2.3 Universal properties and Universal element

Prop 2.3.1. Let $x, y \in \mathrm{ob}\,C$.

$x \cong y$ in $C \iff C(-,x) \cong C(-,y)$ in Set^{C}

$C(x,-) \cong C(y,-)$

pf) ($\Rightarrow$) $y: C^{op} \longrightarrow \mathrm{Set}^{C}$ is a functor

So preserve isomorphism.

($\Leftarrow$) Let $\alpha: C(-,x) \cong C(-,y)$ iso.
Since y is fully faithful, $\exists!\, f: x \to y$ iso.

Rmk; in ($\Leftarrow$), Maybe, there are more iso;
but f is unique among any iso $x \cong y$.
So f is "the" iso.
"the": object $\iff$ Object in question is well-def up to canonical iso.

Corollary 2.3.2 Full category spanned by its terminal object is either empty or contractible groupoid. In particular, any two terminal objects in C are uniquely isomorphic.
Contractible groupoid $G \Rightarrow G \cong \mathbb{1}$.

pf). By Yoneda lemma,

$$\mathrm{Hom}(C(-,t),\ C(-,t')) \overset{\text{bijection}}{\cong} C(t,t')$$

Since t' is terminal, $C(t,t') = \{*\}$ Singleton.

Recall

Def) 1.3. t is terminal

$\Leftrightarrow$ $C(-,t) : C^{op} \longrightarrow Set$ is naturally iso to $* : C^{op} \longrightarrow Set$

$\forall c,\ c \longmapsto \{*\}$.

Thus, $C(-,t) \cong * \cong C(-,t')$.

$\therefore$ $t \cong t'$ by Prop 2.3.1 ▨

Def) (Universal property)

An universal property of $c \in C$ is expressed by representable functor F with universal element $x \in F_c$, that defines a natural iso $C(c,-) \cong F$ via

Yoneda Lemma.

Ex 2.3.4 $U : \mathrm{Ring} \to \mathrm{Set} \cong \mathrm{Ring}(\mathbb{Z}[x], -)$

Since $\mathrm{Ring}(\mathbb{Z}[x], R) \cong UR$

defined by $x \in \mathbb{Z}[x]$.

By Yoneda lemma, $ev : \mathrm{Ring}(\mathbb{Z}[x], R) \to UR$
$$\phi \longmapsto \phi(x)$$

is bijection.

Ex 2.3.6) (There is no ex 2.3.5)

$E : BG \to \mathrm{Set}$ is representable iff $G \cong E$ as left G-set
$\quad \cdot \longmapsto E$ (bijection)

pf) If E is representable, $BG(\cdot, -) \cong E$

Thus, $BG(\cdot, \cdot) \cong E(\cdot) = E$. Since $BG(\cdot, \cdot) \cong G$, $G \cong E$. Conversely, if $E \cong G$, then $BG(\cdot, -) \cong E$

$\Rightarrow$ Action of G on E is (as a left multiplication)

① Free (since every stabilizer is trivial)

② transitive (orbit is entire set.)

③ E is nonempty

pf) If $E \cong G$, then $\forall g \in E$, $\exists g^{-1} \in G$ s.t. $g^{-1}g = e$. So orbit is entire set. Also, any $g \neq e$ permute G, so stabilizer is 0. ③ is clear.

Conversely, any nonempty free and transitive left G-set is representable (omit pf)

By Yoneda Lemma, universal element for universal property of $\bullet \in BG$ is $e \in G$.

$\Rightarrow$ E is just an underlying set G forgetting gp structure

$\therefore$ G-torsor := A representable G-set (like E)

ex) $\mathbb{A}^n$: affine space = forgetting $(0, \cdots, 0)$ as identity in $\mathbb{R}^n$.

$\mathbb{R}^n$ act on $\mathbb{A}^n$ by thinking $\alpha \in \mathbb{R}^n$ as a vector sending pt to pt + α.

This action is

① Free : Every nonzero vector doesn't stabilize pt.

② Transitive : Any two pt in $\mathbb{A}^n$ form a vector. So they are in the same orbit.

Choice of identity 0 in $\mathbb{A}^n$ gives iso on $\mathbb{R}^n \cong \mathbb{A}^n$

So, $0 \in \mathbb{A}^n$ is universal element

Ex 2.3.17 U, W : K-vector space.

$\mathrm{Bilin}(V, W; -) : \mathrm{Vect}_K \longrightarrow \mathrm{Set}$

$U \longmapsto \{ f : V \times W \to U$ K-bilinear $\}$

From bilinearity, i.e, $f(v, -) : W \to U$
$f(-, w) : V \to U$

f is identified as a map $V \to \mathrm{Hom}(W, U)$
$W \to \mathrm{Hom}(V, U)$

We claim $\mathrm{Bilin}(V, W; -) \cong \mathrm{Vect}_K(V \otimes_K W, -)$

(Actually it is known as the universal property of the tensor product.)

By Yoneda Lemma, the isomorphism

$$\mathrm{Bilin}(V, W; U) \cong \mathrm{Vect}_K(V \otimes_K W, U)$$

is determined by universal element of $\mathrm{Bilin}(V, W; V \otimes_K W)$ i.e.

$\otimes : V \times W \longrightarrow V \otimes_K W$ canonical bilinear map.

$\Rightarrow V \otimes_K W$: Universal vector space equipped with a bilinear map from $V \times W$.

What is meaning of it?

Notes that $\mathrm{Vect}_k(V \otimes_k W, U) \cong \mathrm{Bilin}(V, W; U)$

Pick $f \in \mathrm{Bilin}(V, W; U)$ and $\bar{f} : V \otimes_k W \to U$ corresp to f. Then,

$$\begin{tikzcd}
\mathrm{Vect}_k(V \otimes_k W, V \otimes_k W) \arrow[r, "\cong"] \arrow[d, "\bar{f}_*"'] & \mathrm{Bilin}(V, W; V \otimes_k W) \arrow[d, "\bar{f}_*"] \\
\mathrm{Vect}_k(V \otimes_k W, U) \arrow[r, "\cong"'] & \mathrm{Bilin}(V; W; U)
\end{tikzcd}$$

Then

$$\begin{tikzcd}
1_{V \otimes_k W} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & \otimes \arrow[d, mapsto] \\
\bar{f} \arrow[r, mapsto] & f \quad \bar{f} \circ \otimes
\end{tikzcd}$$

i.e,

$$\begin{tikzcd}
V \times W \arrow[r, "\otimes"] \arrow[dr, "f"'] & V \otimes_k W \arrow[d, "\bar{f}"] \\
& U
\end{tikzcd}$$

and $\bar{f}$ is unique by iso.

$\Rightarrow$ Actually $\otimes : V \times W \to V \otimes_k W$ is initial element in some other category

Also, (if we know existence of $V \otimes_k W$) then we can construct it.

pf)
$$\begin{tikzcd}
V \times W \arrow[r, "\otimes"] & V \otimes W \arrow[r, shift left, "\text{quotient}"] \arrow[r, shift right, "0"'] & V \otimes W / \langle v \otimes w \rangle = 0
\end{tikzcd}$$

By universal property, $0 = \text{quotient}$.

Since quotient is surj $V \otimes W = \langle v \otimes w : v \in V, w \in W \rangle$

Prop 2.3.9. $V \otimes_k W \cong W \otimes_k V$

pf) $\exists$ natural iso $\mathrm{Bilin}(V,W;-) \cong \mathrm{Bilin}(W,V;-)$

$$f : V\times W \to U \longmapsto f^{\#} : W \times V \to U.$$

$$f^{\#}(w,v) := f(v,w)$$

Then,

$$\mathrm{Vect}_k(V\otimes_k W, -) \cong \mathrm{Bilin}(V,W;-) \cong \mathrm{Bilin}(W,V;-) \cong \mathrm{Vect}_k(W\otimes V, -)$$

$\Rightarrow$ $V\otimes_k W \cong W \otimes_k V$ by 2.3.1

Also, this give explicit iso $V\otimes W \cong W \otimes V$ since by Yoneda lemma, it is image of 1_C in C ($C = \mathrm{Vect}_k(V\otimes_k W, V\otimes_k W)$)

Thus, Let $\phi : W\otimes_k V \xrightarrow{\cong} V\otimes_k W$: iso. Then

$$\phi^* : \mathrm{Vect}_k(V\otimes_k W, -) \xrightarrow{\cong} \mathrm{Vect}_k(W\otimes V, -)$$

by precomposition.

So

$$\begin{tikzcd} W\times V \arrow[r, "\otimes"] \arrow[d, "(w,v)\mapsto(v,w)"'] & W\otimes V \arrow[d, "\exists!\,\phi"] \\ V\times W \arrow[r, "\otimes"'] & V\otimes_k W \end{tikzcd}$$