3.1. Limit and Colimit as a universal cone

Recall: diagram (or shape) J: in a category C

$\Rightarrow$ $F : J \longrightarrow C$ a functor.

Objective: Introduce lim and colim of diagram as universal cones over and under the diagram

Def 3.1.1 [Cone]

$c \in C$, J: category.

$c : J \longrightarrow C$ is called constant functor.

$$\begin{array}{ccc} x & \longmapsto & c \\ f \downarrow & \longmapsto & \downarrow 1_c \\ y & \longmapsto & c \end{array}$$

$$\Rightarrow \quad \begin{array}{rccl} \Delta : & C & \longrightarrow & C^J \\ & c & \longmapsto & c : J \to C \\ & f \downarrow & & \downarrow \alpha_f \\ & d & \longmapsto & d : J \to C \end{array}$$

α_f: natural transf

$(\alpha_f)_c = f$

is an embedding.

Def 3.1.2. A cone over $F: J \to C$ with summit (apex) $c \in C$.

= Natural transformation $\lambda : c \Rightarrow F$.

- Data : $\lambda = \{\lambda_j : j \in J\}$
- From natural transf, for $f: j \to k \in J$

$$\begin{tikzcd} c \arrow[r, "1_c"] \arrow[d, "\lambda_j"'] \arrow[dr, phantom, "\circlearrowleft"] & c \arrow[d, "\lambda_k"] \\ Fj \arrow[r, "Ff"'] & Fk \end{tikzcd} \iff \begin{tikzcd} & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] \arrow[d, phantom, "\circlearrowleft"] & \\ Fj \arrow[rr, "Ff"'] & {} & Fk \end{tikzcd}$$

- Thus, if we have data satisfying the commuting diagram, then we can define cone.

A cone under F with nadir c

= Natural transf $\lambda : F \Rightarrow c$ s.t.

$\forall f: j \to k$,
$$\begin{tikzcd} Fj \arrow[rr, "Ff"] \arrow[dr, "\lambda_j"'] & & Fk \arrow[dl, "\lambda_k"] \\ & c & \end{tikzcd}$$

Cone under F is called $\boxed{\text{cocone}}$ since it is dual of a cone.

(Cone over $F: J^{op} \to C^{op}$.)

By Yoneda Lemma, $\mathrm{Hom}(\mathcal{C}(-,c), \mathrm{Cone}(-,F)) \cong \mathrm{Cone}(c,F)$

Hence if $\exists \lim F$ s.t. $\mathcal{C}(-, \lim F) \overset{\alpha}{\cong} \mathrm{Cone}(-,F)$ exists.

Then $\exists \lambda \in \mathrm{Cone}(\lim F, F)$ corresp to α

s.t.

$$\begin{tikzcd}
 & \lim F \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] \arrow[d, phantom, "\circlearrowright"] & \\
F_j \arrow[rr, "Ff"'] & & F_k
\end{tikzcd}$$

corresp to α

$\Longrightarrow$ So A limit of F consists of
- $\lim F \in \mathrm{ob}\,\mathcal{C}$
- $\lambda : \lim F \Rightarrow F$ — limit cone

This λ is universal cone.

Since for any $f : c \longrightarrow d$

$$\begin{tikzcd}
\mathcal{C}(d, \lim F) \arrow[r, "\cong"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Cone}(d, F) \arrow[d, "\alpha_f^*"] \\
\mathcal{C}(c, \lim F) \arrow[r, "\cong"] & \mathrm{Cone}(c, F)
\end{tikzcd}$$

Thus if $\lambda^c : c \Rightarrow F$ is a cone,

$\exists u : c \longrightarrow \lim F$ corresp to λ^c s.t.

$\forall f : j \to k \in J$,

$$\begin{tikzcd}
 & \lim F \arrow[ddl, bend right, "\lambda_j"'] \arrow[ddr, bend left, "\lambda_k"] \arrow[d, "(\alpha_u^*)_c = u"] & \\
 & c \arrow[dl, "\lambda^c_j"'] \arrow[dr, "\lambda^c_k"] & \\
F_j \arrow[rr, "Ff"'] & & F_k
\end{tikzcd}$$

(since $\alpha_u^* : \mathrm{Cone}(\lim F, F) \to \mathrm{Cone}(c, F)$ consists of u.

A colimit of F is similar;

- $\operatorname{colim} F \in \mathcal{C}$
- $\lambda : F \Rightarrow \operatorname{colim} F$ s.t

$$\mathcal{C}(\operatorname{colim} F, -) \cong \operatorname{Cone}(F, -)$$

Def 3.1.6. [Second Definition]

$F : J \to \mathcal{C}$ diagram

Limit of F = terminal object of

$$\int \operatorname{Cone}(-, F)$$

Colimit of F = initial object of

$$\int \operatorname{Cone}(F, -)$$

Def [Category of element] $F : \mathcal{C}^{op} \to \mathrm{Set}$

has a category $\int F$

$$\operatorname{Ob} \int F = \{ (c, x) : c \in \mathcal{C},\ x \in Fc \}$$

$$\operatorname{Mor} \int F = \left\{ (c, x) \to (c', x') : \begin{array}{l} f : c \to c' \\ Ff(x') = x \end{array} \right\}$$

And naturally has forgetful functor

$$\pi : \int F \to \mathcal{C} \qquad \begin{array}{ccc} (c, x) & \longmapsto & c \\ f \downarrow & \longmapsto & f \downarrow \\ (c', x) & \longmapsto & c' \end{array}$$

[Working definition]

Limit of F = universal cone over F

Colimit of F = " under F.

i.e. limit of F = representation for some contravariant functor.

= terminal object in its category of element

colimit of F = "

covariant "

= initial "

"

Def 3.1.5 (Def ①)

For any diagram $F : J \longrightarrow C$ $\left(\begin{array}{l} J : \text{small} \\ C : \text{locally small} \end{array} \right.$

define $\mathrm{Cone}(-, F) : C^{op} \longrightarrow \mathrm{Set}$

$$\begin{tikzcd} c \arrow[d, "f"'] \arrow[r, mapsto] & \{\lambda : c \Rightarrow F \text{ natural transf}\} \\ d \arrow[r, mapsto] & \{\phi : d \Rightarrow F \text{ natural transf}\} \arrow[u, "\alpha_f"'] \end{tikzcd}$$

$\alpha_f : c \longrightarrow d$ as natural transf.

Then limit of F = representation of $\mathrm{Cone}(-, F)$

$(\lim F)$ * May not representable

If $\lim F$ is limit of F in the definition,
$= (\lim F, \lambda)$

For any (c, λ^c), with $c \in C$, $\lambda^c \in \mathrm{Cone}(c, F)$

$$\exists! \ (c, \lambda^c) \xrightarrow{u} (\lim F, \lambda)$$

s.t. $c \xrightarrow{u} \lim F$, $\alpha_u : \lambda \Rightarrow \lambda^c$
with $(\alpha_u)_c = u$.

This implies that $\forall f : j \to k \in J$.

$$\begin{tikzcd}
 & \cancel{\lim F} \arrow[ddl, bend right, "\lambda_j"'] \arrow[ddr, bend left, "\lambda_k"] \arrow[d, "(\alpha_u)_c = u"] & \\
 & \cancel{c} \arrow[dl, "\lambda^c_j"'] \arrow[dr, "\lambda^c_k"] & \\
Fj \arrow[rr, "Ff"'] & & Fk
\end{tikzcd}$$

Commutes

Prop 3.1.7 (Essential Uniqueness)
Given any two limit cones $\lambda : l \Rightarrow F$
$\lambda' : l' \Rightarrow F$

over a common diagram F,
$\exists! \ l \cong l'$ that commutes legs of limit cones

pf). (ℓ, λ), (ℓ', λ') are terminal object of $\int \mathrm{Cone}(-, F)$ $\Rightarrow$ uniq. iso.
or $\int \mathrm{Cone}(F, -)$

Rmk: There may be nontrivial automorphism $\ell \cong \ell'$ but this does not commute with limit.

Def 3.1.9. [Product]

Product = limit of a diagram indexed by
$J :=$ a discrete category with only identity morphisms.

Thus, $F : J \to C$ is just collection
$\{F_j\}_{j \in J}$ (since J has no nontrivial morphism)

Thus, cone over J is $\lambda : c \Rightarrow F$
i.e. collection of $\{\lambda_j : c \longmapsto F_j\}_{j \in J}$.

Therefore its limit is $\prod_{j \in J} F_j$
with legs $\pi_k : \prod_{j \in J} F_j \longrightarrow F_k$

By the universal property of $\prod_{j\in J} F_j$ as a cone gives natural isomorphism

$$\mathcal{C}(c, \prod_{j\in J} F_j) \xrightarrow[\cong]{(\pi_k)_*} \mathrm{Cone}(c, F) \cong \prod_{k\in J} \mathcal{C}(c, F_k)$$

(Yoneda lemma) Just....

Ex 3.1.10. $X, Y \in \mathrm{Top}$. Product of $X \times Y$ is a space $X \times Y$ satisfying

$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

cts projection and has universal property: For any $A \in \mathrm{Top}$ with $f: A \to X$, $g: A \to Y$, $\exists! h: A \to X\times Y$ s.t.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, "\exists! h"] & \\ X & X\times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

commute

To construct $X \times Y$ as usual sense (Cartesian product)

By taking $A = \{*\}$, A as a constant functor

says that $X \times Y \supseteq X \underset{\text{cartesian}}{\times} Y$ as a set.

(By similar argument, $X \times Y = X \underset{\text{cartesian}}{\times} Y$ as a set)

This is because $Top(A, X \times Y) \cong Top(A, X) \times Top(A, Y)$ as a set.

To show $X \times Y = \underset{\text{Cartesian}}{X \times Y}$ as a top space

take $A = X \times Y$ as a set with various topology

By the universal property,

$X \times Y$ forced to defined as the coarsest topology on cartesian product set

π_X, π_Y are cts (These are case when $J = \{\bullet, \bullet\}$)

(Do the same thing for any index J.)

Def 3.1.11. Terminal object = product when the indexing category is empty.

i.e, if $J = \{ \}$, Cone over J with summit c is just c. $\Rightarrow \int Cone(-, J \to C)$

$\cong C$.

By def of limit, $\lim (J \to C)$ is terminal obj of $\int Cone(-, J \to C) \cong C$.

Ex 3.1.12 $\mathbb{1} : \{ \cdot \circlearrowleft 1 \cdot \}$ called terminal category

Since it is terminal obj of Cat or CAT

Def 3.1.13 [Equalizer]

Equalizer = Limit of $F : J \longrightarrow C$

where $J = \cdot \overset{f}{\underset{g}{\rightrightarrows}} \cdot$ parallel pair.

$\Rightarrow$ Let $F(J) = A \overset{f}{\underset{g}{\rightrightarrows}} B$. A cone over F with summit C is $(a : C \longrightarrow A,\ b : C \longrightarrow B)$

s.t.

$$\begin{tikzcd} & C \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "\circlearrowleft"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

$\Leftrightarrow$ $fa = b = ga$.

Thus cone over parallel pair $A \overset{f}{\underset{g}{\rightrightarrows}} B$ (with summit C) is represented by a morphism $a : C \longrightarrow A$ s.t. $fa = ga$.

Hence Equalizer is the universal arrow with this property, $h : E \longrightarrow A$. In particular $\forall$ cone $a : C \longrightarrow A$, $\exists! k : C \longrightarrow E$ s.t.

$$\begin{tikzcd} C \arrow[d, "k"'] \arrow[dr, "a"] \arrow[d, phantom, shift left=3, "\circlearrowleft"] & & \\ E \arrow[r, "h"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Commutes.

Ex 3.1.14. $\phi, \psi : G \to H \in \text{Group}$

$\Rightarrow Eq(\phi, 0) = (\ker\phi \longrightarrow G)$

since

$$\begin{tikzcd}
C \arrow[d, "\exists! k"'] \arrow[dr, "a"] & & \\
\ker\phi \arrow[r, hook] \arrow[ur, phantom, "\circlearrowleft"] & G \arrow[r, shift left, "\phi"] \arrow[r, shift right, "0"'] & H
\end{tikzcd}$$

$Eq(\phi, \psi) = (\underline{\{g \in G : \phi(g) = \psi(g)\}} \hookrightarrow G)$

If H is abelian, $(= \ker(\phi - \psi))$

Actually, $Eq(A \underset{g}{\overset{f}{\rightrightarrows}} B) \hookrightarrow C$ is monomorphic

$h : E \to A \longmapsto E$

Def 3.1.15 [Pullback]

Pullback = limit of $(F : \bullet \longrightarrow \bullet \longleftarrow \bullet \longmapsto C)$

Let $F(J) = B \xrightarrow{f} A \xleftarrow{g} C$

Then cone over F with summit D is

$$\begin{tikzcd}
D \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, "a"] & C \arrow[d, "g"] \\
B \arrow[r, "f"'] & A
\end{tikzcd}$$

triple morphism satisfying left comm. diagram

$\Rightarrow \; gc = a = fb.$

Thus we can represent this cone by two morphism $B \xleftarrow{b} D \xrightarrow{c} C$ satisfying rectangular.

Hence pull back: $B \xleftarrow{h} P \xrightarrow{k} C$ is universal cone over $(B \xrightarrow{f} A \xleftarrow{g} C)$

i.e. for any cone $B \xleftarrow{b} D \xrightarrow{c} C$,

$$\begin{tikzcd}
D \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "c"] \arrow[ddr, bend right, "b"'] & & \\
& P \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
& B \arrow[r, "f"'] & A
\end{tikzcd}$$

$\exists!\; D \to P$ satisfying commuting diagram left.

"$\lrcorner$" denote P is pullback, i.e. limit diagram not just commutative square

P is called "fiber product", denoted by $B \times_A C$.

If C is concrete category

thus $B = \{*\} \subseteq A$ as a set

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\ \{*\} \arrow[r, hook, "\iota"'] & A \end{tikzcd}$$

implies $P = \{c \in C : g(c) = *\}$

$= g^{-1}(*)$

thus fiber of map g over $*$ (or $f(*)$)

Ex 3.1.18 $\rho : \mathbb{R} \longrightarrow S^1$ cts

$t \longmapsto e^{2\pi i t}$.

What is P?

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d, dashed] & \mathbb{R} \arrow[d, "\rho"] \\ 1 \arrow[r, hook, "\iota"'] & S^1 \end{tikzcd}$$

$= \rho^{-1}(1)$

$= \{t \in \mathbb{R} : e^{2\pi i t} = 1\}$

$= \mathbb{Z}$

Similar as equalizer

Pullback$\left(B \underset{\text{mono}}{\rightarrowtail} A \leftarrow C\right)$ defines $B \times_A C \underset{\text{mono}}{\rightarrowtail} C$

Ex 3.1.19.

$$\begin{tikzcd} P \arrow[r, dashed, "a"] \arrow[d, dashed, "b"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z} \arrow[d, "n"] \\ \mathbb{Z} \arrow[r, "m"'] & \mathbb{Z} \end{tikzcd}$$

What is P? in Ab.

must $na = mb$.

P : ab gp s.t. $\forall p \in P$

$$n a(p) = m b(p)$$

Moreover, it commutes with any other cones, satisfying $n(a(p)) = m b(p)$

$\Rightarrow$ $n a(p) = m b(p)$ should be l.cm of n, m,

Thus, we may take $P = \mathbb{Z}$ with

$$a(1) = \frac{\mathrm{Lcm}(n,m)}{n}, \quad b(1) = \frac{\mathrm{Lcm}(n,m)}{m}.$$

Def 3.1.21 [Inverse Limit].

Limit of $F : J \to C$ when $J = \omega^{op}$

$\cdots \longrightarrow 3 \longrightarrow 2 \longrightarrow 1 \longrightarrow 0$: ω^{op}.

Then cone over F with summit C is

$$\begin{tikzcd} C \arrow[d] \arrow[dr, "\lambda_3"'] \arrow[drr, "\lambda_2"] \arrow[drrr, bend left, "\lambda_1"] \arrow[drrrr, bend left, "\lambda_0"] & & & & \\ \cdots \arrow[r] & F_3 \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r] & F_0 \end{tikzcd}$$

each triangle commutes

$\varprojlim F_n$: Is the terminal cone.

(Thus direct limit is when $J = \omega$.)
and colimit of $F : J \to C$

Ex 3.1.22

$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n$.

Their dual notions are

Coproduct : Colimit of discrete category

Initial object : " empty "

Coequalizer : " $\cdot \rightrightarrows \cdot$

Pushout : " $\cdot \leftarrow \cdot \rightarrow \cdot$

Direct limit : " ω.
(Sequential colimit)

Ex 3.1.24 Pushout of $S^1 \leftarrow * \rightarrow S^1$
is $S^1 \vee S^1$.

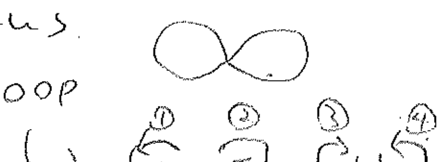
$$\begin{tikzcd} * \arrow[r, "f"] \arrow[d] & S^1 \arrow[d] \\ S^1 \arrow[r] & S^1 \vee S^1 \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

Moreover,

$$\begin{tikzcd} S^1 \arrow[r, "aba^{-1}b^{-1}"] \arrow[d, "i"] & S^1 \vee S^1 \arrow[d] \\ D^2 \arrow[r, dashed] & T \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

T : torus.

$a, ba^{-1}b^{-1}$ loop



Ex). 1.25 Cokernel in Group

$= \text{Coequalizer}(\phi : G \to H,\ 0 : G \to H)$