

When items need to be divided (divorce, roommates moving out, company negotiations, mediation, and other reasons), we want the division of items to be fair. We will discuss ways to achieve that.

“Fair” doesn’t always mean equal monetary value. Some items, for example, may have more sentimental value to one person than another. So, for that person, that particular item is really “worth” more.

These are our goals for “fairness”.

- A fair division procedure is _____ if each player believes he or she received the same fractional part of the total value.
- A fair division procedure is _____ if each player has a strategy that can guarantee him or her a share of whatever is being divided that is, in the eyes of that player, at least as large as that received by any other player, no matter what the other players do.
- A fair division procedure is said to be _____ if it produces an allocation of the property that no other allocation can make one player better off without making some other player worse off.

1. THE ADJUSTED WINNER PROCEDURE

The adjusted winner procedure can be used to divide items between two parties.

- (1) Each party distributes 100 points over the items in a way that reflects their relative worth to that party.
- (2) Each item is initially given to the party that assigns it more points and each party totals up the number of points it has received. If there is a tie for an item, the item goes to the party with fewer total points.
- (3) If the number of points at the end of Step 2 for each party is equal, the procedure is complete. Otherwise, the party with more points is called the “initial winner” and the other party is called the “initial loser”.
- (4) Calculate the point ratio for each item that belongs to the initial winner. The point ratio is given by
$$\frac{\text{the initial winner's point value for the item}}{\text{the initial loser's point value for the item}}$$
- (5) Move items from the initial winner to the initial loser in increasing order of point ratio. Stop when you get to an item whose move will cause the initial winner to have fewer points than the initial loser. This item will need to be shared.
- (6) Let x represent the fractional part of the shared item that will be transferred from the initial winner to the initial loser. Set the initial winner's total points after the sharing of the item equal to the initial loser's total points after the sharing of the item.
- (7) Solve the equation and state the final division of items between the two parties. Notice that the parties now have an equal number of points of value.

Example 1.1.

Suppose a labor union and management are trying to resolve a dispute that involves four issues. The points are assigned as shown below. Use the adjusted winner procedure to resolve this conflict.

Step 1:

Issue	Labor	Management
Base Salary	30	45
Salary increase	20	45
Benefits	35	5
Vacation time	15	5

Step 2:

Step 3:

The initial winner is _____

The initial loser is _____

Step 4

Item	Point Ratio

Step 5,6,7:

Example 1.2.

Matthew and Jennifer must split 6 items between the two of them. There is a car, a piano, a painting, a grandfather clock, a necklace, and a ring. How should these items be divided?

Matthew and Jennifer assign point values to the 6 items as shown in the table. How should these items be divided?

Step 1:

Item	Matthew	Jennifer
Car	20	5
Piano	35	20
Painting	15	15
Clock	10	15
Necklace	15	20
Ring	5	25

Step 2:

Step 3:

The initial winner is _____

The initial loser is _____

Step 4

Item	Point Ratio

Step 5,6,7:

Example 1.3.

Use the adjusted winner procedure to divide the items below between Harry and Ron.

Step 1:

Item	Harry	Ron
Clock	39	31
Radio	41	34
Car	16	14
Tent	4	21

Step 2:

Step 3:

The initial winner is _____

The initial loser is _____

Step 4

Item	Point Ratio

Step 5,6,7:

2. THE KNASTER INHERITANCE PROCEDURE

The Knaster inheritance procedure can be used to divide items among more than two parties. This procedure allocates the items one at a time but requires the parties to have a large amount of cash available.

- (1) The n heirs - independently and simultaneously - submit monetary bids for the item.
- (2) The high bidder is awarded the item and places $\left(\frac{n-1}{n}\right) * (\text{bid})$ in a holding account.
- (3) Each of the other heirs withdraws $\frac{1}{n}$ of his or her own bid from the holding account.
- (4) The money left in the holding account is divided equally among all n heirs.
- (5) The final division of items and cash for the heirs is stated.

Example 2.1.

Abe, Betty and Calvin have inherited a house to share equally. Each person writes a bid for the house on a piece of paper. Who gets the house and how much money does each person get or pay?

Step1: Abe bid \$90,000, Betty bid \$75,000, and Calvin bid \$60,000.

Step2: _____ gets the house and places _____ in a holding account.

Step3-4:

Abe:

Betty:

Calvin:

Step5:

Abe:

Betty:

Calvin:

When there are multiple items to be shared, use the Knaster inheritance procedure for each item individually and then total the results.

Example 2.2.

James, Mary, Phil, and Yaz have inherited a car and a truck to share equally. They each submit sealed bids for both vehicles. Describe a fair division of these items (tell who gets each vehicle and how much money each person gets or pays).

Step1 :

Car	Truck
James' bid: \$20,000	James' bid: \$30,000
Mary's bid: \$18,000	Mary's bid: \$38,000
Phil's bid: \$21,000	Phil's bid: \$41,000
Yaz's bid: \$14,000	Yaz's bid: \$44,000

Step2:

_____ gets the car and places _____ in a holding account.

_____ gets the truck and places _____ in a holding account.

Step3-4:

Car	Truck
James' bid:	James' bid:
Mary's bid:	Mary's bid:
Phil's bid:	Phil's bid:
Yaz's bid:	Yaz's bid:

Step5:

Car	Truck
James' bid:	James' bid:
Mary's bid:	Mary's bid:
Phil's bid:	Phil's bid:
Yaz's bid:	Yaz's bid:

Example 2.3.

Ben and Carlos inherited a boat that they must share equally. They both submit secret bids for the boat. Ben bids \$125,000 and Carlos bids \$172,000. Who gets the boat and how much money does each person get or pay?

3. 13.5 DIVIDE AND CHOOSE

With divide-and-choose, one party divides the object into two parts in any way, and then the second party chooses the part he or she wants.

Example 3.1.

Let's cut a cake and assume that both people want as much cake as possible.

- (1) If you are the divider, what is your strategy for cutting?

- (2) If you are the chooser, what is your strategy for choosing?

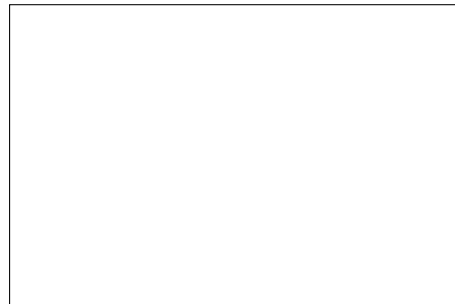
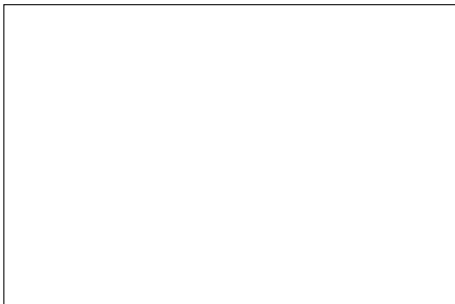
- (3) If part of the cake has more icing than another part, could that affect the strategies?

4. 13.6 CAKE-DIVISION PROCEDURES: PROPORTIONALITY

A *cake-division procedure* for n players is a procedure that the players can use to allocate a cake among them so that each player has a strategy that will guarantee that player a piece with which he or she is "satisfied."

The Steinhaus Proportional Procedure (Lone Divider) for Three Players

- (1) The players (A, B, and C) let player A be the divider.
- (2) Player A divides the cake into three equal pieces: i, ii, and iii
- (3) If players B and C each like different pieces, they get those pieces and A gets the remaining piece.
- (4) If players B and C both want the same piece, they give the least desirable piece to player A. The remaining two pieces are combined. Player B divides the combined pieces and C chooses.



5. 13.8 VICKREY AUCTIONS

In a Vickrey auction, bidders independently submit _____ bids for the object being sold. The winner is the high bidder, but he or she pays only the amount of the second-highest bid. (For our examples, we will assume that ties do not occur.)

Example 5.1.

Four people were bidding for tickets to a musical. Jane bid \$80, Carlos bid \$100, Gary bid \$110, and Amit bid \$120.

- (1) Who wins the tickets?
- (2) How much does he/she pay for the tickets?

eBay uses a variation on Vickrey Auctions for online bidding. An eBay auction has a minimum bid and a bid increment set by the seller before bidding starts. A bidder is free to enter the highest price that he/she is willing to pay for the item, because he/she will only have to pay the amount of the second-highest bid plus the bid increment if he/she wins. Each time a higher bid is placed, the “current eBay bid” is updated to be the second-highest bid plus one bid increment. Bidding continues until time expires.

Example 5.2.

Shelly, John, and Andy are bidding on a painting on eBay. The minimum bid was set at 50, *and the bid increment is 2*. Complete the following chart to show the progress of the auction before time ran out.

(a)

Bidder	Bid	Current Winner	Current eBay bid
Shelly	\$50	Shelly	\$50
John	\$2000		
Shelly	\$60		
Andy	\$83		
Shelly	\$120		

- (b) Who won the auction?
- (c) How much did he/she pay for the painting?