When is the multi-graded module over an affine semigroup ring Cohen–Macaulay?

Byeongsu Yu (joint w. Laura Matusevich), Texas A&M University

Contents

$\mathbb{Z}^d$-graded $\Bbbk[\mathbb{N}A]$-modules

An *affine semigroup ring* $\Bbbk[\mathbb{N}A]$ is a combinatorial friendly ring with a natural $\mathbb{Z}^d$-graded structure.

- Set $A := \{a_i, \cdots, a_n\} \subset \mathbb{Z}^d$ and identify it as a $d \times n$ matrix; then $\mathbb{N}A := \{A \cdot u : u \in \mathbb{N}^n\}$ form a monoid, called an *affine semigroup*.
- An *affine semigroup ring* $\Bbbk[\mathbb{N}A] := \Bbbk[t^{a_i}]_{i=1}^n$ is a subring of the Laurent poly. ring $\Bbbk[t_1^{\pm}, \cdots, t_d^{\pm}]$.
- $\Bbbk[\mathbb{N}A]$ is $\mathbb{Z}^d$-graded by setting $\deg(t^a) = a$.
- Given a $\mathbb{Z}^d$-graded $\Bbbk[\mathbb{N}A]$-module M, *a-graded part* of M is $M_a := \{x \in M : \deg(x) = a\}$.
- $\deg(M) := \{a \in \mathbb{Z}^d : M_a \neq 0\}$.

Examples of $\mathbb{Z}^d$-graded modules

- **Polynomial rings**

Let A be a standard basis of $\mathbb{Z}^d$. Then $\mathbb{N}A = \mathbb{N}^d$. $\Bbbk[\mathbb{N}A] = \Bbbk[x_1, \cdots, x_n]$, graded by $\deg(x_i) = e_i$.

- **$\Bbbk[\mathbb{N}A]/I$; I is a monomial ideal** (Fig. 1(a))

$A = \left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{smallmatrix}\right]$, $\Bbbk[\mathbb{N}A] = \Bbbk[s, st, st^2]$, $I = \langle s^2t^2, s^3t \rangle$

- **$\Bbbk[(\mathbb{N}A)_{\mathrm{sat}}]/\Bbbk[\mathbb{N}A]$ as a module** (Fig. 1(b))

$(\mathbb{N}A)_{\mathrm{sat}} := \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d$ is the *saturation* of $\mathbb{N}A$, and $(\mathbb{N}A)_{\mathrm{sat}} \setminus \mathbb{N}A$ is the *holes* of $\mathbb{N}A$.
ex) $A = \left[\begin{smallmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{smallmatrix}\right]$, then $\Bbbk[(\mathbb{N}A)_{\mathrm{sat}}] = \Bbbk[s, t]$.

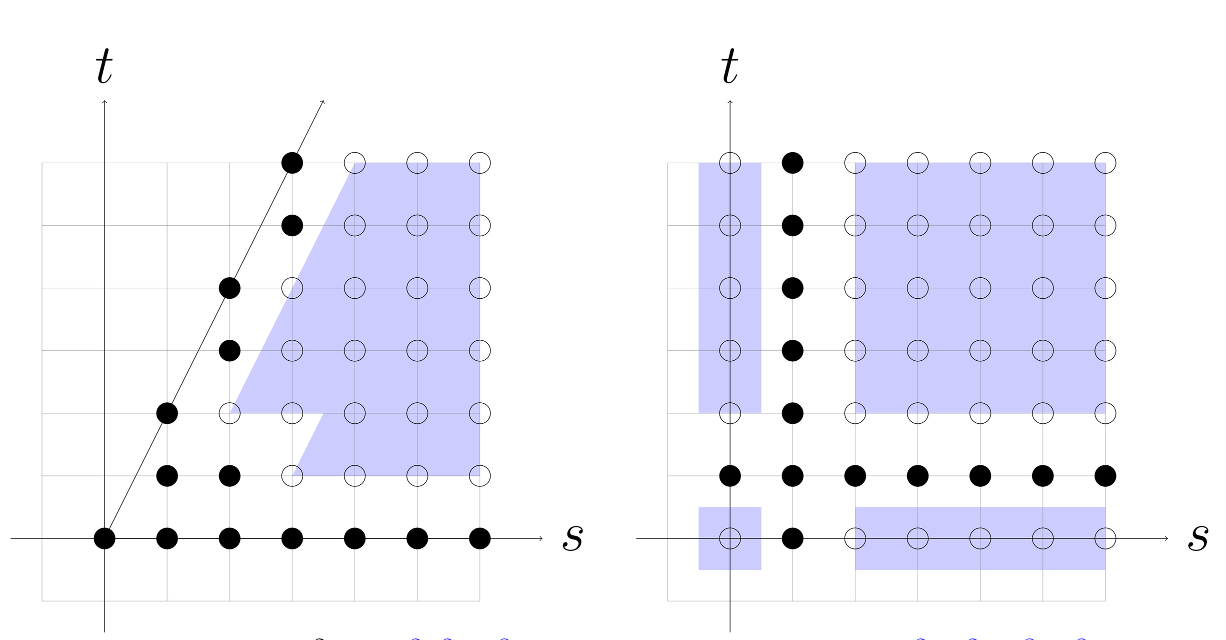

Figure 1: $\mathbb{Z}^d$-graded modules over affine semigroup rings

Degree pairs

Fact: Graded prime ideals $\overset{1\text{-}1}{\leftrightarrow}$ Faces of $\mathbb{R}_{\geq 0}A$ [3]

- $\operatorname{Spec}_{\mathrm{Mon}}(\Bbbk[\mathbb{N}A]) \cong \mathcal{F}(\mathbb{R}_{\geq 0}A)$
- $(a, F) \in \deg(M) \times \mathcal{F}(\mathbb{R}_{\geq 0}A)$ is a *proper pair*.
- $(a, F) < (b, G)$ iff $a + \mathbb{N}F \subseteq b + \mathbb{N}G$
- *Degree pairs* are maximal proper pairs w.r.t. $<$.

Degree pairs are the generalization of *standard pairs* [1,5] of the quotients of affine semigroup rings.

Ishida Complex

Let M_F be the localization by a monomial prime ideal corresponding to $F \in \mathcal{F}(\mathbb{R}_{\geq 0}A)$.

$$L^\bullet : \cdots \xrightarrow{\partial} L^k := \bigoplus_{F \in \mathcal{F}(\mathbb{R}_{\geq 0}A)^{k-1}} \Bbbk[\mathbb{N}A]_F \xrightarrow{\partial} \cdots$$

The differential ∂ is induced by a comp.wise map

$$\partial_{F,G} : \Bbbk[\mathbb{N}A]_F \to \Bbbk[\mathbb{N}A]_G \text{ s.t. } \begin{cases} 0 & \text{if } F \not\subset G \\ \cdot(\pm 1) & \text{if } F \subset G \end{cases}.$$

Thm 1 [4] $H^k_{\mathfrak{m}}(M) \cong H^k(L^\bullet \otimes_{\Bbbk[Q]} M)$

Thm 2 [4] M is CM $\iff \forall k \neq d, H^k_{\mathfrak{m}}(M) = 0$

Combinatorial Cohen–Macaulay criteria (Matusevich, Yu.) [1]

Given a $\mathbb{Z}^{\dim \Bbbk[\mathbb{N}A]}$-graded module M over an affine semigroup ring $\Bbbk[\mathbb{N}A]$,

- M is Cohen–Macaulay module if and only if every chaff attached to grains of $\deg(M)$ is either acyclic or (-1)-dimensional at homological index $\dim \Bbbk[\mathbb{N}A]$.
- Especially, $\Bbbk[\mathbb{N}A]/I$ is Cohen–Macaulay ring if and only if every chaff attached to grains of $\Bbbk[\mathbb{N}A]/I$ is either acyclic or (-1)-dimensional at homological index $\dim \Bbbk[\mathbb{N}A]$.

Example of degree pairs

Given $A = \left[\begin{smallmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & 1 \end{smallmatrix}\right]$ and $I = \langle x, xyz, xyz^2 \rangle$,

(a) $\Bbbk[(\mathbb{N}A)_{\mathrm{sat}}]/\Bbbk[\mathbb{N}A] \cong \Bbbk[x, y, z]/\Bbbk[\mathbb{N}A]$

Deg Pairs: $(y, \{a_1, a_2\})$, $(yz, \{a_1, a_2\})$, $(z, \{a_1, a_2\})$.

(b) $\Bbbk[\mathbb{N}A]/I$

Deg Pairs: $(1, \{a_1, a_2\})$, $(xz, \{a_1, a_2\})$, $(xy, \{a_2\})$.

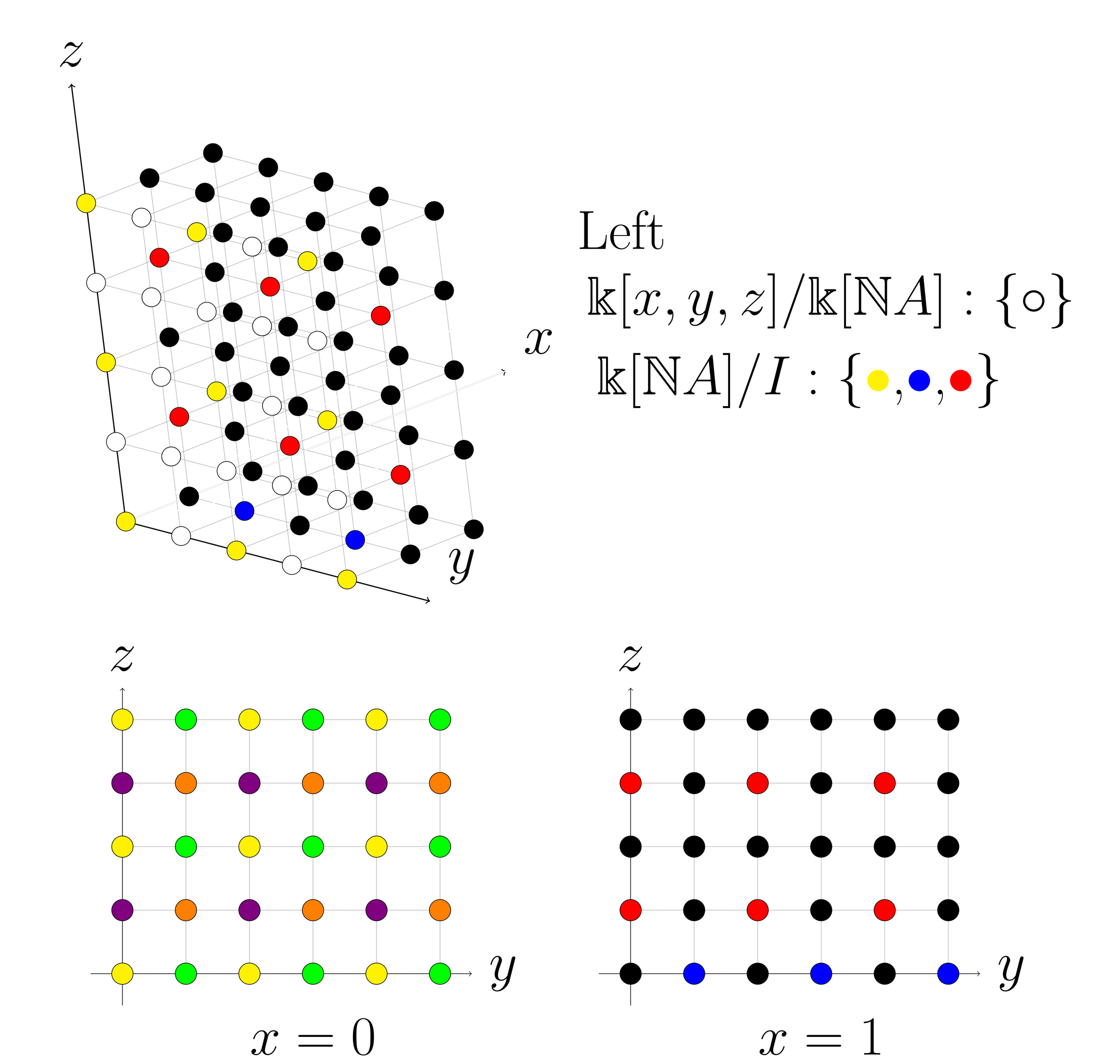

Figure 2: Degree pairs of $\Bbbk[(\mathbb{N}A)_{\mathrm{sat}}]/\Bbbk[\mathbb{N}A]$ and $\Bbbk[\mathbb{N}A]/I$

Hochster-type formula

Degree pairs with the same face (a, F) and (b, F) *overlaps* if $(b + \mathbb{N}F) \cap (a + \mathbb{N}F) \neq \emptyset$. Overlapping is an equivalence relation.

An *overlap class* $[a, F]$ is the equiv. class containing (a, F). Let $\cup[a, F] = \cup_{(a,F)} a + \mathbb{N}F$. Likewise, let

$$\cup \deg(M) := \cup_{F \in \mathcal{F}(\mathbb{R}_{\geq 0}A)} \deg(M_F)$$

The smallest topology on $\cup \deg(M)$ in which $\cup[a, F]$ is clopen for any equiv. class $[a, F]$ from any localizations is called *degree pair topology*.

Grain G is a minimal open set of the degree pair topology. *Chaff* D_{G} of a grain G is the set of faces F such that $\deg(M_F) \supseteq \mathsf{G}$.

Prop. [2] Grains $\mathcal{G}(M)$ of $\cup \deg(M)$ are finite. Indeed, D_{G} is an interval of $\mathcal{F}(\mathbb{R}_{\geq 0}A)$; thus forms an abstract polyhedral complex.

Thm. [2] The multi-graded Hilbert series for $H^i_{\mathfrak{m}}(M)$ of a finely $\mathbb{Z}^d$-graded module M is

$$\operatorname{Hilb}\left(H^i_{\mathfrak{m}}(M), \mathbf{t}\right) = \Sigma_{\mathsf{G} \in \mathcal{G}(M)} \dim_k H^{i+\bullet}_{\mathrm{CW}}(D_{\mathsf{G}}; \Bbbk) \left(\Sigma_{a \in \mathsf{G}} \mathbf{t}^a\right)$$

where $\bullet$ is the dimension of the minimal face in D_{G}.

In other words, $\operatorname{Hilb}(H^i_{\mathfrak{m}}(M), \mathbf{t})$ is a finite rational sum over the homology of polyhedral cell complexes.

Example of Hochster-type formula

$A = \left[\begin{smallmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ with $F_1 := \mathbb{R}_{\geq 0}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $F_2 := \mathbb{R}_{\geq 0}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$.
$I := \langle x^{\left[\begin{smallmatrix} 4 & 5 & 4 \\ 1 & 1 & 2 \end{smallmatrix}\right]} \rangle \subset \Bbbk[\mathbb{N}A]$. Then,

$$L^\bullet : 0 \to \Bbbk[\mathbb{N}A] \to \Bbbk[\mathbb{N}A]_{s^2,s^3} \oplus \Bbbk[\mathbb{N}A]_{st} \to 0.$$

All comps. and degree space are depicted below.

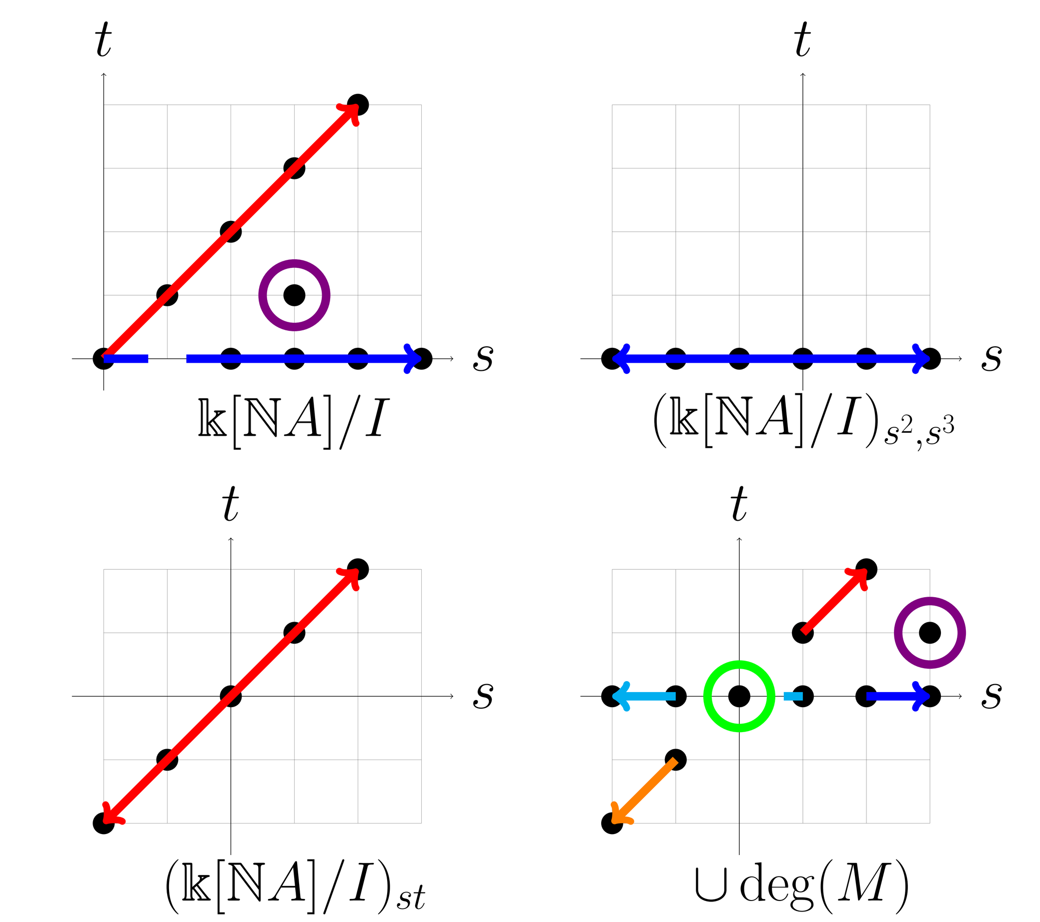

Figure 3: An irreducible decomposition

Grain	$\wedge$	$\wedge$	$\bullet$	$\wedge$	$\wedge$	$\bullet$
Chaff	$\{\emptyset, F_1\}$	$\{\emptyset, F_2\}$	$\{\emptyset\}$	$\{F_1\}$	$\{F_2\}$	$\mathcal{F}(\mathbb{R}_{\geq 0}A)$

Grain	Chain complex of chaff	Exact?
$\wedge\wedge$	$0 \to \mathbb{K} \to \mathbb{K} \to 0$	Exact!
$\bullet$	$0 \to \mathbb{K} \to 0 \to 0$	Not Exact
$\wedge\wedge$	$0 \to 0 \to \mathbb{K} \to 0$	Not Exact
$\bullet$	$0 \to \mathbb{K} \to \mathbb{K}^2 \to 0$	Not Exact

Thus,

$$\operatorname{Hilb}(H^0_{\mathfrak{m}}(\Bbbk[\mathbb{N}A]/I); s, t) = s^3t \implies \text{not CM.}$$

Bibliography

[1] Laura Matusevich and Byeongsu Yu, *Standard pairs for monomial ideals in semigroup rings*, J. Pure Appl. Algebra **226** (2022), no. 9, 107036, DOI 10.1016/j.jpaa.2022.107036.

[2] ______, *Local cohomology of monomial ideals in affine semigroup rings*, preprint (2022).

[3] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005.

[4] Winfried and Herzog Bruns Jürgen, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.

[5] Bernd and Trung Sturmfels Ngô Viêt and Vogel, *Bounds on degrees of projective schemes*, Math. Ann. **302** (1995), no. 3, 417–432, DOI 10.1007/BF01444501.