Ch2 : Univ property, Representability, Yoneda Lemma.

2.1 Representable functor

We will explain what the universal property is.

Ex 2.1.1 $\underset{\text{set}}{(X}, \underset{\text{endo.}}{f: X \to X}, \underset{\substack{\cap \\ X}}{x_0})$

is called a <u>discrete dynamical system</u>

From $(X, f: X \to X, x_0)$ we have $\{x_i\}_{i \in \mathbb{N}}$

s.t. $x_i = f(x_{i-1})$.

If we let $S: \mathbb{N} \xrightarrow{+1} \mathbb{N}$, then

$(\mathbb{N}, S: \mathbb{N} \to \mathbb{N}, 0)$ is universal discrete system s.t. $\forall (X, f: X \to X, x_0)$

$\exists\, r: \mathbb{N} \to X$ s.t. $r(n) = x_n$, thus

$$\begin{tikzcd} \mathbb{N} \arrow[r, "S"] \arrow[d, "r"'] & \mathbb{N} \arrow[d, "r"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

commutes.

Def 2.1.3

(1) $c \in \mathcal{C}$ is initial

$\iff \mathcal{C}(c, -) : \mathcal{C} \longrightarrow \mathrm{Set}$

$\Downarrow$ iso.

$* : \mathcal{C} \longrightarrow \mathrm{Set}$ const functor

$c \longmapsto$ singleton. constant functor

(2) $c \in \mathcal{C}$ is terminal

$\iff \mathcal{C}(-, c) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$

$\Downarrow$ iso

$* : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$ const functor

$c \longmapsto$ singleton

Pf) Let $i \in \mathcal{C}$ initial $\mathcal{C}(i, c)$ is singleton

$$\begin{tikzcd} \mathcal{C}(i, c) \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\curvearrowright"] & \mathcal{C}(i, c') \arrow[d, "\text{iso}"] \\ \text{singleton} \arrow[r] & \text{singleton} \end{tikzcd}$$

Since singletons are iso in Set.

Def 2.14. ① $F: \mathcal{C} \to \mathrm{Set}$ is representable

if $\exists c \in \mathcal{C}$ s.t. $F \Rightarrow \mathcal{C}(c, -)$ is iso

or $F \Rightarrow \mathcal{C}(-, c)$ "

② Representation for $F: \mathcal{C} \to \mathrm{Set}$

$= (c, \alpha: F \Rightarrow \mathcal{C}(c, -))$ where
or $F \Rightarrow \mathcal{C}(-, c)$. α is natural iso.

$c \in \operatorname{obj} \mathcal{C}$.

Def) Universal property of object X

= description of $\mathrm{Hom}(X, -)$ or $\mathrm{Hom}(-, X)$

associate to that object.

Ex 2.15.

(1) $1_{\mathrm{Set}}: \mathrm{Set} \to \mathrm{Set} \overset{\alpha}{\cong} \mathrm{Set}(1, -)$

So representation : 1.

pf) $X \in \mathrm{Set}$, $f: X \to X'$.

$\alpha_X: x \mapsto (1 \mapsto x)$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha_X"'] & X' \arrow[d, "\alpha_{X'}"] \\
\mathrm{Set}(1, X) \arrow[r, "f_*"'] & \mathrm{Set}(1, X')
\end{tikzcd}$$

(2) $U : \mathrm{Group} \longrightarrow \mathrm{Set} \cong \mathrm{Group}(\mathbb{Z}, -)$

pf) G : gp $\quad f : G \to H$ gp Homo

$\alpha_G : g \longmapsto (1 \mapsto g)$

$$\begin{tikzcd}
UG \arrow[r, "f"] \arrow[d, "\alpha_G"'] & UH \arrow[d, "\alpha_H"] \\
\mathrm{Group}(\mathbb{Z}, G) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Group}(\mathbb{Z}, H) \arrow[d, phantom, "\cong" rotate=90] \\
UG \arrow[r, phantom, "f_*"] & UH
\end{tikzcd}$$

$\Rightarrow \mathbb{Z}$: free gp on a single generator

(3) R : unital ring $\quad U : \mathrm{Mod}_R \to \mathrm{Set} \cong \mathrm{Mor}_R(R, -)$

pf) $f : M \longrightarrow N$ module homo

$\alpha_M : m \longmapsto (1 \mapsto m)$

$$\begin{tikzcd}
UM \arrow[r, "f"] \arrow[d, "\alpha_M"'] & UN \arrow[d, "\alpha_N"] \\
\mathrm{Mod}_R(R, M) \arrow[r, "f_*"'] & \mathrm{Mod}_R(R, M)
\end{tikzcd}$$

$\Rightarrow R$: free R-module on a single generator.

(4) $U : \mathrm{Ring} \longrightarrow \mathrm{Set} \cong \mathrm{Ring}(\mathbb{Z}[x], -)$

pf) $f : R \longrightarrow S \quad \alpha_R : r \longmapsto (x \mapsto r)$

$\Rightarrow \mathbb{Z}[x]$: free unital ring on a single generator

(5) $U(-)^n : \text{Group} \to \text{Set} \cong \text{Group}(F_n, -)$

$$\begin{tikzcd} UG^n \arrow[r, "f^n"] \arrow[d, "\alpha_G"'] & UH^n \arrow[d, "\alpha_H"] \\ \text{Group}(F_n, G^n) \arrow[r] & \text{Group}(F_n, H^n) \end{tikzcd}$$

$\alpha_{G^n} : g_i \mapsto (x_i \mapsto g_i)$

$\Rightarrow F_n$: free gp on n generators

$U(-)^n : Ab \to \text{Set} \cong Ab(\bigoplus_n \mathbb{Z}, -)$

(6) For any $G \in$ Group with presentation defines Group $\to$ Set

ex) $G = S_3 = \langle s, t \mid s^2 = t^2 = 1, sts = tst \rangle$

$\Rightarrow S_3 :$ Group $\to$ Set s.t.

$G \longmapsto \{ (g_1, g_2) \in G^2 : g_1^2 = g_2^2 = e, g_1 g_2 g_1 = g_2 g_1 g_2 \}$

$\cong \text{Group}(S_3, -)$

since any $f \in \text{Group}(S_3, G)$ is rep by $\begin{pmatrix} s \mapsto g_1 \\ t \mapsto g_2 \end{pmatrix}$, this is well-def.

"free" : universal property expressed by covariant represented functor

(viii) $(-)^*: \mathrm{Ring} \longrightarrow \mathrm{Set} \cong \mathrm{Ring}(\mathbb{Z}[x^{\pm 1}], -)$

$$\begin{tikzcd}
R^* \arrow[r, "f|_{R^*}"] \arrow[d, "\alpha_R"'] & S^* \arrow[d, "\alpha_S"] \\
\mathrm{Ring}(\mathbb{Z}[x^{\pm 1}], R) \arrow[r, "f_*"'] & \mathrm{Ring}(\mathbb{Z}[x^{\pm 1}], S)
\end{tikzcd}
\qquad
\begin{tikzcd}
y \arrow[r, mapsto] \arrow[d, mapsto] & f(y) \arrow[d, mapsto] \\
\phi_y \arrow[r, mapsto] & f \circ \phi_y = \phi_{f(y)}
\end{tikzcd}$$

Notes that $\phi: \mathbb{Z}[x^{\pm 1}] \longrightarrow R$

is determined by $\phi(x) \in R^*$.

($1 \mapsto 1$ by unital condition.)

Thus, $\alpha_R: \underset{R^*}{\underset{\cap}{y}} \longmapsto \phi_y : \begin{array}{l} 1 \longmapsto 1 \\ x \longmapsto y \end{array}$

viii). $U: \mathrm{Top} \longrightarrow \mathrm{Set}$ (forgetful) $\cong \mathrm{Top}(\{x\}, -)$

$$\begin{tikzcd}
UX \arrow[r, "f"] \arrow[d, "\alpha_X"'] & UY \arrow[d, "\alpha_S"] \\
\mathrm{Top}(\{x\}, X) \arrow[r] & \mathrm{Top}(\{x\}, Y)
\end{tikzcd}$$

$\alpha_X : x' \longmapsto (x \mapsto x')$

$$\Rightarrow \quad \begin{tikzcd}
x' \arrow[r, mapsto] \arrow[d, mapsto] & f(x') \arrow[d, mapsto] \\
x \mapsto x' & x \mapsto f(x') = x \mapsto x' \mapsto f(x')
\end{tikzcd}$$

(9). $ob : Cat \longrightarrow Set \cong Cat(\mathbb{1}, -)$

$$\begin{tikzcd}
ob\, C \arrow[r, "F"] \arrow[d, "\alpha_C"'] & ob\, D \arrow[d, "\alpha_D"] \\
Cat(\mathbb{1}, C) \arrow[r, "F_*"'] & Cat(\mathbb{1}, D)
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r, mapsto] \arrow[d, mapsto] & Fc \arrow[d, mapsto] \\
(\bullet \mapsto c) \arrow[r, mapsto] & (\bullet \mapsto c \mapsto Fc) = (\bullet \mapsto Fc)
\end{tikzcd}$$

$\Rightarrow \alpha_C : c \longmapsto (\bullet \longmapsto c)$

10). $mor : Cat \longrightarrow Set \cong Cat(2, -)$

$$\begin{tikzcd}
mor\, C \arrow[r, "F"] \arrow[d, "\alpha_C"'] & mor\, D \arrow[d, "\alpha_D"] \\
Cat(2, C) \arrow[r, "F_*"'] & Cat(2, D)
\end{tikzcd}
\qquad
\begin{tikzcd}
f : c \to d \arrow[r, mapsto] \arrow[d, mapsto] & Ff : Fc \to Fd \arrow[d, mapsto] \\
\left(\begin{smallmatrix} 0 \\ \downarrow \\ 1 \end{smallmatrix} \mapsto \begin{smallmatrix} c \\ \downarrow f \\ d \end{smallmatrix}\right) \arrow[r, mapsto] & \left(\begin{smallmatrix} 0 \\ \downarrow \\ 1 \end{smallmatrix} \; \begin{smallmatrix} c \\ \downarrow f \\ d \end{smallmatrix} \; \begin{smallmatrix} Fc \\ \downarrow Ff \\ Fd \end{smallmatrix}\right)
\end{tikzcd}$$

11). $Iso : Cat \longrightarrow Set \cong Cat(\mathbb{I}, -)$

$$\begin{tikzcd}
Iso\, C \arrow[r, "F"] \arrow[d] & Iso\, D \arrow[d] \\
Cat(\mathbb{I}, C) \arrow[r] & Cat(\mathbb{I}, D)
\end{tikzcd}
\qquad
\begin{tikzcd}
f : c \xrightarrow{\sim} d \arrow[r, mapsto] \arrow[d, mapsto] & Ff \\
\left(\uparrow\downarrow \mapsto f^{-1}\uparrow\downarrow f\right) \arrow[r, mapsto] & \left(\downarrow\uparrow \mapsto Ff, Ff^{-1}\right)
\end{tikzcd}$$

$\Rightarrow \mathbb{I}$: free (walking) isomorphism.

12) $\mathrm{Comp} : \mathrm{Cat} \longrightarrow \mathrm{Set} \cong \mathrm{Cat}(3, -)$

$$3 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \qquad \mathrm{Comp}\, C := \{ (f,g) \in (\mathrm{mor}\, C)^2 \text{ s.t. } gf \in \mathrm{mor}\, C \}$$

$$\begin{tikzcd} \mathrm{Comp}\, C \arrow[r, mapsto, "F"] \arrow[d] & \mathrm{Comp}\, D \arrow[d] \\ \mathrm{Cat}(3, C) \arrow[r, mapsto, "F_*"'] & \mathrm{Cat}(3, D) \end{tikzcd} \qquad \begin{tikzcd} (f,g) \arrow[r, mapsto] \arrow[d, mapsto] & (Ff, Fg) \arrow[d, mapsto] \\ \left(\begin{tikzcd}[sep=small] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[rr] & & \cdot \end{tikzcd} \right) \arrow[r, mapsto] & \left(\begin{tikzcd}[sep=small] & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "fg"'] & & \cdot \end{tikzcd} \right) \arrow[r, mapsto] & \begin{tikzcd}[sep=small] & \cdot \arrow[dr, "Fg"] & \\ \cdot \arrow[ur, "Ff"] \arrow[rr] & & \cdot \end{tikzcd} \end{tikzcd}$$

13) $U : \mathrm{Set}_* \longrightarrow \mathrm{Set} \cong \mathrm{Set}_*((\{x,y\}, x), -)$

$$(A, a) \xrightarrow{f} (B, b)$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ \mathrm{Set}_*((\{x,y\}, x), (A,a)) \arrow[r, "f_*"'] & \mathrm{Set}_*(\sim, (B,b)) \end{tikzcd} \qquad \begin{tikzcd} a' \arrow[r, mapsto] \arrow[d, mapsto] & f(a') \arrow[d, mapsto] \\ \begin{pmatrix} x \mapsto a \\ y \mapsto a' \end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix} x \mapsto b \\ y \mapsto f(a') \end{pmatrix} \\ A \arrow[u, phantom, "\cong"] & B \arrow[u, phantom, "\cong"] \end{tikzcd}$$

$$\alpha_A : a' \longmapsto \begin{pmatrix} x \mapsto a \\ y \mapsto a' \end{pmatrix}$$

14) $\mathrm{Path} : \mathrm{Top} \longrightarrow \mathrm{Set} \cong \mathrm{Top}(I, -)$

$\mathrm{Loop} : \mathrm{Top}_* \longrightarrow \mathrm{Set} \cong \mathrm{Top}_*(S^1, -)$

$X \longmapsto$ set of paths/loops

Since $I \to X$ defines path (as $S^1 \to X, x_0$ defines loop.)

"Free" object = a representation of covariant functor $C \to Set$

("Free" means that it induce ... desired property on any object by rep. functor)

"Cofree" " for contravariant functor

Ex 2.1.6: (Ex of Contravariant)

(1) $P : Set^{op} \to Set \cong Set(-, \Omega)$ where $\Omega = \{T, \perp\}$

$$\begin{array}{ccc} A & \longmapsto & P(A) \\ f \downarrow & \longmapsto & \uparrow f^{-1} \\ B & \longmapsto & P(B) \end{array}$$

$\alpha_A : P(A) \longrightarrow Set(A, \Omega)$

$A' \longmapsto \chi_{A'} : f(A') = \{T\}$

$f(A'^c) = \{\perp\}$

$\Rightarrow$

$$\begin{array}{ccc} P(A) & \xleftarrow{f^{-1}} & P(B) \\ \alpha_A \downarrow & & \downarrow \alpha_B \\ Set(A, \Omega) & \xleftarrow[f^*]{} & Set(B, \Omega) \end{array} \Rightarrow \begin{array}{ccc} f^{-1}(B') & \longleftarrow & B' \\ \downarrow & & \downarrow \\ \chi_{f^{-1}B'} & \longleftarrow & \chi_{B'} \end{array}$$

$\chi_{B'}(B') = T$

$\chi_{B'}((B')^c) = \perp$

$\chi_{f^{-1}B'}(f^{-1}B') = \{T\}$

$\chi_{f^{-1}B'}((f^{-1}B')^c) = \{\perp\}$

$= \chi_{B'} \circ f$

$\chi_{B'} \circ f(f^{-1}B') = \chi_{B'}(B') = T$

$\chi_{B'} \circ f((f^{-1}B')^c) \subseteq \chi_{B'}((B')^c) = \perp$

(II). $\mathcal{O} : \mathrm{Top}^{op} \longrightarrow \mathrm{Set} \cong \mathrm{Top}(-, S)$

Where S: Sierpinski space.

$S = \{0, 1\}$, $\mathcal{O}(S) = \{\emptyset, S, \{0\}\}$

Given $f : X \to Y$ cts (So $\{1\}$ is closed)

$$\begin{tikzcd} \mathcal{O}(X) \arrow[d, "\alpha_X"'] & \mathcal{O}(Y) \arrow[l, "f^{-1}"'] \arrow[d, "\alpha_Y"] \\ \mathrm{Top}(X, S) & \mathrm{Top}(Y, S) \arrow[l, "f^*"] \end{tikzcd}$$

$\alpha_X : U \longmapsto \chi_U : X \to S$

$\chi_U(U) = \{0\}$

$\chi_U(U^c) = \{1\}$

$$\Rightarrow \begin{tikzcd} f^{-1}(V) \arrow[d, mapsto] & V \arrow[l] \arrow[d, mapsto] \\ \chi_{f^{-1}(V)} \\ \chi_V \circ f & \chi_V \arrow[l, mapsto] \end{tikzcd}$$

To see $\chi_V \circ f = \chi_{f^{-1}(V)}$

let $x \in f^{-1}(U)$

$\Rightarrow \chi_U \circ f(x) = 0 = \chi_{f^{-1}(U)}(x)$

If $x \notin f^{-1}(U)$

$\chi_V \circ f(x) = 1 = \chi_{f^{-1}(U)}(x)$

3) $C : \mathrm{Top}^{op} \longrightarrow \mathrm{Set} \cong \mathrm{Top}(-, S)$

Sierpinski

$\alpha_X : C \longmapsto \chi_C : X \to S$

$\begin{pmatrix} C \longmapsto 1 \\ C^c \longmapsto 0 \end{pmatrix}$

$$\begin{tikzcd} f^{-1}(U) \arrow[d] & U \arrow[l] \arrow[d] \\ \chi_{f^{-1}(U)} = \chi_U \circ f & \chi_U \arrow[l, mapsto] \end{tikzcd}$$

So, $\boxed{\mathcal{O} \cong \mathrm{Top}(-, S) \cong C}$

natural iso.

(4). $\mathrm{Hom}(- \times A, B) : \mathbf{Set}^{op} \to \mathbf{Set} \cong \mathbf{Set}(-, B^A)$

$$X \longmapsto \mathrm{Hom}(X \times A, B)$$

Let $f : X \to Y$.

$$\begin{tikzcd}
\mathrm{Hom}(X \times A, B) \arrow[d, "\alpha_X"'] & \mathrm{Hom}(Y \times A, B) \arrow[l, "(f \times 1_A)^*"'] \arrow[d, "\alpha_Y"] \\
\mathrm{Set}(X, B^A) & \mathrm{Set}(Y, B^A) \arrow[l, "f^*"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \times A \xrightarrow{f \circ 1} Y \times A \xrightarrow{g} B \arrow[d, mapsto] & g : Y \times A \to B \arrow[l, mapsto] \arrow[d, mapsto] \\
x \mapsto \{(x, g \circ (f \times 1_A)(x,a)) : a \in A\} & (y \mapsto \{(b, g(y,b)) : b \in B\}) \arrow[l, mapsto]
\end{tikzcd}$$

$x \mapsto (x, g \circ (f \times 1_A)(y,a))$ //

where $\alpha_X : X \times A \xrightarrow{\phi} B \longmapsto \left(x \longmapsto \{(a, \phi(x,a)) : a \in A\} \in B^A \right)$

It is called "Currying".

(6). $H^n(-;A) : Top^{op} \longrightarrow Ab$.

A : ab gp. $H^n(-;A)$ singular cohomology with coeff in A.

Actually. $H^n(-;A) : Htpy^{op} \longrightarrow Ab$.

Think $H^n(-;A) : Htpy_{CW}^{op} \longrightarrow Set$.

$$\cong Htpy_{CW}^{op}(-, K(A,n))$$

$X \xrightarrow{f} Y$ CW cpx.

$$\begin{tikzcd}
H^n(X;A) \arrow[d, "\cong"'] & H^n(Y,A) \arrow[l, "f^*"'] \arrow[d, "\cong"] \\
Htpy_{CW}^{op}(X, K(A,n)) & Htpy_{CW}^{op}(Y, K(A,n)) \arrow[l, "f^*"]
\end{tikzcd}$$

(7). Classifying space of G : = CW cpx BG

s.t. $Htpy_{CW}^{op} \longrightarrow Set$

$$X \longmapsto Htpy_{CW}^{op}(BG, X)$$

$\cong$ the set of iso classes of principal G-bundle over X.

Remaining Q :

① How unique? i.e. If F is rep by c, c', then $c \cong c'$?

② What data is needed to construct natural iso between F and $C(c, -)$?

③ How do representation related to initial or terminal obj?