Ch2 : Univ property, Representability, Yoneda Lemma.

2.1 Representable functor

We will explain what the universal property is.

Ex 2.1.1 $(\underset{\text{set}}{X}, \underset{\text{endo}}{f: X \to X}, \underset{\underset{X}{\cap}}{x_0})$

is called a <u>discrete dynamical system</u>

From $(X, f: X \to x, x_0)$ we have $\{x_i\}_{i \in \mathbb{N}}$

s.t $X_i = f(x_{i-1})$.

If we let : $S: \mathbb{N} \xrightarrow{+1} \mathbb{N}$, then

$(\mathbb{N}, S: \mathbb{N} \to \mathbb{N}, 0)$ is universal discrete system s.t. $\forall (X, f: X \to X, x_0)$

$\exists\, r: \mathbb{N} \to X$ s.t. $r(n) = X_n$, thus

$$\begin{tikzcd}
\mathbb{N} \arrow[r, "s"] \arrow[d, "r"'] & \mathbb{N} \arrow[d, "r"] \\
X \arrow[r, "f"'] & X
\end{tikzcd}$$

Commutes

Def 2.1.3

(1) $c \in \mathcal{C}$ is initial

$$\Longleftrightarrow \mathcal{C}(c, -) : \mathcal{C} \longrightarrow \mathrm{Set}$$

$$\Downarrow \text{ iso.}$$

$$* : \mathcal{C} \longrightarrow \mathrm{Set} \quad \text{const functor}$$

$$c \longmapsto \text{singleton} \quad \text{constant functor}$$

(2) $c \in \mathcal{C}$ is terminal

$$\Longleftrightarrow \mathcal{C}(-, c) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$

$$\Downarrow \text{ iso}$$

$$* : \mathcal{C}^{op} \longrightarrow \mathrm{Set} \quad \text{const functor}$$

$$c \longmapsto \text{singleton}$$

pf) Let $i \in \mathcal{C}$ initial. $\mathcal{C}(i, c)$ is singleton

$$\begin{tikzcd}
\mathcal{C}(i, c) \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C}(i, c') \arrow[d, "\text{iso.}"] \\
\text{singleton} \arrow[r] & \text{singleton}
\end{tikzcd}$$

Since singletons are iso in Set.

Def 2.14. ① $F : \mathcal{C} \longrightarrow \mathrm{Set}$ is <u>representable</u>

if $\exists\, c \in \mathcal{C}$ s.t. $F \Rightarrow \mathcal{C}(c, -)$ is iso

or $F \Rightarrow \mathcal{C}(-, c)$ 〃

② Representation for $F : \mathcal{C} \longrightarrow \mathrm{Set}$

$$= (\underset{\underset{\text{obj } c}{\uparrow}}{c}\ ,\ \alpha : F \Rightarrow \mathcal{C}(c, -))$$ where

or $F \Rightarrow \mathcal{C}(-, c)$. α is natural iso.

Def) Universal property of object X

= description of $\mathrm{Hom}(X, -)$ or $\mathrm{Hom}(-, X)$

associate to that object.

Ex 2.15.

(1) $1_{\mathrm{Set}} : \mathrm{Set} \longrightarrow \mathrm{Set} \overset{\alpha}{\cong} \mathrm{Set}(1, -)$

So representation : 1

pf) $X \in \mathrm{Set}$, $f : X \to X'$.

$\alpha_X : x \longmapsto (1 \to x)$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha_X"'] & X' \arrow[d, "\alpha_{X'}"] \\
\mathrm{Set}(1, X) \arrow[r, "f_*"'] & \mathrm{Set}(1, X')
\end{tikzcd}$$

(2) $U : \mathrm{Group} \longrightarrow \mathrm{Set} \cong \mathrm{Group}(\mathbb{Z}, -)$

pf) G : gp $\quad f : G \to H$ gp Homo

$$\begin{tikzcd}
UG \arrow[r, "f"] \arrow[d, "\alpha_G"'] & UH \arrow[d, "\alpha_H"] \\
\mathrm{Group}(\mathbb{Z}, G) \arrow[r, "f_*"'] \arrow[d, phantom, sloped, "\cong"] & \mathrm{Group}(\mathbb{Z}, H) \arrow[d, phantom, sloped, "\cong"] \\
UG & UH
\end{tikzcd}$$

$\alpha_G : g \longmapsto (1 \mapsto g)$

$\Rightarrow \mathbb{Z}$: free gp on a single generator

(3) R : unital ring $\quad U : \mathrm{Mod}_R \to \mathrm{Set} \cong \mathrm{Mor}_R(R, -)$

pf) $f : M \longrightarrow N$ module homo

$$\begin{tikzcd}
UM \arrow[r, "f"] \arrow[d, "\alpha_M"'] & UN \arrow[d, "\alpha_N"] \\
\mathrm{Mod}_R(R, M) \arrow[r, "f_*"'] & \mathrm{Mod}_R(R, M)
\end{tikzcd}$$

$\alpha_M : m \longmapsto (1 \mapsto m)$

$\Rightarrow R$: free R-module on a single generator.

(4) $U : \mathrm{Ring} \longrightarrow \mathrm{Set} \cong \mathrm{Ring}(\mathbb{Z}[x], -)$

pf) $f : R \longrightarrow S \qquad \alpha_R : r \longmapsto (x \mapsto r)$

$\Rightarrow \mathbb{Z}[x]$: free unital ring on a single generator

(5) $U(-)^n : \mathrm{Group} \longrightarrow \mathrm{Set} \cong \mathrm{Group}(F_n, -)$

$$\begin{tikzcd}
UG^n \arrow[r, "f^n"] \arrow[d, "\alpha_G"'] & UH^n \arrow[d, "\alpha_H"] \\
\mathrm{Group}(F_n, G^n) \arrow[r] & \mathrm{Group}(F_n, H^n)
\end{tikzcd}$$

$\alpha_{G^n} : g_i \longmapsto (x_i \mapsto g_i)$

$\Rightarrow F_n$: free gp on n generators

$U(-)^n : \mathrm{Ab} \longrightarrow \mathrm{Set} \cong \mathrm{Ab}(\bigoplus_n \mathbb{Z}, -)$

✓

(6) For any $G \in \mathrm{Group}$ with presentation defines $\mathrm{Group} \longrightarrow \mathrm{Set}$

ex) $G = S_3 = \langle s, t \mid s^2 = t^2 = 1,\ sts = tst \rangle$

$\Rightarrow S_3 : \mathrm{Group} \longrightarrow \mathrm{Set}$ s.t

$$G \longmapsto \{ (g_1, g_2) \in G^2 : g_1^2 = g_2^2 = e,\ g_1 g_2 g_1 = g_2 g_1 g_2 \}$$

$\cong \mathrm{Group}(S_3, -)$

since any $f \in \mathrm{Group}(S_3, G)$ is rep by $\begin{pmatrix} s \mapsto g_1 \\ t \mapsto g_2 \end{pmatrix}$, this is well-def.

"free" : universal property expressed by covariant represented functor

(vii) $(-)^* : \text{Ring} \longrightarrow \text{Set} \cong \text{Ring}(\mathbb{Z}[x^{\pm1}], -)$

$$\begin{tikzcd}
R^* \arrow[r, "f|_{R^*}"] \arrow[d, "\alpha_R"'] & S^* \arrow[d, "\alpha_S"] \\
\text{Ring}(\mathbb{Z}[x^{\pm1}], R) \arrow[r, "f_*"'] & \text{Ring}(\mathbb{Z}[x^{\pm1}], S)
\end{tikzcd}
\qquad
\begin{tikzcd}
y \arrow[r, mapsto] \arrow[d, mapsto] & f(y) \arrow[d, mapsto] \\
\phi_y \arrow[r, mapsto] & f \circ \phi_y
\end{tikzcd}$$

$$= \phi_{f(y)}$$

Notes that $\phi : \mathbb{Z}[x^{\pm1}] \longrightarrow R$

is determined by $\phi(x) \in R^*$.

($1 \longmapsto 1$ by unital condition.)

Thus, $\alpha_R : \underset{\substack{\cap \\ R^*}}{y} \longmapsto \phi_y : \begin{array}{l} 1 \longmapsto 1 \\ x \longmapsto y \end{array}$

viii). $U : \text{Top} \longrightarrow \text{Set}$ (forgetful) $\cong \text{Top}(\{x\}, -)$

$$\begin{tikzcd}
UX \arrow[r, "f"] \arrow[d, "\alpha_X"'] & UY \arrow[d, "\alpha_S"] \\
\text{Top}(\{x\}, X) \arrow[r] & \text{Top}(\{x\}, Y)
\end{tikzcd}$$

$\alpha_X : x' \longmapsto (x \mapsto x')$

$$\Rightarrow \quad \begin{tikzcd}
x' \arrow[r, mapsto] \arrow[d, mapsto] & f(x') \arrow[d, mapsto] \\
x \mapsto x' & x \mapsto f(x')
\end{tikzcd}$$

$$= x \mapsto x' \mapsto f(x')$$

(9). $\mathrm{ob} : \mathbf{Cat} \longrightarrow \mathbf{Set} \cong \mathbf{Cat}(\mathbb{1}, -)$

$$\begin{tikzcd}
\mathrm{ob}\, C \arrow[r, "F"] \arrow[d, "\alpha_C"'] & \mathrm{ob}\, D \arrow[d, "\alpha_D"] & c \arrow[r, mapsto] \arrow[d, mapsto] & Fc \arrow[d, mapsto] \\
\mathbf{Cat}(\mathbb{1}, C) \arrow[r, "F_*"'] & \mathbf{Cat}(\mathbb{1}, D) & (\bullet \mapsto c) \arrow[r, mapsto] & (\bullet \mapsto c \mapsto Fc) \\
 & & & = (\bullet \mapsto Fc)
\end{tikzcd}$$

$\Rightarrow \alpha_C : c \longmapsto (\bullet \longmapsto c)$

10). $\mathrm{mor} : \mathbf{Cat} \longrightarrow \mathbf{Set} \cong \mathbf{Cat}(2, -)$

$$\begin{tikzcd}
\mathrm{mor}\, C \arrow[r, "F"] \arrow[d, "\alpha_C"'] & \mathrm{mor}\, D \arrow[d, "\alpha_D"] & f : c \to d \arrow[r, mapsto] \arrow[d, mapsto] & Ff : Fc \to Fd \arrow[d, mapsto] \\
\mathbf{Cat}(2, C) \arrow[r, "F_*"'] & \mathbf{Cat}(2, D) & \left(\begin{smallmatrix} 0 \\ \downarrow \\ 1 \end{smallmatrix} \mapsto \begin{smallmatrix} c \\ \downarrow f \\ d \end{smallmatrix}\right) \arrow[r, mapsto] & \left(\begin{smallmatrix} 0 \\ \downarrow \\ 1 \end{smallmatrix} \quad \begin{smallmatrix} c \\ \downarrow f \\ d \end{smallmatrix} \quad \begin{smallmatrix} Fc \\ \downarrow Ff \\ Fd \end{smallmatrix}\right)
\end{tikzcd}$$

11). $\mathrm{Iso} : \mathbf{Cat} \longrightarrow \mathbf{Set} \cong \mathbf{Cat}(\mathbb{I}, -)$

$$\begin{tikzcd}
\mathrm{Iso}\, C \arrow[r, "F"] \arrow[d] & \mathrm{Iso}\, D \arrow[d] & f : c \xrightarrow{\sim} d \arrow[r, mapsto] \arrow[d, mapsto] & Ff \\
\mathbf{Cat}(\mathbb{I}, C) \arrow[r] & \mathbf{Cat}(\mathbb{I}, D) & \left(\uparrow\downarrow \mapsto {}^{f^{-1}}\uparrow\downarrow^{f}\right) \arrow[r, mapsto] & \left(\downarrow\uparrow \mapsto Ff, Ff^{-1}\right)
\end{tikzcd}$$

$\Rightarrow$ $\mathbb{I}$: free (walking) isomorphism.

12) $\mathrm{Comp} : \mathrm{Cat} \longrightarrow \mathrm{Set} \cong \mathrm{Cat}(3, -)$

$$3 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \qquad \mathrm{Comp}\,\mathcal{C} = \{ (f,g) \in (\mathrm{mor}\,\mathcal{C})^2 \ \text{s.t.}\ gf \in \mathrm{mor}\,\mathcal{C} \}$$

$$\begin{tikzcd} \mathrm{Comp}\,\mathcal{C} \arrow[r, mapsto, "F"] \arrow[d] & \mathrm{Comp}\,\mathcal{D} \arrow[d] \\ \mathrm{Cat}(3,\mathcal{C}) \arrow[r, mapsto, "F_*"'] & \mathrm{Cat}(3,\mathcal{D}) \end{tikzcd} \qquad \begin{tikzcd} (f,g) \arrow[rr, mapsto] \arrow[d, mapsto] & & (Ff,Fg) \arrow[d, mapsto] \\ \left(\begin{tikzcd}[ampersand replacement=\&, column sep=small] \& \cdot \arrow[dr] \& \\ \cdot \arrow[ur] \arrow[rr] \& \& \cdot \end{tikzcd} \right) \arrow[r, mapsto] & \left(\begin{tikzcd}[ampersand replacement=\&, column sep=small] \& \cdot \arrow[dr, "g"] \& \\ \cdot \arrow[ur, "f"] \arrow[rr, "fg"'] \& \& \cdot \end{tikzcd} \right) \arrow[r, mapsto] & \begin{tikzcd}[ampersand replacement=\&, column sep=small] \& \cdot \arrow[dr] \& \\ \cdot \arrow[ur] \arrow[rr] \& \& \cdot \end{tikzcd} \arrow[d, equal] \\ & & \begin{tikzcd}[ampersand replacement=\&, column sep=small] \& \cdot \arrow[dr, "Fg"] \& \\ \cdot \arrow[ur, "Ff"] \arrow[rr] \& \& \cdot \end{tikzcd} \end{tikzcd}$$

✓ 13) $U : \mathrm{Set}_* \longrightarrow \mathrm{Set} \cong \mathrm{Set}_*((\{x,y\}, x), -)$

$(A,a) \xrightarrow{f} (B,b)$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\ \mathrm{Set}_*((\{x,y\},x),(A,a)) \arrow[r, "f_*"'] & \mathrm{Set}_*(\sim,(B,b)) \end{tikzcd} \qquad \begin{tikzcd} a' \arrow[r, mapsto] \arrow[d, mapsto] & f(a') \arrow[d, mapsto] \\ \begin{pmatrix} x \mapsto a \\ y \mapsto a' \end{pmatrix} \arrow[r, mapsto] \arrow[d, phantom, "\cong" rotate=90] & \begin{pmatrix} x \mapsto b \\ y \mapsto f(a') \end{pmatrix} \arrow[d, phantom, "\cong" rotate=90] \\ A & B \end{tikzcd}$$

$\alpha_A : a' \longmapsto \begin{pmatrix} x \mapsto a \\ y \mapsto a' \end{pmatrix}$

✓ 14) $\mathrm{Path} : \mathrm{Top} \longrightarrow \mathrm{Set} \cong \mathrm{Top}(I, -)$

$\mathrm{Loop} : \mathrm{Top}_* \longrightarrow \mathrm{Set} \cong \mathrm{Top}_*(S^1, -)$

$X \longmapsto$ set of paths/loops

Since $I \to X$ defines path (as $S^1 \longrightarrow X, \{x\}$ defines loop.)

"Free" object = a representation of covariant functor $C \longrightarrow Set$

("Free" means that it induce ... desired property on any object by rep. functor.)

"Cofree" " for contravariant functor

Ex 2.1.6: (Ex of Contravariant)

(1) $P: Set^{op} \longrightarrow Set \cong Set(-, \Omega)$ where $\Omega = \{T, \perp\}$

$$\begin{tikzcd}
A \arrow[r, mapsto] \arrow[d, "f"'] \arrow[dr, phantom, "\longmapsto"] & P(A) \\
B \arrow[r, mapsto] & P(B) \arrow[u, "f^{-1}"']
\end{tikzcd}$$

$\alpha_A : P(A) \longrightarrow Set(A, \Omega)$

$A' \longmapsto X_{A'} ;\ f(A') = \{T\}$

$f(A'^c) = \{\perp\}$

$\Rightarrow$

$$\begin{tikzcd}
P(A) \arrow[d, "\alpha_A"'] & P(B) \arrow[l, "f^{-1}"'] \arrow[d, "\alpha_B"] \\
Set(A, \Omega) & Set(B, \Omega) \arrow[l, "f^*"]
\end{tikzcd}$$

$\Rightarrow$

$$\begin{tikzcd}
f^{-1}(B') \arrow[d, mapsto] & B' \arrow[l, mapsto] \arrow[d, mapsto] \\
X_{f^{-1}B'} & X_{B'} \arrow[l, mapsto]
\end{tikzcd}$$

$X_{B'}(B') = T$

$X_{B'}((B')^c) = \perp$

$X_{f^{-1}B'}(f^{-1}B') = \{T\}$

$X_{f^{-1}B'}((f^{-1}B')^c) = \{\perp\}$

$= X_{B'} \circ f$

$X_{B'} \circ f(f^{-1}B') = X_{B'}(B') = T$

$X_{B'} \circ f((f^{-1}B')^c) \leq X_{B'}((B')^c) = \perp$

(II). $\mathcal{O} : \mathrm{Top}^{op} \longrightarrow \mathrm{Set} \cong \mathrm{Top}(-, S)$

Where S: Sierpinski space.

$S = \{0, 1\}$, $\mathcal{O}(S) = \{\emptyset, S, \{0\}\}$

Given $f : X \to Y$ cts (So $\{1\}$ is closed)

$$\begin{tikzcd}
\mathcal{O}(X) \arrow[d, "\alpha_X"'] & \mathcal{O}(Y) \arrow[l, "f^{-1}"'] \arrow[d, "\alpha_Y"] \\
\mathrm{Top}(X, S) & \mathrm{Top}(Y, S) \arrow[l, "f^*"]
\end{tikzcd}$$

$\alpha_X : U \longmapsto \chi_U : X \to S$

$\chi_U(U) = \{0\}$

$\chi_U(U^c) = \{1\}$

$$\begin{tikzcd}
f^{-1}(V) \arrow[d, mapsto] & V \arrow[l] \arrow[d, mapsto] \\
\chi_{f^{-1}(V)} & \\
\chi_V \circ f & \chi_V \arrow[l, mapsto]
\end{tikzcd}$$

$\Rightarrow$

To see $\chi_V \circ f = \chi_{f^{-1}(V)}$

let $x \in f^{-1}(U)$

$\Rightarrow \chi_V \circ f(x) = 0 = \chi_{f^{-1}(U)}(x)$

If $x \notin f^{-1}(U)$

$\chi_V \circ f(x) = 1 = \chi_{f^{-1}(U)}(x)$

3) $C : \mathrm{Top}^{op} \longrightarrow \mathrm{Set} \cong \mathrm{Top}(-, \underset{\text{Sierpinski}}{S})$

$\alpha_X : C \longmapsto \chi_C : X \to S$

$\begin{pmatrix} C \longmapsto 1 \\ C^c \longmapsto 0 \end{pmatrix}$

$$\begin{tikzcd}
f^{-1}(U) \arrow[d] & U \arrow[l] \arrow[d] \\
\chi_{f^{-1}(U)} = \chi_U \circ f & \chi_U \arrow[l, mapsto]
\end{tikzcd}$$

So, $\boxed{\mathcal{O} \cong \mathrm{Top}(-, S) \cong C}$

natural iso.

(4). $\mathrm{Hom}(- \times A, B) : \mathrm{Set}^{op} \to \mathrm{Set} \cong \mathrm{Set}(-, B^A)$

$X \longmapsto \mathrm{Hom}(X \times A, B)$

Let $f : X \to Y$.

$$\begin{tikzcd}
\mathrm{Hom}(X \times A, B) \arrow[d, "\alpha_X"'] & \mathrm{Hom}(Y \times A, B) \arrow[l, "f^* \times 1_A"'] \arrow[d, "\alpha_Y"] \\
\mathrm{Set}(X, B^A) & \mathrm{Set}(Y, B^A) \arrow[l, "f^*"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \times A \xrightarrow{f \circ 1} Y \times A \xrightarrow{g} B \arrow[d, mapsto] & g : Y \times A \longmapsto B \arrow[l, mapsto] \arrow[d, mapsto] \\
x \mapsto (x, g \circ (f \times 1_A)(x, a)) : a \in A & (y \mapsto ((b, g(y, b)) : b \in B \}) \arrow[l, mapsto]
\end{tikzcd}$$

$x \mapsto (x, g \circ (f \times 1_A)(x, a))$. //

where $\alpha_X : X \times A \xrightarrow{\phi} B \longmapsto \left(x \longmapsto \{ (a, \phi(x, a)) : a \in A \} \in B^A \right)$

It is called "Currying".

(6). $H^n(-;A) : \mathrm{Top}^{op} \longrightarrow \mathrm{Ab}$.

A : ab gp. $H^n(-;A)$ singular cohomology with coeff in A.

Actually, $H^n(-;A) : \mathrm{Htpy}^{op} \longrightarrow \mathrm{Ab}$.

Think $H^n(-;A) : \mathrm{Htpy}_{CW}^{op} \longrightarrow \mathrm{Set}$.

$$\cong \mathrm{Htpy}_{CW}^{op}(-, K(A,n))$$

$X \xrightarrow{f} Y$ CW cpx.

$$\begin{tikzcd}
H^n(X;A) \arrow[d, "\cong"'] & H^n(Y,A) \arrow[l, "f^*"'] \arrow[d, "\cong"] \\
\mathrm{Htpy}_{cw}^{op}(X, K(A,n)) & \mathrm{Htpy}_{cw}^{op}(Y, K(A,n)) \arrow[l, "f^*"]
\end{tikzcd}$$

(7). Classifying space of G : = CW cpx BG

s.t. $\mathrm{Htpy}_{CW}^{op} \longrightarrow \mathrm{Set}$

$$X \longmapsto \mathrm{Htpy}_{CW}^{op}(BG, X)$$

$\cong$ the set of iso classes of principal G-bundle over X.

Remaining Q :

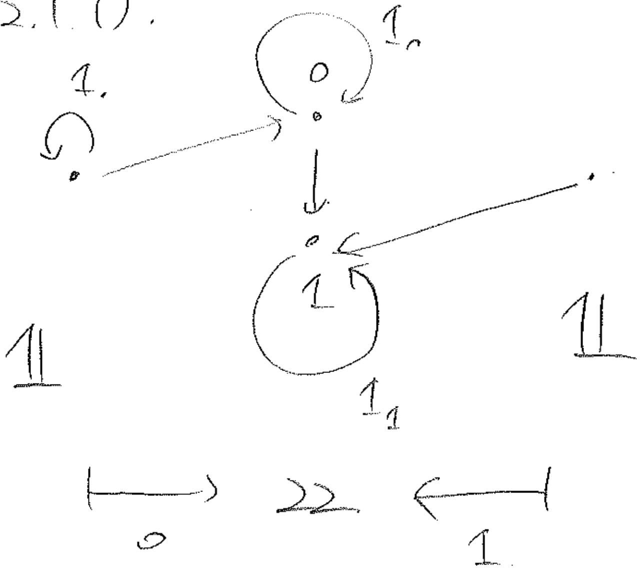
① How unique? i.e. If F is rep by c, c', then $c \cong c'$?

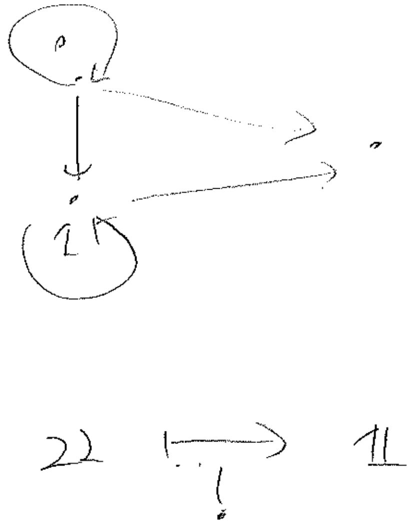
② What data is needed to construct natural iso between F and $\mathcal{C}(c, -)$?

③ How do representation related to initial or terminal obj ?

Exe

Ex 2.1(1).





Now notes that

$$\mathrm{Cat}(\mathbb{1}, C) \cong \{ c \in \mathrm{ob}\, C : c : \overset{1_0}{\bullet} \longmapsto \overset{1_c}{c} \}$$

$$\mathrm{Cat}(\mathbb{2}, C) \cong \{ f : c \to c' \in \mathrm{Mor}\, C : \begin{array}{c} 0 \\ \downarrow \\ 1 \end{array} \longmapsto \begin{array}{c} c \\ \downarrow f \\ c' \end{array} \}$$

Thus, for given functor $F : C \longrightarrow D$.

$$\begin{tikzcd}
\mathrm{Cat}(\mathbb{1}, C) \arrow[d, "\mathrm{Cat}(\mathbb{1}, F)"'] & \mathrm{Cat}(\mathbb{2}, C) \arrow[d, "\mathrm{Cat}(\mathbb{2}, F)"] \\
\mathrm{Cat}(\mathbb{1}, D) & \mathrm{Cat}(\mathbb{2}, D)
\end{tikzcd}$$

where $\mathrm{Cat}(\mathbb{1}, F)$ = a set map

$$\mathrm{Ob}\,C \longrightarrow \mathrm{Ob}\,D$$

$\mathrm{Cat}(\mathbb{2}, F)$ = a set map

$$\mathrm{Mor}\,C \longrightarrow \mathrm{Mor}\,D$$

Hence, natural transformation $\alpha : \mathrm{Cat}(\mathbb{1}, -) \Rightarrow \mathrm{Cat}(\mathbb{2}, -)$

should map $c \in \mathrm{Ob}\,C$ to $f \in \mathrm{Mor}\,C$

One canonical way corresp to $!$ is

$$\mathrm{Cat}(\mathbb{1}, C) \xrightarrow{!_*} \mathrm{Cat}(\mathbb{2}, C)$$ precomposition of $!$

This induces $c \longmapsto 1_c$

In other direction $\beta : \mathrm{Cat}(\mathbb{2}, -) \Rightarrow \mathrm{Cat}(\mathbb{1}, -)$

There are two canonical ways.

$$\begin{tikzcd}\mathrm{Cat}(\mathbb{2}, -) \arrow[r, shift left, "0_*"] \arrow[r, shift right, "1_*"'] & \mathrm{Cat}(\mathbb{1}, -)\end{tikzcd}$$ precomposition of 0 or 1.

For given $C \in \mathrm{Cat}$, this induces

$$f : c \to c' \longmapsto c = \mathrm{dom}\, f.$$

$$\begin{tikzcd}\mathrm{Cat}(\mathbb{2}, C) \arrow[r, shift left, "0_*"] \arrow[r, shift right, "1_*"'] & \mathrm{Cat}(\mathbb{1}, C)\end{tikzcd}$$

$$f : c \to c' \longmapsto c' = \mathrm{cod}\, f.$$

▨

Ex 2.1 ii) If F is representable, $F \overset{\cong}{\underset{\alpha}{}} C(c,-)$

Hence, Let $f : c' \rightarrowtail c''$ be monomorphism.

Then below diagram commutes.

$$\begin{tikzcd}
Fc' \arrow[r, "\alpha_{c'}"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\curvearrowright"] & C(c,c') \arrow[d, "f_*"] \\
Fc'' \arrow[r, "\alpha_{c''}"'] & C(c,c'')
\end{tikzcd}$$

Now we claim f_* is injective;

Let $g, g' \in C(c, c')$

s.t. $f_*(g) = f_*(g') \Rightarrow fg = fg' \Rightarrow g = g'$

since f is mono. Also, by natural iso each $\alpha_{c'}$, $\alpha_{c''}$ are iso. In category of Set, it is bijection. Thus

$Ff = \alpha_{c''}^{-1} \circ f_* \circ \alpha_{c'}$ which is injective

since composition of injective functions are injective

2.1 iii)

In case of (ii) (F is representable)

Since $G = G \cdot 1_D \cong GH \cdot H^{-1} \cong FH^{-1}$. So its answer depends on case (i)

(If (i) is true, then G is representable.)

Before doing that. let $H : C \longrightarrow D$

$H^{-1} : D \longrightarrow C$

be given equiv of categories s.t. $GH \underset{\alpha}{\cong} F$

Suppose $G \cong D(d, -)$

Then let $H^{-1}(d) = c$. We claim that

$GH \cong C(c, -)$ Notes that

given condition shows, for $f : c' \longrightarrow c''$

$$\begin{tikzcd}
GH(c') \cong & D(Hc, Hc') \arrow[d, "(Hf)_*"'] & C(c, c') \arrow[d, "f_*"] \\
& D(Hc, Hc'') & C(c, c'').
\end{tikzcd}$$

Now from equivalence of categories H, H^{-1},

$$D(d_1, d_2) \cong D(HH^{-1}(d_1), HH^{-1}(d_2))$$

bijection

$$C(c_1, c_2) \cong C(H^{-1}H(c_1), H^{-1}H(c_2))$$

(This is because natural iso gives 1-1 correspondence)

Now think H, H^{-1} as functions below.

$$D(d_1, d_2) \xrightarrow{H^{-1}} C(H^{-1}(d_1), H^{-1}(d_2)) \xrightarrow{H} D(HH^{-1}(d_1), HH^{-1}(d_2))$$

$$C(c_1, c_2) \xrightarrow{H} D(H(c_1), H(c_2)) \xrightarrow{H^{-1}} C(H^{-1}H(c_1), H^{-1}H(c_2))$$

Notes that composition of those are bijection.

Thus, from firstline H^{-1} is inj, H is surj.
H is inj H^{-1} is surj

So H^{-1}, H are bijection as a set map between $C(c_1, c_2)$ and $D(H(c_1), H(c_2))$ by setting $d_1 = Hc_1$, $d_2 = Hc_2$.

From this argument, we get

$$\begin{tikzcd}
\cong GH(c') \arrow[r, phantom, "\cong"] & D(Hc, Hc') \arrow[r, "H|_{D(Hc,Hc')}", "\cong"'] \arrow[d, "(H\theta)_*"'] & C(c, c') \arrow[d] \\
GH(c'') & D(Hc, Hc'') \arrow[r, "H|_{D(Hc,Hc'')}", "\cong"'] & C(c, c'')
\end{tikzcd}$$

Thus, $F \cong GH \cong C(c, -)$. F is representable. ☐

Ex 2: iv) Let F be a subfunctor of $C(-, c)$
(Then, F is contravariant functor $C^{op} \to SET$

s.t. $F(c') \hookrightarrow C(c', c)$

① If F, is given by collection of subsets

$F_c \subset G_c$ so that each $Gf : G_c \to G_{c'}$

restricts a function $Ff : F_c \to F_{c'}$, then

α is constructed by inclusion, $Ff = (Gf|_{F_c}$

$$\begin{tikzcd} F_c \arrow[r, "\alpha_c"] \arrow[d, "Gf|_{F_c}"'] \arrow[dr, phantom, "\circlearrowright"] & G_c \arrow[d, "Gf"] \\ F_{c'} \arrow[r, "\alpha_{c'}"'] & G_{c'} \end{tikzcd}$$

So that the diagram commutes

② If F is subfunctor of $C(-, c)$

then $\bigcup_{d \in Ob C} F(d)$ is closed under right-

composition with arbitrary morphism Gf.

Ex 2.1. v).

$$\mathrm{Iso} : \mathrm{Cat} \to \mathrm{Set} \;\cong\; \mathrm{Cat}(\mathbb{I}, -)$$

$C \longmapsto$ Set of iso of C

$F \big\downarrow \longmapsto$ F as a function

$D \longmapsto$ Set of iso of C.

Where $\mathbb{I} : \bullet \underset{\cong}{\overset{\cong}{\rightleftarrows}} \bullet$

Then Iso is subfunctor of Mor; .

)Set of iso is subset of set of morphisms

2) To set $\mathrm{Cat}(\mathbb{I}, -) \overset{\alpha}{\Longrightarrow} \mathrm{Cat}(\mathbb{2}, -)$

Let $\iota : \mathbb{2} \longrightarrow \mathbb{I}$ by $\overset{\bullet}{\underset{\bullet}{\downarrow}} \Longrightarrow \overset{\bullet}{\underset{\bullet}{\downarrow\!\uparrow}}$ natural embedding

Then, for any $F : C \longrightarrow D \in \mathrm{Cat}$,

$$\begin{tikzcd}
f \arrow[rr, mapsto, dashed, "\iota_*"] \arrow[ddd, mapsto, bend right] & & f\circ\iota \arrow[dd, dash] \\
\mathrm{Cat}(\mathbb{I}, C) \arrow[r] \arrow[d, "F^*"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Cat}(\mathbb{2}, C) \arrow[d, "F^*"] & \\
\mathrm{Cat}(\mathbb{I}, D) \arrow[r] & \mathrm{Cat}(\mathbb{2}, D) & F\circ f\circ\iota \\
F\circ f \arrow[rr, dashed, "\iota_*"'] & &
\end{tikzcd}$$

Commutes. So $\alpha = \iota$.

Yoneda Lemma. 2.2

Q: Given a functor $F: \cancel{Se}\, C \longrightarrow Set$,
What data is needed to define $C(c,-) \cong F$?
or $C(c,-) \Rightarrow F$?

Ex 2.2.1 $F: W \longrightarrow Set$ $\quad W$: ordinal Category ~~of ordinal~~

$\Rightarrow (F_n)_{n \in W}$, $f_{n,n+1}: F_n \to F_{n+1}$.

Then $W(k,-): W \longrightarrow Set$

$$\Rightarrow W(k,m) = \begin{cases} \emptyset & m \cancel{\;} < k \\ \{*\} & m \geq k \end{cases} \quad \left(\begin{array}{l} m = k \\ \Rightarrow W(k,k) \\ \quad = \{1_k\} \end{array} \right.$$

If $\alpha: W(k,-) \Rightarrow F$, then

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "\alpha_0"'] & \emptyset \arrow[r] \arrow[d, "\alpha_1"] & \cdots \arrow[r] & \emptyset \arrow[r] \arrow[d, "\alpha_{k-1}"] & * \arrow[r] \arrow[d, "\alpha_k"] & * \arrow[r] \arrow[d, "\alpha_{k+1}"] & \cdots \\
F_0 \arrow[r, "f_{0,1}"'] & F_1 \arrow[r] & \cdots \arrow[r] & F_{k-1} \arrow[r, "f_{k-1,k}"'] & F_k \arrow[r, "f_{k,k+1}"'] & F_{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

Commutes. Notes that $m<k$, α_m is empty as a set of tuple. (So commutes vacuously.)

If $m \geq k$, α_m is identified as an element.

$\Rightarrow \alpha_{n+1} = f_{n,n+1}(\alpha_n) \Rightarrow \alpha$ is determined by choice of $\alpha_k \in F_k$.

Ex 2.2.2 Let G: gp. BG: category of one element with $\mathrm{Mor}\, BG = G$

$\Rightarrow G: BG \longrightarrow \mathrm{Set}$.

$G: BG^{op} \longrightarrow \mathrm{Set}$ are unique as a representable covariant functor (contra ") since $BG(\cdot,\cdot) \cong G$ (as a set)

$$\begin{tikzcd} BG(\cdot,\cdot) \arrow[r, "1."] \arrow[d, "\phi"'] \arrow[rd, phantom, "\circlearrowright"] & G \arrow[d, "\phi"] \\ BG(\cdot,\cdot) \arrow[r, "1."] & G \end{tikzcd}$$

BG has only one element, i.e,

$X: BG \longrightarrow \mathrm{Set}$

$$\begin{tikzcd} \bullet \arrow[r, mapsto] \arrow[d, "g"'] & X \arrow[d, "g_*"] \\ \bullet \arrow[r, mapsto] & X \end{tikzcd}$$

g_*: iso of X satisfying (left) Group action.

i.e, $G \leq \mathrm{Aut}(X)$.

If $\alpha: BG(\cdot, -) \Rightarrow X$

$$\begin{tikzcd} G \arrow[r, "\alpha."] \arrow[d, "g\cdot"'] \arrow[rd, phantom, "\circlearrowright"] & X \arrow[d, "g_*"] \\ G \arrow[r, "\alpha."'] & X \end{tikzcd}$$

$h \mapsto$ $g\cdot\alpha.(h)$ $\alpha.(gh)$

$\Rightarrow$ $\alpha.(g\cdot h) = g\cdot\alpha(h)$

Especially if $h = e$, $\alpha.(g) = g.\phi(e)$

So, choice of $\phi(e) \in X$ forces us to define $\phi(g)$.

And $\phi(e)$ can be any element, since left action of G on G is free. (i.e. every stabilizer gp is trivial.)

Prop 2.2.3: G-equivariant maps $G \to X$
$f(gx) = gf(x)$
Corresponds bijectively to elements of X
Identified as image of identity $e \in G$

In these two example., natural transformation whose domain is a representable functor are determined by the choice of single element which lives in the set def by evaluating codomain functor at the representing object.
Moreover choice is permitted.
I.e Let $F: C \to Set$, $C(c,-)$ be a representable functor. Then, $\alpha: C(c,-) \Rightarrow F$ is determined by choice of elements in Fc.
I.e. $\text{Hom}(C(c,-), F) \cong F_c$ as a set.

Thm 2.2.4 (Yoneda Lemma)
For any functor $F: C \to Set$, C: locally small
$\forall c \in \text{Obj}\, C$ Then, $\exists$ bijection

$$\text{Hom}(C(c,-), F) \cong F_c$$

that associates natural transf: $\alpha: C(c,-) \Rightarrow F$
to the element $\alpha_c(1_c) \in F_c$. This bijection is natural in both c and F.

This bijection is natural in both c and F.

Rmk: Since $\mathcal{C}$ is not small, $\mathrm{Hom}(\mathcal{C}(c,-),F)$ might be large. However, Yoneda Lemma shows that $\mathrm{Hom}(\mathcal{C}(c,-),F)$ is a set.

pf 1: Bijection) construct

$$\Phi : \mathrm{Hom}(\mathcal{C}(c,-),F) \longrightarrow Fc$$

$$\alpha : \mathcal{C}(c,-) \Rightarrow F \longmapsto \alpha_c(1_c)$$

WTS Φ is bijection. It suffices to show that

$\Psi : Fc \longrightarrow \mathrm{Hom}(\mathcal{C}(c,-),F)$ has inverse.

To do this, for each $x \in Fc$, we need to define $\Psi(x)$ as a natural transf.

$\Rightarrow$ Need to define $\Psi(x)_d : \mathcal{C}(c,d) \to Fd$ for any $d \in \mathrm{obj}\,\mathcal{C}$ s.t.

$f : c \to d$.

$$\begin{tikzcd}
\mathcal{C}(c,c) \arrow[r, "\Psi(x)_c"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "Ff"] \\
\mathcal{C}(c,d) \arrow[r, "\Psi(x)_d"'] & Fd
\end{tikzcd}$$

Then, ~~Φ~~

$$\begin{tikzcd}
1_c \arrow[r, mapsto] \arrow[d, mapsto] & \Psi(x)_c(1_c) \arrow[dr] & \\
f \arrow[r, mapsto] & \Psi(x)_d(f) & Ff(\Psi(x)_c(1_c))
\end{tikzcd}$$

Thus, WTS $\Psi(x)_d(f) = Ff(\Psi(x)_c(1_c))$

Since Ψ is intended as a inverse of Φ.

$\Phi(\Psi(x)) := x$. naturally.

Thus, $\Phi(\Psi(x)) \underset{\substack{\uparrow \\ \text{from def of } \Phi}}{=} \Psi(x)_c(1_c) = x$.

Therefore, naturality forces to define

$\Psi : Fc \to \mathrm{Hom}(C(c,-), F)$

$\Psi(x)_d(f) := Ff(x)$.

It determines $\Psi(x)_d$ as a map $C(c,d) \to Fd$.

To see $\Psi(x)$ is natural transformation.

let $g : d \to e$. WTS

$$\begin{tikzcd}
C(c,d) \arrow[r, "\Psi(x)_d"] \arrow[d, "g_*"'] \arrow[dr, phantom, "\circlearrowright"] & Fd \arrow[d, "Fg"] \\
C(c,e) \arrow[r, "\Psi(x)_e"'] & Fe
\end{tikzcd}$$

commutes. Let $f \in C(c,d)$

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & \Psi(x)_d(f) \arrow[d, mapsto] \\
gf \arrow[dr, mapsto] & Fg(\Psi(x)_d(f)) \\
 & \Psi(x)_e(gf)
\end{tikzcd}$$

$= F(gf)(x)$

$= Fg(Ff(x))$

By functoriality of F, $F(gf).x$
$= (Fg)\cdot(Ff)(x)$.

So $\Psi : F_c \longrightarrow \mathrm{Hom}(C(c,-), F)$ is well-def

function.

By construction, $\Phi\Psi(x) = \Psi(x)_c(1_c) = x$.

NTS $\Psi\Phi(\alpha) = \alpha$ for any $\alpha : C(c,-) \Rightarrow F$.

$$\begin{array}{c} \Psi\Phi(\alpha) \\ \| \\ \Psi(\alpha_c(1_c)) \end{array}$$

$\Rightarrow$ It suffices to show that for any $f : c \to d$,

$$\Psi(\alpha_c(1_c))_d(f) = Ff(\alpha_c(1_c))$$

By naturality of α.

$$\begin{tikzcd} C(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright"] & F_c \arrow[d, "Ff"] \\ C(c,d) \arrow[r, "\alpha_d"'] & Fd \end{tikzcd}$$

Thus, $Ff(\alpha_c(1_c))$

$= \alpha_d(f)$.

$\Rightarrow \Psi(\alpha_c(1_c))_d(f) = \alpha_d(f)$

$\Rightarrow \Psi(\alpha_c(1_c))_d = \alpha_d$.

$\Rightarrow \Psi(\Phi(\alpha))_d = \alpha_d$.

$\Rightarrow \Psi\Phi(\alpha) = \alpha$.

So, Ψ and Φ are inverse to eachother.

Hence. $\mathrm{Hom}(C(c,-), F) \cong F_c$

Proof of Naturality) ① Naturality in the functor.

WTS, given $\beta : F \Rightarrow G$, if x represents

$\beta\alpha : C(c,-) \Rightarrow F \Rightarrow G$, i.e. $\Phi_G(\beta\alpha) = x$,

then $x = \beta_c(y)$ s.t. $y \in Fc$ represents

$\alpha : C(c,-) \Rightarrow F$.

i.e. $x = \Phi_F(\beta_c(y))$

In other words,

$$\begin{tikzcd}
\mathrm{Hom}(C(c,-), F) \arrow[r, "\Phi_F", "\cong"'] \arrow[d, "\beta_*"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "\beta_c"] \\
\mathrm{Hom}(C(c,-), G) \arrow[r, "\Phi_G"', "\cong"] & Gc
\end{tikzcd}$$

commutes.

pf) $\Phi_G(\beta \cdot \alpha) = (\beta \cdot \alpha)_c(1_c) = \beta_c(\alpha_c(1_c))$

$= \beta_c(\Phi_F(\alpha))$.

② Naturality in the object

Given $f : c \to d$ in C, if $x \in Fd$ represent

$\alpha f^* : C(d,-) \Rightarrow C(c,-) \Rightarrow F$ i.e. $\Phi_d(\alpha f^*) = x$,

then $x = Ff(y)$ where y represents α, i.e.

$x = Ff(\Phi_c(\alpha))$.

In other words,

$$\begin{tikzcd}
\alpha \arrow[drrr, mapsto, bend left] \arrow[ddrrr, mapsto, dashed, bend right=60] & & & \\
\mathrm{Hom}(C(c,-),F) \arrow[r, "\Phi_c", "\cong"'] \arrow[d, "(f^*)^*"'] \arrow[dr, phantom, "\curvearrowright"] & Fc \arrow[d, "Ff"] & & \\
\mathrm{Hom}(C(d,-),F) \arrow[r, "\cong", "\Phi_d"'] & Fd & & Ff(\alpha_c(1_c)) \\
 & & & (\alpha\cdot f^*)_d(1_d)
\end{tikzcd}$$

Pf) Notes that $(\alpha f^*)_d$ is $\alpha_d \circ f^*$. Hence,

$$C(d,d) \xrightarrow{f^*} C(c,d) \xrightarrow{\alpha_d} Fd$$

Thus $1_d \longmapsto f \longmapsto \alpha_d(f)$

And in the proof ob bijection,

$$\begin{tikzcd}
C(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "Ff"] \\
C(c,d) \arrow[r, "\alpha_d"'] & Fd
\end{tikzcd} \Rightarrow \begin{aligned} &\alpha_d(f) \\ &= Ff(\alpha_c) \end{aligned}$$

$$\Rightarrow (\alpha\cdot f^*)_d(1_d) = \alpha_d(f) = Ff(\alpha_c(1_c))$$

▨

Rmk 2.2.17. If we donot consider size issue, Yoneda lemma can be viewed as natural isomorphism between functors.

Let $(c, F) \in \mathrm{Obj}\ \mathcal{C} \times \mathrm{Set}^{\mathcal{C}}$.

def: $\mathrm{ev}: \mathcal{C} \times \mathrm{Set}^{\mathcal{C}} \longrightarrow \mathrm{Set}$

$(c, F) \longmapsto Fc$ = codomain of Φ

Also, let

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "y"] & \mathrm{Set}^{\mathcal{C}} \\
c \arrow[r, mapsto] \arrow[dd, "f"'] & \mathcal{C}(c,-) \\
 \arrow[r, mapsto] & \\
d \arrow[r, mapsto] & \mathcal{C}(d,-) \arrow[uu, "f^*"']
\end{tikzcd}$$

Then, $\mathrm{Hom}(y(-), -) := \mathcal{C} \times \mathrm{Set}^{\mathcal{C}} \xrightarrow{y \times 1_{\mathrm{Set}^{\mathcal{C}}}} (\mathrm{Set}^{\mathcal{C}})^{op} \times \mathrm{Set}^{\mathcal{C}} \longrightarrow \mathrm{SET}.$

$(c, F) \longmapsto (\mathcal{C}(c,-), F) \longmapsto \mathrm{Hom}(\mathcal{C}(c,-), F)$

$\parallel$

domain of Φ.

If $\mathcal{C}$ is small, No problem

$\mathcal{C}$ is locally small, $\mathrm{Set}^{\mathcal{C}}$ need not be locally small.

Then, $\mathrm{Hom}(y(-), -): \mathcal{C} \times \mathrm{Set}^{\mathcal{C}} \longrightarrow \mathrm{Set}$

$(c, F) \longmapsto \mathrm{Hom}(\mathcal{C}(c,-), F)$

Then, by naturality proof, we see that

$$\begin{tikzcd}
C \times Set^C \arrow[r, bend left=50, "\mathrm{Hom}(Y(-),-)", ""{name=U, below}] \arrow[r, bend right=50, "ev."', ""{name=D, above}] & Set
\arrow[Rightarrow, from=U, to=D, "\Phi \cong"]
\end{tikzcd}$$

Corollary 2.2.8 (Yoneda embedding)

$$\begin{tikzcd}
C \arrow[r, hook, "y"] & Set^{C^{op}} \\
c \arrow[dd, "f"'] \arrow[r, mapsto] & C(-,c) \arrow[dd, "f_*"] \\
{} \arrow[r, mapsto] & {} \\
d \arrow[r, mapsto] & C(-,d)
\end{tikzcd}
\qquad
\begin{tikzcd}
C^{op} \arrow[r, hook, "y"] & Set^{C} \\
c \arrow[dd, "f"'] \arrow[r, mapsto] & C(c,-) \\
{} \arrow[r, mapsto] & {} \\
d \arrow[r, mapsto] & C(d,-) \arrow[uu, "f^*"']
\end{tikzcd}$$

define full and faithful embeddings

Rmk: Co- and contravariant Yoneda embedding are two different incarnation of a common bifunctor

$$\begin{tikzcd}
C(-,-) : C^{op} \times C \arrow[r] & Set \\
(c, d) \arrow[dd, "{(f^{op}, g)}"] \arrow[r, mapsto] & C(c,d) \arrow[dd, "g_* f^*"] \\
{} \arrow[r, mapsto] & {} \\
(e, f) \arrow[r, mapsto] & C(e,f)
\end{tikzcd}$$

f : pre
g : post

pf) WTS for any $f : c \to d$,

$C(c,d) \cong Hom(C(-,c), C(-,d))$

and $C(c,d) \cong Hom(C(d,-), C(c,-))$.

① By exercise 1.4 iv. (distinct parallel morphism define distinct natural trans (pre, post compositi))

it is injective.

Also, Yoneda lemma says that

$$\alpha : C(d,-) \Rightarrow C(c,-) \quad \text{corresponds to}$$

$$\Phi(\alpha) = \alpha_d(1_d) \in C(c,d).$$

If we denote $\alpha_d(1_d) =: f$, then

$f^* : C(d,-) \Rightarrow C(c,-)$ sends $1_d \longmapsto f$.

$\Longrightarrow \boxed{\alpha = f^*}$ by Yoneda lemma. ▨

Cor 2.2.8 implies that natural transformation between represented functor corresponds to morphisms between the representing object.

There are three example, but introduce two first

1). Every row operation on matrix with n rows def by left mult of $n \times n$ matrix.

2). Cayley's theorem.

3). In Vect_k, $V \otimes_k W \cong W \otimes_k V$.

Cor 2.2.9. Matrix mult. (R: unital ring)

pf) Mat_R : obj $:= \mathbb{N}$.

$Mat(m, n) \cong Mat_{n\times m}(R)$.

Row operation define natural endomorphism of $Hom(-, n)$, i.e. if α is a row op.

$f : m \to k$ then
$k\times n$ matrix

$$\begin{tikzcd}
x \arrow[dd, mapsto] & Hom(k, n) \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & Hom(k, n) \arrow[d, "f"] \\
& Hom(m, n) \arrow[r, "\alpha"'] & Hom(m, n) \\
x\cdot f & &
\end{tikzcd}$$

$n\times k$

By Cor 2.2.8, α is rep by element in $Hom(n,n)$
$n\times n$ mat.

Moreover, Thm 2.2.4 Identify what it is

i.e. $\alpha_n(1_n)$ = row op on $n\times n$ identity matrix

Cor 2.2.10 : Any gp is isomorphic to subgp of permutation gp.

pf) Let BG, category of 1 elem with $Mor\,BG \cong G$ for some gp G. Ex 2.2.2 gives $BG \hookrightarrow Set^{BG^{op}}$

as right G-set G.

Corollary 2.2.8 says that

G-equivariant endomorphism of right G-set G are those maps defined by left multiplication, i.e element of $\underline{BG(\cdot,\bullet) \cong G}$.

Thus, $G \cong \mathrm{Aut}_{\substack{\text{right}\\ G\text{-set}}}(G) \leq \mathrm{Sym}(G)$.

Since $\mathrm{Set}^{BG^{op}} \longrightarrow \mathrm{Set}$ is faithful functor.

Ex 2.2-i).

Dual of Thm 2.2.4 (Yoneda Lemma, Contravariant version)

For any functor $F: C^{op} \to Set$, whose domain C is locally small and any object $c \in C$, There is a bijection $Hom(C(-,c), F) \cong Fc$

that associates a natural transformation

$\alpha : C(-,c) \Rightarrow F$ to the element $\alpha_c(1_c) \in Fc$.

Moreover, this correspondence is natural in both c and F.

pf) Let $\Phi : Hom(C(-,c), F) \longrightarrow Fc$

$\alpha \longmapsto \alpha_c(1_c)$

It is well-defined map sending $1_c \in C(c,c)$ to $\alpha_c(1_c)$ since α_c is a function in SET.

① Construct inverse function.

Let $\Psi : Fc \longrightarrow Hom(C(-,c), F)$ be a function. If it is defined well, then $\forall x \in Fc$ $\forall f: b \to c \in Mor\, C$,

$$\begin{tikzcd} C(c,c) \arrow[r, "\Psi(x)_c"] \arrow[d, "f^*"'] & Fc \arrow[d, "Ff"] \\ C(b,c) \arrow[r, "\Psi(x)_b"] & Fb \end{tikzcd}$$

commutes.

Then, for $1_c \in C(c,c)$, $\Psi(x)_b(f) = Ff(\Psi(x)_c(1_c))$.

Thus, if Ψ is inverse of Φ, then

$x = \Phi(\Psi(x)) = \Psi(x)_c(1_c)$. Moreover, for any

$f \in C(b,c)$, $\Psi(x)_b(f) := Ff(\Psi(x)_c(1_c))$

$= Ff(x)$.

Hence define $\Psi : Fc \longrightarrow Hom(C(-,c), F)$

$x \longmapsto \Psi(x)$

$= \{ \Psi(x)_b : b \in Ob\,C \}$

where $\Psi(x)_b(f) := Ff(x)$ for any $f \in C(b,c)$

① It is well-defined; let $g : b \to a \in Mor\,C$.
Then.

$$\begin{tikzcd}
h \arrow[rr, mapsto] \arrow[ddd, mapsto, bend right] & & \Psi(x)_a(h) = Fh(x) \arrow[dd, mapsto, bend left] \\
C(a,c) \arrow[r, "\Psi(x)_a"] \arrow[d, "g^*"'] \arrow[dr, phantom, "\circlearrowleft"] & Fa \arrow[d, "Fg"] & \\
C(b,c) \arrow[r, "\Psi(x)_b"'] & Fb & Fg\cdot Fh(x) = F(h\cdot g)(x) \arrow[d, phantom, "\parallel"] \\
h\cdot g \arrow[rr, mapsto] & & \Psi(x)_b(h\cdot g) = F(h\cdot g)(x)
\end{tikzcd}$$

Thus $\Psi(x)$ is natural transformation.

②. Ψ is right inverse; by construction $\Phi\Psi(x) = x$.

③ Ψ is left inverse

Let $\alpha \in Hom(C(-,c), F)$.

$$\Psi\Phi(\alpha) = \Psi(\alpha_c(1_c))$$

$$= \{\Psi(\alpha_c(1_c))_b : b \in \mathrm{Ob}\, C\}$$

s.t. $\Psi(\alpha_c(1_c))_b(f) = Ff(\alpha_c(1_c))$.

For any $f \in C(b, c)$.

Since α is natural transformation,

$$\begin{tikzcd}
C(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "Ff"] \\
C(b,c) \arrow[r, "\alpha_b"'] & Fb
\end{tikzcd}$$

$\Rightarrow Ff(\alpha_c(1_c)) = \alpha_b(f)$.

$\Rightarrow \Psi(\alpha_c(1_c))_b = \alpha_b$.

$\Rightarrow \Psi(\alpha_c(1_c)) = \alpha$ as a natural transformation

$\Rightarrow \Psi\Phi(\alpha) = \alpha$.

▨

So Ψ is bijection inverse of Φ.

pf of naturality) Φ_G, Φ_F : constructed for functors G, F resp.

Statement 1: Given $\beta : F \Rightarrow G$, let $\Phi_G(\beta\alpha) = x \in Gc$.

Then $x = \beta_c(\Phi_F(\alpha))$

In other words, below commutes.

$$\begin{tikzcd}
\mathrm{Hom}(C(-,c), F) \arrow[r] \arrow[d, "\beta_*"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "\beta_c"] \\
\mathrm{Hom}(C(-,c), G) \arrow[r] & Gc
\end{tikzcd}$$

pf)

$$\Phi_G(\beta\cdot\alpha) = (\beta\cdot\alpha)_c(1_c)$$

$$= \beta_c(\alpha_c(1_c))$$

$$= \beta_c(\Phi_F(\alpha))$$

▨

Statement 2: Given $f: b \to c \in \mathrm{Mor}\, C$, Let

$$\Phi_b(\alpha f^*) = x \Rightarrow x = Ff(\alpha_c(1_c))$$

i.e,

$$\begin{tikzcd}
\mathrm{Hom}(C(-,c), F) \arrow[r, "\Phi_c"] \arrow[d, "(f^*)_*"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "Ff"] \\
\mathrm{Hom}(C(-,b)) \arrow[r, "\Phi_b"'] & Fb.
\end{tikzcd}$$

pf) Notes that

$$\begin{tikzcd}
C(b,b) \arrow[r, "f^*"] & C(b,c) \arrow[r, "\alpha_b"] & Fb \\
1_b \arrow[r, mapsto] & f \arrow[r, mapsto] & \alpha_b(f)
\end{tikzcd}$$

From

$$\begin{tikzcd}
C(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "Ff"] \\
C(b,c) \arrow[r, "\alpha_b"'] & Fb
\end{tikzcd}$$

,

$$\begin{aligned}
\alpha_b(f) &= Ff(\alpha_c(1_c)) \\
&\parallel \\
\alpha f^*&(1_b) \\
&\parallel \\
\Phi_b&(\alpha f^*).
\end{aligned}$$

▨

2.2 ii)

Ex 2.2.11). This question is asking that why Yoneda lemma does not imply that there is a natural bijection

$$\mathrm{Hom}(F, C(c,-)) \cong Fc \quad \cdots ①$$

for a functor $F : C \longrightarrow \mathrm{Set}$ with $c \in \mathrm{ob}\, C$.

If it is natural bijection, then Yoneda Lemma says

$$\mathrm{Hom}(F, C(c,-)) \cong Fc \cong \mathrm{Hom}(C(c,-), F)$$

If we take $F = C(d,-)$ then Yoneda lemma says:

$$\mathrm{Hom}(C(c,-), C(d,-)) \cong C(d,c)$$

$$\mathrm{Hom}(C(d,-), C(c,-)) \cong C(c,d)$$

But ① says that

$$\mathrm{Hom}(C(c,-), C(d,-)) \cong C(c,d)$$

$$\mathrm{Hom}(C(d,-), C(c,-)) \cong C(d,c)$$

$\Rightarrow$ $C(c,d) \cong C(d,c)$ for any $c, d \in C$.

However, this is not true in general.

Ex 2.2. iii)

$$\begin{tikzcd}
y : \omega \arrow[r, hook] & \mathrm{Set}^{\omega^{op}} & \\
n \arrow[d, "f_{n,n+1}"'] \arrow[r, mapsto] & F_n := \omega(-, n) \arrow[d, "(f_{n,n+1})_*"] & \text{post comp.} \\
n+1 \arrow[r, mapsto] & F_{n+1} := \omega(-, n+1) &
\end{tikzcd}$$

In a similar manner in p5.6 we can draw a diagram

$$\begin{tikzcd}
 & & & & & n & n+1 & & \\
F_n(\omega) = & * \arrow[r] \arrow[d, "(f_{n,n+1})_*"] & * \arrow[r] \arrow[d, "(f_{n,n+1})_*"] & \cdots \arrow[r] & & * \arrow[r] \arrow[d, "(f_{n,n+1})_*"] & \emptyset \arrow[r] \arrow[d, "\emptyset"] & \emptyset \arrow[r] \arrow[d, "\emptyset"] & \cdots \\
F_{n+1}(\omega) & * \arrow[r] & * \arrow[r] & \cdots \arrow[r] & & * \arrow[r] & * \arrow[r] & \emptyset \arrow[r] & \cdots
\end{tikzcd}$$

Now let $k < n \in \omega$. Then,

$$\begin{aligned}
\Phi : \omega(k, n) &\longrightarrow \mathrm{Set}^{\omega^{op}}(\omega(-,k), \omega(-,n)) \\
f_{k,k+1} \circ \cdots \circ f_{n-1,n} &\longmapsto (f_{k,k+1} \circ \cdots \circ f_{n-1,n})_*
\end{aligned}$$

is bijective since $\mathrm{Set}^{\omega^{op}}(\omega(-,k), \omega(-,n))$ is a singleton; to see this, let $\alpha : \omega(-,k) \Rightarrow \omega(-,n)$. Then by argument in p56, α is determined by $\alpha_k \in \omega(k,n)$. However, $\alpha_k = f_{k,k+1} \circ \cdots \circ f_{n-1,n}$.

$\Rightarrow \alpha = (f_{k,k+1} \circ \cdots \circ f_{n-1,n})_*$. Thus, the codomain is singleton, therefore Φ is bijection $\Rightarrow$ y is fully faithful..

Ex 2.2 (iv)

(i) ⇒ (ii) It suffices to show that for any $c \in C$ $(f_*)_c$ is isomorphism. Since $C(c,x) \xrightarrow{(f_*)_c} C(c,y)$ is a morphism in SET, it suffices to show that $(f_*)_c$ is bijection. And $(f^{-1})_*$ is inverse of f_*, since

$$C(c,x) \xrightarrow{(f_*)_c} C(c,y) \xrightarrow{(f^{-1}_*)_c} C(c,x)$$

$$g \longmapsto fg \longmapsto f^{-1}fg = g$$

(And other way is similar.)

$\Longrightarrow$ (f_*) is natural iso

(ii) ⇒ (i) If f_* is natural iso, $(f_*)_c$ is an isomorphism. Thus, $C(c,x) \cong C(c,y)$ $\forall c \in C$.

Thus $C(y,x) \cong C(y,y)$. Thus $\exists g \in C(y,x)$ s.t. $fg = 1_y$.

Conversely, $gf \in C(x,x) \overset{(f_*)_x}{\cong} C(y,x)$

$$\Rightarrow \quad f_*(gf) = fgf = f$$
$$f_*(1_x) = f$$

Since $(f_*)_x$ is iso, it is bijective as a function.

$\Longrightarrow$ $gf = 1_x$

1) ⟺ (3) By duality on above argument.

Ex22v) $P \cong Set(-, \Omega)$ by Ex 2.1.6.

Thus, $Hom(P, P) \cong Hom(Set(-,\Omega), Set(-,\Omega))$

$\cong Hom(\Omega, \Omega)$ by Yoneda Lemma.

Now construct natural endomorphism using construction of Yoneda bijection.

$$\Phi : Hom(\Omega, \Omega) \longrightarrow Hom(P, P)$$

where $\Phi(x)_b(f) = Pf(x)$ for any $f \in Set(b, \Omega)$

Then for $g : b \to a \in Set(b, a)$

$$\begin{tikzcd}
Set(a,\Omega) \arrow[r, "\Phi(x)_a"] \arrow[d, "Pg"'] \arrow[dr, phantom, "\circlearrowright"] & Pa \arrow[d, "Pg"] & \Rightarrow & h \arrow[r, mapsto] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Ph(x) = h^{-1}(x) \arrow[d] \\
Set(b,\Omega) \arrow[r, "\Phi(x)_b"'] & Pb & & h \circ g \arrow[r, mapsto] & P(hg)(x)
\end{tikzcd}$$

Thus there are four cases of x

x_1: $$\begin{tikzcd} \perp \arrow[r] & \perp \\ T \arrow[ur] & \end{tikzcd}$$ Then, $x_1 = \emptyset$ in $P\Omega$

x_2: $$\begin{tikzcd} \perp \arrow[r] & \perp \\ T \arrow[r] & T \end{tikzcd}$$ Then $x_2 = \{T\}$ $P\Omega$

x_3: $$\begin{tikzcd} \perp \arrow[dr] & \perp \\ T \arrow[ur] & T \end{tikzcd}$$ Then $x_3 = \{\perp\}$ $P\Omega$

x_4: $$\begin{tikzcd} \perp \arrow[dr] & \perp \\ T \arrow[r] & T \end{tikzcd}$$ " $x_4 = \{\perp, T\}$ $P\Omega$

Thus, $\mathcal{I}(X_i)_b(f) = \left(\begin{array}{ll} f^{-1}(\emptyset) & i=1 \\ f^{-1}(\{T\}) & 2 \\ f^{-1}(\{\perp\}) & 3 \\ f^{-1}(\Omega) & 4. \end{array}\right.$

Thus ~~a~~ let: $\alpha_i = \mathcal{I}(X_i)$ Then for any $g: b \to a$

$$\begin{tikzcd} Pa \arrow[r, "(\ )^{-1}(\emptyset)"] \arrow[d, "g^{-1}=P_g"'] & Pa \arrow[d, "P_g=g^{-1}"] \\ Pb \arrow[r, "(\ )^{-1}(\emptyset)"'] & Pb \end{tikzcd} \Rightarrow$$

Let $h: A \to \Omega$ with $h^{-1}(\{T\}) = A'$. Then

$$\begin{tikzcd} h \arrow[r, mapsto] \arrow[d] & \emptyset \arrow[d] \\ g^{-1}\cdot h^{-1}(\{T\}) \arrow[r, mapsto] & \emptyset \end{tikzcd}$$

or $h\,g \longmapsto g^{-1}h^{-1}(\emptyset) = \emptyset$

Thus α_1 is trivial natural transformation sending everything to $\emptyset$. In case of $i=2$.

$$\begin{tikzcd} h \arrow[r, mapsto] \arrow[d] & h^{-1}(\{T\}) = A' \arrow[d] \\ h\,g \arrow[r, mapsto] & g^{-1}h^{-1}(\{T\}) = g^{-1}(A') \end{tikzcd}$$

But $h^{-1}(\{T\})$ is identified as h.

$\Rightarrow \alpha_2 = 1_P$ in $\mathrm{Set}^{\mathrm{Set}^{op}}$.

In case of $i=3$.

$$\begin{tikzcd} h \arrow[r] \arrow[d] & h^{-1}(\{\perp\}) \arrow[d] \\ h\,g \arrow[r, mapsto] & g^{-1}h^{-1}(\{T\}) = g^{-1}(A-A') \end{tikzcd}$$

So α_3 is conversion from A' to $A-A'$. (or from h: to $\alpha_3 h$. where α_3 as a function

In case of $i=4$

$$\begin{tikzcd} h \arrow[r] \arrow[d] & A \arrow[d] \\ h\,g \arrow[r] & g^{-1}h^{-1}\Omega = g^{-1}(A) \end{tikzcd}$$

α_4 is trivial natural transf sending everything to A.

Note that this doesn't work for covariant power set functor since $P \not\cong \mathrm{Set}(\Omega, -)$

Ex 2.2 vi. This is equivalent to asking there is nontrivial natural endomorphism $\alpha : 1_{Top} \Rightarrow 1_{Top}$.

Answer is no.. Let $\alpha \in \mathrm{Hom}(1_{Top}, 1_{Top})$, For any $* \longrightarrow X$
$* \longmapsto x$

we have

$$\begin{tikzcd} * \arrow[r, "\alpha_*"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & * \arrow[d] \\ X \arrow[r, "\alpha_X"'] & X \end{tikzcd}$$

However, α_* should be trivial identity. (There is no other choices)

Commutativity of the diagram gives $\alpha_X(x) = x$.

The x was chosen arbitrarily, α_X is identity

X " α is trivial natural endo.

Ex 2.2 vii) Observe that for any $X \in Top$

$\mathrm{Path}(X) \cong Top(I, X)$ since every path p X can be identified by $I \xrightarrow{cts} P \subseteq X$. and vice versa.

Thus, $\mathrm{Hom}(\mathrm{Path}, \mathrm{Path}) \cong \mathrm{Hom}(Top(I,-), Top(I,-))$

$$\underset{\text{Yoneda}}{\cong} Top(I, I)$$

Hence any natural automorphism of Path functor is constructed from homeomorphism of I.

2.3 Universal properties and Universal element

Prop 2.3.1 Let $x, y \in \operatorname{ob} C$.

$x \cong y$ in C $\iff$ $C(-,x) \cong C(-,y)$ in Set^{C}

$C(x,-) \cong C(y,-)$

pf) $(\Rightarrow)$ $y: C^{op} \longrightarrow \mathrm{Set}^{C}$ is a functor

So preserve isomorphism

$(\Leftarrow)$ Let $\alpha : C(-,x) \cong C(-,y)$ iso

Since y is fully faithful, $\exists ! \, f : x \to y$ iso.

Rmk; in $(\Leftarrow)$, Maybe, there are more iso!

but f is unique among any iso $x \cong y$.

So f is "the" iso.

"the" : object $\iff$ Object in question is well-def up to canonical iso.

Corollary 2.3.2 Full category spanned by its terminal object is either empty or Contractible groupoid. In particular, any two terminal objects in C are uniquely isomorphic.

Contractible groupoid $G \Rightarrow G \cong \mathbb{1}$.

Pf). By Yoneda lemma,

$$\mathrm{Hom}(C(-,t), C(-,t')) \overset{\substack{\text{bijection}\\ \downarrow}}{\cong} C(t,t')$$

Since t' is terminal, $C(t,t') = \{*\}$
Singleton.

Recall

Def 2.1.3. t is terminal

$\Longleftrightarrow$ $C(-,t) : C^{op} \longrightarrow \mathrm{Set}$ is naturally iso

to $* : C^{op} \longrightarrow \mathrm{Set}$

$\forall c,\ c \longmapsto \{*\}$.

Thus, $C(-,t) \cong * \cong C(-,t')$. 1_*

$\therefore$ $t = t'$ by Prop 2.3.1 ▨

Def) (Universal property)

An <u>universal property</u> of $c \in C$ is expressed by representable function F with <u>universal element</u> $x \in F_c$, that defines a natural iso $C(c,-) \cong F$ via Yoneda Lemma.

Ex 2.3.4. $U: \mathrm{Ring} \to \mathrm{Set} \cong \mathrm{Ring}(\mathbb{Z}[x], -)$

Since $\mathrm{Ring}(\mathbb{Z}[x], R) \cong UR$

defined by $x \in \mathbb{Z}[x]$.

By Yoneda lemma, $ev: \mathrm{Ring}(\mathbb{Z}[x], R) \to UR$
$\phi \longmapsto \phi(x)$

is bijection.

Ex 2.3.6) (There is no ex 2.3.5)

$E: BG \to \mathrm{Set}$ is representable iff $G \overset{\text{bijection}}{\cong} E$
$\cdot \longmapsto E$ as left G-set

Pf) If E is representable, $BG(\cdot, -) \cong E$

Thus, $BG(\cdot, \cdot) \cong E(\cdot) = E$ Since $BG(\cdot,\cdot) \cong G$, $G \cong E$. Conversely, if $E \cong G$, then $BG(\cdot, -) \cong E$

▨

$\Rightarrow$ Action of G on E is (as a left multiplication)

① Free (since every stabilizer is trivial)

② transitive (orbit is entire set.)

③ E is nonempty

pf) If $E \cong G$, then $\forall g \in E$, $\exists g^{-1} \in G$ s.t. $g^{-1}g = e$. So orbit is entire set. Also, any $g \in$ permute G, So St

Stabilizer is 0. ③ is clear.

Conversely, any nonempty free and transitive left G-set is representable. (Omit pf)

By Yoneda Lemma, universal element for universal property of $\bullet \in BG$ is $e \in G$.

$\Rightarrow$ E is just an 'underlying set G forgetting gp structure.

$\therefore$ G-torsor := A representable G-set (like E)

ex) $\mathbb{A}^n$: affine space = forgetting $(0, \cdots, 0)$ as Identity in $\mathbb{R}^n$.

$\mathbb{R}^n$ act on $\mathbb{A}^n$ by thinking $\alpha \in \mathbb{R}^n$ as a vector sending pt to pt + α;

This action is

① Free: Every nonzero vector doesn't stabilize pt.

② Transitive: Any two pt in $\mathbb{A}^n$ form a vector.
So they are in the same orbit.

Choice of Identity $\overset{0}{}$ in $\mathbb{A}^n$ gives iso on $\mathbb{R}^n \cong \mathbb{A}^n$.

So, $0 \in \mathbb{A}^n$ is universal element

Ex 2.3.17 U, W : k-vector space

$\mathrm{Bilin}(V, W; -) : \mathrm{Vect}_k \longrightarrow \mathrm{Set}$

$U \longmapsto \{ f : V \times W \to U$ k-billinear $\}$

From bilinearity, i.e, $f(v, -) : W \to U$
$f(-, w) : V \to U$.

f is identified as a map $V \to \mathrm{Hom}(W, U)$
$W \to \mathrm{Hom}(V, U)$.

We claim $\mathrm{Bilin}(V, W; -) \cong \mathrm{Vect}_k(V \otimes_k W, -)$

(Actually it is known as the universal property of the tensor product.)

By Yoneda Lemma, the isomorphism

$\mathrm{Bilin}(V, W; U) \cong \mathrm{Vect}_k(V \otimes_k W, U)$

is determined by universal element of

$\mathrm{Bilin}(V, W; V \otimes_k W)$ i.e.

$\otimes : V \times W \longrightarrow V \otimes_k W$ canonical billinear map.

$\Rightarrow V \otimes_k W$: Universal vector space equipped with a bilinear map from $V \times W$.

What is meaning of it?

Notes that $\mathrm{Vect}_k(V\otimes_k W, U) \cong \mathrm{Bilin}(U, W; U)$

Pick $f \in \mathrm{Bilin}(U, W; U)$ and $\bar{f}: U\otimes_k W \to U$ corresp to f Then,

$$\begin{tikzcd}
\mathrm{Vect}_k(U\otimes_k W, V\otimes_k W) \arrow[r, "\cong"] \arrow[d, "\bar{f}_*"'] & \mathrm{Bilin}(U, W; U\otimes_k W) \arrow[d, "\bar{f}_*"] \\
\mathrm{Vect}_k(U\otimes_k W, U) \arrow[r, "\cong"'] & \mathrm{Bilin}(U, W; U)
\end{tikzcd}$$

Then

$$\begin{tikzcd}
1_{U\otimes_k W} \arrow[r, mapsto] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \otimes \arrow[d] \\
\bar{f} \arrow[r, mapsto] & f \quad \bar{f}\circ\otimes
\end{tikzcd}$$

i.e,

$$\begin{tikzcd}
U\times W \arrow[r, "\otimes"] \arrow[dr, "f"'] & U\otimes_k W \arrow[d, "\bar{f}"] \arrow[dl, phantom, "\circlearrowright"] \\
 & U
\end{tikzcd}$$

and $\bar{f}$ is unique by iso.

$\Rightarrow$ Actually $\otimes : V\times W \to U\otimes_k W$ is initial element in some other category

Also, (if we know existence of $V\otimes_k W$) then we can construct it.

$$\text{pf)}\quad V\times W \xrightarrow{\otimes} U\otimes W \underset{0}{\overset{\text{quotient}}{\rightrightarrows}} U\otimes V/\langle u\otimes w\rangle = 0.$$

By universal property, $0 =$ quotient.

Since quotient is surj $V\otimes W = (v\otimes w : v \in V, w \in W)$

Prop 2.3.9. $V \underset{k}{\otimes} W \cong W \underset{k}{\otimes} V$

pf) $\exists$ natural iso $\mathrm{Bilin}(V, W; -) \cong \mathrm{Bilin}(W, V; -)$

$f : V \times W \to U \longmapsto f^{\#} : W \times V \to U.$

$f^{\#}(w, v) := f(v, w)$

Then,

$\mathrm{Vect}_k(V \underset{k}{\otimes} W, -) \cong \mathrm{Bilin}(V, W; -) \cong \mathrm{Bilin}(W, V; -) \cong \mathrm{Vect}_k(W \otimes V, -)$

$\Rightarrow$ ~~$W\otimes$~~ $V \underset{k}{\otimes} W \cong W \underset{k}{\otimes} V$ by 2.3.1

Also, this give explicit iso $V \otimes W \cong W \otimes V$ since by Yoneda lemma, it is image of ...

1_C in C $\left(C = \mathrm{Vect}_k(V \underset{k}{\otimes} W, V \underset{k}{\otimes} W) \right)$

Thus, Let $\phi : W \underset{k}{\otimes} V \xrightarrow{\cong} V \underset{k}{\otimes} W$: iso. Then

$\phi^{*} : \mathrm{Vect}_k(V \underset{k}{\otimes} W, -) \xrightarrow{\cong} \mathrm{Vect}_k(W \otimes V, -)$

by precomposition.

So

$$\begin{tikzcd}
W \times V \arrow[r, "\otimes"] \arrow[d, "{(w,v) \mapsto (v,w)}"'] & W \otimes V \arrow[d, "\exists ! \ \phi"] \\
V \times W \arrow[r, "\otimes"'] & V \underset{k}{\otimes} W
\end{tikzcd}$$

Ex 2.3. i) $\mathrm{mor}2 = \{\cdot \longrightarrow \cdot\}$ is singleton.

So universal element is nontrivial morphism in $\mathrm{mor}\,2$

ii) $\mathcal{O}(S) = \{\phi, S, \{1\}\}$ where $S = \{1, 2\}$

Universal element is $\{1\}$ since any $Y' \in \mathcal{O}(X)$ is identified by $f_{X'}(x) = \begin{cases} 1 & x \in X' \\ 2 & x \notin X' \end{cases}$

So that $f_{X'}^{-1}(\{1\}) = X'$.

iii) $C(S) = \{\phi, S, \{2\}\}$ Universal element is $\{2\}$ by the dual argument of above.

2.3. ii)

(1) From bilinearity of $\otimes$, $1 \otimes av = a(1 \otimes v)$
$a \otimes v = a(1 \otimes v)$

by applying linearity on $(1 \otimes -)$ and $(- \otimes v)$.

Let
$$\begin{tikzcd} k \otimes V \arrow[r, "f"] & V \arrow[r, "g"] & k \otimes V. \\ a \otimes v \arrow[r, mapsto] & av \arrow[r, mapsto] & 1 \otimes av = a \otimes v. \end{tikzcd}$$

f is surjective. Since $gf = 1_{k \otimes V}$, f is injective

$\Rightarrow$ f is isomorphism $\Rightarrow$ $k \otimes V \cong V$.

(2) This is just in Atiyah Macdonald Prop 2.14 in case of ring. This proof only use the universal property.

Ex2.3 iii)

Let $ev : B^A \times A \longrightarrow B$

$(f, a) \longmapsto f(a)$.

① $\mathrm{Set}(- \times A, B) \overset{\alpha}{\cong} \mathrm{Set}(-, B^A)$

Let $f : C \longrightarrow D$. Then

$$\begin{tikzcd}
\mathrm{Set}(D \times A, B) \arrow[d, "(f \times 1_A)_*"'] & \mathrm{Set}(D, B^A) \arrow[l, "\alpha_D"'] \arrow[d, "f_*"] \\
\mathrm{Set}(C \times A, B) & \mathrm{Set}(C, B^A) \arrow[l, "\alpha_C"]
\end{tikzcd}$$

where $\alpha_D = (g : d \longmapsto g(d)) \longmapsto g' : (d, a) \longmapsto g(d)(a)$

As we've seen it with naive Currying,

it is natural transformation

② Universal property: Let $f : U \longrightarrow B^A$ Then

$$\begin{tikzcd}
\mathrm{Set}(B^A, B^A) \arrow[r, "\alpha_{B^A}"] \arrow[d, "f_*"'] \arrow[dr, bend right, "①"'] \arrow[dr, bend left, "②"] & \mathrm{Set}(B^A \times A, B) \arrow[d, "(f \times 1_A)_*"] \\
\mathrm{Set}(U, B^A) \arrow[r, "\alpha_U"'] & \mathrm{Set}(U \times A, B)
\end{tikzcd}$$

If we send 1_{B^A} along ①, it is $\alpha_U(f)$

$\Rightarrow \alpha_U(f) = f' : (u, a) \longmapsto f(u)(a)$.

If we send 1_{B^A} along ②, $\alpha_{B^A}(1_{B^A}) = ev$. So $ev(f \times 1_A(-))$

Thus
$$\begin{tikzcd}
1_{B^A} \arrow[r, mapsto] \arrow[dd] & ev \arrow[d] \\
 & ev \circ (f \times 1_A(-)) \\
f \arrow[r, mapsto] & \alpha_U(f) = f'
\end{tikzcd}$$

This implies that for any $f' \in Set(U \times A, B)$
$\exists f$ s.t.

$$\begin{tikzcd}
U \times A \arrow[r, "f \times 1_A"] \arrow[dr, "f'"'] \arrow[dr, phantom, bend left, "\circlearrowright"] & B^A \times A \arrow[d, "ev"] \\
 & B
\end{tikzcd}$$

where $f' = (u,a) \longmapsto f(u)(a)$.

And you can check it by

$$\begin{tikzcd}
(u,a) \arrow[r, mapsto] \arrow[dr] & (f(u), a) \arrow[d] \\
 & f'(u,a) = f(u)(b)
\end{tikzcd}$$

2.4. Category of element.

"c is universal object in C with an x"

$\Rightarrow$ $F \cong C(-, c)$ or $\cong C(c, -)$

and $x \in Fc$ st classifies the natural isomorphism defining the representation by Yoneda Lemma.

Def 2.4.1 $F : C \longrightarrow \mathrm{Set}$. Define $\int F$ as

$\mathrm{Ob} \int F \triangleq \{ (c, x) : c \in C,\ x \in Fc \}$

$\mathrm{Mor} \int F = \{ f : (c, x) \longrightarrow (c', x')$ If $f : c \longrightarrow c'$ and $Ff(x) = x'$ holds. $\}$

This has forgetful function $\pi : \int F \longrightarrow C$

If $Ff(x) = x'$, then
$$\begin{tikzcd}
(c, x) \arrow[r, mapsto] \arrow[d, "f"'] & c \arrow[d, "f"] \\
(c', x') \arrow[r, mapsto] & c'
\end{tikzcd}$$

Def 2.4.2 (contravariant case)

Ex 2.4.3 $n\text{-Color} : \mathrm{Graph}^{op} \longrightarrow \mathrm{Set}$
$$\begin{tikzcd}
G \arrow[r, mapsto] \arrow[d] & \{ c : n\text{-coloring of } G \} \\
H \arrow[r, mapsto] & \{ c : \text{" } H \} \arrow[u]
\end{tikzcd}$$

$\Rightarrow$ $\int n\text{-color}$: Ob : (G, c)

Mor : $\phi\ (G, c) \longrightarrow (G', c')$

$\phi : G \to G'$ graph morphism

$n\text{-Color}(\phi)(c') = c$

Thus, $\int n\text{-color}$ is category of n-colored graphs and color preserving graph homomorphism.

Ex 2.4.4 $U: C \longrightarrow Set$ forgetful functor.

$\int U$: Ob : (c, x)

Mor : $(c, x) \xrightarrow{f} (c', x')$ with $f: c \longrightarrow c'$

$Uf(x) = x'$.

Thus f preserve x.

$\Rightarrow$ $C_{*} = \int U$ category of based objects in C.

Ex 2.4.5 $F: C \longrightarrow Set$, C is discrete. $\Rightarrow$ F is identified by $(F_c)_{c \in C}$

$\Rightarrow$ $\int F$: $Ob \int F = \coprod_{c \in C} F_c$ identified by disj union.

$\pi : \coprod_{c \in C} F_c \longrightarrow C$ has $\pi^{-1}(c) = F_c$.

$(c, x) \longmapsto c$

So, $\coprod_{c \in C} F_c$ is called "dependent sum" of $(F_c)_{c \in C}$

$\prod_{c \in C} F_c$ is " " " product" which is the set of sections of π:

$s: C \longrightarrow \coprod_{c \in C} F_c$

$c \longmapsto s(c) \in F_c$.

Ex 2.4.6. $\int C(c,-)$

Ob: (d,f) where $d\in C$, $f\in C(c,d)$

Mor: $(d,f)\xrightarrow{g}(d',f')$ Iff $\exists g: d\to d'$ and $gf=f'$.

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ d \arrow[rr, "g"'] & & d' \end{tikzcd}$$

$\Rightarrow \int C(c,-) = {}^{c}/C$ slice category under $c\in C$

Dually, $\int C(-,c) = C/_{c}$ " over "

By Exercise 1.2.i, with $C(-,c)\cong C^{op}(c,-)$,

$$\int C^{op}(c,-) = \left({}^{c}/C^{op}\right) \underset{\underset{\text{Ex 1.2.i}}{\uparrow}}{\cong} \left(C/_{c}\right)^{op} = \left(\int C(-,c)\right)^{op}$$

Lemma 2.4.7. $F: C^{op}\longrightarrow Set$

$\Rightarrow \int F \cong y\downarrow F$ where $y: C \hookrightarrow Set^{C^{op}}$

$F: \mathbb{1}\longrightarrow Set^{C^{op}}$.

Pf)

	$\int F$	$y\downarrow F$
Object	(c,x) : $c\in C$, $x\in Fc$	$(C(-,c), F, C(-,c)\Rightarrow F)$
Morphism	$(c,x)\longrightarrow(c',x')$ when $f: c\to c'\in C$. $Ff(x')=x$.	$(C(-,c), F, C(-,c)\Rightarrow F)$ $(f, 1_F)\downarrow$ $(C(-,c'), F, C(-,c')\Rightarrow F)$ when $(C(-,c')\Rightarrow F)\circ f_* = C(-,c)\Rightarrow F$.

Now define $L : \int F \longrightarrow y \downarrow F$.

$$\begin{tikzcd}
(c, x) \arrow[r, mapsto] \arrow[d, "f"'] & (\mathcal{C}(-,c),\ F,\ \Phi(x) : \mathcal{C}(-,c) \Rightarrow F) \arrow[d, "{(f, 1_F)}"] \\
(c', x') \arrow[r, mapsto] & (\mathcal{C}(-,c'),\ F,\ \Phi(x') : \mathcal{C}(-,c') \Rightarrow F)
\end{tikzcd}$$

where $\Phi : Fc \longrightarrow \mathrm{Hom}(\mathcal{C}(-,c), F)$ defined by Yoneda lemma.

1) It is well-defined functor.

It suffices to show that

$$\begin{tikzcd}
\mathcal{C}(-,c) \arrow[r, Rightarrow, "\Phi(x)"] \arrow[d, Rightarrow, "f_*"'] & F \\
\mathcal{C}(-,c') \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Let $g : d \to e \in \mathcal{C}$.

Then,

$$\begin{tikzcd}
\mathcal{C}(d,c) \arrow[r, "\Phi(x)_d"] \arrow[d, "f_*"'] & Fd \\
\mathcal{C}(d,c') \arrow[ur, "\Phi(X')_d"'] &
\end{tikzcd}$$

Notes that by construction, for any $h \in \mathcal{C}(d,c)$

$\Phi(x)_d(h) = Fh(x)$.

Thus we need to show that $Fh(x) = F(fh)(x')$
$= Fh \cdot Ff(x')$

By definition, $Ff(x') = x$. So it is the same.

Hence $\Phi(x)_d = \Phi(x')_d \circ f_* \Rightarrow \Phi(x) = \Phi(x) \circ f_*$.

So L is well-defined. (other functoriality is trivial.)

This L induces bijection between objects;
Since any $(C(-,c), F, \alpha : C(-,c) \Rightarrow F)$
$\Phi(\alpha_c(1_c)) = \alpha$ by Yoneda lemma, so it is image of L. $\Rightarrow$ L is surjective.

Also if two objects of $y \downarrow F$ are the same, then they have same $\alpha : C(-,c) \Rightarrow F$, so Yoneda lemma assures that they are from same image. $\Rightarrow$ L is injective.

Moreover, Corollary 2.2.8 shows that $\mathrm{Hom}(C(-,c), C(-,c')) \cong C(c,c')$, Hence every morphism of $y \downarrow F$ comes from $C(c,c')$ s.t. $Ff(x') = x$. So L has surjectivity on $\mathrm{Mor}(y \downarrow F)$. Also, two morphisms are the same $\Rightarrow$ they are the same as morphism in C, $\Rightarrow$ L is injective on $\mathrm{Mor}(y \downarrow F)$

$\therefore$ L is bijection between Categories
$\Rightarrow$ Equivalent.

▨

Prop 2.4.8 $F: C \longrightarrow \mathrm{Set}$ is representable

$\iff \int F$ has initial object

Dually, $F: C^{op} \longrightarrow \mathrm{Set}$ is rep~.

$\iff \int F$ has terminal object.

Pf) ($\Rightarrow$) $C(c, \sim) \cong F$ for some $c \in C$.

$$\Rightarrow \int F \cong \int C(c, \sim) \underset{\underset{\text{Exercise 2.4 vii}}{\uparrow}}{\cong} {}^{c}/C$$

And $1_c \in {}^{c}/C$ is initial object.

($\Leftarrow$). Let $(c, x) \in \int F$ be initial.

WTS $\Phi(x): C(c, -) \Rightarrow F$ is natural iso.

Let $y \in Fd$. Then $\exists! \, (c, x) \xrightarrow{f} (d, y)$. by initial object $\Rightarrow \exists! \, f: c \longrightarrow d$ s.t. $Ff(x) = y$.

$\Rightarrow \Phi(x)_d(f) := Ff(x) = y$.

Since y was arbitrarily chosen, $\Phi(x)_d(C(c, d)) = Fd$.

$\Rightarrow \Phi(x)_d$ is surjective.

Also, for each $y \in Fd$, $\Phi(x)_d^{-1}(y)$ is singleton by uniqueness. $\Rightarrow \Phi(x)_d$ is injective.

$\Rightarrow \Phi(x)_d$ is iso. Since d was arbitrarily chosen

$\Phi(x)$ is natural iso.

Another way of $(\Rightarrow)$)

Let $\alpha : C(c, -) \cong F$. We claim $(c, \alpha_c(1_c))$ is initial. To see this, for any $d \in C$, $Fd \underset{\alpha_d}{\cong} C(c,d)$

$\Rightarrow \forall y \in Fd \ \exists !\, f : c \longrightarrow d$ s.t. $\alpha_d(f) = y$. $\Rightarrow Ff(\alpha_c(1_c)) = y$.

Since α induces

$$\begin{tikzcd} C(c,c) \arrow[r, "\alpha_c \cong"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\circlearrowright"] & Fc \arrow[d, "Ff"] \\ C(c,d) \arrow[r, "\cong"] & Fd \end{tikzcd}$$

Hence, for any $(d, y) \in \int F$ $\exists$ unique morphism
$f : (c, \alpha_c(1_c)) \longrightarrow (d, y)$.

Representation itself may not unique but any of two are isomorphic

Prop 2.4.9. $F : C \longrightarrow Set$. Full subcategory of $\int F$ spanned by its representation is empty or contractible groupoid.

pf) F is not rep^n $\Rightarrow$ empty
If F is rep., $\int F$ has some initial objects.
By Corollary 2.3.2, the span is contractible groupoid.

Ex. 2.4.10. $TX : BG \longrightarrow Set$

$$\begin{tikzcd} \bullet \arrow[r, mapsto] \arrow[d, "g"'] & X \arrow[d, "g \text{ as an automorphism}"] \\ \bullet \arrow[r, mapsto] & X \end{tikzcd}$$

$T_G X$: Obj : X

Mor: $g: x \to y$ if $g \in G$ s.t. $gx = y$.

Ex 2.3.6: If X is representable, any element in X chosen as a universal element.

By 2.4.9, X is rep $\Leftrightarrow \int X \cong T_G X$ is contractible groupoid.

($\int X \cong T_G X$ is clear since any object in $\int X$ is of form $(\cdot, x)$ and any morphism $(\cdot, x) \xrightarrow{g} (\cdot, y)$ is $g \in G$ s.t. $g.x = y$.)

And, $T_G X$ is contractible iff $\forall x, y \in X, \exists! g \in G$ s.t. $gx = y$. (pf): definition of contractible $\Leftrightarrow \mathrm{Hom}(x,y)$ is singleton $\forall x,y$)

Thus, $T_G X$ is contractible iff X is free and transitive G-set. (Existence $\Rightarrow$ transitive
Uniqueness on $x=y \Rightarrow$ free)

$\Rightarrow$ Any free-transitive G-set is representable. ▨

Rmk)!! For any category E, $E = \int *$

where $* : E \to \mathrm{Set}$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & * \arrow[d, "1_*"] \\ y \arrow[r, mapsto] & * \end{tikzcd}$$

Ex 2.4.11.

Let C : Category of dynamical system

Ob C : (X, f, x_0) ; $X \in SET$ $f : X \to X$ endomorphism

$x_0 \in X$

Mor C :
$$(X, f, x_0) \xrightarrow{h} (Y, g, y_0)$$

s.t. $h : X \longrightarrow Y \in SET$

and $h(x_0) = y_0$ and $h \circ f = g$.

Let $U : End \longrightarrow Set$ where End is a cat. of sets equipped with endomorphism.

Then $\int U = C$; Object is the same and any morphism in $\int U$ is a morphism in C and vice versa.

Votes that $(\mathbb{N}, s : \mathbb{N} \to \mathbb{N})$ is a representation of U as we've shown in Ex 2.1.1. And its initial object is $(\mathbb{N}, s, 0) \in \int U$.

$\Rightarrow$ 0 is universal element.

Ex 2.4.12. (1). Let $P : Set \longrightarrow Set$ contravariant functor.

$$\begin{tikzcd}
A \arrow[rr, mapsto] \arrow[dd, "f"'] & & 2^A \\
{} \arrow[rr, mapsto] & & {} \\
B \arrow[rr, mapsto] & & 2^B \arrow[uu, "f^{-1}"']
\end{tikzcd}$$

Then, $\int P$: Obj = (A, A') $A' \subseteq A$

Mor: $(A, A') \xrightarrow{f} (B, B')$

s.t. $f: A \longrightarrow B$, $f^{-1}(B') = A'$.

Terminal object is $(\{\top, \bot\}, \{\top\})$ since for any (A, A'), $\exists! h: A \longrightarrow \{\bot, \top\}$ so that

$$x \longmapsto \begin{matrix} \bot & x \notin A' \\ \top & x \in A' \end{matrix}$$

$$(A, A') \xrightarrow{h} (\{\top, \bot\}, \{\top\}).$$

And notes that $(\{\top, \bot\}, \{\top\})$ is isomorphic to

$($ " $, \{\bot\})$

(ii) $U: Vect_K \longrightarrow Set$ forgetful functor

$\int U$: obj : (V, x) : $x \in V$

Mor: $(V, x) \xrightarrow{f} (W, y)$ if $f: V \to W$ linear and $f(x) = y$.

Then, $(\mathbb{K}, c)$ is initial for an $c \in K$ since $L!: \mathbb{K} \to V$ is determined by $L(c)$ since c is basis of $\mathbb{K}$ as v.s.

$\forall c, c' \in \mathbb{K}$

$\Rightarrow (\mathbb{K}, c) \cong (\mathbb{K}, c')$

(iii). $\int \mathrm{Bilin}(U, W; -)$ identify it as $f: V\times W \to U$.

Obj: $(U, \underset{\text{bilinear}}{f: V\times W \longrightarrow U}\}$ ~ $f: V\times W \longrightarrow U$.

Mor:
$$\begin{tikzcd}
(U, f: V\times W \to U\} \arrow[d, "L"] \\
(U'; f': V\times W \to U'\}
\end{tikzcd}$$
s.t. $L: U \to U'$ and $L(f) = f'$

By 2.3.7., $\otimes: V\times W \longrightarrow V\otimes_k W$ is initial

in this category.

(iv) $U(-)^n$: Group $\longrightarrow$ Set
$$\begin{tikzcd}
G \arrow[r] \arrow[d, "f"'] & G^n \arrow[d, "f^n"] \\
H \arrow[r, mapsto] & H^n
\end{tikzcd}$$

$\int U(-)^n$: obj : $(G, (g_1, \dots, g_n)))$

Mor: $(G,(g_1, \dots, g_n)) \xrightarrow{f} (H, (h_1, \dots, h_n))$

s.t. $f: G \to H$ gp homomorphism

$f(g_i) = h_i$ $\forall i \in [n]$.

Initial element: $(F^n, (x_1, \dots, x_n))$ where F^n is

a free gp gen by $x_1, \dots, x_n$.

(v). $U(-)^*$: $\mathrm{Vect}_k^{op} \longrightarrow \mathrm{Set}$
$$\begin{tikzcd}
V \arrow[r, mapsto] \arrow[d, "f"'] & V^* \\
W \arrow[r, mapsto] & W^* \arrow[u, "f^*"']
\end{tikzcd}$$

So $\int U(-)^*$

Obj: $(V, f: V \to k)$

~ $f: V \to k$

Mor: $f \xrightarrow{L} g: W \to k$

if $L: V \to W$ s.t. $f = gL$.

Thus, $\int U(-)^* = \mathrm{Vect}_k/k$ since

Obj is the same.

any Morphism induces

$$\begin{tikzcd} V \arrow[rr, "L"] \arrow[dr, "f"'] & & W \arrow[dl, "g"] \\ & k & \end{tikzcd}$$

Hence, $1: k \to k$ is terminal in Vect_k/k

$\Rightarrow U(-)^* \cong \mathrm{Vect}_k(-, k)$.

So $1: k \to k$ is called universal dual vector.

vi) $U: \mathrm{Ring} \longrightarrow \mathrm{Set} \Rightarrow U \cong \mathrm{Ring}(\mathbb{Z}[x], -)$

and $x \in \mathbb{Z}[x]$ is universal element

$\int U$: obj (R, r) $r \in R$

Mor : $f: R \to S$ s.t. $f(r) = s$.

So, $(\mathbb{Z}[x], x)$ is initial element of $\int U$.

Since $(\mathbb{Z}[x], x) \xrightarrow{f} (\mathbb{Z}[x], x)$ implies $f = 1_{\mathbb{Z}[x]}$,

but $\mathrm{Ring}(\mathbb{Z}[x], \mathbb{Z}[x])$ is not a singletone.

If $f \in$ " then $f(x)$ determines

f. $\Rightarrow f(x)$ is a polynomial in $\mathbb{Z}[x]$.

$\Rightarrow$ All maps in $\mathrm{Ring}(\mathbb{Z}[x], \mathbb{Z}[x])$ is classified by polynomials in $\mathbb{Z}[x]$. By Yoneda embedding,

$\therefore \mathrm{Ring}'(\mathbb{Z}[x], \mathbb{Z}[x])' \cong \mathrm{Hom}(U, U)$

$\therefore$ all natural endomorphism α is consists of component

$\alpha_R : r \longmapsto p(r) \in p(x) \in \mathbb{Z}[x]$.

Ex 2.4.1 $F: C \longrightarrow Set$

	$\int F$	$* \downarrow F$
Obj	$(c, x) : x \in Fc$	$(*, c, * \longrightarrow Fc)$
Mor	$(c, x) \xrightarrow{f} (c', x')$ s.t. $f: c \longrightarrow c'$ $Ff(x) = x'$	$(*, c, * \longrightarrow Fc)$ $\Big\downarrow (1_*, f)$ $(*, c', * \longrightarrow Fc')$ s.t.

$$\begin{tikzcd}
* \arrow[r, "1_*"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & * \arrow[d] \\
Fc \arrow[r, "Ff"'] & Fc'
\end{tikzcd}$$

Now identify $* \longrightarrow Fc$ as its image; say x.

Then, $(*, c; * \longrightarrow Fc) \rightsquigarrow (c, x)$

$(*, c'; * \longrightarrow Fc') \rightsquigarrow (c, x')$

$\Rightarrow (1_*, f)$ is $f: c \longrightarrow c'$ s.t. $Ff(x) = x'$

$\therefore (1, f)$ is identified as a morphism of $\int F$

So $\int F = * \downarrow F$. (Do the same thing in reversed direction.)

Ex 2.4.2 C/c : Obj: $f: b \longrightarrow c$

Mor: for any $f: b \to c$, $g: d \to c$

$h: b \longrightarrow d$ s.t.

$$\begin{tikzcd}
b \arrow[rr, "h"] \arrow[dr, "f"'] & \circlearrowright & d \arrow[dl, "g"] \\
& c &
\end{tikzcd}$$

If $g: d \longrightarrow c$ is terminal, then for any $f: b \longrightarrow c$

$\exists!\, h: b \longrightarrow d$ s.t. $gh = f$.

Thus, if $g = 1_c$, then it is terminal. Since h is uniquely determined as f.

Ex. 2.4. iii) $F: C \longrightarrow \mathrm{set}$ representable

$\iff \int F$ has initial object.

$\iff (\int F)^{op}$ has terminal object.

Now it suffices to show that $\int F^{op} \cong (\int F)^{op}$.

$\int F^{op}$: Obj: (c, x)

Mor: $(c, x) \xleftarrow{f^{op}} (c', x')$

when $f: c \longrightarrow c'$

$(F^{op} f^{op})(x') \ni x \iff Ff(x) = x'$.

$(\int F)^{op}$: Obj: (c, x)

$(c, x) \xleftarrow{f^{op}} (c', x')$

when $f: c \longrightarrow c'$

$Ff(x) = x'$.

Since $F^{op}: C^{op} \longrightarrow \mathrm{Set}$

$$\begin{tikzcd}
c \arrow[d, "f^{op}"'] \arrow[r, mapsto] & Fc \\
d \arrow[r, mapsto] & Fd \arrow[u, "Ff"']
\end{tikzcd}$$

Thus, $\iff F^{op}: C^{op} \longrightarrow \mathrm{Set}$ is representable.

2.4 iv) $\mathcal{O} : \mathrm{Top}^{op} \longrightarrow \mathrm{Set} \cong \mathrm{Top}(-, S)$

when S is sierpinski space with $S = \{1, 2\}$

$\{1\}$ is open

Thus $\int \mathcal{O}$ has $(S, \{1\})$ as its initial element and $\{1\}$ is universal element.

This means that for any openset $U \subseteq X$

$\exists ! f : X \longrightarrow S$ cts s.t. $f(x) = \begin{cases} 1 & \text{if } x \in U \\ 2 & \text{o.w} \end{cases}$

$\Rightarrow f^{-1}(\{1\}) = U$.

Ex 2.4. v) $F : \mathrm{Set}^{op} \longrightarrow \mathrm{Set}$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f^{op}"', "\longmapsto"] & \mathrm{Pre}(X) = \{(X, \leq_x) : \leq_x : \text{preorder}\} \\
Y \arrow[r, mapsto] & \mathrm{Pre}(Y) = \{(Y, \leq_y) : \leq_y : \text{preorder} \arrow[u]
\end{tikzcd}$$

s.t. $Ff(Y, \leq_y)$ is a preorder on X

s.t. $x_1 \leq x_2$ in X iff $f(x_1) \leq_y f(x_2)$.

$\int F$: obj : $(X, (X, \leq)) \longleftrightarrow (X, \leq)$ preorder

Mor : $(X, \leq_x) \xrightarrow{f} (Y, \leq_y)$

s.t. $f : X \longrightarrow Y$ and $Ff(Y, \leq_y) = (X, \leq_x)$

i.e. $x_1 \leq_x x_2 \Leftrightarrow f(x_1) \leq_y f(x_2)$.

This is subcategory of the category of Preorder.

Notes that $\int F$ has no terminal object.

Suppose it has, say $(Z, \leq)$.

Then, $([3], \leq)$ with usual order has a map

$\exists! f_3 : [3] \longrightarrow Z$ s.t. $f_3(1) \leq f_3(2) \leq f_3(3)$.

Then, $([2], \leq)$ with usual order has 3 maps,

$$1 \xmapsto{f_{21}} f_3(1) \qquad 1 \xmapsto{f_{22}} f_3(1) \qquad 1 \xmapsto{f_{23}} f_3(2)$$
$$2 \longmapsto f_3(3) \qquad 2 \longmapsto f_3(2) \qquad 2 \longmapsto f_3(3)$$

Since Z is terminal, $f_{21} = f_{22} = f_{23}$

$\Longrightarrow f_3(1) = f_3(2)$, $f_3(2) = f_3(3)$

However, this contradicts the fact that

$f_3(1) = f_3(2) \Rightarrow f_3(1) \geq f_3(2) \Rightarrow 1 \geq 2$;

Hence such Z doesn't exist. ▨

Ex 2.4. vi) $\int \mathrm{Hom}$:

Obj: $((c, c'), f : c \to c')$. $(c, c') \in C^{op} \times C$, $f \in C(c, c')$

Mor:
$$\begin{tikzcd}((c, c'), f : c \longrightarrow c') \arrow[d, "{(h^{op}, l)}"'] \\ ((d, d'), g : d \longrightarrow d')\end{tikzcd}$$

s.t. $(h^{op} : c \to d \Leftrightarrow h : d \to c), (l : c' \to d')$,

and $\mathrm{Hom}(h^{op}, l) f = g$, i.e., $lfh = g$.

By Identifying $(\mathcal{C}(c,c'); f: c \to c')$ as $f: c \to c'$

This category has

Obj := Mor $\mathcal{C}$

Mor: $(f: c \to c') \xrightarrow{(h^{op}, l)} g: d \to d'$

s.t. $h: d \to c$, $l: c' \to d'$ s.t. $g = lfh$

Name comes from twisted arrows (h, l) in

diagram.

$$\begin{tikzcd}
c \arrow[r, "f"] & c' \arrow[d, "l"] \\
d \arrow[u, "h"] \arrow[r, "g"'] & d'
\end{tikzcd}$$

▨

Ex. 2.4. viii)

Notes that $CAT/\mathcal{C}$ has.

Obj: $F: B \to \mathcal{C}$ functor.

Mor: $(F: B \to \mathcal{C}), \xrightarrow{H} (G: D \to \mathcal{C})$

then $H: B \to D$ s.t.

$$\begin{tikzcd}
B \arrow[rr, "H"] \arrow[dr, "F"'] & & D \arrow[dl, "G"] \\
& \mathcal{C} &
\end{tikzcd}$$

Now for any $F: \mathcal{C} \to Set$

$\int F$ sends F to $\Pi_{\int F}: \int F \to \mathcal{C}$, forgetful functor

and $\alpha: F \Rightarrow G$, then $\int \alpha: \int F \to \int G$.

s.t.

$$\begin{tikzcd}
\int F \arrow[rr, "\int \alpha"] \arrow[dr, "\Pi_{\int F}"'] & & \int G \arrow[dl, "\Pi_{\int G}"] \\
& \mathcal{C} &
\end{tikzcd}$$

To define $\int\alpha$ more presisely,

$$\begin{tikzcd}
\int\alpha : (c, x) \arrow[r, mapsto] \arrow[d, "f"'] & (c, \alpha_c(x)) \arrow[d, "f"] \\
(c', x') \arrow[r, mapsto] & (c, \alpha_{c'}(x'))
\end{tikzcd}$$

This well defined since we have commute diagram.

$$\begin{tikzcd}
x \arrow[rrr, mapsto, bend left] \arrow[ddd, bend right] & & & \alpha_c(x) \arrow[ddd, bend left] \\
& Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowright"] & Gc \arrow[d, "Gf"] & \\
& Fc' \arrow[r, "\alpha_{c'}"'] & Gc' & \\
Ff(x) = x' \arrow[rrr, mapsto] & & & \alpha_{c'}(x') = Gf(\alpha_c(x))
\end{tikzcd}$$

Thus,

$$\begin{tikzcd}
\int F \arrow[rr, "\int\alpha"] \arrow[dr, "\pi_{\int F}"'] & \circlearrowright & \int G \arrow[dl, "\pi_{\int G}"] \\
& C &
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
(c, x) \arrow[rr, mapsto] \arrow[dr] & \circlearrowright & (c, \alpha_c(x)) \arrow[dl] \\
& c &
\end{tikzcd}$$

So $\int\alpha$ is a morphism in CAT/C.

Thus, if F, G are naturally isomorphic, then $F \cong G$ in Set^C, thus $\pi_{\int F} \cong \pi_{\int G}$ in CAT/C.

$$\Rightarrow \begin{tikzcd}
\int F \arrow[rr, shift left, "H"] \arrow[dr] & \circlearrowright & \int G \arrow[ll, shift left, "L"] \arrow[dl] \\
& C &
\end{tikzcd}$$

(Functor preserves iso)

Inthe slice category, if $x \xrightarrow{f} y$, $y \xrightarrow{g} x$ are pair of isomorphism, then.

$g \circ f : x \longrightarrow x$ is an identity for $x \to c$,

i.e,

$$\begin{tikzcd} x \arrow[rr, "g \circ f"] \arrow[dr] & \circlearrowleft & x \arrow[dl] \\ & c & \end{tikzcd}$$

Since 1_x is defined as Identity over $x \to c$, $g \circ f = 1_x$. Likewise, $f \circ g = 1_y$. $\Rightarrow$ (f, g are isomorphism $\Rightarrow$ $x \cong y$ in C)

Hence, $\pi_{\int F} \cong \pi_{\int G} \Rightarrow \int F \cong \int G$ in CAT.

Ex 2.4. iii) Let $f : c \longrightarrow d \in \mathrm{Mor}\, C$. $(c, x) \in \int F$.

Then, let $y = Ff(x)$. This induce $\bar{f}$ s.t.

$$(c, x) \xrightarrow{\bar{f}} (d, y) \quad \text{s.t.}$$

$\bar{f}$! $f : c \longrightarrow d$ and $Ff(x) = y$.

Moreover, $\pi(\bar{f}) = f$.

To see it is unique, suppose $\exists \alpha : (c, x) \to (d, y')$ s.t $\pi(\alpha) = f$. Then, $y' = F(\pi(\alpha))(x) = Ff(x) = y$ $\forall x$.

Thus $\alpha = \bar{f}$.

Ex. 2.4. ix) Discrete Right Fibration;

For any $f : d \to c \in C$, $(c, x) \in \int F$,

$\exists!\ \alpha \in \mathrm{Mor} \int F$ s.t. $\pi(\alpha) = f$ and $\mathrm{dom}\, \alpha = (c, x)$.

Ex. 2.4. ✱). Suppose H is rep by Z,

i.e. $C(A,-)\times C(B,-) \cong C(Z,-)$

Then, for any $f: A \to C$, $g: B \to C$.

$\exists!\,(f,g): Z \to C$ Correspond to f and g

$\forall C \in \mathrm{Ob}\,C$. Thus, $C(Z,C) \cong C(A,C)\times C(B,-)$

And, $\int C(A,-)\times C(B,-)$ has

obj: $(C, (f: A\to C,\ g: B \to C))$

Mor $\bar{h}\Big\downarrow$

$(C', (f': A \to C',\ g': B \to C'))$

if $h: C \to C'$ and $hf = f'$, $hg = g'$.

Thus, if $(Z, z_1: A \to Z,\ z_2: B \to Z)$ is initial,

Then. $\forall\ (C, (f,g))$. $\exists!$ morphism

$$(Z,(z_1,z_2)) \xrightarrow{\bar{h}} (C,(f,g))$$

s.t. $h: Z \to C$, $hz_1 = f$, $hz_2\ g$.

In a diagram

$$\begin{tikzcd}
A \arrow[dr, "z_1"] \arrow[ddr, bend right=60, "f"'] & & B \arrow[dl, "z_2"'] \arrow[ddl, bend left=60, "g"] \\
\circlearrowleft & Z \arrow[d, "h"] & \circlearrowleft \\
& C &
\end{tikzcd}$$

looks like Colimit of $J \to C$ over $J = \{\cdot, \cdot\}$ i.e. Coproduct of two object must exists.