Ch4. Adjunctions!

4.1. Adjoint functors

Def 4.1.1 (adjunction)

<u>An adjunction:</u> $F: C \longrightarrow D$, $G: D \longrightarrow C$

with iso $D(Fc, d) \cong C(c, Gd)$ $\forall c \in C, d \in D$

that is natural in both c and d.

$\Rightarrow$ F: left adjoint, G: right adjoint

$$Fc \xrightarrow{f^\sharp} d \quad \longleftrightarrow \quad c \xrightarrow{f^\flat} Gd$$

corresponding under bijection $D(Fc, d) \cong C(c, Gd)$

are called <u>adjunct</u> or <u>transposes</u> of each other

Rmk) If C, D are locally small,

then iso $D(Fc, d) \cong C(c, Gd)$ $\forall c \in C, d \in D$

gives natural iso

$$\begin{tikzcd} C^{op} \times D \arrow[r, bend left, "D(F-,-)"{name=U}] \arrow[r, bend right, "C(-,G-)"'{name=L}] \arrow[from=U, to=L, phantom, "\Downarrow\cong"] & Set \end{tikzcd}$$

In other words, $\forall k: d \longrightarrow d' \in \mathrm{Mor}\, D$,

$$\begin{tikzcd} D(Fc, d) \arrow[r, "\cong"] \arrow[d, "k_*"'] & C(c, Gd) \arrow[d, "Gk_*"] \\ D(Fc, d') \arrow[r, "\cong"'] & C(c, Gd') \end{tikzcd}$$

Thus

$$\begin{tikzcd} f^\sharp \arrow[r, mapsto, "="] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & f^\flat \arrow[d, mapsto] \\ k \circ f^\sharp \arrow[r, mapsto] & (k \circ f^\sharp)^\flat \end{tikzcd} \quad Gk \circ f^\flat = (k \circ f^\sharp)^\flat \quad \Rightarrow \quad \begin{tikzcd} c \arrow[r, "f^\flat"] \arrow[dr, "(k\circ f^\sharp)^\flat"'] & Gd \arrow[d, "Gk"] \\ & Gd' \end{tikzcd}$$

Dually, $\forall\ h: c' \longrightarrow c \in \mathrm{Mor}\,C$,

$$\begin{tikzcd} D(Fc,d) \arrow[r, "\cong"] \arrow[d, "Fh^*"'] \arrow[dr, phantom, "\circlearrowright"] & C(c,Gd) \arrow[d, "h^*"] \\ D(Fc',d) \arrow[r, "\cong"'] & C(c',Gd) \end{tikzcd} \Rightarrow \begin{tikzcd} f^{\#} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & f^{b} \arrow[d, mapsto] \\ f^{\#}\circ Fh \arrow[r, mapsto] & (f^{\#}\circ Fh)^{b} \end{tikzcd}$$

$h\circ f^{b} = (f^{\#}\circ Fh)^{b}$

$$\Rightarrow \begin{tikzcd} c' \arrow[d, "h"'] \arrow[dr, "(f^{\#}\cdot Fh)^{b}"] & \\ c \arrow[r, "f^{b}"'] & Gd \end{tikzcd}$$

Lemma 4.1.3. Let $F: C \rightleftarrows D : G$ with iso $D(Fc,d) \cong C(c,Gd)$ $\forall c \in C,\ d \in D$. Then,

) The collection of iso are natural iso.

$\Longleftrightarrow$ ② (For any $k: d \longrightarrow d'$, $h: c \to c'$ in $\mathrm{Mor}\,D$, $\mathrm{Mor}\,C$

$$\begin{tikzcd} Fc \arrow[r, "f^{\#}"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\circlearrowright"] & d \arrow[d, "k"] \\ Fc' \arrow[r, "g^{\#}"'] & d' \end{tikzcd} \Longleftrightarrow \begin{tikzcd} c \arrow[r, "f^{b}"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & Gd \arrow[d, "Gk"] \\ c' \arrow[r, "g^{b}"'] & Gd' \end{tikzcd}$$
)

pf) Suppose the collection of iso is natural.

Then, as we've seen, $(k\cdot f^{\#}) = Gk\circ f^{b}$ by above
$(g^{\#}\circ Fh) = f^{b}\circ h$

naturality diagrams $\Longleftrightarrow$ right square commutes

since it is natural iso.

Conversely, suppose ②. Then, it suffices to show that

$$\begin{tikzcd} D(Fc,d) \arrow[r, "\cong"] \arrow[d, "k_*"'] \arrow[dr, phantom, "\circlearrowright"] & G(c,Gd) \arrow[d, "Gk_*"] \\ D(Fc,d') \arrow[r, "\cong"'] & C(c,Gd') \end{tikzcd}$$ and

$$
\begin{tikzcd}
D(Fc, d) \arrow[r, "\cong"] \arrow[d, "Fh^*"'] \arrow[dr, phantom, "\circlearrowright"] & C(c, Gd) \arrow[d, "h^*"] \\
D(Fc', d) \arrow[r, "\cong"'] & C(c', Gd)
\end{tikzcd}
$$

To see this,

$$
\begin{array}{ccccc}
D(Fc, d) & \xrightarrow{\cong} & C(c, Gd) & \xrightarrow{Gk_*} & C(c, Gd') \\
f^\sharp & \longmapsto & f^\flat & \longmapsto & Gk \circ f^\flat
\end{array}
$$

$$
\begin{array}{ccc}
D(Fc, d) & \xrightarrow{k_*} & D(Fc, d') \\
f^\sharp & \longmapsto & k \circ f^\sharp
\end{array}
$$

$$
\begin{array}{ccccc}
D(Fc', d) & \xrightarrow{\cong} & C(c', Gd) & \xrightarrow{h^*} & C(c', Gd) \\
g^\sharp & \longmapsto & g^\flat & \longmapsto & g^\flat \circ h
\end{array}
$$

$$
\begin{array}{ccc}
D(Fc, d) & \xrightarrow{Fh^*} & D(Fc', d) \\
g^\sharp & \longmapsto & (g^\sharp \circ Fh)
\end{array}
$$

Now by letting $h = 1_c$, $c' = c$, we get

$$
\left(
\begin{tikzcd}
Fc \arrow[r, "f^\sharp"] \arrow[dr, "g^\sharp"'] & d \arrow[d, "k"] \\
 & d'
\end{tikzcd}
\iff
\begin{tikzcd}
c \arrow[r, "f^\flat"] \arrow[dr, "g^\flat"'] & Gd \arrow[d, "Gk"] \\
 & Gd'
\end{tikzcd}
\right)
\Rightarrow (k \circ f^\sharp)^\flat = Gk \circ f^\flat
$$

Also, by letting $k = 1_d$, $d' = d$, we get

$$
\left(
\begin{tikzcd}
Fc \arrow[d, "Fh"'] \arrow[dr, "f^\sharp"] & \\
Fc' \arrow[r, "g^\sharp"'] & d
\end{tikzcd}
\iff
\begin{tikzcd}
c \arrow[d, "h"'] \arrow[dr, "f^\flat"] & \\
c' \arrow[r, "g^\flat"'] & Gd'
\end{tikzcd}
\right)
\Rightarrow (g^\sharp \circ Fh)^\flat = g^\flat \circ h
$$

Thus, the above two (naturality) diagrams commute. ∎

Notation 4.1.5. $F \dashv G$: F is left adjoint of G
$G \vdash F$ (and G is right adj of F)

$$\begin{tikzcd} C \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & D \arrow[l, shift left=1.5, "G"] \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, shift left=1.5, "F"'] \arrow[r, phantom, "\top"] & D \arrow[l, shift left=1.5, "G"'] \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, shift left=1.5, "G"'] \arrow[r, phantom, "\perp"] & C \arrow[l, shift left=1.5, "F"'] \end{tikzcd}$$

. . . .

Ex 4.1.6.

$$\begin{tikzcd} \mathrm{Top} \arrow[r, "U"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\top"] & \mathrm{Set} \arrow[l, bend right, "D"'] \arrow[l, bend left, "I"] \end{tikzcd}$$

U : forgetful functor.

D : give discrete topology on given set

I : " in "

This is because,

① $\forall f : S \longrightarrow U(T)$ a function, $f : D(S) \longrightarrow T$ is cts

pf) every subset of S is open in $D(S)$

② $\forall f : U(T) \longrightarrow S$ a function, $f : T \longrightarrow I(S)$ is cts.

pf) $I(S)$ has only two open sets, $\{S, \varnothing\}$

and $f^{-1}(S) = T$, $f^{-1}(\varnothing) = \varnothing$.

Thus, $\mathrm{Top}(D(S), T) \cong \mathrm{Set}(S, U(T))$

$\mathrm{Set}(U(T), S) \cong \mathrm{Top}(T, I(S))$

Ex) Adjunction between preorders = Galois connection.

If $F \dashv G$ where dom F ; cod F are preorders then F, G are order preserving function, thus, $Fa \leq b \iff a \leq Gb$.

So in this context, we call F <u>lower adjoint</u>
G <u>upper adjoint</u>

Ex 4.1.7.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook, "\perp"] \arrow[r, bend right=50, "\lfloor\ \rfloor \text{ floor}"'] \arrow[r, bend right=25, phantom, "\perp"] & \mathbb{R} \arrow[l, bend right=50, "\lceil\ \rceil \text{ ceiling}"'] \end{tikzcd}$$

since $\forall n \in \mathbb{N}, \forall r \in \mathbb{R}$
$n \leq \lfloor r \rfloor \iff n \leq r \iff n \leq \lceil r \rceil$

ex) $\lfloor 3.4 \rfloor = 3$
$\lceil 3.4 \rceil = 4$.

Ex 4.1.8 Let $A, B \in$ Set. PA, PB a powerset of A and B resp, are poset.

If $f : A \to B$, then $f^{-1} : PB \to PA$, $b \mapsto f^{-1}(b)$
$f_* : PA \to PB$, $a \mapsto f(a)$ exists.

Let $f_! : PA \longrightarrow PB$
$a \longmapsto \{ b \in B : f^{-1}(b) \subseteq a \}$

then notes that $\forall A' \subset A,\ B' \subseteq B$,

$$f_!(A') \subseteq B' \iff A' \subseteq f^{-1}(B') \iff f(A') \subseteq B'$$

pf) Second $\iff$ is clear. For the first one,

$f_!(A') \subseteq B' \Rightarrow \forall a \in A', f(a) \in B' \Rightarrow A' \subseteq f^{-1}(B')$

$A' \subseteq f^{-1}(B') \Rightarrow f_!(A') \subseteq f_!(f^{-1}(B')) = \{ b \in B : f^{-1}(b) \subseteq f^{-1}(B') \}$
$= B'$

since if $b \notin B' \Rightarrow f^{-1}(b) \cap f^{-1}(B') = \emptyset$.

$$\Rightarrow \quad \begin{tikzcd} PB \arrow[r, "f^{-1}"] \arrow[r, bend left=40, phantom, "\perp"] \arrow[r, bend right=40, phantom, "\perp"'] & PA \arrow[l, bend right=50, "f_*"'] \arrow[l, bend left=50, "f_!"] \end{tikzcd}$$

Ex. 4.1.9. Propositional function : $P : X \longrightarrow \Omega = \{\perp, T\}$

Let $(\Omega, \leq)$: $\perp \leq T$. Then, we define partial order on Ω^X, s.t. $P \leq Q \iff P(x) \leq Q(x) \ \forall x \in X$

$$\iff (P \Rightarrow Q)$$

by identifying T : True
$\perp$: False.

Now, universal and existential quantifier defines function $\forall_X, \exists_X : \Omega^X \rightrightarrows \Omega$

s.t. $\forall_X P = \begin{cases} T & \text{if} \quad P(x) = T \ \forall x \in X \\ \perp & \text{if} \quad P(x) = \perp \text{ for some } x \in X \end{cases}$

$\exists_x P = \begin{cases} T & \text{if} \quad \exists x \in X \text{ s.t. } P(x) = T \\ \perp & \text{if} \quad \forall x \in X,\ P(x) = \perp \end{cases}$

Now define $\Delta_X : \Omega \longrightarrow \Omega^X$

$\perp \longmapsto \Delta_X(\perp) : X \longrightarrow \Omega$
$x \longmapsto \perp$

$T \longmapsto \Delta_X(T) : X \longrightarrow \Omega$
$x \longmapsto T$

kind of dummy variable.

Then, for any $y \in \Omega$, $P \in \Omega^X$

① $\exists_x P \leq y \iff P \leq \Delta_x(y)$: pf) If y = True = T, trivially hold.
If y = False, $\exists_x P = \perp$
$\iff \forall x \in X\ P(x) = \perp$

② $\Delta_x(y) \leq P \iff y \leq \forall_x P$

pf) : y = True, $\Rightarrow P = \Delta_x(y)$; $\Rightarrow y \leq \forall_x P$.
y = False $\Rightarrow$ P : any function $\Rightarrow y \leq \forall_x P$ = False or True. done.

Thus, $\Omega \underset{\forall_x}{\overset{\exists_x}{\rightleftarrows}} \Omega^X$ with $\Delta_X : \Omega \to \Omega^X$, $\exists_x \dashv \Delta_X \dashv \forall_x$

Ex. 4.1.10. "free ⊣ forgetful"

$$\begin{tikzcd} A \arrow[r, shift left, "U"] \arrow[r, phantom, "\top"] & S \arrow[l, shift left, "F"] \end{tikzcd}$$

U: forgetful (right adj)
F: free (left adj)

"free" is used for "universal" in this particular context.

"cofree" construction: Construction via right adjoint of forgetful functor.

(i) $U: Set_* \to Set$.

$F: Set \to Set_*$ by $X \mapsto X_+ := X \sqcup \{x\}$.

Since for $(X, \{x\})$, Y $: U(X) \to Y$ induces $(X,\{x\}) \to Y_+$ and vice versa.

(ii) $U: Monoid \to Set$
$F: Set \to Monoid$ $\coprod_{n \geq 0} X^{\times n}$.) Seen it Ex in Ch1.

(iii) $U: Ring \to Ab$.
$F: Ab \to Ring$. $\bigoplus_{n \geq 0} A^{\otimes n}$, tensor algebra

$f^\#: A \to U(R) \iff f^\flat: \bigoplus_{n \geq 0} A^{\otimes n} \to R$

pf) $f^\flat_n(a_1 \otimes \cdots \otimes a_n) = f^\#(a_1) \cdots f^\#(a_n)$ and extend this.
since direct sum, this $f^\flat$ is well-defined ring homo.
ul

(iv) $U: Ab \to Set$ F:
$F: Set \to Ab$. by $X \mapsto \mathbb{Z}[X] := \bigoplus_X \mathbb{Z}$.

(v) $U: Mod_R \to Set$
$F: Set \to Mod_R$ by $X \mapsto R[X] := \bigoplus_X R$

If $R = v.s.$, then it gives free R-vector space.

(vi) U: Ring $\longrightarrow$ Set as Ring $\to$ Ab $\to$ Set

F: Set $\longrightarrow$ Ring by Set $\to$ Ab $\to$ Ring

$$X \longmapsto \bigoplus_X \mathbb{Z} \longmapsto \bigoplus_{n\geq 0} \left(\bigoplus_X \mathbb{Z}\right)^{\otimes n}$$

vii) $(-)^\times$: Ring $\longrightarrow$ Group by $R \longmapsto R^\times = \{\text{unit in } R\}$

F: Group $\longrightarrow$ Ring by $G \longmapsto \mathbb{Z}[G]$

the group ring.

$f^\sharp: R^\times \longrightarrow G \iff f^\flat: R \to \mathbb{Z}[G]$

[since $f^\flat$ map unit to $g \in G$, otherwise it is not a homomorphism.]

viii) U: Group $\longrightarrow$ Set

F: Set $\longrightarrow$ Group by $X \longmapsto F(X)$

set of words

free gp over X

(ix) U: Ab $\hookrightarrow$ CMonoid

Gr: CMonoid $\hookrightarrow$ Ab by $(M,+,0) \longmapsto Gr(M,+,0)$

where $Gr(M,+,0) = M \times M / (a,b) \simeq (a',b')$

iff $\exists c \in M$ s.t.

$a + b' + c = a' + b + c$.

So Gr: left adjoint

x) U: Group $\longrightarrow$ Monoid.

Gr: Monoid $\longrightarrow$ Group.

(Universal enveloping group).

$(X,+,0)$

$\longmapsto$ Free gp on

$\{\underline{x} : x \in X\}$

modulo $\underline{0} = 0$, $\underline{x+y} = \underline{x} + \underline{y}$

(xi). $\phi^* : \mathrm{Mod}_S \to \mathrm{Mod}_R$ from $\phi : R \to S$ ring homo.

left adjoint: $S \otimes_R - : \mathrm{Mod}_R \to \mathrm{Mod}_S$

is called extension of scalars

i.e. $\mathrm{Hom}_{\mathrm{Mod}_S}(S \otimes_R M, N) \cong \mathrm{Hom}_{\mathrm{Mod}_R}(M, \phi^*(N))$

(xii). $U : \mathrm{Mod}_R \to \mathrm{Ab}$. $\cong \phi^*$, $\phi : \mathbb{Z} \to R$

(xiii). $\phi^* : \mathrm{Set}^{BG} \to \mathrm{Set}^{BH}$ induced by $\phi : H \to G$ gp homo.

left adjoint: induction, say F

Then, for any $X \in \mathrm{Set}^{BG}$, $Y \in \mathrm{Set}^{BH}$,

$$\mathrm{Hom}_{\mathrm{Set}^{BG}}(FY, X) \cong \mathrm{Hom}_{\mathrm{Set}^{BH}}(Y, \phi^*(X))$$

Thus, the equivariant set map $Y \to \phi^*(X)$ has bijection on $FY \to X$.

$$FY = G \times_H Y = G \times Y \big/ (g\phi(h), y) \sim (g, h.y).$$

All these free forgetful functor is example of monadic adjunction. § 5.5.

Ex 4.1.11. (Frobenius reciprocity)

Ex 4.1.10 is denoted as $A \underset{U}{\overset{F}{\rightleftarrows}} S$ ($F \dashv U$) where "free $\dashv$ forgetful". This can be generalized as follow.

Ex 6.2.8 says that, C: complete and cocomplete
$\phi: H \to G$ gp homo $\Rightarrow$ $\phi^*: C^{BG} \to C^{BH}$
admits both right / left adjoint.

So, we have.

$$\begin{tikzcd}
C^{BG} \arrow[rr, "\mathrm{res}_H^G"] \arrow[rr, phantom, bend left=25, "\perp"] \arrow[rr, phantom, bend right=25, "\perp"] & & C^{BH} \arrow[ll, bend right=50, "\mathrm{ind}_H^G"'] \arrow[ll, bend left=50, "\mathrm{coin}_H^G"]
\end{tikzcd}$$

If let $C = \mathrm{Vect}_K$, then Vect_K^{BH}: Category of H-representation
Vect_K^{BG}: Category of G-representation

$\Rightarrow$ $\mathrm{ind}_H^G \dashv \mathrm{res}_H^G$ is called "Frobenius reciprocity".

Ex 4.1.12. (Nonexample)
$\mathrm{Field} \xrightarrow{U} \mathrm{Ring}$, $\mathrm{Field} \xrightarrow{U} \mathrm{Ab}$, $\mathrm{Field} \xrightarrow{(-)^\times} \mathrm{Ab}$, $\mathrm{Field} \xrightarrow{U} \mathrm{Set}$
does not admit left / right adjoint.

pf). In Ring, Ab, Set, all forgetful field has a map from it to $\mathbb{Z}$.
But any field does not admit a morphism with a field with distinct characteristic.

Ex 4.1.13. $U: \mathsf{Cat} \to \mathsf{DirGraph}$. ~~admits~~ a

$F: \mathsf{DirGraph} \to \mathsf{Cat}$

$(E, V) \longmapsto$ a free category over E, V.

Object $= V$.

Mor : E.

Composition : Concatenation of path.

Then, $F(G) \to C \quad \overset{\sim}{\longleftrightarrow} \quad G \to U(C)$.
naturally bijective

since $F(G) \to C$ defines a diagram in C with no commutativity requirements

$\Rightarrow$ determines $G \to U(C)$.

All edges in G form an <u>atomic</u> arrow admitting no nontrivial factorization.

Ex 4.1.14. $\mathbb{n+1} = 0 \to 1 \to 2 \to \cdots \to n$.
freely generated by above.

Then $d^i: \mathbb{n} \to \mathbb{n+1}$ a canonical injective functor.

$$l \longmapsto \begin{pmatrix} l & \text{if} & l < i \\ l+1 & \text{if} & l \geq i \end{pmatrix}$$

$$s^i: \mathbb{n+1} \to \mathbb{n}$$

$$l \longmapsto \begin{pmatrix} l & \text{if} & l \leq i \\ l-1 & \text{if} & l > i \end{pmatrix},$$

So $i \in \mathbb{n}$ has two object in its preimage.

These defines a sequence of $2n+1$ adjoints

$$d^n \dashv s^{n-1} \dashv d^{n-1} \dashv s^{n-2} \dashv \cdots \dashv s^1 \dashv d^1 \dashv s^0 \dashv d^0.$$

ex) $\mathrm{Hom}_{n+1}(d^n(i), j) \cong \mathrm{Hom}_n(i, s^{n-1}(j))$.

If $i, j \le n-1$, done.

$i = n-1,\ j = n \Rightarrow \mathrm{Hom}_{n+1}(n-1, n) \cong \mathrm{Hom}_n(n-1, n-1)$

$$n-1 \to n \longmapsto 1_{n-1}$$

$i = n-1,\ j = n+1 \Rightarrow \mathrm{Hom}_{n+1}(n-1, n+1) \cong \mathrm{Hom}_n(n-1, n)$

$$\underset{n-1}{\cdot} \to \cdot \to \underset{n+1}{\cdot} \longmapsto \underset{n-1}{\cdot} \to \underset{n}{\cdot}$$

$i = n \quad j = n+1 \Rightarrow \mathrm{Hom}_{n+1}(n+1, n+1) \cong \mathrm{Hom}_n(n, n)$

$$1_{n+1} \longmapsto 1_n$$

$\Rightarrow \exists$ Arbitrary long finite sequence of adj functors

Ex 4.1.15

$$\begin{tikzcd} \text{Groupoid} \arrow[r, hook] & \text{Cat} \arrow[l, bend right, "\perp"'] \arrow[l, bend left, "\perp"] \end{tikzcd}$$

Right adjoint $\mathrm{sub}: C \longmapsto$ maximal subgroupoid in Lem 1.1.13

$\mathrm{Cat}(G, C) \cong \mathrm{Groupoid}(G, \mathrm{sub}(C))$

since functor preserves iso.

Left adjoint: Category of fraction of C.

Obj $:= \mathrm{ob}\, C$.

Mor: $\{$finite zig-zag morphisms in $C\} / \sim$.

$$\boxed{1_a \sim a \xrightarrow{f} b \xleftarrow{f} a \sim b \xleftarrow{f} a \xrightarrow{f} b}$$

Ex 4.1.i) In the proof of Lem 4.13 of this lecture notes.

Ex. 4.1. ii)

(i) ob : Cat $\longrightarrow$ Set

L: Set $\longrightarrow$ Cat

$X \longmapsto X$ as a discrete category.

Then, $\mathrm{Cat}(L(X), Y) \cong \mathrm{Set}(X, \mathrm{ob}Y)$

since any functor $L(X) \longrightarrow Y$ is uniquely defined by a set map $\mathrm{ob}(L(X)) = X \longrightarrow \mathrm{ob}Y$ (since there is no morphism.) and vice versa.

R: Set $\longrightarrow$ Cat.

$X \longmapsto X$ as indiscrete category.

i.e. $\forall (x, y) \in X$

$\mathrm{Mor}\,X \ni x \to y$

and any composition $x \to y \to z$ is identified as $x \to z$.

I.e. $|X(x, z)| = 1 \quad \forall x, z \in X$.

Then, $\mathrm{Set}(\mathrm{ob}Y, X) \cong \mathrm{Cat}(Y, R(X))$

since for any set map $f : \mathrm{ob}Y \to X$, we uniquely extend this to functor by $\forall y_1 \to y_2 \in \mathrm{Mor}\,Y$, define $f(y_1 \to y_2) := f(y_1) \to f(y_2)$ which is uniquely in $X(f(y_1), f(y_2))$. Thus, any composition is preserved and $1_y \longmapsto 1_{f(y)}$. So natural.

(ii). $\mathrm{Vert} : \mathrm{Graph} \longrightarrow \mathrm{Set}$

$(V, E) \longmapsto V$.

$L : \mathrm{Set} \longrightarrow \mathrm{Graph}$

s.t. $\mathrm{Graph}(LX, Y) \cong \mathrm{Set}(X, \mathrm{Vert}(Y))$

where ~~x~~ $L(X)$ is ~~a~~ discrete graph, i.e. $E = \phi$.

~~q~~ $R : \mathrm{Set} \longrightarrow \mathrm{Graph}$

$X \longmapsto (X, E)$ where $(X, E) = C_X$, complete graph over vertices X.

Then, $\mathrm{Set}(\mathrm{Vert}(Y), X) \cong \mathrm{Graph}(Y, R(X))$

since if $f : \mathrm{Vert}(Y) \longrightarrow X$ exists as a set morphism, then, for any edge, e, $f(e)$ has incidence in $R(X)$ since $R(X)$ is complete, and vice versa.

iii). $\mathrm{Vert} : \mathrm{DirGraph} \longrightarrow \mathrm{Set}$

$(V, E) \longmapsto V$.

L : discrete graph, as usual

R : Indiscrete graph with $\forall v_1, v_2 \in V$, $v_1 \to v_2$, $v_2 \to v_1$ exists in the edge.

By the same argument as above,

$L \dashv \mathrm{Vert} \dashv R$.

Ex 4.1. iii) Let $c, c' \in \mathrm{Ob}\,C$.

It suffices to show that

$$C(LUc, c') \cong C(c, RUc')$$

From adjointness, $C(LUc, d) \cong D(Uc, Ud) \cong C(c, RUd)$

Since each iso is natural iso, thus their composition is natural iso. $\Rightarrow$ $LU \dashv RU$.

Ex. 4.1. iv) WTS $R[X] := \bigoplus_X R$ is functorial.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be set maps.

Then, $R[f] : \bigoplus_X R \longrightarrow \bigoplus_Y R$ by

$$(a_x)_{x \in R} \longmapsto (b_y)_{y \in Y}$$

where $b_y = a_x$ if $y = f(x)$.

Thus, $R[g] \circ R[f]$ sends (a_x) to $(c_z)_{z \in Z}$ with

$$c_z = a_x \text{ if } z = gf(x).$$

$\Rightarrow$ $R[g] \circ R[f] = R[g \circ f]$.

Also, $X \xrightarrow{1_X} X$ induces $R[1_X] = I_{\bigoplus_X R}$

$\therefore$ $R[-]$ is functor.

$D \downarrow 1_D$

$F : C \longrightarrow D \quad G : D \Rightarrow C$

Ex 4.1. v) $F \downarrow 1_D$

Obj : $(c \in C,\ d \in D,\ f^{\#} : Fc \to d \in D)$

Mor : $(c, d, f^{\#} : Fc \to d) \longrightarrow (c', d', f'^{\#} : Fc' \to d')$

is $(h : c \to c',\ k : d \to d')$ s.t.

$$\begin{tikzcd} Fc \arrow[r, "f^{\#}"] \arrow[d, "Fh"'] \arrow[dr, phantom, "?"] & d \arrow[d, "k"] \\ Fc' \arrow[r, "f'^{\#}"'] & d' \end{tikzcd}$$

$I_C \downarrow G$: Obj $(c \in C, d \in D : f^\flat : c \to Gd \in C)$.

Mor : $(c, d, f^\flat : c \to Gd) \longrightarrow (c', d', f'^\flat : c' \to Gd')$

is $(h : c \to c', k : d \to d')$ s.t.

$$\begin{tikzcd} c \arrow[r, "f^\flat"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & Gd \arrow[d, "k"] \\ c' \arrow[r, "f'^\flat"'] & d' \end{tikzcd}$$

Obviously, if $F \dashv G$, then, take a functor.

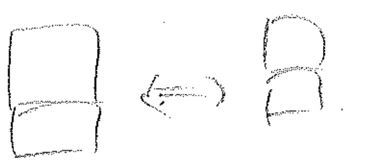
$H : F \downarrow 1_D \longrightarrow I_C \downarrow G$

$$\begin{tikzcd} (c, d, f^\sharp : Fc \to d) \arrow[r, mapsto] \arrow[d, "(h,k)"'] & (c, d, f^\flat : c \to Gd) \arrow[d, "(h,k)"] \\ {} \arrow[r, phantom, "\longmapsto"] & {} \\ (c', d', f'^\sharp : Fc' \to d') \arrow[r, mapsto] & (c', d', f'^\flat : c' \to Gd') \end{tikzcd}$$

This is well defined map for (h, k), since

$$\begin{tikzcd} Fc \arrow[r, "f^\sharp"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\circlearrowright"] & d \arrow[d, "k"] \\ Fc' \arrow[r, "(f')^\sharp"'] & d' \end{tikzcd} \Leftrightarrow \begin{tikzcd} c \arrow[r, "f^\flat"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & Gd \arrow[d, "Gk"] \\ c' \arrow[r, "(f')^\flat"'] & Gd' \end{tikzcd}$$ by Lemma 4.1.3.

Moreover, $(1_c, 1_d) \longmapsto (1_c, 1_d)$ holds by applying and compositionality is also holds by applying Lem 4.1.3 twice on



$\Rightarrow$ H is well-defined bijection $\Rightarrow$ Iso of category.

4.2 Unit, Counit as universal arrow.

Recall. $C \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} D$ with $D(Fc, d) \cong C(c, Gd)$

Since iso is natural, $D(Fc, -) \cong C(c, D-)$.

By Yoneda lemma, ″

is determined by an element of $C(c, GFc)$.

Lem 4.2.2 Given $F \dashv G$, $\exists \eta : 1_C \Rightarrow GF$

which is called <u>unit</u> of the adjunction

s.t. $\eta_c : c \longrightarrow GFc$ at c is defined to be

the transpose of 1_{Fc}, i.e., if $1_{Fc} = f^\sharp$ then $\eta_c = f^\flat$.

pf) To see η is natural, it suffices to

show that $\forall f : c \longrightarrow c' \in C$, left square commutes.

But Lemma 4.1.3 shows that

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & GFc \arrow[d, "GFf"] \\ c' \arrow[r, "\eta_{c'}"'] & GFc' \end{tikzcd} \iff \begin{tikzcd} Fc \arrow[r, "1_{Fc}"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "Ff"] \\ Fc' \arrow[r, "1_{Fc'}"'] & Fc' \end{tikzcd}$$

and right one definitely commutes, done!

Dually, $C(-, Gd) \cong D(F-, d)$ and Yoneda Lemma

says ″ is determined by

an element of $D(FGd, d)$, which is transpose of 1_{Gd},

say ε_d. By duality, $\varepsilon := \{\varepsilon_d : d \in D\}$ is a

natural transf $\varepsilon : FG \Rightarrow 1_D$.

Lemma 4.2.3. ε is called <u>counit</u>.

Ex 4.2.4. $F: \mathrm{Set} \longrightarrow \mathrm{Gp}$ free group

$\perp$

$U: \mathrm{Gp} \longrightarrow \mathrm{Set}$ forgetful functor.

We know $F(S)$ for $S \in \mathrm{Set}$ is a free gp on a set S.

Then, $\eta_S : S \longrightarrow UF(S)$

$x \longmapsto x$ as a singleton word.

$\varepsilon_G : FU(G) \longrightarrow G$.

$g_1 g_2 \cdots \longmapsto g_1 \cdot g_2 \cdots$

as a word — as a product in G.

So η: give singleton. ε: evaluation.

in some sense.

By Def 2.3.3 (Universal element)

each natural iso

$$\mathrm{Group}(F(S), -) \cong \mathrm{Set}(S, U(-))$$

corresponds to $\eta_S \in \mathrm{Set}(S, UF(S))$ and

$$\mathrm{Set}(-, U(G)) \cong \mathrm{Group}(F(-), G)$$

corresponds to $\varepsilon_G \in \mathrm{Group}(FU(G), G)$

By Prop 2.4.8. (Universal element $\Rightarrow$ Initial or terminal in $\int F$)

there is a certain duality. $\Rightarrow$ See Ex 4.2.10.

Conversely $F : C \rightleftarrows D : G$ with $\eta : 1_C \Rightarrow GF$, $\varepsilon : FG \Rightarrow 1_D$

defines adjunction.

Def. 4.2.5. (Adjunction) : $F : C \rightleftarrows D$ with η, ε above satisfying <u>triangle identities</u>.

$$
\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon F"] \\
 & F
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
G \arrow[r, Rightarrow, "\eta G"] \arrow[dr, Rightarrow, "1_G"'] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\
 & G
\end{tikzcd}
$$

Commutes in D^C $\qquad$ Commutes in C^D.

$F\eta$, εF, ηG, $G\varepsilon$: defined by whiskering (Ex 1.7.ii)

So,
$$F\eta := \{ (F\eta)_c = F\eta_c \quad \forall c \in C \}$$
$$\varepsilon F := \{ (\varepsilon F)_c = \varepsilon_{Fc} \quad \forall c \in C \}$$
$$\eta G := \{ (\eta G)_d = \eta_{Gd} \quad \forall d \in D \}$$
$$G\varepsilon := \{ (G\varepsilon)_d = G\varepsilon_d \quad \forall d \in D \}$$

This shows that "the counit is a left inverse of the unit <u>modulo translation</u>" (Not exact inverse) in the sense that $(F\eta_c)^{-1} \Rightarrow \varepsilon_{Fc}$ as left inverse

$(G\varepsilon_D)^{-1} \Rightarrow \eta_{Gd}$ as right " .

Prop 4.2.6. $F: C \rightleftarrows D : G$, $\exists$ natural iso $D(Fc, d) \cong C(c, Gd)$

$\Leftrightarrow$ $\exists \eta : 1_C \Rightarrow GF$, $\varepsilon : FG \Rightarrow 1_D$ satisfying

the triangle identity

i.e. second definition is well-defined.

pf) Lem 4.2.2, 4.2.3 with their dual shows

$\exists \eta, \varepsilon$. So it suffices to show that they satisfy

the triangle id.

By Lemma 4.1.3,

$$\begin{tikzcd} Fc \arrow[r, "1_{Fc}"] \arrow[d, "F\eta_c"'] \arrow[dr, phantom, "\circlearrowleft"] & Fc \arrow[d, "1_{Fc}"] \\ FGFc \arrow[r, "\varepsilon_{Fc}"'] & Fc \end{tikzcd} \iff \begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[d, "\eta_c"'] \arrow[dr, phantom, "\circlearrowleft"] & GFc \arrow[d, "1_{GFc}"] \\ GFc \arrow[r, "1_{GFc}"'] & GFc \end{tikzcd}$$

Since transpose of $\eta_c = 1_{Fc}$. By the same observation,

" $\varepsilon_{Fc} = 1_{GFc}$

Also,

$$\begin{tikzcd} FGd \arrow[r, "1_{FGd}"] \arrow[d, "1_{FGd}"'] \arrow[dr, phantom, "\circlearrowleft"] & FGd \arrow[d, "\varepsilon_d"] \\ FGd \arrow[r, "\varepsilon_d"'] & d \end{tikzcd} \iff \begin{tikzcd} Gd \arrow[r, "\eta_{Gd}"] \arrow[d, "1_{Gd}"'] \arrow[dr, phantom, "\circlearrowleft"] & GFGd \arrow[d, "G\varepsilon_d"] \\ Gd \arrow[r, "1_{Gd}"'] & Gd \end{tikzcd}$$

By Lemma 1.7.1 of vertical composition, $\circlearrowleft$ $=$ $\circlearrowright$

$\Rightarrow$ ε, η satisfies the triangle identity.

Conversely, for given $f^\sharp : Fc \to d$, $g^\flat : c \to Gd$,

define their adjunct as below.

$$f^{b} := c \xrightarrow{\eta_c} GFc \xrightarrow{Gf^{\#}} Gd,$$

$$g^{\#} := Fc \xrightarrow{Fg^{b}} FGd \xrightarrow{\varepsilon_d} d.$$

Then, $$(f^{b})^{\#} = Fc \xrightarrow{Ff^{b}} FGd \xrightarrow{\varepsilon_d} d$$

$$= Fc \xrightarrow{F\eta_c} FGFc \xrightarrow{FGf^{\#}} FGd \xrightarrow{\varepsilon_d} d.$$

Since
$$\begin{tikzcd}
FGd \arrow[r, "\varepsilon_d"] & d \\
FGFc \arrow[u, "FGf^{\#}"] \arrow[r, "\varepsilon_{Fc}"'] \arrow[ur, phantom, "\circlearrowright"] & Fc \arrow[u, "f^{\#}"']
\end{tikzcd}$$
from naturality of ε,

$$= Fc \xrightarrow{F\eta_c} FGFc \xrightarrow{\varepsilon_{Fc}} Fc \xrightarrow{f^{\#}} d.$$

And by the triangle identity, $\varepsilon_{Fc} \circ F\eta_c = 1_{Fc}$.

$$= f^{\#}.$$

And, $$(g^{\#})^{b} = c \xrightarrow{\eta_c} GFc \xrightarrow{Gg^{\#}} Gd$$

$$= c \xrightarrow{\eta_c} GFc \xrightarrow{GFg^{b}} GFGd \xrightarrow{G\varepsilon_d} Gd.$$

Since
$$\begin{tikzcd}
c \arrow[r, "\eta_c"] \arrow[d, "g^{b}"'] \arrow[dr, phantom, "\circlearrowright"] & GFc \arrow[d, "GF(g^{b})"] \\
Gd \arrow[r, "\eta_{Gd}"'] & GFGd
\end{tikzcd}$$
from naturality of η,

$$= c \xrightarrow{g^{b}} Gd \xrightarrow{\eta_{Gd}} GFGd \xrightarrow{G\varepsilon_d} Gd$$

by the triangle identity, $G\varepsilon_d \circ \eta_{Gd} = 1_{Gd}$

$$= g^{b}.$$

So they are inverse to each other. ∎

Rmk 4.2.7. Prop 4.2.6 gives the data of fully specified adjunction. So, $F : C \rightleftarrows D : G$ is adjunction

iff (i) Natural iso $D(Fc, d) \cong C(c, Gd)$ $\forall c \in C, \forall d \in D$.

$\Leftrightarrow$ (ii) $\eta : 1_C \Rightarrow GF$, $\varepsilon : FG \Rightarrow 1_D$ s.t. $G\varepsilon \cdot \eta G = 1_G$, $\varepsilon F \cdot F\eta = 1_F$.

By Yoneda lemma and Prop 4.2.6, each η, ε defines the universal property so.

$\Leftrightarrow$ (iii) $\eta : 1_C \Rightarrow GF$ s.t.

$$D(Fc, d) \xrightarrow{G} C(GFc, Gd) \xrightarrow{(\eta_c)^*} C(c, Gd)$$

is an iso $\forall c \in C, d \in D$

dual $\Leftrightarrow$ (iv) $\varepsilon : FG \Rightarrow 1_D$ s.t.

$$C(c, Gd) \xrightarrow{F} D(Fc, FGd) \xrightarrow{(\varepsilon_d)_*} D(Fc, d)$$

is an iso $\forall c \in C, d \in D$

Corollary 4.2.8. A, B poset, $F : A \to B$, $G : B \to A$ form a Galois connection with $F \dashv G$.

$\Rightarrow$ $FGF = F$, $GFG = G$

pf) Adjunction between preorder = Galois connection

$\Leftrightarrow$ $Fa \leq b \Leftrightarrow a \leq Gb$.

By triangle identity, $\underset{F\eta}{Fa \leq} FGFa \underset{\varepsilon F}{\leq} Fa$.

$$\underset{\eta G}{Ga \leq} GFGa \underset{G\varepsilon}{\leq} Ga$$

$\boxtimes$

Ex). $$\begin{tikzcd} PB \arrow[r, shift right, "f^{-1}"'] & PA \arrow[l, shift right, "f_*"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$$, the above corollary gives

$$f(X) = f(f^{-1}(f(X))) \quad , \quad f^{-1}(f(f^{-1}(Y))) = f^{-1}(Y).$$

(Maybe contraction and extension of ideals...?)

Ex 4.2.i) Let such subcategory $\tilde{C}$, $\tilde{D}$.

Then, $\tilde{\eta} : 1_{\tilde{C}} \Rightarrow \tilde{G}\tilde{F}$ and $\tilde{\varepsilon} : \tilde{F}\tilde{G} \Rightarrow 1_{\tilde{D}}$

are still unit and counit of $\tilde{F}$, $\tilde{G}$, restriction of F and G, and they are natural iso

$\Rightarrow$ $\tilde{C} \cong \tilde{D}$ equiv of category.

Ex 4.2.ii) Did it in the lecture note.

Ex 4.2.iii) $U : \mathrm{Mod}_R \longrightarrow \mathrm{Set}$ $\Rightarrow F \dashv U$.

$$F : \mathrm{Set} \longrightarrow \mathrm{Mod}_R$$
$$X \longmapsto R[X] := \bigoplus_X R$$

thus, $\eta : 1_C \Rightarrow UF$ s.t. for given $f : X \to Y$

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & UFX \arrow[d, "UF(f)"] \\ Y \arrow[r, "\eta_Y"'] & UFY \end{tikzcd} \iff \begin{tikzcd} FX \arrow[r, "1_{FX}"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowright"] & FX \arrow[d, "Ff"] \\ FY \arrow[r, "1_{FY}"'] & FY \end{tikzcd}$$

as we've shown, η_X = sending element x to $(a_{x'})_{x' \in X}$ where $a_{x'} = \begin{cases} 1_R & \text{if } x' = x \\ 0_R & \text{o.w.} \end{cases}$

Conversely,
$$\begin{tikzcd} FU(M) \arrow[r, "\varepsilon_M"] \arrow[d, "FU(f)"'] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[d, "f"] \\ FU(N) \arrow[r, "\varepsilon_N"'] & N \end{tikzcd} \iff \begin{tikzcd} U(M) \arrow[r, "I_{U(M)}"] \arrow[d, "U(f)"'] \arrow[dr, phantom, "\circlearrowright"] & U(M) \arrow[d, "U(f)"] \\ U(M) \arrow[r] & U(N) \end{tikzcd}$$

for any $f: M \to N$, thus, $\varepsilon_M : (a_m)_{m\in M} \mapsto \sum_{m \in M} a_m \cdot m$.

Then,
$$\begin{tikzcd} (a_m)_{m\in M} \arrow[r, mapsto] \arrow[d, mapsto] & \sum_{m\in M} a_m \cdot m \arrow[d] \\ & \sum_{m\in M} a_m f(m) \text{ by homomorphism} \\ (b_n)_{n\in N} \arrow[r, mapsto] & \sum_{n\in N} b_n \cdot n. \end{tikzcd}$$

s.t. $b_n = \begin{cases} a_m & \text{for } n = f(m) \\ 0 & \text{o.w.} \end{cases}$

since $b_n = 0$ if $n \notin \operatorname{Im} f$,
$\sum_{n\in N} b_n \cdot n = \sum_{m\in M} a_m f(m)$.

done

Ex 4.2.iv) Notes that ε_d is the universal element of the functor $C(-, Gd) \cong D(F-, d)$

Thus, ε_d is the terminal object of $\int C(-, Gd) = \int D(F-, d)$

by Prop 2.4.8.

Ex 4.2.v) A <u>morphism of adjunction</u> from $F \dashv G$ to $F' \dashv G'$ = pair of functor H, K s.t.

$$\begin{tikzcd} C \arrow[r, "H"] \arrow[d, shift right, "F"'] \arrow[d, phantom, "\dashv"] & C' \arrow[d, shift right, "F'"'] \arrow[d, phantom, "\dashv"] \\ D \arrow[r, "K"'] \arrow[u, shift right, "G"'] & D' \arrow[u, shift right, "G'"'] \end{tikzcd}$$

s.t. ① left square with left adjoint and right " commutes.

i.e. $KF = F'H$, $HG = G'K$.

and ② following equivalent form.

(i). $H\eta = \eta' H$. for each unit η, η' of F, F'

(ii) $K\varepsilon = \varepsilon' K$ " counit $\varepsilon, \varepsilon'$ of G, G'

iii) $\forall c \in C$, $d \in D$,

$$\begin{tikzcd}
D(Fc, d) \arrow[r, "\cong"] \arrow[d, "K"'] & C(c, Gd) \arrow[d, "H"] \\
D'(KFc, Kd) \arrow[d, equal] \arrow[r, phantom, "\circlearrowright"] & C'(Hc, HGd) \arrow[d, equal] \\
D'(F'Hc, Kd) \arrow[r, "\cong"'] & C'(Hc, G'Kd)
\end{tikzcd}$$

pf)

(i) $\Leftrightarrow$ (iii) By remark 4.2.17, ~~we~~ using def (iii) we know that the iso $D(Fc, d) \xrightarrow{\cong} C(c, Gd)$ can be written by $f \longmapsto Gf \circ \eta_c$

Thus, in the diagram (iii), we have

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & Gf \circ \eta_c \arrow[d, mapsto] \\
Kf \arrow[r, mapsto] & HGf \circ H\eta_c = G'Kf \circ \eta'_{Hc}
\end{tikzcd}$$

since $HGf \circ H\eta_c = HGf \circ \eta'_{Hc}$ by (i) and $G'Kf \circ \eta'_{Hc} = HGf \circ \eta'_{Hc}$ by $G'K = HG$.

Conversely, if (iii) holds, then $HGf \circ H\eta_c = G'Kf \circ \eta'_{Hc}$

$\Rightarrow HGf \circ H\eta_c = HGf \circ \eta'_{Hc}$

By taking $f = 1_{Fc}$, we get $H\eta_c = \eta'_{Hc}$ $\forall c \in C$

$\Rightarrow H\eta = \eta' H$.

(ii) $\Rightarrow$ (iii) Similarly, by Remark 4.2.7, (iv)

$$\begin{array}{ccc} \mathcal{C}(c, Gd) & \xrightarrow{\cong} & \mathcal{D}(Fc, d) \\ g & \longmapsto & \varepsilon_d \circ Fg. \end{array}$$

Thus, the given diagram in (iii) shows that

$$\begin{tikzcd} \varepsilon_d \circ Fg \arrow[d, mapsto] & g \arrow[l, mapsto] \arrow[d, mapsto] \arrow[dl, phantom, "2"] \\ K\varepsilon_d \circ KFg = \varepsilon'_{Kd} \circ F'Hg & Hg \arrow[l, mapsto] \end{tikzcd}$$

Since $K\varepsilon_d = \varepsilon'_{Kd}$ by (ii), $KF = F'H$ by given commutative condition.

For (iii) $\Rightarrow$ (ii) (iii) implies $K\varepsilon_d \circ KFg = \varepsilon'_{Kd} \circ F'Hg$
$= \varepsilon'_{Kd} \circ KFg$

Now take $g := 1_{Gd}$ to get $K\varepsilon_d = \varepsilon'_{Kd}$ $\forall d \in \mathcal{D}$

$\Rightarrow$ $K\varepsilon = \varepsilon' K$.

4.3. Contravariant and multivariate adjoint functor.

Def 4.3.1. $F : C^{op} \longrightarrow D$, $G : D^{op} \longrightarrow C$

are <u>mutually left adjoint</u> if $\exists$ natural iso. on.

$$D(Fc, d) \cong C(Gd, c).$$

| counits : $GF \Rightarrow 1_C$, $FG \Rightarrow 1_D$

are <u>mutually right adjoint</u> if $\exists$ "

$$D(d, Fc) \cong C(c, Gd).$$

| units : $1_C \Rightarrow GF$, $1_D \Rightarrow FG$

And if C, D are preorders, these are called <u>antitone Galois Connection</u>

Ex 4.3.2. $n \in \mathbb{N}$. k: alg. closed field.

$$V : P(k[x_1 \cdots x_n])^{op} \longrightarrow P(k^n)$$

$$S \longmapsto V(S) := \{ (t_1 \cdots t_n) \in k^n : f(t_1 \cdots t_n) = 0 \ \forall f \in S \}$$

$$I : P(k^n)^{op} \longrightarrow P(k[x_1 \cdots x_n])$$

$$T \longmapsto I(T) := \{ f \in k[x_1 \cdots x_n] : f(t) = 0 \ \forall t \in T \}$$

are two functors between the posets.

They are mutually right adjoint since

$$T \subset V(S) \iff S \subset I(T)$$

By unit and counits, we have

$$T \subset V(I(T)) \quad , \quad S \subset I(V(S))$$

By Hilbert's Nullstellensatz, poset of Zariski closed subsets of k^n is iso to opposite of poset of radical ideal

Ex 4.3.3. σ : Signature = a list of function, constant, relation symbol with standard logical symbol.

Axiom_σ : sentences in a first-order language using σ.

Struct_σ : a set of σ-structures

σ-structure : sets with interpretations of given constant, relation, and function symbol.

ex). $\mathbb{N}$: natural number

$\sigma = \{ +, \leq, 0, 1, \cdots \}$

a structure : any set with a specified const, binary relation, and $+$.

one of axiom_σ : ① $\forall x, y, z \ ((x \leq y) \wedge (y \leq z)) \Rightarrow (x \leq z)$

② $\forall x, \ x + 0 = x$

① denotes : transitivity

② " unit of $+$

Let M : a set of σ-structure

A : a set of Axiom

$M \models A$: each axiom in A is satisfied in (is true in) each σ-structure in M.

ex) transitivity is true if its interpretation
$\underset{\text{axiom}}{\wedge}$ regards $\leq$ as transitive one.

Then, we have

$$P(\mathrm{Axiom}_6)^{op} \xrightarrow{\mathrm{True\ in}} P(\mathrm{Struct}_6)$$

$$A \longmapsto M(A)$$, a model which is

$$P(\mathrm{Struct}_6)^{op} \xrightarrow{\mathrm{Satisfying}} P(\mathrm{Axiom}_6)$$

$\Rightarrow$ Galois conn. between syntax and semantics.

Prop 4.3.4. $F: A \to B$ a functor st. $\forall b \in B$
$\exists Gb \in A$ s.t. $(*)\ B(Fa, b) \cong A(a, Gb)$ which
is natural in $a \in A$.

$\Rightarrow$ $\exists!$ unique way to extend $G: \mathrm{ob}B \to \mathrm{ob}A$
to a functor $G: B \to A$ s.t. $(*)$ is natural in
$b \in B$. i.e. $F \dashv G$.

pf) $\forall f: b \to b'$, we have

$$\begin{array}{c} A(a, Gb) \overset{i}{\cong} B(Fa, b) \\ \downarrow f_* \\ A(a, Gb') \underset{j}{\cong} B(Fa, b') \end{array}$$

So, $j \circ f_* \circ i$ defines a natural transformation
$A(-, Gb) \Rightarrow A(-, Gb')$. since $\forall g: a \to a'$,

$$\begin{tikzcd} A(a, Gb) \arrow[r, "i"] \arrow[d, "g_*"] \arrow[dr, phantom, "\circlearrowright"] & B(Fa, b) \arrow[r, "f_*"] \arrow[d, "Fg_*"] \arrow[dr, phantom, "\circlearrowright"] & B(Fa, b') \arrow[r, "j^{-1}"] \arrow[d, "Fg_*"] \arrow[dr, phantom, "\circlearrowright"] & A(a, Gb') \arrow[d, "g_*"] \\ A(a', Gb) \arrow[r, "i"] & B(Fa', b) \arrow[r, "f_*"] & B(Fa', b') \arrow[r, "j^{-1}"] & A(a', Gb') \end{tikzcd}$$

by naturality of i and j.

Thus, by Yoneda lemma, $\exists! Gf \in A(Gb, bb')$

s.t. $(Gf)_* = j \circ f_{1*} \circ i$.

This uniqueness actually shows that this is functorial. (I don't understand comments.)

Prop 4.3.6. $F: A \times B \longrightarrow C$ bifunctor s.t. $\forall a \in A$, $F(a, -): B \longrightarrow C$ admits right adjoint $G_a: C \longrightarrow B$

$\Longrightarrow$ (i). $\exists!\ G: A^{op} \times C \longrightarrow B$ s.t. ① $G(a,c) = G_a(c)$

and ② $C(F(a,b), c) \cong B(b, G(a,c))$

are natural in all three variables.

If furthermore, $\forall b \in B$, $F(-, b): A \to C$ admits a right adjoint $H_b: C \longrightarrow A$, then

$\Rightarrow$ (ii) $\exists!\ H: B^{op} \times C \longrightarrow A$ s.t. $H(b,c) = H_b(c)$

and $C(F(a,b), c) \cong B(b, G(a,c)) \cong A(a, H(b,c))$

are natural in all three variables.

(iii) $\forall c \in C$, $G(-, c): A^{op} \longrightarrow B$, $H(-, c): B^{op} \longrightarrow A$

are mutual right adjoint.

pf)

(i). First of all, define assignment $G: ob(A^{op} \times C) \to ob B$ by $G(a,c) := G_a(c)$.

Then, at least, $G(a, -): C \longrightarrow B = G_a$ is a functor for each $c \in C$.

By Ex.(1) viii, it suffices to show that $\forall f: a' \to a \in A$

$$G(f^{op}, -) : G(a,-) \Rightarrow G(a',-) \quad \text{is defined}$$

functorially in A.

Now from adjunct $F(a,-) \dashv G_a = G(a,-)$,
we have $\forall b \in B$, $c \in C$, $f: a' \to a \in A$,

$$B(b, G(a,c)) \overset{i}{\cong} C(F(a,b), c) \xrightarrow{F(f,b)^*} C(F(a',b), c) \overset{j}{\cong} B(b, G(a',c))$$

Claim 1: $j \circ (F(f,1))^* \circ i$ is a natural transformation
$\varepsilon_{b,f} : B(b, G(a,-)) \Rightarrow B(b, G(a',-))$

Claim 2: $j \circ (F(f,b))^* \circ i$ is a natural transformation
$\eta_{c,f} : B(-, G(a,c)) \Rightarrow B(-, G(a',c))$

Then, by construction, $(\varepsilon_{b,f})_c = (\eta_{c,f})_b$. $\forall c \in C, \forall b \in B$

Also, Yoneda Lemma gives $(\eta_{c,f})_b = ((\eta_{c,f})_{G(a,c)}(1_{G(a,c)}))_*$
a post composition of an element $\phi_f = (\eta_{c,f})_{G(a,c)}(1_{G(a,c)})$.

Then the remaining job is by defining $G(f^{op}, c) = \phi_f$,

claim 3: The defined $G : A^{op} \times C \to B$ is well-defined functor s.t. $C(F(a,b), c) \cong B(b, G(a,c))$ are natural in all three variables.

Claim 1 pf) $\forall g: c \to c' \in C$,

$$\begin{tikzcd}
B(b, G(a,c)) \arrow[r] \arrow[d, "G_a(g)_*"'] \arrow[dr, phantom, "\circlearrowright"] & C(F(a,b), c) \arrow[r, "F(f,b)^*"] \arrow[d, "g_*"] \arrow[dr, phantom, "\circlearrowright"] & C(F(a',b), c) \arrow[r] \arrow[d, "g_*"] \arrow[dr, phantom, "\circlearrowright"] & B(b, G(a',c)) \arrow[d, "G_{a'}(g)_*"] \\
B(b, G(a,c')) \arrow[r] & C(F(a,b), c') \arrow[r, "F(f,b)^*"] & C(F(a',b), c') \arrow[r] & B(b, G(a',c'))
\end{tikzcd}$$

Left and Rights commutes since $F(a,-) \dashv G_a$, $F(a',-) \dashv G_{a'}$

Middle one commutes since the one is pre comp, the other is post comp.

pf of claim 2) $\forall g : b' \to b \in B$,

$$\begin{tikzcd}
B(b, G(a,c)) \arrow[r, "\cong"] \arrow[d, "g^*"] & C(F(a,b),c) \arrow[r, "F(f,b)^*"] \arrow[d, "F(a,g)^*"] & C(F(a',b),c) \arrow[r] \arrow[d, "F(a',g)^*"] & B(b, G(a',c)) \arrow[d, "g^*"] \\
B(b', G(a,c)) \arrow[r, "\cong"'] & C(F(a,b'),c) \arrow[r, "F(f,b')^*"'] & C(F(a',b'),c) \arrow[r] & B(b', G(a',c))
\end{tikzcd}$$

Left, right commutes by adjunction. Middle from functoriality of F $\Rightarrow$ whole diagram commutes.

pf of claim 3)

To see it is a bifunctor, we need to check functoriality. First of all, $G(1_a, 1_c) = G(1_a, c)$

$= (\eta_{c,1_a})_{G(a,c)}(1_{G(a,c)})$ but $\eta_{c,1_a}$ = identity

by construction. $B(b, G(a,c)) \to C(\sim \xrightarrow{1_a} C(\sim \to B(b, G(a,c))$

So, $G(1_a, 1_c) = 1_{G(a,c)}$

Also, we need to check that $\forall f : a' \to a \in A$, $g : c \to c' \in C$,

$G(f^{op}, g) := G(1_a, g) \cdot G(f^{op}, 1_{\mathrm{dom}\, g})$ is well-defined.

$= G(f^{op}, 1_c)\, G(1_{a'}, g)$

Notes that it suffices to show that

$$\begin{tikzcd}
G(a,c) \arrow[r, "G(f^{op}, c)"] \arrow[d, "G_a(g)"'] & G(a',c) \arrow[d, "G_{a'}(g)"] \\
G(a,c') \arrow[r, "G(f^{op}, c')"'] & G(a',c')
\end{tikzcd}$$

As we've defined above,

$$G(f^{op}, c) = \phi_f = (\eta_{c,f})_{G(a,c)}(1_{G(a,c)})$$

$$= (\varepsilon_{G(a,c),f})_c(1_{G(a,c)})$$

Thus, from naturality of ε, we have

$$\begin{tikzcd}
B(G(a,c), G(a,c)) \arrow[r, "(\varepsilon_{G(a,c),f})_c"] \arrow[d, "(G_a(g))_*"'] \arrow[dr, phantom, "\circlearrowright"] & B(G(a,c), G(a',c)) \arrow[d, "(G_{a'}(g))_*"] \\
B(G(a,c), G(a',c')) \arrow[r, "(\varepsilon_{G(a,c),f})_{c'}"'] & B(G(a,c), G(a',c'))
\end{tikzcd}$$

post

$G_{a'}(g) \circ G(f^{op}, c)$

$$\Rightarrow = G_{a'}(g) \circ (\varepsilon_{G(a,c),f})_c(1_{G(a,c)})$$

$$= (\varepsilon_{G(a,c),f})_{c'}(G_a(g)) = (\eta_{c',f})_{G(a,c)}(G_a(g))$$

then, naturality of η,

$$\begin{tikzcd}
B(G(a,c'), G(a,c')) \arrow[r, "(\eta_{c',f})_{G(a,c')}"] \arrow[d, "(G_a(g))_*"'] \arrow[dr, phantom, "\circlearrowright"] & B(G(a,c'), G(a',c')) \arrow[d, "(G_a(g))_*"] \\
B(G(a,c), G(a,c')) \arrow[r, "(\eta_{c',f})_{G(a,c)}"'] & B(G(a,c), G(a',c'))
\end{tikzcd}$$

gives

$$= (\eta_{c',f})_{G(a,c')}(1_{G(a,c')}) \circ G_a(g) = G(f^{op}, c') \circ G_a(g).$$

Thus, $G(f^{op}, g)$ is well-defined.

Now, since we know $G_a(g_1 \cdot g_2) = G_a(g_1) \cdot G_a(g_2)$ by assumption and $G(f_1^{op} \cdot f_2^{op}, c) = G(f_1^{op}, c) \cdot G(f_2^{op}, c)$ by construction of η, since RHS is a map of form

$$B(\sim) \to C(\sim) \xrightarrow{F(f_1,b)^*} \underline{C(\sim) \to B(\sim) \to C(\sim)} \xrightarrow{F(f_2,b)^*} C(\sim) \to \dots$$

and the middle this part is just canceledout.

$$\Rightarrow G(f_1^{op} \cdot f_2^{op}, g_1 g_2) = G(1_c, g_1 g_2) \cdot G(f_1^{op} \cdot f_2^{op}, cod(g_1 g_2))$$

$$= G(1_c, g_1)\, G(1_c, g_2) \cdot G(f_1^{op}, \sim)\, G(f_2^{op}, \sim)$$

$$= \quad '' \quad G(f_1^{op}, \sim)\, G(1_c g_2) \quad ''$$

$$= G(f_1^{op}, g_1) \cdot G(f_2^{op}, g_2)$$

$\therefore$ G is functorial.

So G is well-defined.

To see $C(F(a,b), c) \cong B(b, G(a,c))$ is natural

Claim 1 shows that it is natural for c

" 2 " for b.

by applying them on the identity map 1_a.

So it suffices to show that it is natural for a.

Let $f: a' \to a \in A$. Then it suffices to show that

$$\begin{tikzcd} C(F(a,b), c) \arrow[r, "\cong"] \arrow[d, "F(f,b)^*"'] \arrow[dr, phantom, "\circlearrowright"] & B(b, G(a,c)) \arrow[d, "G(f^{op}, c)_*"] \\ C(F(a',b), c) \arrow[r, "\cong"'] & B(b, G(a',c)) \end{tikzcd}$$

pre post.

Notes that

$$G(f^{op}, c)_* = \left((\eta_{c,f})_{G(a,c)}(1_{G(a,c)})\right)_* = (\eta_{c,f})_b$$

and $(\eta_{c,f})_b = B(b, G(a,c)) \xrightarrow{\cong} C(F(a,b),c) \xrightarrow{F(f,b)^*} C(F(a',b),c) \xrightarrow{\cong} B(b, G(a',c))$

$\Rightarrow$ The diagram commutes. ▨

(b). Similarly, define $H(b,c) := H_b(c)$ and

from adjunction $F(-,b) \dashv H_b$ gives $\forall f: b' \to b \in B$

$$A(a, H_b(c)) \xrightarrow[i]{\cong} C(F(a,b),c) \xrightarrow{F(a,f)^*} C(F(a,b'),c) \xrightarrow[j]{\cong} A(a, H_{b'}(c))$$

Claim 1: $j \circ F(a,f)^* \circ i$ is a natural transformation

$$\varepsilon_{a,f}: A(a, H(b,-)) \Rightarrow A(a, H(b',-))$$

pf) For $g: c \to c' \in C$.

$$\begin{tikzcd}
A(a, H(b,c)) \arrow[r, "\cong"] \arrow[d, "H_b(g)_*"'] \arrow[dr, phantom, "\circlearrowleft"] & C(F(a,b),c) \arrow[r, "F(a,f)^*"] \arrow[d, "g_*"] \arrow[dr, phantom, "\circlearrowleft"] & C(F(a,b'),c) \arrow[r, "\cong"] \arrow[d, "g_*"] \arrow[dr, phantom, "\circlearrowleft"] & A(a, H(b',c)) \arrow[d, "H_{b'}(g)_*"] \\
A(a, H(b,c')) \arrow[r, "\cong"'] & C(F(a,b),c') \arrow[r, "F(a,f)^*"'] & C(F(a,b'),c') \arrow[r, "\cong"'] & A(a, H(b',c'))
\end{tikzcd}$$

since left, right: from adjunction,

middle: post and pre composition are commutative.

Claim 2. $j \circ F(a,f)^* \circ i$ is a natural transformation

$$\eta_{c,f}: A(-, H(b,c)) \Rightarrow A(-, H(b',c))$$

pf) For $g: a' \to a$,

Left Right: Adjunction

Middle: Functoriality of F.

$$\begin{tikzcd}
A(a, H(b,c)) \arrow[r, "\cong"] \arrow[d, "g^*"'] \arrow[dr, phantom, "\circlearrowleft"] & C(F(a,b),c) \arrow[r, "F(a,f)^*"] \arrow[d, "F(g,b)^*"] \arrow[dr, phantom, "\circlearrowleft"] & C(F(a,b'),c) \arrow[r] \arrow[d, "F(g,b)^*"] \arrow[dr, phantom, "\circlearrowleft"] & A(a, H(b',c)) \arrow[d, "g^*"] \\
A(a', H(b,c)) \arrow[r, "\cong"'] & C(F(a',b),c) \arrow[r, "F(a,f)^*"] & C(F(a',b'),c) \arrow[r] & A(a', H(b',c))
\end{tikzcd}$$

Thus, $(\varepsilon_{a,f})_c = (\eta_{c,f})_a \quad \forall c \in C, b \in B, a \in A.$

By Yoneda Lemma, $(\eta_{c,f})_a = \left((\eta_{c,f})_{H(b,c)}(1_{(b,c)})\right)_*$

Define $H(f^{op}, 1_c) := (\eta_{c,f})_{H(b,c)}(1_{H(b,c)})$.

Then, for any $g : c \longrightarrow c'$, $f : b' \longrightarrow b$

$$H(f^{op}, g) := H(b', g)\, H(f^{op}, 1_c)$$
$$= H(f^{op}, 1_{c'})\, H(b, g)$$

is well-defined

pf) $H(b', g)\, H(f^{op}, 1_c) = H_{b'}(g) \circ (\eta_{c,f})_{H(b,c)}(1_{H(b,c)})$

$= H_{b'}(g)\, (\varepsilon_{H(b,c),f})_c (1_{H(b,c)})$

From naturality of ε,

$$\begin{tikzcd}
A(H(b,c), H(b,c)) \arrow[r, "(\varepsilon_{H(b,c),f})_c"] \arrow[d, "H_b(g)_*"'] \arrow[dr, phantom, "\circlearrowright"] & A(H(b,c), H(b',c)) \arrow[d, "H_{b'}(g)_*"] \\
A(H(b,c), H(b,c')) \arrow[r, "(\varepsilon_{H(b,c),f})_{c'}"'] & A(H(b,c), H(b',c'))
\end{tikzcd}$$

$= (\varepsilon_{H(b,c),f})_{c'}(H_b(g)) = (\eta_{c',f})_{H(b,c)}(H_b(g))$

From naturality of η,

$$\begin{tikzcd}
A(H(b,c'), H(b,c')) \arrow[r, "(\eta_{c',f})_{H(b,c')}"'] \arrow[d, "H_b(g)^*"'] \arrow[dr, phantom, "\circlearrowright"] & A(H(b,c'), H(b',c')) \arrow[d, "H_b(g)^*"] \\
A(H(b,c), H(b,c')) \arrow[r, "(\eta_{c',f})_{H(b,c)}"'] & A(H(b,c), H(b',c'))
\end{tikzcd}$$

$= (\eta_{c',f})_{H(b',c)}(1_{H(b',c)}) \cdot H_b(g) = H(f^{op}, 1_{c'})\, H_b(g)$

done.

Now $H(1_b, 1_c) = (\eta_{c,1_b})_{H(b,c)} (1_{H(b,c)})$

$= 1_{H(b,c)}$ since $(\eta_c, 1_b) =$ Identity by construction

And by the same argument as in (i), H satisfies functoriality.

Now, $A(a, H(b,c)) \cong C(F(a,b),c)$ are natural in all three variables from claim 1, claim 2, and for $f: b' \to b \in B$

$$
\begin{tikzcd}
C(F(a,b),c) \arrow[d, "F(a,f)^*"'] & A(a, H_b(c)) \arrow[l, "\cong"'] \arrow[d, "H(f^{op}, 1_c)_*"] \\
C(F(a,b'),c) \arrow[r, "\cong"'] & A(a, H_{b'}(c))
\end{tikzcd}
$$

and $H(f^{op}, 1_c)_* = (\eta_{c,f})_a = A(a, H_b(c)) \xrightarrow{\cong} C(F(a,b),c) \xrightarrow{F(a,f)^*} C(F(a,b'),c) \xrightarrow{\cong} A(a, H_{b'}(c))$

by construction.

iii) $F: C^{op} \to D$, $G: D^{op} \to C$ are mutually right adjoint if $\exists$ natural iso $D(d, Fc) \cong C(c, Gd)$.

Let $C = A$, $D = B$, $F = G(-,c)$, $G = H(-,c)$

$\Rightarrow B(b, G(a,c)) \cong A(a, H(b,c))$ from (ii)

$\Rightarrow$ They are mutually right adjoint. ∎

Def. 4.3.17. $A \times B \xrightarrow{F} C$, $A^{op} \times C \xrightarrow{G} B$, $B^{op} \times C \xrightarrow{H} A$

with natural iso.

$$C(F(a,b), c) \cong B(b, G(a,c)) \cong A(a, H(b,c))$$

Called <u>two-variable adjunction</u>.

Let $F: C \times C \to C$ monoidal product.

left closure of F: ptwise defined right adjoint H / G

in the sense of def 4.3.17

If $H \cong G$ $\Rightarrow$ F is called <u>closed</u>.

Ex 4.3.8. $Set \times Set \xrightarrow{\times} Set$, $Set^{op} \times Set \to Set$, $Set^{op} \times Set \to Set$

has iso $\{A \times B \xrightarrow{f} C\} \cong \{A \xrightarrow{f} C^B\} \cong \{B \xrightarrow{f} C^A\}$

$\Rightarrow$ a two variable adjunction

Ex. 4.3.9. A cartesian closed category:

a category C with finite products bifunctor

$C \times C \xrightarrow{\times} C$ is closed.

Ex 1.7.vii shows that $C \times C \xrightarrow{\times} C$ is closed.

Ex. 4.3.10. (Steenrod) Convenient Category of top spaces.

cgHaus: obj = cptly generated Hausdorff space X.

a subset A is closed in X

$\Leftrightarrow$ $A \cap K$ is closed in K

for all cpt subspace $K \subseteq X$.

i: cgHaus $\hookrightarrow$ Haus is inclusion.

k: Haus $\hookrightarrow$ cgHaus is called <u>k-ification</u>

$k(X) :=$ A space X with refined topology $\tau_{k(X)}$
s.t. adding a subset $A \subset X$ s.t.
$A \cap K$ is closed in K $\forall K$ cpt $\subseteq X$
in the collection of closed sets.

$i \dashv k$; and by Exercise 4.6 ii), ~~&~~ Prop 4.5.15,
cgHaus is complete / cocomplete.

$\Rightarrow$ Let $X, Y \in$ cgHaus. Then
$$X \underset{cgHaus}{\times} Y = k(X \underset{Haus}{\times} Y).$$

The ptwise right adjoint of cgHaus $\times$ cgHaus $\xrightarrow{\times}$ cgHaus
is given by the function spaces $\mathrm{Map}(X,Y)$,
where $\mathrm{Map}(X,Y)$ as a set $= \{f: X \to Y \text{ cts}\}$
〃 as a top space := k-ification of
the cpt open topology.

Ex 4.3.11) ($\otimes$ and Hom)
$$Ab^{op} \times Ab \xrightarrow{Hom} Ab \text{ is a bifunctor}$$
$$(A, B) \longmapsto Hom(A,B)$$ with addition defined ptwise in B.

From algebra, we know that
$$Ab(A \otimes_{\mathbb{Z}} B, C) \underset{T.F}{\cong} Ab(B, Hom(A,C)) \quad \forall B, C \in Ab$$
Thus, by dual of Prop 4.3.4
$Ab \xrightarrow{A \otimes_{\mathbb{Z}} -} Ab$ is a functor adjoint to $Hom(A,-)$

Then by applying Prop 4.3.6(i) we have $Ab \times Ab \xrightarrow{\otimes_{\mathbb{Z}}} Ab$ 〃
a tensor product bifunctor.

So that tensor and Hom gives a two-variable adj.

This process can be reversed, i.e., from $-\underset{\mathbb{Z}}{\otimes}-$, we get $\mathrm{Hom}(-,-)$ as two variable adj.

(In this case we need $Ab(A, \mathrm{Hom}(B,C)) \cong Ab(A\underset{\mathbb{Z}}{\otimes}B, C)$.) but the way of construction is same) apply Prop 4.3.4 and prop 4.3.6(i).

Ex 4.3.14. CTop : Cartesian closed category of top space.

Then $\mathrm{CTop} \underset{\mathrm{Map}(S^1,-)}{\overset{S^1\times -}{\rightleftarrows}} \mathrm{Top}$ where S^1 : unit circle.

($\mathrm{Map}(S^1, X)$: free loop space on X)
since pt in $\mathrm{Map}(S^1,X)$ is $f: S^1 \to X$, a loop in X)

Let $*$: singleton space,

$\Rightarrow */\mathrm{CTop} \cong \mathrm{CTop}_*$ a Ctop with base pt.

By Ex 4.3 iv (ii). $\mathrm{Top}_* \times \mathrm{Top}_* \xrightarrow{\wedge} \mathrm{Top}_*$, $\mathrm{Top}_*^{op} \times \mathrm{Top}_* \xrightarrow{\mathrm{Map}_*(,)} \mathrm{Top}_*$

"

forms a two variable adjunction
where $\mathrm{Map}_*((X,x),(Y,y))$ = base pt preserving cts functions $X \to Y$.

$\wedge$ = <u>smash product</u>.

$\Omega X := \mathrm{Map}_*(S^1,(X,x))$ = <u>based loop space on (X,x)</u>

Then $\Omega : \mathrm{CTop}_* \longrightarrow \mathrm{CTop}_*$. It has left adjoint
$\Sigma X := S^1 \wedge X$ = reduced suspension on (X,x).

$\Omega \dashv \Sigma$ is called "loops $\dashv$ suspension adjunction".

$$\begin{tikzcd} CTop_* \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & CTop_* \arrow[l, bend left, "\Omega"] \end{tikzcd} \quad \text{i.e. } CTop_*(\Sigma X, Y) \cong CTop_*(X, \Omega Y)$$

$\boxtimes$

By applying 4.3.6 again, we get n-variable adjunctions determined by $F: A_1 \times \cdots \times A_n \to B$ admitting pointwise right adjoint when any $(n-1)$ variables are fixed.

Ex. 4.3.i)

Def) A mutual left adjunction consists of an $F: C^{op} \to D$, $G: D^{op} \to C$ with natural transformation $\eta: GF \Rightarrow 1_C$, $\varepsilon: FG \Rightarrow 1_D$

s.t.

$$\begin{tikzcd} F \arrow[r, Rightarrow, "F\eta"] \arrow[d, Rightarrow, "1_F"'] & FGF \arrow[dl, Rightarrow, "\varepsilon F"] \\ F & \end{tikzcd} ; \quad \begin{tikzcd} G \arrow[r, Rightarrow, "G\varepsilon"] \arrow[d, Rightarrow, "1_G"'] & GFG \arrow[dl, Rightarrow, "\eta G"] \\ G & \end{tikzcd}$$

pf) In the original definition, naturality gives that $\forall f: c \to c' \in C$, $\forall g: d \to d' \in D$,

$$\begin{tikzcd} D(Fc, d) \arrow[r, "\cong"] \arrow[d, "(Ff)^*"'] & C(Gd, c) \arrow[d, "f_*"] \\ D(Fc', d) \arrow[r, "\cong"] & C(Gd, c') \end{tikzcd} , \quad \begin{tikzcd} D(Fc, d) \arrow[r, "\cong"] \arrow[d, "g_*"'] & C(Gd, c) \arrow[d, "(Gg)^*"] \\ D(Fc, d') \arrow[r, "\cong"] & C(Gd', c) \end{tikzcd}$$

Thus, $\forall h^\#: Fc \to d$,

$$\begin{tikzcd} Gd \arrow[r, "h^b"] \arrow[dr, "(h^\# \cdot Ff)^b"'] & c \arrow[d, "f"] \arrow[dl, phantom, "\circlearrowleft"] \\ & c' \end{tikzcd} , \quad \begin{tikzcd} Gd \arrow[r, "h^b"] & c \\ Gd' \arrow[u, "Gg"] \arrow[ur, "(gh^\#)^b"'] \arrow[ur, phantom, "\circlearrowleft"] & \end{tikzcd}$$

and $\forall\, h^b : Gd \to c$,

$$\begin{tikzcd} Fc \arrow[r, "h^\#"] & d \\ Fc' \arrow[u, "Ff"] \arrow[ur, "(f h^b)^\#"'] \arrow[u, phantom, shift left=4ex, "\circlearrowleft"] & \end{tikzcd} \qquad \begin{tikzcd} Fc \arrow[r, "h^\#"] \arrow[dr, "(h^b Gg)^\#"'] \arrow[dr, phantom, shift left=2ex, "\circlearrowleft"] & d \arrow[d, "g"] \\ & d' \end{tikzcd}$$

Now, unit η and ε as defined follow;

By letting $d = Fc$, $h^\# = 1_{Fc}$, $\eta_c := h^b$ s.t.

$$\begin{tikzcd} GFc \arrow[r, "\eta_c"] \arrow[dr, "(Ff)^b"'] \arrow[dr, phantom, shift left=2ex, "\circlearrowleft"] & c \arrow[d, "f"] \\ & c' \end{tikzcd}, \qquad \begin{tikzcd} GFc \arrow[r, "\eta_c"] & c \\ Gd' \arrow[u, "Gg"] \arrow[ur, "(g)^b"'] \arrow[u, phantom, shift left=4ex, "\circlearrowleft"] & \end{tikzcd} \quad \Rightarrow \quad \begin{array}{l} f\eta_c = (Ff)^b \\ \eta_c Gg = (g)^b \end{array}$$

for $g : Fc \to d'$
$f : c \to c'$.

By letting $c = Gd$, $h^b = 1_{Gd}$, $\varepsilon_d := h^\#$ s.t.

$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"] & d \\ Fc' \arrow[u, "Ff"] \arrow[ur, "(f^\#)"'] \arrow[u, phantom, shift left=4ex, "\circlearrowleft"] & \end{tikzcd} \qquad \begin{tikzcd} FGd \arrow[r, "\varepsilon_d"] \arrow[dr, "(Gg)^\#"'] \arrow[dr, phantom, shift left=2ex, "\circlearrowleft"] & d \arrow[d, "g"] \\ & d' \end{tikzcd} \quad \Rightarrow \quad \begin{array}{l} \varepsilon_d \circ Ff = f^\# \\ g \cdot \varepsilon_d = (Gg)^\# \end{array}$$

for $f : Gd \to c'$
$g : d \to d'$

Now let $d = Fc$, $f = \eta_c$. Then we have

$$\begin{tikzcd} \mathcal{D}(FGFc, Fc) \arrow[r, phantom, "\cong"] \arrow[d, "(F\eta_c)^*"'] & \mathcal{C}(GFc, GFc) \arrow[d, "(\eta_c)_*"] \\ \mathcal{D}(Fc, Fc) \arrow[r, phantom, "\cong"] & \mathcal{C}(GFc, Fc) \end{tikzcd} \Rightarrow \begin{tikzcd} \varepsilon_{Fc} \arrow[d] & 1_{GFc} \arrow[l, mapsto] \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ \varepsilon_{Fc} \cdot F\eta_c = 1_{Fc} & \eta_c \arrow[l, mapsto] \end{tikzcd}$$

$\therefore$ Thus $\varepsilon_{Fc} \cdot F\eta_c = 1_{Fc} \Rightarrow \varepsilon F \cdot F\eta = 1_F$

as a natural transform.

Also by letting $c = Gd$, $g := \varepsilon_d$, we get.

$$\begin{tikzcd} D(FGd, FGd) \arrow[r, phantom, "\cong"] \arrow[d, "(\varepsilon_d)_*"'] & C(GFGd, Gd) \arrow[d, "(G\varepsilon_d)^*"] \\ D(FGd, d) \arrow[r, phantom, "\cong"] & C(Gd, Gd) \end{tikzcd} \Rightarrow \begin{tikzcd} 1_{FGd} \arrow[r, mapsto] \arrow[d] & \eta_{Gd} \arrow[d] \\ \varepsilon_d \arrow[r, mapsto] & 1_{Gd} = \eta_{Gd} \cdot G\varepsilon_d \end{tikzcd}$$

$\Rightarrow \eta G \cdot G\varepsilon = 1_G$ as a natural transformation

Conversely, if we have such η and ε,

given $f^\# : Fc \to d$, $g^b : Gd \to c$,

define $f^b : Gd \xrightarrow{Gf^\#} GFc \xrightarrow{\eta_c} c$.

$g^\# : Fc \xrightarrow{Fg^b} FGd \xrightarrow{\varepsilon_d} d$.

Then, we need to show that transpose of this definition gives original one. Actually

$$\begin{tikzcd} (f^b)^\# = Fc \arrow[r, "F\eta_c"] \arrow[dr, bend right, "1_{Fc}"'] & FGFc \arrow[r, "FGf^\#"] \arrow[d, "\varepsilon_{Fc}"] \arrow[dl, phantom, "\text{①} \circlearrowleft"] & FGd \arrow[r, "\varepsilon_d"] \arrow[dl, phantom, "\circlearrowleft \text{②}"] & d = f^\# \\ & Fc \arrow[urr, bend right, "f^\#"'] & & \end{tikzcd}$$

where ① from triangle identity and ② from naturality of ε. Similarly,

$$\begin{tikzcd} (g^\#)^b : Gd \arrow[r, "G\varepsilon_d"] \arrow[dr, bend right, "1_{Gd}"'] & GFGd \arrow[r, "GFg^b"] \arrow[d, "\eta_{Gd}"] \arrow[dl, phantom, "\text{①} \circlearrowleft"] & GFc \arrow[r, "\eta_c"] \arrow[dl, phantom, "\text{②} \circlearrowleft"] & c = g^b \\ & Gd \arrow[urr, bend right, "g^b"'] & & \end{tikzcd}$$

where ① is from triangle idez. and ② is from naturality of η.

Def) A mutual right adjunction consists of an $F: C^{op} \longrightarrow D$, $G: D^{op} \longrightarrow C$ with natural transformation $\eta: 1_C \Rightarrow GF$ $\varepsilon: 1_D \Rightarrow FG$

s.t.

$$\begin{tikzcd} FGF \arrow[r, Rightarrow, "F\eta"] & F \\ F \arrow[u, Rightarrow, "\varepsilon F"] \arrow[ur, Rightarrow, "1_F"'] & \end{tikzcd} \quad , \quad \begin{tikzcd} GFG \arrow[r, Rightarrow, "G\varepsilon"'] & G \\ G \arrow[u, Rightarrow, "\eta G"] \arrow[ur, Rightarrow, "1_G"'] & \end{tikzcd}$$

pf) $\forall f: c' \longrightarrow c \in C$, $\forall g: d' \longrightarrow d \in D$, if original def holds,

$$\begin{tikzcd} D(d, Fc) \arrow[r, phantom, "\cong"] \arrow[d, "(Ff)_*"'] \arrow[dr, phantom, "\circlearrowright"] & C(c, Gd) \arrow[d, "f^*"] \\ D(d, Fc') \arrow[r, phantom, "\cong"] & C(c', Gd) \end{tikzcd} \qquad \begin{tikzcd} D(d, Fc) \arrow[r, phantom, "\cong"] \arrow[d, "g^*"'] \arrow[dr, phantom, "\circlearrowright"] & C(c, Gd) \arrow[d, "(Gg)_*"] \\ D(d', Fc) \arrow[r, phantom, "\cong"] & C(c, Gd') \end{tikzcd}$$

Thus, $\forall h^\#: d \longrightarrow Fc$,

$$\begin{tikzcd} c \arrow[r, "h^b"] & Gd \\ c' \arrow[u, "f"] \arrow[ur, "(Ff \cdot h^\#)^b"'] \arrow[u, phantom, "\circlearrowright", shift left=4ex] & \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "h^b"] \arrow[dr, "(h^\# g)^b"'] \arrow[dr, phantom, "\circlearrowright", shift left=2ex] & Gd \arrow[d, "Gg"] \\ & Gd' \end{tikzcd}$$

and $\forall h^b: c \longrightarrow Gd$,

$$\begin{tikzcd} d \arrow[r, "h^\#"] \arrow[dr, "(h^b f)^\#"'] \arrow[dr, phantom, "\circlearrowright", shift left=2ex] & Fc \arrow[d, "Ff"] \\ & Fc' \end{tikzcd} \quad , \quad \begin{tikzcd} d \arrow[r, "h^\#"] & Fc \\ d' \arrow[u, "g"] \arrow[ur, "(Gg \cdot h^b)^\#"'] \arrow[u, phantom, "\circlearrowright", shift left=4ex] & \end{tikzcd}$$

Now define unit η and counit ε as follows,

By letting $d = Fc$, $h^\# = 1_{Fc}$, $\eta_c = h^b$ s.t.

$$\begin{tikzcd} c \arrow[r, "\eta_c"] & GFc \\ c' \arrow[u, "f"] \arrow[ur, "(Ff)^b"'] \arrow[u, phantom, "\circlearrowright", shift left=4ex] & \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[dr, "(g)^b"'] & GFc \arrow[d, "Gg"] \\ & Gd' \end{tikzcd} \Rightarrow$$

$\eta_c f = (Ff)^b$

$Gg\,\eta_c \overset{?}{=} (g)^b$

for $g: d' \longrightarrow Fc$

$f: c' \longrightarrow c$

Also, by letting $c = Gd$, $h^\flat = 1_{Gd}$, $\varepsilon_d = h^\#$ s.t.

$$\begin{tikzcd} d \arrow[r, "\varepsilon_d"] \arrow[dr, "(f)^\#"'] & FGd \arrow[d, "Ff"] \\ & Fc' \end{tikzcd} \quad \begin{tikzcd} d \arrow[r, "\varepsilon_d"] & FGd \\ d' \arrow[u, "g"] \arrow[ur, "(Gg)^\#"'] & \end{tikzcd} \Rightarrow \begin{array}{l} Ff \cdot \varepsilon_d = (f)^\# \\ \varepsilon_d \cdot g = (Gg)^\# \end{array}$$

for $f : c' \longrightarrow Gd$.
$g : d' \longrightarrow d$.

Now let $d = Fc$, $f = \eta_c$. Then,

$$\begin{tikzcd} D(Fc, FGFc) \arrow[r, phantom, "\cong"] \arrow[d, "(F\eta_c)_*"'] & C(GFc, GFc) \arrow[d, "(\eta_c)^*"] \\ D(Fc, Fc) \arrow[r, phantom, "\cong"] & C(c, GFc) \end{tikzcd} \Rightarrow \begin{tikzcd} \varepsilon_{Fc} \arrow[d, mapsto] & 1_{GFc} \arrow[l, mapsto] \arrow[d, mapsto] \\ F\eta_c \cdot \varepsilon_{Fc} = 1_{Fc} & \eta_c \arrow[l, mapsto] \end{tikzcd}$$

$\Rightarrow F\eta_c \cdot \varepsilon_{Fc} = 1_{Fc} \Rightarrow F\eta \cdot \varepsilon F = 1_F$ as a natural transformation

Also, let $c = Gd$, $g = \varepsilon_d$ we get

$$\begin{tikzcd} D(FGd, FGd) \arrow[r, phantom, "\cong"] \arrow[d, "\varepsilon_d^*"'] & C(Gd, GFGd) \arrow[d, "(G\varepsilon_d)_*"] \\ D(d, FGd) \arrow[r, phantom, "\cong"] & C(Gd, Gd) \end{tikzcd} \Rightarrow \begin{tikzcd} 1_{FGd} \arrow[r, mapsto] \arrow[d, mapsto] & \eta_{Gd} \arrow[d, mapsto] \\ \varepsilon_d \arrow[r, mapsto] & 1_{Gd} = G\varepsilon_d \eta_{Gd} \end{tikzcd}$$

$\Rightarrow G\varepsilon_d \cdot \eta_{Gd} = 1_{Gd} \Rightarrow G\varepsilon \cdot \eta G = 1_G$ as a natural transf.

Thus the triangle identity holds.

Conversely,

for given $f^\#: d \longrightarrow Fc$, $g^b: c \longrightarrow Gd$,

define $f^b: c \xrightarrow{\eta_c} GFc \xrightarrow{Gf^\#} Gd$

$g^\#: d \xrightarrow{\varepsilon_d} FGd \xrightarrow{Fg^b} Fc$

Then,

$$
(f^b)^\#:\quad
\begin{tikzcd}
d \arrow[r, "\varepsilon_d"] \arrow[drr, "f^\#"'] & FGd \arrow[r, "FGf^\#"] \arrow[d, phantom, "① \circlearrowright"] & FGFc \arrow[r, "F\eta_c"] \arrow[dr, phantom, "\circlearrowright ②"] & Fc \\
& & Fc \arrow[u, "\varepsilon_{Fc}"] \arrow[ur, "1_{Fc}"'] &
\end{tikzcd}
= f^\#
$$

Since ① commute by naturality of ε

② commute by triangle ineq.

$$
(g^\#)^b:\quad
\begin{tikzcd}
c \arrow[r, "\eta_c"] \arrow[drr, "g^b"'] & GFc \arrow[r, "GFg^b"] \arrow[d, phantom, "① \circlearrowright"] & GFGd \arrow[r, "G\varepsilon_d"] \arrow[dr, phantom, "\circlearrowright ②"] & Gd \\
& & Gd \arrow[u, "\eta_{Gd}"] \arrow[ur, "1_{Gd}"'] &
\end{tikzcd}
= g^b.
$$

Since ① commute by naturality of η,

② ,, triangle ineq. ∎

Ex 4.3 ii) Done in the lecture note.

Ex 4.3 iii) It suff to show that

$Set(X, PY) \cong Set(Y, PX)$ is natural

for each variable X and Y.

So for X,

let $f: X_1 \longrightarrow X_2 \implies P(f): P(X_2) \longrightarrow P(X_1)$

$A \subseteq X_2 \longmapsto f^{-1}(A)$

Thus,

$$
\begin{tikzcd}
Set(X_2, PY) \arrow[r, phantom, "\cong"] \arrow[d, "(f)^*"'] & Set(Y, PX_2) \arrow[d, "(Pf)_*"] \\
Set(X_1, PY) \arrow[r, phantom, "\cong"] & Set(Y, PX_1)
\end{tikzcd}
\implies
\begin{tikzcd}
\phi^\# \arrow[r, mapsto] \arrow[d] & \phi^b \arrow[d, "Pf \cdot \phi^b"] \\
\phi^\# \cdot f \arrow[r, mapsto] & (\phi^\# \cdot f)^b
\end{tikzcd}
$$

Let $\phi^{\#} \in Set(X, PY)$. Natural definition of Iso is $\phi^{b} := Y \longrightarrow PX$ by $y \longmapsto \{x \in X : y \in \phi^{\#}(x)\}$

Similarly if we know ϕ^{b}, then

$$\phi^{\#} := X \longrightarrow PY \text{ by } x \longmapsto \{y \in Y : x \in \phi^{b}(y)\}$$

Thus, let $\phi^{\#} \in Set(X_2, PY)$. Then, it suffice to show that $Pf \cdot \phi^{b} = (\phi^{\#} f)^{b}$.

For given $y \in Y$,

$$(\phi^{\#} f)^{b}(y) = \{x \in X_1 : y \in \phi^{\#} \cdot f(x)\}$$

$$Pf \cdot \phi^{b}(y) = f^{-1}(\{z \in X_2 : y \in \phi^{\#}(z)\})$$

If $k \in \{x \in X_1 : y \in \phi^{\#} f(x)\}$

$\Rightarrow \phi^{\#} f(k) \ni y$. Thus, $f(k) \in \{z \in X_2 : y \in \phi^{\#}(z)\}$

$\Rightarrow k \in f^{-1}(\{z \in X_2 : y \in \phi^{\#}(z)\})$

Conversely, if $k \in f^{-1}(\{z \in X_2 : y \in \phi^{\#}(z)\})$

$$f(k) \in \{z \in X_2 : y \in \phi^{\#}(z)\}$$

$$\Rightarrow k \in \{x \in X_1 : y \in \phi^{\#} f(x)\}$$

$\Rightarrow (\phi^{\#} f)^{b} = Pf \cdot \phi^{b}(y)$.

Thus this Iso is natural for variable X.

For $g: Y_1 \longrightarrow Y_2$, $Pg: PY_2 \longrightarrow PY_1$, thus we need to show that

$$\begin{tikzcd} \mathrm{Set}(X, PY_2) \arrow[r, phantom, "\cong"] \arrow[d, "(Pg)_*"'] & \mathrm{Set}(Y_2, PX) \arrow[d, "g^*"] \\ \mathrm{Set}(X, PY_1) \arrow[r, phantom, "\cong"] & \mathrm{Set}(Y_1, PX) \end{tikzcd}$$

i.e. $\forall\, \phi^\# \in \mathrm{Set}(X, PY_2)$,

$$\begin{tikzcd} \phi^\# \arrow[r, mapsto] \arrow[d, mapsto] & \phi^b \arrow[d, mapsto] \\ Pg\,\phi^\# \arrow[r, mapsto] & \phi^b \cdot g \\ & (Pg \cdot \phi^\#)^b \end{tikzcd}$$

we need to show

$$(Pg \cdot \phi^\#)^b = \phi^b \cdot g.$$

For given $y \in Y_1$,

$$(Pg \cdot \phi^\#)^b(y) = \{ x \in X : Pg \cdot \phi^\#(x) \ni y \}$$
$$= \{ x \in X : g^{-1}(\phi^\#(x)) \ni y \}$$

$$\phi^b \cdot g(y) = \{ x \in X : \phi^\#(x) \ni g(y) \}$$

Let $k \in \phi^b \cdot g(y)$. $\Rightarrow g^{-1}(\phi^\#(x)) \ni y \Rightarrow k \in (Pg \cdot \phi^\#)^b$

Conversely, $k \in (Pg \cdot \phi^\#)^b \Rightarrow g^{-1}(\phi^\#(k)) \ni y$

$\Rightarrow \phi^\#(k) \ni g(y)$.

$\Rightarrow k \in \phi^b \cdot g(y)$

$\Rightarrow (Pg \cdot \phi^\#)^b = \phi^b \cdot g$

So it is natural iso for Y

$\Rightarrow$ P is right adjoint to itself.

Ex 4.3.iv) Let $F_{(X,x)} : Set_* \longrightarrow Set_*$ be a left adjoint to $Hom_*((X,x),-)$. Then,

$$Set_*(F_{(X,x)}(Y,y),(Z,z)) \cong Set_*((Y,y), Hom_*((X,x),(Z,z)))$$

Now think $\phi \in$ RHS. Then, for each $y' \in Y$,

$\phi(y') : (X,x) \longrightarrow (Z,z)$ a function.

s.t. $\phi(y,x') = z \quad \forall x' \in X$

$\phi(y',x) = z \quad \forall y' \in Y$.

Thus, naturally, ϕ maps $y \times X$ and $Y \times x$ to z.

Define $X \wedge Y := X \times Y / (x,y') \sim (x',y) \quad \forall y' \in Y, x' \in X$.

then, $(X \wedge Y, (x,y))$ is a pointed set.

We claim $F_{(x,y)}(Y,y) = (X \wedge Y, (x,y))$ is a well-defined functor.

If $f : (Y,y) \longrightarrow (A,a) \in Set_*$,

$$F_{(x,y)}(f) : (X \wedge Y, (x,y)) \longrightarrow (X \wedge A, (x,a))$$
$$(x',y') \longmapsto (x', f(y')).$$

Then, $F_{(x,y)}(f)(x,y') \longmapsto (x, f(y')) \equiv (x,a)$

$\quad\quad\quad\quad // \quad (x',y) \longmapsto (x',a) \equiv (x,a)$

So well-defined.

Moreover, it satisfies the pointwise left adjoint.

For $\phi^\# \in Set_*((Y,y), Hom_*(X,x), (Z,z))$

define $\phi^b : (X \wedge Y, (x,y)) \longrightarrow (Z,z)$

$$(x',y') \longmapsto \phi^\#(y')(x').$$

Conversely, For $\phi^b \in Set((X \wedge Y, (x,y)), (Z,z))$

define $\phi^\# : (Y,y) \longrightarrow Hom_*((X,x),(Z,z))$

$$y' \longmapsto f_{y'} : x' \longmapsto \phi^b(x',y')$$

Then, given $\phi^\#$,

$$(\phi^b)^\#(y') = f_{y'} : x' \longmapsto \phi^b(x',y')$$
$$= \phi^\#(y')(x')$$

$$\Rightarrow f_{y'} = \phi^\#(y')$$

$\Rightarrow (\phi^b)^\# = \phi^\#$.

and $(\phi^\#)^b(x',y') = \phi^\#(y')(x')$

$$= f_{y'}(x')$$
$$= \phi^b(x',y')$$

$\Rightarrow (\phi^\#)^b = \phi^b$.

Thus, $\#$ and b has an iso. Moreover, this is natural iso, since for $f:(Y_2,y_2) \longrightarrow (Y_1,y_1)$

we have

$$\begin{tikzcd}
Set_*((X\wedge Y,(x,y)),(Z,z)) \arrow[r, phantom, "\cong"] \arrow[d, "(1_X;f)^*"'] & Set_*((Y,y),Hom_*((X,x),(Z,z))) \arrow[d, "f^*"] \\
Set_*((X\wedge Y_1,(x,y_1)),(Z,z)) \arrow[r, phantom, "\cong"] & Set_*((Y_1,y_1),Hom_*((X,x),(Z,z)))
\end{tikzcd}$$

Then,
$$\begin{tikzcd}
\phi^b \arrow[r, mapsto] \arrow[d, mapsto] & \phi^\# \arrow[dr, mapsto] & \\
\phi^b\cdot(1_X,f) \arrow[r, mapsto] & (\phi^b\cdot(1_X,f))^\# & \phi^\#\cdot f
\end{tikzcd}$$

Then, for $y_1' \in Y_1$,

$$\phi^\#\cdot f(y_1') = \phi^\#(f(y_1')) : x' \longmapsto \phi^b(x', f(y_1'))$$

$$(\phi^b\cdot(1_X,f))^\#(y_1') = f_{y_1'} : x' \longmapsto \phi^b(1_X,f)(x',y_1')$$
$$= \phi^b(x', f(y_1'))$$

thus, $\phi^\#(f(y_1')) = f_{y_1'} \quad \forall y_1' \in Y_1$

$\Rightarrow \phi^\# = (\phi^b\cdot(1_X f))^\#$.

Conversely, for $g : (Z,z) \longrightarrow (Z_1,z_1)$.

$$\begin{tikzcd}
Set_*((X\wedge Y,(x,y)),(Z,z)) \arrow[r, phantom, "\cong"] \arrow[d, "g_*"'] & Set_*((Y,y),Hom_*((X,x),(Z,z))) \arrow[d, "Hom_*((X,x),g)_*"] \\
Set_*((X\wedge Y,(x,y)),(Z_1,z_1)) \arrow[r, phantom, "\cong"] & Set_*((Y,y),Hom_*((X,x),(Z_1,z_1)))
\end{tikzcd}$$

Thus, we need to check that

$$\begin{tikzcd} \phi^b \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & \phi^\# \arrow[dr, mapsto] & \\ g\cdot\phi^b \arrow[r, mapsto] & (g\cdot\phi^b)^\# & \mathrm{Hom}_*((X,x),g)\cdot\phi^\# : y' \longmapsto g\cdot\phi^\#(y') \end{tikzcd}$$

Then,

$\mathrm{Hom}_*((X,x),g)(\phi^\#)(y') := g\cdot\phi^\#(y') : x' \longmapsto g\phi^\#(y')(x')$

$= g\cdot\phi^b(x',y')$

$(g\cdot\phi^b)^\#(y') = f_{y'} : x' \longmapsto g\cdot\phi^b(x',y')$

$\Rightarrow$ They commute.

Thus, by dual of prop 4.3.6, $Set_* \times Set_* \longrightarrow Set_*$

$((X,x),(Y,y)) \longmapsto (X\wedge Y, (x,y))$

is the unique bi-functor s.t.

$Set_*((X\wedge Y,(x,y)),(Z,z)) \cong Set_*((Y,y),\mathrm{Hom}_*((X,x),(Z,z)))$

are natural in all three variables.

ii) In a Cartesian closed category, notes that

$C(a\times b, c) \cong C(a, c^b) \cong C(b, c^a)$

with a function $C^{op}\times C \xrightarrow{(-)^{(-)}} C$, $C^{op}\times C \xrightarrow{(-)^{(-)}} C$

are natural.

And, instead of C_*, we have a generalized notion of pointed object. (Refer nLab.)

Def) Let $1\in C$ be a terminal element of C.
Then, a "global element" of object X is a morphism

$1 \longrightarrow X$

In this case, we say $*/\mathcal{C}$, a slice category under the object $*$ as "pointed category".

ex) If we let $\mathcal{C} = \mathrm{Set}$, then $\{*\}$ is terminal object in Set, so we can identify Set_* as $*/\mathrm{Set}$

Since $\mathcal{C}$ has a pull back $\forall$ diagram, it has a terminal object, say $1 \in \mathrm{ob}\,\mathcal{C}$ (See Def 3.1.11)

Lemma) $\mathcal{C}$ has a finite product and terminal object 1, then $X \cong X \times 1$ $\forall X \in \mathrm{ob}\,\mathcal{C}$

pf)

$$\begin{tikzcd} & X \arrow[dl, "1_X"'] \arrow[d, dashed, "u"] \arrow[dr, "T_X"] & \\ X \arrow[r, phantom, "\circlearrowright"] & X \times 1 \arrow[l, "\pi_1"] \arrow[r, "T_{X\times 1}"'] & 1 \arrow[l, phantom, "\circlearrowright"] \end{tikzcd}$$

$\exists!\, u : X \to X \times 1$ by the universal property of $X \times 1$ s.t. $\pi_1 \circ u = 1_X$.

To see $u \circ \pi_1 = 1_{X\times 1}$, notes that

$$\begin{tikzcd} & X \times 1 \arrow[dl, "\pi_1"'] \arrow[d, "u\circ\pi_1"] \arrow[dr, bend left, "T_{X\times 1}"] & \\ X \arrow[r, phantom, "\circlearrowright"] & X \times 1 \arrow[l, "\pi_1"] \arrow[r, "T_{X\times 1}"'] & 1 \arrow[l, phantom, "\circlearrowright"] \end{tikzcd}$$

left diagram commutes since $\pi_1 \circ u \circ \pi_1 = 1_X \circ \pi_1 = \pi_1$ and the universal property of 1.

However, we already know that

$$\begin{tikzcd} & X \times 1 \arrow[dl, "\pi_1"'] \arrow[d, "1_{X\times 1}"] \arrow[dr, bend left] & \\ X \arrow[r, phantom, "\circlearrowright"] & X \times 1 \arrow[l, "\pi_1"] \arrow[r, "T_{X\times 1}"'] & 1 \arrow[l, phantom, "\circlearrowright"] \end{tikzcd}$$

$\Rightarrow u \circ \pi_1 = 1_{X\times 1}$ by the uniqueness of universal property's inducing map. ▨

Notes that by Thm 3.4.18, our $\mathcal{C}$ has a finite product. Thus by the lemma, $X \times 1 \cong X$ $\forall X \in \mathrm{ob}\,\mathcal{C}$.

From Cartesian closedness,

$$\mathcal{C}(1, B^X) \cong \mathcal{C}(X \times 1, B) \cong \mathcal{C}(X, B)$$

where last bijection induced by $X \times 1 \cong X$

Thus, for each $c_b^{\#} : 1 \longrightarrow B^X$, $\exists c_b^b : X \longrightarrow B$ which corresponds to $c_b^{\#}$ under the above iso.

Now, using the universal property of product, $\forall C \in \mathrm{Ob}\,\mathcal{C}$,

$$\begin{tikzcd}
 & X \times C \arrow[dl, "\pi_X"'] \arrow[ddd, dashed, "\exists ! \eta_1"] \arrow[dddr, bend left, "\pi_C"] & \\
X \arrow[d, "\tau_X"'] & & \\
1 \arrow[d, "x"'] & & \\
X & X \times C \arrow[l, "\pi_X"] \arrow[r, "\pi_C"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X \times C \arrow[dddl, bend right, "\pi_X"'] \arrow[ddd, dashed, "\exists ! \eta_2"] \arrow[dr, "\pi_C"] & \\
 & & C \arrow[d, "\tau_C"] \\
 & & 1 \arrow[d, "c"] \\
X & X \times C \arrow[l, "\pi_X"] \arrow[r, "\pi_C"'] & C
\end{tikzcd}$$

we can get $\eta_1, \eta_2 : X \times C \longrightarrow X \times C$

(So, if $\mathcal{C} = \mathrm{Set}$, $\eta_1(x', c') = (x, c')$, $\forall x' \in X, c' \in C$)
$\eta_2(x', c') = (x', c)$

Then, take the coequalizer $X \wedge C$ over η_1, η_2

$$X \times C \underset{\eta_2}{\overset{\eta_1}{\rightrightarrows}} X \times C \xrightarrow{\psi} X \wedge C$$

In a $\mathcal{C} = \mathrm{Set}$, $X \wedge C = \{(x', c') : x' \in X, c' \in C\} / \eta_1(x', c') \sim \eta_2(x', c')$

and $r_1(x', c') \sim r_2(x', c') \iff (x, c') \sim (x', c)$ $\forall x' \in X$, $c' \in C$

$\Rightarrow X \wedge C = {}^? X \times C / \underset{\substack{\forall x' \in X \\ c' \in C}}{(x, c') \sim (x', c)}$

which is equivalent to definition of Set.)

Then, $1 \xrightarrow{(x,c)} X \times C \xrightarrow{\psi} X \wedge C$ work as a base element of $X \wedge C$. (In $C = $ Set, this map is just (x, c))

So we have a bifunctor

$$\begin{array}{ccc} - \wedge - : (1/C) \times (1/C) & \longrightarrow & (1/C) \\ 1 \xrightarrow{x} X \quad 1 \xrightarrow{c} C & \longmapsto & 1 \xrightarrow{\psi \circ (x,c)} X \wedge C \end{array}$$

(This needs condition that $\forall x \in C, \exists 1 \to X$ to be well-defined function)

And note that $\forall C \in \mathrm{Ob}\, C$

$$C(X, C^1) \cong \mathrm{Hom}(X \times 1, C) \cong \mathrm{Hom}(X, C)$$

Thus $C(-, C^1) \cong C(-, C)$ natural iso. as a functor.

By Exercise 2.2 IV, $C^1 \cong C$

Thus, $1^1 \cong 1$. Hence,

$$\begin{array}{ccccc} 1 & & 1 & & 1^1 & & 1 \\ \downarrow & \times & \downarrow & \xmapsto{(-)^{(-)}} & \downarrow & = & \downarrow \\ X & & C & & C^X & & C^X \end{array}$$

Hence, in a $*/C$, we have a functor

$$_*(-)^{(-)} : (1/C)^{op} \times (1/C) \longrightarrow C \longrightarrow 1/C$$

$$\begin{array}{c} 1 \\ \downarrow \\ X \end{array} \times \begin{array}{c} 1 \\ \downarrow \\ C \end{array} \longmapsto \begin{array}{c} 1^1 \\ \downarrow \\ C^X \end{array} \cong \begin{array}{c} 1 \\ \downarrow \\ C^X \end{array}$$

Thus, it suffices to show that $_*(-)^{(-)}$ and $-\underset{*}{\wedge}-$ form a left adjunction, i.e,

$$1/C\left(\begin{array}{c} 1 \\ {\scriptstyle \psi\circ(x,y)}\downarrow \\ X\wedge Y \end{array}, \begin{array}{c} 1 \\ {\scriptstyle z}\downarrow \\ Z \end{array}\right) \cong 1/C\left(\begin{array}{c} 1 \\ {\scriptstyle y}\downarrow \\ Y \end{array}, \begin{array}{c} 1 \\ {\scriptstyle z^x}\downarrow \\ Z^X \end{array}\right) \cong 1/C\left(\begin{array}{c} 1 \\ {\scriptstyle x}\downarrow \\ X \end{array}, \begin{array}{c} 1 \\ {\scriptstyle z^y}\downarrow \\ Z^Y \end{array}\right)$$

are natural iso for all three variables.

pf) To use proposition 4.3.6, it suffices to show that

$_*(1\to X)\wedge -$ is left adjoint of $_*(-)^{(1\to X)}$

and $-\underset{*}{\wedge}(1\to Y)$ " of $_*(-)^{(1\to Y)}$

Before direct proof, investigate two variable adjunction of Cartesian closed categories

Let $f : X \to X'$. Then,

$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_X"'] \arrow[d, dashed, "\exists!\,(f,1_Y)"] \arrow[ddr, "\pi_Y"] & \\ X \arrow[d, "f"'] & & \\ X' & X'\times Y \arrow[l, "\pi'_{X'}"] \arrow[r, "\pi'_Y"'] & Y \end{tikzcd}$$

by the universal property of $X'\times Y$. Thus,

$$\begin{tikzcd} C(X'\times Y, Z) \arrow[r, phantom, "\cong"] \arrow[d, "(f,1_Y)^*"'] \arrow[dr, phantom, "\circlearrowright"] & C(X', Z^Y) \arrow[d, "f^*"] \\ C(X\times Y, Z) \arrow[r, phantom, "\cong"] & C(X, Z^Y) \end{tikzcd} \Longrightarrow \begin{tikzcd} g^\flat \arrow[r, mapsto] \arrow[d, mapsto] & g^\sharp \arrow[d, mapsto] \\ (g^\flat\circ(f,1_Y)) \arrow[r, mapsto] & \underset{(g^\flat\circ(f,1_Y))^\sharp}{=} g^\sharp\circ f \end{tikzcd}$$

i.e. $(g^b \circ (f, 1_Y))^\# = g^\# \circ f$

Also,

$$\begin{tikzcd} \mathcal{C}(X' \times Y, Z) \arrow[r, phantom, "\cong"] \arrow[d, "(f,1_Y)^*"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C}(Y, Z^{X'}) \arrow[d, "(1_Z)^f"] \\ \mathcal{C}(X \times Y, Z) \arrow[r, phantom, "\cong"] & \mathcal{C}(Y, Z^X) \end{tikzcd}$$

where $(1_Z)^f$ induced by $\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{(-)^{(-)}} \mathcal{C}$

from $Z^{(-)} : \mathcal{C}^{op} \longrightarrow \mathcal{C}$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & X \\ X' \arrow[r, mapsto] & X' \arrow[u, "(1_Z)^f"'] \end{tikzcd}$$

Thus,

$$\begin{tikzcd} g^b \arrow[r, mapsto] \arrow[d, mapsto] & g^\# \arrow[d, mapsto] \\ g^b \circ (f, 1_Y) \arrow[r, mapsto] & (1_Z)^f \circ g^\# = (g^b \circ (f, 1_Y))^\# \end{tikzcd}$$

i.e, $(1_Z)^f \cdot g^\# = (g^b \cdot (f, 1_Y))^\#$.

Now we claim that

$$\begin{tikzcd} & 1 \arrow[dl, "(x,y)"'] \arrow[dr, "z"] \arrow[d, phantom, "\circlearrowright"] & \\ X \times Y \arrow[rr, "f^b"'] & & Z \end{tikzcd} \iff \begin{tikzcd} & 1 \arrow[dl, "x"'] \arrow[dr, "z^y"] \arrow[d, phantom, "\circlearrowright"] & \\ X \arrow[rr, "f^\#"'] & & Z^Y \end{tikzcd}$$

(-)

$$\begin{tikzcd} \mathcal{C}(X \times Y, Z) \arrow[r, phantom, "\cong"] & \mathcal{C}(X, Z^Y) \\ \mathcal{C}(1 \times Y, Z) \arrow[r, phantom, "\cong"] \arrow[u, "(x, 1_Y)^*"] \arrow[ur, phantom, "\circlearrowright"] & \mathcal{C}(1, Z^Y) \arrow[u, "(x)^*"'] \\ \mathcal{C}(1 \times 1, Z) \arrow[r, phantom, "\cong"] \arrow[u, "(1, y)^*"] \arrow[ur, phantom, "\circlearrowright"] & \mathcal{C}(1, Z^1) \arrow[u, "(1_Z)^y"'] \end{tikzcd} \Rightarrow \begin{tikzcd} f^b \arrow[r, mapsto] \arrow[d, mapsto] & f^\# \arrow[d, mapsto] \\ f^b(x, 1_Y) \arrow[d, mapsto] & f^\# \cdot x \arrow[d, mapsto] \\ f^b(x, 1_Y)(1, y) & (1_Z)^y \cdot f^\# x \end{tikzcd}$$