

Feel free to ask  
me question!

Here's what you need to know to get the perfect grade.

(1) 5.6 Optimization: Be familiar with these two problems.

(a) Example: Suzie can sell 20 bracelets each day when the price is \$10 for a bracelet. If she raises the price by \$1, then she sells 2 fewer bracelets each day. If it costs \$8 to make each bracelet, find the selling price that will maximize Suzie's profit.

$$\text{maximize } R(x) = (\text{net price}) \cdot \text{quantity} = \text{price} - \text{cost} - \text{total cost.}$$

$$\begin{aligned} R(p) &= (p - 8)Q(p) \\ &= (p - 8)(-2p + 40) \\ &= -2p^2 + 16p + 40p - 320 \end{aligned}$$

$$\text{dom } R(p) = [0, \infty)$$

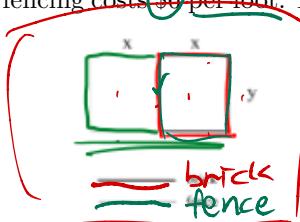
Critical values:

$$R'(x) = 0 \text{ or DNE. Selling price} = 14\text{\$}$$

$$-4p + 56 = 0 \Rightarrow 4p = 56 \Rightarrow p = 14$$

$R''(p) = -4 < 0$

(b) Example: Ben needs to enclose two rectangular regions that share one side, and each has an area of 1400 ft<sup>2</sup>. One of the regions needs to be enclosed on all four sides by a brick wall, and the other region just needs fencing on its remaining three sides (see diagram below). The brick wall costs \$18 per foot, and the fencing costs \$6 per foot. Find the dimensions of each region that would be the most economical for Ben.



$$x = \frac{35}{\sqrt{2}}$$

$$y = \frac{1400}{2 \cdot \frac{35}{\sqrt{2}}} = \dots$$

$$C(x) = (2x+2y) \cdot 18 + (2x+y) \cdot 6$$

Minimize  $C(x)$ .

$$\text{s.t. } 2xy = 1400 \text{ ft}^2 \quad y = \frac{1400}{x}$$

$$C(x) = (2x + 2 \cdot \frac{1400}{x}) \cdot 18 + (2x + \frac{1400}{x}) \cdot 6$$

$$= 36x + \frac{1400 \cdot 36}{x} + 12x + \frac{6 \cdot 1400}{x}$$

$$= 48x + \frac{1400}{x}(36+6)$$

$$= 48x + \frac{42 \cdot 1400}{x}$$

$$\text{Minimize } C(x) = 48x + \frac{42 \cdot 1400}{x}$$

$$\text{s.t. } x \geq 0 \quad (0, \infty)$$

$$\Rightarrow C'(x) = 48 - 42 \cdot 1400 \cdot x^{-2}$$

$$x^2 = \frac{1}{x^2}$$

Find  $x$  where  $C'(x) = 0$  or DNE  $\Rightarrow$

$$\sqrt{48} = \sqrt{3 \cdot 16} = \sqrt{3} \cdot \sqrt{16} = 4\sqrt{3}$$

$$42 = 3 \cdot 7 \cdot 2$$

$$1400 = 2 \cdot 7 \cdot 10 \cdot 10$$

$$42 \cdot 1400 = 3 \cdot 7 \cdot 10^2 \cdot 10^2$$

$$\sqrt{11} = \sqrt{3} \cdot 2 \cdot 7 \cdot 10$$

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$$48 - 42 \cdot 1400 \cdot x^{-2} = 0$$

$$48x^2 = 42 \cdot 1400$$

$$x^2 = \frac{42 \cdot 1400}{48}$$

$$x = \pm \frac{\sqrt{42 \cdot 1400}}{\sqrt{48}}$$

$$= \pm \frac{\sqrt{3} \cdot 140}{\sqrt{3} \cdot 4} = \pm \frac{140}{4} = \boxed{\pm 35}$$

35.

5.6. problem:

1. Find an equation.

(1) it is function of price or quantity ...

revenue  
= (price - cost) # of product  $\Rightarrow$  By second derivative test,  $C$  has local minimum on  $x = 35$

# product  
= function of price

(2) two variables  
 $x, y$  but  
there is some relationship between  $x, y$ .

$$C''(x) = -21 \cdot 1400 \cdot x^{-3} (-2)$$

$$= 42 \cdot 1400 x^{-3}$$

$\Rightarrow$  Since  $x = 35$  is  
the only critical value  
and there is no boundary  
points, it is absolute  
minimum.

$$x = 35 \quad y = \frac{1400}{25 \cdot 5} = 40$$

Ex 2

- (2) 6.1 Antiderivatives
- Antiderivative of a function  $f(x)$ : a function  $F(x)$  such that  $F'(x) = f(x)$ .
  - Indefinite integral:  $\int f(x) dx = F(x) + C$  such that  $C$  is constant,  $F$  is an antiderivative.

(c) Properties of indefinite integral: For constant  $C$  and  $k$ ,

- $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$ , where  $n \neq -1$
- $\int k dx = kx + C$
- $\int e^x dx = e^x + C$
- $\int \frac{1}{x} dx = \ln|x| + C$ , where  $x \neq 0$
- $\int kf(x) dx = k \int f(x) dx$
- $\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$

2. Find critical values

3. Derivative  
second derivative

4. Second derivative  
test +  
conclude local  
max or local min.

$$\boxed{(\ln|u|)' = \frac{1}{u}} \quad n=1 \quad \int u^n du = \frac{1}{n+1} u^{n+1}$$

(d) Example2, i): Find an indefinite integral  $\int \frac{4u+u^3-3u^{-7}}{5u^2} du$

$$\begin{aligned} &= \int \left( \frac{4u}{5u^2} + \frac{u^3}{5u^2} - \frac{3u^{-7}}{5u^2} \right) du = \int \left( \frac{4}{5}u^{-1} + \frac{1}{5}u^1 - \frac{3}{5}u^{-9} \right) du \\ (6) \quad &= \int \frac{4}{5}u^{-1} du + \int \frac{1}{5}u^1 du + \int \left( -\frac{3}{5}u^{-9} \right) du = \frac{4}{5} \int u^{-1} du + \frac{1}{5} \int u^1 du - \frac{3}{5} \int u^{-9} du \\ &= \frac{4}{5} \cdot (\ln|u| + C) + \frac{1}{5} \cdot (u^2 + C) - \frac{3}{5} \cdot \frac{1}{-8} \cdot u^{-8} \\ &= \frac{4}{5} \ln|u| + \frac{1}{5}u^2 + \frac{3}{40}u^{-8} + C \end{aligned}$$

(e) Example4: The marginal revenue of selling  $x$  watches each day is given by  $R'(x) = 30 - 0.0003x^2$  dollars per watch for  $0 \leq x \leq 540$ . If the revenue is \$1487.50 when 50 watches are sold, find the revenue function.

$$\begin{aligned} (1) \quad R(50) &= 1487.50 \quad R(x) = 30x - 0.0001x^3 \\ (1) \quad \int R'(x) dx &= \int (30 - 0.0003x^2) dx \quad (6) \quad \int (30 dx) + \int (-0.0003x^2 dx) \\ &= 30x + C - 0.0003 \int x^2 dx \quad \downarrow (2) \quad \downarrow (5) \\ &= 30x + C - 0.0003 \left( \frac{1}{3}x^3 + C \right) \quad \downarrow (1) \\ &= 30x + C - 0.0001x^3 + C \\ &\text{Q) } 50 \text{ into } Y(4) = 30x - 0.0001x^3 + C \\ (4) \quad 1487.5 &= Y(50) = 30 \cdot 50 - 0.0001 \cdot 50^3 + C = 1487.50 \quad \Rightarrow C = 0 \end{aligned}$$

(f) Also see Example 5 in the previous lecture notes.

(3) 6.2: Substitution

(a) Reversing the chain rule!

$$(b) \text{ Example 1: } \int e^{x^3-1} \cdot 3x^2 dx = \int e^u du = e^u + C$$

$u = x^3 - 1$

$du = 3x^2 dx$

$$= e^{x^3-1} + C$$

$$du = f'(x)dx$$

(c) General Indefinite Integral Formulas

$$(i) \int (f(x))^n \cdot f'(x) dx = \frac{(f(x))^{n+1}}{n+1} + C, n \neq -1$$

$$\text{Ex1} (ii) \int e^{f(x)} \cdot f'(x) dx = e^{f(x)} + C$$

$$(iii) \int \frac{1}{f(x)} \cdot f'(x) dx = \ln |f(x)| + C$$

$$(d) \text{ Example 3 c): } \int \frac{2e^{5/x^4}}{3x^5} dx$$

$$\begin{aligned} & \text{(ii) } c) \int \frac{2e^{5/x^4}}{3x^5} dx = \int e^u \cdot \frac{5}{3x^5} dx \quad \text{or } \int e^u du \quad \text{or } \int \frac{e^u}{f} du \\ & \quad \text{diff } u = \left(\frac{5}{x^4}\right) = 5 \cdot x^{-4} \\ & \quad \frac{du}{dx} = (-4) \cdot 5 \cdot x^{-5} = -20 \cdot x^{-5} \\ & \quad du = -20 \cdot x^{-5} dx \\ & \quad \left(-\frac{1}{20}\right) du = x^{-5} dx \\ & \quad \frac{1}{x^5} dx \\ & \quad \int e^u \cdot \frac{5}{3x^5} dx = \frac{5}{3} \int e^u du \\ & \quad \stackrel{(5)}{=} \frac{5}{3} \cdot \left(-\frac{1}{20}\right) \cdot \int e^u du = -\frac{1}{30} \cdot \int e^u du \\ & \quad \stackrel{(3)}{=} -\frac{1}{30} \cdot (e^u + C) \\ & \quad \stackrel{6.1}{=} -\frac{1}{30} e^u + C \\ & \quad \boxed{= -\frac{1}{30} \cdot e^{5/x^4} + C} \end{aligned}$$

$$\begin{aligned} & \text{(e) Example 3 d) } \int \frac{8t^3}{\sqrt[4]{2-5t^4}} dt = \int \frac{8t^3 \cdot (2-5t^4)^{-\frac{1}{4}}}{u} dt \\ & \quad \text{diff } u = 2-5t^4 \\ & \quad \frac{du}{dt} = -5 \cdot 4t^3 = -20t^3 \\ & \quad du = -20t^3 dt \\ & \quad \left(-\frac{1}{20}\right) du = t^3 dt \\ & \quad \int \frac{8t^3}{\sqrt[4]{2-5t^4}} dt = \int \frac{8t^3 \cdot (2-5t^4)^{-\frac{1}{4}}}{u} \cdot \frac{1}{-20} du \\ & \quad = 8 \int u^{-\frac{1}{4}} \cdot \left(-\frac{1}{20}\right) du \\ & \quad = \frac{8}{-20} \cdot \int u^{-\frac{1}{4}} du \end{aligned}$$

$$\begin{aligned}
 \int \frac{8t^3}{\sqrt[7]{2-5t^4}} dt &= 8 \int \frac{t^3}{\sqrt[7]{2-5t^4}} dt = 8 \int \frac{1}{u^{\frac{1}{7}}} du \\
 &\stackrel{(5)}{=} -\frac{8}{20} \cdot \int \frac{1}{u^{\frac{1}{7}}} du = -\frac{2}{5} \cdot \int u^{-\frac{1}{7}} du \\
 &\stackrel{(6,1)}{=} -\frac{2}{5} \left( \frac{1}{-\frac{1}{7}+1} \cdot u^{-\frac{1}{7}+1} + C \right) \\
 &\stackrel{(6,1)}{=} -\frac{2}{5} \left( \frac{1}{\frac{6}{7}} \cdot u^{\frac{6}{7}} + C \right) = -\frac{2}{5} \left( \frac{7}{6} u^{\frac{6}{7}} \right) + C \\
 &\stackrel{(6,1)}{=} -\frac{7}{15} \cdot u^{\frac{6}{7}} + C = -\frac{7}{15} \cdot (2-5t^4)^{\frac{6}{7}} + C
 \end{aligned}$$

(4) 6.3: Estimating Distance Traveled



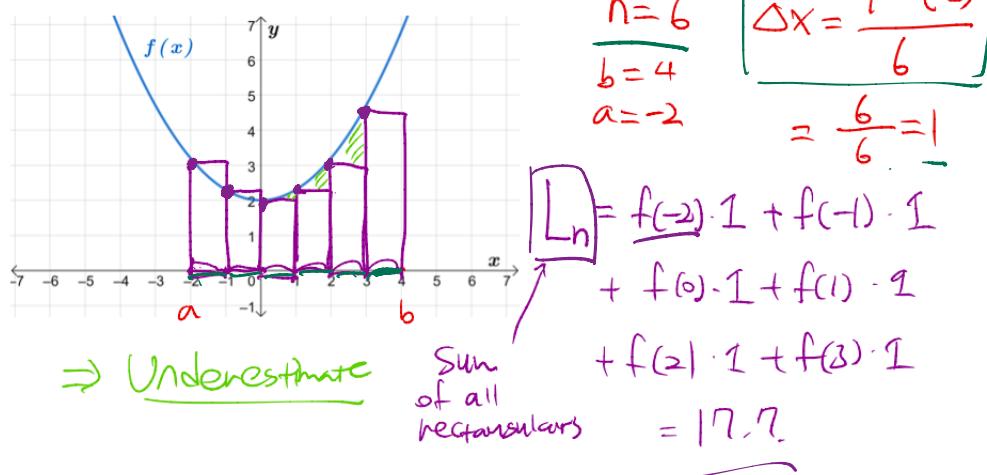
- (a) We will estimate the area under a curve from  $x = a$  to  $x = b$  by dividing the region into subintervals (rectangles) of equal width.

$$\text{width of each subinterval} = \Delta x = \frac{b - a}{n}$$

where  $n$  is the number of subintervals (rectangles).

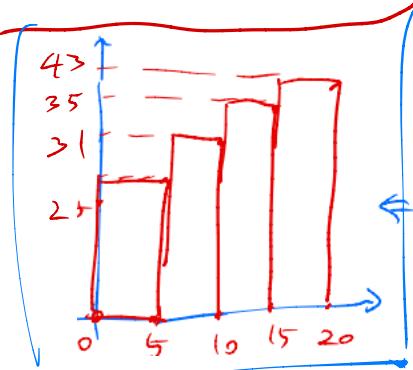
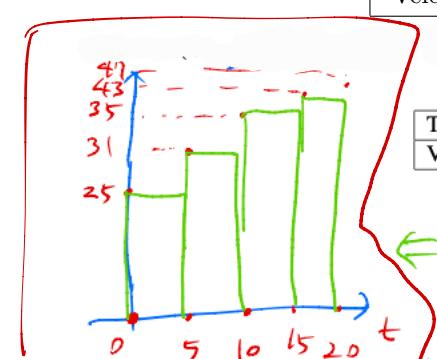
- (b) (Left Sum) Example1: For the function  $f(x) = 0.3x^2 + 2$ , estimate the area of the region that lies under the graph of  $f(x)$  between  $x = -2$  to  $x = 4$  using a left-hand sum with six subintervals of equal width.

Example 1: For the function  $f(x) = 0.3x^2 + 2$ , estimate the area of the region that lies under the graph of  $f(x)$  between  $x = -2$  to  $x = 4$  using a left-hand sum with six subintervals of equal width.

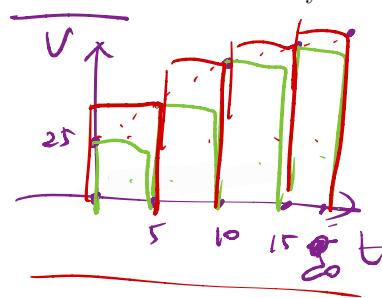


- (c) Example 6: The table below shows the velocity (ft/s) of an object every five seconds over a 20 second time interval. Estimate the total distance the object travels over the 20 second time interval by finding upper and lower estimates (i.e., right and left sums).

| Time (s)        | 0  | 5  | 10 | 15 | 20 |
|-----------------|----|----|----|----|----|
| Velocity (ft/s) | 25 | 31 | 35 | 43 | 47 |



| Time (s)        | 0  | 5  | 10 | 15 | 20 |
|-----------------|----|----|----|----|----|
| Velocity (ft/s) | 25 | 31 | 35 | 43 | 47 |



RIGHTMOST

left most up

(5) 6.4: The Definite Integral

(a) In general, we can use any  $x$ -coordinate,  $x_i^*$ , to find the height of the rectangle in the  $i^{\text{th}}$  subinterval.

Using summation notation, we can write the sum of the areas of the rectangles as

$$f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_n^*) \Delta x = \sum_{i=1}^n f(x_i^*) \Delta x = \text{approximate area between } f(x) \text{ and } x\text{-axis}$$

The sum  $\sum_i^n f(x_i^*) \Delta x$  is called a Riemann sum.

(b) Then, the **definite integral** of  $f(x)$  from  $a$  to  $b$  is  $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$

(c) Example 1: Use the graph of  $f(x)$  below to find the following. Note that the graph consists of three straight lines and a semicircle.

In  $x \in (-5, 2)$ ,  $f(x) < 0$

a)  $\int_{-5}^2 f(x) dx = - (b-a) \cdot \text{height of triangle} = -(2-(-5)) \cdot \frac{7}{2} = -14 \frac{1}{2} = -7$

$f(x) = \begin{cases} -\frac{2}{5}x - 2 & -5 < x < 0 \\ 0 & 0 \leq x < 2 \end{cases}$

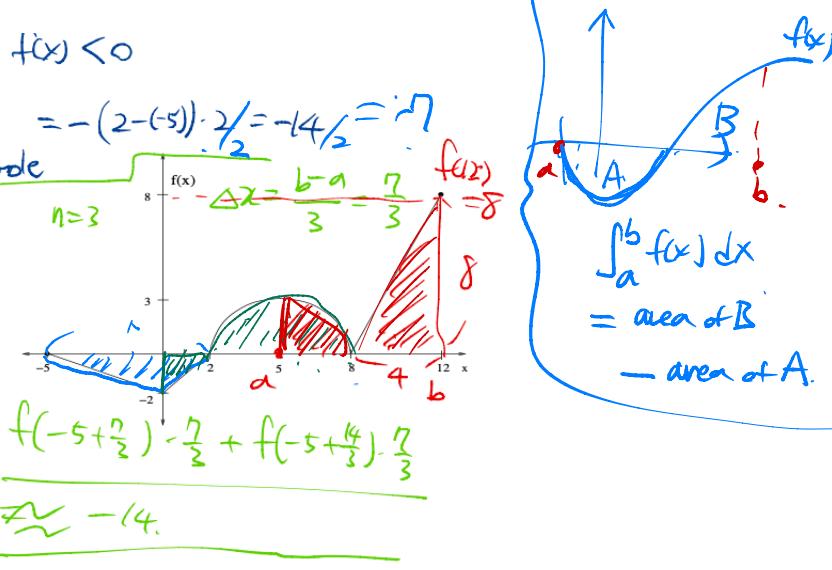
b)  $\int_0^8 f(x) dx = \text{Area of Semicircle} - \text{Area of triangle}$

$$\begin{aligned} &= \pi \cdot 3^2 \cdot \frac{1}{2} - 2 \cdot 2 \cdot \frac{1}{2} \\ &= \frac{9}{2}\pi - 2 \end{aligned}$$

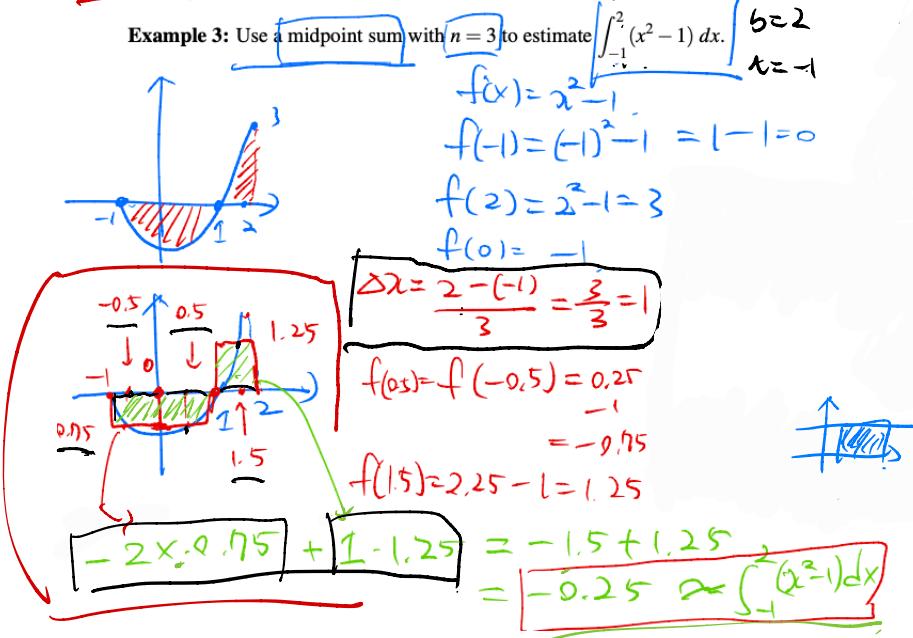
$\int_0^8 f(x) dx = f(-5 + \frac{7}{3}) \cdot \frac{7}{3} + f(-5 + \frac{4}{3}) \cdot \frac{4}{3} \approx -14$

c)  $\int_5^{12} f(x) dx = \text{Area of quarter of the circle} + \text{Area of triangle.}$

$$\begin{aligned} &= \pi \cdot 3^2 \cdot \frac{1}{4} + 4 \cdot 8 \cdot \frac{1}{2} \\ &= \frac{9}{4}\pi + 16 \end{aligned}$$



(d) Example 3: Use a midpoint sum with  $n = 3$  to estimate  $\int_{-1}^2 (x^2 - 1) dx$



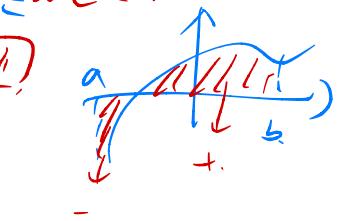
### (6) 6.5. The Fundamental Theorem of Calculus

(a)  $\int_a^b f(x) dx$  gives an exact value and "counts" area above the  $x$ -axis positively and area below the  $x$ -axis negatively.

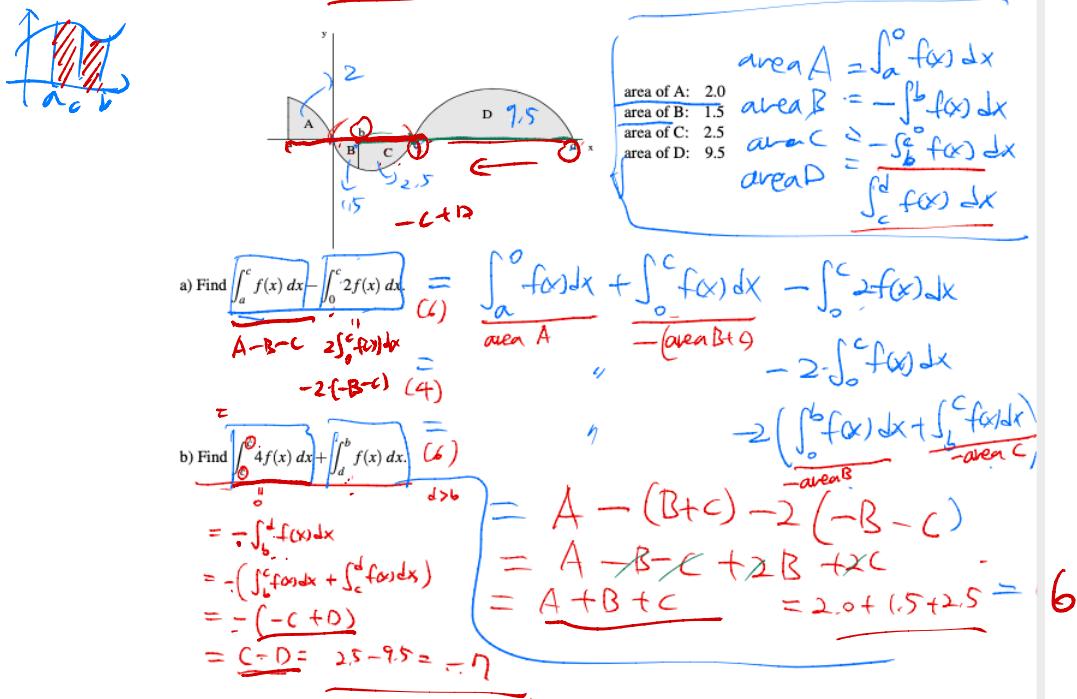
(b) Properties of Definite Integral:

- (i)  $\int_a^b m dx = m(b-a)$ , where  $m$  is a constant
- (ii)  $\int_a^a f(x) dx = 0$
- (iii)  $\int_a^b f(x) dx = - \int_b^a f(x) dx$
- (iv)  $\int_a^b kf(x) dx = k \int_a^b f(x) dx$ , where  $k$  is a constant
- (v)  $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- (vi)  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ , where  $a < c < b$

$$\int_a^b f(x) dx = \text{area of } \boxed{\text{region}}$$



(c) Example 2: Use the graph of  $f(x)$  with the indicated areas below to answer the following.



- (d) The Fundamental Theorem of Calculus, Part 2 - Suppose  $f$  is continuous on  $[a, b]$

$$\int_a^b f(x) dx = F(b) - F(a)$$

where  $F$  is any antiderivative of  $f$ , that is,  $F' = f$

- (e) Example 5: Evaluate  $\int_2^k (t^2 + 4) dt$

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$$\text{Example 5: Evaluate } \int_2^k (t^2 + 4) dt = \int_2^k t^2 dt + \int_2^k 4 dt$$

$$= \frac{1}{3} t^3 \Big|_2^k + 4t \Big|_2^k$$

$$= \frac{1}{3} k^3 - \frac{1}{3} 2^3 + 4(k-2)$$

$$= \frac{1}{3} k^3 - \frac{8}{3} + 4k - 8$$

$$= \frac{1}{3} k^3 + 4k - \frac{32}{3}$$

- (f) Ex6:

**Example 6:** A honeybee population starts with 200 honeybees and increases at a rate of  $n'(t) = 100e^{2t}$  bees per week, where  $t$  is in weeks and  $t \geq 0$ .

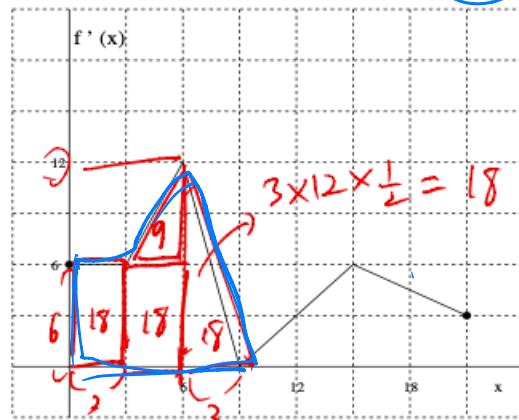
a) Find the change in the honeybee population over the first 4 weeks. Round to the nearest integer, if necessary

$$\int_0^4 n'(t) dt = \int_0^4 100e^{2t} dt$$

(1) Use calculator  
+ calculate  $\approx 148997.8994$ .  
definite integral.

- (g) Example 7: Consider the graph of  $f'(x)$  shown below. If  $f(0) = 50$ , find  $f(9)$ .

**Example 7:** Consider the graph of  $f'(x)$  shown below. If  $f(0) = 50$ , find  $f(9)$ .



$$f(9) - f(0) = \int_0^9 f'(x) dx$$

$\int$  FTC  $\int$

$$50 = 63$$

$$\begin{aligned} f(9) - 50 &= 63 \\ f(9) &= 113 \end{aligned}$$

$$\therefore 18 + 18 + 18 + 9 = 63$$

- (h) Average Value of a Continuous Function  $f$  over  $[a, b]$

$$\frac{1}{b-a} \int_a^b f(x) dx$$

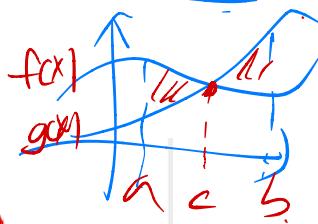
$$\begin{aligned} f(a) - f(0) &= \int_0^9 f'(x) dx \\ &= \int_0^9 1 dx \end{aligned}$$

(7) 6.6: Area Between Two Curves

(a) Theorem: If  $f(x)$  and  $g(x)$  are two continuous functions with  $f(x) \geq g(x)$  on  $[a, b]$ , then the area between the two curves on  $[a, b]$  is given by

$$\int_a^b (f(x) - g(x)) dx.$$

(b) Example 5: Find the area that is bounded by  $y = -x^2$  and  $y = 2x^3 - 5x$



$$-x^2 = 2x^3 - 5x$$

$$\Rightarrow 0 = 2x^3 + x^2 - 5x \\ = x(2x^2 + x - 5) \\ = x$$

*quadratic formula  
to figure out  
intersection*

$$x = \frac{1.3508}{0} \\ -1.8508$$

$$\text{(3)} \quad \int_{-1.8508}^0 g(x) - f(x) dx = \int_{-1.8508}^0 (2x^3 - 5x + x^2) dx \\ + \int_0^{1.3508} f(x) - g(x) dx = \int_0^{1.3508} (-x^2 - 2x^3 + 5x) dx \quad [6.8854]$$

(c) Example 7: Set up the definite integral(s) representing the area bounded by  $y = -x^2 + 10x - 17$  and the  $x$ -axis on  $[5, B]$ , where  $B > 8$

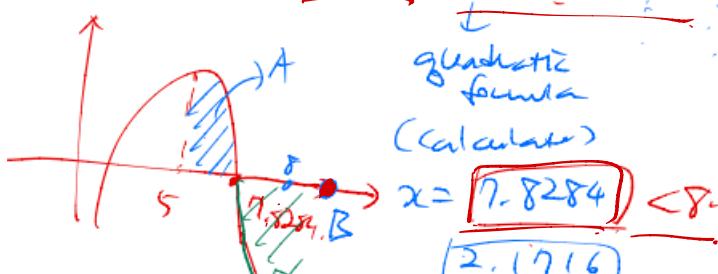
Example 7: Set up the definite integral(s) representing the area bounded by  $y = -x^2 + 10x - 17$  and the  $x$ -axis on  $[5, B]$ , where  $B > 8$ .

$$g(x) = 0 \quad f(x) = -x^2 + 10x - 17$$

$$= -x^2 + \frac{10x}{2} - \frac{17}{2}$$

$$= \frac{4x}{3} + \frac{17}{2} + \frac{17}{3} + \frac{7}{2}$$

$$= 21 + 9 = 30$$



*quadratic formula  
(calculator)*

$$x = 7.8284 < 8$$

$$A = \int_5^{7.8284} f(x) dx = \int_5^{7.8284} (-x^2 + 10x - 17) dx$$

$$B = \int_{7.8284}^B -f(x) dx = \int_{7.8284}^B (x^2 - 10x + 17) dx$$

$$= \left[ \frac{1}{3}x^3 - 5x^2 + 17x \right]_{7.8284}^B \\ = \left[ \frac{1}{3}B^3 - 5B^2 + 17B - \left( \frac{1}{3}(7.8284)^3 - 5(7.8284)^2 + 17(7.8284) \right) \right]$$