(1.5, i). Let $\alpha : F \Rightarrow G$ be a natural t.f.

$\Rightarrow \quad \forall\, f : c \longrightarrow c'$,

$$
\begin{tikzcd}
Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowright"] & Gc \arrow[d, "Gf"] \\
Fc' \arrow[r, "\alpha_{c'}"'] & Gc'
\end{tikzcd}
$$

Now define H as

$$H : C \times \underline{2} \longrightarrow D.$$

$$
\begin{array}{lcl}
(c,0) & \longmapsto & Fc \\
f\cdot 1_0 \downarrow & \longmapsto & \downarrow Ff \\
(c',0) & \longmapsto & Fc'
\end{array}
\qquad
\begin{array}{lcl}
(c,1) & \longmapsto & Gc \\
f\cdot 1_1 \downarrow & \longmapsto & \downarrow Gf \\
(c',1) & \longmapsto & Gc'
\end{array}
\qquad \text{and}
$$

$$
\begin{array}{lcl}
(c,0) & \longmapsto & Fc \\
f\cdot(0\to 1) \downarrow & \longmapsto & \downarrow \; Gf\circ\alpha_c = \alpha_{c'}\circ Ff \\
(c',1) & \longmapsto & Gc'
\end{array}
$$

1) It satisfies functoriality.

All we needs are 4 cases. ① $\xrightarrow{f\cdot 1_0}\;\xrightarrow{g\,(0\to 1)}$,

② $\xrightarrow{f(0\to 1)}\;\xrightarrow{g\,1_1}$,

for $f : c \to c'$, $g : c' \longrightarrow c''$

①
$$
\begin{tikzcd}
(c,0) \arrow[r, mapsto, "f\cdot 1_0"] \arrow[d, mapsto] & (c',0) \arrow[r, mapsto, "g(0\to1)"] \arrow[d, mapsto] & (c'',1) \arrow[d, mapsto] \\
Fc \arrow[r, mapsto, "Ff"] & Fc' \arrow[r, mapsto, "Gg\circ\alpha_{c'}"] & Gc''
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(c,0) \arrow[r, mapsto, "gf(0\to1)"] \arrow[d, mapsto] & (c'',1) \arrow[d, mapsto] \\
Fc \arrow[r, mapsto, "G(gf)\circ\alpha_c"] & Gc''
\end{tikzcd}
$$

Then,

$$\begin{tikzcd}
Fc \arrow[r, "Ff"] \arrow[d, "\alpha_c"] \arrow[drr, bend left, "a"] \arrow[drr, bend right=60, "b"'] & Fc' \arrow[r] \arrow[d, "\alpha_{c'}"] & Fc'' \arrow[d, "\alpha_{c''}"] \\
Gc \arrow[r, "Gf"'] & Gc' \arrow[r, "Gg"'] & Gc''
\end{tikzcd}$$

a is left diagram, b: right diagram

$\Longrightarrow$ Commutes by natural transformation.

Other case is similar.

And $H((c,i) \xrightarrow{1_{(c,i)}} (c,i)) = \begin{cases} Fc \xrightarrow{F1_c} Fc & i=0 \\ Gc \xrightarrow{G1_c} Gc & i=1 \end{cases}$

$$= 1_{Fc} \text{ or } 1_{Gc}$$

$$= 1_{H(c,i)}.$$

So H is a functor

2) H satisfies

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, "F"'] & C\times 2 \arrow[d, "H"] & C \arrow[l] \arrow[dl, "G"] \\
& D &
\end{tikzcd}$$

$$\begin{tikzcd}
c \arrow[r, mapsto] \arrow[d, "f"'] & Fc \arrow[d, "Ff"] \\
c' \arrow[r, mapsto] & Fc'
\end{tikzcd}
=
\begin{tikzcd}
c \arrow[r, mapsto] \arrow[d, "f"'] & (c,0) \arrow[r, mapsto] \arrow[d, "f\cdot 1_0"] & Fc \arrow[d, "Ff"] \\
c' \arrow[r, mapsto] & (c',0) \arrow[r, mapsto] & Fc'
\end{tikzcd}$$

and the other way is similar.

1.5. ii.)

Γ = Obj : finite set

Mor : $S \longrightarrow T$

$\Rightarrow \ \theta : S \to P(T)$ s.t.

① $\theta(\alpha) \cap \theta(\beta) = \phi$ when $\alpha \neq \beta$.

$$S \xrightarrow{\theta} T \xrightarrow{\phi} U = S \xrightarrow{\psi} U$$

s.t. $\alpha \overset{\psi}{\longmapsto} \bigcup_{\beta \in \theta(\alpha)} \phi(\beta)$

$$F : \Gamma \longrightarrow (\mathrm{Fin}_*)^{op} \qquad G : \mathrm{Fin}_*^{op} \longrightarrow \Gamma$$

$$\begin{array}{ccc} S & \longrightarrow & (S \cup \{s\}, s) \\ \theta \downarrow & & \uparrow \theta^{-1} \\ T & \longrightarrow & (T \cup \{t\}, t) \end{array} \qquad \begin{array}{ccccc} (S, s) & \longmapsto & S \setminus s \\ f \downarrow & \longmapsto & \uparrow f^{-1} \\ (T, t) & \longmapsto & T \setminus t \end{array}$$

where

$$\theta^{-1}(\beta) = \begin{cases} \alpha & \text{if } \beta \in \theta(\alpha) \\ s & \text{o.w.} \end{cases} \qquad f^{-1} : \beta \longmapsto f^{-1}(\beta)$$

First of all, $\theta^{-1}(\beta)$ is well-defined since no element in T is contained in two preimage by ①.

Also, f^{-1} is well-defined since it is a map from $T \setminus t$ to $P(S \setminus s)$ with distinct preimage.

To see $\Gamma \cong Fin_*^{op}$.

Notes that $FG: Fin_*^{op} \longrightarrow \Gamma \longrightarrow Fin_*^{op}$

$$\begin{tikzcd}
(S, s) \arrow[r, mapsto] \arrow[d, "f"'] & S\setminus s \arrow[r, mapsto] & (S\setminus s \cup \{S\setminus s\}, S\setminus s) \arrow[d, "h"] \\
(T, t) \arrow[r, mapsto] & T\setminus t \arrow[u, "f^{-1}"'] \arrow[r, mapsto] & (T\setminus t \cup \{T\setminus t\}, T\setminus t)
\end{tikzcd}$$

To figure out h, notes that

If $x \in S\setminus s$, let $y = f(x)$.

$\Rightarrow \quad f^{-1}(y) = \{x \in S\setminus s : f(x) = y\}$

Thus, for any $x \in f^{-1}(y)$

$$h(x) = y.$$

Hence, $f|_{S\setminus s} = h|_{S\setminus s}$ and $h(S\setminus s) = T\setminus t$

Thus, define $\alpha_{(S,s)}$ as usual incl with $S\setminus s \mapsto s$, $T\setminus t \mapsto t$

$$\begin{tikzcd}
(S\setminus s \cup \{S\setminus s\}, S\setminus s) \arrow[r, "\alpha_{(S,s)}"] \arrow[d, "h"'] & (S, s) \arrow[d, "f"] \\
(T\setminus t \cup \{T\setminus t\}, T\setminus t) \arrow[r] & (T, t)
\end{tikzcd}$$

Then, if $x \in S \setminus s$ with $f(x) = y$.

$$\alpha_{(T,t)} \circ h(x) = \alpha_{(T,t)}(y) = y.$$

$$f \circ \alpha_{(S,s)}(x) = f(x) = y.$$

and for $x = S \setminus s$,

$$\alpha_{(T,t)} \circ h(S \setminus s) = \alpha_{(T,t)}(T \setminus t) = t.$$

$$f \circ \alpha_{(S,s)}(S \setminus s) = f(s) = t.$$

$\therefore$ diagram commutes.

Likewise

$$\begin{array}{ccccc} GF: \Gamma & \longrightarrow & Fin_*^{op} & \longrightarrow & \Gamma \\ S & \longmapsto & (S \cup \{s\}, s\} & \longmapsto & S, \\ \theta \downarrow & & \uparrow \theta^{-1} & & \downarrow h \\ T & \longmapsto & (T \cup \{T\}, T) & \longmapsto & T \end{array}$$

s.t. for $x \in S$, $\theta(x) \subseteq T$. Hence, $\forall y \in \theta(x)$

$\theta^{-1}(y) = x. \Rightarrow h(x) = $ preimage of x under θ^{-1}
$= \theta(x)$

$\Rightarrow GF = 1_\Gamma.$

Hence $\Gamma \cong Fin_*^{op}.$

M: monoid

In particular, $Fin_* \xrightarrow{M} Set$ is a functor in 1.3.2(xi) and G induce a func $\hat{G}: \Gamma^{op} \to Fin_*$.

$\Rightarrow \Gamma^{op} \xrightarrow{\hat{G}} Fin_* \xrightarrow{M} Set$ are presheaves on Γ.

1.5. iii) Any morphism $f: a \to b$ and (fixed morphisms $a \cong a'$, $b \cong b'$)

$\Rightarrow$ determine $f': a' \to b'$ so that

$$\begin{tikzcd} a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, "f'"] \\ b \arrow[r, "\cong"'] & b' \end{tikzcd} \quad \begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\ b \arrow[r] & b' \end{tikzcd} \quad \begin{tikzcd} a \arrow[d, "f"'] & a' \arrow[l] \arrow[d, "f'"] \\ b & b' \arrow[l] \end{tikzcd} \quad \begin{tikzcd} a \arrow[r] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\ b & b' \arrow[l] \end{tikzcd}$$

pf) Define $f' = BfA$

$\Rightarrow$ $Bf = f'A^{-1}$, $B^{-1}f' = fA$, $B^{-1}f'A^{-1} = f$

1.5. iv) $F: C \to D$ full and faithful

(i) $f: c \to c'$ mor in C s.t. Ff is iso in D

$\Rightarrow$ f is iso.

(ii) $x, y \in C$ s.t. $Fx \cong Fy$ in D

$\Rightarrow$ $x \cong y$ in C

pf) Since F is full and faithful,

$C(x,y)$ and $D(Fx, Fy)$ are bijective.

So, $\exists!$ $Fg \in D(Fy, Fx)$ s.t. $Fg \cdot Ff = 1_{Fx}$

$Ff\ Fg = 1_{Fy}$.

then $F(gf) = 1_{Fx}$ since $C(x,x)$ and $D(Fx,Fx)$

$F(fg) = 1_{Fy}$. $\quad$ $C(y,y)$ " $D(Fy,Fy)$ are bijective, $gf = 1_x$, $fg = 1_y$.

Similarly, If $Fx \cong Fy$, $\exists\ Ff \in D(Fx, Fy)$ which is isomorphism. By (i), f is iso in C, thus $x \cong y$.

Lem 1.3.8 (Functor preserve iso) is converse of this statement.

Ex 1.5.v.

$$\begin{tikzcd} 2 \arrow[r] & D \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, "f"] & y \end{tikzcd} \qquad \begin{tikzcd} c_x \arrow[r, bend left, "f"] & c_y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

It is faithful since $2(x,y) \hookrightarrow D(c_x, c_y)$ but not reflect since $x \to y$ in 2 is not iso, but in D it is.

Ex 1.5. vi)

(i) Composite of full, faithful, ess surj again same. $F: C \to D$, $G: D \to E$

pf) $C(x,y) \overset{\twoheadrightarrow}{\hookrightarrow} D(c_x, c_y) \overset{\twoheadrightarrow}{\hookrightarrow} E(Dc_x, Dc_y)$

$\Rightarrow$ $C(x,y) \overset{\twoheadrightarrow}{\hookrightarrow} E(Dc_x, Dc_y)$

And for essential surj, $\forall e \in E$, $\exists d \in D$ st. $e \cong Gd$.
and for $d \in D$ $\exists c \in C$ st. $d \cong Fc$

$\Rightarrow$ $e \cong Gd \cong GFc$ implies $\forall e \in E\ \exists c$ st. $GFc \cong e$.

(ii). It is just from 1.5.9. Thm.
if we assume axiom of choice

Also, $C \cong C$, $C \cong D \Rightarrow D \cong C$ trivially.

Ex 1.5. vii) Connected groupoid $\Rightarrow$ all objects are isomorphic
if $G = G(g, g)$

$$\begin{array}{ccc} G & \longrightarrow & BG \\ h & \longmapsto & \bullet \\ f \downarrow \cong & & \downarrow g \xrightarrow{\alpha_h} h \xrightarrow{f} h' \xrightarrow{(\alpha_{h'})^{-1}} g \\ h' & \longmapsto & \bullet \end{array} \qquad \begin{array}{ccc} BG & \longrightarrow & G \\ \bullet & \longmapsto & g \\ \phi \downarrow & \longmapsto & \downarrow \phi \\ \bullet & \longmapsto & g \end{array}$$

$\phi \in G(g, g)$

Then,

$$BG \longrightarrow G \longrightarrow BG = 1_{BG}$$

$$\begin{array}{ccccc} \bullet & \longmapsto & g & \longmapsto & \bullet \\ \phi \downarrow & \longmapsto & \downarrow \phi & & \downarrow \phi \\ \bullet & \longmapsto & g & \longmapsto & \bullet \end{array}$$

(or take $\alpha : 1_{BG} \Rightarrow BG \to G \to BG$ as identity)

$$\begin{array}{ccccc} G & \longrightarrow & BG & \longrightarrow & G \\ h & \longmapsto & \bullet & \longmapsto & g \\ \psi \downarrow & \longmapsto & g \to h \xrightarrow{\psi} h' \to g & & \downarrow g \to h \xrightarrow{\psi} h' \to g \\ h' & \longmapsto & \bullet & \longmapsto & g \end{array}$$

Thus, we need to find iso. satisfyin.

$$\begin{tikzcd} g \arrow[r, "\alpha_h"] \arrow[d, "g \to h \xrightarrow{\psi} h' \to g"'] \arrow[dr, phantom, "\circlearrowright"] & h \arrow[d, "\psi"] \\ g \arrow[r] & h' \end{tikzcd}$$

And take

$\alpha_h = g \to h$

$\alpha_{h'} = (h' \to g)^{-1}$

Since such α_h, $\alpha_{h'}$ is fixed by construction of $G \to BG$, It is welll-def and satisfy the Commuting diagram

Ex 1.5 viii) Later

Ex 1.5 ix) Any category equiv to locally small category is locally small.

pf) Let $C \underset{G}{\overset{F}{\rightleftarrows}} D$. equiv of category

By Thm 1.5.9, F, G : full, faithful.

$\Rightarrow$ If assume D locally small,

then $C(X,Y)$ is a set since it is bijectn to a set $D(X,Y)$. $\Rightarrow$ C is locally small

1.5. viii).

Affine : Obj = affine planes, $\mathbb{A}^2 k$
where k is a field.

Mor = affine linear map.

$L - c$: L : linear isomorphism

c : constant function

Proj$'$: Obj = projective planes $(\mathbb{P}^2 k, \ell)$
w/ ℓ : line of infinity.

Mor = projective linear isomorphism

F : Proj$' \longrightarrow$ Affine.

$(\mathbb{P}^2 k, \ell) \longmapsto \mathbb{A}^2 k$ by deleting ℓ in $\mathbb{P}^2 k$

Notes that $\mathbb{P}^2_k = k^2 \sqcup (k^1 \sqcup pt)$

where $k^2 = \{[x:y:z] : z \neq 0\} \longmapsto \{(\frac{x}{z}, \frac{y}{z}) \in \mathbb{A}^2 k\}$

$k = \{[x:y:0] : y \neq 0\}$

$pt = \{[x:0:0]\}$.

with $\ell = k \sqcup pt$.

Thus, G : Affine $\longrightarrow$ Proj$'$.

is embedding $\mathbb{A}^2 k$ into $(\mathbb{P}^2 k, \ell)$

by identifying $\mathbb{A}^2 k$ as k^2 part.

$(x, y) \longmapsto [x : y : 1]$

Claim 1: F, G maps pt to pt, line to line. (except ℓ).

Let L be a line in $\mathbb{P}^2_k$ generated by

$\vec{a} = [a_0 : a_1 : a_2]$, $\vec{b} = [b_0 : b_1 : b_2]$

Then, $L = \{ u\vec{a} + v\vec{b} : u, v \in k, (u,v) \neq (0,0) \}$

Case 1: If $\vec{a}, \vec{b} \in k^2$,

F maps $u\vec{a} + v\vec{b}$ into $\left(\dfrac{u\frac{a_0}{a_2} + v\frac{b_0}{b_2}}{u+v}, \dfrac{u\frac{a_1}{a_2} + v\frac{b_1}{b_2}}{u+v} \right)$

Thus, $F(L) = \{ t\left(\frac{a_0}{a_2}, \frac{a_1}{a_2}\right) + (1-t)\left(\frac{b_0}{b_2}, \frac{b_1}{b_2}\right) \}$

Since $\frac{u}{u+v}, \frac{v}{u+v}$ can be mapped into t and $1-t$. for any $t \in k$,

$= \{ \left(\frac{b_0}{b_2}, \frac{b_1}{b_2}\right) + t\left(\frac{a_0}{a_2} - \frac{b_0}{b_2}, \frac{a_1}{a_2} - \frac{b_1}{b_2}\right) \}$

Hence $F(L)$ is a line in $\mathbb{A}^2 k$.

Case 2: If one of $\vec{a}, \vec{b}$ is in ℓ.

Then, by adding suitable $u\vec{a}$ to $\vec{b}$,

We can change this as a case 1.

Case 3: Both $\vec{a}, \vec{b}$ are in ℓ

Then, $L = \ell$ and F drops ℓ.

Now, let L be a line in $\mathbb{A}^2_k$.

$\Rightarrow$ $L = \{tA + (1-t)B : t \in k\}$

for some $A = (a_0, a_1)$, $B = (b_0, b_1)$

$\Rightarrow$ $G(L) = \{t[a_0 : a_1 : 1] + (1-t)[b_0 : b_1 : 1]$

$t \in k\}$.

To see that $G(L)$ is contained in a line in $\mathbb{P}^2_k$, let L' be a line gen by $[a_0 : a_1 : 1]$, $[b_0 : b_1 : 1]$. Then $L' \supseteq G(L)$

So $G(L)$ matches with only one line L'

Thus, $FG(L) \subseteq F(L') = L$.

and $GF(L) = G\left(t\left(\frac{a_0}{a_2}, \frac{a_1}{a_2}\right) + (1-t)\left(\frac{b_0}{b_2}, \frac{b_1}{b_2}\right)\right)$

$\subseteq L'$ where L' is gen by

$\left[\frac{a_0}{a_2} : \frac{a_1}{a_2} : 1\right]$, $\left[\frac{b_0}{b_2} : \frac{b_1}{b_2} : 1\right]$

$= [a_0 : a_1 : a_2]$, $[b_0 : b_1 : b_2]$

$\Rightarrow$ $= L$.

Hence $FG(\mathbb{A}^2_k) = \mathbb{A}^2_k$ since they

$GF(\mathbb{P}^2_k) = \mathbb{P}^2_k$ preserves pt and lines.

Thus, equivalence is induced by α as identity for each object.

1.5. xi) See Hame, P. "Examples of some properties of functors"

	Full	Faithful	Essentially Surj
Ab → Gp	✓	✓	×
Ring → Ab	×	✓	×
Ring $\xrightarrow{(-)^\times}$ Gp	×	×	×
Ring → Rng	×	✓	×
Field → Ring	✓	✓	×
Mod_R → Ab	dep on R	✓	✓

(b) Ring → Ab: ① Not full since. No ring homo $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}$

Since $f(\underbrace{1+\cdots+1}_{n}) = 0$ but $\underbrace{f(1)+\cdots+f(1)}_{n} = n$

Contradiction.

(However, Rng on Ab ∃ trivial homo.)

② Not essentially Surj!

Let $\mathbb{Z}(p^\infty) := \{ e^{2\pi i m/p^n} : m, n \in \mathbb{N} \}$

Prüfer p-group.

If $U(R) \cong \mathbb{Z}(p^\infty)$ then $1 \mapsto a \in \mathbb{Z}(p^\infty)$

$\Rightarrow |1| = n$ in $R \Rightarrow \forall r \in R, \ nr = 0$

Since $r(1+\cdots+1) = r \cdot 0 = 0$.

Thus every element in $U(R)$ has order at most n. But $\mathbb{Z}(p^\infty)$ has element with greater order ☑

(c) $(-)^{\times}$: Ring $\longrightarrow$ Grp.

① Not full

Since $\mathbb{Z}$ is initial in Ring, $\forall R \in$ Ring

$\exists !\ f: \mathbb{Z} \longrightarrow R \Rightarrow |\mathrm{Ring}(\mathbb{Z}, R)| = 1$

But $\mathbb{Z}^{\times} \cong \mathbb{Z}/2$. Take $R = \mathbb{F}_p$ finite p-field.

$\Rightarrow R^{\times} \cong \mathbb{Z}/(p-1)$

And $|\mathrm{Grp}(\mathbb{Z}^{\times}, R^{\times})| = 2$ since 0

and $\mathbb{Z}/2 \longrightarrow \mathbb{Z}(p-1)$ by $1 \longmapsto \frac{p-1}{2}$

are the homomorphisms.

② Not faithful.

Claim 1: $\phi: k[t] \longrightarrow k[t]$ automorphis fixing k $\Rightarrow \phi(t) = at + b$..

pf) $\phi(t)$ cannot be degree more than one otherwise it is not automorphism.

Also, $\phi(t)$ cannot have degree 0 by same reas.

And all $\phi(t) = at + b$ is auto since

$\phi(\frac{1}{a}t - b) = t$.

$\boxed{//}$

Thus by Claim 1,

Ring $(k[t], k[t])$ is determined by $\{ at+b : a \in k^\times, b \in k \}$

And $k[t]^\times = k^\times$

But $\mathrm{Grp}(k^\times, k^\times)$ is determined by generators of $k^\times$ (since $k^\times$ is cyclic)

and any map in Ring $(k[t], k[t])$ induces map $k^\times, k^\times$ fixing all $k^\times$.

Thus it is not faithful.

③ Not ess. surj.

[Pearson, Schneider, 1970]

Not every cyclic gp is isomorphic to the gp of units of some ring.

$(\mathbb{Z}/5 \ncong R^\times \quad \forall R \in \mathrm{Ring})$

(d) Ring $\longrightarrow$ Rng.

① Not full : Zero homomorphism is not a homomorphism in Ring

② Faithful : Any unital ring homo is also ring homo and distinct unital ring homos are " ring homos.

③ Not ess. surj. : Ring without mult. identity is not iso to unital ring.

(e) Field $\hookrightarrow$ Ring

① fully faithful since every field homo is ring homo.

② Not ess. surj : Not every ring is a field.

(f) $U : \mathrm{Mod}_R \longrightarrow \mathrm{Ab}$.

① faithful : Any distinct R-homo is also distinct gp homo.

(Since distinction determined by value, not property)

② Ess surj : $\forall A \in \mathrm{Ab}$, make trivial R-module structure s.t. $\forall a \in A$, $\forall r \in R$ $ra = a$.

③ Not necessarilly full.

$\mathrm{End}(R) \cong R$ in Mod_R

but $\mathrm{End}(R) \not\cong R$ in Ab. in general.

(If $R = \mathbb{Z}$, then it is full.)