1.5. Equivalence of Categories.

$2 : \underset{0}{\cdot} \longrightarrow \underset{1}{\cdot}$

① :
$$\begin{tikzcd} C \arrow[r, "i_0"] \arrow[dr, "F"'] & C\times 2 \arrow[d, "H"] & C \arrow[l, "i_1"'] \arrow[dl, "G"] \\ & D & \end{tikzcd}$$

$c \longmapsto (c,0)$, $(c,1) \longmapsfrom c$

Lem 1.5.1. $F, G : C \rightrightarrows D$. Then,

$\{\alpha : F \Rightarrow G \mid$ natural transformation$\}$

$\updownarrow$ $\boxed{\text{bijection}}$

$\{H :$ satisfying diagram $\}$

pf) Construction of H is in Ex 1.5.1)

To see bijective corresp, notes that given H, with $(c, 0) \xrightarrow{f\cdot(0\to 1)} (c', 1)$

$H(c,0) = Fc$, $H(c',1) = Gc'$ by diagram. Now define

$$\alpha_c = H\left((c,0) \xrightarrow{1_c\cdot(0\to 1)} (c,1)\right).$$

Then,

② :
$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \end{tikzcd}$$

If we show this diag. commutes, done.

Actually it is taking H on

$$\begin{tikzcd}
(C,0) \arrow[r, "1_C\cdot(0\to 1)"] \arrow[d, "f\cdot 1_0"'] \arrow[dr, phantom, "\circlearrowright"] & (C,1) \arrow[d, "f\cdot 1_1"] \\
(C',0) \arrow[r, "1_{C'}\cdot(0\to 1)"'] & (C',1)
\end{tikzcd}$$

$\Rightarrow$ ② commutes!

Thus, H induces a natural transformation α.

Hence the bijection occur ▨

If $C = 2$, 2×2 is depicted as

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] \arrow[dr] & (1,0) \arrow[d] \\
(0,1) \arrow[r] & (1,1)
\end{tikzcd}$$

then H sends it to

$$\begin{tikzcd}
F_0 \arrow[r, "F(\to)"] \arrow[d, "\alpha_0"'] \arrow[dr] & F_1 \arrow[d, "\alpha_1"] \\
G_0 \arrow[r, "G(\to)"'] & G_1
\end{tikzcd}$$

If we change 2 to $\mathbb{I}$: $\bullet \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \bullet$ in the lemma, then

Functor satisfying comm. diagram

$\updownarrow$ bijection

Natural Isomorphism.

Def 1.5.4 Equivalence of Categories. Consists of.

① $F : C \rightleftarrows D : G$

② $\eta : 1_C \cong GF$, $\varepsilon\, FG \cong 1_D$

natural isomorphism.

In, this case write $C \simeq D$.

(cf. $C \cong D$: isomorphism of category)

Lem 1.5.5 $C \simeq D$ is equivalence relation

pf) Ex 1.5.vi.

Ex 1.5.6.

(i) $(-)_+ : Set^\partial \longrightarrow Set_*$ in 1.3.

$U : Set_* \longrightarrow Set^\partial$

are actually equiv of category by

$1_{Set^\partial} = U(-)_+$ and

$\eta : 1_{Set_*} \cong (U-)_+$ with

$$\eta_{(X,x)} : (X, x) \longrightarrow (X \setminus \{x\} \cup \{X \setminus \{x\}\}, X \setminus \{x\})$$
$$y \longmapsto y \quad \text{if } y \neq x.$$
$$x \longmapsto X \setminus \{x\}$$

(2) $$\mathrm{Mat}_{\mathbb{k}} \underset{H}{\overset{\mathbb{k}^{(-)}}{\rightleftarrows}} \mathrm{Vect}_{\mathbb{k}}^{\text{basis}} \underset{C}{\overset{U}{\rightleftarrows}} \mathrm{Vect}_{\mathbb{k}}^{\text{f.d.}}$$

Where $\mathbb{k}^{(-)} \; (n \xrightarrow{M} m) \Rightarrow \mathbb{k}^n \xrightarrow{L_M} \mathbb{k}^m$.

U : forgetful functor

C : Sending V. space by choosing a basis,

H : Sending V.S to dim and linear transformation to matrix over given bases.

Aim : WTS

$$\mathrm{Mat}_{\mathbb{k}} \simeq \mathrm{Vect}_{\mathbb{k}}^{\text{basis}} \simeq \mathrm{Vect}_{\mathbb{k}}^{\text{f.d}}$$

where $\mathrm{Vect}_{\mathbb{k}}^{\text{basis}}$: Category of f.d. V.S with a chosen basis.

Def: 1.5.7. $F : \mathcal{C} \to \mathcal{D}$ a functor is

① full if $\forall x, y \in \mathcal{C}$, $\mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$ is surjective

② faithful " " is injective

③ Essentially surjective if $\forall d \in \mathcal{D}$, $\exists c \in \mathcal{C}$ s.t. $Fc \cong d$.

Rem 1.5.8

(1) Full, faithfull : local conditions

(2) Faithfull and injective on object = embedding

(3) Full + faithful = fully faithful

fully faithful + injective on object = full embedding

In case of full embedding, image of domain = full subcategory of the codomain

Thm 1.5.9. $F : C \simeq D : G \iff F, G$ are full + faithful + ess. surj.

(under axiom of choice)

Lem 1.5.10. For $f : a \to b$ with $a \cong a'$, $b \cong b'$ $\exists\, f' : a' \longrightarrow b'$ s.t. any arrow in the left square makes it commutes

$$\begin{tikzcd} a \arrow[d, "f"'] & a' \arrow[d, "f'"] \\ b & b' \end{tikzcd}$$

pf) Ex 1.5.iii)

pf of Thm) Let $f, g : c \rightrightarrows c'$ with $Ff = Fg$.

$$\Rightarrow \begin{tikzcd} c \arrow[r, "\eta_c", "\cong"'] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & GFc \arrow[d, shift right, "GFf"'] \arrow[d, shift left, "GFg"] \arrow[dr, phantom, "\circlearrowleft"] & c \arrow[l, "\eta_c"', "\cong"] \arrow[d, "g"] \\ c' \arrow[r, "\cong", "\eta_{c'}"'] & GFc' & c' \arrow[l, "\cong"] \end{tikzcd}$$

with $GFf = GFg$. (since $Ff = Fg$)

By 1.5.10,
$$\begin{tikzcd} c \arrow[r, "\eta_c", "\cong"'] \arrow[d, "f"'] & GFc \arrow[r, "\eta_c^{-1}", "\cong"'] & c \arrow[d, "g"] \\ c' \arrow[r, "\cong", "\eta_{c'}"'] & GFc' \arrow[r, "\cong", "\eta_{c'}^{-1}"'] & c' \end{tikzcd}$$
commutes

$\Rightarrow \quad g = f. \quad \Rightarrow \quad F$ is faithful.

(So is G by applying same argument on FG
$\Rightarrow$ "by symmetry".)

Let $g \in D(Fc, Fc')$

$$f := \eta_{c'}^{-1}(Gg)\,\eta_c \in C(c, c')$$

$$\Rightarrow \quad \begin{tikzcd} GFc \arrow[r, "\eta_c^{-1}"] \arrow[d, "GFf"'] \arrow[dr, phantom, "\circlearrowleft"] & c \arrow[r, "\eta_c"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & GFc \arrow[d, "Gg"] \\ GFc' \arrow[r, "\eta_{c'}^{-1}"'] & c' \arrow[r, "\eta_{c'}"'] & GFc' \end{tikzcd}$$
commutes from equivalence and def of f and g.

$\Rightarrow \quad Gg = GFf.$ From G is faithful,

$g = Ff \quad \Rightarrow \quad F$ is full.

Now, for $d \in D$, $\quad \varepsilon_d : FGd \xrightarrow{\cong} d$

$\Rightarrow \quad F$ is essentially surjective.

By symmetry, G is also full and ess. sur.

Conversely let $F : C \longrightarrow D$, full faithful and ess. sur. Want to construct $G : D \to C$ equiv.

By axiom of choice and essential surj.

$\forall d \in D, \ \exists ! c \in C$ s.t. $d \cong Fc$

Let $Gd := c \implies \varepsilon_d : d \cong FGd$

Then, given $f \in D(d, d')$

We have
$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"] \arrow[d, dashed, "g"] & d \arrow[d, "f"] \\ FGd' \arrow[r, "\varepsilon_{d'}"'] & d' \end{tikzcd}$$
$\implies$ Let $g = \varepsilon_{d'}^{-1} f \varepsilon_d$

$\implies$ diagram commutes

Then, from F is fully faithful, $\exists h \in C(Gd, Gd')$ s.t. $Fh = g$. Let $Gf := h$

Hence, by this definition
$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"] \arrow[d, "FGf"'] \arrow[dr, phantom, "\circlearrowright"] & d \arrow[d, "f"] \\ FGd' \arrow[r, "\varepsilon_{d'}"'] & d' \end{tikzcd}$$
$\implies \varepsilon_d : FG \Rightarrow 1_D$

Claim 1: G is functorial

For first condition,

$$\begin{tikzcd}
FGd \arrow[r, "\varepsilon_d"] \arrow[d, "FG1_d"'] \arrow[dr, phantom, "①"] & d \arrow[r, "\varepsilon_d^{-1}"] \arrow[d, "1_d"] \arrow[dr, phantom, "②"] & FGd \arrow[d, "F(1_{Gd})"] \\
FGd \arrow[r, "\varepsilon_d"'] & d \arrow[r, "\varepsilon_d^{-1}"'] & FGd
\end{tikzcd}$$

① commutes by definition of $G1_d$

② 〃 functoriality of F

$$\Longrightarrow \quad FG1_d = F(1_{Gd}) \Rightarrow G(1_d) = 1_{Gd}$$

by faithfullness of F

For second condition,

let $f: d \to d'$, $f': d' \to d''$

$\Rightarrow$

$$\begin{tikzcd}
FGd \arrow[r, "\varepsilon_d"] \arrow[dd, bend right, "FG(f'f)"'] \arrow[ddr, phantom, "?"] & d \arrow[r, "\varepsilon_d^{-1}"] \arrow[d, "f"] \arrow[dd, bend right, "f'f"'] & FGd \arrow[d, "F(Gf)"] \\
 & d' \arrow[r, "\varepsilon_{d'}^{-1}"] \arrow[d, "f'"] & FGd' \arrow[d, "F(Gf')"] \\
FGd'' \arrow[r, "\varepsilon_{d''}"'] & d'' \arrow[r, "(\varepsilon_{d''})^{-1}"'] & FGd''
\end{tikzcd}$$

commutes by definition of $G(f'f)$, Gf, and Gf' (we define these to commute such diagram)

$$\Rightarrow FG(f'f) = F(G(f')) \cdot F(G(f))$$
$$= F(G(f')\, G(f))$$

$\Rightarrow$ $G(f'f) = G(f')\, G(f)$
by faithfullness of F.

$\Rightarrow$ G is functor

Claim 2: $\eta : 1_C \Rightarrow GF$ is natural iso.

Define η_c is preimage of ε^{-1}_{Fc} by F

(It is well-def since F is fully faithful.)

i.e. $F\eta_c = \varepsilon^{-1}_{Fc}$. For any $f : c \to c'$,

$$\begin{tikzcd}
Fc \arrow[r, "F\eta_c"] \arrow[d, "Ff"'] \arrow[dr, phantom, "①"] & FGFc \arrow[r, "\varepsilon_{Fc}"] \arrow[d, "FGFf"] \arrow[dr, phantom, "②"] & Fc \arrow[d, "Ff"] \\
Fc' \arrow[r, "\varepsilon_{\eta_{c'}}"'] & FGFc' \arrow[r, "\varepsilon_{Fc'}"'] & Fc'
\end{tikzcd}$$

② commutes by naturality of ε.

Since ε is natural iso, ① also commutes. Converse... ?

$\Rightarrow$ $FGFf \circ F\eta_c = \varepsilon_{\eta_{c'}} \circ Ff = F\eta_{c'} \circ Ff$

$\Rightarrow$ $F(GFf \circ \eta_c) = F(\eta_{c'} \circ f)$
by

$\Rightarrow$ $GFf \circ \eta_c = \eta_{c'} \circ f$ by faithfullness of F

and by Ex 1.5 iv, $\eta_c, \eta_{c'}$ are iso $\Rightarrow$ η is natural iso. $\boxed{/\!/\!/}$

Corollary 1.5.11. $\mathrm{Mat}_k \simeq \mathrm{Vect}_k^{fd}$ for any field k.

pf.) $\mathrm{Mat}_k \rightleftarrows \mathrm{Vect}_k^{basis} \rightleftarrows \mathrm{Vect}_k^{fd}$

Those are ess sur, full, and faithful.

(full : set of matrix $\xleftrightarrow{\text{biject}}$ all lin transf of given basis
faithful $\xleftrightarrow{\text{biject}}$ lin transform.)

(ess surj : objects are 1-1 corresp.)

Def) Category is connected if $\forall x, y \in C$
x, y connected by a finite zig-zag
morphisms.

Prop 1.5.12. Any connected groupoid is equiv to automorphism gp of any of its object as a category.

pf) G : groupoid Fix $g \in \mathrm{Obj}\, G$.

Let $H = G(g, g)$.

$\Rightarrow$ $BG \hookrightarrow G$ is fully faithful since $BG(\cdot, \cdot) \cong G(g, g)$ by def.

$$\begin{tikzcd} \cdot \arrow[r, mapsto] \arrow[d, "\phi"'] & g \arrow[d, "\phi"] \\ \cdot \arrow[r, mapsto] & g \end{tikzcd}$$

and ess. surj.
(Groupoid is connected. (all mor = iso)
$\Rightarrow$ every pair of objects is isomorphic)

Cor 1.5.13. X : path-connected
x : base pt of X

$\Longrightarrow$ $\pi_1(X, x)$

pf) Let $\Pi_1(X)$: fundametal groupoid

By prop 1.5.12, for any $x, x' \in X$

$$\pi_1(X, x) \overset{\simeq}{\hookrightarrow} \Pi_1(X) \overset{\sim}{\hookleftarrow} \pi_1(X, x')$$

This gives $\pi_1(X, x) \simeq \pi_1(X, x')$

Claim : $C \simeq D$ with C, D : 1 object category
i.e. C, D are group.
$\Longrightarrow$ $C \cong D$ as a group.

pf) Let $F : C \longrightarrow D$ be equivalence

$\Longrightarrow$ $C(\cdot, \cdot) \cong D(\cdot, \cdot)$ as bijection.

By 1.3.i) Exercise, F is gp homo.

on $C(\cdot, \cdot)$ and $D(\cdot, \cdot)$

From bijection, F is bijective homo

$\Longrightarrow$ Gp Isomorphism

$\Longrightarrow$ $C \cong D$ as a group.

(Moreover $C \cong D$ as category since
gp iso F^{-1} induces a functor s.t. $1_C = FF^{-1}$
$1_D = F^{-1}F$.)

Rem 1.5.14 Top_*^{pc} : path conn spaces

$$\pi_1 : \mathrm{Top}_*^{pc} \xrightarrow{\pi_1} \mathrm{Group} \hookrightarrow \mathrm{Cat}$$

$$\Pi_1 : \mathrm{Top}_*^{pc} \xrightarrow{U} \mathrm{Top} \xrightarrow{\Pi_1} \mathrm{Groupoid} \hookrightarrow \mathrm{Cat}.$$

Inclusion of $\pi_1(X,x) \leq \Pi_1(Y)$ gives

natural transformation

$$\pi_1 \Rightarrow \Pi_1$$

$$\left(\begin{tikzcd} \pi_1(X,x) \arrow[r, hook] \arrow[d, "\pi_1(f)"'] \arrow[dr, phantom, "\circlearrowright"] & \Pi_1(X) \arrow[d, "\Pi_1(f)"] \\ \pi_1(Y,y) \arrow[r, hook] & \Pi_1(Y) \end{tikzcd}\right)$$

And this inclusion is a functor.

Moreover, " is equivalence of category

since fully faithful (as 1 obj subcategory)

and ess. surj. (from connected groupoid)

However, $\Pi_1(Y) \to \pi_1(X,x)$ inverse equivalence

requires axiom of Choice for its construction.

(In this case, $\forall p \in X$, choose a path p to x)

And these chosen paths $(p \to x)$ need not be

preserved by morphism in Top_*^{pc}.

Def 1.5.15 C : Category is <u>skeletal</u>

if it contains 1-object in each isomorphism class.

skC : a skeletal category equiv to C

(Unique up to iso)

Rem 1.5.16

skC construction : Choose 1 object in each iso classes and skC : full subcategory of C having these objects.

By thm 1.5.9, $skC \hookrightarrow C$ is full (by def) and faithful (by construction) and ess surj. by choice of representative of iso class

$\Rightarrow skC \simeq C$.

But $sk(-) : CAT \longrightarrow CAT$ is not a functor.

since $sk(F)$ may not be a functor

(ex) $C : \underset{0}{\cdot} \longrightarrow \underset{1}{\cdot} \cong \underset{2}{\cdot}$ $F : 0 \to 0$, $1 \to 2$, $2 \to 1$

$skC : \dot{0} \quad \dot{1}$.

$\Rightarrow skF := F|_{skC}$ sends 1 to 2 but 2 is not in skC.)

Ex 1.5.17)

(I) G: Connected Groupoid

$\Rightarrow$ $skG = G(g,g)$ (by construction)

(II) $sk(P, \geq)$ = poset! (since we delete any isomorphic but not equal one)

$\uparrow$ preorder

(III) $sk(Vect_{\mathbb{K}}^{fd}) \cong Mat_{\mathbb{K}}$

(every v.s with same dim is isomorphic)

(IV) $sk\, Fin_{iso}$ (Fin_{iso} : obj: finite set, Mor: bijection)

= obj: positive integer

Mor: $Hom(n,n) = S_n$ permutation of n

$Hom(n,m) = \phi$ if $m \neq n$.

Ex 1.5.18

$X: BG \to Set$: a left G-set.

Translation Groupoid : $T_G X$: obj = $X(\cdot) \in Set$

(Mor: $g: x \to y$ if $\exists g \in G$ s.t. $gx = y$.)

$\Rightarrow$ $skT_G X$: Obj: connected components in $T_G X$

i.e. orbit of G-action

Mor: for two distinct orbit, $\emptyset$.

Let $x \in X$, O_x : orbit of x

$$\mathrm{Hom}_{sk\mathcal{T}_G X}(O_x, O_x) \underset{\uparrow}{\cong} \mathrm{Hom}_{\mathcal{T}_G X}(x, x) =: G_x$$

from $sk\mathcal{T}_G X \simeq \mathcal{T}_G X$ equivalence

$\Rightarrow$ fully faithful

i.e, $\mathrm{Hom}_{sk\mathcal{T}_G X}(O_x, O_x)$ is stabilizer G_x of x.

$\Rightarrow$ Any pair in the same orbit should have isomorphic stabilizer.

Also, for any fixed $x \in X$, it has disj union

$$\bigcup_{y \in O_x} \mathrm{Hom}_{\mathcal{T}_G X}(x, y) = G$$

Since $|\mathrm{Hom}_{\mathcal{T}_G X}(x, y)| = |G_x| \ \forall y$,

$$|O_x| \cdot |G_x| = |G|.$$

i.e, orbit - stabilizer Thm.

Def) C : essentially small $\Rightarrow$ $C \simeq D$, D is small category

C : " discrete $\Rightarrow$ $C \simeq D$, D is discrete category

C: locally small, $D \simeq C$ $\Rightarrow$ D is locally small

groupoid 〃 $\Rightarrow$ 〃 groupoid.

〃 $\Rightarrow$ $D^{op} \simeq C^{op}$

$C \simeq D$, $C' \simeq D'$ $\Rightarrow$ $C \times C' \simeq D \times D'$

$\left(\begin{array}{c} f: x \to y \in C \text{ iso.} \\ F: C \simeq D \end{array} \right) \Leftrightarrow Ff \text{ is iso.}$

$F: C \hookrightarrow D$ fully faithful.

Then "Essential image" of F

= full subcategory of objects isomorphic to some Fc for $c \in C$.