COMBINATORIAL COHEN–MACAULAY CRITERION: RESEARCH STATEMENT

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1. INTRODUCTION

My area of study is combinatorial commutative algebra. The goal of combinatorial commutative algebra is to study the interplay between commutative algebra and various subfields of combinatorics such as enumerative combinatorics and discrete geometry. Two cornerstones of this field were Hochster's work [Hoc72] on Cohen–Macaulayness of normal affine semigroup rings using the underlying polyhedral cones and Stanley's work [Sta75] on proving Upper Bound Conjecture using Reisner's criterion showing that k-algebras associated to certain simplicial complexes are Cohen–Macaulay. These works provided a framework of the combinatorial commutative algebra.

Especially, my main objects of study are monomial ideals in affine semigroup rings, which generalize both monomial ideals in polynomial rings and semigroup rings. In the viewpoint of affine semigroup rings, polynomial rings are special cases of normal rings when the underlying polyhedral cones are simplexes. Simultaneously, as subrings of polynomial rings, $\mathbb{Z}^d$-graded structure lets monomial ideals of affine semigroup rings exist. Therefore, it seems natural to generalize known results for monomial ideals in the polynomial ring to the affine semigroup ring case.

Among plenty of algebraic properties of affine semigroup rings, my interest was focused on a combinatorial criterion for the Cohen–Macaulayness of quotients of affine semigroup rings by monomial ideals. Not only this criterion is meaningful as a generalization of Hochster and Reisner's result about the Stanley-Reisner ring [Hoc77], but also the criterion is interesting for algebraists, geometers, and combinatorialists as model examples of their fields. For example, varieties having affine semigroups as their coefficient rings are called (non-normal) toric varieties which algebraic geometers and combinatorialists still actively work on. Moreover, algebraists and combinatorialists working on A-hypergeometric D-module are also concerned with the Cohen–Macaulayness of related affine semigroup rings over $\mathbb{C}$. Studying such a combinatorial criterion would help them approach more intensive and complicate examples for their study.

2. CURRENT RESULT

2.1. **Affinity between monomial ideals and polyhedral geometry.** Given a collection $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\}\mathbb{Z}^d$, the *affine semigroup* $Q = \mathbb{N}\mathcal{A}$ is a span of $\mathcal{A}$ by non-negative integers $\mathbb{N}$. The *affine semigroup ring* over a field $\mathbb{K}$ associated with $\mathcal{A}$ is defined as $\mathbb{K}[Q] := \mathbb{K}[\mathbf{t}^{\alpha_1}, \cdots, \mathbf{t}^{\alpha^n}] \subset \mathbb{K}[t_1, \cdots, t_d]$ with notation $\mathbf{t}^{\alpha_i} := t_1^{\alpha_{i,1}} \cdots t_d^{\alpha_{i,d}}$.

Monomial ideals in affine semigroup rings are closely related to polyhedral geometry, especially polyhedral cone $\mathbb{R}_{\geq 0}Q := \{\sum_{i=1}^d c_i\alpha_i : c_i \in \mathbb{R}_{\geq 0}\}$ as a set of all non-negative real linear combinations over $\mathcal{A}$. For example, a poset $\operatorname{Spec}_{\text{mon}} \mathbb{K}[Q]$ of all prime monomial ideals of $\mathbb{K}[Q]$ with an order by inclusion has an order reversing isomorphism with the face lattice $\mathcal{F}(\mathbb{R}_{\geq 0}Q)$ of the cone $\mathbb{R}_{\geq 0}Q$ [MS05]. Since this isomorphism is induced by the span of the complement of a monomial prime ideal over $\mathbb{R}_{\geq 0}$, a set $\mathcal{F}(Q) := \{Q \setminus \deg \mathfrak{p} : \mathfrak{p} \in \operatorname{Spec}_{\text{mon}} \mathbb{K}[Q]\}$ of complements of monomial prime ideals are called *faces* of Q. This set coincides with $\{F \cap Q : F \in \mathcal{F}(\mathbb{R}_{\geq 0}Q)\}$ [MS05]. Likewise, all radical ideals correspond to polyhedral subcomplexes of $\mathbb{R}_{\geq 0}Q$. These facts justify the notation of I_Δ for a radical monomial ideal, where Δ is a polyhedral subcomplex of $\mathbb{R}_{\geq 0}Q$.

On the other hand, many affine semigroups share the same cone. Indeed, the map $\mathbb{R}_{\geq 0}(-)$ sending an affine semigroup Q to the cone $\mathbb{R}_{\geq 0}Q$ is not injective but surjective. In other words, there are affine semigroups not containing all integral points of $\mathbb{R}_{\geq 0}Q$. Thankfully, these *holes*, integral points of $\mathbb{R}_{\geq 0}Q$ not belonging to Q, can be decomposed into a union of translated faces [HTY09, Kat15, MY21]. All of these connections naturally lead me to study monomial ideals as polyhedral-geometric objects.

2.2. **Standard pairs as polyhedral geometric objects.** The notion of *standard pair* was introduced by [STV95] as a combinatorial structure to calculate bounds on geometric multiplicities of monomial ideals in a polynomial ring. This can be generalized for monomial ideals of affine semigroup rings [MY20]. For a monomial ideal I of $\mathbb{K}[Q]$, a *proper pair* $(\mathbf{t}^\alpha, F)$ for some $\mathbf{t}^\alpha \in \mathbb{K}[Q]$ and $F \in \mathcal{F}(Q)$ is a pair satisfying $\mathbf{t}^{\alpha+\varphi} \notin I$ for any $\varphi \in F$. Give an order between proper pairs $(\mathbf{t}^\alpha, F) > (\mathbf{t}^\beta, G)$ if $\{\mathbf{t}^{\alpha+\varphi} : \varphi \in F\} \supset \{\mathbf{t}^{\beta+\varphi} : \varphi \in G\}$. The *standard pairs* $\operatorname{Std}(I)$ of I is the set of maximal proper pairs of I. Moreover, a pair $(\mathbf{t}^\alpha, F)$ divides $(\mathbf{t}^\beta, F)$ if there exists $\gamma \in Q$ such that $(\mathbf{t}^{\alpha+\gamma}, F) < (\mathbf{t}^\beta, F)$. The set of *overlap classes* $\overline{\operatorname{Std}}(I)$ of I is defined as $\operatorname{Std}(I)/\sim$ where two pairs of $(\mathbf{t}^\alpha, F), (\mathbf{t}^\beta, F)$ are equivalent if they divide each other.

Standard pairs and overlap classes are useful when studying a monomial ideal I of an affine semigroup ring. For example, I is primary if and only if all standard pairs of I share the same face. Moreover, I is irreducible if and only if it is primary and has the unique overlap class that is maximal with respect to divisibility. Thus, the number of maximal overlap classes of I is the same as the number of components of its irredundant irreducible decomposition [MY20]. Moreover, standard pairs and overlap classes are finitely many.

2.3. **Generalized Ishida complex.** Matusevich and I have extended the *Ishida complex* to calculate the local cohomology of quotients of pointed affine semigroup rings supported on radical monomial ideals. Originally, the Ishida complex calculates such cohomology supported on the maximal graded ideal only [Ish88, BH93, ILL$^+$07]. Indeed, we observed that the chain complex used in the original Ishida complex is not $\mathcal{F}(\mathbb{R}_{\geq 0}Q)$ but the unbounded rays of the convex hull of the maximal monomial ideal. Precisely, let J be a monomial ideal of $\mathbb{K}[Q]$. Let T be a crosssection of the convex hull $\operatorname{conv}(\sqrt{J})$ of $\sqrt{J}$ as a subset of $\mathbb{R}^d$ containing all unbounded faces [Zie95]. The *generalized Ishida complex* over J is a chain complex

$$L^\bullet : 0 \to L^0 = \mathbb{K}[Q] \xrightarrow{\delta} L^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} L^d \to 0$$

such that $L^i = \oplus_{F \in \mathcal{F}(T)^i} \mathbb{K}[Q]_{\hat{F}}$ is a direct sum of all localizations of $\mathbb{K}[Q]$ by a monomial prime ideal whose corresponding face $\hat{F}$ is the minimal face of Q containing $F \cap Q$ over all i-dimensional faces of T. Also, $\delta : L^i \to L^{i+1}$ is a direct sum of all canonical maps of localizations between components of L^i and L^{i+1} with changes of signs. This sign is determined by the orientation of T. Then, the local cohomology $H^i_I(M)$ of M with support J is isomorphic to $H^i(L^\bullet \otimes_{\mathbb{K}[Q]} M)$ for any $\mathbb{K}[Q]$-module M [MY21].

2.4. **Hochster-type formula for the Hilbert series of the local cohomology.** One of our goals was to produce a Hochster-type formula for the Hilbert series of local cohomology in this context. Fortunately, the *standard pair topology* allows us to classify the graded parts of the generalized Ishida complex that share the same polyhedral chain complex. Indeed, overlap classes of a monomial ideal I with face F are restrictions of overlap classes of the localization $(\mathbb{K}[Q]/I)_F$ by a monomial prime ideal $\mathfrak{p}_F := \{\mathbf{t}^\alpha : \alpha \in Q \setminus F\}$ to $\mathbb{K}[Q]/I$. In other words, the set of all overlap classes over all localizations cover the *standard monomial space* $\operatorname{stdm}(I) = \{\mathbf{t}^\alpha : \mathbf{t}^\alpha \in (\mathbb{K}[Q]/I)_F \text{ for some } F \in \mathcal{F}(Q)\}$ consisting of all monomials appearing as standard monomials for some localization. Hence, by thinking of each overlap class as a clopen set, $\mathrm{stdm}(I)$ enriches a topology generated by overlap classes. Since the number of standard pairs (and equivalence classes) is always finite, the topology is finite. We saw that two monomials whose degrees are in the same minimal open set appear on the same localizations. Therefore, two monomials in the same minimal open set have the same graded parts of Ishida complex tensored with. We call a set of faces of T whose corresponding localizations containing the given minimal open set the *chaff* of the minimal open set; such chaffs determine the graded parts of the Ishida complex. Hence, the Hilbert series of $H^i_J(\mathbb{K}[Q]/I)$ is

$$\mathrm{Hilb}(H^i_J(\mathbb{K}[Q]/I,\mathbf{t})) = \sum_{\sigma\in\mathrm{chaff}(I)} \dim_{\mathbb{K}} H^i(\sigma,\mathbb{K}) \left(\sum_{\alpha\in G_\sigma\in\mathrm{grain}(I)} \mathbf{t}^\alpha \right)$$

where $\mathrm{chaff}(I)$ and $\mathrm{grain}(I)$ are sets of all chaffs or grains of I respectively. Since grains are intersections of translations of faces of Q, $\sum_{\alpha\in G_\sigma\in\mathrm{grain}(I)} \mathbf{t}^\alpha$ can be replaced with a rational function.

2.5. **Cohen–Macaulayness criterion of the quotients by monomial ideals.** The Hochster type formula from the standard pair topology gives a Cohen–Macaulayness criterion for a quotient of an affine semigroup by a monomial ideal. It is a celebrated result that Stanley-Reisner rings are Cohen–Macaulay if and only if their corresponding simplicial complex Δ has links with vanishing non-top homology. Likewise, quotients of affine semigroup rings by monomial ideals are Cohen–Macaulay if and only if their chaffs have vanishing non-top homology [MY21]. In other words, the chaffs play the same role as links in the Stanley-Reisner case. This result generalizes the combinatorial Cohen–Macaulayness criterion for an affine semigroup, given in [TH86].

2.6. **Duality between local cohomologies of Stanley-Reisner ring.** Our Hochster type formula for quotients of affine semigroup rings elucidates a hidden duality between the local cohomologies of the Stanley-Reisner rings. To see this, fix a Stanley-Reisner ring $\mathbb{K}[\mathbf{t}]/I_\Delta$ for a monomial radical ideal I_Δ of the polynomial ring $\mathbb{K}[\mathbf{t}]$ corresponding to a simplicial complex Δ. Then Reisner's criterion can be reformulated as follows; for each face F of d-simplex, there exists the unique grain G_F whose graded part of the Ishida complex with maximal ideal support is equal to the link of F in Δ if $F\in\Delta$, or 0 if $F\notin\Delta$ and chaffs from all other grains are acyclic [MY21]. Thus, the Hilbert series of $H^\bullet_{\mathfrak{m}}(\mathbb{K}[\mathbf{t}]/I_\Delta)$ can be decomposed as a finite sum over grains G_F for each $F\in\Delta$. Also, overlap classes of $\mathbb{K}[\mathbf{t}]/0 = \mathbb{K}[\mathbf{t}]$ are equal to sets of lattice points from orthants of $\mathbb{K}[\mathbf{t}]$, therefore we may label each set of lattice points of orthants as $\mathfrak{R}_F$ for each F in the d-simplex. Then, for each F in the d-simplex,

$$H^\bullet_{\mathfrak{m}}(\mathbb{K}[\mathbf{t}]/I_\Delta)_\alpha \cong H^{\dim\mathbb{K}[\mathbf{t}]-\bullet}_{I_\Delta}(\mathbb{K}[\mathbf{t}])_\beta$$

for any $\alpha\in G_F$ and $\beta\in\mathfrak{R}_{F^c}$ [MY21].

Indeed, [Hun07, Ric] mentioned that $H^\bullet_{\mathfrak{m}}(\mathbb{K}[\mathbf{t}]/I_\Delta)=0$ if and only if $H^{\dim\mathbb{K}[\mathbf{t}]-\bullet}_{I_\Delta}(\mathbb{K}[\mathbf{t}])=0$ for the maximal ideal $\mathfrak{m}$ and a radical monomial ideal I_Δ, which is a corollary of this duality.

2.7. **Computational package.** All results above are constructive in the sense that algorithms computing them exist. For example, Matusevich and I devised an algorithm to find standard pairs of a monomial ideal [MY20]. Using this, I implemented a `SageMath` package `StdPair` to calculate algebraic invariants of an affine semigroup and its (monomial) ideal symbolically, such as standard pairs, associated primes, multiplicity, minimal generators, and primary decomposition [Yu20].

Moreover, we observed that transverse sections of the convex hull of ideals are polytopes generated by "cutting" faces out of other polytopes. These are all combinatorially determined regardless of their geometric realization [MY21]. Using this, we suggest an algorithm to calculate our Hochster type formula for a quotient of an affine semigroup ring. This generalizes an algorithm calculating the local cohomology of modules over normal affine semigroup rings [HM05].

3. Future studies

3.1. **Finding a combinatorial Cohen–Macaulay criterion for general (unsaturated) binomial ideals.** It seems that the Ishida complex method can be expanded to a non-prime binomial ideals. Indeed, for an unsaturated binomial ideal I whose saturation is I_{sat}, the Ishida complex from the polyhedral structure of the affine semigroup ring $\mathbb{K}[Q] \cong \mathbb{K}[\mathbf{x}]/I_{sat}$ calculates the local cohomology of $\mathbb{K}[\mathbf{x}]/I$. Using this we hope that we may calculate the Hilbert series of the binomial coefficient ring $\mathbb{K}[\mathbf{x}]/I$ and obtain the Hochster type formula.

3.2. **Classifying acyclic chaffs using hyperplane arrangement.** Chaffs are of the intersection of the upper set of the poset of regions and face lattice. Here, the *poset of regions* is a poset generated by hyperplanes of $\mathbb{R}_{\geq 0}Q$ which embeds the face lattice of affine semigroups [BEZ90]. Indeed, the condition when poset of regions are lattice is well studied [Rea03b, Rea16, DHMP20, Rea03a]. However, it is still unclear whether the intersection between the upper set of the poset of regions and the face lattice as a sub-lattice of the poset of regions is acyclic or not. All we know is that the complement of the face lattice in the poset of region is a subset of regions whose rank is less than $n-3$ where n is the rank of the poset of regions.

Moreover, this classification naturally asks the classification of acyclic polyhedral complexes, which is still an open problem, unlike the case of simplicial complexes solved by Stanley and Duval [Duv94, Sta93] in relation to Kalai's conjecture [BK88].

3.3. **Characterizing local cohomology modules with infinite dimensional socles.** The graded parts of local cohomology modules $H^{\bullet}_{J}(\mathbb{K}[Q]/I)$ are covered by grains. Thus, $H^{\bullet}_{J}(\mathbb{K}[Q]/I)$ can be regarded as a union of lattices that came from the convex hull or polyhedral cones. On the other hand, the graded parts of the socle of a module M can be regarded as lattice points in the face whose outer normal vector equals the generators of Q. Hence, it seems natural to ask whether one can find a combinatorial condition when $H^{\bullet}_{J}(\mathbb{K}[Q]/I)$ has an infinite-dimensional socle [MS05][Problem 13.18]. This problem was inspired by Hartshorne's counter-example for Grothendieck's conjecture.

3.4. **Class groups of non-normal toric varieties.** Standard pairs may help to study class groups of non-normal toric varieties via standard pairs. Indeed, Matusevich and I constructed *void pairs* which organize all non-lattice points of the polyhedral cone $\mathbb{R}_{\geq 0}Q$ as translations of faces [MY21]. Void pairs might allow one to calculate monomial fractional ideals of non-normal affine semigroup rings. If it was true, then we could calculate the class group of non-normal toric varieties by generalizing the fact that class groups of the varieties are the direct sum of the class groups of the field and those of normal monoid [BG09][Theorem 4.60].

3.5. **Irreducible resolution of quotients of non-normal affine semigroup rings.** An ideal W of $\mathbb{K}[Q]$ is *irreducible* if W cannot be expressed as an intersection of two distinct ideals [MS05]. The *irreducible resolution* of a module M is an exact sequence $0 \to M \to \overline{W}^0 \to \cdots$ such that each $\overline{W}^i$ is a direct sum of quotients of $\mathbb{K}[Q]$ by irreducible ideals. Unless Q is normal, there is no known algorithm for constructing irreducible decompositions and resolutions. Although an algorithm for normal cases was known [HM05] but not implemented in software like `Macaulay2` or `SageMath`. Void pairs introduced in Section 3.4 seems to shed light on finding an implementable algorithm over non-normal affine semigroup rings for `Macaulay2` or `SageMath`.

3.6. **Find a combinatorial criterion for Gorensteinness.** For a Noetherian local ring A, A is regular implies that A is a complete intersection, A is a complete intersection implies that A is Gorenstein, and A is Gorenstein implies that A is Cohen–Macaulay [Mat89][p.171]. Thus, it seems natural to ask when affine semigroup rings, as a $*$-local rings, have such properties. From [Mat89][Theorem 14.4], $\mathbb{K}[Q]/I$ is regular if it is isomorphic to a polynomial ring. Hence, the combinatorial criterion for regularity is to check whether $\mathbb{K}[Q]/I$ has the unique standard pair isomorphic to $\mathbb{N}^d$ for some d. Also, [FMS97] provides a combinatorial criterion when affine semigroup rings are complete intersections. The Koszul complex of quotients of affine semigroup gave a combinatorial criterion for quotients of affine semigroups which are complete intersections.

A combinatorial criterion for Gorensteinness of affine semigroup rings is provided by [TH86], while it is still open for the cases of quotients of affine semigroups. In this case, the local cohomology of canonical modules of affine semigroup rings may answer whether there is a combinatorial criterion for the Gorensteinness of affine semigroup rings. This generalizes the criterion when affine semigroup rings are normal [BG09][Theorem 6.33]. The challenge of this problem is to find a finite cover of the canonical module compatible with the Ishida complex.

3.7. **Software Development.** I plan to distribute libraries for calculating algebraic properties of affine semigroup rings and their quotients. Toward this goal, an algorithm the Hilbert series of local cohomologies of affine semigroup rings over a radical monomial ideal will be integrated in the package `StdPair` [MY20]. Moreover, this package will be redistributed as a C++ library so that not only `SageMath` users but also `Macaulay2` users have benefits from `StdPair`.

3.8. **Other interests.**

3.8.1. *Categorifying collections of quotients of affine semigroups.* Recent works [Gub19, BGG16, BCG13] studied polytopes as objects for categorical/homological analysis. Since polyhedrons can be regarded as the homogenization of polytopes, and the collection of affine semigroups corresponds to their underlying polyhedral cones, we may ask what kind of properties the collection of affine semigroups have as a category. Definitely, as a subcategory of the category of monoids, it inherits lots of nice properties such as complete and co-completeness [Flo15, CLS12]. However, it is still unclear that whether Hom set between two affine semigroups forms an affine semigroup like Hom-polytope [BCG13], or that the category of affine semigroups forms an abelian category.

3.8.2. *Interdisciplinary research.* Although my main interest lies in affine semigroup rings and their relative fields, I also have interest on working with statisticians or scientists in other disciplines. Recently, Kisung You and I suggested a gradient-free dimension reduction algorithm finding a linear projection preserving the persistent homology of the given data [YY21] as much as possible. In this paper, I suggested measures showing how much portions of the filtration of Rips complexes over the original data are quasi-isomorphic (resp. homotopy equivalent) to the filtration of Rips complexes over the projected data.

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