1.6. Art of the diagram chase

diagram (informal): directed graph

commutes ("): any two composable arrows are the same.

Def 1.6.2. (Monoid) $M \in \mathrm{Set}$ with

$\mu : M \times M$, $\eta : 1 \to M$ s.t.

$$\begin{tikzcd} M \times M \times M \arrow[r, "1_M \times \mu"] \arrow[d, "\mu \times 1_M"'] \arrow[dr, phantom, "\circlearrowright"] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"'] & M \end{tikzcd}, \quad \begin{tikzcd} M \arrow[r, "\eta \times 1_M"] \arrow[dr, "1_M"'] & M \times M \arrow[d] & M \arrow[l, "1_M \times \eta"'] \arrow[dl, "1_M"] \\ & M & \end{tikzcd}$$

μ: multiplication

$\eta : 1 \longrightarrow M$ $\quad$ 1: singleton.

thus $\eta(1)$ is multiplicative identity

Def 1.6.3 (Topological monoid) $M \in \mathrm{Top}$ with the same commutative diagram (so μ is cts)

(Unital ring) $R \in \mathrm{Ab}$ with $R \otimes_{\mathbb{Z}} R$ instead of $R \times R$ (Monoidal Structure)

(k-algebra) $R \in \mathrm{Vect}_k$ with $R \otimes_k R$ instead of $R \times R$ (")

Def 1.6.4 Diagram of C = $F : J \to C$
where J : indexing category is small.

Diagram is commutative
$\Rightarrow$ Any composite relation in J
must holds at C via F.

Ex 1.6.6 2×2 : Obj $(0,0)$, $(0,1)$ $(1,0)(1,1)$
Morphisms

$$
\begin{tikzcd}
(0,0) \arrow[r, "{(0\to 1, 1_0)}"] \arrow[d, "{(1_0, 0\to 1)}"'] \arrow[dr] & (1,0) \arrow[d, "{(1_1, 0\to 1)}"] \\
(0,1) \arrow[r, "{(0\to 1, 1_1)}"'] & (1,1)
\end{tikzcd}
$$

Notes that $(1_1, 0\to 1) \circ (0\to 1, 1_0)$
$= (0\to 1, 0\to 1)$
$= (0\to 1, 1_1) \circ (1_0, 0\to 1)$

So diagonal arrow is unique
"Commutative square!"

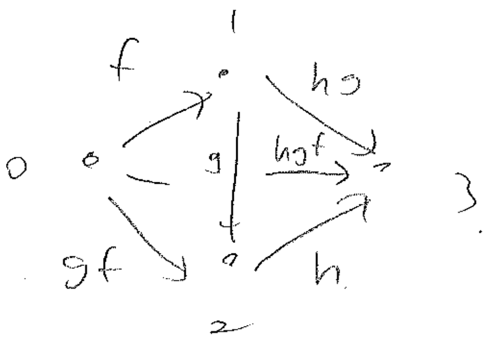
Rem 1.6.7 : "Shape" as indexing category
Shape = directed graph with
specified commutativity relation.

ex) 2×2

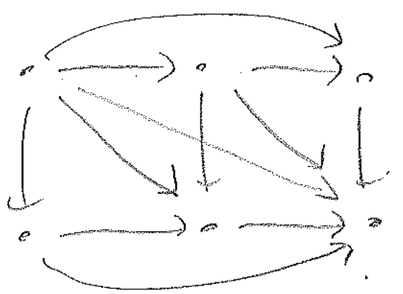
$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "h"] \\ c \arrow[r, "k"'] & d \end{tikzcd}$$

with $hf = kg$.

4



2×3



is
$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "g"'] & \cdot \arrow[r, "j"] \arrow[d, "h"] & \cdot \arrow[d, "l"] \\ \cdot \arrow[r, "k"'] & \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

with $hf = kg$, $lj = mh$

Lemma 1.6.11. $f_1 \cdots f_n$: composable path.

$$f_k f_{k-1} \cdots f_1 = g_m \cdots g_1$$

$$\Rightarrow \quad f_n \cdots f_1 = f_n \cdots f_{k+1} g_m \cdots g_1$$

pf) $g = h \Rightarrow fg = fh$ for any composable f.

Diagram chasing : showing two paths are equal

Lem 1.6.12

$$\begin{tikzcd} \cdot \arrow[d, "f"'] \arrow[dr, "h"] & \\ \cdot \arrow[r, phantom, "\circlearrowleft"] \arrow[r, "g"'] & \cdot \end{tikzcd}, \quad f \text{ iso} \Rightarrow \begin{tikzcd} \cdot \arrow[dr, "h"] & \\ \cdot \arrow[u, "f^{-1}"] \arrow[r, "g"'] & \cdot \end{tikzcd}$$

(Dually,

$$\begin{tikzcd} \cdot \arrow[d, leftarrow, "k"'] & \\ \cdot & \cdot \arrow[ul, "i"'] \arrow[l, "j"] \end{tikzcd} \text{ and } k \text{ iso} \Rightarrow \begin{tikzcd} \cdot \arrow[d, "k^{-1}"'] & \\ \cdot & \cdot \arrow[ul, "i"'] \arrow[l, "j"] \end{tikzcd} \Bigg)$$

pf) $gf = h \Rightarrow gff^{-1} = hf^{-1} \Rightarrow g = hf^{-1}$

Lem 1.6.13

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\circlearrowleft"] & \cdot \arrow[d, "\beta"] \\ \cdot \arrow[r, "\delta"'] & \cdot \end{tikzcd} \Rightarrow \alpha^{-1}\beta^{-1} = \gamma^{-1}\delta^{-1}$$

pf) $\beta\alpha = \delta\gamma \Rightarrow \alpha^{-1}\beta^{-1}\cdot(\beta\alpha)\gamma^{-1}\delta^{-1} = \alpha^{-1}\beta^{-1}(\delta\gamma)\gamma^{-1}\delta^{-1}$

$\Rightarrow \gamma^{-1}\delta^{-1} = \alpha^{-1}\beta^{-1}$.

Def 1.6.14 $i \in C$ is initial if $\forall c \in C$
$\exists! \; i \longrightarrow c$.
$t \in C$ is terminal if $\forall c \in C$
$\exists! \; c \longrightarrow t$.

Ex 1.6.15

Category	initial	terminal
Set	$\varnothing$	singleton.

Category	Initial	Terminal
Set	$\emptyset$	Singleton
Top	"	"
Set_*	Singleton	
Mod_R	0	
Group	0	
Ring	$\mathbb{Z}$	0
Rng (non unital)	0	0
Field	Do not exist (diff characteristic $\Rightarrow$ No homo)	
Cat	$\emptyset$	$\mathbb{1}$
$(P, \leq)$	global minima	global maxima
	(if exist)	

Lem 1.6.16. $f_1 \cdots f_n$ composable seq.
$g_1 \cdots g_m$

s.t $\text{dom}(f_1) = \text{dom}(g_1)$, $\text{cod}(f_n) = \text{cod}(g_m)$

If either $\text{dom}(f_1) =$ initial or $\text{cod}(f_n)$ terminal

$\Rightarrow f_n \cdots f_1 = g_m \cdots g_1$. pf) Uniqueness of morphism from / to initial / terminal.

Def 1.6.17. C : Concrete category

if $U : C \longrightarrow \mathrm{Set}$ a faithful functor exist.

Ex 1.6.18 : (Concrete Category) = Ex 1.1.3.

Graph) : $V \sqcup E$: Graph $\longrightarrow$ Set is faithful

Lem 1.6.19. : $U : C \to D$ faithful then

for any diagram in C whose image

commutes in D also commutes in C.

pf) Let $f_1 \cdots f_n$, $g_1 \cdots g_m$ parallel seq

of composable morphism s.t.

$$Uf_1 \cdots Uf_n = Ug_1 \cdots Ug_m$$

$\Rightarrow$ $U(f_1 \cdots f_n) = U(g_1 \cdots g_m)$ by functoriality

$\Rightarrow$ $f_1 \cdots f_n = g_1 \cdots g_m$ by faithfulness.

Rem 1.6.20. Even outer rectangular commutes.

inner rectangular may not commutes

ex)

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "1_{\mathbb{Z}}"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "1_{\mathbb{Z}}"] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "1_{\mathbb{Z}}"'] & \mathbb{Z}
\end{tikzcd}$$

Lem 1.6.21

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[r, "j"] \arrow[d, "h"] & c \arrow[d, "l"] \\ a' \arrow[r, "k"'] & b' \arrow[r, "m"'] & c' \end{tikzcd}$$

and $ljf = mkg$.

If either ① $\boxed{}\boxed{\circlearrowleft}$ (m: mono) or ② (f: epi) $\boxed{\circlearrowleft}\boxed{}$

then the diagram commutes

pf) Assume ①: $\underset{\underset{mkg}{\parallel}}{ljf} = mhf \Rightarrow hf = kg$ by mono.

② is dual case of ①.

Ex 1.6.i) Let i, initial, t: terminal

$\Rightarrow \exists! g : i \longrightarrow t$ and ~~∃!~~

If $\exists f : t \longrightarrow i$. $\Rightarrow gf : t \to t$

Since $t \to t$ is unique, $gf = 1_t$

~~Pf~~ Also, $fg : i \longrightarrow i$ is unique $\Rightarrow fg = 1_i$.

Ex 1.6.ii) Let t_1, t_2 are the terminal object. $\Rightarrow \exists! f : t_1 \longrightarrow t_2$ and $\exists! g : t_2 \longrightarrow t_1$

Thus $fg : t_2 \longrightarrow t_2$ unique $\Rightarrow fg = 1_{t_2}$

$gf \quad t_1 \longrightarrow t_1$ " $\Rightarrow gf = 1_{t_1}$

Ex 1.6.iii) Let $f : c \longrightarrow c'$ s.t.

$Ff : c \longrightarrow c'$ is mono in $\mathbb{D}$.

Let $g_1, g_2 \in C(b, c)$ s.t.

$fg_1 = fg_2$

$\Rightarrow F(fg_1) = F(fg_2) \Rightarrow Ff\,Fg_1 = Ff\,Fg_2$

$\Rightarrow Fg_1 = Fg_2$ by Ff is mono

$\Rightarrow g_1 = g_2$ by faithfulness

$\Rightarrow f$ is mono in C.

Thus, if C is concrete category, then faithful $U : C \longrightarrow Set$ exists. thus if $f \in Mor C$ s.t. Uf = injection, then f is mono.

By duality, faithful functor reflects epi.

Ex 1.6. iv) $\mathcal{C}$: category. $\underline{f : c \to c'}$
not epi or mono.

$\mathbf{2}$: $\overset{\cdot}{0} \longrightarrow \overset{\cdot}{1}$: iso.

$F : \mathbf{2} \longrightarrow \mathcal{C}$. $\quad F(0 \to 1) = f$.

$$\begin{array}{ccc} 0 & & c \\ \downarrow & \longmapsto & \downarrow f \\ 1 & & c' \end{array}$$

$\Rightarrow$ neither epi or mono.

Ex 1.6. v) DIV : Category of divisible group

$(G, +)$ is divisible if $\forall n \in \mathbb{N}$, $g \in G$,

$\exists y \in G$ s.t. $ny = g$.

Let $\pi : \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$. $f, g : G \to \mathbb{Q}$ s.t.

$\pi \circ f = \pi \circ g$. Let $x \in G$.

$\Rightarrow \pi \circ f(x) = \pi \circ g(x) \Rightarrow f(x) - g(x) = n \in \mathbb{Z}$.

If $n \neq 0$,
By divisibility, $\exists y \in G$ s.t. $2ny = x$.

$\Rightarrow f(2ny) = f(x) \Rightarrow \frac{1}{2n} f(x) = f(y)$

$\therefore f(y) - g(y) = \frac{1}{2n}(f(x) - g(x)) = \frac{1}{2}$,

contradiction.

$\Rightarrow n = 0$. $\therefore f(x) = g(x)$. $\forall x$.

So π is mono. but not injective.

Also, $\pi : \mathbb{Z} \longrightarrow \mathbb{Q}$ in Ring is epi
but not surjective.

tsu-tmr

Suppose $f, g : \mathbb{Q} \longrightarrow R$ s.t. $f \circ \pi = g \circ \pi$.

Then, $f(\frac{a}{b}) = f(\frac{1}{b}) f(a) = f(\frac{1}{b}) \cdot g(a)$

$$= g(b \tfrac{a}{b}) f(\tfrac{1}{b}) = g(\tfrac{a}{b}) g(b) f(\tfrac{1}{b})$$

$$= g(\tfrac{a}{b}) f(b) f(\tfrac{1}{b})$$

$$= g(\tfrac{a}{b})$$

(Assume comm ring; but it holds for any associative unital ring)

$\therefore f = g$.

Ex.

1.6.vi) Let (c, γ) be a terminal

Then for any (d, ϕ) coalgebra.

$\exists ! f : (d, \phi) \longrightarrow (c, \gamma)$ s.t.

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & c \arrow[d, "\gamma"] \\ Td \arrow[r, "Tf"'] & Tc \end{tikzcd}$$

Thus $C \xrightarrow{\gamma} TC \Rightarrow TC \xrightarrow{T\gamma} T^2C$

has a coalgebra map (unique)

$$f : (TC, T\gamma) \longrightarrow (C, \gamma)$$

s.t.

$$\begin{tikzcd}
TC \arrow[r, "f"] \arrow[d, "T\gamma"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "\gamma"] \\
T^2C \arrow[r, "Tf"'] & TC
\end{tikzcd}$$

Thus, $Tf \circ T\gamma = \gamma \circ f$

And

$$\begin{tikzcd}
C \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\circlearrowright"] & TC \arrow[d, "T(\gamma)"] \\
TC \arrow[r, "T(\gamma)"'] & TC^2
\end{tikzcd}$$

by functoriality

$\Rightarrow T\gamma : (C, \gamma) \longmapsto (TC, T\gamma)$ is a morphism

$\Rightarrow f \circ \gamma : C \longrightarrow C$ s.t.

$$\begin{tikzcd}
C \arrow[r, "f\circ\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "\gamma"] \\
TC \arrow[r, "T(f\circ\gamma)"'] & TC
\end{tikzcd}$$

Since (C, γ) is terminal, $f\circ\gamma = 1_C$

Hence, $\gamma \circ f = Tf \circ \underline{T\gamma = T(f \circ \gamma)}$ (first square)

$$= T(1_C)$$

$$= 1_{TC}$$

$\Rightarrow$ γ is iso.