}{rcl} \Delta : C & \longrightarrow & C^J \\ c & \longmapsto & c : J \to C \\ f \downarrow & & \downarrow \alpha_f \\ d & \longmapsto & d : J \to C \end{array}$$

α_f : natural transf

$(\alpha_f)_c = f$

$$\begin{tikzcd} c \arrow[r, "1_c"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & c \arrow[d, "f"] \\ d \arrow[r, "1_d"'] & d \end{tikzcd}$$

is an embedding

Def 3.1.2. A cone over $F: J \to C$ with summit (apex) $c \in C$.

= Natural transformation $\lambda: c \Rightarrow F$.

- Data: $\lambda = \{\lambda_j : j \in J\}$
- From natural transf, for $f: j \to k \in J$

$$\begin{tikzcd} c \arrow[r, "1_c"] \arrow[d, "\lambda_j"'] \arrow[dr, phantom, "\circlearrowleft"] & c \arrow[d, "\lambda_k"] \\ Fj \arrow[r, "Ff"'] & Fk \end{tikzcd} \iff \begin{tikzcd} & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] \arrow[d, phantom, "\circlearrowleft"] & \\ Fj \arrow[rr, "Ff"'] & {} & Fk \end{tikzcd}$$

- Thus, if we have data satisfying the commuting diagram, then we can define cone.

A cone under F with nadir c

= Natural transf $\lambda: F \Rightarrow c$ s.t.

$\forall f: j \to k$,

$$\begin{tikzcd} Fj \arrow[rr, "Ff"] \arrow[dr, "\lambda_j"'] & & Fk \arrow[dl, "\lambda_k"] \\ & c & \end{tikzcd}$$

Cone under F is called $\boxed{\text{cocone}}$ since it is dual of a cone.

(Cone over $F: J^{op} \to C^{op}$.)

lim F : universal cone over F

Def) $F : J \to C$ diagram.

$\mathrm{Cone}(-, F) : C^{op} \to \mathrm{Set}$

$$
\begin{tikzcd}
c \arrow[d, "f"'] \arrow[r, mapsto] & \{\lambda : \underset{J \to C}{c} \Rightarrow \underset{J \to C}{F} \text{ : natural transformation}\} \\
d \arrow[r, mapsto] & \{\lambda : d \Rightarrow F \text{ natural transformation}\} \arrow[u, "\alpha_f : c \to d \text{ as natural transformation}"']
\end{tikzcd}
$$

lim F : representation of $\mathrm{Cone}(-, F)$, i.e.

$\mathrm{Cone}(-, F) \cong C(-, \lim F) \Leftrightarrow$ terminal object of $\int \mathrm{Cone}(-, F)$

limit cone : universal element in this.

Similarly,

$\mathrm{Cone}(F, -) : C \to \mathrm{Set}$

$$
\begin{tikzcd}
c \arrow[d, "f"'] \arrow[r, mapsto] & \{\lambda : c \Rightarrow F\} \arrow[d, "\alpha_f"] \\
d \arrow[r, mapsto] & \{\lambda : d \Rightarrow F\}
\end{tikzcd}
$$

Colimit cone : universal element

Colim F : rep of $\mathrm{Cone}(F, -)$. $\Leftrightarrow$ initial object of $\int \mathrm{Cone}(F, -)$

Notes that if $\lambda_c : c \Rightarrow F$, $\lambda_d : d \Rightarrow F$, $f : c \to d$.

Then α_f sends λ_d to λ_c via below diagram.

$$
\begin{tikzcd}
 & d \arrow[ddl, bend right, "\lambda_d(j)"'] \arrow[ddr, bend left, "\lambda_d(k)"] & \\
 & c \arrow[u, "f"] \arrow[dl, "\lambda_c(j)"'] \arrow[dr, "\lambda_c(k)"] & \\
Fj \arrow[rr] & & Fk
\end{tikzcd}
$$

So : $\mathrm{Cone}(-, F)$ is well-defined.

(Similar to $\mathrm{Cone}(F, -)$)

Prop 3.1.7. $\lambda : \ell \Rightarrow F$, $\lambda' : \ell' \Rightarrow F$ are two limit cones, $\Rightarrow$ $\ell \overset{\exists!}{\cong} \ell'$. Unique iso. commutes with legs of the limit cones.

pf) $(\lim F, \lambda)$, $(\lim F, \lambda')$ are terminal

$\Rightarrow$ " $\cong$ " . by Corollary 2.3.2.

And isomorphism in $\int \mathrm{Cone}(-, F)$ gives commuting diagram.

Rmk: There may be nontrivial automorphism between ℓ and ℓ'. but by Corollary 2.3.2 if it commutes with cone, then it is the iso in prop 3.1.7.

Def 3.1.9. [Product]

Product = limit of a diagram indexed by J := a discrete category with only identity morphisms.

Thus, $F : J \to C$ is just collection $\{F_j\}_{j \in J}$. (Since J has no nontrivial morphisms)

Thus, cone over J is $\lambda : c \Rightarrow F$

i.e. collection of $\{\lambda_j : c \longmapsto F_j\}_{j \in J}$.

Therefore its limit is $\prod_{j \in J} F_j$

with legs $\pi_k : \prod_{j \in J} F_j \longrightarrow F_k$

By the universal property of $\prod_{j \in J} F_j$ as a cone gives natural isomorphism

$$C(c, \prod_{j \in J} F_j) \xrightarrow[\cong]{(\pi_k)_*} \mathrm{Cone}(c, F) \cong \prod_{k \in J} C(c, F_k)$$

(Yoneda lemma) Just...

Ex 3.1.10. $X, Y \in \mathrm{Top}$. Product of $X \times Y$ is a space

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

satisfying cts projection and has universal property: For any $A \in \mathrm{Top}$ with $f: A \to X$, $g: A \to Y$, $\exists! h: A \to X \times Y$ s.t.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, "\exists! h"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

commute.

To construct $X \times Y$ as usual sense (Cartesian product) By taking $A = \{*\}$, A as a constant function says that $X \times Y \supseteq X \times_{\text{Cartesian}} Y$ as a set.

(By similar argument, $X \times Y = X \times_{\text{Cartesian}} Y$ as a set)

This is because $\mathrm{Top}(A, X \times Y) \cong \mathrm{Top}(A, X) \times \mathrm{Top}(A, Y)$
as a set.

To show $X \times Y = \underset{\text{Cartesian}}{X \times Y}$ as a top space

take $A = X \times Y$ as a set with
various topology.

By the universal property,
$X \times Y$ forced to defined as the coarsest topology on cartesian product s.t.
π_X, π_Y are cts. (These are case when $J = \{\bullet, \bullet\}$)

(Do the same thing for any index J.)

Def 3.1.11. Terminal object = product when the indexing category is empty.

i.e, if $J = \{\}$, Cone over J with summit c is just c. $\Rightarrow \int \mathrm{Cone}(-, J \to C)$

$$\cong C.$$

By def of limit, $\lim(J \to C)$ is terminal obj of $\int \mathrm{Cone}(-, J \to C) \cong C$.

Ex 3.1.12 $\mathbb{1} : \{ \cdot \circlearrowright 1 \cdot \}$ called terminal category

Since it is terminal obj of Cat or CAT

Def 3.1.13 [Equalizer]

Equalizer = Limit of $F : J \longrightarrow C$

where $J = \cdot \overset{f}{\underset{g}{\rightrightarrows}} \cdot$ parallel pair.

$\Rightarrow$ Let $F(J) = A \overset{f}{\underset{g}{\rightrightarrows}} B$. A cone over F with summit C is $(a : C \longrightarrow A,\ b : C \longrightarrow B)$

s.t.

$$\begin{tikzcd} & C \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "\circlearrowleft"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

$\Leftrightarrow$ $fa = b = ga$.

Thus cone over paralle pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ (with summit C) is represented by a morphism $a : C \longrightarrow A$ s.t. $fa = ga$.

Hence Equalizer is the universa arrow with this property, $h : E \longrightarrow A$. In particular

$\forall$ cone $a : C \longrightarrow A$, $\exists! k : C \longrightarrow E$ s.t.

$$\begin{tikzcd} C \arrow[d, "k"'] \arrow[dr, "a"] \arrow[dr, phantom, "\circlearrowleft", near end, shift right=3] & & \\ E \arrow[r, "h"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Commutes.

Ex 3.1.14. $\phi, \psi : G \to H \in \text{Group}$

$\Rightarrow \mathrm{Eq}(\phi, 0) = (\ker\phi \longrightarrow G)$

since

$$\begin{tikzcd}
C \arrow[d, "\exists! k"'] \arrow[drr, "a"] & & & \\
\ker\phi \arrow[rr, hook] & & G \arrow[r, shift left, "\phi"] \arrow[r, shift right, "0"'] & H
\end{tikzcd}$$

$\mathrm{Eq}(\phi, \psi) = (\{g \in G : \phi(g) = \psi(g)\} \hookrightarrow G)$

If H is abelian, $(= \ker(\phi - \psi))$

Actually, $\mathrm{Eq}(A \overset{f}{\underset{g}{\rightrightarrows}} B) \hookrightarrow C$ is monomorphism

$h : E \to A \longmapsto E$

Def 3.1.15 [Pullback]

Pullback = limit of $(F : \bullet \longrightarrow \bullet \longleftarrow \bullet \longmapsto C)$

Let $F(J) = B \xrightarrow{f} A \xleftarrow{g} C$

Then cone over F with summit D is triple morphism satisfying left comm. diagram

$$\begin{tikzcd}
D \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, "a"] & C \arrow[d, "g"] \\
B \arrow[r, "f"'] & A
\end{tikzcd}$$

$\Rightarrow$ $gc = a = fb$.

Thus we can represent this cone by two morphism $B \xleftarrow{b} D \xrightarrow{c} C$ satisfying rectangular.

Hence pull back: $B \xleftarrow{h} P \xrightarrow{k} C$ is universal cone over $(B \xrightarrow{f} A \xleftarrow{g} C)$

i.e. for any cone $B \xleftarrow{b} D \xrightarrow{c} C$,

$$
\begin{tikzcd}
D \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "c"] \arrow[ddr, bend right, "b"'] & & \\
& P \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
& B \arrow[r, "f"'] & A
\end{tikzcd}
$$

$\exists!$ $D \longrightarrow P$ satisfy commuting diagram left.

"$\lrcorner$" denote P is pullback, i.e. limit diagram not just commutative square

P is called "fiber product", denoted by $B \times_A C$.

If C is concrete category

thus $B = \{*\} \subseteq A$ as a set

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ \{*\} \arrow[r, hook, "i"'] & A \end{tikzcd}$$

implies $P = \{c \in C : g(c) = *\}$

$= g^{-1}(*)$

thus fiber of map g over $*$ (or $f(*)$)

Ex 3.1.18 $\rho : \mathbb{R} \longrightarrow S^1$ cts.

$t \longmapsto e^{2\pi i t}$.

What is P?

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d, dashed] & \mathbb{R} \arrow[d, "\rho"] \\ 1 \arrow[r, hook, "i"'] & S^1 \end{tikzcd}$$

$= \rho^{-1}(1)$

$= \{t \in \mathbb{R} : e^{2\pi i t} = 1\}$

$= \mathbb{Z}$

Similar as equalizer

Pullback$\left(B \underset{\text{mono}}{\rightarrowtail} A \leftarrow C\right)$ defines $B \times_A C \underset{\text{mono}}{\rightarrowtail} C$

Ex 3.1.19.

$$\begin{tikzcd} P \arrow[r, dashed, "a"] \arrow[d, dashed, "b"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{Z} \arrow[d, "n"] \\ \mathbb{Z} \arrow[r, "m"'] & \mathbb{Z} \end{tikzcd}$$

What is P? In Ab.

must $na = mb$.

P : ab gp s.t. $\forall p \in P$

$$na(p) = mb(p)$$

Moreover, it commutes with any other cones, satisfying $n(a(p)) = mb(p)$

$\Rightarrow$ $na(p) = mb(p)$ should be l.cm

of n, m,

Thus, we may take $P = \mathbb{Z}$ with

$$a(1) = \frac{\mathrm{Lcm}(n,m)}{n}, \quad b(1) = \frac{\mathrm{Lcm}(n,m)}{m}.$$

Def 3.1.2.1 [Inverse Limit].

Limit of $F : J \to C$ when $J = \omega^{op}$

$\cdots \to 3 \to 2 \to 1 \to 0$: ω^{op}

Then cone over F with summit C is

$$\begin{tikzcd} C \arrow[d] \arrow[r, "\lambda_3"] \arrow[rr, bend left, "\lambda_2"] \arrow[rrr, bend left, "\lambda_1"] \arrow[rrrr, bend left, "\lambda_0"] & F_3 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \\ \cdots \arrow[ur] & & & & \end{tikzcd}$$

each triangle commutes.

$\varprojlim F_n$: Is then terminal cone.

(Thus direct limit is when $J = \omega$.)
and colimit of $F : J \to C$

Ex 3.1.22

$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$.

Their dual notions are

Coproduct : Colimit of discrete category

Initial object : " empty "

Coequalizer : " $\bullet \rightrightarrows \bullet$

Pushout : " $\bullet \leftarrow \bullet \rightarrow \bullet$

Direct limit : " ω.

(Sequential colimit)

Ex 3.1.24 Pushout of $S^1 \leftarrow * \rightarrow S^1$ is $S^1 \vee S^1$

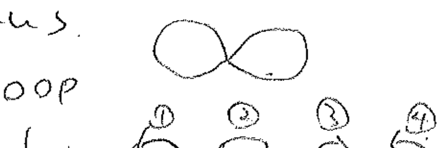
$$\begin{tikzcd} * \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner"] & S^1 \arrow[d] \\ S^1 \arrow[r] & S^1 \vee S^1 \end{tikzcd}$$

Moreover,

$$\begin{tikzcd} S^1 \arrow[r, "aba^{-1}b^{-1}"] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner"] & S^1 \vee S^1 \arrow[d] \\ D^2 \arrow[r, dashed] & T \end{tikzcd}$$

T : torus.

$a, ba^{-1}b^{-1}$ loop



Ex3.1.25 Cokernel in Group
$= \text{Coequalizer}(\phi: G \to H, 0: G \to H)$

Ex3.1.26. (n-skeleta)

$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$

Colimit of sequence $= \bigcup_{n \geq 0} X_n$.

ex) CW complex = colimit of its n-skeleta.

Rmk 3.1.27) $f: \coprod_{i \in I} A_i \longrightarrow X \iff \{f_i : A_i \to X \mid i \in I\}$

Thus, $C(\coprod_{i \in I} A_i, X) \xrightarrow{\cong} \prod_{i \in I} C(A_i, X)$

$f \longmapsto (f_i)_{i \in I}$.

by letting $\iota_i : A_i \to \coprod A_i$ and $f_i := f \circ \iota_i$.

Dually, $g: X \longrightarrow \prod_{j \in J} B_j \iff \{g_j : X \to B_j \mid j \in J\}$

$\Rightarrow C(X, \prod_{j \in J} B_j) \xrightarrow{\cong} \prod_{j \in J} C(X, B_j)$

$g \longmapsto g_i$.

by letting $\pi_j : \prod_{j \in J} B_j \to B_j$ with $g_i = \pi_j \circ g$.

Then, if $\coprod_{i \in I} A_i \xrightarrow{f} \prod_{j \in J} B_j$, it gives left diagram.

$$\begin{tikzcd}
\coprod_{i \in I} A_i \arrow[r, "f"] & \prod_{j \in J} B_j \arrow[d, "\pi_{j'}"] \\
A_{i'} \arrow[u, "\iota_{i'}"] \arrow[r, dashed, "f(i',j')"'] \arrow[ur, phantom, "\circlearrowright"] & B_{j'}
\end{tikzcd}$$

i.e., a map from coproduct to product is determined by a matrix of component maps.

In certain categories, Ab, Mod_R, Ch_R, have a notion of zero homomorphism between any pair of object.

In any finite collection I,

$$\coprod_{i \in I} A_i \xrightarrow{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}} \prod_{i \in I} A_i$$ make sense;

This induces isomorphism in "Abelian Category".

$\Rightarrow \coprod A_i \cong \prod_{i \in I} A_i$. So use notation $\bigoplus_{i \in I} A_i$.

a <u>direct sum</u>.

Then any composition of maps between direct sums is matrix multiplication.

Ex 3.1.i) Let $f: c \longrightarrow d \in \mathrm{Hom}\, C$. Then

$$\begin{tikzcd} \mathrm{Cone}(c,F) \\ \mathrm{Cone}(d,F) \arrow[u, "\alpha_f"] \end{tikzcd}$$

$\alpha_f = \mathrm{Cone}(f, F)$ s.t.

If $\lambda \in \mathrm{Cone}(d, F)$ then

$$\begin{tikzcd} & c \arrow[d, "f"] \arrow[ddl, bend right, "\lambda_j \circ f"'] \arrow[ddr, bend left, "\lambda_k \circ f"] & \\ & d \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ F_j \arrow[rr] & & F_k \end{tikzcd}$$

Implies that

then,

$$\overset{\alpha_f}{\Rightarrow} d \overset{\lambda}{\Rightarrow} F \in \mathrm{Cone}(c, F)$$

Dually,

$$\mathrm{Cone}(F, f) : \mathrm{Cone}(F, c) \xrightarrow{\alpha_f} \mathrm{Cone}(F, d)$$

s.t. if $\lambda \in \mathrm{Cone}(F, c)$ $f \circ \lambda$ is a cone in $\mathrm{Cone}(F, d)$.

$$\begin{tikzcd} F_j \arrow[rr] \arrow[dr, "\lambda_j"'] \arrow[ddr, bend right, "f \circ \lambda_j"'] & & F_k \arrow[dl, "\lambda_k"] \arrow[ddl, bend left, "f \circ \lambda_k"] \\ & c \arrow[d, "f"] & \\ & d & \end{tikzcd}$$

$\therefore F \Rightarrow c \Rightarrow d \in \mathrm{Cone}(d, F)$

Ex 3.1.ii) Notes that for any $\lambda \in \mathrm{Cone}(c, F)$
$\lambda : c \Rightarrow F$ natural transformation, thus $\lambda \in \mathrm{Hom}(\Delta(c), F)$

$\Rightarrow \mathrm{Cone}(c, F) \subseteq \mathrm{Hom}(\Delta(c), F)$.

Conversely, let $\lambda \in \mathrm{Hom}(\Delta(c), F)$. By definition, λ is a cone over F with summit c $\Rightarrow \mathrm{Cone}(c, F) = \mathrm{Hom}(\Delta(c), F)$

Also, $\mathrm{Hom}(\Delta(f), F) \cong \alpha_f = \mathrm{Cone}(f, F)$

Thus, $\mathrm{Cone}(-, F) \cong \mathrm{Hom}(\Delta(-), F)$.

Ex. 3.1. iii) WTS, $\int \mathrm{Cone}(-, F) \cong \Delta \downarrow F$

	$\int \mathrm{Cone}(-, F)$	$\Delta \downarrow F$
Obj	(c, λ), $c \in \mathcal{C}$, $\lambda \in \mathrm{Cone}(c, F)$	$(\Delta(c), F, \lambda : \Delta(c) \Rightarrow F)$. So $\lambda \in \mathrm{Cone}(c, F)$
Mor	$(c, \lambda) \xrightarrow{\bar{f}} (c', \lambda')$ if $f : c \to c'$ $\in \mathcal{C}$ and $\alpha_f \circ \lambda' = \lambda$	$(\Delta(c), F, \lambda) \xrightarrow{(f, 1_F)} (\Delta(c'), F, \lambda')$ s.t. $\begin{array}{ccc} \Delta(c) & \xrightarrow{\alpha_f} & \Delta(c') \\ {\scriptstyle \lambda} \downarrow & \circlearrowleft & \swarrow {\scriptstyle \lambda'} \\ F & & \end{array}$ $\Rightarrow \alpha_f \circ \lambda' = \lambda$.

Hence, there is bijection between them.

Now, for any $F \in (\mathcal{C}^{op})^J$;

$$\int \mathrm{Cone}^{op}(-, F) \cong \Delta^{op} \downarrow F^{op}$$

Where $\Delta^{op} : \mathcal{C}^{op} \to (\mathcal{C}^{op})^J$, $F^{op} : 1 \to (\mathcal{C}^{op})^J$.

And observe $\int \mathrm{Cone}^{op}(-, F) = \int \mathrm{Cone}(F, -)$

and $\Delta^{op} \downarrow F^{op} = F \downarrow \Delta$.

Ex 3.1. iv) From universal property, below diagram commute, for any of triangle. Then, $\lambda'_j a = \lambda_j$

$$\begin{tikzcd} & \ell \arrow[d, shift right, "a"'] \arrow[ddl, bend right, "\lambda_j"'] \arrow[ddr, bend left, "\lambda_k"] & \\ & \ell' \arrow[u, shift right, "b"'] \arrow[dl, "\lambda'_j"] \arrow[dr, "\lambda'_k"'] & \\ F_j \arrow[rr] & & F_k \end{tikzcd}$$

$\lambda_j b = \lambda'_j \Rightarrow \lambda_j b a = \lambda_j \Rightarrow ba = 1_\ell$.

Since $\mathrm{Hom}((\ell, \lambda), (\ell, \lambda)) = \{1_{(\ell, \lambda)}\}$.

and 1_ℓ induces $1_{(\ell,\lambda)}$.
By the similar argument, $ab = 1_{\ell'}$ $\Rightarrow$ a, b are isomorphisms $\Rightarrow$ $\ell \cong \ell'$.

Ex 3 (v) If $\lim F$ exists with limit cone λ;
then for any cone $\phi : p \Rightarrow F$, $\exists!\, a : p \to \lim F$ s.t. left diagram commutes

$$\begin{tikzcd}
 & p \arrow[d, "\exists! a"] \arrow[ddl, bend right, "\phi_j"'] \arrow[ddr, bend left, "\phi_k"] & \\
 & \lim F \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\
F_j \arrow[rr] & & F_k
\end{tikzcd}$$

$\Rightarrow$ Let $\mathrm{summit}(P, \leq)$ be a subcategory of $(P, \leq)$ spanned by elements in $(P, \leq)$ which is a summit of a cone.

Then, $\lim F$ is "a" maximum of $\mathrm{Summit}(P, \leq)$
(Not "the" maximum since preorder has two distinct element on which both $\geq$ and $\leq$ hold.)

Conversely, colimit is a minimum of subcategory $\mathrm{nadir}(P, \leq)$, spawed by elements which is nadir of some co cone over F.

Ex 3.1, vi) Let $f', g' : C \to E$ s.t. $hf' = hg'$.

Let $a = hf' = hg'$. Then, $fa = fhf' = (gh)f' = g(a)$. Thus universal property gives unique $k : C \to E$ s.t. $a = hk$.

i.e.

$$\begin{tikzcd} C \arrow[d, "\exists!\, k"'] \arrow[dr, "a"] & & \\ E \arrow[r, "h"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

This implies $f' = g'$.

Since uniqueness of k s.t. $a = hk$ implies $f' = k = g'$.

$\Rightarrow$ h is mono.

3.1, vii) Let $a, b : D \to P$ s.t. $ka = kb$.

$$\begin{tikzcd} D \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[rr, dashed, bend left, "c"] \arrow[dr, dashed, bend right, "d"'] & P \arrow[r, "k"] \arrow[d, "h"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ & B \arrow[r, tail, "f"'] & A \end{tikzcd}$$

Then, $gka = gkb$

$\Rightarrow fha = fhb$

$\Rightarrow ha = hb$

Now let $c = ka = kb$

$d = ha = hb$.

Then, $\exists!\, l : D \to P$ s.t. $c = kl$, $d = hl$.

This implies $a = l = b$ by uniqueness. $\Rightarrow$ k is mono.

3.1 viii) Suppose composite (big) rectangle is pull back for bg and h, then, the "big" pullback gives unique l s.t. $h' = cl$, $bg = bal$.

$$\begin{tikzcd} \bullet \arrow[dr, "\exists!\, l"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h'"'] & & & \\ & \bullet \arrow[r, "a"] \arrow[d, "c"'] & \bullet \arrow[r, "b"] \arrow[d, "i"] & \bullet \arrow[d, "d"] \\ & \bullet \arrow[r, "e"'] & \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

Apply "small" (right) pull back on bg and eh

we get unique k st. $bk = bg$ and $eh = ik$.

$$\begin{tikzcd}
\bullet \arrow[dr, "\exists! l"'] \arrow[drr, "\exists! k"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & & \\
& \bullet \arrow[r, "a"] \arrow[d, "c"'] & \bullet \arrow[r, "b"] \arrow[d, "i"] & \bullet \arrow[d, "d"] \\
& \bullet \arrow[r, "e"] & \bullet \arrow[r, "f"] & \bullet
\end{tikzcd}$$

By uniqueness of k. $al = k = g$.

Thus, $h = cl$, $g = al$. This shows the existence of a map l to make

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "c"'] & \bullet \arrow[d, "i"] \\
\bullet \arrow[r, "e"'] & \bullet
\end{tikzcd}$$

(left) pull back. To see uniqueness suppose $\exists l'$ st. $g = al'$, $h = cl'$. Then, applying Lis pull back on bg and h gives $h = cl = cl'$. Since l is unique, $l' = l$. $\blacksquare$

Conversely, both left and right one are pullbacks.

$$\begin{tikzcd}
\bullet \arrow[dr, "\exists! l"'] \arrow[drr, "\exists! k"] \arrow[drrr, bend left, "g"] \arrow[ddr, bend right, "h"'] & & & \\
& \bullet \arrow[r, "a"] \arrow[d, "c"'] & \bullet \arrow[r, "b"] \arrow[d, "i"] & \bullet \arrow[d, "d"] \\
& \bullet \arrow[r, "e"'] & \bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}$$

Then right pull back induce k, for g and eh st $g = bk$, $eh = ik$. And left pull back for k and h induces l s.t. $k = al$ $h = cl$.

Thus, $g = bal$, $h = cl$. Also if $\exists l'$ st. $g = bal'$, $h = cl'$ then, by uniqueness of k and l, $al' = k = al \Rightarrow l = l'$.

$\therefore$ Composite rectangular is pull back

Ex 3.1.9(x) Let i be initial in J. Then, for any $j \to k \in J$.

$$\begin{tikzcd} & i \arrow[dl] \arrow[dr] \arrow[d, phantom, "\circlearrowleft"] & \\ j \arrow[rr] & & k \end{tikzcd}$$

by uniqueness of $\mathrm{Hom}(i,k)$.

Thus, let $\lambda_j = F(i \to j)$ $\forall j \in J$.

Then, λ is a cone over F with summit Fi.

Let $\phi : C \Rightarrow F$ a cone with summit C.

Then,

$$\begin{tikzcd} & C \arrow[d, "\phi_i"] \arrow[ddl, bend right, "\phi_j"'] \arrow[ddr, bend left, "\phi_k"] & \\ & Fi \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ Fj \arrow[rr] & & Fk \end{tikzcd}$$

commutes since property of cone ϕ make the triangles in each side commute.

Now suppose $f : C \to Fi$ s.t. $\alpha_f(\lambda) = \phi$.

Then

$$\begin{tikzcd} C \arrow[d, "f"] \arrow[dd, bend right, "\phi_i"'] \\ Fi \arrow[d, "1_{Fi}"] \\ Fi \end{tikzcd}$$

$\Rightarrow$ $f = \phi_i$. Hence in $\int \mathrm{Cone}(-,F)$

(Fi, λ) is terminal element.

$\therefore$ $Fi = \lim F$.

Now suppose J is indexed by ω, i.e.

$\exists\, G : \omega \to J$ exists.

If J has terminal i, then $F : J \to C$ has colimit

Fi.

Then, If $\emptyset$ is a cocone with nadir c and λ is colimit cone,

$$\begin{tikzcd}
F_0 \arrow[r] \arrow[d, "\lambda_0"] \arrow[dd, bend right, "\phi_0"'] & F_1 \arrow[r] \arrow[dl, "\lambda_1"] \arrow[ddl, bend left, "\phi_1"] & \cdots \\
F_n \arrow[d, "\phi_n"] & & \cdots \\
C & &
\end{tikzcd}$$

implies that

$$\phi_n \circ \lambda_k = \phi_k$$

Since J is indexed by ω, this implies that for $k > n$, $\lambda_k \circ (F_n \to \cdots \to F_k) = 1_{F_n}$.

$\Rightarrow F_k = F_n$. Hence $J = \mathbb{n}$, a subcategory of ω spanned by $G(0), G(1), \cdots G(n)$,

And colimit of F is F_n.

Ex 3.1.x) Let $A \overset{c}{\underset{d}{\rightrightarrows}} \mathbb{Z}$ be a map s.t.

$$\begin{tikzcd}
A \arrow[dr, "\exists ! k"] \arrow[drr, bend left, "d"] \arrow[ddr, bend right, "c"'] & & \\
& \mathbb{Z} \arrow[r, "b"] \arrow[d, "a"'] & \mathbb{Z} \arrow[d, "n"] \\
& \mathbb{Z} \arrow[r, "m"'] & \mathbb{Z}
\end{tikzcd}$$

Then pull back induces k s.t. $d = bk$
$c = ak$

$$\begin{tikzcd}
A \arrow[dr, "-k"] \arrow[drr, bend left, "d"] \arrow[ddr, bend right, "c"'] & & \\
& \mathbb{Z} \arrow[r, "-b"] \arrow[d, "-a"'] & \mathbb{Z} \arrow[d, "n"] \\
& \mathbb{Z} \arrow[r, "m"'] & \mathbb{Z}
\end{tikzcd}$$

Now, we define $-k$, a composition of k with inversion. Then $(-k)\circ(-b) = k \circ b$ and $-a \circ -k = ak$.

Thus, $(-a, -b)$ also has a universal property.

However,

$$\begin{tikzcd} \mathbb{Z} \arrow[dr, "-1"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\ & \mathbb{Z} \arrow[r, "-b"] \arrow[d, "-a"] & \mathbb{Z} \arrow[d, "n"] \\ & \mathbb{Z} \arrow[r, "m"'] & \mathbb{Z} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[dr, "-1"] \arrow[drr, bend left, "-b"] \arrow[ddr, bend right, "-a"'] & & \\ & \mathbb{Z} \arrow[r, "b"] \arrow[d, "a"] & \mathbb{Z} \arrow[d, "n"] \\ & \mathbb{Z} \arrow[r, "m"'] & \mathbb{Z} \end{tikzcd}$$

Shows that $(\mathbb{Z}, (a,b)) \cong (\mathbb{Z}, (-a,-b))$ in the category of cones. So, this $(-a,-b)$ is just another limit one naturally isomorphic to (a,b)

Ex 3.1.xi) Note, that each leg is denoted by

$$i_j : A_j \longrightarrow \coprod_{j \in I} A_j$$

Since this category has a morphism for each pair of objects, let $f_{ij} : A_i \longrightarrow A_j \;\; \forall i \in I$, $f_{jj} = 1_{A_j}$ we can pick such f_{ij} by given assumption.

Then, A_j is coproduct over I, so universal property gives that

$$\forall i \in I, \quad \begin{tikzcd} A_i \arrow[r, "i_i"] \arrow[dr, "f_{ij}"'] & \coprod_{j \in I} A_j \arrow[d, "\exists! k_{ji}"] \\ & A_j \end{tikzcd}$$

Now let $i = j$. Then,

$$\begin{tikzcd} A_j \arrow[r, "i_j"] \arrow[dr, "1_{A_j}"'] & \coprod A_j \arrow[d, "k_{jj}"] \\ & A_j \end{tikzcd} \Rightarrow i_j \text{ is split mono.}$$

3.1.xii) Let $\lambda : c \Rightarrow F$ is J-shaped Cone

$\eta : c \Rightarrow FE$ is I-shaped cone.

Then, for $f' : i' \to j' \in J$ essential surjectivity gives $E(i) \overset{a_i}{\underset{\to}{\cong}} i'$, $E(j) \overset{a_j}{\underset{\to}{\cong}} j'$ for some $i, j \in I$.

Hence, $J(E(i), E(j)) \cong J(i', j')$ since composition of isomorphism induce bijection on $J(-,-)$. Thus, $\exists f : i \to j$ s.t.

$$E(f) = E(i) \xrightarrow{\cong} i' \xrightarrow{f'} j' \xrightarrow{\cong} E(j).$$

Hence

$$\begin{tikzcd}
& c \arrow[dl, "\eta_i"'] \arrow[dr, "\lambda_j"] \arrow[d, phantom, "\circlearrowleft"] & \\
FE(i) \arrow[rr, "FE(f)"] \arrow[d, "\cong"', "F(a_i)"] & \circlearrowleft & FE(j) \\
F(i') \arrow[rr, "Ff'"] & \circlearrowleft & F(j') \arrow[u, "\cong F(a_j^{-1})"'] \\
& c \arrow[ul, "\lambda_{i'}"] \arrow[ur, "\lambda_{j'}"'] &
\end{tikzcd}$$

Thus, sending η to $\eta a := \{F a_i \circ \eta_i\}$ sends $(c, \eta) \in \int Cone(-, FE)$ to $(c, \eta a) \in \int Cone(-, F)$

Conversely, sending λ to $\lambda a^{-1} := \{F(a_i^{-1}) \circ \lambda_i\}$ sends $(c, \lambda) \in \int Cone(-, F)$ to $(c, \lambda a^{-1}) \in \int Cone(-, FE)$

And this two map is inverse to each other.

So if we show these are functor, then $\int Cone(-, F)$ and $\int Cone(-, FE)$ are isomorphism of category.

If $(c, \eta) \xrightarrow{g} (c', \eta') \in \int \mathrm{Cone}(-, FE)$

then, $g : c \to c'$ and $\int \mathrm{Cone}(g, FE)(\eta') = \eta$

And $\int \mathrm{Cone}(g, FE) = \alpha_g : c \Rightarrow c'$.

So, $\eta' \circ \alpha_g = \eta$.

$$\Rightarrow \begin{tikzcd}
 & c \arrow[d, "f"] \arrow[ddl, bend right, "\eta_i"'] \arrow[ddr, bend left, "\eta_j"] & \\
 & c' \arrow[dl, "\eta'_i"'] \arrow[dr, "\eta'_j"] & \\
FE(i) \arrow[rr, "FE(f)"] \arrow[d, "F(a_i)"] & & FE(j) \arrow[d, "F(a_j)"] \\
F(i') \arrow[rr, "Ff'"] & & F(j')
\end{tikzcd}$$

Now sending η to ηa and η' to ηa.

has

$$\eta' a \circ \alpha_g = \eta a$$

So $\eta \longmapsto \eta a$

$\alpha_g \downarrow \quad \longmapsto \alpha_g$ is functor.

$\eta' \longmapsto \eta' a$.

In the similar way, we can conclude that

$\int \mathrm{Cone}(-, FE) \cong \int \mathrm{Cone}(-, F)$.

Hence if $\lim FE$ exits, $(\lim FE, \lambda)$ is initial in $\int \mathrm{Cone}(-, FE)$. Thus, $(\lim FE, \lambda a)$ is initial since they are isomorphic category.

$\Rightarrow \lim F = \lim FE$.

Ex 3.1. xiii). Coproduct of CRing.

Let $\coprod_{i \in J} R_j$ be a coproduct of CRing

Then, $i_{j'}: R_{j'} \longrightarrow \coprod_{j \in J} R_j$ exists and

for any $f_j : R_j \longrightarrow S$ $\forall j$, $\exists! \, k : \coprod R_j \to S$

s.t.

$$\begin{tikzcd} R_{j'} \arrow[r, "i_{j'}"] \arrow[dr, "f_{j'}"'] & \coprod_{j \in J} R_j \arrow[d, "k"] \\ & S \end{tikzcd}$$

When $J = 2$, then right diagram holds.

$$\begin{tikzcd} R_1 \arrow[r, "i_1"] \arrow[dr, "f_1"'] & R_1 \amalg R_2 \arrow[d, "k"] & R_2 \arrow[l, "i_2"'] \arrow[dl, "f_2"] \\ & S & \end{tikzcd}$$

Notes that every comm. ring is $\mathbb{Z}$-module

So,

$$\begin{tikzcd} R_1 \times R_2 \arrow[r, "\otimes"] \arrow[dr, "f_1(r_1) \cdot f_2(r_2)"'] & R_1 \otimes_{\mathbb{Z}} R_2 \arrow[d, "\exists! \bar{f}"] \\ & S \end{tikzcd}$$

from the universal property of tensor product

We claim that $R_1 \otimes_{\mathbb{Z}} R_2 \cong R_1 \amalg R_2$

$R_i \xrightarrow{i_i} R_1 \amalg R_2$ induces induces $R_1 \times R_2 \longrightarrow R_1 \amalg R_2$

So universal property of tensorproduct gives unique $\bar{i} : R_1 \otimes_{\mathbb{Z}} R_2 \longrightarrow R_1 \amalg R_2$. Also, Universal property of Coproduct on $R_1 \xrightarrow{- \otimes 1} R_1 \otimes R_2$, $R_2 \xrightarrow{1 \otimes -} R_1 \otimes R_2$

induces unique $k : R_1 \amalg R_2 \longrightarrow R_1 \otimes_{\mathbb{Z}} R_2$

$\Rightarrow$

$$\begin{tikzcd}
 & R_1 \amalg R_2 \arrow[d, shift right, "k"'] & \\
R_1 \arrow[ur, "i_1"] \arrow[dr, "- \otimes 1"'] & & R_2 \arrow[ul, "i_2"'] \arrow[dl, "1 \otimes -"] \\
 & R_1 \otimes_{\mathbb{Z}} R_2 \arrow[u, shift right, "\bar{i}"'] &
\end{tikzcd}$$

Now $\bar{i} \circ k \circ i_1 = \bar{i}(- \otimes 1) = i_1$

$\Rightarrow$ Thus $R_1 \xrightarrow{\bar{i} \circ k \circ i_1} R_1 \amalg R_2$ with universal property
$R_2 \xrightarrow{\bar{i} \circ k \circ i_2} R_1 \amalg R_2$

of coproduct induces $\exists! \, R_1 \amalg R_2 \longrightarrow R_1 \amalg R_2$

which is identity. $\Rightarrow$ $\bar{i} \circ k = 1_{R_1 \amalg R_2}$

Conversely, $k \circ \bar{i} \circ (1 \otimes -) = k \circ i_2 = 1 \otimes -$

Hence,

$$\begin{tikzcd}
R_1 \times R_2 \arrow[r, "\otimes"] \arrow[dr, "k \circ \bar{i} \circ (- \otimes -) = \otimes"'] & R_1 \otimes R_2 \arrow[d, "\exists! \, \overline{k \circ i (1 \otimes -)}"] \\
 & R_1 \otimes R_2
\end{tikzcd}$$

$\Rightarrow k \circ \bar{i} = 1_{R \otimes R}$

Hence, k and $\bar{i}$ are isomorphism.

3.2 Limits in the category of sets.

Def 3.2.1. $F: J \to C$ diagram is small if J is small category.

C is complete $\iff$ $\forall F: J \to C$ small diagram $\exists \lim F$

" cocomplete $\iff$ $\forall F: J \to C$ " $\exists \operatorname{colim} F$.

Claim: $F: J \to \mathrm{Set}$ small diagram. If $\lim F$ exists, then $\lim F \cong \mathrm{Cone}(1, F)$

pf) $\mathrm{Set}(X, \lim F) \cong \mathrm{Cone}(X, F)$ $\forall X \in \mathrm{Set}$.

Take $X = 1$, singleton. Then

$$\lim F \cong \mathrm{Set}(1, \lim F) \cong \mathrm{Cone}(1, F)$$

Def 3.2.3 $\forall F: J \to \mathrm{Set}$ small diagram,

$\lim F := \mathrm{Cone}(1, F)$ and define $\lambda: \lim F \Rightarrow F$

by $\lambda_j: \lim F \to F_j$

$$\mu: 1 \Rightarrow F \longmapsto \mu_j: 1 \to F_j.$$

(Notes that μ_j can be identified as element of F_j)

Thm 3.2.6. 3.2.3 is well-defined, i.e, set is complete.

pf) We need to show that λ is a cone and limit cone.

Let $f: j \longrightarrow k \in J$. $\Rightarrow$

$$\begin{tikzcd} & \lim F \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ Fj \arrow[rr, "Ff"] & & Fk \end{tikzcd}$$

induces $Ff(\lambda_j(\mu)) = Ff(\mu_j)$

Now, $\mu : 1 \Rightarrow F$. Hence

$$\begin{tikzcd} & 1 \arrow[dl, "\mu_j"'] \arrow[dr, "\mu_k"] \arrow[d, phantom, "\circlearrowleft"] & \\ Fj \arrow[rr, "Ff"'] & {} & Fk \end{tikzcd} \Rightarrow Ff(\mu_j) = \mu_k$$

$\Rightarrow Ff(\lambda_j(\mu)) = Ff(\mu_j) = \mu_k = \lambda_k(\mu)$.

So the triangle commute. $\Rightarrow$ λ is a cone.

To see λ is universal cone, let $\xi : X \Rightarrow F$ be a cone. For each $x \in X$, $\xi x : 1 \Rightarrow F$ in the sense that $(\xi x)_j := \xi_j|_{\{x\}}$.

Hence let $r: X \longrightarrow \lim F$ This induces
$x \longmapsto \xi x$.

$$\begin{tikzcd} & X \arrow[d, "r"] \arrow[ddl, bend right, "\xi_j"'] \arrow[ddr, bend left, "\xi_k"] & \\ & \lim F \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ Fj \arrow[rr] & & Fk \end{tikzcd}$$

s.t. $\lambda_j(r(x)) = \lambda_j(\xi x)$
$= (\xi x)_j = \xi_j|_{\{x\}} = \xi_j(x)$

Moreover, any map $X \longrightarrow \lim F$ should sends x to ξx, otherwise the above diagram doesn't commute

$\Rightarrow$ r is unique. $\Rightarrow$ $\lim F$ is well-defined.

Ex 3.2.7) Let $F: J \to C$ with J is discrete.
Then $\lim F = \prod_{j\in J} A_j$. By thm 3.2.6 $\lim F \cong \mathrm{Cone}(1,F)$

Notes that $\lambda : 1 \Rightarrow F$ consists of $\lambda_j : 1 \to A_j$, i.e. elements of A_j for each $j \in J$.

$\Rightarrow \prod_{j\in J} A_j := \{ (a_j \in A_j)_{j\in J} \}$ a Cartesian product.

Ex 3.2.8) Terminal object of Set = Cone with summit 1 over empty diagram

$= 1$.

Ex 3.2.9.) $F : (\overset{j}{\cdot} \rightrightarrows \overset{k}{\cdot}) \longrightarrow C$: equalizer.

$\mathrm{Cone}(1,F) := \{ \lambda : 1 \Rightarrow F \}$

$$\begin{tikzcd}
 & 1 \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\
Fj \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & Fk
\end{tikzcd}$$

$\Rightarrow$ So identifying λ_j as an element of Fj,

$\mathrm{Cone}(1,F) = \{ x \in Fj : fx = gx \}$

Ex.3.2.10. $F : \omega^{op} \longrightarrow \mathrm{Set}$. If $\lambda \in \mathrm{Cone}(1,F)$

$$\Rightarrow \quad \begin{tikzcd}
 & 1 \arrow[d] \arrow[dr, "\lambda_3"] \arrow[drr, "\lambda_2"] \arrow[drrr, "\lambda_1"] \arrow[drrrr, "\lambda_0"] & & & & \\
 \cdots \arrow[r] & \cdots \arrow[r] & F_3 \arrow[r, "f_{32}"'] & F_2 \arrow[r, "f_{21}"'] & F_1 \arrow[r, "f_{10}"'] & F_0
\end{tikzcd}$$

By identifying each λ_i as element of F_i,

$\mathrm{Cone}(1,F) = \{ (x_n) \in \prod_{n\in\omega} F_n \mid f_{n,n-1}(x_n) = x_{n-1} \}$

Ex. 3.2.11. For pull back, $\lambda \in \mathrm{Cone}(1; F)$ implies

$$\begin{tikzcd} 1 \arrow[r, "\lambda_1"] \arrow[d, "\lambda_2"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

So by identifying λ_1 and λ_2 as element of C and B resp.,

$$\mathrm{Cone}(1, F) =: B \times_A C := \{(b, c) \in B \times C : f(b) = g(c)\}.$$

Ex 3.2.12 $X : BG \longrightarrow \mathrm{Set}$. Then $\lambda \in \mathrm{Cone}(1, X)$ satisfy

$$\begin{tikzcd} & 1 \arrow[dl, "\lambda"'] \arrow[dr, "\lambda"] & \\ X \arrow[rr, "g_*"'] & \circlearrowright & X \end{tikzcd}$$

for all $g \in G$.

By identifying $\lambda \in X$, $g\lambda = \lambda$ $\therefore \lambda$ is fixed $\forall g$

$\Rightarrow$ $\lim F = \mathrm{Cone}(1, X) = X^G$ set of G-fixed pt of X.

Rmk) Let

$$E \rightarrowtail B \times C \underset{(b,c) \mapsto g(c)}{\overset{(b,c) \mapsto f(b)}{\rightrightarrows}} A.$$

Then, its equalizer is

$$E = \{(x, y) \in B \times C \text{ s.t. } f(x) = g(y)\}$$

$$= B \times_A C.$$

This is not a coincidence.

Thm. 3.2.13. $\forall F : J \to \mathrm{Set}$ small diagram,

$$\lim_J F \rightarrowtail \prod_{j \in \mathrm{ob} J} F_j \underset{d}{\overset{c}{\rightrightarrows}} \prod_{f \in \mathrm{mor} J} F(\mathrm{cod} f)$$

is an equilizer diagram, i.e. $\lim_J F$ is an equilizer of c and d.

Pf) $\lim_J F = \mathrm{Cone}(1, F)$. Pick $\lambda : 1 \Rightarrow F$.

Then for any $f \in \mathrm{Mor}\,J$,

$$\begin{tikzcd} & 1 \arrow[dl, "\lambda_{\mathrm{dom} f}"'] \arrow[d, phantom, "\circlearrowleft"] \arrow[dr, "\lambda_{\mathrm{cod} f}"] & \\ F(\mathrm{dom} f) \arrow[rr, "Ff"] & & F(\mathrm{cod} f) \end{tikzcd}$$

Hence we may identify λ_j be element of Fj.

$\Rightarrow$ λ is identified by $(\lambda_j) \in \prod_{j \in \mathrm{Obj}} Fj$

Now, define

$$c : (\lambda_j)_{j \in \mathrm{Obj}} \longmapsto (\lambda_{\mathrm{cod} f})_f \in \prod_{f \in \mathrm{Mor} J} F(\mathrm{cod} f)$$

by copying existing components of $(\lambda_j)_{j \in \mathrm{Obj}}$.

$$d : (\lambda_j)_{j \in \mathrm{Obj}} \longmapsto (Ff(\lambda_{\mathrm{dom} f}))_f \in \prod_{f \in \mathrm{Mor} J} F(\mathrm{cod} f)$$

Then, If $c(\lambda) = d(\lambda)$, then for each $f \in \mathrm{Mor}\,J$,

$Ff(\lambda_{\mathrm{dom} f}) = \lambda_{\mathrm{cod} f}$ $\Rightarrow$ $\lambda \in \mathrm{Cone}(1, F)$.

$\Rightarrow$ Equalizer $\subseteq \lim_J F$.

And every element in $\lim_J F$ satisfy $c(\lambda) = d(\lambda)$

$\Rightarrow$ $\lim_J F$ is Equalizer of c and d. $\blacksquare$

rmk) c, d can be obtained by universal property of $\prod_{j \in \mathrm{Obj} J} F_j$.

$$\begin{tikzcd} & & F(\mathrm{cod} f) \\ \lim_J F \arrow[r, tail] & \prod_{j \in \mathrm{Obj} J} F_j \arrow[ur, "\pi_{\mathrm{cod} f}"] \arrow[r, "\exists ! c"] & \prod_{f \in \mathrm{Mor} J} F(\arrow[u, "\pi_f"'] \end{tikzcd}$$

$$\begin{tikzcd} \lim_J F \arrow[r, tail] & \prod_{j \in \mathrm{Ob} J} F_j \arrow[r, "\exists! \, l"] \arrow[d, "\pi_{\mathrm{dom} f}"'] & F(\mathrm{cod} f) \arrow[d, "\pi_f"] \\ & F(\mathrm{dom} f) \arrow[r, "Ff"'] & F(\mathrm{cod} f) \end{tikzcd}$$

Then, we can use the same argument showing that $\lim_J F$ is equalizer.

Ex. 3.2.16 (Idempotent) $e: A \to A$ is idempotent if $e \cdot e = e$. So $F = (\overset{A}{\circlearrowleft} \cdot \, \circlearrowright e) \to \mathrm{Set}$ s.t. $e \cdot e = e$

If $\lambda: 1 \Rightarrow F$, then,

$$\begin{tikzcd} & 1 \arrow[dl, "\lambda"'] \arrow[d, "\lambda"] \arrow[dr, "\lambda"] & \\ A \arrow[r, "e"'] & A \arrow[r, "e"'] & A \end{tikzcd}$$

$e\lambda = \lambda$ and $e \cdot e\lambda = e\lambda$.

Thus $\lim F = \{a \in A : ea = a\}$
$= A^e$: fixed pt of idempotent.

Using Exercise 3.2.11, $A^e \overset{s}{\rightarrowtail} A \overset{1}{\underset{e}{\rightrightarrows}} A$ as an equalizer. Then, $\exists! r$ s.t below diagram commute.

$$\begin{tikzcd} A \arrow[d, dashed, "r"'] \arrow[dr, "e"] & & \\ A^e \arrow[r, tail, "s"'] & A \arrow[r, shift left, "1"] \arrow[r, shift right, "e"'] & A \end{tikzcd}$$

$\Rightarrow sr = e$
i.e. sr splits e.

Now $srs = es = s$. Thus

$$\begin{tikzcd} A^e \arrow[d, shift right, "rs"'] \arrow[d, shift left, "1_{A^e}"] \arrow[dr, "s"] & & \\ A^e \arrow[r, tail, "s"] & A \arrow[r, shift left, "1"] \arrow[r, shift right, "e"'] & A \end{tikzcd}$$

$\Rightarrow$ $rs = 1_{A^e}$ by uniqueness

Also notes that r is surjective: $\forall a \in A^e$, $ea = a$ $\Rightarrow$ $sr(a) = a$ and s is canonical injection, thus $r(a) = a$.

Conversely, $B \overset{s}{\rightarrowtail} A \overset{r}{\twoheadrightarrow} B$ with $rs = 1_B$ gives an idempotent sr on A.

($sr\,sr = s\,1_B\,r = sr$) which is split by B.

Ex. 3.2 i) Take a pullback P of below diagram.

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d, dashed] & \mathrm{mor}\,C \arrow[d, "\mathrm{dom}"] \\ \mathrm{mor}\,C \arrow[r, "\mathrm{Cod}"'] & \mathrm{Ob}\,C \end{tikzcd}$$

Then $P := \mathrm{mor}\,C \times_{\mathrm{Ob}C} \mathrm{mor}\,C$
$= \{ (g, f) \in \mathrm{mor}\,C \times \mathrm{mor}\,C \mid \mathrm{dom}\,g = \mathrm{cod}(f) \}$

Thus if $(g, f) \in P$, then $g \circ f$ is possible composition.

And we have $m: P \longrightarrow \mathrm{mor}\,C$, $(g, f) \longmapsto gf$. It is well-defined function.

Now we have

$$\begin{tikzcd} P \arrow[r, "m"] & \mathrm{mor}\,C \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{Cod}"'] & \mathrm{Ob}\,C \arrow[l, "\mathrm{id}"] \end{tikzcd}$$

Then $m(gf)$ has dom $\mathrm{dom}\,f$ cod $\mathrm{cod}\,g$.

We claim that
$$\begin{tikzcd} B\times_A B \arrow[r, "m"] & B \arrow[r, shift left=3, "f"] \arrow[r, shift right=3, "h"'] & A \arrow[l, "g"'] \end{tikzcd}$$
with ① $f(m(b_1, b_2)) = f(b_2)$
② $h(m(b_1, b_2)) = h(b_1)$ has a category structure
③ $m(b_3, m(b_2, b_1)) = m(m(b_3, b_2), b_1)$ ④ $m(g(a), b) = b =$ [illegible]
⑤ $m(b, g(a)) = b$

① Object $= A$

② Morphism $= B$ i.e. by identifying $b \in B$

as $f(b) \longrightarrow h(b)$.

③ Composition m. Suppose b_1, b_2, b_3 are

composable, i.e. $(b_2, b_1), (b_3, b_2) \in P$.

Then, $f \circ m(b_2, b_1) = f(b_1)$ $\quad f \circ m(b_3, b_2) = f(b_2)$

$h \circ m(b_2, b_1) = h(b_2)$ $\quad h \circ m(b_3, b_2) = h(b_3)$

Hence, $f(b_3) = h(b_2) = h \circ m(b_2, b_1)$

implies $(b_3, m(b_2, b_1)) \in B \times_A B$.

Also, $f \circ m(b_3, b_2) = f(b_2) = h(b_1) \Rightarrow (m(b_3, b_2), b_1) \in B \times_A B$.

Then,

$$m(b_3, m(b_2, b_1)) = m(m(b_3, b_2), b_1)$$

shows associativity.

Also, if $g(a)$ has a role of identity morphism

Thus, the diagram with ①, ②, ③, ④, 5

gives a small category structure J.

with $\mathrm{Ob}\, J = A$

$\mathrm{Mor}\, J = B$.

Ex 3.2 ii) Change $\prod_{f \in \mathrm{Mor} J'} F(\mathrm{cod} f)$, with $\prod_{f \in \mathrm{Mor} J / \mathrm{ob} J} F(\mathrm{cod} f)$ in the diagram p. 88. In this case, If J is not a discrete category then we still have c and d which is the same as above. So equalizer must be same.

In case of J is discrete, then $c = d$. thus $\lim_J F = \prod_{j \in \mathrm{ob} J} F_j$, a product, as be expected.

Ex 3.2 iii) If
$$\begin{tikzcd} a \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & c \arrow[d, "g"] \\ b \arrow[r, "l"'] & d \end{tikzcd}$$
, then $lf = gh$.

$$\Rightarrow \begin{tikzcd} P \arrow[r] \arrow[d] & \mathrm{Hom}(a,c) \arrow[d, "g^*"] \\ \mathrm{Hom}(b,d) \arrow[r, "f_*"'] & \mathrm{Hom}(a,d) \end{tikzcd}$$

let P be a pull back. then,

$$P = \mathrm{Hom}(a,c) \times_{\mathrm{Hom}(a,d)} \mathrm{Hom}(b,d)$$

s.t $(\cong \{ (h, l) \in \mathrm{Hom}(a,c) \times \mathrm{Hom}(b,d)$
$g^*(h) = f_* l$.
i.e. $gh = lf \}$

Thus, $Sq(f,g) := \mathrm{Hom}(a,c) \times_{\mathrm{Hom}(a,d)} \mathrm{Hom}(b,d)$

Ex 3.2 iv Let $J^{\S}$ be a category s.t.

$\mathrm{Ob}\, J^{\S} = \mathrm{Mor}\, J$. $\mathrm{Mor}\, J^{\S}$: for each $f: x \to y \in J$

$$\left.\begin{matrix} 1_x \to f \\ 1_y \to f \end{matrix}\right\} \in \mathrm{Mor}\, J^{\S}$$

Let $\alpha \in \mathrm{Hom}(F, G)$. Then, for any $f: x \to y \in J$

$$\begin{tikzcd} Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowleft"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\alpha_y"'] & Gy \end{tikzcd}$$

Now construct $H: J^{\S} \to \mathrm{Set}$

$$\begin{tikzcd} 1_x \arrow[d] \arrow[r, mapsto] & \mathrm{Hom}(Fx, Gx) \arrow[d, "(Gf)_*"] \\ f: x \to y \arrow[r, mapsto] & \mathrm{Hom}(Fx, Gy) \\ 1_y \arrow[u] \arrow[r, mapsto] & \mathrm{Hom}(Fy, Gy) \arrow[u, "(Ff)^*"'] \end{tikzcd}$$

If $\lambda \in \mathrm{Cone}(1, H)$ then,

$$\begin{tikzcd} & 1 \arrow[dl, "\lambda_x"'] \arrow[d, "\lambda_f"] \arrow[dr, "\lambda_y"] & \\ \mathrm{Hom}(Fx, Gx) \arrow[r, "(Gf)_*"'] & \mathrm{Hom}(Fx, Gy) & \mathrm{Hom}(Fy, Gy) \arrow[l, "(Ff)^*"] \end{tikzcd}$$

By identification, this shows that

$(Gf)_*(\lambda_x): Fx \to Gy = \lambda_f = Ff^*(\lambda_y)$

i.e.

$$\begin{tikzcd} x \arrow[r, "Ff"] \arrow[d, "\lambda_x"'] \arrow[dr, phantom, "\circlearrowleft"] & y \arrow[d, "\lambda_y"] \\ x \arrow[r, "Gf"'] & y \end{tikzcd}$$

Hence we identify λ as $\lambda \in \mathrm{Hom}(F, G)$

Conversely, if $\lambda \in \mathrm{Hom}(F, G)$

then define $\lambda_f = Gf \circ \lambda_x = \lambda_y \circ Ff$

for any $f : x \to y \in J$ nontrivial morphism

Then, by identifying λ_x and λ_y as a map

$1 \to \mathrm{Hom}(Fx, Gx)$ and $1 \to \mathrm{Hom}(Fy, Gy)$

we have a diagram

$$\begin{tikzcd} & 1 \arrow[dl, "\lambda_x"'] \arrow[d, "\lambda_f"] \arrow[dr, "\lambda_y"] & \\ \sim \arrow[r, "(Gf)^*"'] & \sim & \sim \arrow[l, "(Ff)_*"] \end{tikzcd}$$

which commutes every triangle.

Hence such λ is a cone.

So, $\mathrm{Hom}(F, G) \cong \mathrm{Lim}\, H$ as a set.

Ex. 3.2.4) It suffices to show that equalizer is $\mathrm{Hom}(F, G)$. Let E be equalizer of diagram. Then,

$$E = \left\{ ((\lambda_j)_{j \in \mathrm{Ob}J} \in \prod_{j \in \mathrm{Ob}J} C(Fj, Gj) \right.$$

$$\text{s.t. } (Ff^*) \circ \pi_{j'} ((\lambda_j)_{j \in \mathrm{Ob}J})$$

$$= Gf_* \circ \pi_j ((\lambda_j))_{j \in \mathrm{Ob}J}$$

i.e, $\lambda_{j'} \circ Ff = Gf \circ \lambda_j$, i.e.

$$\begin{tikzcd} Fj \arrow[r, "\lambda_j"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowright"] & Gj \arrow[d] \\ Fj' \arrow[r, "\lambda_{j'}"'] & Gj' \end{tikzcd}$$

Thus $E \subseteq \mathrm{Hom}(F,G)$. Conversely, any $\lambda \in \mathrm{Hom}(F,G)$ as identified by $(\lambda_j)_{j\in \mathrm{ob}J} \in \prod_{j\in\mathrm{ob}J} C(Fj,Gj)$ satisfy the diagram $\Rightarrow$ $\lambda \in E$. $\Rightarrow$ $E = \mathrm{Hom}(F,G)$.

Ex 3.2.vi)

$\lim F \cong \mathrm{Cone}(1,F)$. Let $\lambda \in \lim F$.

Then, $\lambda_c : 1 \to Fc$ can be identified as an element of Fc.

Thus, for each $\lambda \in \lim F$,

let
$$G : \mathrm{Cone}(1,F) \longrightarrow \mathrm{Section}(\pi)$$
$$\lambda \longmapsto F_\lambda : C \longrightarrow \int F$$
$$c \longmapsto (c, \lambda_c).$$

Conversely, let $H \in \mathrm{Section}(\pi)$.

Then, $\forall c \in \mathrm{Ob}C$, $Hc = (c, \eta_c)$ for some $\eta_c \in Fc$. And, for all $\exists\, c \xrightarrow{f} d \in C$,

$$Hf : (c, \eta_c) \longrightarrow (d, \eta_d)$$

s.t. $f : c \to d$ and $Ff(\eta_c) = \eta_d$.

Hence.
$$\begin{tikzcd} & 1 \arrow[dl, "\eta_c"'] \arrow[dr, "\eta_d"] \arrow[d, phantom, "\Omega"] & \\ Fc \arrow[rr, "Ff"'] & {} & Fd \end{tikzcd}$$
$\Rightarrow$ $\eta = \{\eta_c \mid c \in \mathrm{Ob}C\}$ can be identified as a cone $1 \Rightarrow F$.

And this is inverse of G.

Recall limit cone λ is $\lambda : \lim F \Rightarrow F$ s.t.

$\forall\, \mu : X \Rightarrow F$ a cone, $\exists!\, f : X \to \lim F \in \mathbf{Set}$.

s.t. $\lambda \circ \alpha_f = \mu$.

Then, $\forall x \in X$, $\mu(x) : 1 \Rightarrow F$. Thus,

$\lambda \circ \alpha_{f(x)}$: "

Since $f|_{\{x\}}$ is an iso, we know that

$\mu(x) = \lambda(f(x))$ as a natural transformation.

Thus, every cone $\mu : X \Rightarrow F$ is just set of

$\mu(x) : 1 \Rightarrow F$ s.t. $\mu(x)$ is a section.

Thus limit cone is just a cone S induced by

Section (π) s.t. for $f : c \to d \in \mathrm{Mor}\, C$,

when $F : C \to \int F$
$c \mapsto (c, \lambda_c)$

$$\begin{tikzcd}
 & & \text{Section}(\pi) \arrow[dll, "S_c"'] \arrow[drr, "S_d"] & & \\
 & & F \arrow[dl] \arrow[dr] \arrow[d, phantom, "\circlearrowleft"] & & \\
F_c \arrow[rrrr, bend right, "Ff"'] & \lambda_c \arrow[rr, mapsto] & \, & \lambda_d & F_d
\end{tikzcd}$$

left diagram is induced, thus $S : \text{Section}(\pi) \Rightarrow F$

can be regarded as a limit. For any $\mu : X \Rightarrow F$,

$\mu(x) : 1 \Rightarrow F$ so $\mu(x) \in \text{Section}(\pi)$, Thus the map

$X \to \text{Section}(\pi)$ by $x \mapsto \mu(x)$ induces desired diagram.

3.3. Preservation, reflection, creation

Def 3.3.1. For any class $\mathcal{L}$ of diagrams $K : J \to C$ and $F : C \to D$

① F preserves those limits :

$\forall K : J \to C$ in $\mathcal{L}$ and $\kappa : \lim K \Rightarrow K$ is a limit cone $\Rightarrow$ $F\kappa$ is a limit cone of FK

i.e. $F(\lim K) = \lim FK$ and $F\kappa : \lim FK \Rightarrow FK$.

② F reflects 〃

$\forall \kappa : X \Rightarrow K$ in C, and $F\kappa$ is a limit cone of FK $\Rightarrow$ κ is a limit cone, i.e. $X = \lim K$ and κ has universal property

③ F creates 〃

If $\lim FK$ exists, $\exists\, \kappa : \lim FK \Rightarrow FK$ a limit cone s.t. it can be lifted to a limit cone over K and F reflects the limits.

(For colimit, dualize these.)

In otherwords, if $\kappa : \lim K \Rightarrow K$ is a limit cone, $\kappa' : X \Rightarrow K$ is a cone,

① preserve : " $F\kappa$ is a limit cone (Thus $F(\lim K) = \lim FK$)

② reflects " $F\kappa'$ is a limit cone $\Rightarrow$ κ' is limit cone

③ creates FK has a limit cone $\Rightarrow$ K has a limit cone and F ~~is~~ reflects.

Prop 3.3.3. ① $F: C \to D$ creates limit ② D has a limit.

$\Rightarrow$ C has a limit and F preserves it.

pf) Let $K: J \to C$ be in the class.

$\mu: d \Rightarrow FK$ is a limit cone.

Since F creates these limits, $\exists \lambda: c \Rightarrow K$ a limit cone s.t. $F\lambda$ is a limit cone.

By Ex. 3.1.iv, $F\lambda \cong \mu$.

To see F preserves them, let $\lambda': c' \Rightarrow K$ be another limit cone. By prop 3.1.7, $\lambda' \cong \lambda$

$\Rightarrow F\lambda' \cong F\lambda \cong \mu$. $\Rightarrow$ $F\lambda'$ is a limit cone.

Def 3.3.4 (Sheaf) $\mathcal{O}(X)$: poset of open subsets of X ordered by inclusion. Let I be a discrete small category.

$\{U_i\}_{i \in I}$ cover U if U_is are image of I

and colimit $(\{U_i\} \bigcup_{i \in I} \{ U_i \leftarrow U_i \cap U_j \to U_j \}) = U$.

Notes that colimit in this diagram is an open set U' with $U_i \to U'$ and $U_i \cap U_j \to U'$.

$\Rightarrow U' = \bigcup_{i \in I} U_i$. So this definition is consistent with usual definition of open coverings.

Let $F: \mathcal{O}(X)^{op} \to Set$. By def in 1.3.7(v) (since $\mathcal{O}(X)$ is small) F is presheaf. Sheaf is a presheaf preserving these colimits $\forall$ pair i, j and $\forall$ index I covering U.

Now let I' be a small category s.t. $I' \supseteq I$ s.t.

① I indexes $\{U_i\}_{i \in I}$ a cover of U.

② I' has objects (i,j) corresponds to $U_i \cap U_j$ $\forall i \in I$.

Let $K : (I')^{op} \hookrightarrow \mathcal{O}(X)^{op}$ and $F : \mathcal{O}(X)^{op} \to Set$ a sheaf. Notes that any morphism $U_i \to U_j$ actually induces $U_i \cap U_j \rightrightarrows U_i, U_j$.

Then, Thm 3.2.13 yields

$$\lim_{I'^{op}} FK \rightarrowtail \prod_{(i,j) \in I'} FK(i,j) \underset{d}{\overset{c}{\rightrightarrows}} \prod_{f^{op} \in mor(I')^{op}} F(\mathrm{cod} f^{op})$$

Let f^{op} be an arbitrary morphism in $(I')^{op}$.

We may regards it as $U_i \cap U_j \hookrightarrow U_i$ by the above argument.

Then, c is just sends $F(U_i \cap U_j)$ component in the middle to the same component in the last, by seeing p. 88 diagraph

Now, d is a little different. $\pi_{\mathrm{dom} f^{op}} = \pi_{\mathrm{cod} f}$.

So if $f : U_i \cap U_j \hookrightarrow U_i$ then d sends $F(U_i)$ in the middle to $F(U_i \cap U_j)$ in the last via $F(U_i \cap U_j \hookrightarrow U_i)$.

Now Let $\lambda \in \prod_{(i,j) \in I'} FK(i,j)$ If $c(\lambda) = d(\lambda)$, then,

$Ff(\lambda|_{U_i}) = \lambda|_{U_i \cap U_j} = Ff(\lambda|_{U_j})$ by applying d twice and c once.

This shows that if $Ff(\lambda|_{U_i}) = Ff(\lambda|_{U_j})$ for all i, j, then $\lambda \in \lim_{I'} FK$.

Thus, instead of C, we may use this condition.

$$\lim_{I'} FK \rightarrowtail \prod_{(i,j)\in I'} FK(i,j) \underset{Ff(\lambda|_{U_j})\circ\pi_j}{\overset{Ff(\lambda|_{U_i})\circ\pi_i}{\rightrightarrows}} \prod_{f^{op}\in \mathrm{Mor}(I')^{op}} F(\mathrm{cod} f^{op})$$

Since those two maps are only depends on $FK(i,j)$ with $i=j$, we may delete component with $i\neq j$, so.

$$\lim_{I'} FK \rightarrowtail \prod_{i\in I} FK(i) \underset{''}{\overset{''}{\rightrightarrows}} \prod \quad ''$$

Also, image of those two map only came of $FK(i,j)$ with $i\neq j$. So we may delete $F(U_i)$ parts in the last

$$\lim_{I'} FK \rightarrowtail \prod_{i\in I} FK(i) \underset{''}{\overset{''}{\rightrightarrows}} \prod_{\substack{f\in \mathrm{Mor}(I')^{op} \\ f: U_i\cap U_j \to U_i \\ \quad\;\; \to U_j}} F(U_i\cap U_j)$$

then, by change of notation, we may get

$$\lim_I F \rightarrowtail \prod_{i\in I} F(U_i) \underset{''}{\overset{''}{\rightrightarrows}} \prod_{i,j\in I} F(U_i\cap U_j)$$

as desired.

Lemma 3.3.5 $F : C \longrightarrow D$ fully faithful.
$\Rightarrow$ F reflects limits and colimits which are in codomain.

pf) Let $\mu : c \Longrightarrow K$ be a cone of $K \longrightarrow C$

s.t. $F\mu : Fc \Longrightarrow FK$ is a limit cone.

Let $\mu' : c' \Longrightarrow K$, WTS. $\exists ! \, f : c' \longrightarrow c$ s.t.

$$\begin{tikzcd}
 & c' \arrow[d, "\exists ! f"] \arrow[ddl, bend right, "\mu_i'"'] \arrow[ddr, bend left, "\mu_j'"] & \\
 \arrow[r, phantom, "\circlearrowright"] & c \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] \arrow[r, phantom, "\circlearrowright"] & {} \\
K_i \arrow[rr] & & K_j
\end{tikzcd}$$

left diagram commutes.

Now notes that $F\mu'$ is still a cone; this comes from functoriality of F.

Hence $\exists ! \; \bar{f} : Fc' \longrightarrow Fc$ s.t. $F\mu' = \alpha_{\bar{f}} \circ F\mu$.
By fully faithfulness $\exists \, f : c' \longrightarrow c$ s.t. $Ff = \bar{f}$.

Then, $F\mu_i' = F(\mu_i \circ f) \Rightarrow \mu_i' = \mu_i \circ f$
since fully faithfull implies $C(c, K_i) \xrightarrow{F(-)} D(Fc, FK_i)$ is bijection.
Thus, the above diagram commutes. So F reflects limit.
For the colimit, apply this on contravariant functor and get reflection of colimit by duality.

Lemma 3.3.6 Equivalence of Categories preserves, reflects, and creates any limit and colimits that are present in either its domain or codomain.

pf) Let $F : C \longrightarrow D$ $G : D \longrightarrow C$ be equivalence of categories. Then, F, G are fully faithful and essentially surjective.

It suffices to show that each F and G creates limit in D or C. Suppose we showed it

① reflectivity: done by Lemma 3.3.5

② preserves limit/colimit: If λ is a limit cone in C, then, G creates a limit cone λ' in D s.t. $G\lambda' \cong \lambda$. Thus, $G\lambda'$ is a limit cone.

Hence $F(\lambda) \cong FG(\lambda') \cong \lambda'$ implies F preserves a limit.

Thus, all we have to do is show that F creates limit/colimit.

Let λ be a limit cone in D, i.e. $\lambda : d \Rightarrow FK$.

By essential surjectivity, $\exists\, c \in C$ s.t. $Fc \xrightarrow[\cong]{g} d$.

Now we claim $\lambda_g : Fc \Rightarrow FK$ is also a limit cone; to see this, let $\mu : d' \Longrightarrow FK$ be a cone. Then,

$$\begin{tikzcd}
& d' \arrow[d, "\exists! h"] \arrow[ddl, bend right, "\mu_i"'] \arrow[ddr, bend left, "\mu_j"] & \\
& d \arrow[d, "g^{-1}"] \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
Fk_i \arrow[rr] & Fc \arrow[l, "\lambda_i \circ g"'] \arrow[r, "\lambda_j \circ g"] & Fk_j
\end{tikzcd}$$

left diagram commutes

Thus $\exists$ a map $g^{-1}h : d' \longrightarrow Fc$ s.t. $\mu = \alpha_{g^{-1}h} \circ \lambda g$.

Moreover this $g^{-1}h$ is unique; If $\exists h' : d' \longrightarrow Fc$ s.t. the above one commute, then left diagram

$$\begin{tikzcd}
& d' \arrow[d, "h'"] \arrow[dddl, bend right, "\mu_i"'] \arrow[dddr, bend left, "\mu_j"] & \\
& Fc \arrow[d, "g"] \arrow[ddl, bend right, "\lambda_i \circ g"'] \arrow[ddr, bend left, "\lambda_j \circ g"] & \\
& d \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
K_i \arrow[rr] & & Fk_j
\end{tikzcd}$$

commute, thus, universal property of $\lambda : d \Rightarrow FK$ shows that

$$gh' = h$$
$$\Rightarrow h' = g^{-1}h.$$

Now, fully faithfulness of F shows that any $\lambda_i \circ g$ comes from $\overline{\lambda_i \circ g} \in C(c, k_i)$ via F. Thus, $\exists \bar{\lambda} : c \Rightarrow K$ s.t. $F\bar{\lambda} = \lambda g$.

Since F reflects a limit, $\bar{\lambda}$ is a limit cone.

Hence, (since F is already reflects the limit)

F creates a limit. Since F was chosen arbitrarily, both F and G creates limits

By duality, // colimits.

$\square$

$K := \Pi K'$ and $k : c \Rightarrow K$ is a cone induced by a diagram $K' : J \to {}^{C}/_{c}$ with summit c.

WTS if $K : J \xrightarrow{(K,k)} {}^{C}/_{c} \xrightarrow{\Pi} C$ admits a limit cone in C, then this cone lifts to define a unique limit cone over (K,k) in ${}^{C}/_{c}$.

Let $\lambda : l \Rightarrow K$ be a limit cone in C. Also we have $k : c \Rightarrow K$. Hence $\exists!\, t : c \to l$ s.t. $k = \lambda \circ \alpha_t$.

Now, we can identify $\lambda_i : l \to K_i$ as a morphism from $t : c \to l$ to $(K,k)_i : c \to K_i$ in ${}^{C}/_{c}$. Thus below left $\triangle$ commutes since right triangle commutes in C.

$$
\begin{tikzcd}
 & (l,\, t : c \to l) \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] \arrow[d, phantom, "\circlearrowright"] & \\
(K,k)_i \arrow[rr, "(K,k)(i \to j)"'] & & (K,k)_j
\end{tikzcd}
\qquad
\begin{tikzcd}
 & c \arrow[d, "t"] \arrow[dl, bend right, "(K,k)_i"'] \arrow[dr, bend left, "(K,k)_j"] & \\
K_i \arrow[rr, "(K,k)(i \to j)"'] & l \arrow[l, "\lambda_i"'] \arrow[r, "\lambda_j"] & K_j
\end{tikzcd}
$$

So, λ is a cone $t \Rightarrow (K,k)$.

To see it is a limit cone, suppose that $\exists \mu : r \Rightarrow (K,k)$ in ${}^{C}/_{c}$. Then by letting $r : c \to r'$

Def) 3.3.7. $F: C \to D$ strictly creates limits.

If $\forall K: J \to C$ in the class and $\forall$ limit cone λ over FK,

① $\exists!$ lift $\bar{\lambda}$ of λ over K in C as a cone

② $\bar{\lambda}$ is limit cone

Def) J: category is connected if $J \neq \phi$ and in the underlying directed graph, every pair of objects is connected by a finite zig-zag of morphism.

A diagram $K: J \to C$ is connected $\Leftrightarrow$ J is connected

Prop 3.3.8 $\forall c \in \mathrm{Ob}C$, $\Pi: {}^{c}/_{C} \to C$ strictly creates

1) all limits C admit

2) all connected colimits C admit

$$\left(\begin{array}{lcl} & (d, f: c \to d) \longmapsto & d \\ h: d \to e \text{ s.t. } \begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd} & \downarrow \quad\quad \longmapsto & \downarrow h \\ & (e, g: c \to e) \longmapsto & e \end{array}\right)$$

pf) Let $K': J \to {}^{c}/_{C}$ be a diagram. Then $\forall i \to j \in \mathrm{Mor}J$, $K'(i \to j)$ is a morphism $h: \mathrm{Cod}(K'_i) \to \mathrm{Cod}(K'_j)$ s.t.
$$\begin{tikzcd} & c \arrow[dl, "K'_i"'] \arrow[dr, "K'_j"] & \\ \mathrm{Cod}(K'_i) \arrow[rr, "h"'] & & \mathrm{Cod}(K'_j) \end{tikzcd}$$
in C.

Hence K' always induces a cone $k: c \Rightarrow (J \to {}^{c}/_{C} \Rightarrow C)$ in C.

Thus, we use notation $K' = (K, k): J \to {}^{c}/_{C}$ s.t.

we have

$$
\begin{tikzcd}
 & (r', r: c \to r') \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] \arrow[d, phantom, "\circlearrowleft"] & \\
(k,k)_i \arrow[rr, "(k,k)(i\to j)"'] & & (k,k)_j
\end{tikzcd}
\implies
\begin{tikzcd}
 & c \arrow[d, "r"] \arrow[ddl, bend right, "(k,k)_i"'] \arrow[ddr, bend left, "(k,k)_j"] & \\
 & r' \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] \arrow[d, phantom, "\circlearrowleft"] & \\
k_i \arrow[rr, "(k,k)(i\to j)"'] & & k_j
\end{tikzcd}
$$

Since λ is a limit cone, $\exists!\, t': r' \to l$ s.t.

$$
\begin{tikzcd}
 & r' \arrow[d, "t'"] \arrow[ddl, bend right, "\mu_i"'] \arrow[ddr, bend left, "\mu_j"] & \\
 & l \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] \arrow[d, phantom, "\circlearrowleft"] & \\
k_i \arrow[rr] & & k_j
\end{tikzcd}
$$

left diagram commutes, hence $r \circ t' = t$ by uniqueness of t.

Thus, t' is a morphism in c/C.

$\boxed{\text{Figure 1}}$

Hence,

$$
\begin{tikzcd}
 & (r', r: c \to r) \arrow[d, "t'"] \arrow[ddl, bend right, "\mu_i"'] \arrow[ddr, bend left, "\mu_j"] & \\
 & (l, t: c \to l) \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] \arrow[d, phantom, "\circlearrowleft"] & \\
(k,k)_i \arrow[rr] & & (k,k)_j
\end{tikzcd}
$$

left diagram commutes by Figure 1

$\implies \lambda$ is a limit cone in c/C

In case of colimit, let $\mu: K \Rightarrow p$ is a colimit cone and J is connected. Still we have $k: c \Rightarrow p$ a cone.

Then, let η be a morphism

$$\eta_j : c \xrightarrow{k_j} k_j \xrightarrow{\mu_j} p.$$

We claim that $\eta : c \longrightarrow p$ as an object of $^{c}/\mathcal{C}$ is independent of choice of $j \in J$. To see this

for any $j, i \in \mathrm{ob}J$, $\exists$ zig zag morphism. Thus

$$\begin{tikzcd}
 & c \arrow[dl, "{(k,k)_j}"'] \arrow[d, "{(k,k)_{j'}}"] \arrow[dr, "{(k,k)_{i'}}"] \arrow[drrr, bend left, "{(k,k)_i}"] & & & \\
k_j \arrow[r] \arrow[dr, "\mu_j"'] & k_{j'} \arrow[d, "\mu_{j'}"] & k_{i'} \arrow[l] \arrow[r] \arrow[dl, "\mu_{i'}"] & \cdots \arrow[r] & k_i \arrow[dlll, "\mu_i"] \\
 & p & & &
\end{tikzcd}$$

left diagram with diagram chasing shows that $\mu_j \circ k_j = \mu_i \circ k_i$.

Now let $\phi : (J \to {}^{c}/\mathcal{C}) \Rightarrow f$ be a cone in $^{c}/\mathcal{C}$ under $(l, f : c \longrightarrow l)$. Then,

$$\begin{tikzcd}
 & (l, f : c \longrightarrow l) & \\
(k_i : c \longrightarrow k_i) \arrow[ur, "\phi_i"] \arrow[rr] & & (k_j : c \longrightarrow k_j) \arrow[ul, "\phi_j"']
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & l & \\
 & c \arrow[u, "f"] \arrow[dl, "k_i"'] \arrow[dr, "k_j"] & \\
k_i \arrow[uur, bend left, "\phi_i"] \arrow[rr] \arrow[dr, "\mu_j"'] & & k_j \arrow[uul, bend right, "\phi_j"'] \arrow[dl, "\mu_j"] \\
 & p &
\end{tikzcd}$$

Now,

$$\begin{tikzcd}
 & (l, f : c \longrightarrow l) & \\
 & (p, \eta : c \longrightarrow p) \arrow[u, "u"] & \\
(k_i : c \longrightarrow k_i) \arrow[uur, bend left, "\phi_i"] \arrow[ur, "\mu_j"] \arrow[rr] & & (k_j : c \longrightarrow k_j) \arrow[ul, "\mu_i"'] \arrow[uul, bend right, "\phi_j"']
\end{tikzcd}$$

Since μ is a colimit cone, $\exists! \, u : p \longrightarrow l$ s.t. $u \circ \mu_i = \phi_i$

So it suffices to show that u is a morphism in $^{c}/\mathcal{C}$.

Since $f = \phi_i \circ k_i$,
$\eta = \mu_i \circ k_i$

$$\begin{tikzcd} p \arrow[r, "u"] & l \\ c \arrow[u, "\eta"] \arrow[ur, "f"'] & \end{tikzcd}$$

Thus $u \circ \eta = u \circ \mu_i \circ k_i = \phi_i \circ k_i = f$.

$\Rightarrow$ u is a morphism in c/C.

Thus $\mu : \eta_j \Rightarrow (J \to c/C)$ is a colimit cone.

These colimit and lim construction is unique since it depends only on cone (or cocone) structure of C. $\Rightarrow$ Π strictly creates connected colimits or limits. $\square$

Rmk: If $K : J \to C^A$ is a diagram, then $\lim_J K$ is a functor from $A \to C$ s.t. is a limit of $J \to C$ whose evaluation at a by $j \mapsto F_j(a)$. (This is prop 3.3.9)

Prop 3.3.9.) Let A: small category. $\Pi : C^A \to C^{obA}$ be a forgetful functor where obA is a discrete subcategory of A. Then Π strictly creats all limit and colimit that exist in C.

These limits are defined objectwise, meaning that $\forall a \in A$, $ev_a : C^A \to C$ preserves all limits and colimits existing in C.

pf) C^{obA} is isomorphic to $\prod_{obA} C$; since any functor in C^{obA} is just choosing obA indexed elements in Ob C.

By applying universal property of product in CAT, any diagram $K: J \to C^{obA} \cong \prod_{obA} C$ is just obA-indexed family of diagram $J \to C$.

Let $\lim K$ be a limit in $\prod_{obA} C$, λ is a limit cone. Then, for any $\mu: (c_i)_{i \in obA} \Rightarrow K$, with $i \to j \in J$.

$$\begin{tikzcd}
 & (c_i)_{i\in obA} \arrow[d] \arrow[ddl, bend right, "\mu_i"'] \arrow[ddr, bend left, "\mu_j"] & \\
 & \lim K \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
K i \arrow[rr, "K(i\to j)"'] & & K j
\end{tikzcd}$$

but in the product of category, every homomorphism can be decomposed into components indexed by obA. So for any index $t \in obA$,

$$\begin{tikzcd}
 & C_t \arrow[d] \arrow[ddl, bend right, "\mu_{it}"'] \arrow[ddr, bend left, "\mu_{jt}"] & \\
 & (\lim K)_t \arrow[dl, "\lambda_{it}"'] \arrow[dr, "\lambda_{jt}"] & \\
(K_i)_t \arrow[rr, "K(i\to j)"'] & & (K_j)_t
\end{tikzcd}$$

Hence, $\lim K$ is assemble of limit in each C and so is its limit cone.

Thus, C admit lim iff C^{obA} does. (dually for colimit)

Also, $ev_a: C^{obA} \to C$ can be viewed as a projection, thus it preserves limit. (dually colimit)

To show $\Pi : C^A \longrightarrow C^{\mathrm{ob}A}$ strictly creates lim/colim,

Let $F : J \longrightarrow C^A$ a diagram s.t. ΠF has a limit and limit cone $\lambda : (C_i)_{i \in \mathrm{ob}A} \Rightarrow \Pi F$.

This equivalent to say that C admits a limit to $J \xrightarrow{F} C^A \xrightarrow{ev_a} C$ admits a limit for any $a \in \mathrm{ob}A$. Let $\lambda^a : C_a \Rightarrow ev_a F$ be a limit cone.

Then, for any $\mu : d \Rightarrow ev_a F$, $i \longrightarrow j \in J$

$$
\begin{tikzcd}
 & d \arrow[d, "!"] \arrow[ddl, bend right, "\mu_i"'] \arrow[ddr, bend left, "\mu_j"] & \\
 & C_a \arrow[dl, "\lambda^a_i"'] \arrow[dr, "\lambda^a_j"] & \\
ev_a F(i) \arrow[rr] & & ev_a F(j)
\end{tikzcd}
$$

Now let $a \longrightarrow a'$ be a morphism / C.

Then, since $F(i)$ is a functor, $\exists$ a morphism

$F(i)(a \longrightarrow a') : ev_a F(i) \longrightarrow ev_{a'} F(i)$

Thus, we have

$$
\begin{tikzcd}
 & C_a \arrow[dl, bend right, "\lambda^a_i"'] \arrow[d] \arrow[dr, bend left, "\lambda^a_j"] & \\
ev_a F(i) \arrow[d, "(a \to a')"'] & C_{a'} \arrow[dl, "\lambda^{a'}_i"'] \arrow[dr, "\lambda^{a'}_j"] & ev_a F(j) \arrow[d, "F(j)(a \to a')"] \\
ev_{a'} F(i) \arrow[rr] & & ev_{a'} F(j)
\end{tikzcd}
$$

the unique map $C_a \longrightarrow C_{a'}$ induced by limit s.t. left diagram commute.

Now define $G : A \longrightarrow C$ s.t. $G(a) = C_a$.

$G(a \longrightarrow a') = (C_a \longrightarrow C_{a'})$ induced by the universal property of limit cone $\lambda^{a'}$.

Then functoriality is satisfied by the uniqueness of universal property; 1_a goes to 1_{C_a} by uniqueness and $a \to a' \to a''$ goes to $a \to a''$ by uniqueness too.

Thus G is a functor. Now we claim G is $\lim(J \to C^A)$.

Let $n : H \Rightarrow F$ be a cone. Then,

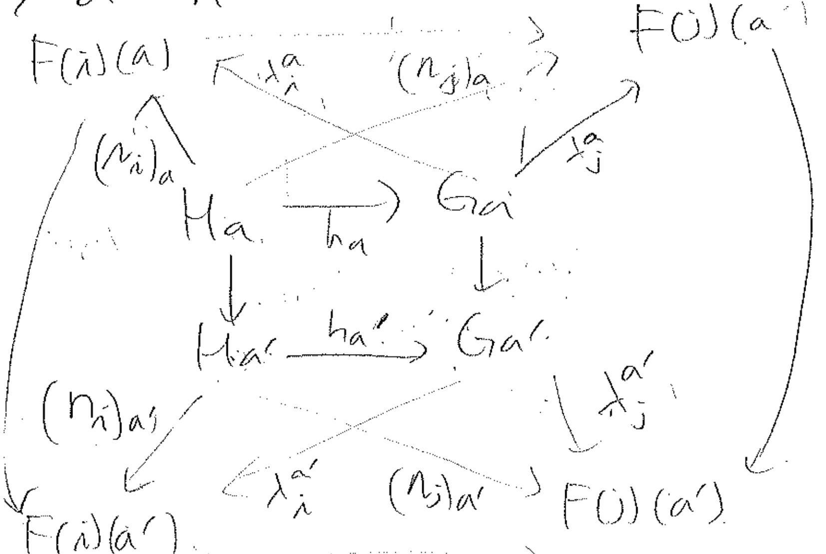
$$\begin{tikzcd} & H \arrow[dl, "n_i"'] \arrow[dr, "n_j"] & \\ F(i) \arrow[rr, "F(i\to j)"'] & & F(j) \end{tikzcd}$$

Thus, by ev_a, we get

$$\begin{tikzcd} & H(a) \arrow[d, dashed, "\exists! h_a"] \arrow[ddl, "(n_i)_a"'] \arrow[ddr, "(n_j)_a"] & \\ & C_a \arrow[dl, "\lambda_i^a"'] \arrow[dr, "\lambda_j^a"] & \\ ev_a F(i) \arrow[rr, "ev_a F(i\to j)"'] & & ev_a F(j) \end{tikzcd}$$

Then, the universal property gives left diagram with $h_a : H(a) \to C_a = G(a)$.

Now we claim that $h = \{h_a : a \in Ob(A)\}$ is a natural transformation because of uniqueness;



$Ha \to Ga \to Ga'$ and $Ha \to Ha' \to Ga'$ gives the commuting cone structure $\Rightarrow$ uniqueness of limit cone gives the same map.

Finally, this natural transformation make cones commute via left diagram

$\Rightarrow G$ is $\lim_J F$.

Ex.3.3.i) Let $i \to j \in J$. $\lambda : K \Rightarrow \mathrm{colim}\,K$ be a colimit cone.

(i)

$$\begin{tikzcd}
 & F\mathrm{colim}\,K & \\
FK_i \arrow[ur, "F\lambda_i"] \arrow[rr] & \Omega & FK_j \arrow[ul, "F\lambda_j"']
\end{tikzcd}$$

Then left diagram commutes by functoriality $\Rightarrow$ $F\lambda$ is a cone in D.

Thus, the universal property of $\mathrm{colim}\,FK$ gives $\exists!\, t : \mathrm{colim}\,FK \to F\mathrm{colim}\,K$ s.t. left diagram commutes ($\mu : FK \Rightarrow \mathrm{colim}\,FK$ is a colimit cone)

$$\begin{tikzcd}
 & F\mathrm{colim}\,K & \\
 & \mathrm{colim}\,FK \arrow[u, "\exists!\, t"'] & \\
FK_i \arrow[uur, bend left, "F\lambda_i"] \arrow[ur, "\mu_i"] \arrow[rr] & & FK_j \arrow[ul, "\mu_j"'] \arrow[uul, bend right, "F\lambda_j"']
\end{tikzcd}$$

ii) If t is an isomorphism, then $F\lambda$ is also a colimit cone since $\mu_i = t^{-1} F\lambda_i$ $\forall i$.

So if $\eta : FK \Rightarrow X$ is a cocone then, $\exists!\, t' : \eta \to \mathrm{colim}\,FK$ s.t. $\eta_i = t' \circ \mu_i$ $\forall i$

$\Rightarrow \eta_i = t' \circ t^{-1} \circ F\lambda_i$ $\forall i$. And $t' \circ t^{-1}$ is uniquely determined by the universal property of $\mathrm{colim}\,FK$.

Thus, F preserves colimit.

3.3 ii), 3.3 iii) already done in lecture note.

Ex 3.3 (iv). Let $K : J \to C$ be in class and

By (i), $\exists\, \mu : c \Rightarrow K$ is a limit cone and $F\mu$ is a limit cone in D. Thus FK has a limit cone over FK ($F\mu$) which can be lifted to a limit cone (μ) over K.

Thus we need to show that F reflects limits.

Let $\lambda : c' \Rightarrow K$ s.t. $F\lambda$ is a limit cone.

Then, $F\lambda \cong F\mu$ by Prop 3.1.7 (Uniqueness of limit cone)

$\Rightarrow Fc \cong Fc' \Rightarrow c \overset{t}{\cong} c'$ by (ii) (F reflects iso.)

Then, it suffices to show that λ is a limit cone.

Notes that for any $\eta : c'' \Rightarrow K$, then left diagram commutes

$$\begin{tikzcd}
 & c'' \arrow[d, "\exists! t'"] \arrow[ddl, bend right, "h_i"'] \arrow[ddr, bend left, "\eta_j"] & \\
 & c \arrow[d, "t^{-1}"] \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] & \\
K_i \arrow[rr] & c' \arrow[l, "\lambda_i"] \arrow[r, "\lambda_j"'] & K_j
\end{tikzcd}$$

$\Rightarrow \exists!\, t^{-1} \circ t' : c'' \to c'$ s.t. cones commutes

$\Rightarrow \lambda$ is a limit cone.

$\boxed{\text{Q.E.D.}}$

Ex 3.3. v) Let $(A, *)$, $(B, \#) \in Set_*$.

Then their coproduct is disjoint union of A, B

but picking a pt from A or B.

Thus, if point picked from B, then

$(A, \#) \longrightarrow (A, *) \amalg (B, \#)$ is not a function.

$A \longrightarrow A \amalg B$ in Set. Hence it is not a

coproduct in Set, since it is possible that

in Set, $\#$ and $*$ can be sent to distinct

objects.

(This happen in Top_* too.)

If J was connected, then $\exists$ morphism s.t.

sending $*$ to $\#$ or vice versa in limit.

Ex 3.3 vi) done in lecture note.

3.4. Representable nature of lim and colim.

Thm 3.4.2 For any $F: J \to C$ s.t. $\lim F$ exists. and $C(-, \lim_J F) \cong \lim_J C(-, F-)$.

pf) First of all, Let $X \in$ Set, Then, we claim

$$\lim_J C(X, F-) \cong \mathrm{Cone}(X, F) \quad (\cong C(-, \lim_J F)$$

by definition of limit.

If $\lambda \in \mathrm{Cone}(X, F)$ then λ satisfies equalizer in Thm 3.2.13 $\Rightarrow \lambda \in \lim_J (X, F-)$.

Conversely, any $\lambda \in \lim_J (X, F-)$ is an element of $\prod_{j \in J} C(X, Fj)$, and $\forall f: i \to j \in J$

$$\begin{tikzcd} & X \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] \arrow[d, phantom, "\circlearrowleft"] & \\ Fi \arrow[rr, "Ff"'] & & Fj \end{tikzcd}$$

$\Rightarrow \lambda$ is a cone over F with summit X.

$\Rightarrow \exists$ isomorphism.

To see this is natural, think when $g: X \to X'$ is a set map. Then, for any $\lambda \in \lim_J (X', F)$

$$\begin{tikzcd} & X' \arrow[dl, "g"'] & \\ X \arrow[d, "\lambda_i"'] \arrow[drr, phantom, "\circlearrowleft"] & & X \arrow[ul, "g"'] \arrow[d, "\lambda_j"] \\ Fi \arrow[rr, "Ff"'] & & Fj \end{tikzcd}$$

$\lambda \in \prod_{j \in J} C(X', Fj)$

thus, $\lambda g \in \prod_{j \in J} C(X, Fj)$

But as a cone, λg is left diagram which is the same one.

So

$$\begin{tikzcd} \lim_J C(X', F-) \arrow[r, "\cong"] \arrow[d, "g_*"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Cone}(X', F) \arrow[d, "g_*"] \\ \lim_J C(X, F-) \arrow[r, "\cong"] & \mathrm{Cone}(X, F) \end{tikzcd}$$

Thm 3.4.3', natural iso expresses
"representable universal property of the limit"
$\lim_J F$ is defined representably as $\lim_J C(X,F-)$
for chosen X.

Ex. 3.4.4) $F: I \to C$ is an iterated product,
i.e. $\lim F = A^I := \prod_I A$, then the property
gives $C(X, A^I) \cong \prod_I C(X,A)$.
i.e. any morphism $h: X \to A^I$ is determined by $h_i: X \to A$
for all $i \in I$ with each product projection $\varepsilon_i: A^I \to A$.
In a category of set, this is already known.

Prop 3.4.5: $C(X,-): C \to Set$ preserves limit
if C is locally small.
pf) Thm 3.4.2 shows $C(X, \lim F) \cong \lim(J \xrightarrow{F} C \xrightarrow{C(X,-)} Set)$
Moreover, remember that the natural isomorphism
is constructed via $\prod_{j \in J} C(X, F_j)$. Hence,

$$
\begin{tikzcd}
\lim_J C(X,F-) \arrow[r, phantom, "\cong"] \arrow[dr, "\text{cone } \mu \text{ as } \prod_{i \in J} \mu_i"'] & \mathrm{Cone}(X,F) \arrow[d] \arrow[r, phantom, "\cong"] & C(X, \lim_J F) \arrow[dl, "C(X,\lambda) \text{ as } \prod_{j \in I} C(X,\lambda_j)"] \\
& \prod_{j \in J} C(X, F_j) &
\end{tikzcd}
$$

μ: limit cone in Set
λ: limit cone in C

And if $\lambda_j: \lim_J F \to F_j$ is a leg of limit cone,
$C(X,\lambda) = \prod_{j \in J} C(X,\lambda_j)$ is RHS diagonal map, and LHS is
product of legs of limit cone.

More precisely,

$\overset{u}{\searrow}$ is constructed by Thm 3.2.13;

Actually Thm 3.2.13 construction shows that $\overset{\lambda}{\searrow}$ is a limit cone.

Then

$$\begin{tikzcd}
\mathrm{Cone}(X,F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C(X, \lim_J F) \arrow[dl, "C(X,\lambda)"] \\
\prod_{j\in J} (X, F_j) &
\end{tikzcd}$$

is just application of the universal property of limit cone λ;

any cone $\lambda' : X \Rightarrow F$ has a map $\exists! k \; X \to \lim_J F$

commutes

s.t.

$$\begin{tikzcd}
& X \arrow[d, "k"] \arrow[ddl, bend right, "\lambda'_i"'] \arrow[ddr, bend left, "\lambda'_j"] & \\
& \lim_J F \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\
F_i \arrow[rr] & & F_j
\end{tikzcd}$$

So, $\lambda' \in \mathrm{Cone}(X,F)$ can be sent to k, the unique map.

and $k \circ \lambda_i = \lambda'_i$ $\forall i \in J$ shows that the triangle commutes via $C(X,\lambda)$

Hence, by identifying $C(X,\lambda)$ as a cone

the $\bigtriangledown$ diagram shows that it acts as a limit cone; i.e. for any cone $\eta : A \Rightarrow C(X, F-)$ in Set

∃! unique map $p: A \to \lim_J \mathcal{C}(X, F-)$ s.t. ① commutes

$$\begin{tikzcd}
A \arrow[d, "p"'] \arrow[dr, bend left, "\prod_{j\in J}\eta_j"] \arrow[dr, phantom, "① \circlearrowleft"] & \\
\lim_J \mathcal{C}(X,F-) \arrow[d, "k"', "\cong"] \arrow[r, "\prod_{j\in J}\mu_j"] \arrow[dr, phantom, "②"] & \prod_{j\in J}\mathcal{C}(X,Fj) \\
\mathcal{C}(X,\lim_J F) \arrow[ur, "\mathcal{C}(X,\lambda)"'] &
\end{tikzcd}$$

Moreover, ② is also commutes by above ▽ diagram

$\Rightarrow$ ∃! unique map kp s.t. $A \xrightarrow{kp} \mathcal{C}(X, \lim_J F) \xrightarrow{\mathcal{C}(X,\lambda)} \prod_{j\in J}\mathcal{C}(X,Fj)$ is equal to $\prod_{j\in J}\eta_j$

This implies $\eta_j = \mathcal{C}(X,\lambda_j)\circ kp \Rightarrow \mathcal{C}(X,\lambda)$ is a limit cone. ∎

Thm 3.4.6 (Second interpretation of Thm 3.4.2)
$\mathcal{C}$: locally small
(i) $\mathcal{C}(X,-)$ preserves all limits that exist in $\mathcal{C}$
(ii) The covariant Yoneda embedding $y: \mathcal{C} \hookrightarrow \mathrm{Set}^{\mathcal{C}^{op}}$ preserves and reflects limits.

pf) (i) is just result of Prop 3.4.5.
(ii) Yoneda embedding is fully faithful $\Rightarrow$ reflects limits.
Also, $y(\lim_J F) = \mathcal{C}(-, \lim_J F)$
And for any $X \to X' \in \mathcal{C}$,

$$\begin{tikzcd}
\mathcal{C}(X', \lim_J F) \arrow[r, phantom, "\cong"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \lim_J \mathcal{C}(X', F-) \arrow[d] \\
\mathcal{C}(X, \lim_J F) \arrow[r, phantom, "\cong"] & \lim_J \mathcal{C}(X, F-)
\end{tikzcd}$$

by Thm 3.4.2

This shows. $y(\lim_J F) = \mathcal{C}(-, \lim_J F) \cong \lim_J \mathcal{C}(-, F-)$
as we've constructed in Prop 3.3.9. ∎

Dually, for any $F : J \to \mathcal{C}$, $X \in \mathcal{C}$, let

$$\mathcal{C}(F-, X) := J^{op} \xrightarrow{F} \mathcal{C}^{op} \xrightarrow{\mathcal{C}(-,X)} \mathrm{Set}.$$

Thm 3.2.6 shows that a limit exists.
Thm 3.2.13 shows construction of limit;
By the very similar argument of pf of
thm 3.4.2, $\lim_{J^{op}} \mathcal{C}(F-, X) \cong \mathrm{Cone}(F, X)$.

Thm 3.4.7) For any $F : J \to \mathcal{C}$,

∃natural iso $\mathcal{C}(\mathrm{colim}_J F, X) \cong \lim_{J^{op}} \mathcal{C}(F-, X)$

"Representable uni. pro. of colimit"

Ex. 3.4.8 $X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$. thm 3.4.7 implies

$$\mathcal{C}(X \sqcup Y, Z) \cong \mathcal{C}(X, Z) \times \mathcal{C}(Y, Z).$$

Ex 3.4.9. Iterated Coproduct of $A \in \mathcal{C}$: "copowers" or "tensors".

$$\Rightarrow \mathcal{C}(\textstyle\coprod_I A, X) \cong \mathcal{C}(A, X)^I$$

i.e. $h : \coprod A \to X$ is determined by $h_i : A \to X$.
with coproduct inclusion $\iota_i : A \to \coprod_I A$.

Ex 3.4.10. By rep universal property.

ish. out.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "g"'] & V \arrow[d, "h"] \\ W \arrow[r, "k"'] & P \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \Longrightarrow \begin{tikzcd} \mathcal{C}(P, X) \arrow[r, "\mathcal{C}(h,X)"] \arrow[d, "\mathcal{C}(k,X)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}(V, X) \arrow[d, "\mathcal{C}(f,X)"] \\ \mathcal{C}(W, X) \arrow[r, "\mathcal{C}(g,X)"'] & \mathcal{C}(U, X) \end{tikzcd}$$

Pull back.

Thm 3.4.11. C: locally small

(i) $C(-, X)$ carry colimits to limit.

(ii) $y: C^{op} \hookrightarrow Set^{C}$ Yoneda embedding preserves and reflects limits in C^{op}

pf) Dualize the argument for covariant case.

Thm 3.4.12 (Generalization of 3.2.13)

$F: J \to C$ small diagram. Suppose C has coproduct and coequalizer.

$$\coprod_{f \in \mathrm{Mor}J} F(\mathrm{dom} f) \overset{d}{\underset{c}{\rightrightarrows}} \coprod_{j \in \mathrm{obj}J} F_j \twoheadrightarrow \mathrm{colim}_J F$$

In particular, if C admits coproduct and coequalizer then it is cocomplete. Dually,

$$\lim_J F \rightarrowtail \prod_{j \in \mathrm{obj}J} F_j \overset{c}{\underset{d}{\rightrightarrows}} \prod_{f \in \mathrm{Mor}J} F(\mathrm{cod} f)$$

and if C admits product and equalizer, then C is complete.

pf) Using Remark 3.2.15, we have below diagram.

$$\begin{tikzcd}
 & & F(\mathrm{dom} f) \arrow[dl, "i_{\mathrm{dom} f}"'] \arrow[d, "i_f"] \\
C & \coprod_{j \in \mathrm{obj}J} F_j \arrow[l, two heads] & \coprod_{f \in \mathrm{Mor}J} F(\mathrm{dom} f) \arrow[l, shift right, "d"'] \arrow[l, shift left, "c"] \\
 & F(\mathrm{cod} f) \arrow[u, "i_{\mathrm{cod} f}"] & F(\mathrm{dom} f) \arrow[l] \arrow[u, "i_f"']
\end{tikzcd}$$

Fh

where $(d)_f = i_{\mathrm{dom} f}$

$(c)_f = i_{\mathrm{cod} f} \circ Ff$.

By hypothesis C (a coequalizer of c and d) exists.

By Thm. 3.4.11 (ii), Yoneda embedding carries coequalizer diagram to an equalizer diagram. Thus for any $X \in C$,

$$\begin{tikzcd}
 & & C(F(\mathrm{dom} f), X) \\
C(c,Y) \arrow[r, tail] & C(\coprod_{j \in \mathrm{Ob} J} Fj, X) \arrow[ur, bend left, "C(i_{\mathrm{dom} f}, X)"] \arrow[r, shift left, "C(d,X)"] \arrow[r, shift right, "C(c,X)"'] \arrow[d, "C(i_{\mathrm{cod} f}, X)"'] & C(\coprod_{f \in \mathrm{Mor} J} F(\mathrm{dom} f), X) \arrow[u, "C(i_f, X)"'] \arrow[d, "C(i_f, X)"] \\
 & C(F(\mathrm{cod} f), X) \arrow[r] & C(F(\mathrm{dom} f), X)
\end{tikzcd}$$

Applying 3.4.11 (i) carrying colimit to a limit, we have

$$\begin{tikzcd}
 & & C(F(\mathrm{cod} f), X) \\
C(c,X) \arrow[r, tail] & \prod_{j \in \mathrm{Ob} J^{op}} C(Fj, X) \arrow[ur, bend left, "\pi_{\mathrm{cod} f}"] \arrow[r, shift left, "c"] \arrow[r, shift right, "d"'] \arrow[d, "\pi_{\mathrm{dom} f}"'] & \prod_{f \in \mathrm{Mor} J^{op}} C(F(\mathrm{cod} f), X) \arrow[u, "\pi_f"'] \arrow[d, "\pi_f"] \\
 & C(F(\mathrm{dom} f), X) \arrow[r, "C(Ff, X)"'] & C(F(\mathrm{cod} f), X)
\end{tikzcd}$$

(Change from $F(\mathrm{dom} f)$ to $F(\mathrm{cod} f)$ occur by indexation with J^{op}) where $c := C(c, X)$, $d := C(d, X)$.

If X is fixed, by Prop 3.3.9, with $ev_X : Set^C \to Set$, the above one is an equalizer diagram in Set.

By Thm 3.2.13, The diagram's equalizer is equal to $\lim_{J^{op}} C(F-, X)$

$\Rightarrow \lim_{J^{op}} C(F-, X) \cong C(C, X) \quad \forall X \in C$.

Now these are assembled into a functor in Set^{obC}. Using forgetful functor $Set^{C} \to Set^{obC}$ with prop 3.3.9, $C(C, -) \cong \lim_{j \in J^{op}} C(Fj, -)$

By Thm 3.4.7, $C(C, X) \cong \lim_{j \in J^{op}} C(Fj, -) \cong C(\mathrm{colim}_J F, X)$

$\forall X \in C. \Rightarrow C \cong \mathrm{colim}_J F$ ▨

Rmk 3.4.15. (Generalized element)

Direct pf of above Thm is just checking c and d has the universal property that defines $\mathrm{colim}_J F$. This can be viewed as the same thing of what we actually did for getting proof, via "philosophy of generalized elements".

Ex) $\forall a \in A$, (Set) we can identify a as $a : 1 \to A$.

In this case, $\forall f : A \to B$, $f(a)$ is identified as fa.

In any category C, a "generalized element" of A is a morphism $a : X \to A$ with codomain A.

Then, $f : A \to B$ act on X-shaped element by composition

$\Rightarrow$ $Hom(-, A) : C^{op} \to Set$ send object X to the set of X-shaped generalized element of A.

Since $C \hookrightarrow Set^{C^{op}}$ Yoneda embedding is fully faithful, no information about A is lost by by considering not A but $Hom(-, A)$.

For example, if we want to show $f, g : A \rightrightarrows B$ are equal in this philosophy, it suff to show that

$fa = ga \quad \forall a : X \to A. \in Hom(X, A)$

If we want to show a cone is limit cone, then show that induced fuction in Set is a cone in Set, etc...

Lemma 3.4.16 Let C : category with terminal object 1. $\Rightarrow$

$$\begin{tikzcd} A \times B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!"] \\ A \arrow[r, "!"'] & 1 \end{tikzcd}$$

pull back defines product $A \times B$ of $A, B \in C$.

pf) Let $A \xleftarrow{\pi_A'} C \xrightarrow{\pi_B'} B$ a diagram.

Then, universal property of pull back gives a map $\exists ! k : C \to A \times B$ st.

$$\begin{tikzcd} & C \arrow[dl, "\pi_A'"'] \arrow[d, "k"] \arrow[dr, "\pi_B'"] & \\ A \arrow[dr, bend right, "!"'] & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \arrow[dl, bend left, "!"] \\ & 1 & \end{tikzcd}$$

left diagram commute.

By deleting $!$ and 1, this is just universal property of product of two elements.

Lem 3.4.17. C : category with binary products
$\Rightarrow$ pullback defines an equalizer, i.e.

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d, "l"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "(f,g)"] \\ B \arrow[r, "(1_B, 1_B)"'] & B \times B \end{tikzcd} \rightsquigarrow E \rightarrowtail^{e} A \overset{f}{\underset{g}{\rightrightarrows}} B$$

pf) The representable universal property of the product
$\Rightarrow \operatorname{Hom}(X, B\times B) \cong \operatorname{Hom}(X,B) \times \operatorname{Hom}(X,B)$
Also, for pullback we have (by Thm 3.4.6(i))

$$\begin{tikzcd} \operatorname{Hom}(X,E) \arrow[r, "e_*"] \arrow[d, "l_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Hom}(X,A) \arrow[d, "(f_*, g_*)"] \\ \operatorname{Hom}(X,B) \arrow[r, "(1_*, 1_*)"'] & \operatorname{Hom}(X,B) \times \operatorname{Hom}(X,B) \end{tikzcd}$$

By concrete construction of pullback in Set as we've seen in Ex 3.2.11,

$\operatorname{Hom}(X,E) \cong \{ (h_1, h_2) \in \operatorname{Hom}(X,A, X\times B)$
s.t. $(1_*, 1_*)(h_2) = (f_*, g_*)(h_1) \}$
i.e. $\Leftrightarrow f h_1 = h_2 = g h_1$

Thus, $\operatorname{Hom}(X,E) \overset{e_*}{\cong} \{ h \in \operatorname{Hom}(X,A) : fh = gh \} \subseteq \operatorname{Hom}(X,A)$
which is equalizer of $\boxed{f_* \text{ and } g_*}$
Thus $\operatorname{Hom}(X,E) \overset{e_*}{\rightarrowtail} \operatorname{Hom}(X,A) \overset{f_*}{\underset{g_*}{\rightrightarrows}} \operatorname{Hom}(X,B)$
is an equalizer diagram. ~~Note Thm 3.4.6(ii) shows~~

Now Thm 3.4.6 (ii) shows that E is an equalizer of f, g. □

Ex 3.4. i) done in lecture note.

Ex 3.4. ii) Yoneda embedding preserves limit via Thm 3.4.2

" reflects limit since it is fully faithful.

But it does not create limit since limit may be not representable;

Ex 3.4 iii) $F: C \to Set$, $\Pi: \int F \to C$

(i) Π strictly creates all limits that C admit and F preserves.

pf) Let $K': J \to \int F$ a small diagram s.t $K'i = (Ki, x_i)$

By this notation, we may define $K: \mathrm{ob}J \to C$ by $i \mapsto Ki$.

Also, if $i \to j \in \mathrm{Mor}J$, $K'(i \to j): (Ki, x_i) \xrightarrow{f} (Kj, x_j)$

and f comes from a morphism $Ki \to Kj$ s.t $Ff(x_i) = x_j$. Hence by defining $K(i \to j) := f$,

we may define $K: J \to C$ a functor.

Now, let $\lambda: (c, x) \Rightarrow K'$ be a limit cone.

Then
$$\begin{tikzcd} & (c,x) \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ (Ki, x_i) \arrow[rr] & & (Kj, x_j) \end{tikzcd} \Rightarrow \begin{tikzcd} & c \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ Ki \arrow[rr] & & Kj \end{tikzcd}$$
thus this λ

induces a cone λ in C, which is a unique lift, since other cone in C must have at least one distinct λ_i, which induces distinct cone's in $\int F$.

Now it suffices to show that λ is a limit cone in C.

Let $\mu : d \Rightarrow K$ be a cone. Then,

$$
\begin{tikzcd}
 & d \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] \arrow[d, phantom, "\circlearrowright"] & \\
K_i \arrow[rr] & {} & K_j
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & (d, y) \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] \arrow[d, phantom, "\circlearrowright"] & \\
(K_i, \mu_i(y)) \arrow[rr] & {} & (K_j, \mu_j(y))
\end{tikzcd}
\quad \forall y \in Fd.
$$

By universal property of λ, $\exists!\ f : (d, y) \to (c, x)$ s.t. left diagram commute.

$$
\begin{tikzcd}
 & (d, y) \arrow[d, "f"] \arrow[ddl, bend right, "\mu_i"'] \arrow[ddr, bend left, "\mu_j"] & \\
 & (c, x) \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] \arrow[d, phantom, "\circlearrowright"] & \\
(K_i, \mu_i(y)) \arrow[rr] & {} & (K_j, \mu_j(y))
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & d \arrow[d, "f"] \arrow[ddl, bend right, "\mu_i"'] \arrow[ddr, bend left, "\mu_j"] & \\
 & c \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] \arrow[d, phantom, "\circlearrowright"] & \\
K_i \arrow[rr] & {} & K_j
\end{tikzcd}
$$

$\therefore$ λ is a limit cone in C.

ii) Π strictly creates connected colimit.

Let λ be a colimit cone. Then

$$
\begin{tikzcd}
(K_i, x_i) \arrow[rr] \arrow[dr, "\lambda_i"'] & \circlearrowright & (K_j, x_j) \arrow[dl, "\lambda_j"] \\
 & (c, x) &
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
K_i \arrow[rr] \arrow[dr] & \circlearrowright & K_j \arrow[dl] \\
 & c &
\end{tikzcd}
$$

So λ induces a cocone in C.

To see λ in C is a colimit in C,

let $\mu : K \Rightarrow d$ a cocone. Then, for any $i \to j \in J$

$$
\begin{tikzcd}
K_i \arrow[rr] \arrow[dr, "\mu_i"'] & \arrow[d, phantom, "\circlearrowleft"] & K_j \arrow[dl, "\mu_j"] \\
& d &
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
(K_i, x_i) \arrow[rr] \arrow[dr, "\mu_i"'] & & (K_j, x_j) \arrow[dl, "\mu_j"] \\
& \begin{matrix} y \\ \| \\ (d, \mu_i(x_i)) \\ \| \\ \mu_j(x_j) \end{matrix} &
\end{tikzcd}
\quad \text{NB!}
$$

Notes that this μ is unique cone in $\int F$ since connectedness fixes y regardless of choice of i.

Precisely, if we fix x_i for some $i \in J$, x_j is fixed via zig zag map over the cone $\mu_i \Rightarrow y$ is fixed.

Now universal property of d shows that

$$
\begin{tikzcd}
(K_i, x_i) \arrow[rr] \arrow[dr, "\lambda_i"'] \arrow[ddr, bend right, "\mu_i"'] & \arrow[d, phantom, "\circlearrowleft"] & (K_j, x_j) \arrow[dl, "\lambda_j"] \arrow[ddl, bend left, "\mu_j"] \\
& (c, x) \arrow[d, "!"] & \\
& (d, \mu_i(x_i) = \mu_j(x_j)) &
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
K_i \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & K_j \arrow[dl] \arrow[ddl, bend left] \\
& c \arrow[d] & \\
& d &
\end{tikzcd}
$$

Thus λ is a colimit cone in C.

Ex. 3.4.iv) done in lecture note.

3.5. Completes and cocompletes categories

A functor is continuous / cocontinuous if it preserves all small limits or colimits.

Prop 3.5.1. Set is cocomplete.

pf) By thm 3.4.12, it suffices to show that Set has coproducts and coequalizer.

Coproduct = disj. union.

Coequalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is $B/_{f(a)\sim g(a)}$

Prop 3.5.2. Top is co/complete.

pf) $U: \text{Top} \to \text{Set} \cong \text{Top}(*, -)$

By prop 3.4.5, if Top has any limits, then its underlying set is limit in the category Set.

① Product : $\prod_\alpha X_\alpha$: Cartesian product with product topology.

(by definition, it is the coarest topology s.t. $\prod_\alpha X_\alpha \to X_\alpha$ is cts.)

② Equalizer: the equalizer $E \rightarrowtail X \rightrightarrows Y$ with subspace topology inherited from X.

i.e. The coarsest topology s.t. $E \hookrightarrow X$ cts.

$\Rightarrow$ Thm 3.4.12 shows that Top is complete.

Dually, ① Coproduct : disj union as Top space the finest topology s.t. $X_\alpha \to \coprod X_\alpha$ is cts.

② Coequalizer of $X \rightrightarrows Y \Rightarrow Y/_{f(a) \sim g(a)}$ with quotient topology.

Hence Thm 3.4.12 shows that Top is cocomplete ▨

Ex 3.5.3. Any limit/colimit can be constructed via diagram in prop 3.5.2 by constructing $\left(\begin{array}{ll}\text{the coarsest} & (\text{limit}) \\ \text{the finest} & (\text{colimit})\end{array}\right)$ topology

on the limit/colimit in Set.

ex) $J : \omega^{op} \longrightarrow \mathrm{Top}$

$$\begin{array}{ccc} n & \longrightarrow & S^1 \\ \downarrow & \longmapsto & \downarrow p \\ n+1 & \longrightarrow & S^1 \end{array}$$

$p : S^1 \longrightarrow S^1$ is pth power map.
$e^{2\pi i x} \longmapsto (e^{2\pi i x})^p$

So we have $\cdots \longrightarrow S^1 \xrightarrow{p} S^1 \xrightarrow{p} S^1$.

$\Rightarrow \lim J$ = p-adic solenoid.

Ex 3.5.4

Space of ordered configurations of n points

: $PConf_n(X) := \{(x_1, \cdots x_n) \in X^n : x_i \neq x_j \ \forall i \neq j\}$
with subspace topology $\subseteq X^n$.

Let Σ_n : symmetric group.

$$\begin{array}{ccc} Conf_n(X) : B\Sigma_n & \longrightarrow & \mathrm{Top}. \\ \bullet & \longmapsto & PConf_n(X) \\ g \downarrow & \longmapsto & \downarrow \text{permutation by } g. \\ \bullet & \longmapsto & \text{"} \end{array}$$

$\Rightarrow Conf_n(X) := \mathrm{Colim}\left(B\Sigma_n \xrightarrow{PConf_n(X)} \mathrm{Top}\right)$

Prop. 3.5.5. $\mathcal{C}$ is complete / cocomplete

$\Rightarrow$ $c/\mathcal{C}$ and $\mathcal{C}/c$ are complete / cocomplete

$\forall c \in \mathcal{C}$.

pf) Prop 3.3.8. shows that $\mathcal{C}$ complete / cocomplete

$\Rightarrow$ $c/\mathcal{C}$ has all limits and all connected limit

created by $\Pi : c/\mathcal{C} \to \mathcal{C}$.

Now 1_c is initial in $c/\mathcal{C}$. Dual statement

of Lem 3.4.16 creates coproduct in $c/\mathcal{C}$.

(since pushout is connected colimit)

And dual of Lem 3.4.17 generates coequalizer

So, by thm 3.4.12, $c/\mathcal{C}$ is cocomplete ▨

In this proof, coproduct of $(c \to x_a)$ is a

map $c \longrightarrow (x_\alpha \sqcup x_{\alpha'}) \sqcup \cdots)$

defined by $c \to x_\alpha$ s.

Rmk) ob, mor $Cat \rightrightarrows Set$ are representable by

Ex 2.1.5 (ix) and (x).

By Thm 3.4.6 (i), Both preserves limits when

limit of $K : J \to Cat$ exists.

Furthermore, domain, codomain identities are

natural transformation between ob and mor.

$\Rightarrow$ They are preserved. via ob and mor

Also Composition as a $Cat(3,-)$ or $CAT(3,-)$

presered. These gives a candidate of product and Thus, equalizer in Cat or CAT.

Prop 3.5.6. Cat and CAT are complete.

pf) ① Product : product of category

② Equalizer: $E \rightarrowtail C \overset{F}{\underset{G}{\rightrightarrows}} D$

where E is a subcategory of C

$$\text{s.t. } \operatorname{ob} E = \{ c \in \operatorname{ob} C : Fc = Gc \}$$
$$\operatorname{Mor} E = \{ f \in \operatorname{Mor} C : Ff = Gf \}.$$

Apply Thm 3.4.12. $\boxed{/\!/}$

Ex 3.5.7. Let $F : C \longrightarrow \text{Set}$. Pullback in CAT of $C \xrightarrow{F} \text{Set} \xleftarrow{U} \text{Set}_*$ is a category s.t.

Object: $(c, (X, x))$ s.t. $Fc = X$.

Mor : $(c, (X, x)) \xrightarrow{(f,g)} (c', (X', x'))$

s.t. $f : c \longrightarrow c' \in C$.
$g : (X, x) \longrightarrow (X', x') \in \text{Set}_*$

$$Ff = Ug.$$

Notes that $(c, (X, x))$ can be denoted as (c, x) with $x \in Fc$.

And morphism can be denoted as $(c, x) \xrightarrow{f} (c', x')$ s.t. $Ff(x) = x'$.

$\Rightarrow$ This pullback is $\int F$.

Claim) Cat and CAT are cocomplete.
But proof is a little bit delicate;

① Coproduct: just disjoint union of category

(Notes that any category is coproduct of its connected components)

② To use Lemma 3.4.17, we need pushout. But it is difficult.

Ex 3.5.8.

$$\begin{array}{ccc} 1 \sqcup 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & P \end{array}$$

we want to have pushout P.

Let $P \to C$ a functor.

Then

$$\begin{array}{ccccc} 1 \sqcup 1 & \longrightarrow & 2 & & \\ \downarrow & & \downarrow & & \\ 1 & \longrightarrow & P & \longrightarrow & C \end{array}$$

gives a cocone over diagram $1 \leftarrow 1 \sqcup 1 \to 2$ with nadir C.

thus, $1 \to P \to C = 1 \sqcup 1 \to 1 \to P \to C$
$= 2 \to P \to C$

implies that all these map denote $\overset{f}{\circlearrowleft} C$.

Thus, P should have an endomorphism without any relation on the composite $f \cdot f$

$\Rightarrow P =$ free category with one object and one non-identity morphism

By identifying $Mor(\cdot,\cdot) = \{1_c, f, f^2, \cdots\}$

as a monoid, we know that $P = B\mathbb{N}$

where $\mathbb{N}$ is monoid of natural number under addition

Similarly,
$$\begin{tikzcd} 1 \sqcup 1 \arrow[r] \arrow[d] & \square \arrow[d] \\ 1 \arrow[r] & B\mathbb{Z} \end{tikzcd}$$
since 2 is changed to walking iso $\square$ thus f should be iso.

Cocompleteness proved at chapter 4.

Notes that equivalence of categories preserve complete / cocomplete

Prop. 3.5.9. P: poset is complete and cocomplete as a category $\Longleftrightarrow$ P is a complete lattice

i.e. every subset $A \subset P$ has both infimum and a supremum.

pf) In any poset or preorder, if limit (or colimit) exists, it should be infimum of its object (or supremum of its object)

Since
$$\begin{tikzcd} & d \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & c \arrow[dl] \arrow[dr] & \\ F_i \arrow[rr] & & F_j \end{tikzcd}$$
implies that any $d \le F_i \ \forall i$ must $d \le c$, as a term of order.

(Similar to colimit)

It doesn't matter when J is discrete or not. ///.

Ex 3.5.10. Let C: category with finite limits

kernel pair of $f: X \to Y$ = pullback over f as,

$$\begin{tikzcd} R \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

. Thus pullback gives $(s,t): R \to X \times X$.

So R is always a subobject of $X \times X$

(= monomorphism with codomain $X \times X$)

In Set, $R \subset X \times X$ defines relation on X.

This is always "equivalence relation" in the below sense.

(i)

$$\begin{tikzcd} X \arrow[dr, "\exists! \rho"] \arrow[drr, bend left, "1_X"] \arrow[ddr, bend right, "1_X"'] & & \\ & R \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\boxed{\text{a reflexivity map } \rho}$ induced by left diagram s.t.

$$\begin{tikzcd} X \arrow[rr, "(1_X, 1_X)"] \arrow[dr, "\rho"'] & \circlearrowright & X \times X \\ & R \arrow[ur, tail, "(s,t)"'] & \end{tikzcd}$$

ρ factors $(1_X, 1_X)$

i.e. ρ is a section of (s,t)

(ii) $\boxed{\text{Symmetry map } \sigma}$

$$\begin{tikzcd} R \arrow[dr, "\sigma"] \arrow[drr, bend left, "s"] \arrow[ddr, bend right, "t"'] & & \\ & R \arrow[r, "t"] \arrow[d, "s"'] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

s.t. $s\sigma = t$, $t\sigma = s$

(iii) $\boxed{\text{Transitivity map } \tau}$

$$\begin{tikzcd} R \times_X R \arrow[r, "\bar{t}"] \arrow[d, "\bar{s}"'] & R \arrow[r, "t"] \arrow[d, "s"] & X \\ R \arrow[r] \arrow[d, "s"'] & X & \\ X & & \end{tikzcd}$$

with

$$\begin{tikzcd} R \times_X R \arrow[dr, "\tau"] \arrow[drr, bend left, "t\bar{t}"] \arrow[ddr, bend right, "s\bar{s}"'] & & \\ & R \arrow[r, "t"] \arrow[d, "s"'] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$\Rightarrow s\tau = s\bar{s}$, $t\tau = t\bar{t}$.

An <u>equivalence relation</u> in a category C with finite limits

$= (s,t) : R \rightarrowtail X \times X$

with ρ, σ, τ in a manner of (i), (ii), (iii)

In this case, Coequalizer of $s, t : R \rightrightarrows X \xrightarrow{e} X/_R$

gives $e : X \twoheadrightarrow X/_R$ a quotient object.

In Set, $X/_R$ is the set of R-equivalence classes of elements of X. Moreover, $\exists!$ factorization

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "e"'] & Y \\ X/_R \arrow[ur, dashed, "m"'] & \end{tikzcd}$$

In a good situation (e.g. Grothendieck Topos) m is monomorphism. In this case, The factorization is called <u>image factorization</u> and m is identified as the image $X/_R$ as a subobject of Y.

Ex 3.5.1) BG : 1-object category

Let $X : BG \to$ Set. Colimit of X can be decomposed as

$$\begin{tikzcd} & & X \arrow[dll, "1_X"'] \arrow[d, "i_g"] \\ \varinjlim X & X \arrow[l, two heads] & \coprod_{g \in G} X \arrow[l, shift right, "d"'] \arrow[l, shift left, "c"] \\ & X \arrow[u, "1_X"] & X \arrow[l, "g_*"] \arrow[u, "i_g"'] \end{tikzcd}$$

Thus $d_g = 1_X$, $c_g = g_*$

Hence,

$\operatorname{colim} X = X/_{x \sim g(x)}$

$= \{ G \cdot x : x \in X \}$

a set of orbits of G action.

Ex 3.5.ii) Use the diagram in (i) again;

$$\begin{tikzcd}
& & & F(\mathrm{dom} f) \arrow[dll, "i_{\mathrm{dom} f}"'] \arrow[d, "i_f"] \\
\mathrm{colim} F & \coprod_{j \in \mathrm{ob} J} F(j) \arrow[l, two heads] & & \coprod_{f \in \mathrm{Mor} J} F(\mathrm{dom} f) \arrow[ll, shift right, "d"'] \arrow[ll, shift left, "c"] \\
& F(\mathrm{cod} f) \arrow[u, "i_{\mathrm{cod} f}"] & & F(\mathrm{dom} f) \arrow[ll, "Ff"] \arrow[u, "i_f"']
\end{tikzcd}$$

Then $\mathrm{colim} F = \{ [\lambda] : \lambda \in \coprod_{j \in \mathrm{ob} J} F(j) \}$

where $[\lambda] := \{ \lambda' \in \coprod_{j \in \mathrm{ob} J} F_j : \exists\, \eta \in \coprod_{f \in \mathrm{Mor} J} F(\mathrm{dom} f)$

s.t. $\lambda' = c(\eta)$, $\lambda = d(\eta)$

or vice versa $\}$

Since d is just inclusion, c is Ff for each component $F(\mathrm{dom} f)$, this means that

$[\lambda] := \{ \lambda' \in \coprod_{j \in \mathrm{ob} J} F_j : \exists\, \eta \in \coprod_{f \in \mathrm{Mor} J} F(\mathrm{dom} f)$ s.t. $\lambda' = \eta$, $\lambda = Ff(\eta)$ or vice versa. $\}$

Now we claim that (j, λ) and (i, η) is connected in $\int F$ $\Leftrightarrow$ $[\lambda] = [\eta]$.

Suppose LHS. Then $\exists$ a finite length zigzag morphism, $(j, \lambda) \xrightarrow{f_1} (j_1, \lambda_1) \xleftarrow{f_2} (j_2, \lambda_2) \to \cdots \xleftarrow{f_n} (i, \eta)$

Say $j_n = i$; $\lambda_n = \eta$. Then, $[\lambda] = [\lambda_1]$ since $Ff_1(\lambda) = \lambda_1$. $[\lambda_1] = [\lambda_2]$ since $Ff_2(\lambda_2) = \lambda_1$ ...

So $[\lambda_{n-1}] = [\lambda_n]$ since either $Ff_n(\lambda_{n-1}) = \lambda_n$

or $Ff_n(\lambda_n) = \lambda_{n-1}$

Is well-defined and true.

Conversely, let $[\lambda] = [\eta]$. Then $\exists \lambda' \in [\lambda]$ s.t. $\eta = \lambda'$ and $\lambda = Ff(\lambda')$ or $\lambda = \lambda'$ and $\eta = Ff(\lambda')$ for some $f \in \mathrm{Mor} J$

This shows $i \to j$ or $j \to i$, so (i, λ) (j, η) is connected in $\int F$.

Thus, if $\lambda : F \Rightarrow \mathrm{Colim} F$ is a colimit cone,

$$\begin{tikzcd} Fi \arrow[rr, "Ff"] \arrow[dr, "\lambda_i"'] & & Fj \arrow[dl, "\lambda_j"] \\ & \mathrm{colim} F & \end{tikzcd}$$

Thus, if $x \in Fi$, then $\lambda_i(x)$ denotes the orbit of x under Ff $\forall f \in \mathrm{Mor} J$.

Thus colimit core is just set of functions sending element in Fi to its orbit in Set.

Ex. 3.5 (ii) Let $A \doteq \{ E \overset{s}{\underset{t}{\rightrightarrows}} V \}$ be a category with two parallel nontrivial morphism.

Then,

① Set^A is the category of directed multigraph.

pf) Let $F : A \to \mathrm{Set}$ as $E_F \overset{s}{\underset{t}{\rightrightarrows}} V_F$

$G : A \to \mathrm{Set}$ $\quad E_G \overset{s}{\underset{t}{\rightrightarrows}} V_G$

By thinking $e \in E_F$ is an arrow from $s(e)$ to $t(e)$ = gives a mult digraph. Moreover, if $\exists \alpha : F \Rightarrow G$

Then,

$$\begin{array}{ccc} E_F & \overset{s}{\underset{t}{\rightrightarrows}} & U_F \\ {\scriptstyle \alpha_E}\downarrow & \circlearrowright & \downarrow{\scriptstyle \alpha_G} \\ E_G & \overset{s}{\underset{t}{\rightrightarrows}} & U_G \end{array}$$

Thus, if $v = s(e)$ with $w = t(e)$,

then α_G maps $s(e)$ as $s(\alpha_E(e))$ and w as $t(\alpha_E(e))$

so that incidence relation is preserved.

② Set^A is complete / cocomplete.

This is direct application of Thm 3.3.9 and the fact that Set is co/complete.

③ DirGraph is a full subcategory of Set^A.

pf) any morphism between digraph is a morphism as multi-digraph.

Thus $\text{DirGraph} \overset{i}{\hookrightarrow} Set^A$ is a fully faithful functor.

④ There is a functor $G: Set^A \to \text{DirGraph}$

s.t. $\text{DirGraph} \overset{i}{\hookrightarrow} Set^A \xrightarrow{G} \text{DirGraph} = 1_{\text{DirGraph}}$

pf). Define $G: Set^A \to \text{DirGraph}$ as

$$E \rightrightarrows V \longmapsto E/\sim \; \rightrightarrows V$$

Where $e_1 \sim e_2 \iff s(e_1) = s(e_2),\ t(e_1) = t(e_2)$

Then, $\sim$ is equivalence relation. And for each $[e] \in E/\sim$

$s([e]) := s(e)$ is well-defined. Moreover, $[e] \neq [e']$,

$t([e]) := t(e)$

then $s(e) \neq s(e')$ or $t(e) \neq t(e')$

Thus, $E/\sim \underset{t}{\overset{s}{\rightrightarrows}} V$ is a directed graph.

If
$$\begin{tikzcd} E_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\alpha_E, \alpha_G)"] & V_1 \\ E_2 \arrow[r, shift left] \arrow[r, shift right] & V_2 \end{tikzcd}$$
is a morphism.

notes that $\alpha_{E/\sim}([e]) := [\alpha_E(e)]$

gives.
$$\begin{tikzcd} E_1/\sim \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\alpha_{E/\sim}"'] \arrow[dr, phantom, "\circlearrowright"] & V_1 \arrow[d, "\alpha_G"] \\ E_2/\sim \arrow[r, shift left] \arrow[r, shift right] & V_2 \end{tikzcd}$$

Since if $e, e' \in [e]$, then

$$s(\alpha_E(e)) = \alpha_G(s(e)) = \alpha_G(s(e')) = s(\alpha_E(e'))$$
$$t(\quad " \quad) = \quad " \quad (t(e)) = \quad " \quad (t(e')) = t(\quad " \quad)$$

$\Rightarrow$ $\alpha_E(e)$ and $\alpha_E(e')$ are equivalent.

So $\alpha_{E/\sim}$ is well-defined

So,
$$\begin{tikzcd} {[e]} \arrow[r, mapsto] \arrow[r, mapsto, shift right=2] \arrow[dd] \arrow[ddr, phantom, "\circlearrowright"] & t([e]) = t(e) \quad s([e]) = s(e) \arrow[d] \\ & \alpha_G(s(e)) \quad \alpha_G(t(e)) \arrow[d, equal] \\ {[\alpha_E(e)]} \arrow[r, mapsto] \arrow[r, mapsto, shift right=2] & s(\alpha_E(e)) \quad t(\alpha_E(e)) \end{tikzcd}$$

Commutes.

Hence $\mathrm{Set}^A \xrightarrow{G} \mathrm{DirGraph}$ is a functor.

Notes that $\mathrm{DirGraph} \hookrightarrow \mathrm{Set}^A \longrightarrow \mathrm{DirGraph}$

$$= 1_{\mathrm{DirGraph}}.$$

5) DirGraph has colimit if Set^A has a corresponding colimit.

pf) Let $K: J \longrightarrow$ DirGraph.

Then, $iK : J \longrightarrow \text{DirGraph} \longrightarrow Set^A$.

is also a small diagram.

Since Set^A is complete / cocomplete, so

$\lim iK$ and $\operatorname{colim} iK$ exists.

Now we claim that $G(\lim iK) \cong \lim(GiK)$
$= \lim K$.

Let $\lambda : \lim iK \Rightarrow iK$ be a limit cone. Then, for any cone $\eta : D \Rightarrow iK$, $f: i \longrightarrow j \in J$, we have

$\exists!\, q : D \longrightarrow \lim iK$ s.t. left diagram commutes

$$\begin{tikzcd}
 & D \arrow[d, "q"] \arrow[ddl, bend right, "\eta_i"'] \arrow[ddr, bend left, "\eta_j"] & \\
 & \lim iK \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
Ki \arrow[rr, "Kf"'] & & Kj
\end{tikzcd}$$

We claim that G preserves limit cone.

Let $\eta : D \Rightarrow GiK = K$ be a cone in DirGraph.

Then, $i\eta : D \Rightarrow iK$ is a cone in Set^A

Since the function i does not change component of natural transformations η_i $\forall i \in J$, as a sub category. Thus $\exists!\, q$ s.t. above diagram holds.

By applying G,

$$\begin{tikzcd}
 & G_i D = D \arrow[d, "G_i q"] \arrow[ddl, bend right, "G_i \eta_i = \eta_i"'] \arrow[ddr, bend left, "G_i \eta_j = \eta_j"] & \\
 & G \lim iK \arrow[dl, "G\lambda_i"'] \arrow[dr, "G\lambda_j"] & \\
G_i K_i = K_i \arrow[rr, "G_i Kf = Kf"'] & & G_i K_j = K_j
\end{tikzcd}$$

left diagram commutes from functoriality of G.

Moreover, such t is unique; If there is another $q' : D \to G \lim iK$ s.t. left diagram commutes

then, define $q' : D \to \lim iK$ as a section;
i.e, $q'_E(x) = e'$ for some fixed representation $e' \in [\underset{E}{t}(x)]$.
$q'_V(x) = q_V(x)$.

Then,
$$\begin{tikzcd}
E_D \arrow[r, "q'_E"] \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, phantom, "\circlearrowleft"] & E_{\lim iK} \arrow[d, shift left] \arrow[d, shift right] \\
V_D \arrow[r, "q'_V = q_V"'] & V_{\lim iK}
\end{tikzcd}$$
since
$$\begin{tikzcd}
e \arrow[r, mapsto] \arrow[d, shift left] \arrow[d, shift right] & q'_E(e) \in [q_E(e)] \arrow[d, shift left] \arrow[d, shift right] \\
s(e), t(e) \arrow[r, mapsto] & (s(q'_E(e)), t(q'_E(e))) \\
 & (q_V(s(e)), q_V(t(e)))
\end{tikzcd}$$

but $q'_E(e) \in [q_E(e)] \Rightarrow$
$$s(q'_E(e)) = s(q_E(e)) = q_V(s(e))$$
$$t(q'_E(e)) = t(q_E(e)) = q_V(t(e))$$

gives commuting property. Hence q' is well-defined natural transformation.

Now,
$$\begin{tikzcd}
E_D \arrow[r, "q'_E"] \arrow[rr, bend left, "(\eta_i)_E"] \arrow[d, shift left] \arrow[d, shift right] & E_{\lim iK} \arrow[r, "(\lambda_i)_E"] \arrow[d, shift left] \arrow[d, shift right] & E_{K_i} \arrow[d, shift left] \arrow[d, shift right] \\
V_D \arrow[r, "q'"'] \arrow[rr, bend right, "(\eta_i)_V"'] & V_{\lim iK} \arrow[r, "(\lambda_i)_V"'] & V_{K_i}
\end{tikzcd}$$

It suffices to show that left diagram commutes.

For $e \in E_D$,

$$e \longmapsto q'_E(e) \in [q_E(e)] \longrightarrow (\lambda_i)_E(q'_E(e))$$

Then, $s(q'_E(e)) = s(q_E(e))$, $t(q'_E(e)) = t(q_E(e))$

Implies that $(\lambda_i)_E(q_E(e)) = (\lambda_i)_E(q'_E(e))$

since K_i is not a multigraph, so for any two ordered vertices, $\exists$ at most 1 arrow.

By universal property, $(\lambda_i)_E(q_E(e)) = (\eta_i)_E(e)$

$\Rightarrow q'_E \circ (\lambda_i)_E = (\eta_i)_E$.

And by definition, $q'_V \circ (\lambda_i)_V = q_V \circ (\lambda_i)_V = (\eta_i)_V$.

$\Rightarrow q' \circ \lambda = \eta$.

Then, universal property shows that $q' = q$

Thus, $\varphi = Gq' = Gq$ $\therefore$ Gq is unique.

This shows that $G\lambda : G\lim iK \Rightarrow GiK = K$

is a limit cone in Di Graph.

$\Rightarrow$ G preserves limit cone.

Thus, $G(\lim iK) \cong \lim(GiK) = \lim K$

$\Rightarrow$ Di Graph admit limits if it is admitted by Set^A

$\Rightarrow$ " is complete since Set^A is complete.

By the duality, doing the all argument dually,

we can get Di Graph admits colimit if it is in Set^A

$\Rightarrow$ Di Graph is cocomplete.

Ex. 3.5.N) Since we proved Cat is complete/cocomplete using this set structure so we can make $J^{\triangleleft}$ concretely: Define $i_0 : J \to J\times 2$, $j \mapsto (j,0)$. By Ob,

$$\begin{tikzcd} \mathrm{Ob}J \arrow[r, "!"] \arrow[d, "i_0"'] & \mathrm{Ob}\,1 \arrow[d] \\ \mathrm{Ob}J\times 2 \arrow[r] & \mathrm{Ob}J^{\triangleleft} \end{tikzcd}$$

is a pushout in Set

$\Rightarrow$ $\mathrm{Ob}J^{\triangleleft} = \mathrm{Ob}J\times 2 \amalg 1 /\sim$ where $(j,0) \sim 1$ $\forall j \in J$.

$\Rightarrow$ $\mathrm{Ob}J^{\triangleleft} \cong \{(j,1) : j \in \mathrm{Ob}J\} \cup \{1\}$

Now by Mor forgetful functor,

$$\begin{tikzcd} \mathrm{Mor}J \arrow[r, "!"] \arrow[d, "i_0"'] & \mathrm{Mor}\,1 \arrow[d] \\ \mathrm{Mor}(J\times 2) \arrow[r] & \mathrm{Mor}J^{\triangleleft} \end{tikzcd}$$

induces that $\mathrm{Mor}J^{\triangleleft} = \mathrm{Mor}(J\times 2)/\sim$ where $(f, 1_0) = 1_1$.

Thus $\mathrm{Mor}J^{\triangleleft} \cong \{1 \to (j,1) : \forall j \in \mathrm{Ob}J\}$
$\cup \{(f, 0\to 1) : f \in \mathrm{Mor}J\}$
$\cup \{(f, 1_1) : f \in \mathrm{Mor}J\}$

Now, we claim that $J^{\triangleleft}$ is a cone over J with summit 1, by identifying $\{(j,1)\}$ as image of J.

To see that, let $f: i \to j \in \mathrm{Mor}\, J$. Then its image in $J^{\triangleleft}$ is $(i,1) \xrightarrow{(f,1_1)} (j,1)$. It suffices to show that

$$\begin{tikzcd} & 1 \arrow[dl, "a"'] \arrow[dr, "b"] & \\ (i,1) \arrow[rr, "{(f,1_1)}"'] & & (j,1) \end{tikzcd}$$

commutes, where a is induced by $(i,0) \to (i,1)$, b 〃 $(j,0) \to (j,1)$.

Since
$$\begin{tikzcd} (i,0) \arrow[r, "{(1_i, 0\to 1)}"] \arrow[d, "{(f,1_0)}"'] \arrow[dr, phantom, "\circlearrowright"] & (i,1) \arrow[d, "{(f,1_1)}"] \\ (j,0) \arrow[r, "{(1_j, 0\to 1)}"'] & (j,1) \end{tikzcd}$$
in $J \times 2$, we have

$$(f,1_1) \circ a = (f,1_1) \circ (1_i, 0 \to 1) = (1_j, 0 \to 1) \circ (f, 1_0)$$

$$= b \circ 1_1 = b. \quad \blacksquare$$

Ex. 3.5.4) Let $K: J \to (\mathbb{N}, \leq)$ be a small diagram. If $\lambda: \lim K \Rightarrow K$ a limit cone exists, then $\forall i \to j \in J$, for any cone $\eta: X \Rightarrow K$

$$\begin{tikzcd} & X \arrow[d] \arrow[ddl, bend right, "\eta_i"'] \arrow[ddr, bend left, "\eta_j"] & \\ & \lim K \arrow[dl, "\lambda_i"] \arrow[dr, "\lambda_j"'] & \\ Fi \arrow[rr] & & Fj \end{tikzcd}$$

From this, notes that

① If $X \Rightarrow K$ is a cone, then $X \mid Fi$ $\forall i \in J$.

② If $\lim K$ exists, then $X \mid \lim K$.

Thus, $\lim K$ is g.c.d$(Fi;\ i \in \mathrm{Ob} J)$.

Conversely, suppose $Y = \text{g.c.d}(Fi,\ i \in \mathrm{Ob}J)$. Then $Y \mid Fi$ $\forall i \Rightarrow Y \xrightarrow{y_i} Fi$ exists.

Since $(\mathbb{N}, \leq)$ is a poset as a category

these $\{\lambda_i : i \in \mathrm{Obj}\}$ form a cone., since every diagram commutes:

If $X \Rightarrow F$ is a cone, then $X \mid F_i \ \forall i \Rightarrow X \mid Y$

since Y is gcd thus the above limit diagram holds $\Rightarrow$ Y is lim k

In case of colimit, every arrows are reversed, so, by duality argument, colim k is $\mathrm{l.c.m}(F_i, i \in J)$

Ex. 3.5. vi)

Fin^{op}_{mono} : Obj : finite sets $\qquad$ Fix $X \in \mathrm{Top}$.

Morphisms: injections.

$$\mathrm{Fin}^{op}_{mono} \longrightarrow \mathrm{Top}.$$

$$\begin{array}{ccccc} [n] & \longmapsto & P\mathrm{Conf}_n(X) & : & \{(x_1, \dots, x_n) \in X^n : x_i \neq x_j \ \forall i \neq j\} \\ & & & & (x_{f(1)} \cdots x_{f(n)}) \text{ partial function} \\ f \big\uparrow & \longmapsto & \big\uparrow & & \big\uparrow \ f^{-1} = \begin{array}{c} [m] \longrightarrow [n] \\ k \longmapsto f^{-1}(k) \end{array} \\ [m] & \longmapsto & P\mathrm{Conf}_m(X) & : & (x_1 \cdots x_m) \end{array}$$

this is well-defined since f^{-1} is just a projection map. (with permutation of coordinate.)

However, $[n] \longmapsto \mathrm{Conf}_n(X)$ cannot be a functor!

since $[(x_1 \cdots x_n)] \longmapsto [(x_{f^{-1}(1)}, \cdots x_{f^{-1}(n)})]$ is not well-defined. For example $f : \{1,2\} \hookrightarrow \{1,2,3\}$ with canonical injection cannot determine $[(4,5,6)] \longrightarrow [(4\,5)]$, $[(5\,6)]$, $[(4,6)]$

Since $[(456)] = [(5,6,4)] = [(4\,6\,5)] \dots$

Ex 3.5. vii) a morphism.

Fiber Space : $p : E \to B \in Top$

Map between fiber spaces : a commutable square, i.e, $(p : E \to B) \to (p' : E' \to B')$ is $(g : E \to E', f : B \to B')$ s.t.

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowright"] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Thus, Category of fiberspace is $\subseteq$ diagram category Top^2.

And one of its non-full subcategory is

Top/B, a slice category over B.

i.e Objects are $p : E \to B$ fiber space over B

and Morphisms are

$$\begin{tikzcd} E \arrow[rr, "g"'] \arrow[dr, phantom, "\circlearrowright"] \arrow[r] & B & E' \arrow[l] \end{tikzcd}$$

identified as $(g, 1_B)$ in Top^2.

1) In a morphism,

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

let $p^{-1}(b)$ a fiber of $b \in B$

Then, $g^{-1}(p^{-1}(b)) = (p')^{-1}(f^{-1}(b))$

by commuting diagram

$\Rightarrow g^{-1} : p^{-1}(b) \longmapsto f^{-1}(b)$

is a canonical map.

2) WTS

fiber of product of fiber spaces = product of fibers.

First of all we need to figure out product of fiber spaces in Top^2

Let $p: E \to B$, $p': E' \to B'$.

then product is $q: E'' \to B''$ with projections, $\pi_p: q \mapsto p$, $\pi_{p'}: q \to p'$ s.t.

$$\begin{tikzcd}
 & s \arrow[dl] \arrow[d] \arrow[dr] & \\
p & q \arrow[l] \arrow[r] & p'
\end{tikzcd}$$

$\forall s \in Top^2$ with $s \to p$, $s \to p'$

Likewise Ex 3.5 iv, we can generate q via Set.

$$\begin{tikzcd}
 & E_s \arrow[d, "s"] \arrow[dl, dashed] \arrow[dr, dashed] \arrow[dd, bend left] & \\
 & B_s \arrow[ddl] \arrow[dd, bend left] \arrow[ddr, dashed] & \\
E \arrow[d, "p"'] & E'' \arrow[l, "(\pi_p)_E"'] \arrow[r, "(\pi_{p'})_E"] \arrow[d, "q"] & E' \arrow[d, "p'"] \\
B & B'' \arrow[l, "(\pi_p)_B"] \arrow[r, "(\pi_{p'})_{B'}"'] & B'
\end{tikzcd}$$

From left picture which holds in Set we know $E'' = E \times E'$, $B'' = B \times B'$

Then $q = (p, p')$

Thus, the unique map $s \to q$ is $(E_s \to E'', B_s \to B'')$ a tuple of unique map induced by E'' and B'' as a product. Then it is just trivial to see that above diagram commutes.

Now, take a fiber in q over $(b, b') \in B''$. Then it is $p^{-1}(b) \times (p')^{-1}(b')$ which is product of fiber

(iii) Let $\pi: B \times F \to B$. Then fiber of π over $b \in B$ is $\pi^{-1}(b) = \{b\} \times F$.

As a subspace $\{b\} \times F$ is homeomorphic to F since in finite product, box topology = product topology. So any basic open set in $B \times F$ is $O_B \times O_F$ form, thus $(\{b\} \times F) \cap (O_B \times O_F) = \{b\} \times O_F$ is basic open set in $\{b\} \times F$, thus $f: \{b\} \times F \to F$ induces ...

$f^{-1}(O_F) = \{b\} \times O_F$ $\forall$ all open set O_F in F.

$\Rightarrow \{b\} \times F \cong F$ in Top.

(iv) Let $\pi_F: B \times F \to B$, $\pi_G: B \times G \to B$ be two trivial fiber space. Then, if it is iso.

$\exists f, g$ s.t.

$$\begin{tikzcd} & B & \\ B \times F \arrow[ur, "\pi_1"] \arrow[rr, shift left, "f"] & \circlearrowleft & B \times G \arrow[ul, "\pi_1'"'] \arrow[ll, shift left, "g"] \end{tikzcd}$$

Let $\pi_G: B \times G \to G$
$\pi_F: B \times F \to F$
canonical morphism.

Then, $f(b, f) = (b, y)$ for some $y \in G$
$g(b, g) = (b, y')$ $y' \in F$.

$\Rightarrow f' = \pi_G \circ f$, $g' = \pi_F \circ g$ are cts.

Conversely, we claim that given $f': B \times F \to G$ and $g': B \times G \to F$ s.t. $g'(b, f'(b, x)) = x$ and $f'(b, g'(b, y)) = y$, this induces a map f and g satisfying the above diagram

Notes that $f := B\times F \xrightarrow{\ f'\ } B\times G$
$$(b,x) \longmapsto (b, f'(b,x))$$
$$g := B\times G \longrightarrow B\times F$$
$$(b,y) \longmapsto (b, g'(b,y))$$

are cts; notes that for open set $O_B \times O_G$ in $B\times G$,

$$f^{-1}(O_B \times O_G) = (f')^{-1}(O_G) \cap (O_B \times F)$$

since if $f(b,x) \in O_B \times O_G$, then

$f'(b,x) \in O_G$ and $b \in O_B$

$\Rightarrow (b,x) \in f'^{-1}(O_G) \cap (O_B \times G)$

Conversely, if $(b,x) \in (f')^{-1}(O_G) \cap (O_B \times F)$

then $f(b,x) = (b, f'(b,x)) \in O_B \times O_G$.

So the equality holds.

Moreover, $g(f(b,x)) = g(b, f'(b,x)) = (b, g'(b, f'(b,x))$

$= (b,x)$

and $f(g(b,y)) = f(b, g'(b,y)) = (b, f'(b, g'(b,y)) =$

$= (b,y)$.

$\Rightarrow f, g$ satisfy the diagram

(v) Let $p: E \to B$ a fiber space

Then a section $s: B \to E$ is a cts map

s.t. $p \circ s = 1_B$.

Now define $\text{Sect}: \text{Top}/B \longrightarrow \text{Set}$

$$\begin{tikzcd} p: E \to B \arrow[r, mapsto] \arrow[d, "f: E \to E'"'] & \{\text{sections of } p\} \arrow[d, "f_* \text{ post comp}"] \\ p': E' \to B' \arrow[r, mapsto] & \{\text{sections of } p'\} \end{tikzcd}$$

then ① $\forall s: B \to E$ a fiber of $p: E \to B$,

$f \circ s: B \to E \to E'$ is a fiber of $p': E' \to B'$

since $p' \circ f \circ s(x) = p \circ s(x) = x$. $\forall x \in B$

Thus, as a set map, f_* is well-defined.

② Functoriality trivially holds by argument in ①

$\Rightarrow$ Sect is well-defined functor.

(vi). Notes that

Top^2_{pb} : obj: $p: E \to B$ any fiber space

Mor: $(p: E \to B) \xrightarrow{(f,g)} (p': E' \to B')$

iff the commuting square is actually

a pull back; i.e.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[d, "p'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

Now, $\mathrm{Sect} : \left(\mathrm{Top}^2_{pb}\right)^{op} \longrightarrow \mathrm{Set}$

$$\begin{tikzcd}
p : E \to B \arrow[r, mapsto] \arrow[d, "{(f,g)}"'] & \text{Section of } p \\
p' : E' \to B' \arrow[r, mapsto] & \text{Section of } p' \arrow[u, "\phi"']
\end{tikzcd}$$

from the fact that for any $s' : B' \longrightarrow E'$ of Section of p',

$s' \circ g : B \longrightarrow B' \longrightarrow E'$

$$\begin{tikzcd}
B \arrow[dr, dotted, "s"] \arrow[drr, bend left, "s' \circ g"] \arrow[ddr, bend right, "I_B"'] & & \\
& E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\
& B \arrow[r, "s"'] & B'
\end{tikzcd}$$

$\Rightarrow$ induces $\exists ! s : E \longrightarrow B$ in left diagram. and s is a section of p; i.e., $p \circ s = I_B$.

thus ϕ is well-defined as the unique map induced by I_B and $(-) \circ g$ on the pullback of f and p.

Also, if $p \xrightarrow{(f,g)} p' \xrightarrow{(f',g')} p''$, then,

$$\begin{tikzcd}
B \arrow[dr, "s_0"] \arrow[drr, bend left, "s'' \circ s' \circ g"] \arrow[ddr, bend right, "I_B"'] & & \\
& E \arrow[r, "f' \circ f"] \arrow[d, "p"'] & E'' \arrow[d, "p''"] \\
& B \arrow[r, "g' \circ g"'] & B''
\end{tikzcd}$$

it induces $s_0 : B \to E$ and let s_2, s_1 are maps induced by

$$\begin{tikzcd}
B' \arrow[dr, "s_2"] \arrow[drr, bend left, "s'' \circ g'"] \arrow[ddr, bend right, "I_B"'] & & \\
& E' \arrow[r, "f'"] \arrow[d, "p'"'] & E'' \arrow[d, "p''"] \\
& B' \arrow[r, "g'"'] & B''
\end{tikzcd}$$

and

$$\begin{tikzcd}
B \arrow[dr, "s_1"] \arrow[drr, bend left, "s_2 \circ g"] \arrow[ddr, bend right, "I_B"'] & & \\
& E \arrow[r, "f"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\
& B \arrow[r, "g"'] & B'
\end{tikzcd}$$

You notes that

$s'' \circ g' = f' \circ s_2$, $\quad s_2 \circ g = f \circ s_1$

$\Rightarrow \; s'' \circ s' \circ g = f' \circ s_2 \circ g = (f' \circ f) \circ s_1$ and $p \circ s_0 = I_B = p s_1$

thus universal property of pull back
$$\begin{tikzcd} E \arrow[r] \arrow[d] & E'' \arrow[d] \\ B \arrow[r] & B'' \end{tikzcd}$$
shows that $S_0 = S_2'$

This implies functoriality of Sect.

($I_p \longmapsto 1_{Sect(p)}$ is trivial)

(vii)
$$\begin{tikzcd}
E' \arrow[rr, "l"] \arrow[ddr, "p'"'] & & E \arrow[dr, "g"] \arrow[ddr, "p"] & \\
& F' \arrow[rr, "r"] \arrow[d, "q'"'] & & F \arrow[d, "q"] \\
& B' \arrow[rr, "f"'] & & B
\end{tikzcd}$$

From
$$\begin{tikzcd} F' \arrow[r] \arrow[d] & F \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$
with $gl : E' \longrightarrow F$ and $p' : E' \longrightarrow B'$.

We get unique map $g' : E' \longrightarrow F'$.

s.t. $q' \circ g' = p'$ and $r g' = lg$ from the universal property of pull back.

So, g and g' can be identified as a morphism in $Top/_B$ and $Top/_{B'}$

Also, (r, f) and (l, f) are morphisms in Top^2_{pb}. Now it suffices to show that.

$$\begin{tikzcd}
Sect(p) \arrow[r, "Sect_{pb}(l,f)"] \arrow[d, "g_*"'] \arrow[dr, phantom, "\circlearrowright"] & Sect(p') \arrow[d, "g'_*"] \\
Sect(q) \arrow[r, "Sect_{pb}(r,f)"'] & Sect(q')
\end{tikzcd}$$

Let $s \in \mathrm{Sect}(p)$. Then, $\mathrm{Sect}_{pb}(l,f)(s) = s_q$ in the left diagram

$$\begin{tikzcd}
B' \arrow[dr, "\exists! s_q"] \arrow[drr, bend left, "s \circ f"] \arrow[ddr, bend right, "1_{B'}"'] & & \\
& E' \arrow[r, "l"] \arrow[d, "p'"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "p"] \\
& B' \arrow[r, "f"'] & B
\end{tikzcd}$$

So, $g' \circ s_q : B' \to F'$ is in $\mathrm{Sect}(q')$

Conversely, $g \circ s \in \mathrm{Sect}(q)$, thus

$$\begin{tikzcd}
B' \arrow[dr, "\exists! s'_q"] \arrow[drr, bend left, "g \circ s \circ f"] \arrow[ddr, bend right, "1_{B'}"'] & & \\
& F' \arrow[r, "r"] \arrow[d] & F \arrow[d] \\
& B' \arrow[r] & B
\end{tikzcd}$$

$s'_q \in \mathrm{Sect}(q')$ by left diagram

$$\Rightarrow \quad \begin{tikzcd}
s \arrow[r, mapsto] \arrow[d, mapsto] & s_q \arrow[d, mapsto] \\
g \circ s \arrow[r, mapsto] & s'_q \qquad g' \circ s_q
\end{tikzcd}$$

Now notes that

$$\begin{tikzcd}
B' \arrow[dr, "g' \circ s_q"] \arrow[drr, bend left, "g \circ s \circ f"] \arrow[ddr, bend right, "1_{B'}"'] & \circlearrowleft & \\
& F' \arrow[r, "r"] \arrow[d] & F \\
& B' &
\end{tikzcd}$$

Since $p' \circ g' \circ s_q = 1_{B'}$ because $g' \circ s_q$ is a section

And $r \circ g' \circ s_q = g \circ l \circ s_q = g \circ s \circ f$. Thus universal property says that $s'_q = g' \circ s_q$.

from
$$\begin{tikzcd}
E' \arrow[r, "l"] \arrow[d, "p'"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "p"] \\
F' \arrow[r, "r"'] & F
\end{tikzcd}$$

from
$$\begin{tikzcd}
B' \arrow[dr, "\exists! s_q"'] \arrow[drr, bend left, "s \circ f"] & \circlearrowleft & \\
& E' \arrow[r, "l"'] & E
\end{tikzcd}$$

$\square$

3.6. Functoriality of limit and colimits

Functoriality $\neq$ Canonicity

Lots of isomorphic objects can define limit.

Prop 3.6.1. Suppose C has all J-shaped limits. Then choice of limit for each diagram defines $\lim_J : C^J \to C$ as a functor.

Rmk) $\lim_J$ is not canonically defined.

pf) By choice of limit and limit cone, we define $\lim_J$ on $\mathrm{Ob}\, C$. To define $\lim_J$ on morphism,

Suppose $F, G \in C^J$ and $\lambda : \lim_J F \Rightarrow F$ are
$\eta : \lim_J G \Rightarrow G$
chosen limit cone,

Then if $\alpha : F \Rightarrow G \in \mathrm{Hom}(F, G)$, then $\forall i \to j \in J$,

$$\begin{tikzcd} & \lim_J F \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ Fi \arrow[rr] \arrow[d, "\alpha_i"'] & & Fj \arrow[d, "\alpha_j"] \\ Gi \arrow[rr] & & Gj \end{tikzcd}$$

$\Rightarrow$ $\alpha \circ \lambda$ is a cone over G,

thus $\exists ! \, k : \lim_J F \to \lim_J G$

s.t. $\alpha \circ \lambda = \eta \circ k$.

thus, define $\lim_J(\alpha) := k$

The uniqueness of k implies functoriality.

Notes that $1_F : F \Rightarrow F$ is sent to $1_{\lim_J F}$,

And for $F \overset{\alpha}{\Rightarrow} G \overset{\beta}{\Rightarrow} H$,

$$\begin{tikzcd} & \lim_J F \arrow[d, "t"] \arrow[ddl, "\beta_i \circ \alpha_i \circ \lambda_i"'] \arrow[ddr, "\beta_j \circ \alpha_j \circ \lambda_j"] & \\ & \lim_J H \arrow[dl] \arrow[dr] & \\ Hi \arrow[rr] & & Hj \end{tikzcd}$$

we get t by universal property.

And if $t' : \lim_J F \longrightarrow \lim_J G$, $t'' : \lim_J G \longrightarrow \lim_J H$ derived from universal property of limit,

$$\begin{tikzcd}
& \lim_J F \arrow[dl, bend right, "\alpha_i \circ \lambda_i"'] \arrow[dr, bend left, "\alpha_j \circ \lambda_j"] \arrow[d, "t'"] & \\
G_i \arrow[dd, "\beta_i"'] & \lim_J G \arrow[l] \arrow[r] \arrow[d, "t''"] & G_j \arrow[dd, "\beta_j"] \\
& \lim_J H \arrow[dl] \arrow[dr] & \\
H_i \arrow[rr] & & H_j
\end{tikzcd}$$

$t'' \circ t'$ gives commuting (left) diagram. $\Rightarrow t'' \circ t' = t$

So composition is preserved.

$\boxtimes$

Ex 3.6.2 Path : $\mathrm{Top} \longrightarrow \mathrm{Set} \cong \mathrm{Top}(I, -)$

$X \longmapsto \mathrm{Top}(I, X)$. with $I = [0,1]$

Let $0, 1 : * \rightrightarrows I$ endpoints. Define

$P : \mathrm{Top} \longrightarrow \mathrm{Set}^{\bullet \rightrightarrows \bullet}$

$$X \longmapsto \left(\begin{array}{ccc} \mathrm{Top}(I, X) & \overset{ev_0}{\underset{ev_1}{\rightrightarrows}} & \mathrm{Top}(*, X) \\ \wr\wr & & \wr\wr \\ \mathrm{Path}(X) & & Pt(X) \end{array} \right)$$

This is a functor. And its coequalizer is

$$\mathrm{Path}(X) \overset{ev_0}{\underset{ev_1}{\rightrightarrows}} Pt(X) \longrightarrow\!\!\!\!\to \pi_0(X)$$

the set of path components of X, since $\pi_0(X)$ is quotient of $Pt(X)$ by equivalence relation s.t. $x \sim y$ if x and y are path connected.

$\Rightarrow \pi_0 : \mathrm{Top} \xrightarrow{P} \mathrm{Set}^{\bullet \rightrightarrows \bullet} \xrightarrow{\mathrm{colim}} \mathrm{Set}$ is well defined by Prop 3.6.1. (We just choose specific coequalizer $\pi_0(X)$)

(Notes that any set with the same cardinality with $\pi_0(Y)$ can be chosen. But we choose it. and it defines π_0 as a functor.
Other choice gives another functor.)

Rmk) Prop 3.6.1 shows any natural transformation in C^J are sent to morphism between limits (or colimits) regardless of complete/cocompleteness of domain category.

Corollary 3.6.3) A natural iso between diagram induces a naturally defined iso between their limits or colimits whenever they exists.

pf) $\lim_J$. in proof of 3.6.1 can be defined from the full subcategory of C^J spanned by functors having limits in C. Now apply Lem 1.3.8.
(all functor preserve iso.)

Rmk) Cor 3.6.3 is actually motivation of Eilenberg and MacLane.

Ex 3.6.4) G : gp. $X, Y : BG \rightrightarrows \mathrm{Set}$
X and Y are objectwise isomorphic $\Rightarrow$ $X \cong Y$ as a set.
" naturally " $\Rightarrow$ There exists an equivariant bijection $t : X \xrightarrow{\cong} Y$ s.t.

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "g_*"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g_*"] \\ X \arrow[r, "t"'] & Y \end{tikzcd}$$

Let $\{a,b\}$ has the nontrivial $\mathbb{Z}/2$ action by sending $\begin{matrix} a \longmapsto a \\ b \longmapsto b \end{matrix}$

and $\{c,d\}$ has " by sending $\begin{matrix} c \longrightarrow d \\ d \longrightarrow c \end{matrix}$

Then $\begin{matrix} \{a,b\} : B(\mathbb{Z}/2) \longrightarrow Set \\ \{c,d\} : \quad " \quad \longrightarrow Set \end{matrix}$ are objectwise iso but not natural iso.

since there is no function $f : \{a,b\} \to \{c,d\}$ satisfying natural iso.

(pf) If $f(b) = c$, then,
$$\begin{tikzcd} b \arrow[r, mapsto] \arrow[d] \arrow[dr, phantom, "\not\circlearrowleft"] & c \arrow[d] \\ b \arrow[r, mapsto] & d \end{tikzcd}$$

If $f(b) = d$, then
$$\begin{tikzcd} b \arrow[r, mapsto] \arrow[d] \arrow[dr, phantom, "\not\circlearrowleft"] & d \arrow[d] \\ b \arrow[r, mapsto] & c \end{tikzcd}$$

Or in viewpoint of limit, $\lim BG \to X$ is X^G.
$\operatorname{colim} BG \to X$ is G-orbit sets.

$\lim \{a,b\} = \{a\}$, $\operatorname{colim} \{a,b\} = \{a,b\}$

$\lim \{c,d\} = \emptyset$, $\operatorname{colim} \{c,d\} = \{*\}$

Ex. 3.6.5. X: finite set.

$\left.\begin{matrix} Sym(X) : \text{all permutations of } X \\ Ord(X) : \text{all total ordering on } X \end{matrix}\right) \Rightarrow |X|!$ elements

But $Sym : Fin_{iso} \longrightarrow Fin$ acts by conjugation

$Ord : \quad " \quad$ acts by translation

ex). $(1\,3\,2) \in Sym(\{1,2,3\})$ $\quad$ $1 \geq 3 \geq 2$ $\quad$ $Ord(\{1,2,3\})$

$\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \qquad\qquad \downarrow$

$(b\,c\,a) \in Sym\{b,a,c\}$. $\quad$ $\quad$ $Ord(\{b,a,c\})$

But this is not natural. (even objectwise iso.)
$\Rightarrow$ limit / colimits need not be iso.

Species := a functor $F : \mathrm{Fin}_{\mathrm{iso}} \to \mathrm{Fin}$.

labeled F-structures on n := $F(n)$

unlabeled " := colimit of
$$B\Sigma_n \hookrightarrow \mathrm{Fin}_{\mathrm{iso}} \xrightarrow{F} \mathrm{Fin}$$
$$\bullet \longmapsto [n] \longrightarrow F(n)$$

objectwise iso $\Rightarrow$ labeled Sym structure
= labeled Ord "

But unlabeled Sym-structure : set of conjugacy classes of permutation of n-elements

(since any bijection is permutation)

unlabeled Ord-structure : trivial

since all linear orders on n are isomorphic.

Rmk) ① Even limit is unique, there might be distinct limit cones.

② Choice of limit cone is usually not compatible.

ex) Freyd shows that C with pullback has a "canonical pullback" with "horizontal composite" But it seems not possible to find such canonical one for "vertical composite"

③ Even chosen limit objects are equal, natural iso might not be identity.

Lemma 3.6.6 For any triple $X, Y, Z \in C$
when C has binary product,
$\exists! \, \alpha : X \times (Y \times Z) \longrightarrow (X \times Y) \times Z$ natural
iso commuting with projections to X, Y, Z.

pf)

$$\begin{tikzcd} X \times (Y \times Z) \arrow[r, "\pi_{1,X}"] \arrow[d, "\pi_{1,Y\times Z}"'] & X \\ Y \times Z \arrow[r, "\pi_{1,Y}"] \arrow[d, "\pi_{1,Z}"'] & Y \\ Z & \end{tikzcd}$$

, The universal property of $X \times Y$ induces the unique

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_{2,X}"'] \arrow[dr, "\pi_{2,Y}"] & \\ X & & Y \end{tikzcd}$$

$t : X \times (Y \times Z) \longrightarrow X \times Y$

s.t.

$\pi_{1,X} = \pi_{2,X} \circ t$

$\pi_{1,Y} \circ \pi_{1,Y\times Z} = \pi_{2,Y} \circ t$.

Now, t and $\pi_{1,Z} \circ \pi_{1,Y\times Z}$ induces

$\alpha : X \times (Y \times Z) \longrightarrow (X \times Y) \times Z$

$$\begin{tikzcd} & (X \times Y) \times Z \arrow[dl, "\pi_{3,X\times Y}"'] \arrow[dr, "\pi_{3,Z}"] & \\ X \times Y & & Z \end{tikzcd}$$

s.t.

$t = \pi_{3,X\times Y} \circ \alpha$

$\pi_{1,Z} \circ \pi_{1,Y\times Z} = \pi_{3,Z} \circ \alpha$.

Conversely, $\pi_{2,Y} \circ \pi_{3,X\times Y}$ and $\pi_{3,Z}$ induces

$s : (X \times Y) \times Z \longrightarrow Y \times Z$

$$\begin{tikzcd} & Y \times Z \arrow[dl, "\pi_{1,Y}"'] \arrow[dr, "\pi_{1,Z}"] & \\ Y & & Z \end{tikzcd}$$

s.t. $\pi_{2,Y} \circ \pi_{3,X\times Y} = \pi_{1,Y} \circ s$

$\pi_{3,Z} = \pi_{1,Z} \circ s$.

Also, s and $\pi_{2,X} \circ \pi_{3,X\times Y}$ induces

$\beta : (X \times Y) \times Z \longrightarrow X \times (Y \times Z)$ s.t. $\pi_{1,X} \circ \beta = \pi_{2,X} \circ \pi_{3,X\times Y}$

$\pi_{1,Y\times Z} \circ \beta = s$.

Then, $\pi_{3,X\times Y}(\alpha \circ \beta) = t \circ \beta : (X \times Y) \times Z \to (X \times Y)$

Since $\pi_{2,X}(t \circ \beta) = \pi_{1,X}(\beta) = \pi_{2,X} \circ \pi_{3,X\times Y}$

$\pi_{2,Y}(t \circ \beta) = \pi_{1,Y} \circ \pi_{1,Y\times Z} \circ \beta = \pi_{1,Y} \circ s$
$= \pi_{2,Y} \circ \pi_{3,X\times Y}$

$\Rightarrow$ $t \circ \beta = \pi_{3,X\times Y}$ by the universal property of $X \times Y$.

$\Rightarrow$ $\pi_{3,X\times Y}(\alpha \circ \beta) = \pi_{3,X\times Y}$

Also, $\pi_{3,Z}(\alpha \circ \beta) = \pi_{1,Z} \circ \pi_{1,Y\times Z} \circ \beta$
$= \pi_{1,Z} \circ s = \pi_{3,Z}$

$\Rightarrow$ $\alpha \circ \beta = 1_{(X\times Y)\times Z}$ by the univ. property of $(X \times Y) \times Z$

Conversely,

$\pi_{1,X}(\beta \circ \alpha) = \pi_{2,X} \circ \pi_{3,X\times Y} \circ \alpha = \pi_{2,X} \circ t = \pi_{1,X}$

$\pi_{1,Y\times Z} \circ (\beta \circ \alpha) = s \circ \alpha$ and

$\pi_{1,Y} \circ (s \circ \alpha) = \pi_{2,Y} \circ \pi_{3,X\times Y} \circ \alpha = \pi_{2,Y} \circ t = \pi_{1,Y} \circ \pi_{1,Y\times Z}$

$\pi_{1,Z}(s \circ \alpha) = \pi_{3,Z} \circ \alpha = \pi_{1,Z} \circ \pi_{1,Y\times Z}$

$\Rightarrow$ $\pi_{1,Y\times Z} = s \circ \alpha$ by universal property

thus $\pi_{1,Y\times Z} \circ (\beta \circ \alpha) = \pi_{1,Y\times Z}$.

$\Rightarrow$ $\beta \circ \alpha = 1_{X\times(Y\times Z)}$ by the universal property

$\Rightarrow$ α, β are isomorphism in C.

This is natural iso; to see this, let $A = \{\cdot \ \cdot\}$ category without nontrivial morphism.

$$\begin{array}{rcl} \text{let } (\underset{A}{(X, -)}) : C & \longrightarrow & C^A \\ Y & \longmapsto & (X, Y) : A \to C \\ \downarrow & \longmapsto & \downarrow \\ Z & \longmapsto & (X, Z) : A \to C \end{array}$$

Then, $Y \to Z$ induces natural transformation between (X, Y) and (Y, Z) since A has no nontrivial morphism.

Then, $(X, \lim_A (Y, -)) \overset{\alpha}{\Longrightarrow} (\lim_A (X, Y), -)$ by our construction of α, β.

$\therefore$ Natural! ∎

Ex. 3.6.17. $sk(Set)$ a skeletal category of Set.
Then actually for each cardinal κ, there is only one object in $sk(Set)$.
$sk(Set)$ is complete; (just use isomorphisms for limit are in Set)

$\Rightarrow$ It has a product

Let C: ctble infinite $\Rightarrow$ $C \times C = C$
and $\pi_1, \pi_2 : C \to C$ are epi.

If $C \times (C \times C) \underset{\cong}{\xrightarrow{i}} (C \times C) \times C$ is identity, then

$$\begin{array}{ccc} C \times (C \times C) & = & (C \times C) \times C \\ {\scriptstyle f \times (g \times h)} \downarrow & & \downarrow {\scriptstyle (f \times g) \times h} \\ C \times (C \times C) & = & (C \times C) \times C \end{array}$$

for any $f, g, h : C \to C$.

Now, we get below commute diagram

$$\begin{tikzcd}
C \arrow[r, shift left, "g\times h"] \arrow[r, shift right] & C \\
C\times C \arrow[u, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[r, shift left, "f\times(g\times h)"] \arrow[r, shift right, "(f\times g)\times h"'] & C\times C \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] \\
C \arrow[r, shift left, "f"] \arrow[r, shift right, "f\times g"'] & C
\end{tikzcd}$$

$\Rightarrow$ Since π_1, π_2 are epi, $f = f\times g$, $g\times h = h$

$\Rightarrow$ $f, g : C \rightrightarrows C$ equal

Since

$$\begin{tikzcd}
C\times C \arrow[r, "f\times f"] \arrow[d, equal] & C \\
C \arrow[r, "f\times g"'] & C
\end{tikzcd}$$

$\Rightarrow$ $f = g$.

But this is absurd! since $sk(Set)(C,C) = Set(C,C)$

Ex 3.6.i) Special case of prop 3.6.1

" ii) shown in the above.

3.7. Size Matters

Lem 3.7.1. C: category. $1_C : C \to C$ admits limit $\Leftrightarrow$ C has initial object.

pf) $(\lambda_c : l \to c)_{c\in C}$ a limit cone.

We can say that l is "weakly initial object" in the sense that $\forall c \in C$, $\exists \lambda_c : l \to c$.

To see λ_c is only morphism, let $f : l \to c$.

Then,

$$\begin{tikzcd}
& l \arrow[dl, "\lambda_l"'] \arrow[dr, "\lambda_c"] \arrow[d, phantom, "\circlearrowright"] & \\
l \arrow[rr, "f"'] & {} & c
\end{tikzcd}$$

$\Rightarrow f\lambda_l = \lambda_c$. So WTS λ_l is identity

To see this, we have
$$\begin{tikzcd} & l \arrow[dl, "\lambda_l"'] \arrow[dr, "\lambda_c"] & \\ l \arrow[rr, "\lambda_c"'] & \Omega & c \end{tikzcd} \Rightarrow \lambda_c \lambda_l = \lambda_c.$$

Thus, $\lambda \circ \lambda_l : (\lambda_c \circ \lambda_l : l \to c)_{c \in C}$

is another cone. By the universal property of λ,

we have
$$\begin{tikzcd} & l \arrow[d, "\lambda_l = 1_l"] \arrow[ddl, bend right, "\lambda_c \circ \lambda_l"'] \arrow[ddr, bend left, "\lambda_d \circ \lambda_l"] & \\ & l \arrow[dl, "\lambda_c"'] \arrow[dr, "\lambda_d"] & \\ c \arrow[rr] & & d \end{tikzcd}$$

$\lambda_l = 1_l$ since both λ_l and 1_l make the left diagram commutes.

$\Rightarrow f = \lambda_c$.

Conversely, if C has initial element l,

then, $\lambda_c : l \to c$ form a cone over 1_C,

and it is limit cone. Let $\eta : t \Rightarrow 1_C$.

Then, $\forall f : d \to c \in C$,
$$\begin{tikzcd} & t \arrow[dl, "\eta_d"'] \arrow[dr, "\eta_c"] & \\ d \arrow[rr, "f"'] & \Omega & c \end{tikzcd}$$
This implies,

$$\Rightarrow \begin{tikzcd} & t \arrow[dl, "\eta_l"'] \arrow[dr, "\eta_c"] & \\ l \arrow[rr, "\lambda_c"'] & \Omega & c \end{tikzcd} \quad \forall c \Rightarrow \eta_l : \eta \Rightarrow \lambda \text{ exists.}$$

So, λ is a limit cone. ∎

Def 3.2.7. Cardinality of a small category
= 〃 the set of its morphism

a κ-small category = a category whose cardinality is less than κ.

κ-small diagram : a diagram whose indexing category is κ-small.

Prop 3.7.3 (Freyd) Any κ-small category admits all κ-small limits or κ-small colimits is a preorder.

pf). Let C be a κ-small category with $|C| = \lambda$.

$f, g : B \rightrightarrows A \in \mathrm{Mor}\, C$ with $f \neq g$.

By Ex 3.4.4, $B \to A^{\lambda}$ is determined by λ indexed morphisms $B \to A$.

$\Rightarrow$ The collection of all $B \to A^{\lambda}$ whose component is either f or g has cardinality 2^{λ}.

$$|C(B, A^{\lambda})| \geq 2^{\lambda} > \lambda = |\mathrm{Mor}\, C|$$

from Cantor's diagonalization argument

But this contradicts the fact that $C(B, A^{\lambda}) \subseteq \mathrm{Mor}\, C$

$\Rightarrow$ every hom set in C is singleton

$\Rightarrow$ C is preorder.

Do the same thing using cotensor for κ-cocomplete case.

2nd) Prop 3.5.9 shows that there is a preorder to have all limit/colimit of unrestricted size

Since P poset has this property

Ex 3.7.1) done in the above $\Leftrightarrow$ P = Complete lattice

8. Interaction between limits and colimits

Let $F: I \times J \to C$. By Ex (1.7. vii), F can be regarded as $F: I \to C^J$ or $F: J \to C^I$.

If $\forall i \in I$, $\exists \lim_{j \in J} F(i, j)$, then prop 3.3.9 gives $\lim_{j \in J} F(-, j): I \to C$. Moreover,

Thm 3.8.1. If $\lim_{i \in I} \lim_{j \in J} F(i, j)$, $\lim_{j \in J} \lim_{i \in I} F(i, j)$ exist, then they are iso and define the $\lim_{I \times J} F$.

pf) From Yoneda Lemma, it suff to prove

$$C(X, \lim_{i \in I} \lim_{j \in J} F(i, j)) \cong C(X, \lim_{(i, j) \in I \times J} F(i, j)) \cong C(X, \lim_J \lim_I F(i, j))$$

By Thm 3.4.6(i), covariant representable functor preserves limit

$$\Rightarrow C(X, \lim_I \lim_J F(i, j)) \cong \lim_I C(X, \lim_J F(i, j)) \cong \lim_I \lim_J C(X, F(i, j))$$

$$C(X, \lim_{(i, j)} F(i, j)) \cong \lim_{(i, j)} F(i, j)$$

$$C(X, \lim_J \lim_I F(i, j)) \cong \lim_J \lim_I C(X, F(i, j))$$

thus, it suffices to show that $\forall H: I \times J \to Set$ a set valued functor, the limits

$$\lim_I \lim_J H \cong \lim_{I \times J} H \cong \lim_J \lim_I H \quad \text{iso.}$$

Isms iso of category $I\times J \cong J\times I$, only left iso is needed to show.

By 3.2.3, let $\lambda : 1 \Rightarrow \lim_{j\in J} H(i,j) \in \lim_I \lim_J H$.

Then,
$$\begin{tikzcd} & 1 \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_{i'}"] \arrow[d, phantom, "\circlearrowleft"] & \\ \lim_J H(i,j) \arrow[rr] & & \lim_J H(i',j) \end{tikzcd}$$
$\forall\, i \to i' \in I$.

By prop 3.6.1, since we already chosen limit cone of $\lim_J H(i,j)$, $\forall i\in I$, $i\to i'$ as a natural transformation of $H(i,-) \xrightarrow{\ \ } H(i',-)$ determines $\lim_J H(i,j) \longrightarrow \lim_J H(i',j)$.

Denote $(\eta_i) : \lim_J H(i,j) \Rightarrow H(i,-)$.

Then, $\forall i\in I$, $\phi_{(i,j)} : 1 \xrightarrow{\lambda_i} \lim_J H(i,-) \xrightarrow{\eta_{i,j}} H(i,j)$ is a cone over $F: I\times J \longrightarrow C$; To see this notes that $\forall f : i \to i'$,

$$\begin{tikzcd} & 1 \arrow[dl, bend right, "\lambda_i"'] \arrow[dr, "\lambda_{i'}"] \arrow[d, phantom, "\circlearrowleft"] & \\ \lim_J H(i,j) \arrow[rr] \arrow[d, "\eta_{i,j}"'] & \circlearrowleft & \lim_J H(i',j) \arrow[d, "\eta_{i',j}"] \\ H(i,j) \arrow[rr, "H(f,-)"'] & & H(i',j) \end{tikzcd}$$

left rectangular comutes by proof of prop 3.6.1.

(Actually, $k : \lim_J H(i,j) \to \lim_J H(i',j)$ is given by the universal property of cone over $H(i',-) \Rightarrow H(f,-)\circ \eta_{i,j} = k\circ \eta_{i',j}$.)

Thus,
$$\begin{tikzcd} & 1 \arrow[dl, "\phi_{i,j}"'] \arrow[d, "\phi_{i',j}"] \arrow[dr, bend left, "\phi_{i',j'}"] & \\ H(i,j) \arrow[r] & H(i',j) \arrow[r] & H(i',j') \end{tikzcd}$$

(Last triangle is from η_i)

This shows that $\phi : I \Rightarrow F \in \lim_{I\times J} F$.

Conversely, if $\phi \in \lim_{I\times J} F$, then, for fixed i,

$\phi_i : 1 \Rightarrow H(i,-)$, thus, $\phi_i \in \lim_J H(i,j)$

So we get $\lambda_i : 1 \longrightarrow \lim_J (H(i,j))$

$1 \longmapsto \phi_i$

Hence, this induces $\lambda : 1 \Rightarrow \lim_J (H(i,-))$ ← elements coincide.

These shows that $\lim_I \lim_J H \cong \lim_{I\times J} H$.

By duality, colim case also holds ▨

Rmk). This shows that $\exists$ natural iso

$$\begin{tikzcd} C^{I\times J} \arrow[r, "\lim_J"] \arrow[d, "\lim_I"'] \arrow[dr, phantom, "\cong"] & C^I \arrow[d] \\ C^J \arrow[r, "\lim_J"'] & C \end{tikzcd} \qquad \begin{tikzcd} C^{I\times J} \arrow[r, "\mathrm{colim}_J"] \arrow[d, "\mathrm{colim}_I"'] \arrow[dr, phantom, "\cong"] & C^I \arrow[d, "\mathrm{colim}_I"] \\ C^J \arrow[r, "\mathrm{colim}_J"'] & C \end{tikzcd}$$

for any choice of $\lim_I$, $\lim_J$ or colim_I, colim_J.

Lemma 3.8.3. $F : I \times J \to C$ bifunctor s.t. $\lim F$, $\mathrm{colim}\, F$ exist

$\Rightarrow \exists\, K : \mathrm{colim}_{i\in I} \lim_{j\in J} F(i,j) \longrightarrow \lim_{j\in J} \mathrm{colim}_{i\in I} F(i,j)$

a canonical map exists.

pf) By the univ. prop. of colimit, if k is canonical, then $\exists$

$$\left(k_i : \lim_{j \in J} F(i,j) \longrightarrow \lim_{j \in J} \operatorname{colim}_{i' \in I} F(i',j) \right)_{i \in I}$$

that defines a cone.

By the univ prop of limit, k_i is defined by

$$\left(k_{i,j} : \lim_{j' \in J} F(i,j') \longrightarrow \operatorname{colim}_{i' \in I} F(i',j) \right)_{j \in J}$$

that defines a cone over $\operatorname{colim}_{i \in I} F(i,-)$.

So, define $k_{i,j} : \lim_{j' \in J} F(i,j') \xrightarrow{\pi_{i,j}} F(i,j) \xrightarrow{\iota_{i,j}} \operatorname{colim}_{i' \in I} F(i',j)$

Where π_i, ι_j are limit and colimit cone over $F(i,-)$ and $F(-,j)$ respectively.

Then, $k_{i,j}$ is a cone over $\operatorname{colim}_{i' \in I} F(i',$!

thus we have k_i ; thus k by above argument. ☐

Corollary 3.8.4. $f : X \times Y \longrightarrow \mathbb{R}$ a function

$\Rightarrow \sup_{x \in X} \inf_{y \in Y} f(x,y) \leq \inf_{y \in Y} \sup_{x \in X} f(x,y)$

pf) In a poset Category, $\operatorname{colim}_X = \sup_X$, $\lim_Y = \inf_Y$. ☐

Rmk. 3.8.5. Direct proof of the corollary, in Real analysis is actually categorical.

pf) $\sup_X \inf_Y f$ is bdd above by M. $\Leftarrow$ $\inf_Y f(x,-)$ bdd above by M. $\forall x \in X$

$\inf_Y \sup_X f$ is bdd below N. $\Leftarrow$ $\sup_X f(-,y)$ bdd below by N. $\forall y \in Y$.

Then, we have $\inf_Y f(x,-) \le f(x,y) \le \sup_X f(-,y)$

Ex). 3.8.6 Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ with $-\infty \le x \le \infty$ poset.

$\Rightarrow$ $\overline{\mathbb{R}}$ is complete and cocomplete. as a category (or poset).

any function gives diagram, and its limit = infimum
co " = supremum.

Now let $x : \mathbb{N} \to \mathbb{R}$ is a sequence.

Then, we have: $\mathbb{N} \times \mathbb{N} \xrightarrow{+} \mathbb{N} \xrightarrow{x} \mathbb{R}$

$(m, n) \longmapsto m+n \longmapsto x_{m+n}$.

So. $\liminf x_n = \sup_{n\ge 0} \inf_{m \ge n} x_m = \sup_{n\ge0} \inf_{m\ge0} x_{n+m} = \operatorname{colim}_n \lim_m x_{n+m}$

By Lem 3.8.3 $\Rightarrow$ $\liminf x_n \le \limsup x_n$.

In Set and some related category there exists an invertible K.

Def! 3.8.7. J is <u>filtered</u> if there is a cone.

$J' \to J$ $\forall J'$ finite indexing category.

Ex) ω, $\bullet \to \bullet$: filtered, $\bullet \leftarrow \bullet \to \bullet$, $\bullet \rightrightarrows \bullet$, discrete category : not filtered

Thm 3.8.9. Filtered colimit commute with finite limit in Set

Observation: $F: J \to \text{Set}$, J is small filtered category. Then, from diagram in p. 917

$$\operatorname{colim}_{j\in J} F_j = \coprod_{j\in J} F_j \Big/ \sim$$

where $x \sim y$ for $x \in F_j$, $y \in F_k$

$\iff \exists f: j \to t$, $g: k \to t$ s.t. $Ff(x) = Ff(y)$.

Thus, let A be a finite subset of $\coprod_{j\in J} F_j$ s.t. $A/\sim := \{\lambda_i\}_{i\in I}$ where I is finite.

Then, $\exists t \in J$ s.t. $A' \subseteq Ft$ and $A'/\sim = A/\sim$.

pf) A is a finite subset of $\coprod_{j\in J} F_j$, so $B := \{j \in J : F_j \cap A \neq \phi\}$ is a finite set. Let $\overline{B}$ be a full subcategory spanned by B. By definition, $\overline{B} \hookrightarrow J$ has a cocone with nadir, say t. Thus, by sending $F_j \cap A \to Ft$ using cone map, we get such A'.

pf of Thm) WTS $k: \operatorname{colim}_J \lim_I F \to \lim_I \operatorname{colim}_J F$ is a bijection when I: finite, J: small filtered category.

By def 3.2.3, as a set,

$$\lim_I \operatorname{colim}_J F = \{\lambda : 1 \Rightarrow \operatorname{colim}_J F(-, J)\}$$

Since I is finite, $\lambda = \{\lambda_i\}_{i\in I}$ is finite.

Also, only finitely many compatibility conditions λ satisfy

Then, each $\lambda_i : 1 \longrightarrow \operatorname{colim}_J F(i, J)$ can be viewed as a element of $\operatorname{colim}_J F(i, J)$.

Now fix $i \in I$, $t \in J$. Choose $\lambda_i' \in F(i,t)$ s.t. $[\lambda_i'] = \lambda_i$ as equivalence class. Then, for any $f : i \to i'$,

Since $\operatorname{colim}_J F(f, J)(\lambda_i) = \lambda_{i'}$;

$F(f, J)(\lambda_i') \in \lambda_{i'}$ as an equivalence class.

Thus we can choose $\lambda_{i'}' \in F(i', t)$ s.t.

$$\begin{tikzcd} & 1 \arrow[dl, "\lambda_i'"'] \arrow[dr, "\lambda_{i'}'"] \arrow[d, phantom, "\circlearrowleft"] & \\ F(i,t) \arrow[rr, "F(f,t)"'] & & F(i',t) \end{tikzcd}$$

So if I is connected, then we have a cone $\{\lambda' : 1 \Rightarrow F(\, , t)\}$.

Otherwise, if i'' is not connected with i in I, then still $\lambda_{i''}' \in F(i'', t')$ for some t' s.t. $[\lambda_{i''}'] = \lambda_{i''}$. Since J is filtered, $\exists$ a cone under t and t', say t'', i.e.

$$\begin{tikzcd} t \arrow[dr, "n_1"'] \arrow[rr, leftrightarrow] & \Omega & t' \arrow[dl, "n_2"] \\ & t'' & \end{tikzcd}$$

$\Rightarrow$ We may regard both λ_i' and its related $\lambda_{i'}'$ and $\lambda_{i''}'$ as an element of $F(i, t'')$, $F(i', t'')$ and $F(i'', t'')$ respectively via the map n_1, n_2.

Thus, we can say that $\exists t \in J$ s.t. $\lambda' : 1 \Rightarrow F(- , t)$ and $[\lambda_i'] = \lambda_i \quad \forall i \in I$

Now this λ' is inside of $\lim_{i \in I} F(-, t)$.

By p. 97 diagram, dom k is

$$\operatorname{colim}_{j \in J} \lim_I F(i,j) = \coprod_{j \in J} \lim_I F(I, j) \Big/ \sim .$$

thus, $[\lambda']$ is in dom k and by construction, k is induced by $k_{j', i} : \lim_{i \in I} F(i, j) \to F(i, j) \to \operatorname{colim}_{j' \in J} F(i, j')$

and by setting $j = t$, we get $k_{j,i}(\lambda') = [\lambda']$ in cod k.

$\Rightarrow$ k is surjective.

Now, to see k is injective,

let $\left(1 \xrightarrow{\alpha_i} F(i,j)\right)_{i \in I}$, $\left(1 \xrightarrow{\beta_i} F(i,k)\right)_{i \in I}$

a cone $\alpha : 1 \Rightarrow \lim_I F(-, j)$, $\beta = 1 \Rightarrow \lim_I F(-, k)$

for some $j, k \in \operatorname{Ob} J$, then for each $i \in I$,

If $k([\alpha]) = k([\beta])$,

$\exists t_i \in J$ s.t. α_i and β_i have a common image in $F(i, t_i)$. Since I is finite and J is filtered, by the same argument we did for λ, (i.e., since $\{t_i\}$ is finite) $\exists t \in J$ s.t. $\forall i$, α_i and β_i have a common image in $F(i, t)$. Then, as an element of $F(i,j)$ or $F(i,k)$

~~α_A~~ Image of $\alpha_{\hat{i}}$ in $F(\hat{i},t)$ = Image of $\beta_{\hat{i}}$ in $F(\hat{i},t)$

$\Rightarrow [\alpha] = [\beta]$ in dom k

$\therefore$ k is injective.

Rmk) Proof is not categorical, since choosing $t_{\hat{i}}$ is arbitrary. In large class of category (Ch 5) these are created in Set.

Ex. 3.8.i) Let s be a coequalizer map

$$\begin{tikzcd} A \arrow[r, shift left, "1"] \arrow[r, shift right, "e"'] & A \arrow[r, two heads, "s"] \arrow[dr, "e"'] & A/e \arrow[d, dashed, "r"] \\ & & A \end{tikzcd}$$

Then the universal property (left diagram) gives unique r s.t. $rs = e$.

Thus we can set below diagram since

$$\begin{tikzcd} A \arrow[r, shift left, "1"] \arrow[r, shift right, "e"'] & A \arrow[r, "s"] \arrow[dr, "s"'] & A/e \arrow[d, dashed, shift right, "sr"'] \arrow[d, dashed, shift left, "1_{A/e}"] \\ & & A/e \end{tikzcd}$$

$srs = se = s$.

The universal property gives $sr = 1_{A/e}$.

Thus, A/e is a retract of A. Conversely, any retract of the form $B \xrightarrow{r} A \xrightarrow{s} B$ gives an idempotent sr which is split by B.

To see any functor preserves idempotent, if F is a functor, then $Fe\, Fe = F(e \circ e) = Fe$

Now suppose that $j : FA \longrightarrow B \in \mathrm{cod}\, F$ s.t. $j = j \circ Fe$.

Then we have a diagram.

$$\begin{tikzcd} FA \arrow[r, shift left, "1_{FA}"] \arrow[r, shift right, "Fe"'] & FA \arrow[r, "Fs"] \arrow[dr, phantom, "\circlearrowleft"] \arrow[ddr, "j"'] & F(A/e) \arrow[d, "Fr"] \\ & & FA \arrow[d, "j"] \\ & & B \end{tikzcd}$$

We claim that left triangle is commute since

$j\,Fr\,Fs = j \circ Fe = j$.

Moreover, since Fs is still split epi, (since functor preserves split epi) thus epi, therefore if $\exists\, b : F(A/e) \to B$ s.t. $b \circ Fs = j$, then $(j \circ Fr) \circ Fs = j$ implies $b = j \circ Fr$.

Thus, $F(A/e)$ is a coequalizer in cod F.

Now, by duality, idempotent as a limit also preserved by any functor.

Thus, by Thm 3.8.1, they commute with both limits and colimits of any shape.

Ex. 3.8.ii) It suff to show that $\operatorname{colim}_{H}(\lim_{G} X)$
$= \lim_{G}(\operatorname{colim}_{H} X)$.

And from previous lecture, we know that

LHS = The set of H-orbits on X^{G}.

RHS = The set of " consists of G-fixed element.

Suppose $BG \times BH \to X$ exists

$$(g,h) \;\; \bullet \to \bullet \quad \longmapsto \quad x \to x$$

(at least $X = G \times H$ exists in Set)

Then, by identifying $gx := (g, 1_h)x$
$hx := (1_g, h)x$,

X is G-set and H-set.

Moreover, if x, y are in the same H-orbit, then $x = (1, h)y$ for some $h \in H$, thus

$\forall g \in G$, $gx = (g, 1_h)x = (g, 1_h)(1, h)y = (g, h)x$
$= (1, h)(g, 1)x$.

$\Rightarrow gx$ and gy are in the same orbit.

Thus, $\exists G$ action on the set of H-orbits.

Let A be an H-orbit s.t. $gA = A$ $\forall g$.

Then, ~~for any~~ $\forall x \in A$, $\forall g \in G$ $\exists h \in H$ s.t.

$gx \in A \Rightarrow gx = hx$ for some h.

But this implies $g^n x = h^n x$ $\forall n \Rightarrow \exists n > 0$ s.t.

$g^n x = x$. and $n \mid |g|$ and $n \mid |h|$.

$\Rightarrow n = 1$ since $|g|$ and $|h|$ are coprime

$\therefore$ all elements in A are in X^G.

Conversely, if we take an H-orbit on X^G, then these are orbits of X consisting of X^G elements. done