0. Preface

"Natural" $\longleftrightarrow$ defined without arbitrary choice

ex) $V \xrightarrow{\cong} V^*$: need choice of the basis

$V \xrightarrow{\cong} V^{**}$: doesn't needed.

Eilenberg MacLane 45' : Natural Transformation

Functor (as source/Target)

Category (")

Objectives ① Categorifying math. objects ⇒ isomorphism

② study itself ⇒ self-dual

In Chapter 1, ① Def ② Duality ③ Functor : Today!

④ Naturality ⑤ Equivalence (⑥ Diagram Chase ⑦ 2-Categories)

Def 1.1.1 C : category consists of

· Obj(C) : Collection of objects, $X, Y, Z, \ldots$

· Hom(C) " morphisms $f, g, h \ldots$

(or Mor(C))

1) Each morphism has domain and codomain.

2) Each object X has 1_X : identity morphism

3) Two morphism $f : X \to Y$ $g : Y \to Z$ s.t.

$\mathrm{cod}(f) = \mathrm{dom}(g)$; $gf : X \to Z$ exists.

a) (unital) $\forall f : X \to Y \in \mathrm{Hom}(C)$, $f 1_X = f = 1_Y f$

b) (associates) for any composable triple,

$f(gh) = (fg)h$.

In abstract Category : Morphism = arrow or map
(they're not function)

Rem 1.1.5 Size issue
Russell's Paradox $\Rightarrow$ No collection contains itself.
To avoid,

Def 1.1.6. Small category : $\mathrm{Hom}(C)$ is a set.
(Then, from $\mathrm{Obj}(C) \hookrightarrow \mathrm{Hom}(C)$, $\mathrm{Obj}(C)$ is also a set)
$x \mapsto 1_x$

$$\Rightarrow \quad \begin{tikzcd} \mathrm{Hom}\, C \arrow[r, shift left=2, "\mathrm{dom}"] \arrow[r, shift right=2, "\mathrm{cod}"'] & \mathrm{Obj}\, C \arrow[l, "\mathrm{id}"'] \end{tikzcd}$$
are functions.

But none of examples of concrete category are small.

Def 1.1.7 locally small category : $\forall X, Y \in \mathrm{obj}\, C$.
$C(Y,X) = \mathrm{Hom}(X,Y)$ is a set (set of all morphisms from X to Y)
$= \mathrm{Hom}(X,Y)$

Q: When is one thing the same as another thing?
$\Rightarrow$ iso (actually we will define notion of equivalence)

Def: 1.1.9. Iso in Category C is a morphism $f: X \to Y$
for which $\exists\, g: Y \to X$ st $fg = 1_Y$, $gf = 1_X$.
In this case write $X \cong Y$.
Endo: $f: X \to X$. Auto: endo + iso.

Ex 1.1.10.
Set: bijection. (Grp, Ring, Field), Mod_R : bijective ~
Top: Homeo. Htpy: homotopy equiv. $(P, \leq)$: identity by antisymmetry

Q: In a concrete category, Is every iso induced by bijection of underlying set? A: Lem 5.6.1 (Yes)

Def 1.1.11. Groupoid : a category in which every morphism is iso.

Cf) In algebra, groupoid : a group changing binary operation to partial function.
i.e. mult is not def on all ~~two~~ objects pairs of

Ex 1.1.12 G : (Group) : groupoid with 1 object
(it is def of gp in category theory.)

$\pi_1(X)$: fundamental groupoid.
Obj : points in X Mor : Endpt preserving homotopy classes of path.

Def) Subcategory D of C.
Obj D $\subseteq$ Obj C, Mor D $\subseteq$ Mor C s.t.
1) Obj D contains any domain or codomain of $f \in Mor(D)$
2) Mor D " any identity morphism of $X \in obj(D)$
3) Closed under composition.

Ex) CRing $\subset$ Ring $\subset$ Rng
(com unital) (unital) (maybe nonunital).

Lem 1.1.13 Any category C contains maximal groupoid.
is subcategory of C
pf) Show collection of isomorphisms of C is subcategory.

Ex). Fin : Obj : finite set, Mor : functions
Fin_{iso} " , Mor : bijection,
Fin_{iso} is maximal groupoid of Fin

(말하면: Objects X can be identified with identity 1_X.
So Cat def by morphism.

What we care: $\boxed{\text{Morphism}}$

Ex 1.1.3. (Concrete Category)

Name	Obj	Hom	Composite is
Set	set X	function f	composition of f.
Top	top. space X	cts function f.	〃
Set_*, Top_*	$(X; x)$, $x \in X$ base pt	base pt preserving ~	〃
Group	groups	gp homo	〃
Ring	rings	ring 〃	〃
Field	fields	field 〃	〃
Mod_R	left R-module	module 〃	〃
${}_R Mod$	right 〃		
$Vect_{\Bbbk} = Mod_R$	with $R = \Bbbk$		
$Ab = Mod_R$	with $R = \mathbb{Z}$		
Graph	graphs	graph morphism (sending vertex to vertex, edge to edge, preserving incidence relation)	〃
Digraph	di 〃		
Man	manifolds	smooth map.	〃
Meas	measurable space	measurable function	
Poset	Posets	order preserving function	〃
Ch	chain cpx of R-module	chain homo.	

$Model_T$	σ-structure. morphism preserving structure.

Concrete Category (Precise : 1.6.17)

Obj has underlying set
Morph are functions between underlying set.

1.1.4 (Abstract Category)

Name	Obj	Morph	Composition ?
① Mat_R (R: unital ring) ↳ needed for identity morph.	$\mathbb{P} = \{1, 2, 3, \dots\}$	$A : n \to m$ is $(m \times n)$ R-valued matrix	matrix multiplication.
② BG G: group.	$\{\cdot\}$	$g \in G$.	group product.
③ $(P, \leq)$ P: poset	P	$x \leq y \iff x \to y$ (identity from reflexivity)	clear by transitivity

So we can do for preorder (transitivity and reflexivity)

Name	Obj	Morph	Composition
④ Ordinals			
$\mathbb{0}$	$\emptyset$	$\emptyset$	
$\mathbb{1}$	$\{\emptyset\}$	$1_\emptyset$.	
$\mathbb{2}$.	$\{\emptyset, \{\emptyset\}\}$	$1_\emptyset$, $1_{\{\emptyset\}}$, $\emptyset \to \{\emptyset\}$. $(0 \to 1)$	
ω	$\{0, 1, 2, \dots\}$	$0 \to 1 \to 2 \to 3 \to \cdots$	all morphism $n \to m$ equal to $n \to n+1 \to n+2 \to \cdots$
⑤ A (set).	A	$1_a \ \forall a \in A$.	

"Discrete Category": Every morphism is an identity.

Name	Obj	Morph	Composition
⑥ Htpy	top space	homotopy class.	
Htpy_*	(X, x)	base preserving ...	
⑦ ...	measure spaces	equiv class of msble f ... (a.e.)	