1.7. 2-categories of categories

C, D : categories

D^C : functor category

Obj : $F : C \longrightarrow D$ functor

Mor :

$$\begin{tikzcd} C \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=L}] & D \arrow[from=U, to=L, Rightarrow, "\alpha"'] \end{tikzcd}$$

α: natural iso.

$$1_F : F \Rightarrow F \quad \text{by } (1_F)_c := 1_{Fc}$$

is identity morphism.

Lem 1.7.1 (Vertical Composition)

$\alpha : F \Rightarrow G,\quad \beta : G \Rightarrow H$

$\Rightarrow \exists\, \beta \cdot \alpha : F \Rightarrow H \quad$ s.t. $(\beta\cdot\alpha)_c := \beta_c \cdot \alpha_c$

pf)

$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowright"] & Gc \arrow[r, "\beta_c"] \arrow[d, "Gf"] \arrow[dr, phantom, "\circlearrowleft"] & Hc \arrow[d] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \arrow[r, "\beta_{c'}"'] & Hc' \end{tikzcd}$$

from def of natural transf β, α.

$\Rightarrow (\beta\cdot\alpha)$ is natural transf.

Cor 1.7.2 D^C is well-def.

Rem 1.7.3

Size of C	Size of D	Size of D^C
small	small	small
small	locally small	locally small
locally small	"	(?)

$$\Rightarrow \quad \mathrm{Cat}^{op} \times \mathrm{Cat} \to \mathrm{Cat}.$$
$$\mathrm{Cat}^{op} \times \mathrm{CAT} \to \mathrm{CAT}.$$

Vertical composition: composition in Lem 1.7.1

i.e.

$$\begin{tikzcd} C \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] & D \end{tikzcd} \quad (\alpha \Downarrow,\ \beta \Downarrow) \quad = \quad \begin{tikzcd} C \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow \beta\alpha"] & D \end{tikzcd}$$

Horizontal composition:

$$\begin{tikzcd} C \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & D \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\Downarrow \beta"] & E \end{tikzcd} = \begin{tikzcd} C \arrow[r, bend left=50, "HF"] \arrow[r, bend right=50, "KG"'] \arrow[r, phantom, "\Downarrow \beta * \alpha"] & E \end{tikzcd}$$

def by

$$\begin{tikzcd} HF_c \arrow[r, "\beta_{Fc}"] \arrow[d, "H\alpha_c"'] \arrow[dr, dashed, "(\beta*\alpha)_c"] & KF_c \arrow[d, "K\alpha_c"] \\ HG_c \arrow[r, "\beta_{Gc}"'] & KG_c \end{tikzcd}$$

pf)

$$\begin{tikzcd} HF_c \arrow[r, "\beta_{F_c}"] \arrow[d, "H\alpha_c"'] & KF_c \arrow[d, "K\alpha_c"] \\ HG_c \arrow[r, "\beta_{G_c}"'] & KG_c \end{tikzcd}$$

Commutes by $\beta : H \Rightarrow K$

$\Rightarrow$ $(\beta * \alpha)_c$ is well-def.

To show $\beta * \alpha$ is natural transf.

Let $f : c \longrightarrow c' \in \mathrm{Mor}\, C$. Then,

$$\begin{tikzcd} HF_c \arrow[r, "H\alpha_c"] \arrow[d, "HFf"'] & HG_c \arrow[r, "\beta_{G_c}"] \arrow[d, "HGf"] & KG_c \arrow[d, "KGf"] \\ HF_{c'} \arrow[r, "H\alpha_{c'}"'] & HG_{c'} \arrow[r, "\beta_{G_{c'}}"'] & KG_{c'} \end{tikzcd}$$

Commute,

Com by naturality of α and H preserves comm diagram

Comm by $\beta : H \Rightarrow K$

$\Rightarrow$ Also note that $\xrightarrow{H\alpha_c} \xrightarrow{\beta_{G_c}} = (\beta * \alpha)_c$

$\xrightarrow[H\alpha_{c'}]{} \xrightarrow[\beta_{G_{c'}}]{} = (\beta * \alpha)_{c'}$

$\Rightarrow$ $KGf\, (\beta * \alpha)_c = (\beta * \alpha)_{c'}\, HFf$. done.

Lem 1.7.7. (Middle four interchange)

$$
\begin{tikzcd}[row sep=large, column sep=huge]
C \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & D \arrow[r, bend left=60, "J"] \arrow[r, "K" description] \arrow[r, bend right=60, "L"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & E
\end{tikzcd}
$$

$\Rightarrow$

$$
\begin{tikzcd}[column sep=huge]
C \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow\beta\cdot\alpha"] & D \arrow[r, bend left=50, "J"] \arrow[r, bend right=50, "L"'] \arrow[r, phantom, "\Downarrow\delta\cdot\gamma"] & E
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge]
C \arrow[r, bend left=60, "JF"] \arrow[r, "KG" description] \arrow[r, bend right=60, "LH"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma * \alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\delta * \beta"] & E
\end{tikzcd}
$$

Def 1.7.8 A 2-category

Obj : (ex) Categories

1-Morphisms (ex) Functors

2- " (ex) Natural transf.

1-Mor: morphism between pair of objects

2 " " functors

s.t. 1) Obj + 1-Mor : category

2) For any $C, D \in$ Obj

D^C with 2-mor between the elem in D^C

form a category.