[1.2] Fin-dim dist $(X_{t_1}, \dots, X_{t_n})$ $t_1 < \dots < t_n \in T$, X_t: Sto. Pro. | Gaussian Process: Every Fin dim dist. is mult-G.

exp function: $\mu_X(t) := \mathbb{E}[X_t]$ cov function: $C(t,s) = \mathrm{cov}(X_t, X_s) = \mathbb{E}[(X_t - \mathbb{E}[X_t])(X_s - \mathbb{E}[X_s])]$ $\mathrm{Var}(X_t) = C(t,t)$

X_t is "stationary strict" if $(X_{t_1}, \dots, X_{t_n}) = (X_{t_1+h}, \dots, X_{t_n+h})$ $\forall h \geq 0$ | "indep increment" if $(X_{t_1}, \dots, X_{t_n})$

X_t is "stationary increment" if $\mu_X(t) = \mu_X(t+h)$, $C(t+h, s+h) = C(t,s)$ $\forall h \geq 0$ | their diff are indep.

[1.3] BM

def 1: ① $B_0 = 0$, ② Stationary, indep increment ③ $B_t \sim N(0,t)$ $\forall t > 0$ ④ cts sample path a.s.

def 2: Gaussian Proc. with $\mu_t = 0$, $C(t,s) = \min(t,s)$ (i.e. $(B_{t_1}, \dots, B_{t_n}) \sim N(\vec{0}, \Sigma)$, $\begin{pmatrix} B_{t_1} \\ B_{t_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} B_{t_1} \\ B_{t_2 - t_1} \end{pmatrix}$)

Properties ① Gaussian Proc. ② $\mu_t = 0$, $C(t,s) = \min(t,s)$ ③ 0.5 self-similar: $B_t = \frac{1}{\sqrt{c}} B_{ct}$ $\forall c > 0$

pf) Gauss def.

④ BM sample path are nowhere diff. (from indep incre) $\Rightarrow$ $TV = \infty$ a.s. ⑤ Reflection prin. $P(\sup_{0 \le s \le t} \{B_s\} \geq a) = 2P(B_t \geq a)$

[1.4] Cond. Exp: $Z = \mathbb{E}[X|\mathcal{F}]$ if ① $\sigma(Z) \subseteq \mathcal{F}$, ② $\mathbb{E}[X \cdot 1_A] = \mathbb{E}[Z \cdot 1_A]$ $\forall A \in \mathcal{F}$.

Properties ⓪ Unique, exist ① Linear ② $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}]]$ ③ X indep $\mathcal{F}$ $\Rightarrow$ $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$

X, Y indep $\Rightarrow$ $\mathbb{E}[X|Y] = \mathbb{E}[X]$

④ X is $\mathcal{F}$-meas $\Rightarrow$ $\mathbb{E}[X|\mathcal{F}] = X$. $\Rightarrow$ $\mathbb{E}[f(Y)|Y] = f(Y)$. ⑤ X is $\mathcal{F}$-meas: $\mathbb{E}[XG|\mathcal{F}] = X\mathbb{E}[G|\mathcal{F}]$ $\Rightarrow$ $\mathbb{E}[f(Y)G|Y] = f(Y)\mathbb{E}[G|Y]$

⑥ $\mathcal{F} \subseteq \mathcal{F}'$ $\Rightarrow$ $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}']|\mathcal{F}] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}]|\mathcal{F}']$ | ⑧ Projection: if $\mathbb{E}X^2 < \infty$, then

⑦ X indep $\mathcal{F}$, G-$\mathcal{F}$-meas $\Rightarrow$ $\mathbb{E}[h(X,G)|\mathcal{F}] = \mathbb{E}_X[h(X,G)]$ | $\mathbb{E}[(X - \mathbb{E}[X|\mathcal{F}])^2] \leq \mathbb{E}[(X-Z)^2]$ $\forall Z \in L^2(\mathcal{F})$

exp taken by regarding G as constant.

[1.5] Martingale (cts) def: X_t is M. wrt $\mathcal{F}_t$ if ① $\mathbb{E}|X_t| < \infty$ $\forall t$ ② X_t adapted to $\mathcal{F}_t$ (X_t is $\mathcal{F}_t$-meas)

Martingale (discrete) def: ① $\mathbb{E}|X_n| < \infty$ $\forall n$ ② $\sigma(Y_n) \subseteq \mathcal{F}_n$ ③ $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ $\forall n$ ③ $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ $\forall 0 \le s \le t$.

Martingale Transform: X_n: M. wrt $\mathcal{F}_n$ $Y_n = \begin{cases} 0 & n=0 \\ X_n - X_{n-1} & n \geq 1 \end{cases}$ $\mathbb{E}Y_n^2 < \infty$ C_n: $\mathcal{F}_{n-1}$-meas, $\mathbb{E}C_n^2 < \infty$.

$Z_n := \begin{cases} 0 & n=0 \\ C_1 Y_1 + \dots + C_n Y_n & n \geq 1 \end{cases}$ $\Rightarrow$ Z_n is Martingale wrt $\mathcal{F}_t$. pf) ① CS-ineq ② $\mathcal{F}_n$-meas aut. ③ $\mathbb{E}[Z_{n+1}|\mathcal{F}_n] - Z_n = 0$ 보이기.

[2.2] Ito int. def: $\int_0^T f(t,\omega)\,dB_t := \lim_{|\Delta t| \to 0} \sum_{i=1}^n f(t_{i-1}, \omega)(B_{t_i} - B_{t_{i-1}})$ (L^2 sense)

Properties ① Martingale ② $\mathbb{E}[\int_0^t f(t,\omega)\,dB_t] = 0$ $\forall t$ (from Martingale) ③ Ito Isometry: $\mathbb{E}[(\int_0^T f(s,\omega)\,dB_s)^2] = \int_0^T \mathbb{E}[f(s,\omega)^2]\,ds$ ④ Linear ⑤ I_t is cts.

[2.3] Ito Lemma: $f(t,x) \in C^{1,2}$ $\Rightarrow$ $df(t,X_t) = f_t(t,X_t)\,dt + f_x(t,X_t)\,dX_t + \frac{1}{2} f_{xx}(t,X_t)(dX_t)^2$

(변수 X_1, X_2 일 때 $\Rightarrow$ f_t, f_x, f_{xx} 가 $\frac{1}{2}(f_{x_1x_1} + 2f_{x_1x_2} + f_{x_2x_2})$) | Ito prod rule: $d(X_tY_t) = (dX_t)Y_t + X_t(dY_t) + (dX_t)(dY_t)$.

[2.4] Stratonovich: $\int_0^T f(B_t) \circ dB_t = \lim_{|\Delta t| \to 0} \sum_{i=1}^n f(B_{t_i^*})(B_{t_i} - B_{t_{i-1}})$ ($t_i^* = \frac{t_i + t_{i-1}}{2}$) | Traditional Ch. rule: $\int_0^T f'(B_t) \circ dB_t = f(B_T) - f(B_0)$

Conversion: If $dX_t = a(t,X_t)\,dt + b(t,X_t)\,dB_t$ $\Rightarrow$ $\int_0^T f(t,X_t) \circ dB_t = \int_0^T f(t,X_t)\,dB_t + \frac{1}{2}\int_0^T b(t,X_t) f_x(t,X_t)\,dt$

[3.2] Ito process $X_t \Leftrightarrow dX_t = a(t,X_t)\,dt + b(t,X_t)\,dB_t$ (*). Unique strong sol exists if a, b are Lipschitz. $|a(t,x) - a(t,y)| \leq K|x-y|$.

X_t strong sol of (*): $X_t - X_0 = \int_0^t a(s,X_s)\,ds + \int_0^t b(s,X_s)\,dB_s$ a.s. | X_t (Unique) strong sol: For any strong sol Y_t of (*) $\forall \varepsilon, \delta > 0$, $P(\sup_{0 \le t \le T} |X_t - Y_t| > \varepsilon) < \delta$ $\Leftrightarrow$ $P(\sup_{0 \le t \le T} |X_t - Y_t| \neq 0) = 0$.

weak sol: $(X_t, \tilde{B}_t, \tilde{\mathcal{F}}_t)$ s.t. $X_t - X_0 = \int_0^t a(s,X_s)\,ds + \int_0^t b(s,X_s)\,d\tilde{B}_s$ ($\tilde{B}_s$ not necessarily B_s)

Stratonovich Conv: If X_t is sol of (*), $dX_t = \underbrace{(a(t,X_t) - \frac{1}{2} b(t,X_t) b_x(t,X_t))}_{= \tilde{a}(t,X_t)}\,dt + b(t,X_t) \circ dB_t$ also has the same solution

ex) $u(t,x)$ cts diff $\Rightarrow$ $du(t,X_t) = u_t(t,X_t)\,dt + u_x(t,X_t) \circ dX_t$ if $dX_t = \tilde{a}(t,X_t)\,dt + b(t,X_t) \circ dB_t$

[3.3] Case 1 (Langevin eq. or additive noise) $dX_t = (C_1(t)X_t + C_2(t))\,dt + \sigma_2(t)\,dB_t$, $X_0 = x_0$ | Solve: let $y_t := e^{-\int_0^t C_1(s)\,ds}$

Let $Y_t := y_t X_t$ apply Ito prod rule: $dY_t = \dots = C_2(t) y_t\,dt + \sigma_2(t) y_t\,dB_t$ $\overset{적분}{\Rightarrow}$ $Y_t = Y_0 + \int_0^t C_2(s) y_s\,ds + \int_0^t \sigma_2(s) y_s\,dB_s$

Case 2 (Mult. noise) $dX_t = C_1(t)X_t\,dt + \sigma_1(t)X_t\,dB_t$ | Solve $Y_t = \ln X_t$ $\Rightarrow$ Ito Lemma $\Rightarrow$ $dY_t = \dots = (C_1(t) - \frac{\sigma_1^2(t)}{2})\,dt + \sigma_1(t)\,dB_t$

$\Rightarrow$ 적분, exp 취하기.

Case 3 $dX_t = (C_1(t)X_t + C_2(t))dt + (\sigma_1(t)X_t + \sigma_2(t))dB_t$ | ~~Solve: ① Y_t = sol of~~

Solve ① Let Y_t : sol of $dY_t := C_1(t)Y_t dt + \sigma_1(t) Y_t dB_t$, $Y_0 = 1$. ← use case 2 to solve it.

② Let $Z_t = \frac{1}{Y_t}$: apply Ito Lem. $\Rightarrow dZ_t = (-C_1(t) + \sigma_1^2(t))Z_t dt - \sigma_1(t) Z_t dB_t$, $Z_0 = 1$.

③ Ito prod rule on $Z_t X_t$: $d(Z_t X_t) := \cdots = (C_2(t) - \sigma_1(t)\sigma_2(t)) Z_t dt + \sigma_2(t) Z_t dB_t$ ← use case 2 to find sol!

④ Z_t로 나누기 (= Y_t 곱하기) $X_t = Y_t \left(X_0 + \int_0^t (C_2(s) - \sigma_1(s)\sigma_2(s)) Y_s^{-1} ds + \int_0^t \sigma_2(s) Z_s dB_s \right)$.

where $Y_t = \exp\left(\int_0^t (C_1(s) - \frac{\sigma_1^2(s)}{2}) ds + \int_0^t \sigma_1(s) dB_s \right)$.

(If $\sigma_1 = \sigma_2 = 0$, then $dx(t) = (C_1(t)x(t) + C_2(t))dt$ has sol $x(t) = x_0 + e^{\int_0^t C_1(s)ds} \left(\int_0^t C_2(s) e^{-\int_0^s C_1(h)dh} ds \right)$

Gen form

$\mathbb{E}[X_t] = \mathbb{E}[X_0 + \int_0^t (C_1(s)X_s + C_2(s)) ds + \int_0^t (\sigma_1(s)X_s + \sigma_2(s)) dB_s] = \mathbb{E}X_0 + \int_0^t (C_1(s)\mathbb{E}[X_s] + C_2(s)) ds + 0$.

$\mathbb{E}[X_t^2]$: use Ito Lemma $\Rightarrow d(X_t^2) = 2X_t dX_t + \frac{1}{2} \cdot 2(dX_t)^2 = \cdots =$

Let $q_t = \mathbb{E}[X_t^2]$, 두 모멘트 다 (***) 이용해서 풀기!

4.1 BS Assumption : X_t : stock price. $dX_t = cX_t dt + \sigma X_t dB_t$, β_t : bond price : $d\beta_t = r\beta_t dt$.

a_t : # of stock, b_t : # of bond : $V_t = a_t X_t + b_t \beta_t$ self-financing : $dV_t = a_t dX_t + b_t d\beta_t$. Call option ($k$: strike price) $C_T = \max(X_T - k, 0) = (X_T - k)^+$

No Arbitrage Principle : $V_T = a_T X_T + b_T \beta_T = C_T$. Let $V_t = u(T-t, X_t)$. By Ito Lemma,

$dV_t = (-u_t + c u_x X_t + \frac{\sigma^2}{2} u_{xx} X_t^2)dt + (\sigma u_x X_t) dB_t$. ← compare it with self-financing

$\Rightarrow a_t = u_x(T-t, X_t)$, $b_t = \frac{u(T-t, X_t) - u_x(T-t, X_t)X_t}{\beta_t}$, dt-term 비교 → $u_t(\tau, x) = \frac{\sigma^2}{2} x^2 u_{xx}(\tau, x) + rx u_x(\tau, x) - ru(\tau, x)$, $\boxed{u_0(0, x) = (x-k)^+}$.

4.2 Girsanov Thm Ver 1 When $dY_t = a(t,\omega)dt + dB_t$ with $\mathbb{E}[\exp(\frac{1}{2}\int_0^T a(s,\omega)^2 ds)] < \infty$.

$\Rightarrow$ ① $M_t = \exp\left(-\int_0^t a(s,\omega) dB_s - \frac{1}{2}\int_0^t a^2(s,\omega) ds \right)$ is Martingale wrt $\mathcal{F}_t^{(n)}$ (gen by B_t)

② $Q(A) = \int_A M_T(\omega) dP(\omega)$ measure $\Rightarrow Y_t$ is BM wrt Q.

Ver 2 $dY_t = \beta(t,\omega)dt + \theta(t,\omega)dB_t$, $\exists u, \alpha$ s.t. $\theta u = \beta - \alpha$) $\Rightarrow$ ① $M_t = \exp\left(-\int_0^t u(s,\omega)dB_s - \frac{1}{2}\int_0^t u(s,\omega)^2 ds\right)$

② $\tilde{B}_t := \int_0^t u(s,\omega) ds + B_t$ is BM wrt Q ③ $dY_t = \alpha(t,\omega)dt + \theta(t,\omega) d\tilde{B}_t$.

Ineq. ① Markov : $P(X \geq c) \leq \mathbb{E}[X]/c$ ($\mathbb{E}[X] < \infty$, $c > 0$) ② Chebyshev : $P(|X - \mathbb{E}X| \geq c) \leq \frac{Var(X)}{c^2}$ ($\mathbb{E}[X^2] < \infty$, $c > 0$)

③ Concentration : $P(X - \mu \geq c) \leq e^{-\frac{1}{2} \cdot \frac{c^2}{\sigma^2}}$ ($X \sim N(\mu, \sigma^2)$, $c \geq 0$) ④ Tail dist of NRV : $P(X - \mu \geq c) \leq \frac{1}{\sqrt{2\pi}} \frac{\sigma}{c} e^{-\frac{c^2}{2\sigma^2}}$ ($c \geq 0$, $X \sim N(\mu, \sigma^2)$)

* $(a+b)^2 \leq 2a^2 + 2b^2$, $(a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2$ pf) use $(a-b)^2$ iteratively. ⑤ Jensen : $f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$ ∀ f convex (also conditional)

⑥ Young's : $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ ($p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a, b \geq 0$) ⑦ Holder : $\mathbb{E}|XY| \leq \mathbb{E}[|X|^p]^{\frac{1}{p}} \mathbb{E}[|Y|^q]^{\frac{1}{q}}$.

ex) $\mathbb{E}[X^2] \leq \mathbb{E}[|XY|] \Rightarrow \mathbb{E}[X^2] \leq \mathbb{E}|XY| \underset{Young}{\leq} \frac{1}{2}\mathbb{E}[X^2] + \frac{1}{2}\mathbb{E}[Y^2]$ | $\frac{1}{2}\mathbb{E}[X^2] \leq \mathbb{E}[|XY|] = \mathbb{E}[|\varepsilon X| \cdot |\frac{Y}{\varepsilon}|] = \cdots$ ($\varepsilon > 0$)

⑧ BDG Ineq. $\forall p > 0$, $\exists c_p, C_p$ s.t. $c_p \mathbb{E}[\langle M \rangle_t^{\frac{p}{2}}] \leq \mathbb{E}[(M_t^*)^p] \leq C_p \mathbb{E}[\langle M \rangle_t^{\frac{p}{2}}]$ where $M_t := \int_0^t a(s,\omega)dB_s$ $\Rightarrow \langle M \rangle_t = \int_0^t a(s,\omega)^2 ds$

ex) $\mathbb{E}[\sup(\int_0^t \sin(B_s)dB_s)^2] \underset{BDG}{\leq} C\mathbb{E}[\int_0^t \sin^2(B_s)ds] \leq \cdots \leq Ct$. $M_t^* = \sup_{0 \leq s \leq t} |M_s|$.

⑨ Gronwall's Ineq : (β, u : cts, α : nondec, integrable) $u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds \Rightarrow u(t) \leq \alpha(t)\exp(\int_a^t \beta(s)ds)$

ex) $dX_t = f(X_t)dB_t$, $X_0 = x_0$, $|f(x)| \leq a|x|$. Find upper bdd of $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t|^2]$

pf) $|X_t|^2 \leq |X_0 + \int_0^t f(X_s)dB_s|^2 \leq 2|X_0|^2 + 2|\int_0^t f(X_s)dB_s|^2 \xrightarrow{\mathbb{E}\sup} 2\mathbb{E}|X_0|^2 + 2\mathbb{E}[\sup_{0 \leq s \leq t}|\int_0^s f(X_s)dB_s|]$

$\underset{BDG}{\leq} 2\mathbb{E}|X_0|^2 + C\mathbb{E}[\int_0^T f(X_t)^2 dt] = 2\mathbb{E}|X_0|^2 + C\int_0^T \mathbb{E}[f^2(X_t)]dt \leq 2\mathbb{E}|X_0|^2 + Ca^2 \int_0^T \mathbb{E}[X_t^2]dt + \cdots$

$\leq 2\mathbb{E}|X_0|^2 + \tilde{C}a^2 \int_0^T \mathbb{E}(\sup|X_s|^2)dt \xRightarrow{Gronwall} \mathbb{E}[\sup_{0 \leq t \leq T}|X_t|^2] \leq 2\mathbb{E}|X_0|^2 e^{Ca^2T}$.