# 0. Preface

"Natural" $\longleftrightarrow$ defined without arbitrary choice

ex) $V \xrightarrow{\cong} V^*$ : need choice of the basis

$\cdot$ $V \xrightarrow{\cong} V^{**}$ : doesn't needed.

Eilenberg MacLane '45 : Natural Transformation

Functor (as source/Target)

Category ( " )

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Objectives ① Categorifying math. objects $\Rightarrow$ isomorphism

② study itself $\Rightarrow$ self-dual

In Chapter 1, ① Def ② Duality ③ Functor. Today!

④ Naturality ⑤ Equivalence (⑥ Diagram Chase, ⑦ 2-Categories)

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Def 1.1.1 $C$ : category consists of

- $Obj(C)$ : Collection of objects, $X, Y, Z, \cdots$
- $Hom(C)$ " morphisms $f, g, h \cdots$

(or $Mor(C)$)

1) Each morphism has domain and codomain.

2) Each object $X$ has $1_X$ : identity morphism

3) Two morphism $f : X \to Y$, $g : Y \to Z$ s.t.

$Cod(f) = dom(g)$ ; $gf : X \to Z$ exists.

a) (unital) $\forall f : X \to Y \in Hom(C)$, $f 1_X = f = 1_Y f$

b) (associates) for any composable triple, $f(gh) = (fg)h$.

In abstract Category : Morphism = arrow or map
(they're not function)

Rem 1.1.5 Size issue
Russell's Paradox :⇒ No collection contains itself.
To avoid,

Def 1.1.6. Small Category : Hom(C) is a set.
(Then, from $Obj(C) \hookrightarrow Hom(C)$ $\quad X \mapsto 1_X$ $\quad$ Obj(C) is also a set.)

$$\Rightarrow \quad \mathrm{Hom}\,C \; \underset{cod}{\overset{dom}{\rightrightarrows}} \; \mathrm{Obj}\,C \quad (\xleftarrow{id})$$
are functions.

But none of example of concrete category are Small.

Def 1.1.7 locally small category : $\forall X, Y \in obj\, C$.
$C(Y,X) = Hom(X, Y)$ is a set (set of all morphisms from X to Y)
$= Hom(X, Y)$

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Q: when is one thing the same as another thing?
⇒ iso (actually we will define notion of equivalence)

Def: 1.1.9. Iso in Category C is a morphism $f: X \to Y$
for which $\exists g: Y \to X$ st $fg = 1_Y$ $\quad gf = 1_X$
In this case write $X \cong Y$.
Endo: $f: X \to X$. $\quad$ Auto: endo + iso.

Ex 1.1.10.
Set bijection $\quad$ Grp, Ring, Field, Mod$_R$ : bijective ~
Top: Homeo $\quad$ Htpy homotopy equiv. $\quad$ $(P, \leq)$ identity
by antisymmetry

Q: In a concrete category, Is every iso induced by bijection of underlying set? $\quad$ A: Lem 5.6.1 (Yes)

Def 1.1.11. Groupoid : a category in which every morphism is iso.

Cf) In algebra, groupoid : a group changing binary operation to partial function.
i.e. mult is not def on all ~~two~~ objects pairs of

Ex 1.1.12 G : (Group) : groupoid with 1 object.
(It is def of gp in category theory.)

$\pi_1(X)$ : fundamental groupoid.
Obj : points in X  Mor : Endpt preserving homotopy classes of path.

Def) Subcategory D of C.

Obj D ⊆ Obj C, Mor D ⊆ Mor C s.t.

1) Obj D contains any domain or codomain of $f \in Mor(D)$
2) Mor D " any identity morphism of $X \in Obj(D)$
3) Closed under composition.

Ex) CRing ⊆ Ring ⊂ Rng
(Com, unital) (unital) (maybe nonunital).

Lem 1.1.13 Any category C contains maximal groupoid is subcategory.
pf) Show collection of isomorphisms of C

Ex) Fin : Obj : finite set, Mor : functions
$Fin_{iso}$ " Mor : bijection,
$Fin_{iso}$ is maximal groupoid of Fin

(말장난! Objects $X$ can be identified with identity $1_X$.
So Cat def by morphism.

What we care: $\boxed{\text{Morphism}}$

Ex 1.1.3. (Concrete Category)

| Name | Obj | Hom | Composite is |
|---|---|---|---|
| Set | set $X$ | function $f$ | composition of $f$. |
| Top | top. space $X$ | cts function $f$. | 〃 |
| $Set_*$, $Top_*$ | $(X, x)$ <br> $x \in X$ base pt | base pt preserving ~ | 〃 |
| Group | groups | gp homo | 〃 |
| Ring | rings | ring 〃 | 〃 |
| Field | fields | field 〃 | 〃 |
| $Mod_R$ | left $R$-module | module 〃 | 〃 |
| ${}_R Mod$ | right 〃 | | |
| $Vect_{\mathbb{K}} = Mod_R$ | with $R = \mathbb{K}$ | | |
| $Ab = Mod_R$ | with $R = \mathbb{Z}$ | | |
| Graph | graphs | graph morphism (sending vertex to vertex, edge to edge, preserving incidence relation) | 〃 |
| Digraph | di 〃 | | |
| Man | manifolds | smooth map | |
| Meas | measurable space | measurable function | 〃 |
| Poset | Posets | order preserving function | 〃 |
| $Ch_R$ | chain cpx of $R$-module | chain homo. | |

$Model_{\pi}$ — $\sigma$-structure. morphism preserving structure.

Concrete Category (Precise: 1.6.17)
Obj has underlying set
Morph are functions between underlying set.

1.1.4 (Abstract Category)

| Name | Obj | Morph | Composition? |
| --- | --- | --- | --- |
| ① $Mat_R$ (R: unital ring) ↳ needed for identity morph. | $\mathbb{P} = \{1,2,3,\dots\}$ | $A: n \to m$ is $(m \times n)$ R-valued matrix | matrix multiplication. |
| ② $BG$ G: group. | $\{\cdot\}$ | $g \in G$. | group product |
| ③ $(P, \leq)$ P: poset | $P$ | $x \leq y \Leftrightarrow x \to y$. (Identity from reflexivity) | clear by transitivity |

So we can do for preorder (transitivity and reflexivity)

④ Ordinals

| | | | |
| --- | --- | --- | --- |
| $\mathbb{0}$ | $\emptyset$ | $\emptyset$ | |
| $\mathbb{1}$ | $\{\emptyset\}$ | $1_\emptyset$. | |
| $\mathbb{2}$ | $\{\emptyset, \{\emptyset\}\}$ | $1_\emptyset$, $1_{\{\emptyset\}}$, $\emptyset \to \{\emptyset\}$. ($0 \to 1$) | |
| $\omega$ | $\{0,1,2,\dots\}$ | $0 \to 1 \to 2 \to 3 \to \cdots$ | all morphism $n \to m$ equal to $n \to n+1 \to n+2 \to \cdots$ |
| ⑤ $A$ (set). | $A$ | $1_a \ \forall a \in A$. | |

"Discrete Category": Every morphism is an identity.

| | | | |
| --- | --- | --- | --- |
| ⑥ Htpy | top space | homotopy class. | |
| $Htpy_*$ | $(X, x)$ | base preserving ... | |
| ⑦ Measure | measure spaces | equiv class of msble function ($=$ a.e.). | |

Exercise: 1.1 i) morphism can have at most 1 inverse

pf) If $g, h : x \to y$ are inverse of $f : y \to x$

then, $g = g 1_x = gfh = 1_y h = h$.

1.1. ii). Maximal groupoid. $\to$ done.

1.1. iii) Slice Category.

Let $C$ : category $c \in \text{obj}\, C$.

$C/C$ : $\text{obj} := \bigcup_{x \in \text{obj}\, C} \text{Hom}(c, x)$.

under

$\text{Mor} :=$

$\text{Mor}(f : c \to x,\ g : c \to y)$.

$= \{ h : x \to y :$
$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\circlearrowleft"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$
commutes$\}$

i.e. $g = hf$.

$C/c$ : $\text{Obj} := \bigcup_{x \in \text{obj}\, C} \text{Hom}(x, c)$

over. $\text{Mor}(f : x \to c,\ g : y \to c)$

$= \{ h : x \to y :$
$$\begin{tikzcd} & c & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & \arrow[u, phantom, "\circlearrowleft"] & y \arrow[ul, "g"'] \end{tikzcd}$$
commutes, i.e. $f = gh$. $\}$

Duality

Def 1.2.1. From $C$, $C^{op}$ is opposite category

- $\text{Obj}(C^{op}) = \text{obj}(C)$
- $f^{op} \in \text{Hom}(C^{op})$ for each $f \in C$

with $f^{op}: \text{cod}(f) \longrightarrow \text{dom}(f)$

Then, from structure of $C$,

1) $1_X^{op}$ is id in $C^{op}$

2) $f^{op}: X \longrightarrow Y \quad g^{op}: Y \longrightarrow Z$

$$\Rightarrow g^{op} f^{op} : X \longrightarrow Z \quad \longleftrightarrow \quad fg : Z \rightarrow X$$

Thus $C^{op}$ is category iff $C$ is category

Ex 1.2.2. $\text{Mat}_R^{op}$ : $f^{op}$ is transpose of $f$

$(P, \leq)^{op}$ : $x \longrightarrow y \iff y \leq x$

$\omega^{op}$ : $\cdots 3 \longrightarrow 2 \longrightarrow 1 \longrightarrow 0$

$(BG)^{op} \cong B(G^{op})$ where $G^{op}$ : opposite gp.

gp with right multiplication

i.e. $f \cdot g = gf$ in $G$

Thus, any statement of proof about $C$ also applies to opposite : " dual theorem "

Ex).

Lemma 1.2.3 For any $C$, TFAE.

(1) $f: X \to Y$ is iso in $C$.

(2) $f_* : C(c, x) \longrightarrow C(c, y)$ is bijection $\forall c \in \mathrm{Obj}\, C$
$g \longmapsto fg$.

(3) $f^* : C(y, c) \longrightarrow C(x, c)$ 〃
$g \longmapsto gf$.

$f_*$ : post-composition, $f^*$ : pre-composition.

pf)

(i) ⇒ (ii) $f$: iso, ⇒ $\exists g: Y \to x$ inverse of $f$.

Thus, $g_* f_*$ and $f_* g_*$ are identity $\forall c \in \mathrm{Obj}\, C$

(ii) ⇒ (i) Take $c = Y$. $\exists g \in C(Y, X)$ s.t.

$f_*(g) = 1_Y$ ⇒ $fg = 1_Y$

Also,

$f_*(gf) = fgf = 1_Y f = f$.

$f_*(1_X) = f 1_X = f$.

By bijectivity, $1_X = gf$

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(i) ⇔ (iii) Apply (i) ⇔ (ii) on $C^{op}$ to get.

$f^{op}: y \to x$ is iso ⇔ $f^{op}_* : C^{op}(c, y) \to C^{op}(c, x)$
$g^{op} \longmapsto f^{op} g^{op}$.
is bijection $\forall c \in C$.

Now, $C^{op}(c, y) \leftrightarrow C(y, c)$ $C^{op}(c, x) \leftrightarrow C(x, c)$

$g^{op} \leftrightarrow g$, $f^{op} g^{op} \longleftrightarrow gf$

Thus, $f^{op}_*$ sends $g$ to $gf$ ⇒ $f^{op}_* = f^*$.

① 1a

$a + 0b + 2c = 3$

$a + 2b + 0c = 3$

$2c = 2b$

$\therefore \boxed{c = b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$2 \times 3 \quad 3 \times 3 \quad 3 \times 2$

$$= \begin{bmatrix} a & c \\ d & f \end{bmatrix}$$

$$\begin{bmatrix} \quad \end{bmatrix} A \begin{bmatrix} \quad \end{bmatrix}$$

$a \times k \qquad k \times n \qquad n \times b$

$n \times 1$

$\Rightarrow$ $f^{op}: Y \to X$ is iso $\Longleftrightarrow$ $f^*: \mathcal{C}(Y, c) \to \mathcal{C}(X, c)$ bij. $\forall c \in \mathcal{C}$.

$\Updownarrow$

$f: X \to Y$ is iso. done.

Def: 1.2.7. $f: X \to Y$ morphism is

(1) monomorphism if $\forall h, k: W \rightrightarrows X$ for some $W$

mono (noun) $fh = fk \Rightarrow h = k$.

monic (adj).

(left cancellable)

(2) epimorphism if $\forall h, k: Y \rightrightarrows W$ for any $W$,

epi

epic $hf = kf \Rightarrow h = k$

(right cancellable)

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$f$ is mono. $\Longleftrightarrow$ $f_*: \mathcal{C}(c, X) \to \mathcal{C}(c, Y)$ $\forall c \in \mathcal{C}$.

injective

" epi $\Longleftrightarrow$ " "

surjective

Ex 1.2.8. $f: X \to Y$ mono in Set. take $x \in X$

Let $\{x\} \overset{g}{\underset{h}{\rightrightarrows}} X$, suppose $fg = fh$. $\Rightarrow$ $g = h$.

Thus $f$ is injective.

Ex 1.2.9. (split epi / split mono) $s$: section (right inverse)

$X \xrightarrow{s} Y \xrightarrow{r} X$ s.t. $rs = 1_X$. Then

$r$: retraction (retract) (left inverse)

$\begin{pmatrix} s \text{ is mono} \\ r \text{ is epic.} \end{pmatrix}$

In this case,

<u>$s$: split mono</u>

<u>$r$: split epi.</u>

Exercise
1.2.ii) (i) If $f$ is split epimorphism, then $\exists\, s : Y \to X$ s.t. $fs = 1_Y$. Fix $c \in C$,

let $g \in C(c, Y)$. $\Rightarrow$ $sg \in C(c, X)$

$\Rightarrow$ $f_*(sg) = fsg = 1_Y g = g$. surjective.

Conversely, take $c = Y$. $\exists\, s \in C(Y, X)$ s.t. $fs = 1_Y$. $\therefore$ $f$ is split epi.

1.2.ii) ii) In $C^{op}$, apply (i); we get

$f^{op} : Y \to X$ is split epi $\iff$ $\forall c \in C^{op}$, $f^{op}_* : C^{op}(c, Y) \to C^{op}(c, X)$ is surj.

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$.

$f : X \to Y$ is split mono. $\qquad$ $\forall c \in C$, $f^* : C(Y, c) \to C(X, c)$ is surj.

1.2.iii) done.

1.2.iv) What are monomorphism of Field?

ans) Every morphism is monomorphism.

$E \overset{h}{\underset{k}{\rightrightarrows}} F \xrightarrow{g} G$. $\Rightarrow$ $g$ is injective function, thus,
$g(h(x)) = g(k(x)) \Rightarrow h(x) = k(x)$.

But maybe not all epi is surjection. (e.g. $G \to E \rightrightarrows E$ sending distinct roots of $f$ but ~~not~~ fixing $G$.)

Ex 1.2.vi) Let $f : X \rightarrowtail Y$. $\exists\, g : Y \to X$ s.t. $fg = 1_X$. By Lemma 1.2.11 ii) $f$ is epic.

Apply it on $C^{op}$. Then $f^{op} : Y \rightarrowtail X$, $g^{op} : X \to Y$ s.t. $f^{op} g^{op} = 1_X$. By above, $f^{op}$ is iso.

$\Rightarrow$ in $C$, $f : X \twoheadrightarrow Y$, $g : Y \to X$ s.t. $gf = 1_X$.
Then $f$ is iso.

Ex 1.2. i)

By def, $^{c}/C^{op}$ : $Obj = \bigcup_{x \in Obj\, C} Hom_{op}(c, x)$

$$Mor_{op}(f^{op} : c \to x,\ g^{op} : c \to y)$$

$$= \left\{ h^{op} : x \to y \quad \text{s.t.} \quad \begin{tikzcd} & c \arrow[dl, "f^{op}"'] \arrow[dr, "g^{op}"] \arrow[d, phantom, "\circlearrowleft"] & \\ x \arrow[rr, "h^{op}"'] & & y \end{tikzcd} \right\}$$

By duality, its opposite category is

$$(^{c}/C^{op})^{op} : Obj = \bigcup_{x \in Obj\, C} Hom_{op}(c, x)$$

$$Mor_{op_{op}}(f^{op} : c \to x,\ g^{op} : c \to y)$$

$$= \left\{ (h^{op})^{op} : (h^{op}) \in Mor(g^{op} : c \to y, f^{op} : c \to x) \right\}$$

(Forget about commuting diagram: it is just abstract opposite.)

Thus, $(h^{op})^{op} \in Mor_{op,op}(f^{op} : c \to x,\ g^{op} : c \to y)$

$$\Leftrightarrow \quad h^{op} \in Mor_{op}(g^{op} : c \to y,\ f^{op} : c \to x)$$

$$\Leftrightarrow \quad \begin{tikzcd} & c \arrow[dl, "f^{op}"'] \arrow[dr, "g^{op}"] \arrow[d, phantom, "\circlearrowleft"] & \\ x & & y \arrow[ll, "h^{op}"] \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} & c & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & \circlearrowleft & y \arrow[ul, "g"'] \end{tikzcd}$$

Thus by sending $g \longmapsto g^{op}$ for object,

$$h \longmapsto (h^{op})^{op}$$

we can identify $C/c$ by $(^{c}/C^{op})^{op}$.

Ex 1.2.10 Not all epi or monic is surj or inj.

ex) [1.2.v] $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ canonical inclusion.

$f$ is monic since $f$ is injective ( $fh(r) = fk(r) \Rightarrow h(r) = k(r)$ )

$f$ is epic since ( $h(1) = k(1) \Rightarrow h(b) \cdot h(\frac{1}{b}) = k(b) \cdot k(\frac{1}{b})$

and $h(\frac{1}{b}) = h(b)^{-1}$, $k(\frac{1}{b}) = k(b)^{-1}$

since $h(b) = k(b)$ by $f$, $h(b)^{-1} = k(b)^{-1}$

$\Rightarrow h(\frac{1}{b}) = k(\frac{1}{b})$ )

But $f$ is not ring isomorphism! it is not bijective

Lemma 1.2.11. Use $\rightarrowtail$ monic $\twoheadrightarrow$ epic.

(i) $f : X \rightarrowtail Y$ $g : Y \rightarrowtail Z$ $\Rightarrow gf \; X \rightarrowtail Z$.

(ii) $f : X \longrightarrow Y$ $g : Y \longrightarrow Z$ s.t. $gf : X \rightarrowtail Z$

$\Rightarrow f : X \rightarrowtail Y$.

pf) (i) : obvious (ii). suppose $W \overset{h}{\underset{k}{\rightrightarrows}} X$ s.t. $fh = fk$.

Then $gfh = gfk \Rightarrow h = k$.

Dually,

(i) $f^{op} : Y \twoheadrightarrow X$. $g^{op} : Z \twoheadrightarrow Y \Rightarrow f^{op} g^{op} : Z \twoheadrightarrow X$.

(ii) $f^{op} : Y \longrightarrow X$ $g^{op} : Z \longrightarrow Y$ and $f^{op} g^{op} : Z \twoheadrightarrow X$.

$\Rightarrow f^{op} : Y \twoheadrightarrow$

Delete op.

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Exercise 1.2 vii) Define sup, inf on Poset $(P, \leq)$ categorical sense, i.e. dual statement define inf.

Def : $x \in P$ is sup if $\forall y \in P$, $y \to x$ exists.

( " inf " $x \to y$ " )

If $x, y$ are sup, $x \rightleftarrows y \Rightarrow$ by anti symmetry $\underline{x = y}$.

Functoriality.

Def. 1.3.1. $F: C \longrightarrow D$ functor (morphism of category)

① $Fc \in D \quad \forall c \in C$

② $Ff: Fc \longrightarrow Fc' \in D. \quad \forall f: c \to c' \in C.$

Satisfying "functoriality axioms"

a) $f, g$ composable pair, $Fg\,Ff = F(gf)$

b) $\forall c \in C, \quad F(1_c) = 1_{Fc}.$

Ex 1.3.2.

1) $P$: Set $\longrightarrow$ Set $\quad A \longmapsto PA \quad (f: A \to B) \longmapsto f_*: \underset{\subset A}{A'} \longmapsto \underset{\subset B}{f(A')}$

2) Forgetful functor for concrete categories

$U$: Group $\to$ Set $\quad \underset{\text{as gp}}{G} \longmapsto \underset{\text{as a set}}{G}. \quad \underset{\text{as hom}}{f} \longmapsto \underset{\text{as function.}}{f}$

Similar for Ring, Field, Top, ....

$U, E$: Graph $\to$ Set $\quad G \longmapsto \begin{matrix} V(G) \\ E(G) \end{matrix} \quad f: G \to H \longmapsto \begin{pmatrix} V(G) \to V(H) \\ E(G) \to E(H) \end{pmatrix}$

$V \sqcup E$: " $\quad V(G) \underset{\text{disj}}{\sqcup} E(G) \quad V(G) \sqcup E(G) \to V(H) \sqcup E(H)$

3) Another forgetful functors

$$\begin{array}{ccccc} & \mathrm{Mod}_R & \longrightarrow & Ab & \hookleftarrow \mathrm{Group} \\ \mathrm{Field} \hookrightarrow & \mathrm{Ring} & \nearrow & & \end{array}$$

4) Rings $\longrightarrow$ Set$_*$
Group $\longrightarrow$

$R \longmapsto (R, e) \quad e$: identity

functorial because homomorphism preserves identity.

(5). $\mathrm{Top} \longrightarrow \mathrm{Htpy}$ $\quad (\mathrm{Top}_* \longrightarrow \mathrm{Htpy}_*)$

$X \longmapsto X$ $\quad$ base pt preseving ...

$f \longmapsto \tilde{f}$ homotopy class of cts function.

6) $\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Group}$ $\quad X \longmapsto \pi_1(X)$

$f : (X, x) \longrightarrow (Y, y) \longmapsto f_* : \pi_1(X, x) \to \pi_1(Y, y)$

"Fundamental group is homotopy invariants"

$\Longrightarrow$ $\exists$ function $\mathrm{Htpy}_* \longrightarrow \mathrm{Group}$ s.t.

$$\pi_1 = \mathrm{Top}_* \longrightarrow \mathrm{Htpy}_* \longrightarrow \mathrm{Group}.$$

7). $\Pi_1 : \mathrm{Top} \longrightarrow \mathrm{Groupoid}$. $\quad X \mapsto \Pi_1(X)$, $\quad f \longmapsto f_*$

since cts preserves path and path homotopy.

8) $\forall n \in \mathbb{Z}$, $\begin{matrix} Z_n \\ B_n \\ H_n \end{matrix} : Ch_R \rightrightarrows \mathrm{Mod}_R$

n-cycle $Z_n C_\cdot = \ker(d : C_n \longrightarrow C_{n-1})$

n-bdy $B_n C_\cdot = \ker(d : C_{n+1} \to C_n)$

n-homology $H_n C_\cdot = Z_n C_\cdot / B_n C_\cdot$

If you collect ~~for~~ all $n \in \mathbb{Z}$,

$\begin{matrix} Z_* \\ B_* \\ H_* \end{matrix} : Ch_R \rightrightarrows \mathrm{Gr\,Mod}_R$ $\quad$ graded R-module

ex) $Z_* : C_\cdot \longmapsto \bigoplus_{n \in \mathbb{Z}} Z_n C_\cdot$

Singular homology of Top space

$$\mathrm{Top} \longrightarrow Ch_R \xrightarrow{H_*} \mathrm{Gr\,Mod}_R$$

(ix) $F: \mathrm{Set} \longrightarrow \mathrm{Group}$.

$X \longmapsto$ Free gp gen by $X$. $f \longmapsto$ induced gp homo.

example of "free functor".

x) $\mathrm{Euclid}_*$ Obj: $(\mathbb{R}^n, a)$

Mor: $f: (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ differentiable

Consists of $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ "

and $f(a) = b$.

Let $D: \mathrm{Euclid}_* \longrightarrow \mathrm{Mat}_{\mathbb{R}}$

$(\mathbb{R}^n, a) \longmapsto n$.

$f: (\mathbb{R}^n, a) \to (\mathbb{R}^m, b) \longmapsto n \xrightarrow{df_a} m$

$df_a$: Jacobian

$m \times n$ matrix

(Say correction on $\mathrm{Mat}_{\mathbb{R}}$ : )

It satisfy functoriality axiom due to chainrule.

xi). $\mathrm{Fin}_*$: Obj: $(\{x_1, \cdots x_n, x_{n+1}\}, x_i)$

Mor: function preserving basept.

$M^- : \mathrm{Fin}_* \longrightarrow \mathrm{Set}$

Let $n_+ := ([n] \cup \{a\}, \{a\})$ ($M$ : Commutative monoid)

$M^{n_+} = M^{\times n}$ cartesian product ($M^{0_+}$ : Singleton)

$\forall f : m_+ \longrightarrow n_+$, def $M^f : M^m \longrightarrow M^n$

$$(a_1, \cdots, a_m) \longmapsto (b_1 \cdots b_n)$$

where $b_i = \begin{cases} \prod_{j \in f^{-1}(i)} a_j & \text{if } f^{-1}(i) \neq \emptyset \\ 1 & \text{if empty} \end{cases}$

Then, $M^f$ preserves unit. So,

$$\begin{array}{cccc} m_+ \to n_+ & n_+ & Fin_* & \xrightarrow{M^-} Set \\ \downarrow & \downarrow & \downarrow & \nearrow \; M^n \\ M^m \to M^n & (M^n, (1, \cdots, 1)) & Set_* & \end{array}$$

$(a_i)_i^m \mapsto (b_i)_i^n$

$b_i = \left( \prod_{j \in f^{-1}(i)} a_j \right)$.

Segal '74
: Cohomology on some suitable category
(Alg. K-theory from Quillen).

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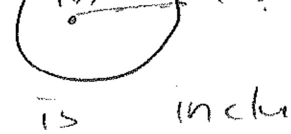
Ex 1.3.3 (Brouwer fixed pt Theorem)

Any cts endo $f : D^2 \longrightarrow D^2$ has a fixed pt.

pf) Let $r : D^2 \longrightarrow S^1$ by




$r$ is cts and $i : S^1 \longrightarrow D^2$ is inclusion

$\Rightarrow ri = 1_{S^1}$ : $r$: split epi (retract) $i$: split mono (section)

Since $\pi_1$ is functor, $Top_* \longrightarrow Group$,

$$\pi_1(S_1, x) \xrightarrow{\pi_1(i)} \pi_1(D^2, x) \xrightarrow{\pi_1(r)} \pi_1(S^1, x)$$

$\pi_1(r)\cdot\pi_1(i) = \pi_1(r.i) = \pi_1(1_{S^1}) = 1_{\pi_1(S^1)}$

But $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(D^2) = 0$.

$\Rightarrow$ $\mathbb{Z} \to 0 \to \mathbb{Z}$ is identity, contradiction

Def 1.3.5 (Contravariant functor) $F: C^{op} \to D$

① $Fc \in D$, $\forall c \in \mathrm{obj}\, C$

② $Ff: Fc' \to Fc$ $\quad \forall f: c \to c' \in \mathrm{Hom}\, C$

s.t. 1) $f: x \to y$ $g: y \to z$, $\Rightarrow$ $Ff \cdot Fg = F(gf)$

$(Fy \to Fx,\ Fz \to Fy)$

2) $F1_c = 1_{Fc}$

Thus

$$\begin{tikzcd} C^{op} \arrow[r, "F"] & D \\ c \arrow[r, mapsto] \arrow[d, "f"'] & Fc \\ c' \arrow[r, mapsto] & Fc \arrow[u, "Ff"'] \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "F"] & D \\ c \arrow[r, mapsto] \arrow[d, "f"'] & Fc \arrow[d, "Ff"] \\ c' \arrow[r, mapsto] & Fc \end{tikzcd}$$

( Contravariant Functor $\qquad$ (Covariant) Functor

---

Ex 1.3.7

(1) $P: Set^{op} \to Set$

$$\begin{tikzcd} A \arrow[r, mapsto] \arrow[d, "f"'] & PA & f^{-1}(B') \subset A \\ B \arrow[r, mapsto] & PB \arrow[u] & B' \subset B \arrow[u, mapsto] \end{tikzcd}$$

(2) $(-)^* : \mathrm{Vect}_k^{op} \longrightarrow \mathrm{Vect}_k$

$$\begin{tikzcd}
V \arrow[r, mapsto] \arrow[d, "\phi"'] & V^* & w \circ \phi \\
{} \arrow[r, mapsto] & {} & {} \\
W \arrow[r, mapsto] & W^* \arrow[uu, "\phi^*"'] & w \arrow[uu, mapsto]
\end{tikzcd}$$

(3) $\mathcal{O} : \mathrm{Top}^{op} \longrightarrow \mathrm{Poset}$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathcal{O}(X) & f^{-1}(U_Y) \\
{} \arrow[r, mapsto] & {} & {} \\
Y \arrow[r, mapsto] & \mathcal{O}(Y) \arrow[uu, "f^{-1}"'] & U_Y \arrow[uu, mapsto]
\end{tikzcd}$$

$\mathcal{O}(X)$: topology as poset by inclusion

$C : \mathrm{Top}^{op} \longrightarrow \mathrm{Poset}$ by collection of closed sets.

(4) $\mathrm{Spec} : \mathrm{CRing}^{op} \longrightarrow \mathrm{Top}$

$$\begin{tikzcd}
R \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Spec}(R) & f^{-1}(P) \\
{} \arrow[r, mapsto] & {} & {} \\
S \arrow[r, mapsto] & \mathrm{Spec}(S) \arrow[uu, "f_*"'] & P \arrow[uu, mapsto]
\end{tikzcd}$$

$\mathrm{Spec}(R)$ with Zariski top.

(5) (set-valued) Presheaf: any functor $C \longrightarrow \mathrm{Set}$ from small category $C$.

ex) $\mathcal{O}(X)^{op} \longrightarrow \mathrm{Set}$

$$\begin{tikzcd}
U \arrow[r, mapsto] \arrow[d, hook, "\mathrm{incl}"'] & C(U) = \{f : U \to \mathbb{R} \text{ cts}\} & f|_U \\
{} & {} & {} \\
V \arrow[r, mapsto] & C(U) = \{f : V \to \mathbb{R} \text{ cts}\} \arrow[uu, "\mathrm{res}(V,U)"'] & f \arrow[uu, mapsto]
\end{tikzcd}$$

(6) Presheaves on $\Delta$ : Simplex Category

$\Delta$ : Obj : $[n] = \{0, 1, \cdots n\}$, $n \in \mathbb{N}$ (finite nonempty ordinal)

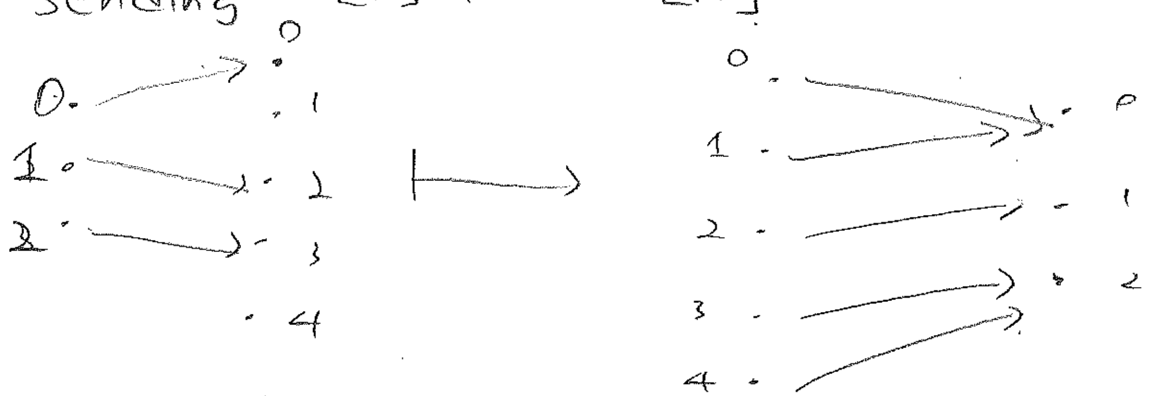
Mor : $f : [n] \longrightarrow [m]$ order preserving function.

$$( i \leq j \longmapsto f(i) \leq f(j) )$$

A simplicial set : any presheaf on $\Delta$, i.e.

$$X : \Delta^{op} \longrightarrow Set$$

ex) $X$ : sending $[n] \longmapsto [n]$



Lem 1.3.8 Functor preserves iso.

pf) $F : C \longrightarrow D \Rightarrow F(g)F(f) = F(gf) = F(1_x) = 1_{Fx}$

$\begin{pmatrix} f : x \longrightarrow y \\ g : y \longrightarrow x \end{pmatrix}$ iso. $F(f)F(g) = F(fg) = F(1_y) = 1_{Fy}$

So does contravariant functor

Ex 1.3.9. $G$ : gp. $BG$ : group as a category.

Let $X : BG \longrightarrow C$ : any category

$\bullet \longmapsto X$ some obj $C$.

$g \longmapsto g_* : X \longrightarrow X$

By functoriality axiom. ① $h_* g_* = (hg)_*$ $\forall g, h \in G$

② $e_* = 1_X$ $e$ : id in $G$.

Thus $X : BG \to C$ defines an left action of the gp $G$ on the object $X \in C$.

ex) If $C = \begin{pmatrix} \text{Set} \\ \text{Vect}_k \\ \text{Top} \end{pmatrix} \Rightarrow X : BG \to C$ is $\begin{pmatrix} G\text{ set} \\ G\text{-rep.} \\ G\text{-Space} \end{pmatrix}$.

And contravariant functor $BG^{op} \to C$ defines right action similar way.

Since Functor preserves iso, and all morphisms in $BG$ are iso, so $g_*$ is automorphism in that category.

Corollary 1.3.10 When $G$ acts functorially on an object $X$ in a category $C$ (i.e. $\exists X : BG \to C$ exists)

$g$ must act by automorphism $g_* : X \to X$.

and $(g_*)^{-1} = (g^{-1})_*$

Remark: Functor may not presere mono or epic but preserves split mono / split epic.

pf) Almost same as 1.3.8.

Def 1.3.11. $C$: locally small. $c \in \mathrm{Obj}\, C$.
Functor represented by $c$ : $C(c, -)$, $C(-, c)$

s.t.

so contravariant functor $B(G) \to C$ defines right action.

Also, since functor preserves isomorphism in G-rep, any morphism mapped into automorphism $g_*: U \to U$

Corollary 1.3.10. G act functorially on an object X in category C, $g \in G$ act by automorphism $g_*: X \to X$ and $(g_*)^{-1} = (g^{-1})_*$.

---

Rmk! Functor preserves split mono / split epi.

Def 1.3.11 C: locally small. $\forall c \in C$ define <u>function rep by c</u>

$$C(c, -), \quad C(-, c) : \underline{C \to Set}$$

(covariant) $C \to Set$

$$\begin{array}{ccc} x & \longmapsto & C(c, x) \\ f \downarrow & \longmapsto & \downarrow f_* \\ y & \longmapsto & C(c, y) \end{array}$$

(contravariant) $C \to Set$

$$\begin{array}{ccc} x & \longmapsto & C(x, c) \\ f \downarrow & \longmapsto & \uparrow f^* \\ y & \longmapsto & C(y, c) \end{array}$$

Thus, left action = covariant action

right " = contravariant "

$C(c, -)$ $\qquad\qquad\qquad\qquad$ $C(-, c)$

$$\begin{tikzcd} C \arrow[r, mapsto] & Set \\ x \arrow[d, "f"'] \arrow[r, mapsto] & C(c,x) \arrow[d, "f_*"] \\ y \arrow[r, mapsto] & C(c,y) \end{tikzcd} \qquad \begin{tikzcd} C^{op} \arrow[r] & Set \\ x \arrow[d, "f"'] \arrow[r, mapsto] & C(x,c) \\ y \arrow[r, mapsto] & C(y,c) \arrow[u, "f^*"'] \end{tikzcd}$$

post composition $\qquad\qquad$ pre composition

"post" composition = Covariant action = left action

pre 〃 = Contravariant 〃 = right action

Bifunctor: ~~$C(-,-)$~~ or functor of two variable

Def 1.3.12 $C, D$: Category $\quad C \times D$ is category

Obj $C \times D$: $(c, d)$, $c \in \operatorname{obj} C$, $d \in \operatorname{obj} D$

Mor: $(f, g) : (c, d) \longrightarrow (c', d')$

for $f : c \longrightarrow c' \in \operatorname{Mor} C$
$\quad g : d \longrightarrow d' \in \operatorname{Mor} D$.

Def 1.3.13, Two sided represented functor

$$C(-, -) : C^{op} \times C \longrightarrow Set$$

( $C$: locally small )

$$\begin{tikzcd} (x, y) \arrow[d, "f^{op}"', shift right] \arrow[d, "h", shift left] \arrow[r] & C(x, y) \arrow[d, "(f^*, h_*)"] & g \arrow[d, mapsto] \\ (w, z) \arrow[r] & C(w, z) & hgf \end{tikzcd}$$

(So it is covariant functor)

In general $C^{op} \not\cong C$.

Ex 1.3.15. $E/F$ : Galois ext. $\Leftrightarrow$ $E/F$ finite ext. $|Aut(E/F)| = (E:F)$

Galois gp : $G = Aut(E/F)$

$\mathcal{O}_G$ : Obj : $G/H$ : a left coset def by subgp $H$ of $G$ (set of all left cosets of $H$)

Mor : $G$ function $G/H \to G/K$ commute with left $G$-action on the cosets.

exercise : every $G/H \to G/K$ has the form $gH \mapsto g\gamma K$ where $\exists \gamma \in G$ s.t. $\gamma^{-1} H \gamma \subset K$.

$Field_F^E$ = obj : intermediate fields

morphism : $K \to L$ field homo fixing $F$.

Thus, $Aut(E/F) = \{ f \in Hom(E,E) : f \text{ is automorphism} \}$

$\Phi : \mathcal{O}_G^{op} \to Field_F^E$

$G/H \longmapsto G \subseteq E$ subfield fixed by $H$ under the action of $G$.

$(G/H \xrightarrow{\gamma} G/K) \longmapsto (x \mapsto \gamma x)$

F.T.C. of Galois Theory : $\Phi$ is bijection

in fact $\Phi$ is iso.

Thus we need a little bit relaxed concept morphism of functor, i.e. natural transformation.

Exercise 1.3. i) Functor between groups?

$$\begin{tikzcd} G \arrow[r, "F"] & H \\ \bullet \arrow[d, "g"'] \arrow[r, mapsto] & \bullet \arrow[d] \\ \bullet \arrow[r, mapsto] & \bullet \end{tikzcd} \qquad g \longmapsto h$$

$F(g) = h$ s.t.

a) $F(gg') = F(g)F(g')$

b) $F(e_G) = e_H$.

$\Rightarrow$ If $F$ is functor, then it is gp homo.

Conversely, gp homo: satisfy a) $\Rightarrow$ b) $\therefore$ by adding $\bullet \longrightarrow \bullet$. it is functor.

1.3. ii) Functor between preorders?

$(P, \leq)$ $\quad$ $(Q, \leq)$

$$\begin{tikzcd} x \arrow[d, "x \leq y"'] \arrow[r, mapsto] & F(x) \arrow[d] \\ y \arrow[r, mapsto] & F(y) \end{tikzcd}$$

$F(x) \leq F(y)$ ⇢ a)

b): done

$F$ is order preserving function.

3. iii) Pr. Hauesmeyer Notation.

[illegible]

$C$: $\overset{\circ}{a} \longrightarrow \overset{\cdot}{b} \qquad \overset{\cdot}{c} \longrightarrow \overset{\cdot}{d}$

$F: C \longrightarrow D$

$a \to b \longmapsto x \longmapsto y$

$c \to d \longmapsto y \longmapsto z$.

$$D: \begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur] \arrow[rr] & & z \end{tikzcd}$$

$$F(C): \begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur] & & z \end{tikzcd}$$

We have composable morphisms in $F(C)$. But composite doesn't exists.

Ex 1.3. iv) $f = 1_x \Rightarrow$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "1_x"'] & C(c,x) \arrow[d, "(1_x)_*"] & g \arrow[d, mapsto] \\ x \arrow[r, mapsto] & C(c,x) & (1_x g) = g \end{tikzcd}$$

identity

$f: x \to y, \quad g: y \to z$

$$\Rightarrow \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & C(c,x) \arrow[d, "f_*"] & h \arrow[d, mapsto] \\ y \arrow[r] \arrow[d, "g"'] & C(c,y) \arrow[d, "g_*"] & fh \arrow[d, mapsto] \\ z \arrow[r] & C(c,z) & gfh \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r] \arrow[d, "g.f"'] & C(c,x) \arrow[d] & h \arrow[d, mapsto] \\ z \arrow[r] & C(c,z) & gfh \end{tikzcd}$$

Ex 1.3.v) Claim: $F: C^{op} \to D \overset{\text{iff}}{\Longleftrightarrow} G: C \to D^{op}$

Let $F: C^{op} \to D$. Construct $G: C \to D^{op}$ s.t.

$G(c) := F(c)$. From $\mathrm{Hom}_D(Fb, Fa) \leftrightarrow \mathrm{Hom}_{D^{op}}(Fa, Fb)$

So given $f \in \mathrm{Hom}_C(a, b)$

define $G(f) = (Ff)^{op}$ in $\mathrm{Hom}_{D^{op}}(Fa, Fb)$

Thus, $F: C \to C^{op} \to D \qquad G: C \xrightarrow{G} D^{op} \to D$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & x \arrow[r, mapsto] & Fx \\ y \arrow[r, mapsto] & y \arrow[u, "f^{op}"'] \arrow[r, mapsto] & Fy \arrow[u, "Ff"'] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"'] & Fx \arrow[r, mapsto] \arrow[d, "(Ff)^{op}"] & Fx \\ y \arrow[r, mapsto] & Fy \arrow[r, mapsto] & Fy \arrow[u, "Ff"'] \end{tikzcd}$$

$F, G$ are the same functor. Also, $F: C \to D \Leftrightarrow G: C^{op} \to D^{op}$

$$\begin{tikzcd} C \arrow[r] & D \\ x \arrow[d] & x \arrow[d, "Ff"] \\ y \arrow[r, mapsto] & y \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r] & C^{op} \arrow[r] & D^{op} \arrow[r] & D \\ x \arrow[d, "f"'] & x & Fx & Fx \arrow[d, "Ff"] \\ y & y \arrow[u, "f^{op}"'] & Fy \arrow[u, "(Ff)^{op}"'] & Fy \end{tikzcd}$$

1.3.vi) $F: D \to C$, $G: E \to C$.

$F \downarrow G$ : Obj := $(d, e, f)$

$d \in D$, $e \in E$, $f: Fd \to Ge \in C$

Mor $((d,e,f), (d',e',f'))$

$= \{ (h, k) \in \mathrm{Mor}\,D \times \mathrm{Mor}\,E$ :

$h: d \to d'$, $k: e \to e'$ s.t.

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\circlearrowright"] & Ge \arrow[d, "Gk"] \\ Fd' \arrow[r, "f'"'] & Ge' \end{tikzcd}$$

in $C$, i.e $f' \cdot Fh = Gk \cdot f$.

It is category since,

① $1_{(d,e,f)} = (1_d, 1_k)$.

② $(d, e, f) \xrightarrow{(h_1, k_1)} (d', e', f') \xrightarrow{(h_2, k_2)} (d'', e'', f'')$

$$\Rightarrow \begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "Fh_1"'] \arrow[dr, phantom, "\circlearrowright"] & Ge \arrow[d, "Fk_1"] \\ Fd' \arrow[r, "f'"] \arrow[d, "Fh_2"'] \arrow[dr, phantom, "\circlearrowright"] & Ge' \arrow[d, "Fk_2"] \\ Fd'' \arrow[r, "f''"'] & Ge'' \end{tikzcd}$$

$\Rightarrow (Fk_2)(Fk_1) \cdot f = Fk_2(f') \cdot Fh_1 = f'' Fh_2 Fh_1$

$$\begin{tikzcd} Fd \arrow[r, "f"] \arrow[d, "F(h_2 h_1)"'] & Ge \arrow[d, "Fk_2 k_1"] \\ Fd'' \arrow[r, "f''"'] & Ge'' \end{tikzcd}$$

$\Rightarrow f'' F(h_2 h_1) = F(k_2 k_1) f$

Functor: $dom : F \downarrow G \longrightarrow D$ $\quad$ $cod : F\downarrow G \longrightarrow E$

$$\begin{tikzcd} (d,e,f) \arrow[d, "(h,k)"'] \arrow[r, mapsto] & d \arrow[d, "h"] \\ (d',e',f') \arrow[r, mapsto] & d' \end{tikzcd} \qquad \begin{tikzcd} (d,e,f) \arrow[d, "(h,k)"'] \arrow[r, mapsto] & e \arrow[d, "k"] \\ (d',e',f') \arrow[r, mapsto] & e' \end{tikzcd}$$

Ex 1.3.vii) $D = (\{\cdot\}, 1_\cdot)$ $\quad F : D \longrightarrow C$ $\quad G = 1_C$
$E = C$ $\qquad \cdot \longmapsto c$

$c/C$ under:

Obj $= (\cdot, x, f : c \longrightarrow x)$

Mor $: ((\cdot, x, f : c \longrightarrow x), (\cdot, y, g : c \longrightarrow y))$

$= \{ (1_\cdot, h : x \longrightarrow y) \in \text{Mor}\, D \times \text{Mor}\, E$

s.t.

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "1_c"'] \arrow[dr, phantom, "\circlearrowright"] & x \arrow[d, "Gh = h"] \\ c \arrow[r, "g"'] & y \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\circlearrowright"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

---

$C/c$ over: $D = C$ $\quad F = 1_C$ $\quad G = \cdot \longmapsto c$
$E = (\{\cdot\}, 1_\cdot)$

Obj : $(x, \cdot, f : x \longrightarrow c)$

Mor $((x, \cdot, f : x \to c), (y, \cdot, g : y \to c))$

$= \{ (h : x \to y, 1_\cdot) \in \text{Mor}\, D \times \text{Mor}\, E$

s.t.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & c \arrow[d, "1_c"] \\ y \arrow[r, "g"'] & c \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} & c & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & \arrow[u, phantom, "\circlearrowright"] & y \arrow[ul, "g"'] \end{tikzcd}$$

projection functor:

$$\mathrm{dom} : {}^{c}/_{C} \longrightarrow D = (\{\cdot\}, 1) \qquad \mathrm{Cod} : {}^{c}/_{C} \longrightarrow E = C$$

$$\begin{array}{ccc} (\cdot, x, f : c \to x) & \longmapsto & \cdot \\ (1, h) \downarrow & & \downarrow 1. \\ (\cdot, y, g : c \to y) & \longmapsto & \cdot \end{array} \qquad \begin{array}{ccc} (\cdot, x, f : c \to x) & \longmapsto & x \\ (1, h) \downarrow & \longmapsto & \downarrow h. \\ (\cdot, y, g : c \to y) & \longmapsto & y \end{array}$$

Thus: $\mathrm{Cod} : {}^{c}/_{C} \longrightarrow C \qquad \mathrm{dom}\ {}^{C}/_{c} \longrightarrow C.$

$$\begin{array}{ccc} f : c \to x. & \longmapsto & x \\ h \downarrow & \longmapsto & \downarrow h. \\ g : c \to y & \longmapsto & y \end{array} \qquad \begin{array}{ccc} f : x \to c & \longmapsto & x \\ \downarrow h & \longmapsto & \downarrow h. \\ g : y \to c & \longmapsto & y \end{array}$$

Similarly., $\mathrm{dom}\ {}^{C}/_{c} \longrightarrow D = C. \qquad \mathrm{Cod} : {}^{C}/_{c} \longrightarrow E = (\{\cdot\}, 1)$

$$\begin{array}{ccc} (x, \cdot, f : x \to c) & \longmapsto & x \\ (h, 1) \downarrow & \longmapsto & \downarrow h. \\ (y, \cdot, g : y \to c) & \longmapsto & y \end{array} \qquad \begin{array}{ccc} (x, \cdot, f : x \to c) & \longmapsto & \cdot \\ (h, 1) \downarrow & \longmapsto & \downarrow 1. \\ (y, \cdot, g : y \to c) & \longmapsto & \cdot \end{array}$$

Ex 1.3. viii) Ex: Functor need not reflect isomorphism

Find $F : C \longrightarrow D$, $f \in \mathrm{Mor}(C)$ s.t. $Ff$ is iso in $D$

$f$ is not iso in $C$.

$$\cdot \xrightarrow{f} \cdot \qquad\qquad \cdot \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \cdot$$

$C \qquad\qquad D.$

ex) Homology

Homotopy

Functor:

(Quasi-iso

but not homeo.)

More generically, $\mathbb{1} = (\{\cdot\}, 1_\cdot)$

$C \longrightarrow \mathbb{1}$ by $c \longmapsto \cdot \qquad \forall f, \forall c.$

$f \longmapsto 1.$

Ex 1.3. ix)

| Map \ Source | $Group_{iso.}$ | $Group_{epi}$ | Group |
|---|---|---|---|
| $Z(-)$ | Yes. (iso) | Yes | No [1] |
| $C(-)$ | Yes | Yes | Yes [2] |
| $Aut(-)$ | Yes | Don't know. | No [3] |

1). If $G \overset{f}{\cong} H \Rightarrow Z(G) \cong Z(H)$

pf)

Claim 1: $G \overset{f}{\twoheadrightarrow} H$ surj, then $f(Z(G)) \leq Z(H)$

pf) $h f(g) = f(h'g) = f(gh') = f(g) h$. $\forall h \in H$,

$g \in Z(G)$, $h'$: preimage of $h$.

Claim 2: $G \overset{\cong f}{\hookrightarrow\!\!\!\twoheadrightarrow} H$ then $f(Z(G)) = Z(H)$

pf)

If $h \in Z(H)$, $\exists g \in G$ st. $f(g) = h$.

$g'g = f^{-1}(h'h) = f^{-1}(hh') = f^{-1}(h) f^{-1}(h') = gg'$.

$\Rightarrow g \in Z(G) \Rightarrow f(Z(G)) \supseteq Z(H)$. done.

Claim 3: $Z(-)$: $\begin{matrix} Group_{iso} \\ Group_{epi} \end{matrix} \longrightarrow Group$ is functorial

pf)

$$\begin{array}{ccc} G & \xrightarrow{Z(-)} & Z(G) \\ \downarrow 1_G & & \downarrow Z(1_G) \\ G & \longrightarrow & Z(G) \end{array}$$

, Let $G \xrightarrow{\phi} H \xrightarrow{\psi} K$ Iso (or epi)

Then from $\phi(Z(G)) \subseteq Z(H)$,
$\psi(Z(H)) \subseteq Z(K)$

we have map

$$Z(G) \xrightarrow{\phi|_{Z(G)}} Z(H) \xrightarrow{\psi|_{Z(H)}} Z(K)$$

Thus, $\psi|_{Z(H)} \circ \phi|_{Z(G)} = (\psi \circ \phi)|_{Z(G)}$
implies first functoriality axiom.

Claim 4: $Z(-)$ Group $\to$ Group is not a functor.

pf) $1_{C_2} = C_2 \longrightarrow S_3 \longrightarrow S_3/A_3 \cong C_2$

$$1 \longmapsto (12) \longmapsto (12)+A_3 \mapsto 1$$

where $C_2$: cyclic group of order 2,
$S_3$: Sym. " 3.
$A_3$: alternating gp of $S_3$

Then, If $Z$ is a functor,
$Z(C_2) \longrightarrow Z(S_3) \longrightarrow Z(C_2) = Z(C_2) \xrightarrow{1_{Z(G)}} Z(C_2)$
$= C_2 \longrightarrow 0 \longrightarrow C_2$, contradiction! $C_2 \xrightarrow{I_{C_2}} C_2$

2). Claim 1 : $G \xrightarrow{f} H \Rightarrow C(f(G)) = f(C(G))$
$\leq C(H)$

pf) $C(f(G)) = \{ f(x)f(y)f(x)^{-1}f(y)^{-1} : x, y \in G \}$

$= \{ f(xyx^{-1}y^{-1}) : x\,y \in G \}$

$= f(C(G)) \leq \{ xyx^{-1}y^{-1} : x, y \in H \}$
$= C(H)$

Claim 2 : $C(-)$ : $\mathrm{Group}_{Iso} \longrightarrow \mathrm{Group}$ are
$\mathrm{Group}_{epi} \longrightarrow$
$\mathrm{Group} \longrightarrow \mathrm{Group}$

functors

pf). $G \xrightarrow{f} H. \Rightarrow C(H) \supseteq f(C(G))$

$\Rightarrow C(G) \xrightarrow{f|_{C(G)}} C(H)$ is gp homo.

If $f = 1_G \Rightarrow C(f) = 1_{C(G)}$

$G \xrightarrow{\phi} H \xrightarrow{\psi} K$

$\Rightarrow C(G) \xrightarrow{\phi|_{C(G)}} C(H) \xrightarrow{\psi|_{C(H)}} C(K) = (\psi \circ \phi)|_{C(G)}$

3)
Claim 1. $\mathrm{Group}_{epi} \xrightarrow{Aut} \mathrm{Group}$ is not a functor
$\mathrm{Group} \longrightarrow \mathrm{Group}$

pf) $G = \mathbb{F}_{11} \rtimes \mathbb{F}_{11}^{\times}$ by multiplication.

$G = \{(a,b) : a \in \mathbb{F}_{11},\ b \in \mathbb{F}_{11}^{\times}\}$

and $(a,b)\cdot(c,d) = (a+bc, bd)$

Also, let $N = \{(a,1) : a \in k\} \cong \mathbb{F}_{11}$

$K = \{(0,b) : b \in k^{\times}\} \cong \mathbb{F}_{11}^{\times}$

$N, K$ are subgp of $G$.

and by construction $N$ is normal;

$$\left( (b,c)(a,1)(b,c)^{-1} = (b+ca, c)(b, c^{-1}) \right.$$

$$\left. = (b+ca-bc, 1) \in N \right)$$

$\Rightarrow NK = G$ (by construction)

By Schur-Zassenhaus lemma,

Any subgp of $G$ with order $|K|$

Conjugate to each other, i.e. if $K_1, K_2$

has order $|K|$, then $\exists g \in G$ s.t.

$gK_1g^{-1} = K_2$

$(a,b)(c,d) = (0,1)$
$= (a+bc, bd)$

Note, that $(0,1)$ is identity in $G$.

and $(a,b)^{-1} = (-\frac{a}{b}, b^{-1})$

Hence let $\alpha \in Aut(G)$

$\Rightarrow$ $\alpha(K)$ is a subgp of $G$ with order $|K|$

$\Rightarrow$ $\exists\, g \in G$ s.t.

$$g\alpha(K)g^{-1} = K$$

$\Rightarrow$ Let $\varphi_g \in Inn(G)$ s.t. $\varphi_g(g') = g g' g^{-1}$

$\Rightarrow$ $\varphi_g \circ \alpha$ fix $K$.

Also, Notes that $\varphi_g \circ \alpha$ fix $N$ since $N$ is unique 11-Sylow sub gp of $G$.

Thus, if $\varphi_g \circ \alpha(1,1) = (a, 1)$

then, $a \neq 0$ since $\varphi_g \circ \alpha^{-1}(0,1) = (0,1)$
(identity)

thus $\exists\, b \in \mathbb{F}_{11}^{\times}$ s.t. $b a = 1$

$\therefore$ $(0,b)(a,1)(0,b^{-1}) = (ab, b)(0,b^{-1}) = (ab,1)$
$= (1,1)$

$\therefore$ $\varphi_b \varphi_g \circ \alpha(1,1) = (1,1)$

And since $(0,b)(0,c)(0,b^{-1}) = (0,c)$,

$\varphi_b$ still fix $K$.

Let $\beta = \varphi_b \circ \varphi_g \circ \alpha$.

Then, $\beta(K) = K$, $\beta((1,1)) = (1,1)$.

and $\beta$ is auto. $(\beta(N) = N)$

If $a \neq 0$ in $\mathbb{F}_{11}$

$\Rightarrow$ $\beta((a,1)) = \beta((0,a)(1,1)(0,a^{-1}))$

$= (\beta(0,a))(1,1)(\beta(0,a))^{-1}$

Since $\beta$ fix $N$, $\beta(a,1) = (b,1)$

And $\beta$ fix $K \Rightarrow \beta(0,a) = (0,c)$. for some $b \in \mathbb{F}_{11}^{\times}$, $c \in \mathbb{F}_{11}^{\times}$

Thus, $(b,1) = (0,c)(1,1)(0,c^{-1})$

$= (c,c)(0,c^{-1}) = (c,1)$

$\Rightarrow b = c$. And

$\beta(0,a) = (0,c) = (0,b)$

Thus, $\gamma: \mathbb{F}_{11} \longrightarrow N \xrightarrow{\beta} N \longrightarrow \mathbb{F}_{11}$

$a \longmapsto (a,1) \longmapsto (b,1) \longrightarrow b$.

$0 \longmapsto (0,1) \longmapsto (0,1) \longmapsto 0$.

induces a map on $\mathbb{F}_{11}$ (we didn't show it is homo)

Similarly, $\mathbb{F}_{11}^{\times} \longrightarrow K \xrightarrow{\beta} K \longrightarrow \mathbb{F}_{11}^{\times}$

$a \longmapsto (0,a) \longmapsto (0,b) \longrightarrow b$.

We claim that $\gamma$ is field homo.

$\sigma(a)\sigma(b) = (\sigma(a), 1)\,(\sigma(b), 1)$ in $G$

$( \ldots , = (0, \sigma(a))\,(\sigma(b), 1)\,(0, \sigma(a)^{-1})$

$= \beta(0, a)\,\beta(b, 1)\,\beta(0, a^{-1})$

Since $\sigma(a) = ! \; a \longmapsto (0, a) \longmapsto \beta(0, a)$

$= (0, c)$

for some $c$

$\longmapsto c$

thus $\beta(0, a) = (0, c) = (0, \sigma(a))$

and $b \longmapsto (b, 1) \longmapsto \beta(b, 1) = (c, 1)$

for some $c$.

$\Rightarrow \sigma(b) = c$

$\Rightarrow \beta(b, 1) = (c, 1) = (\sigma(b), 1)$

$= \beta(ab, 1) = (c, 1)$ for some $c \in \mathbb{F}_{11}$

$\Rightarrow \sigma(a)\sigma(b) = c$

and

$ab \longmapsto (ab, 1) \longmapsto \beta(ab, 1) = (c, 1)$

$\Rightarrow \sigma(ab) = c$

Hence $\sigma(ab) = \sigma(a)\sigma(b)$. Thus $\beta$ induces a field homomorphism on $\mathbb{F}_{11}$.

Thus $\mathrm{Aut}(G) \cong \mathrm{Inn}(G) \cdot \mathrm{Aut}(\mathbb{F}_{11})$.

$$\cong \mathrm{Inn}(G)$$
$$\cong G.$$

Since any automorphism of $G$ is field homomorphism times inner automorphism.

and $\mathrm{Aut}(F_{11}) = ①$ since there are only $F_{11}^{\times}$ gp homomorphisms and only 1 homomorphism fix $1 \mapsto 1$.

---

Now Think about a map.

$\phi : F_{11}^{\times} \longrightarrow G \longrightarrow G/N \cong F_{11}^{\times}$ ! isomorphism.

$\mathrm{Aut}(-)$ induces $\mathrm{Aut}(F_{11}^{\times}) \to \mathrm{Aut}(G) \to \mathrm{Aut}(F_{11}^{\times})$

But $|\mathrm{Aut}(F_{11}^{\times})| = (\mathbb{Z}/4\mathbb{Z}) = 4$ and $(4, 110) = 1$.
$|\mathrm{Aut}(G)| = |G| = 110.$

$\Longrightarrow$ $\mathrm{Aut}(F_{11}^{\times}) \to \mathrm{Aut}(G)$ is zero.

But $\phi = 1_{F_{11}^{\times}}$, thus

$\mathrm{Aut}(\phi) = 1_{\mathrm{Aut}(F_{11}^{\times})} \neq 0$. So Aut doesn't satisfy functoriality axiom.

1.3.X.) Let $G$, $H$ gp, $f: G \to H$ homo,

$X_G$, $X_H$, set of conjugacy classes of $G$, $H$.

$g: G \to X_G$, $h: H \to X_H$ class function

i.e. $g(a) = g(bab^{-1})$ $\forall b \in G$.

Let $\mathrm{Conj}: \mathrm{Group} \to \mathrm{Set}$

$$\begin{tikzcd} G \arrow[r, mapsto] \arrow[d, "\phi"'] & X_G \arrow[d, "\mathrm{Conj}(\phi):"] \\ H \arrow[r, mapsto] & X_H \end{tikzcd} \qquad \begin{tikzcd} \bar{g} \arrow[d, mapsto] \\ \bar{h} \end{tikzcd}$$

① $\mathrm{Conj}(\phi)$ is well-defined;

if $g' \in \bar{g}$, then, $g' = bgb^{-1}$ for some $b \in G$.

Thus, $\phi(g') = \phi(b)\phi(g)\phi(b)^{-1} \Rightarrow \phi(g') \in \overline{\phi(g)}$.

② It is functorial; If $\psi: H \to J$ morphism,

$\mathrm{Conj}(\psi) \cdot \mathrm{Conj}(\phi) = \mathrm{Conj}(\psi \circ \phi)$

and $\mathrm{Conj}\left(\begin{smallmatrix} G \\ 1\downarrow \\ G \end{smallmatrix}\right) = I_{X_G}$.

Thus, if $|X_G| \neq |X_H|$, then $\mathrm{Conj}(\phi)$ is not isomorphism, since functor preserves iso, $\phi: G \to H$ is not iso.

Natural Transformation

Ex). $(-)^* \circ (-)^*$ where $(-)^* : \mathrm{Vect}_k^{op} \to \mathrm{Vect}_k$
$V \mapsto V^*$

!Induces: $V \cong V^{**}$

but this iso comes from $ev_v$ $\forall v \in V$.
No unnatural choise of basis is needed.

Def 1.4.1 (Natural Transformation)

$C, D$: categories $F, G : C \rightrightarrows D$

$\alpha$: Natural Transformation $F \Rightarrow G = \{\alpha_c : c \in C\}$

s.t. $\alpha_c : F_c \to G_c \in \mathrm{Mor}\, D$. $\forall c \in C$

↑ component of $\alpha$.

$$\begin{tikzcd} F_c \arrow[r] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowright"] & G_c \arrow[d, "Gf"] \\ F_{c'} \arrow[r] & G_{c'} \end{tikzcd}$$

s.t. $\forall f : c \to c' \in \mathrm{Mor}\, C$

commutes.

Natural Isomorphism: $\alpha$: natural transformation
s.t. $\alpha_c$ is iso $\forall c \in C$.

Then $\alpha : F \cong G$.

"The arrows are natural" $\Rightarrow$ Collection of arrows define natural transf.

$$\begin{tikzcd} C \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd}$$

Ex 1.4.3

(i) $1_{Vect_k} \xrightarrow{ev} (-)^* \cdot (-)^*$

since $\forall \phi : V \to W$ morphism in $Vect_k$

$$\begin{tikzcd} v \arrow[d, mapsto, "\phi"] & V \arrow[r, "ev"] \arrow[d] & V^{**} \arrow[d, "\phi^{**}"] \\ \phi(v) & W \arrow[r, "ev"'] & W^{**} \end{tikzcd}$$

$v \longmapsto f_v : g \in V^* \longmapsto g(v)$

$\phi^{**}(W^* \to k) = V^* \xrightarrow{\phi^*} W^* \to k$

$\phi^{**}(f_v)$ !

$\phi(v) \longmapsto f_{\phi(v)} : g \in W^* \longmapsto g(\phi(v))$

$\phi^{**}(f_v) := W^* \xrightarrow{\phi^*} V^* \xrightarrow{f_v} k$

$g \longmapsto g \circ \phi \longmapsto g \circ \phi(v)$

$f_{\phi(v)} : g \longmapsto g(\phi(v))$

Same.

(ii) $1_{Vect_k} \not\Rightarrow (-)^*$.

Since $1_{Vect_k}$ : Covariant

$(-)^*$ : Contravariant.

More significantly $V \cong V^*$ cannot be defined without choice of basis (…) which is not preserved by any nonidentity linear endomorphism.

(iii) $1_{Set} \Rightarrow P$.

$$\begin{array}{c} A \longmapsto P(A) \\ P, 1_{Set} : Set \rightrightarrows Set \\ A \longmapsto A \end{array}$$

s.t.

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & P(A) \arrow[d, "f"] \\
B \arrow[r, "\eta_B"'] & P(B)
\end{tikzcd}$$

by $\eta_A : A \to P(A)$
$a \mapsto \{a\}$

(iv) $X, Y : BG \rightrightarrows C$ with $\bullet \longmapsto X$, $\bullet \longmapsto Y$.

What is natural transf $\alpha$?

$\Rightarrow \alpha : X \to Y$ a single morphism in C.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "g_*"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g_*"] \\
X \arrow[r, "\alpha"'] & Y
\end{tikzcd}$$

s.t. left diagram commutes

We say $\alpha$ is "G-equivariant".

(V) $\mathcal{O} : Top^{op} \Rightarrow Set$ by $(\ )^c$, taking complement
$\quad C$

i.e., $\mathcal{O} \overset{(\ )^c}{\Rightarrow} C$

To see this, any cts $f : Y \to X \in Top^{op}$.

$$\begin{tikzcd}
U \arrow[rrr, mapsto] \arrow[ddd, mapsto] & \mathcal{O}(Y) \arrow[r, "(\ )^c"] \arrow[d, "f^{-1}"'] & C(Y) \arrow[d, "f^{-1}"] & U^c \arrow[dd, mapsto] \\
& \mathcal{O}(X) \arrow[r] & C(X) & \\
& & & f^{-1}(U^c) \\
f^{-1}(U) \arrow[rr, mapsto, "(\ )^c"] & & (f^{-1}(U))^c &
\end{tikzcd}$$

Then, $(f^{-1}(U))^c = X - f^{-1}(U) = f^{-1}(U^c)$.

Also, it is natural iso since $(\ )^c$ is bijection.

(VI) $(-)^{op} : Group \to Group$

$$G \longmapsto G^{op} : \begin{cases} obj : \bullet \\ Mor := G \text{ s.t. } \\ \quad g^{op} \cdot h^{op} = (hg)^{op}. \end{cases}$$

Thus $\phi : G \to H$ induces $\phi^{op} : G^{op} \to H^{op}$ by $g \mapsto \phi(g)$.

Let $\eta_G : G \longrightarrow G^{op}$
$g \longmapsto g^{-1}$.

it is not automorphism $G \to G$
but it is homo $G \to G^{op}$.

since $gh \longmapsto (gh)^{-1} = (h^{-1} \cdot g^{-1})^{op} = \eta_G(g) \cdot \eta_G(h)$

(since multiple order is reversed.

Thus
$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[d, "\phi"'] & G^{op} \arrow[d, "\phi^{op}"] \\ H \arrow[r, "\eta_H"'] & H^{op} \end{tikzcd}$$

$g \longmapsto g^{-1}$

$\phi(g) \longmapsto \phi(g)^{-1}$. $\phi(g^{-1})$

and $\phi(g^{-1}) = \phi(g)^{-1}$ in $H$.
$\Rightarrow \phi(g^{-1}) = \phi(g)^{-1}$ in $H^{op}$.

---

(vii). $F : Vect_k \longrightarrow Vect_k$
$V \longmapsto V \otimes V$.

Then,
$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "\phi"'] & V \otimes V \arrow[d, "\phi \otimes \phi"] \\ W \arrow[r, "\eta_W"'] & W \otimes W \end{tikzcd}$$
commute, when $\eta_V = 0$ $\forall V \in Vect_k$

However, there is no basis independent definable lin. map $V \to V \otimes V$.

Ex) $Ab_{fg}$ : Category of fin. gen. ab gp.

For given ab gp $A$, let $TA$ : torsion subgp of $A$.

By classification Thm of ab gp,

$$A \cong TA \oplus (A/TA)$$

Prop 1.4.4. $A \cong TA \oplus (A/TA)$ are not natural in $A \in Ab_{fg}$.

pf) If it were natural,

$$\alpha : A \twoheadrightarrow A/TA \rightarrowtail TA \oplus (A/TA) \cong A$$

acting as a natural endomorphism of the $1_{Ab_{fg}}$, i.e. $1_{Ab_{fg}} \xRightarrow{\alpha} 1_{Ab_{fg}}$.

Claim I: If $\eta$ : natural endomorphism of $1_{Ab_{fg}}$

$\Rightarrow$ $\eta : x \longmapsto nx$ for some $n \in \mathbb{Z}$

pf). Since $\eta_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ is multiple by $n \in \mathbb{Z}$.

and $\forall A \in Ab_{gp}$ with map $\mathbb{Z} \xrightarrow{a} A$, $1 \longmapsto a$.

$$\begin{tikzcd} x \arrow[r, mapsto] & nx \\ \mathbb{Z} \arrow[r, "\eta_{\mathbb{Z}}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z} \arrow[d] \\ A \arrow[r, "\eta_A"] & A \\ ax \arrow[r, mapsto] & anx. \quad \eta_A(ax) = x\eta_A(a) \end{tikzcd}$$

By choosing $x=1$, $\eta_A(a) = na$.

Thus, $\eta_A : A \longrightarrow A$
$\qquad a \longmapsto na.$ $\square$

Now, if $A = \mathbb{Z}$, $\alpha_{\mathbb{Z}} : a \longmapsto na$ by claim. Then $\alpha : A \twoheadrightarrow A/TA = A \to A \xrightarrow{\cong} A$

is iso, thus, $n \neq 0$.

If $A = \mathbb{Z}/2n\mathbb{Z}$, $TA = A$, thus

$$\alpha_{\mathbb{Z}/2n\mathbb{Z}} : A \longrightarrow 0 \longrightarrow A \longrightarrow A$$

$\therefore$ zero map.

But $n \neq 0$ in $\mathbb{Z}/2n\mathbb{Z}$ thus

$$\begin{tikzcd}
1 \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & n \arrow[ddd, mapsto] \\
& \mathbb{Z} \arrow[r, "\alpha_{\mathbb{Z}}"] \arrow[d, "\pi"'] & \mathbb{Z} \arrow[d, "\pi"] & \\
& \mathbb{Z}/2n\mathbb{Z} \arrow[r, "\alpha_{\mathbb{Z}/2n\mathbb{Z}}"'] & \mathbb{Z}/2n\mathbb{Z} & \\
1 \arrow[rrr, mapsto] & & & 0
\end{tikzcd}$$

$n \neq 0$,

doesn't commute, contradiction.

Ex 1.4.6. (Riesz Representation Thm)

$\mathcal{I} : \text{cHaus} \longrightarrow \text{Ban}$

cpt Hausdorff   Banach space

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[r, mapsto] & \mathcal{I}(X) := \{\text{signed Baire measure}\} \arrow[d, "f^{-1}"] & \mu \arrow[d, mapsto] \\ Y \arrow[r, mapsto] & \mathcal{I}(Y) & \mu \circ f^{-1} \end{tikzcd}$$

is a functor.

$C^* : \text{cHaus} \longrightarrow \text{Ban}$

$X \longmapsto C(X)^*$ : dual of the Banach space $C(X)$ of cts realvalued function on $X$

For each $\mu \in \mathcal{I}(X)$, $\phi_\mu : C(X) \longrightarrow \mathbb{R} \in C(X)^*$.
$$g \longmapsto \int_X g \, d\mu$$

Let $\eta : \mathcal{I} \Rightarrow C^*$ as

$$\begin{tikzcd} \mu \arrow[rrr, mapsto] \arrow[ddd] & & & \phi_\mu \arrow[ddd] \\ & \Sigma(X) \arrow[r, "\eta_X"] \arrow[d, "(-\circ f^{-1})"] & C(X)^* \arrow[d, "(-\circ f)"] & \\ & \Sigma(Y) \arrow[r, "\eta_Y"'] & C(Y)^* & \\ \mu\circ f^{-1} & & & \phi_\mu \circ f \end{tikzcd}$$

$$\longmapsto \phi_{\mu\circ f^{-1}}$$

s.t. $\phi_{\mu\circ f^{-1}}(g) = \int_Y g \, d(\mu\circ f^{-1})$, $\phi_\mu \circ f = \int_X g\circ f \, d\mu$.

But from real analysis, they are equal.

thus $\eta : \mu \longmapsto \phi_\mu$ is natural transf of $\Sigma \Rightarrow C^*$.

RRT: $\boxed{\text{This is natural iso.}}$

---

Ex 1.4.7. $\mathcal{C}$: locally small

$f : w \to z$, $h : y \to z$.

$$\Rightarrow \begin{tikzcd} C(x,y) \arrow[r, "h\cdot -"] \arrow[d, "-\cdot f"'] \arrow[dr, phantom, "\circlearrowright"] & C(x,z) \arrow[d, "-\cdot f"] \\ C(w,y) \arrow[r, "h\cdot -"'] & C(w,z) \end{tikzcd}$$

Where $h\cdot - = h_*$
$-\cdot f = f^*$
(post and pre composition)

Since $C(-,y)$, $C(-,z)$ are functors,

$$h_x : C(-,y) \Rightarrow C(-,z)$$

is natural transformation. (n.t)

Similarly,

$$f^* : C(x,-) \Rightarrow C(w,-) \quad \text{n.t.}$$

Ex 1.4.9. For $A, B \in$ Sets. $A+B :=$ disjunion of $A, B$.

$A^B := \{B \to A\}$

$$\Rightarrow \quad A \times (B+C) \cong (A \times B) + (A \times C), \quad (A \times B)^C \cong A^C \times B^C$$

$$A^{B+C} \cong A^B \times A^C \qquad (A^B)^C \cong A^{B \times C}$$

Actually, this is natural iso between

Set $\times$ Set $\times$ Set $\to$ Set functors.

or (Contravariant)

Now, restrict this natural iso in Fin iso,

(Obj: finite set Mor: bijection)

Then let $|-| : \mathrm{Fin}_{iso} \longrightarrow \mathbb{N}$ be functor.

$$\begin{array}{ccc} A & \longmapsto & |A| \\ f \downarrow & & \downarrow 1_{|A|} \\ B & \longmapsto & |B| = |A| \end{array}$$

($\mathbb{N}$: discret category of natural #)

Then this induces

$$a(b+c) = ab + ac \,, \qquad (ab)^c = a^c b^c$$

$$a^{b+c} = a^b a^c \qquad (a^b)^c = a^{bc}$$

a laws of mult and add in $\mathbb{N}$.

decategorification of $Fin_{iso} \Rightarrow \mathbb{N}$ and these law.

Categorification of $\mathbb{N} = Fin_{iso}$

---

Exercise 1.4.i

Since $\alpha$ is natural iso, $\forall c \in C$,

$\exists\ \alpha_c^{-1}$ s.t. $\alpha_c \circ \alpha_c^{-1} = 1_{Gc}$

$\alpha_c^{-1} \circ \alpha_c = 1_{Fc}$

Thus, let $\alpha^{-1}$: collection of $\alpha_c^{-1}$. Then

$\forall f : c \to c'$ WTS

$$\begin{tikzcd} Gc \arrow[r, "\alpha_c^{-1}"] \arrow[d, "Gf"'] & Fc \arrow[d, "Ff"] \\ Gc' \arrow[r, "\alpha_{c'}^{-1}"'] & Fc' \end{tikzcd}$$

$Ff \circ \alpha_c^{-1} = \alpha_{c'}^{-1} \circ Gf$

Since $\alpha_{c'} \circ Ff = Gf \circ \alpha_c$,

$\Rightarrow \alpha_{c'} \circ Ff \circ \alpha_c^{-1} = Gf \circ 1_{Gc}$

$\Rightarrow 1_{Fc'} \circ Ff \circ \alpha_c^{-1} = \alpha_{c'}^{-1} \circ Gf$.

---

1.4.ii. If $\phi : G \to H$ be a functor,

then, $\phi(e_G) = e_H$ . $\phi(g) \cdot \phi(g') = \phi(gg')$

Thus, $\phi$ is any homomorphism

If $\phi, \psi : G \rightrightarrows H$, then, if $\phi \overset{\alpha}{\Rightarrow} \psi$ exists,

① $\alpha \in H$

$\forall g \in G$,

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] \arrow[d, "\phi(g)"'] & \cdot \arrow[d, "\psi(g)"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

② $\psi(g)\alpha = \alpha\phi(g)$

$\Rightarrow \alpha^{-1}\psi(g)\alpha = \phi(g)$.

Hence, $\alpha \in \mathrm{Inn}(M)$ s.t. $\alpha \circ \psi = \phi$.

Ex 1.4. iii) $(P, \leq) \underset{G}{\overset{F}{\rightrightarrows}} (Q, \leq)$,

then $F, G$ are order preserving function.

If $F \overset{\alpha}{\Rightarrow} G$, then $\forall f : p \to p'$ $(p \leq p')$

$$
\begin{tikzcd}
Fp \arrow[r, "\alpha_p"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Gp \arrow[d] \\
Fp' \arrow[r, "\alpha_{p'}"'] & Gp'
\end{tikzcd}
$$

i.e. $Gp' \geq Fp' \geq Fp$.

iff $G_{p'} \geq G_p \geq F_p$.

$\Rightarrow G_p \geq F_p \quad \forall p$.

Thus, natural transformation of $F, G$ order preserving function is $F \leq G$.

i.e. $\forall p \in P, \quad F_p \leq G_p$.

(Notes that $F \leq G \Leftrightarrow$ Natural transf holds.)

1.4. iv)

Ex 1.4 iv) Each $f_*, g_*$, $f^*, g^*$ defines n.t. by Ex 1.4.7.

If $f_* = g_*$ $\Rightarrow$ ✓ for $h: y \to x$.

$$\begin{tikzcd} C(x,c) \arrow[r, "f_*"] \arrow[d, "-\cdot h"'] \arrow[dr, phantom, "\circlearrowright"] & C(x,d) \arrow[d, "-\cdot h"] \\ C(y,c) \arrow[r, "f_*"'] & C(y,d) \end{tikzcd}$$

$\Rightarrow \forall l \in C(y,c)$. $\quad l \longmapsto flh$

$flh = glh$. Now pick $h = 1_x$

$\Rightarrow$ ∴ $fl = gl$, Pick $x = c$, $l = 1_c$.

$\Rightarrow f = g$, contradiction ∴ $f_* \neq g_*$ as natural transformation. ($f^*, g^*$: similar manner)

Ex 1.4 v.

$$\begin{tikzcd} F \downarrow G \arrow[r, "dom"] & D \arrow[r, "F"] & C \\ (d,e,f) \arrow[r, mapsto] \arrow[d, "{(h,k)}"'] & d \arrow[r] \arrow[d, "h"] & Fd \arrow[d, "Fh"] \\ (d',e',f') \arrow[r, mapsto] & d' \arrow[r] & Fd' \end{tikzcd}$$

$$\begin{tikzcd}
F\downarrow G \arrow[r, "\mathrm{cod}"] & E \arrow[r, "G"] & C \\
(d,e,f) \arrow[r, mapsto] \arrow[d, "{(h,k)}"'] & e \arrow[r, mapsto] \arrow[d, "k"] & Ge \arrow[d, "Gk"] \\
(d',e',f') \arrow[r, mapsto] & e' \arrow[r, mapsto] & Ge'
\end{tikzcd}$$

Thus, For each $(d,e,f) \xrightarrow{(h,k)} (d',e',f')$

we need to find $\alpha_{(d,e,f)}$, $\alpha_{(d',e',f')}$

s.t.

$$\begin{tikzcd}
Fd \arrow[r, "\alpha_{(d,e,f)}"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\circlearrowright"] & Ge \arrow[d, "Gk"] \\
Fd' \arrow[r, "\alpha_{(d',e',f')}"'] & Ge'
\end{tikzcd}$$

Commute,

Set $\alpha_{(d,e,f)} = f$. By Construction of $F\downarrow G$, it holds.

---

Ex 1.4 vi) If $F, G$ has different target category

i.e.) $F: A \times B \times B^{op} \to D$

$G: A \times C \times C^{op} \to D'$ with $D \neq D'$.

then, we ~~don't know how to~~ Cannot have a morphism $F(a,b,b) \to G(a,c,c)$ in general

1.5. Equivalence of Categories.

$2 : \circ \longrightarrow \circ$
$\quad\; 0 \qquad\quad 1$

$c \mapsto (c,0)$, $\quad (c,1) \mapsfrom c$

① :
$$\begin{tikzcd} C \arrow[r, "i_0"] \arrow[dr, "F"'] & C \times 2 \arrow[d, "H"] & C \arrow[l, "i_1"'] \arrow[dl, "G"] \\ & D & \end{tikzcd}$$

Lem 1.5.1. $F, G : C \rightrightarrows D$ Then,

$\{ \alpha : F \Rightarrow G \mid$ natural transformation $\}$

$\updownarrow$ bijection

$\{ H :$ Satisfying diagram $\}$

pf) Construction of $H$ is in Ex 1.5.1 )

To see bijective corresp, notes that given $H$, with $(c, 0) \xrightarrow{f \cdot (0 \to 1)} (c', 1)$

$$H(c,0) = Fc \quad , \quad H(c',1) = Gc' \text{ by}$$

diagram. Now define

$$\alpha_c = H\left( (c,0) \xrightarrow{1_c \cdot (0 \to 1)} (c,1) \right).$$

Then, $Fc \xrightarrow{\alpha_c} Gc$. If we show this diag. commutes, done.

② :
$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \end{tikzcd}$$

Actually it is taking $H$ on

$$\begin{tikzcd}
(C,0) \arrow[r, "1_C\cdot(0\to 1)"] \arrow[d, "f\cdot 1_0"'] \arrow[dr, phantom, "\circlearrowright"] & (C,1) \arrow[d, "f\cdot 1_1"] \\
(C',0) \arrow[r, "1_{C'}\cdot(0\to 1)"'] & (C',1)
\end{tikzcd}$$

$\Rightarrow$ ② commutes!

Thus, $H$ induces a natural transformation $\alpha$.

Hence the bijection occur ▨

---

If $C=2$, $2\times 2$ is depicted as..

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] \arrow[dr] & (1,0) \arrow[d] \\
(0,1) \arrow[r] & (1,1)
\end{tikzcd}$$

then $H$ sends it to

$$\begin{tikzcd}
F_0 \arrow[r, "F(\sim)"] \arrow[d, "\alpha_0"'] \arrow[dr] & F_1 \arrow[d, "\alpha_1"] \\
G_0 \arrow[r, "G(\sim)"'] & G_1
\end{tikzcd}$$

If we change $2$ to $\mathbb{I}$: $\bullet \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \bullet$ in the lemma, then

Functor satisfying comm. diagram

$\updownarrow$ bijection

Natural Isomorphism.

Def 1.5.4 Equivalence of categories, consists of.

① $F : C \rightleftarrows D : G$

② $\eta : 1_C \cong GF$, $\varepsilon FG \cong 1_D$

natural isomorphism.

~~Wh~~ in, this case write $C \simeq D$.

( cf. $C \cong D$ : isomorphism of category )

Lem 1.5.5 $C \simeq D$ is equivalence relation

pf) Ex 1.5.vi.

Ex 1.5.6.

(i) $(-)_+ : Set^\partial \longrightarrow Set_*$

in 1.3.

$U : Set_* \longrightarrow Set^\partial$

are actually equiv of category by

$1_{Set^\partial} = U(-)_+$ and

$\eta : 1_{Set_*} \cong (U-)_+$ with

$\eta_{(X,x)} : (X, x) \longrightarrow (X \setminus \{x\} \cup \{X \setminus \{x\}\}, X \setminus \{x\})$

$y \longmapsto y$ if $y \neq x$.

$x \longmapsto X \setminus \{x\}$.

(2) $$\begin{tikzcd} \mathrm{Mat}_{\mathbb{k}} \arrow[r, shift left, "\mathbb{k}^{(-)}"] & \mathrm{Vect}_{\mathbb{k}}^{\text{basis}} \arrow[l, shift left, "M"] \arrow[r, shift left, "U"] & \mathrm{Vect}_{\mathbb{k}}^{\text{f.d.}} \arrow[l, shift left, "C"] \end{tikzcd}$$

Where $\mathbb{k}^{(-)}(n \xrightarrow{M} m) \Rightarrow \mathbb{k}^n \xrightarrow{L_M} \mathbb{k}^m$.

$U$: forgetful functor

$C$: Sending V. space by choosing a basis.

$M$: Sending V.S to dim

and linear transformation to matrix over given bases.

Aim: WTS

$$\mathrm{Mat}_{\mathbb{k}} \simeq \mathrm{Vect}_{\mathbb{k}}^{\text{basis}} \simeq \mathrm{Vect}_{\mathbb{k}}^{\text{f.d}}$$

Where $\mathrm{Vect}_{\mathbb{k}}^{\text{basis}}$: Category of f.d. V.S with a chosen basis.

Def: 1.5.7. $F: \mathcal{C} \to \mathcal{D}$ a functor is

① full if $\forall x, y \in \mathcal{C}$, $\mathcal{C}(x,y) \longrightarrow \mathcal{D}(Fx, Fy)$ is surjective

② faithful " " is injective.

③ Essentially surjective if $\forall d \in \mathcal{D}$, $\exists c \in \mathcal{C}$ s.t. $Fc \cong d$.

Rem 1.5.8

(1) Full, faithfull : local conditions

(2) Faith-full and injective on object = embedding

(3) Full + faithful = fully faithful

fully faithful + injective on object = full embedding

In case of full embedding, image of domain = full subcategory of the codomain

Thm 1.5.9. $F : C \simeq D : G \iff F, G$ are full + faithful + ess. surj.

(under axiom of choice)

Lem 1.5.10. For $f : a \to b$ with $a \cong a'$, $b \cong b'$

$\exists f' : a' \longrightarrow b'$ s.t. any arrow in the left square makes it commutes

$$\begin{tikzcd} a \arrow[d, "f"'] & a' \arrow[d, "f'"] \\ b & b' \end{tikzcd}$$

pf) Ex 1.5.iii)

pf of Thm) Let $f, g : c \rightrightarrows c'$ with $Ff = Fg$.

$$\Rightarrow \begin{tikzcd} c \arrow[r, "\eta_c", "\cong"'] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & GFc \arrow[d, shift right, "GFf"'] \arrow[d, shift left, "GFg"] \arrow[dr, phantom, "\circlearrowright"] & c \arrow[l, "\eta_c"', "\cong"] \arrow[d, "g"] \\ c' \arrow[r, "\cong", "\eta_{c'}"'] & GFc' & c' \arrow[l, "\cong"] \end{tikzcd}$$

with $GFf = GFg$ (since $Ff = Fg$)

By 1.5.10,
$$\begin{tikzcd} c \arrow[r, "\eta_c", "\cong"'] \arrow[d, "f"'] & GFc \arrow[r, "\eta_c^{-1}", "\cong"'] & c \arrow[d, "g"] \\ c' \arrow[r, "\cong", "\eta_{c'}"'] & GFc' \arrow[r, "\cong", "\eta_{c'}^{-1}"'] & c' \end{tikzcd}$$
commutes

$\Rightarrow$ $g = f$. $\Rightarrow$ F is faithful.

(So is G by applying same argument on FG
$\Rightarrow$ "by symmetry")

Let $g \in D(Fc, Fc')$

$$f := \eta_{c'}^{-1}(Gg)\,\eta_c \in C(c, c')$$

$$\Rightarrow \begin{tikzcd} GFc \arrow[r, "\eta_c^{-1}"] \arrow[d, "GFf"'] \arrow[dr, phantom, "\circlearrowleft"] & c \arrow[r, "\eta_c"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & GFc \arrow[d, "Gg"] \\ GFc' \arrow[r, "\eta_{c'}^{-1}"'] & c' \arrow[r, "\eta_{c'}"'] & GFc' \end{tikzcd}$$
commutes from equivalence and def of f and g.

$\Rightarrow$ $Gg = GFf$. From G is faithful,

$g = Ff$ $\Rightarrow$ F is full.

Now, ~~for~~ $d \in D$, $\varepsilon_d : FGd \xrightarrow{\cong} d$

$\Rightarrow$ F is essentially surjective.

By symmetry, G is also full and ess. sur.

Conversely let $F : C \longrightarrow D$ ; full faithful and ess. sur. Want to construct $G : D \to C$ equiv.

By axiom of choice and essential surj.

$\forall d \in D$, $\exists c \in C$ s.t. $d \cong Fc$

Let $Gd := c$ $\Rightarrow$ $\varepsilon_d : d \cong FGd$

Then, given $f \in D(d, d')$

We have

$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"] \arrow[d, dashed, "g"] & d \arrow[d, "f"] \\ FGd' \arrow[r, "\varepsilon_{d'}"'] & d' \end{tikzcd}$$

$\Rightarrow$ Let $g = \varepsilon_{d'}^{-1} f \varepsilon_d$

$\Rightarrow$ diagram commute

Then, from $F$ is fully faithful, $\exists h \in C(Gd, Gd')$ s.t. $Fh = g$. Let $Gf := h$

Hence, by this definition

$$\begin{tikzcd} FGd \arrow[r, "\varepsilon_d"] \arrow[d, "FGf"'] \arrow[dr, phantom, "\circlearrowleft"] & d \arrow[d, "f"] \\ FGd' \arrow[r, "\varepsilon_{d'}"'] & d' \end{tikzcd}$$

$\Rightarrow$ $\varepsilon_d : FG \Rightarrow 1_D$

Claim 1: $G$ is functorial

For first condition,

$$\begin{tikzcd}
FGd \arrow[r, "\varepsilon_d"] \arrow[d, "FG1_d"'] \arrow[dr, phantom, "①"] & d \arrow[r, "\varepsilon_d^{-1}"] \arrow[d, "1_d"] \arrow[dr, phantom, "②"] & FGd \arrow[d, "F(1_{Gd})"] \\
FGd \arrow[r, "\varepsilon_d"'] & d \arrow[r, "\varepsilon_d^{-1}"'] & FGd
\end{tikzcd}$$

① commutes by definition of $G1_d$

② 〃 functoriality of $F$

$$\Longrightarrow \quad FG1_d = F(1_{Gd}) \Rightarrow G(1_d) = 1_{Gd}$$

by faithfullness of $F$

For second condition,

let $f: d \to d'$, $f': d' \to d''$

$$\Rightarrow \begin{tikzcd}
FGd \arrow[r, "\varepsilon_d"] \arrow[dd, "FG(f'f)"'] & d \arrow[r, "\varepsilon_d^{-1}"] \arrow[d, "f"] \arrow[dd, bend right=60, "f'f"'] & FGd \arrow[d, "F(Gf)"] \\
{} \arrow[r, phantom, "?"] & d' \arrow[r, "\varepsilon_{d'}^{-1}"] \arrow[d, "f'"] & FGd' \arrow[d, "F(Gf')"] \\
FGd'' \arrow[r, "\varepsilon_{d''}"'] & d'' \arrow[r, "(\varepsilon_{d''})^{-1}"'] & FGd''
\end{tikzcd}$$

① commutes by definition of $G(f'f)$, $Gf$, and $Gf'$ (We define these to commute such diagram)

$\Rightarrow FG(f'f) = F(G(f')) \cdot F(G(f))$

$= F(G(f')G(f))$

$\Rightarrow G(f'f) = G(f')\,G(f)$

by faithfullnes of $F$.

$\Rightarrow G$ is functor.

Claim 2: $\eta : 1_C \Rightarrow GF$ is natural iso.

Define $\eta_c$ is preimage of $\varepsilon^{-1}_{Fc}$ by $F$.

(It is well-def since $F$ is fully faithful.)

i.e. $F\eta_c = \varepsilon^{-1}_{Fc}$, For any $f : c \to c'$,

$$\begin{tikzcd}
Fc \arrow[r, "F\eta_c"] \arrow[d, "Ff"'] \arrow[dr, phantom, "①"] & FGFc \arrow[r, "\varepsilon_{Fc}"] \arrow[d, "FGFf"] \arrow[dr, phantom, "②"] & Fc \arrow[d, "Ff"] \\
Fc' \arrow[r, "\varepsilon_{\eta_{c'}}"'] & FGFc' \arrow[r, "\varepsilon_{Fc'}"'] & Fc'
\end{tikzcd}$$

② commutes by naturality of $\varepsilon$.

Since $\varepsilon$ is natural iso, ① also commutes.

Converse..?

$\Rightarrow FGFf \circ F\eta_c = \varepsilon_{\eta_{c'}} \circ Ff = F\eta_{c'} \circ Ff$

$\Rightarrow F(GFf \circ \eta_c) = F(\eta_{c'} \circ f)$

$\Rightarrow GFf \circ \eta_c = \eta_{c'} \circ f$ by faithfullness of $F$

and by Ex 1.5 iv, $\eta_c, \eta_{c'}$ are iso $\Rightarrow \eta$ is natural iso. ▨

Corollary 1.5.11. $\mathrm{Mat}_k \simeq \mathrm{Vect}_k^{fd}$ for any field $k$.

pf.) $\mathrm{Mat}_k \rightleftarrows \mathrm{Vect}_k^{basis} \rightleftarrows \mathrm{Vect}_k^{fd}$

Those are ess sur, full, and faithful.

( full: set of matrix $\xleftrightarrow{\text{bijection}}$ all lin transf of given basis

faithful $\xleftrightarrow{\text{bijection}}$ lin transform. )

(ess surj: objects are 1-1 corresp. )

Def) Category is connected if $\forall x, y \in C$.

$x, y$ connected by a finite zig-zag

morphisms.

Prop 1.5.12. Any connected groupoid is equiv to automorphism gp of any of its object as a category.

pf) $G$: groupoid. Fix $g \in \mathrm{Obj}\, G$.

Let $H = G(g, g)$.

$\Rightarrow$ $BG \longleftrightarrow G$ is fully faithful since $BG(\bullet, \bullet) \cong G(g, g)$ by def.

$$\begin{tikzcd} \bullet \arrow[r, mapsto] \arrow[d, "\phi"'] & g \arrow[d, "\phi"] \\ \bullet \arrow[r, mapsto] & g \end{tikzcd}$$

and ess. surj.

( Groupoid is connected, (all mor = iso)

$\Rightarrow$ every pair of objects is isomorphic.)

Cor 1.5.13. $X$ : path-connected
$x$ : base pt of $X$

$\Rightarrow \pi_1(X, x)$

pf) Let $\Pi_1(X)$ : fundamental groupoid

By prop 1.5.12, for any $x, x' \in X$

$$\pi_1(X, x) \overset{\cong}{\hookrightarrow} \Pi_1(X) \overset{\sim}{\hookleftarrow} \pi_1(X, x')$$

This gives $\pi_1(X, x) \simeq \pi_1(X, x')$

Claim : $C \simeq D$ with $C, D$ : 1 object category
i.e. $C, D$ are group.
$\Rightarrow C \cong D$ as a group.

pf) Let $F : C \to D$ be equivalence

$\Rightarrow C(\cdot, \cdot) \cong D(\cdot, \cdot)$ as bijection.

By 1.3.1) Exercise, $F$ is gp homo.

on $C(\cdot, \cdot)$ and $D(\cdot, \cdot)$

From bijection, $F$ is bijective homo.

$\Rightarrow$ Gp Isomorphism

$\Rightarrow C \cong D$ as a group.

(Moreover $C \cong D$ as category since gp iso $F^{-1}$ induces a functor s.t. $1_C = FF^{-1}$, $1_D = F^{-1}F$.)

Rem 1.5.14. $\mathrm{Top}_*^{pc}$ : path conn spaces

$$\pi_1 : \mathrm{Top}_*^{pc} \xrightarrow{\pi_1} \mathrm{Group} \hookrightarrow \mathrm{Cat}$$

$$\Pi_1 : \mathrm{Top}_*^{pc} \xrightarrow{U} \mathrm{Top} \xrightarrow{\Pi_1} \mathrm{Groupoid} \hookrightarrow \mathrm{Cat}.$$

Inclusion of $\pi_1(X,x) \subseteq \Pi_1(Y)$ gives

natural transformation $\pi_1 \Rightarrow \Pi_1$

$$\left(\begin{tikzcd} \pi_1(X,x) \arrow[r, hook] \arrow[d, "\pi_1(f)"'] \arrow[dr, phantom, "\circlearrowright"] & \Pi_1(X) \arrow[d, "\Pi_1(f)"] \\ \pi_1(Y,y) \arrow[r, hook] & \Pi_1(Y) \end{tikzcd}\right)$$

And this inclusion is a functor.

Moreover, " is equivalence of category

since fully faithful (as 1 obj subcategory)

and ess. surj. (from connected groupoid)

However, $\Pi_1(Y) \to \pi_1(X,x)$ inverse equivalence

requires axiom of Choice for its construction.

(In this case, $\forall p \in X$, choose a path $p$ to $x$)

And these chosen paths ($p \to x$) need not be

preserved by morphism in $\mathrm{Top}_*^{pc}$.

Def 1.5.15 $C$: Category is <u>skeletal</u>

If it contains 1-object in each isomorphism class.

$\mathrm{sk}\,C$: a skeletal category equiv to $C$ (Unique up to iso)

Rem 1.5.16

$\mathrm{sk}\,C$ construction: Choose 1 object in each iso classes and $\mathrm{sk}\,C$: full subcategory of $C$ having these objects.

By thm 1.5.9, $\mathrm{sk}\,C \hookrightarrow C$ is full (by def) and faithful (by construction) and ess surj. by choice of representative of iso class

$\Rightarrow \mathrm{sk}\,C \simeq C$.

But $\mathrm{sk}(-): CAT \longrightarrow CAT$ is not a functor.

since $\mathrm{sk}(F)$ may not be a functor

(ex) $C$: $\underset{0}{\cdot} \longrightarrow \underset{1}{\cdot} \cong \underset{2}{\cdot}$ $\qquad F$: $0 \to 0$, $1 \to 2$, $2 \to 1$

$\mathrm{sk}\,C$: $\overset{\cdot}{0}$ $\overset{\cdot}{1}$.

$\Rightarrow \mathrm{sk}\,F := F|_{\mathrm{sk}\,C}$ sends 1 to 2 but 2 is not in $\mathrm{sk}\,C$. )

Ex 1.5.17)

(i) $G$: Connected Groupoid

$\Rightarrow$ $skG = G(g,g)$ (by construction)

(ii) $sk(P, \geq)$ = poset! (since we delete any isomorphic but not equal one)

↑ preorder

(iii) $sk(Vect^{fd}_{\mathbb{K}}) \cong Mat_{\mathbb{K}}$

(every v.s with same dim is isomorphic)

(iv) $sk\, Fin_{iso}$ ($Fin_{iso}$ : obj: finite set, Mor: bijection)

= obj: positive integer

Mor: $Hom(n,n) = S_n$ permutation of $n$

$Hom(n,m) = \phi$ if $m \neq n$.

Ex 1.5.18

$X: BG \rightarrow Set$ : a left $G$-set.

Translation Groupoid : $T_G X$ : obj = $X(\cdot) \in Set$

(Mor: $g: x \rightarrow y$ if $\exists g \in G$ s.t. $gx = y$.

$\Rightarrow$ $sk T_G X$ : Obj: connected components in $T_G X$

i.e. orbit of $G$-action.

Mor: for two distinct orbit, $\phi$.

Let $x \in X$, $O_x$ : orbit of $x$

$$\operatorname{Hom}_{sk\mathcal{T}_G X}(O_x, O_x) \cong \operatorname{Hom}_{\mathcal{T}_G X}(x, x) =: G_x$$

$\uparrow$ from $sk\mathcal{T}_G X \simeq \mathcal{T}_G X$ equivalence.

$\Rightarrow$ fully faithful

i.e, $\operatorname{Hom}_{sk\mathcal{T}_G X}(O_x, O_x)$ is Stabilizer $G_x$ of $x$.

$\Rightarrow$ Any pair in the same orbit should have isomorphic stabilizer.

Also, for any fixed $x \in X$, it has disj union

$$\coprod_{y \in O_x} \operatorname{Hom}_{\mathcal{T}_G X}(x, y) = G$$

Since $|\operatorname{Hom}_{\mathcal{T}_G X}(x, y)| = |G_x| \quad \forall y$,

$$|O_x| \cdot |G_x| = |G|.$$

$\therefore$ orbit - stabilizer Thm.

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Def) $C$ : essentially small $\Rightarrow$ $C \simeq D$, $D$ is small category

$C$ : " discrete $\Rightarrow$ $C \simeq D$, $D$ is discrete category

$C$ : locally small, $D \simeq C$ $\Rightarrow$ $D$ is locally small

groupoid " $\Rightarrow$ " groupoid.

" $\Rightarrow$ $D^{op} \simeq C^{op}$

$C \simeq D$, $C' \simeq D'$ $\Rightarrow$ $C \times C' \simeq D \times D'$

$$\left( \begin{array}{c} f : x \to y \in C \text{ iso.} \\ F : C \simeq D \end{array} \right) \Leftrightarrow Ff \text{ is iso.}$$

---

$F : C \to D$ fully faithful.

Then "Essential image" of $F$

= full subcategory of objects isomorphic to some $Fc$ for $c \in C$.

(1.5, i). Let $\alpha : F \Rightarrow G$ be a natural t.f.

$\Rightarrow \ \forall\, f : c \to c'$,

$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowright"] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \end{tikzcd}$$

Now define $H$ as

$H : C \times 2 \longrightarrow D$.

$$\begin{array}{ccc} (c,0) & \longmapsto & Fc \\ f\cdot 1_0 \downarrow & \longmapsto & \downarrow Ff \\ (c',0) & \longmapsto & Fc' \end{array} \qquad \begin{array}{ccc} (c,1) & \longmapsto & Gc \\ f\cdot 1_1 \downarrow & \longmapsto & \downarrow Gf \\ (c',1) & \longmapsto & Gc' \end{array} \qquad \text{and}$$

$$\begin{array}{ccc} (c,0) & \longmapsto & Fc \\ f\cdot(0\to 1) \downarrow & \longmapsto & \downarrow \ Gf\circ\alpha_c = \alpha_{c'}\circ Ff \\ (c',1) & \longmapsto & Gc' \end{array}$$

1) It satisfies functoriality

All we needs are 4 cases. ① $\xrightarrow{f\cdot 1_0}\ \xrightarrow{g(0\to 1)}$,

② $\xrightarrow{f(0\to 1)}\ \xrightarrow{g 1_1}$,

for $f : c \to c'$, $g : c' \to c''$.

① $$\begin{array}{ccccc} (c,0) & \xmapsto{f\cdot 1_0} & (c',0) & \xmapsto{g(0\to 1)} & (c'',1) \\ \downarrow & & \downarrow & & \downarrow \\ Fc & \xmapsto{Ff} & Fc' & \xmapsto{Gg\circ\alpha_{c'}} & GC'' \end{array} \quad \text{and} \quad \begin{array}{ccc} (c,0) & \xmapsto{gf(0\to 1)} & (c'',1) \\ \downarrow & & \downarrow \\ Fc & \xmapsto{G(gf)\circ\alpha_c} & GC'' \end{array}$$

Then,

$$\begin{tikzcd}
Fc \arrow[r, "Ff"] \arrow[d, "\alpha_c"] \arrow[dr, phantom, "a"] & Fc' \arrow[r] \arrow[d, "\alpha_{c'}"] \arrow[dr, phantom, "b"] & Fc'' \arrow[d, "\alpha_{c''}"] \\
Gc \arrow[r, "Gf"'] & Gc' \arrow[r, "Gg"'] & Gc''
\end{tikzcd}$$

a is left diagram, b: right diagram

$\Rightarrow$ Commutes by natural transformation.

Other case is similar.

And $H((c,i) \xmapsto{1_{(c,i)}} (c,i)) = \begin{cases} Fc \xrightarrow{F1_c} Fc & i=0 \\ Gc \xrightarrow{G1_c} Gc & i=1 \end{cases}$

$$= 1_{Fc} \text{ or } 1_{Gc}$$

$$= 1_{H(c,i)}.$$

So $H$ is a functor.

2) $H$ satisfies

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, "F"'] & C\times 2 \arrow[d, "H"] & C \arrow[l] \arrow[dl, "G"] \\
& D &
\end{tikzcd}$$

$$\begin{matrix} c & \mapsto & Fc \\ f \downarrow & & \downarrow Ff \\ c' & \mapsto & Fc' \end{matrix} \quad = \quad \begin{matrix} c & \mapsto & (c,0) & \mapsto & Fc \\ f \downarrow & & \downarrow f\cdot 1_0 & \mapsto & \downarrow Ff \\ c' & \mapsto & (c',0) & \mapsto & Fc' \end{matrix}$$

and the other way is similar.

1.5. ii.)

$\Gamma$ = Obj : finite set

Mor: $S \longrightarrow T$

$\Rightarrow \theta : S \longrightarrow P(T)$ s.t.

① $\theta(\alpha) \cap \theta(\beta) = \phi$ when $\alpha \neq \beta$.

$$S \xrightarrow{\theta} T \xrightarrow{\phi} U = S \xrightarrow{\psi} U$$

s.t. $$\alpha \overset{\psi}{\longmapsto} \bigcup_{\beta \in \theta(\alpha)} \phi(\beta)$$

$F : \Gamma \longrightarrow (\mathrm{Fin}_*)^{op}$ $\qquad$ $G : \mathrm{Fin}_*^{op} \longrightarrow \Gamma$

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "\theta"'] & (S \cup \{s\}, s) \\
T \arrow[r] & (T \cup \{\tau\}, \tau) \arrow[u, "\theta^{-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(S, s) \arrow[r, mapsto] \arrow[d, "f"'] & S \setminus s \\
(T, t) \arrow[r, mapsto] & T \setminus t \arrow[u, "f^{-1}"']
\end{tikzcd}$$

where

$$\theta^{-1}(\beta) = \begin{cases} \alpha & \text{if } \beta \in \theta(\alpha) \\ s & \text{o.w.} \end{cases} \qquad f^{-1} : \beta \longmapsto f^{-1}(\beta)$$

First of all, $\theta^{-1}(\beta)$ is well-defined since no element in $T$ is contained in two preimage by ①.

Also, $f^{-1}$ is well-defined since it is a map from $T \setminus t$ to $P(S \setminus s)$ with distinct preimage.

To see $T \cong \mathrm{Fin}_*^{op}$.

Notes that $FG: \mathrm{Fin}_*^{op} \longrightarrow T \longrightarrow \mathrm{Fin}_*^{op}$

$$\begin{tikzcd}
(S, s) \arrow[d, "f"'] \arrow[r, mapsto] & S \setminus s \arrow[r, mapsto] & (S\setminus s \cup \{S\setminus s\}, S\setminus s) \arrow[d, "h"] \\
(T, t) \arrow[r, mapsto] \arrow[u, phantom, "\mapsto"] & T\setminus t \arrow[u, "f^{-1}"'] \arrow[r, mapsto] & (T\setminus t \cup \{T\setminus t\}, T\setminus t)
\end{tikzcd}$$

To figure out $h$, notes that

If $x \in S\setminus s$, let $y = f(x)$.

$\Rightarrow \quad f^{-1}(y) = \{ x \in S\setminus s : f(x) = y \}$

Thus, for any $x \in f^{-1}(y)$

$$h(x) = y.$$

Hence, $f|_{S\setminus s} = h|_{S\setminus s}$ and $h(S\setminus s) = T\setminus t$

Thus, define $\alpha_{(S,s)}$ as usual incl with.
$S\setminus s \mapsto s$.
$T\setminus t \mapsto t$.

$$\begin{tikzcd}
(S\setminus s \cup \{S\setminus s\}, S\setminus s) \arrow[r, "\alpha_{(S,s)}"] \arrow[d, "h"'] & (S, s) \arrow[d, "f"] \\
(T\setminus t \cup \{T\setminus t\}, T\setminus t) \arrow[r] & (T, t)
\end{tikzcd}$$

Then, if $x \in S \setminus s$ with $f(x) = y$.

$$\alpha_{(T,t)} \circ h(x) = \alpha_{(T,t)}(y) = y.$$

$$f \circ \alpha_{(S,s)}(x) = f(x) = y.$$

and for $x = S \setminus s$,

$$\alpha_{(T,t)} \circ h(S \setminus s) = \alpha_{(T,t)}(T \setminus t) = t.$$

$$f \circ \alpha_{(S,s)}(S \setminus s) = f(s) = t.$$

$\therefore$ diagram commutes.

Likewise

$$\begin{tikzcd}
GF : \Gamma \arrow[r] & Fin_*^{op} \arrow[r] & \Gamma \\
S \arrow[r, mapsto] \arrow[d, "\theta"'] & (S \cup \{s\}, s\} \arrow[r, mapsto] & S, \arrow[d, "h"] \\
T \arrow[r, mapsto] & (T \cup \{T\}, T) \arrow[u, "\theta^{-1}"] \arrow[r, mapsto] & T
\end{tikzcd}$$

s.t. for $x \in S$, $\theta(x) \subseteq T$. Hence, $\forall y \in \theta(x)$

$\theta^{-1}(y) = x. \Rightarrow h(x) =$ preimage of $x$ under $\theta^{-1}$

$= \theta(x)$.

$\Rightarrow GF = 1_\Gamma$.

Hence $\Gamma \cong Fin_*^{op}$.

M: monoid

In particular, $Fin_* \xrightarrow{M^-} Set$ is a functor in 1.3.2 (xi) and $G$ induce a func $\hat{G} : \Gamma^{op} \to Fin_*$.

$\Rightarrow \Gamma^{op} \xrightarrow{\hat{G}} Fin_* \xrightarrow{M^-} Set$ are presheaves on $\Gamma$.

1.5. iii) Any morphism $f : a \to b$ and (fixed morphism $a \cong a'$, $b \cong b'$)

$\Rightarrow$ determine $f' : a' \to b'$ so that

$$
\begin{tikzcd}
a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, "f'"] \\
b \arrow[r, "\cong"'] & b'
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\
b \arrow[r] & b'
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[d, "f"'] & a' \arrow[l] \arrow[d, "f'"] \\
b & b' \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\
b & b' \arrow[l]
\end{tikzcd}
$$

pf). Define $f' = BfA$

$\Rightarrow$ $Bf = f'A^{-1}$, $B^{-1}f' = fA$, $B^{-1}f'A^{-1} = f$.

1.5. iv) $F : C \to D$ full and faithful

(i) $f : c \to c'$ mor in $C$ s.t. $Ff$ is iso in $D$

$\Rightarrow$ $f$ is iso.

(ii) $x, y \in C$ s.t. $Fx \cong Fy$ in $D$

$\Rightarrow$ $x \cong y$ in $C$

pf) Since $F$ is full and faithful,

$C(x,y)$ and $D(Fx, Fy)$ are bijective.

So, if $Fg \in D(Fy, Fx)$ s.t. $Fg \cdot Ff = 1_{Fx}$

$Ff\, Fg = 1_{Fy}$.

then $F(gf) = 1_{Fx}$ since $C(x,x)$ and $D(Fx,Fx)$

$F(fg) = 1_{Fy}$. $C(y,y)$ " $D(Fy,Fy)$ are bijective, $gf = 1_x$, $fg = 1_y$.

Similarly, if $Fx \cong Fy$, $\exists\ Ff \in D(Fx, Fy)$ which is isomorphism. By (i), $f$ is iso in $C$, thus $x \cong y$.

Lem 1.3.8 (Functor preserve iso) is converse of this statement.

Ex 1.5.v.

$$2 \longrightarrow D$$

$$\overset{\circ}{x} \xrightarrow{f} \overset{\circ}{y} \qquad \overset{\circ}{C_x} \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \overset{\circ}{C_y}$$

It is faithful since $2(x,y) \hookrightarrow D(C_x, C_y)$

but not reflect since $x \to y$ in $2$ is not iso, but in $D$ it is.

Ex 1.5. vi)

(i) Composite of full, faithful, ess surj again same. $F: C \to D$, $G: D \to E$.

pf)
$$C(x,y) \overset{\twoheadrightarrow}{\hookrightarrow} D(Cx, Cy) \overset{\twoheadrightarrow}{\hookrightarrow} E(DCx, DCy)$$

$$\Rightarrow C(x,y) \overset{\twoheadrightarrow}{\hookrightarrow} E(DC_x, DC_y)$$

And for essential surj, $\forall e \in E$, $\exists d \in D$ s.t. $e \cong Gd$

and for $d \in D$ $\exists c \in C$ s.t. $d \cong Fc$.

$\Rightarrow e \cong Gd \cong GFc$ implies $\forall e \in E\ \exists c$ s.t. $GFc \cong e$.

(ii). It is just from 1.5.9. Thm.
if we assume axiom of choice

Also, $C \cong C$, $C \cong D \Rightarrow D \cong C$ trivially.

Ex 1.5.vii) Connected groupoid $\Rightarrow$ all objects are isomorphic
If $G = G(g,g)$.

$$
\begin{array}{ccc}
G & \longrightarrow & BG \\
h & \longmapsto & \bullet \\
f \big\downarrow \cong & & \big\downarrow \; g \xrightarrow{\alpha_h} h \xrightarrow{f} h' \xrightarrow{(\alpha_{h'})^{-1}} g \\
h' & \longmapsto & \bullet
\end{array}
\qquad
\begin{array}{ccc}
BG & \longrightarrow & G \\
\bullet & \longmapsto & g \\
\phi \big\downarrow & \longmapsto & \big\downarrow \phi \\
\bullet & \longmapsto & g
\end{array}
$$

$\phi \in G(g,g)$

Then,

$$
\begin{array}{ccccc}
BG & \longrightarrow & G & \longrightarrow & BG \\
\bullet & \longmapsto & g & \longmapsto & \bullet \\
\phi \big\downarrow & \longmapsto & \big\downarrow \phi & & \big\downarrow \phi \\
\bullet & \longmapsto & g & \longmapsto & \bullet
\end{array}
= 1_{BG}
$$

(or take $\alpha : 1_{BG} \Rightarrow BG \to G \to BG$ as identity)

$$
\begin{array}{ccccc}
G & \longrightarrow & BG & \longrightarrow & G \\
h & \longmapsto & \bullet & \longmapsto & g \\
\psi \big\downarrow & \longmapsto & g \to h \xrightarrow{\psi} h' \to g & & \big\downarrow \; g \to h \xrightarrow{\psi} h' \to g \\
h' & \longmapsto & \bullet & \longmapsto & g
\end{array}
$$

Thus, we need to find iso. satisfying,

$$\begin{tikzcd} g \arrow[r, "\alpha_h"] \arrow[d, "g \to h \xrightarrow{\psi} h' \to g"'] \arrow[dr, phantom, "\circlearrowright"] & h \arrow[d, "\psi"] \\ g \arrow[r] & h' \end{tikzcd}$$

And take

$\alpha_h = g \to h$

$\alpha_{h'} = (h' \to g)^{-1}$

Since such $\alpha_h$, $\alpha_{h'}$ is fixed by construction of $G \to BG$, it is well-def and satisfy the commuting diagram

Ex 1.5 viii) Later

Ex 1.5 ix) Any category equiv to locally small category is locally small.

pf) Let $C \underset{G}{\overset{F}{\rightleftarrows}} D$. equiv of category

By Thm 1.5.9, $F, G$ : full, faithful.

$\Rightarrow$ If assume $D$ locally small,

then $C(x,y)$ is a set since it is bijection to a set $D(x,y)$. $\Rightarrow$ $C$ is locally small

1.5. viii).

Affine : Obj = affine planes, $\mathbb{A}^2 k$
where $k$ is a field.

Mor = affine linear map.

$L - c$ : $L$ : linear isomorphism

$c$ : constant function.

$\text{Proj}'$ : Obj = projective planes $(\mathbb{P}^2 k, \ell)$
w/ $\ell$ : line of infinity.

Mor = projective linear isomorphism

$F : \text{Proj}' \longrightarrow \text{Affine}$

$(\mathbb{P}^2 k, \ell) \longmapsto \mathbb{A}^2 k$ by deleting $\ell$ in $\mathbb{P}^2 k$.

Notes that $\mathbb{P}^2_k = k^2 \sqcup (k^1 \sqcup pt)$

where $k^2 = \{[x:y:z] : z \neq 0\} \longmapsto \{(\frac{x}{z}, \frac{y}{z}) \in \mathbb{A}^2 k\}$

$k = \{[x:y:0] : y \neq 0\}$

$pt = \{[x:0:0]\}$

with $\ell = k \sqcup pt$.

Thus, $G : \text{Affine} \longrightarrow \text{Proj}'$.
is embedding $\mathbb{A}^2 k$ into $(\mathbb{P}^2 k, \ell)$

by identifying $\mathbb{A}^2 k$ as $k^2$ part.

$(x, y) \longmapsto [x : y : 1]$

Claim 1: F, G maps pt to pt, line to line. (except $l$).

Let L be a line in $\mathbb{P}^2_k$ generated by

$\vec{a} = [a_0 : a_1 : a_2]$, $\vec{b} = [b_0 : b_1 : b_2]$

Then, $L = \{ u\vec{a} + v\vec{b} : u, v \in k, (u,v) \neq (0,0) \}$

Case 1: If $\vec{a}, \vec{b} \in k^2$,

F maps $u\vec{a} + v\vec{b}$ into $\left( \dfrac{u\frac{a_0}{a_2} + v\frac{b_0}{b_2}}{u+v}, \dfrac{u\frac{a_1}{a_2} + v\frac{b_1}{b_2}}{u+v} \right)$

Thus, $F(L) = \{ t\left(\frac{a_0}{a_2}, \frac{a_1}{a_2}\right) + (1-t)\left(\frac{b_0}{b_2}, \frac{b_1}{b_2}\right) \}$

Since $\frac{u}{u+v}$, $\frac{v}{u+v}$ can be mapped into $t$ and

$1-t$. for any $t \in k$,

$= \{ \left(\frac{b_0}{b_2}, \frac{b_1}{b_2}\right) + t\left(\frac{a_0}{a_2} - \frac{b_0}{b_2}, \frac{a_1}{a_2} - \frac{b_1}{b_2}\right) \}$

Hence $F(L)$ is a line in $\mathbb{A}^2 k$.

Case 2: If one of $\vec{a}, \vec{b}$ is in $l$.

Then, by adding suitable $u\vec{a}$ to $\vec{b}$,

We can change this as a case 1.

Case 3: Both $\vec{a}, \vec{b}$ are in $l$

Then, $L = l$ and F drops $l$.

Now, let $L$ be a line in $\mathbb{A}^2 k$.

$\Rightarrow$ $L = \{tA + (1-t)B : t \in k\}$

for some $A = (a_0, a_1)$, $B = (b_0, b_1)$

$\Rightarrow$ $G(L) = \{t[a_0 : a_1 : 1] + (1-t)[b_0 : b_1 : 1]$
$t \in k\}$.

To see that $G(L)$ is contained in a line in $\mathbb{P}^2 k$, let $L'$ be a line gen by $[a_0 : a_1 : 1]$, $[b_0 : b_1 : 1]$. Then $L' \supseteq G(L)$

So $G(L)$ matches with only one line $L'$

Thus, $FG(L) \subseteq F(L') = L$.

and $GF(L) = G\left(t\left(\frac{a_0}{a_2}, \frac{a_1}{a_2}\right) + (1-t)\left(\frac{b_0}{b_2}, \frac{b_1}{b_2}\right)\right)$

$\subseteq L'$ where $L'$ is gen by

$\left[\frac{a_0}{a_2} : \frac{a_1}{a_2} : 1\right]$, $\left[\frac{b_0}{b_2} : \frac{b_1}{b_2} : 1\right]$

$= [a_0 : a_1 : a_2]$, $[b_0 : b_1 : b_2]$

$\Rightarrow$ $= L$.

Hence $FG(\mathbb{A}^2 k) = \mathbb{A}^2 k$ since they
$GF(\mathbb{P}^2 k) = \mathbb{P}^2 k$ preserves pt and lines.

Thus, equivalence is induced by $\alpha$ as identity for each object.

1.5. xi) See Hame, P. "Examples of some properties of functors"

| | Full | Faithful | Essentially Surj. |
|---|---|---|---|
| Ab → Gp | ✓ | ✓ | × |
| Ring → Ab | × | ✓ | × |
| Ring $\xrightarrow{(-)^\times}$ Gp | × | × | × |
| Ring → Rng | × | ✓ | × |
| Field → Ring | ✓ | ✓ | × |
| $Mod_R$ → Ab | dep on R | ✓ | ✓ |

(b) Ring → Ab: ① Not full since...
No ring homo $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}$

Since $f(\underbrace{1+\cdots+1}_{n}) = 0$ but $\underbrace{f(1)+\cdots+f(1)}_{n} = n$

Contradiction.

(However, Rng on Ab ∃ trivial homo.)

② Not essentially Surj:

Let $\mathbb{Z}(p^\infty) := \{ e^{2\pi i m/p^n} : m, n \in \mathbb{N} \}$

Prüfer p-group.

If $U(R) \cong \mathbb{Z}(p^\infty)$ then $1 \mapsto a \in \mathbb{Z}(p^\infty)$

$\Rightarrow |1| = n$ in $R \Rightarrow \forall r \in R,\ nr = 0$

Since $r(1+\cdots+1) = r \cdot 0 = 0$.

Thus every element in $U(R)$ has order at most $n$. But $\mathbb{Z}(p^\infty)$ has element with greater order. ☒

(c) $(-)^{\times}$: Ring $\longrightarrow$ Grp.

① Not full

Since $\mathbb{Z}$ is initial in Ring, $\forall R \in$ Ring

$\exists! \ f: \mathbb{Z} \longrightarrow R \Rightarrow |\text{Ring}(\mathbb{Z}, R)| = 1$

But $\mathbb{Z}^{\times} \cong \mathbb{Z}/_2$. Take $R = \mathbb{F}_p$

finite p-field.

$\Rightarrow R^{\times} \cong \mathbb{Z}/(p-1)$

And $|\text{Grp}(\mathbb{Z}^{\times}, R^{\times})| = 2$ since $0$

and $\mathbb{Z}/_2 \longrightarrow \mathbb{Z}/(p-1)$ by $1 \longmapsto \frac{p-1}{2}$

are the homomorphisms.

② Not faithful.

Claim 1: $\phi: k[t] \longrightarrow k[t]$ automorphism

fixing $k$ $\Rightarrow \phi(t) = at + b$.

pf) $\phi(t)$ cannot be degree more than one

otherwise it is not automorphism.

Also, $\phi(t)$ cannot have degree 0 by same reas.

And all $\phi(t) = at + b$ is auto since

$\phi(\frac{1}{a}t - b) = t$.

$\boxed{//}$

Thus by Claim 1,

Ring $(k[t], k[t])$ is determined by

$\{ at+b : a \in k^\times, b \in k \}$.

And $k[t]^\times = k^\times$.

But $\mathrm{Grp}(k^\times, k^\times)$ is determined by

generators of $k^\times$ (since $k^\times$ is cyclic)

and any map in $\mathrm{Ring}(k[t], k[t])$

induces map $k^\times, k^\times$ fixing all $k^\times$.

Thus it is not faithful.

③ Not ess. surj.

[Pearson, Schneider, 1970]

Not every cyclic gp is isomorphic to

the gp of units of some ring.

$(\mathbb{Z}/5 \not\cong R^\times \quad \forall R \in \mathrm{Ring})$

(d) Ring $\longrightarrow$ Rng.

① Not full. : Zero homomorphism is not a homomorphism in Ring

② Faithful : Any unital ring homo is also ring homo and distinct unital ring homos are " ring homos.

③ Not ess. surj. : Rings without mult. identity is not iso to unital rings.

(e) Field $\hookrightarrow$ Rng.

① fully faithful.
since every field homo is ring homo.

② Not ess. surj : Not every ring is a field.

(f) $U : \mathrm{Mod}_R \longrightarrow \mathrm{Ab}$.

① faithful : Any distinct R-homo is also distinct gp homo.

(Since distinction determined by value, not property)

② Ess surj : $\forall A \in \mathrm{Ab}$, make trivial R-module structure s.t. $\forall a \in A$, $\forall r \in R$ $ra = a$.

③ Not necessarilly full.

$\mathrm{End}(R) \cong R$ in $\mathrm{Mod}_R$

but $\mathrm{End}(R) \ncong R$ in $\mathrm{Ab}$. in general.

(If $R = \mathbb{Z}$, then it is full.)

1.6. Art of the diagram chase

diagram (informal) : directed graph

commutes ( " ) : any two composable arrows are the same.

Def 1.6.2. (Monoid) $M \in \mathrm{Set}$ with

$\mu : M \times M$ , $\eta : 1 \to M$ s.t.

$$
\begin{tikzcd}
M \times M \times M \arrow[r, "1_M \times \mu"] \arrow[d, "\mu \times 1_M"'] \arrow[dr, phantom, "\circlearrowright"] & M \times M \arrow[d, "\mu"] \\
M \times M \arrow[r, "\mu"'] & M
\end{tikzcd}
,\quad
\begin{tikzcd}
M \arrow[r, "\eta \times 1_M"] \arrow[dr, "1_M"'] & M \times M \arrow[d] & M \arrow[l, "1_M \times \eta"'] \arrow[dl, "1_M"] \\
& M &
\end{tikzcd}
$$

$\mu$: multiplication

$\eta : 1 \longrightarrow M$ $\quad$ $1$ : singleton.

thus $\eta(1)$ is multiplicative identity

Def 1.6.3. (Topological monoid) $M \in \mathrm{Top}$ with the same commutative diagram (So $\mu$ is cts)

(Unital ring) $R \in \mathrm{Ab}$ with $R \otimes_{\mathbb{Z}} R$ instead of $R \times R$ (Monoidal Structure)

($k$-algebra) $R \in \mathrm{Vect}_k$ with $R \otimes_k R$ instead of $R \times R$ ( " )

Def 1.6.4 Diagram of $C$ = $F : J \to C$
where $J$ : indexing category is small.

Diagram is commutative.

$\Rightarrow$ Any composite relation in $J$
must holds at $C$ via $F$.

Ex 1.6.6. $2 \times 2$ : Obj $(0,0)$, $(0,1)$ $(1,0)$ $(1,1)$

Morphisms

$$
\begin{tikzcd}
(0,0) \arrow[r, "{(0\to 1, 1_0)}"] \arrow[d, "{(1_0, 0\to 1)}"'] \arrow[dr] & (1,0) \arrow[d, "{(1_1, 0\to 1)}"] \\
(0,1) \arrow[r, "{(0\to 1, 1_1)}"'] & (1,1)
\end{tikzcd}
$$

Notes that $(1_1, 0\to 1) \circ (0\to 1, 1_0)$
$= (0 \to 1, 0 \to 1)$
$= (0\to 1, 1_1) \circ (1_0, 0 \to 1)$

So diagonal arrow is unique

"Commutative square!"

Rem 1.6.7 : "shape" as indexing category

Shape = directed graph with
specified commutativity relation.

ex) $2 \times 2$

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "h"] \\ c \arrow[r, "k"'] & d \end{tikzcd}$$

with $hf = kg$.

$4$

$$\begin{tikzcd} & 1 \arrow[dd, "g"] \arrow[dr, "hg"] & \\ 0 \arrow[ur, "f"] \arrow[dr, "gf"'] \arrow[rr, "hgf"] & & 3 \\ & 2 \arrow[ur, "h"'] & \end{tikzcd}$$

$2 \times 3$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left] \arrow[d] \arrow[dr] \arrow[drr] & \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[rr, bend right] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

is

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g"'] & \bullet \arrow[r, "j"] \arrow[d, "h"] & \bullet \arrow[d, "l"] \\ \bullet \arrow[r, "k"'] & \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

with $hf = kg$, $lj = mh$

Lemma 1.6.11 $f_1 \cdots f_n$ : composable path.

$$f_k f_{k-1} \cdots f_i = g_m \cdots g_1$$

$$\Rightarrow \quad f_n \cdots f_1 = f_n \cdots f_{k+1} g_m \cdots g_1$$

pf) $g = h \Rightarrow fg = fh$ for any composable $f$.

Diagram chasing : showing two paths are equal

Lem 1.6.12

$$\begin{tikzcd} \cdot \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] \arrow[r, "h"] & \cdot \\ \cdot \arrow[ur, "g"'] & \end{tikzcd}, \quad f \text{ iso} \Rightarrow \begin{tikzcd} \cdot \arrow[r, "h"] \arrow[dr, phantom, "\circlearrowleft"] & \cdot \\ \cdot \arrow[u, "f^{-1}"] \arrow[ur, "g"'] & \end{tikzcd}$$

(Dually,

$$\begin{tikzcd} \cdot & \\ & \cdot \arrow[ul, "i"'] \arrow[dl, "j"] \arrow[l, phantom, "\circlearrowleft"] \\ \cdot \arrow[uu, "k"] & \end{tikzcd} \text{ and } k \text{ iso} \Rightarrow \begin{tikzcd} \cdot \arrow[dd, "k^{-1}"'] & \\ & \cdot \arrow[ul, "i"'] \arrow[dl, "j"] \arrow[l, phantom, "\circlearrowleft"] \\ \cdot & \end{tikzcd}$$ )

pf). $gf = h \Rightarrow gff^{-1} = hf^{-1} \Rightarrow g = hf^{-1}$

Lem 1.6.13.

$$\begin{tikzcd} \cdot \arrow[r, "\alpha"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\circlearrowleft"] & \cdot \arrow[d, "\beta"] \\ \cdot \arrow[r, "\delta"'] & \cdot \end{tikzcd} \Rightarrow \alpha^{-1}\beta^{-1} = \gamma^{-1}\delta^{-1}$$

pf) $\beta\alpha = \delta\gamma \Rightarrow \alpha^{-1}\beta^{-1}\cdot(\beta\alpha)\gamma^{-1}\delta^{-1} = \alpha^{-1}\beta^{-1}(\delta\gamma)\gamma^{-1}\delta^{-1}$

$\Rightarrow \gamma^{-1}\delta^{-1} = \alpha^{-1}\beta^{-1}$.

Def 1.6.14. $i \in C$ is initial if $\forall c \in C$ $\exists! \ i \longrightarrow c$.

$t \in C$ is terminal if $\forall c \in C$ $\exists! \ c \longrightarrow t$.

Ex 1.6.15.

| Category | initial | terminal |
| --- | --- | --- |
| Set | $\emptyset$ | singleton. |

| Category | Initial | Terminal |
|---|---|---|
| Set | $\emptyset$ | Singleton |
| Top | 〃 | 〃 |
| $Set_*$ | Singleton | |
| $Mod_R$ | 0 | |
| Group | 0 | |
| Ring | $\mathbb{Z}$ | 0 |
| Rng (non unital) | 0 | 0 |
| Field | Do not exist (diff characteristic $\Rightarrow$ No homo) | |
| Cat | $\emptyset$ | $\mathbb{1}$ |
| $(P, \leq)$ | global minima | global maxima |
| | (if exist) | |

Lem 1.6.16. $f_1 \cdots f_n$ composable seq.
$g_1 \cdots g_m$

s.t $\mathrm{dom}(f_1) = \mathrm{dom}(g_1)$, $\mathrm{Cod}(f_n) = \mathrm{Cod}(g_m)$

If either $\mathrm{dom}(f_1) =$ initial or $\mathrm{Cod}(f_n)$ terminal

$\Rightarrow f_n \cdots f_1 = g_m \cdots g_1$ pf) Uniqueness of morphism from / to initial / terminal.

Def 1.6.17. $C$ : Concrete Category
If $U : C \longrightarrow \mathrm{Set}$ a faithful functor exist.

Ex 1.6.18 : ( Concrete Category ) = Ex 1.1.3

Graph) : $V \sqcup E$ : Graph $\longrightarrow$ Set is faithful

Lem 1.6.19. : $U : C \to D$ faithful then
for any diagram in $C$ whose image
commutes in $D$ also commutes in $C$.

pf) Let $f_1 \cdots f_n$ , $g_1 \cdots g_m$ parallel seq
of composable morphism s.t.

$$Uf_1 \cdots Uf_n = Ug_1 \cdots Ug_m$$

$$\Rightarrow U(f_1 \cdots f_n) = U(g_1 \cdots g_m) \quad \text{by functoriality}$$

$$\Rightarrow f_1 \cdots f_n = g_1 \cdots g_m \quad \text{by faithfulness.}$$

Rem 1.6.20. Even outer rectangular commutes.
inner rectangular may not commutes

ex)

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "1_{\mathbb{Z}}"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "1_{\mathbb{Z}}"] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "1_{\mathbb{Z}}"'] & \mathbb{Z}.
\end{tikzcd}$$

Lem 1.6.21

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[r, "j"] \arrow[d, "h"] & c \arrow[d, "l"] \\
a' \arrow[r, "k"'] & b' \arrow[r, "m"'] & c'
\end{tikzcd}$$

and $ljf = mkg$.

If either ① $\boxed{\phantom{x}}\boxed{\circlearrowright}$ ($m$: mono) or ② ($f$: epi) $\boxed{\circlearrowright}\boxed{\phantom{x}}$

then the diagram commutes

pf) Assume ①: $ljf = mhf \Rightarrow hf = kg$ by mono.

$ljf = mkg$

② is dual case of ①.

Ex 1.6.i) Let $i$, initial, $t$: terminal

$\Rightarrow \exists! g : i \to t$ and

If $\exists f : t \to i. \Rightarrow gf : t \to t$

Since $t \to t$ is unique, $gf = 1_t$.

~~Pf~~ Also, $fg : i \to i$ is unique $\Rightarrow fg = 1_i$.

Ex 1.6.ii) Let $t_1, t_2$ are the terminal object. $\Rightarrow \exists! f: t_1 \longrightarrow t_2$ and $\exists! g: t_2 \longrightarrow t_1$

Thus $fg: t_2 \longrightarrow t_2$ unique $\Rightarrow fg = 1_{t_2}$

$gf: t_1 \longrightarrow t_1$ " $\Rightarrow gf = 1_{t_1}$

Ex 1.6.iii) Let $f: c \longrightarrow c'$ s.t.

$Ff: c \longrightarrow c'$ is mono in $D$.

Let $g_1, g_2 \in C(b, c)$ s.t.

$fg_1 = fg_2$

$\Rightarrow F(fg_1) = F(fg_2) \Rightarrow Ff\, Fg_1 = Ff\, Fg_2$

$\Rightarrow Fg_1 = Fg_2$ by $Ff$ is mono

$\Rightarrow g_1 = g_2$ by faithfulness

$\Rightarrow f$ is mono in $C$.

Thus, if $C$ is concrete category, then faithful $U: C \longrightarrow Set$ exists. thus if $f \in MorC$ s.t. $Uf$ = injection, then $f$ is mono.

By duality, faithful functor reflects epi.

Ex 1.6.iv) $\mathcal{C}$ : category, $\underline{f : c \to c'}$ not epi or mono.

$\mathbf{2}$ : $\overset{\cdot}{0} \longrightarrow \overset{\cdot}{1}$ : iso.

$F : \mathbf{2} \longrightarrow \mathcal{C}$ $\quad F(0 \to 1) = f$.

$$\begin{tikzcd} \overset{\cdot}{0} \arrow[d] & \longmapsto & c \arrow[d, "f"] \\ \overset{\cdot}{1} & & c' \end{tikzcd}$$

$\Rightarrow$ neither epi or mono.

Ex 1.6.v) Div : Category of divisible group.

$(G, +)$ is divisible if $\forall n \in \mathbb{N}$, $g \in G$,

$\exists y \in G$ s.t. $ny = g$.

Let $\pi : \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z}$, $f, g : G \longrightarrow \mathbb{Q}$ s.t.

$\pi \circ f = \pi \circ g$. Let $x \in G$.

$\Rightarrow \pi \circ f(x) = \pi \circ g(x) \Rightarrow f(x) - g(x) = n \in \mathbb{Z}$

If $n \neq 0$,
By divisibility, $\exists y \in G$ s.t. $2ny = x$.

$\Rightarrow f(2ny) = f(x) \Rightarrow \frac{1}{2n} f(x) = f(y)$

$\therefore f(y) - g(y) = \frac{1}{2n}(f(x) - g(x)) = \frac{1}{2}$,

contradiction.

$\Rightarrow n = 0$. $\therefore f(x) = g(x)$, $\forall x$.

So $\pi$ is mono, but not injective.

Also, $\pi : \mathbb{Z} \longrightarrow \mathbb{Q}$ in Ring is epi

but not surjective.

tsn-tur.

Suppose $f, g : \mathbb{Q} \longrightarrow R$ s.t. $f \circ \pi = g \circ \pi$.

Then, $f(\frac{a}{b}) = f(\frac{1}{b}) f(a) = f(\frac{1}{b}) \cdot g(a)$

$$= g(b \tfrac{a}{b}) f(\tfrac{1}{b}) = g(\tfrac{a}{b}) g(b) f(\tfrac{1}{b})$$

$$= g(\tfrac{a}{b}) f(b) f(\tfrac{1}{b})$$

$$= g(\tfrac{a}{b})$$

(Assume comm ring; but it holds for any associative unital ring)

$\therefore f = g$.

Ex

1.6.vi) Let $(c, \gamma)$ be a terminal

Then for any $(d, \phi)$ coalgebra.

$\exists ! f : (d, \phi) \longrightarrow (c, \gamma)$ s.t.

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & c \arrow[d, "\gamma"] \\ Td \arrow[r, "Tf"'] & Tc \end{tikzcd}$$

Thus $C \xrightarrow{\gamma} TC \Rightarrow TC \xrightarrow{T\gamma} T^2C$

has a coalgebra map (unique)

$f : (TC, T\gamma) \longrightarrow (C, \gamma)$

s.t.

$$\begin{tikzcd}
TC \arrow[r, "f"] \arrow[d, "T\gamma"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "\gamma"] \\
T^2C \arrow[r, "Tf"'] & TC
\end{tikzcd}$$

Thus, $Tf \circ T\gamma = \gamma \circ f$

And

$$\begin{tikzcd}
C \arrow[r, "\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\circlearrowright"] & TC \arrow[d, "T(\gamma)"] \\
TC \arrow[r, "T(\gamma)"'] & TC^2
\end{tikzcd}$$

by functoriality

$\Rightarrow T\gamma : (C, \gamma) \longmapsto (TC, T\gamma)$ is a morphism

$\Rightarrow f \circ \gamma : C \longrightarrow C$ s.t.

$$\begin{tikzcd}
C \arrow[r, "f\circ\gamma"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\circlearrowright"] & C \arrow[d, "\gamma"] \\
TC \arrow[r, "T(f\circ\gamma)"'] & TC
\end{tikzcd}$$

Since $(C, \gamma)$ is terminal, $f \circ \gamma = 1_C$

Hence, $\gamma \circ f = \underline{Tf \circ T\gamma = T(f \circ \gamma)}$ (first square)

$= T(1_C)$

$= 1_{TC}$

$\Rightarrow$ $\gamma$ is iso.

1.7. 2-categories of categories

$C, D$ : categories

$D^C$ : functor category

Obj : $F : C \longrightarrow D$ functor

Mor :

$$\begin{tikzcd} C \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\alpha \Downarrow"] & D \end{tikzcd}$$

$\alpha$: natural iso.

$\Rightarrow$ $1_F : F \Rightarrow F$ by $(1_F)_c := 1_{Fc}$

is identity morphism.

Lem 1.7.1 (Vertical Composition)

$\alpha : F \Rightarrow G$, $\beta : G \Rightarrow H$.

$\Longrightarrow \exists \beta \cdot \alpha : F \Rightarrow H$ s.t. $(\beta \cdot \alpha)_c := \beta_c \cdot \alpha_c$.

pf)

$$\begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowright"] & Gc \arrow[r, "\beta_c"] \arrow[d, "Gf"] \arrow[dr, phantom, "\circlearrowright"] & Hc \arrow[d] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \arrow[r, "\beta_{c'}"'] & Hc' \end{tikzcd}$$

from def of natural transf $\beta, \alpha$.

$\Longrightarrow (\beta \cdot \alpha)$ is natural transf.

Cor 1.7.2. $D^C$ is well-def.

Rem 1.7.3.

| Size of $C$ | Size of $D$ | Size of $D^C$ |
|---|---|---|
| small | small | small |
| small | locally small | locally small |
| locally small | " | (?) |

$\Rightarrow$ $Cat^{op} \times Cat \to Cat$.

$Cat^{op} \times CAT \to CAT$.

Vertical composition: composition in Lem 1.7.1

i.e.

$$\begin{tikzcd} C \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\alpha\Downarrow"] \arrow[r, phantom, shift right=3, "\beta\Downarrow"] & D \end{tikzcd} = \begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\beta\alpha"] & D \end{tikzcd}$$

Horizontal composition:

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & E \end{tikzcd} = \begin{tikzcd} C \arrow[r, bend left, "HF"] \arrow[r, bend right, "KG"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & E \end{tikzcd}$$

def by

$$\begin{tikzcd} HF_c \arrow[r, "\beta_{F_c}"] \arrow[d, "H\alpha_c"'] \arrow[dr, dashed, "(\beta*\alpha)_c"] & KF_c \arrow[d, "K\alpha_c"] \\ HG_c \arrow[r, "\beta_{G_c}"'] & KG_c \end{tikzcd}$$

Lem 1.7.7. (Middle four interchange)

$$\begin{tikzcd}[row sep=large]
C \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & D \arrow[r, bend left=60, "J"] \arrow[r, "K" description] \arrow[r, bend right=60, "L"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & E
\end{tikzcd}$$

$$\Rightarrow \quad
\begin{tikzcd}
C \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, "\Downarrow\beta\cdot\alpha"] & D \arrow[r, bend left=50, "J"] \arrow[r, bend right=50, "L"'] \arrow[r, phantom, "\Downarrow\delta\cdot\gamma"] & E
\end{tikzcd}
=
\begin{tikzcd}[row sep=large]
C \arrow[r, bend left=60, "JF"] \arrow[r, "KG" description] \arrow[r, bend right=60, "LH"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma * \alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\delta * \beta"] & E
\end{tikzcd}$$

Def 1.7.8 A 2-category

Obj: (ex) Categories

1-Morphisms (ex) Functors

2- " (ex) Natural transf.

1-Mor: morphism between pair of objs
functors

2 " "

s.t. 1) Obj + 1Mor : category

2) For any $C, D \in$ Obj

$D^C$ with 2-mor between the elem in $D^C$ form a category.

pf)

$$\begin{tikzcd} HF_c \arrow[r, "\beta_{F_c}"] \arrow[d, "H\alpha_c"'] & KF_c \arrow[d, "K\alpha_c"] \\ HG_c \arrow[r, "\beta_{G_c}"'] & KG_c \end{tikzcd}$$

Commutes by $\beta : H \Rightarrow K$

$\Rightarrow$ $(\beta * \alpha)_c$ is well-def.

To show $\beta * \alpha$ is natural transf.

Let $f : c \to c' \in \mathrm{Hom}\, C$. Then,

$$\begin{tikzcd} HF_c \arrow[r, "H\alpha_c"] \arrow[d, "HFf"'] & HG_c \arrow[r, "\beta_{G_c}"] \arrow[d, "HGf"] & KG_c \arrow[d, "KGf"] \\ HF_{c'} \arrow[r, "H\alpha_{c'}"'] & HG_{c'} \arrow[r, "\beta_{G_{c'}}"'] & KG_{c'} \end{tikzcd}$$

Commute,

Com by naturality of $\alpha$. and $H$ preserves comm diagram

Comm by $\beta : H \Rightarrow K$

Also note that $\xrightarrow{H\alpha_c} \xrightarrow{\beta_{G_c}} = (\beta * \alpha)_c$

$\xrightarrow[H\alpha_{c'}]{} \xrightarrow[\beta_{G_{c'}}]{} = (\beta * \alpha)_{c'}$

$\Rightarrow$ $KGf\, (\beta * \alpha)_c = (\beta * \alpha)_{c'}\, HFf$. done

3) Obj = Obj

$$\text{Mor} := \begin{tikzcd} C \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & D \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

composition: horizontal composition

form a category

4) Law of middle four interchange holds.

Ex 1.7. i)

Ex 1.7. ii) Let $F, G : C \rightrightarrows D$

Then $F(\mathrm{ob}\, C)$, $G(\mathrm{ob}\, C)$ / $F(\mathrm{Mor}\, C)$, $G(\mathrm{Mor}\, C)$ is a set

since $| \quad " \quad | \leq |C|$

Thus $\bigoplus_{c \in C} D(Fc, Gc)$ is a set

since $C$ is set and $D(Fc, Gc)$ is a set.

Now take a map

$$D^C(F, G) \longrightarrow \bigoplus_{c \in C} D(Fc, Gc)$$

$$\alpha \longmapsto \{(\alpha_c)_{c \in C}\}$$

It is mono, since $(\alpha_c)_{c \in C} = (\beta_c)_{c \in C}$

$\Rightarrow \alpha = \beta$.

Thus $D^C(F, G)$ is a set

$\Rightarrow D^C$ is locally small.

1.7.ii Let $f : c \longrightarrow c'$.

$$\begin{tikzcd} LHF_c \arrow[r, "L\beta_{F_c}"] \arrow[d, "LHFf"'] & LKF_c \arrow[d, "LKFf"] \\ LHF_{c'} \arrow[r, "L\beta_{F_{c'}}"'] & LKF_{c'} \end{tikzcd}$$

WTS $LKFf \circ L\beta_{F_c} = L\beta_{F_{c'}} \circ LHFf$

$$\begin{array}{ccc} LKFf \circ L\beta_{F_c} & & L\beta_{F_{c'}} \circ LHFf \\ \| & & \| \\ L(KFf \circ \beta_{F_c}) & & L(\beta_{F_{c'}} \circ H(Ff)) \end{array}$$

Since for $Ff : F_c \longrightarrow F_{c'}$, from $\beta$ natural,

$$\begin{tikzcd} HF_c \arrow[r, "\beta_{F_c}"] \arrow[d, "HFf"'] \arrow[dr, phantom, "\circlearrowright"] & KF_c \arrow[d, "KFf"] \\ HF_{c'} \arrow[r, "\beta_{F_{c'}}"'] & KF_{c'} \end{tikzcd} \Rightarrow \quad KFf \circ \beta_{F_c} = \beta_{F_{c'}} \circ HFf$$

done.

Thus $L\beta_F$ : Natural transf.

1.7.iii)

$$\begin{tikzcd} C \arrow[r, "1_C"] & C \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & E \arrow[r, "1_E"] & E \end{tikzcd}$$

By whiskering we have

$H\alpha : HF \Rightarrow HG$

$K\alpha : KF \Rightarrow KG$

$\beta F : HF \Rightarrow KF$

$\beta G : HG \Rightarrow KG$

Then,

$$\begin{tikzcd} C \arrow[r, bend left=60, "HF"] \arrow[r, "HG" description] \arrow[r, bend right=60, "KG"'] \arrow[r, phantom, shift left=3, "\Downarrow H\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow \beta G"] & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r, bend left=60, "HF"] \arrow[r, "KF" description] \arrow[r, bend right=60, "KG"'] \arrow[r, phantom, shift left=3, "\Downarrow K\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow \beta F"] & E \end{tikzcd}$$

By vertical composition, we have

$$\beta G \cdot H\alpha : HF \Rightarrow KG, \quad K\alpha\, \beta F : HF \Rightarrow KG$$

s.t.

$$\begin{tikzcd} HF_c \arrow[r, "H\alpha_c"] \arrow[d, "\beta F_c"'] \arrow[dr, bend left=40, "(\beta G H\alpha)_c"] \arrow[dr, bend right=40, "(K\alpha\beta F)_c"'] & HG_c \arrow[d, "\beta G_c"] \\ KF_c \arrow[r, "K\alpha_c"'] & KG_c \end{tikzcd}$$

And we already showed in the proof of Lem 1.7.4 that this diagram commute

$\Rightarrow \beta G H\alpha = K\alpha\,\beta F$

$\Rightarrow (\beta * \alpha)_c = \text{''} = \text{''}$

1.7. iv) Given

$$\begin{tikzcd}[row sep=large, column sep=huge]
C \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & D \arrow[r, bend left=60, "J"] \arrow[r, "K" description] \arrow[r, bend right=60, "L"'] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & E
\end{tikzcd},$$

RHS)

From 1.7. iii) we know that

$$\gamma * \alpha = \gamma G \cdot J\alpha = K\alpha\,\gamma F$$

$$\delta * \beta = \delta H \cdot K\beta = L\beta\,\delta G$$

$$\Rightarrow (\delta * \beta)\cdot(\gamma * \alpha) = L\beta\,\delta G \cdot K\alpha\,\gamma F$$

$$= \quad '' \quad \cdot \gamma G \cdot J\alpha$$

$$= \delta H \cdot K\beta \quad K\alpha\,\gamma F$$

$$= \quad '' \quad \gamma G \cdot J\alpha$$

---

LHS)

$$(\beta \cdot \alpha) * (\delta \cdot \gamma) = (\delta \cdot \gamma\, H)\cdot(J\beta \cdot \alpha)$$

$$= (L\,\beta \cdot \alpha)\cdot(\delta \cdot \gamma\, F)$$

Notes that $\delta \cdot \gamma H = (\delta \cdot \gamma)_{Hc} = \delta_{Hc} \cdot \gamma_{Hc}$

$$J\beta \cdot \alpha = J(\beta \cdot \alpha)_c = J(\beta_c \cdot \alpha_c)$$

$$= J\beta_c \cdot J\alpha_c$$

$\Rightarrow$ one expression of LHS $= \delta_{Hc}\,\gamma_{Hc}\,J\beta_c\,J\alpha_c$

and

$$(\delta * \beta)(\gamma * \alpha) = \delta H \cdot K\beta \;\; \gamma G \cdot J\alpha$$

$$'' \quad {}_c = (\delta H)_c \, (K\beta)_c \, (\gamma G)_c \, (J\alpha)_c$$

$$= \underbrace{\delta_{H_c} \cdot K\beta_c \cdot \gamma_{G_c} \cdot J\alpha_c}_{\text{LHS}}$$

$$\Rightarrow \begin{tikzcd}
JF_c \arrow[d, "J\alpha_c"] \arrow[dddr, bend left=60, "\text{LHS}"] \arrow[dddr, bend right=60, "\text{RHS}"'] & \\
JG_c \arrow[r, "J\beta_c"] \arrow[d, "\gamma_{G_c}"'] & JH_c \arrow[d, "\gamma_{H_c}"] \\
KG_c \arrow[r, "K\beta_c"] & KH_c \arrow[d, "\delta_{H_c}"] \\
 & LH_c
\end{tikzcd}$$

It suffices to show that middle rectangle commutes.

Since $\beta_c : G_c \to H_c$ and $\gamma : J \Rightarrow K$

we have

$$\begin{tikzcd}
JG_c \arrow[r, "J(\beta_c)"] \arrow[d, "\gamma_{G_c}"'] \arrow[dr, phantom, "\circlearrowright"] & JH_c \arrow[d, "\gamma_{H_c}"] \\
KG_c \arrow[r, "K(\beta_c)"'] & KH_c
\end{tikzcd}$$

Thus, RHS = LHS.

∎

Ex 1.7 v.

Let $Z(C) := \mathcal{C}^{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$

Then : $\forall \alpha, \beta \in Z(C)$

① $\alpha \cdot \beta = \beta \cdot \alpha$

Note that $\forall \alpha \in Z(C)$, take $f : c \to c'$ then

$$\begin{tikzcd} c \arrow[r, "\alpha_c"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & c \arrow[d, "f"] \\ c' \arrow[r, "\alpha_{c'}"'] & c' \end{tikzcd}$$

Thus, choose $f$ as

$f = \beta_c : c \to c$

Then,

$$\begin{tikzcd} c \arrow[r, "\alpha_c"] \arrow[d, "\beta_c"'] & c \arrow[d, "\beta_c"] \\ c \arrow[r, "\alpha_c"'] & c \end{tikzcd}$$

$\Rightarrow \alpha_c \cdot \beta_c = \beta_c \alpha_c \quad \forall c$

$\Rightarrow \alpha \cdot \beta = \beta \cdot \alpha$

② $1 \in Z(C)$ $\qquad$ $1$ : natural iso s.t $(1)_c = 1_c$

Then $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$.

③ closed under vertical composition

Rmk : actually ① follows from Horizontal composition.

(1.7, vi) First we can do is use

$$\begin{tikzcd} D \arrow[r, bend left=60, "FG"] \arrow[r, phantom, shift left=3, "\Downarrow\varepsilon"] \arrow[r, "I_D" description] \arrow[r, phantom, shift right=3, "\Downarrow\eta'"] \arrow[r, bend right=60, "G'F'"'] & D \end{tikzcd}$$

to get another natural transf. $\varepsilon * \eta'$. Then,

$$\begin{tikzcd} D \arrow[r, bend left, "FG"] \arrow[r, phantom, "\Downarrow\varepsilon*\eta'"] \arrow[r, bend right, "G'F'"'] & D \end{tikzcd}$$

exists and natural equivalence since both $\varepsilon$, $\eta'$ are natural equivalence.

Then, $GG'F'F \underset{①}{\cong} GFGF \underset{②}{\cong} 1_C \cdot 1_C \cong 1_C$

$F'FGG' \underset{①}{\cong} F'G'F'G' \underset{②}{\cong} 1_E 1_E \cong 1_E$.

For example

$$\begin{tikzcd} C \arrow[r, "G"] & D \arrow[r, bend left, "G'F'"] \arrow[r, bend right, "FG"'] & D \arrow[r, "F"] & C \end{tikzcd}$$

is whiskered composite, thus Ex 1.7.ii gives ①. And

$$\begin{tikzcd} C \arrow[r, bend left, "GF"] \arrow[r, phantom, "\Downarrow\eta"] \arrow[r, bend right, "1_C"'] & C \arrow[r, bend left, "GF"] \arrow[r, phantom, "\Downarrow\eta"] \arrow[r, bend right, "1_C"'] & C \end{tikzcd} = \begin{tikzcd} C \arrow[r, bend left, "GFGF"] \arrow[r, phantom, "\Downarrow\eta*\eta"] \arrow[r, bend right, "1_C \cdot 1_C"'] & C \end{tikzcd}$$

is natural equiv by Horizontal composition.

$\Rightarrow$ ② holds.

By ①, ②, we get desired natural equivalence.

1.7 vii)

Let $F : C \times D \longrightarrow E$ be bifunctor. Then let

$$\begin{aligned} F' : C &\longrightarrow E^{D} \\ c &\longmapsto F(c, -) \\ f \downarrow &\longmapsto F(f, -) \\ c' &\longmapsto F(c', -) \end{aligned}$$

where $F(f, -)$ is natural transformation between $F(c, -)$ and $F(c', -)$.

To define $F(f, -)$, notes that for any $d \xrightarrow{g} d'$,

$$\begin{tikzcd} F(c, d) \arrow[r, "F(f, d)"] \arrow[d, "F(c, g)"'] & F(c', d) \arrow[d, "F(c', g)"] \\ F(c, d') \arrow[r, "F(f, d')"'] & F(c', d') \end{tikzcd}$$

with $F(c, g) = F(1_c, g)$ $\quad F(f, d) = F(f, 1_d)$

$F(c', g) = F(1_{c'}, g)$ $\quad F(f, d') = F(f, 1_{d'})$

Then, since $F$ is bifunctor,

$$F(c', g) \circ F(f, d) = F(1_{c'}, g) \circ F(f, 1_d) = F(f, g)$$

$$F(f, d') \circ F(c, g) = F(f, 1_{d'}) \circ F(1_c, g) = F(f, g)$$

Thus $F(f, -)$ is well-def natural transformation.

Thus $(\ )' : \mathrm{obj}\, E^{C\times D} \longmapsto \mathrm{obj}\,(E^D)^C$ is well-def map.

Conversely, for

$$\begin{tikzcd} F' : C \arrow[r] & E^D \\ c \arrow[d, "f'"'] \arrow[r, mapsto] & F'(c) \arrow[d, "F'(f)"] \\ c' \arrow[r, mapsto] & F'(c') \end{tikzcd}$$

: natural transf

Let $UF : C\times D \longrightarrow E$

$$\begin{tikzcd} (c,d) \arrow[d, "(f,g)"'] \arrow[r, mapsto] & F'(c)(d) \arrow[d, "F'(f)(g)"] \\ (c',d') \arrow[r, mapsto] & F'(c')(d') \end{tikzcd}$$

where $F'(f)(g)$ is defined as follow;

$$\begin{tikzcd} F'(c)(d) \arrow[r, "F'(f)_d"] \arrow[d, "F'(c)(g)"'] & F'(c', d) \arrow[d, "F'(c')(g)"] \\ F'(c)(d') \arrow[r, "F'(f)_{d'}"'] & F'(c', d') \end{tikzcd}$$

Since $F'(f)$ is natural transformation, let.

$$\begin{aligned} F'(f)(g) &:= F'(c')(g) \circ F'(f)_d \\ &= F'(f)_{d'} \circ F'(c)(g) \end{aligned}$$

It is well-defined morphism.

Thus $U(\quad) : \mathrm{Obj}(\mathcal{E}^D)^C \longrightarrow \mathrm{Obj}(\mathcal{E}^{C\times D})$

is well-defined map.

(Actually we need to check compositions;

Identity is clear. For $F'$, apply it on

$c \xrightarrow{f} c' \xrightarrow{f'} c''$ gives

$F(c,-) \xrightarrow{F(f,-)} F(c',-) \xrightarrow{F(f',-)} F(c'',-)$

and $F(f'f,-) = F(f',-) \circ F(f,-)$ since

for any $d \xrightarrow{g} d'$,

$$\begin{tikzcd}
F(c,d) \arrow[r, "F(f,1_d)"] \arrow[d, "F(c,g)"'] & F(c',d) \arrow[r, "F(f',1_d)"] \arrow[d, "F(c',g)"'] & F(c'',d) \arrow[d, "F(c'',g)"'] \\
F(c,d') \arrow[r, "F(f,1_{d'})"'] & F(c',d') \arrow[r, "F(f',1_{d'})"'] & F(c'',d')
\end{tikzcd}$$

and functoriality of $F$ gives $F(f',1_d) \circ F(f,1_d)$
$= F(f'f, 1_d)$

Thus $F(f'f,-)_d = F(f',-)_d \circ F(f,-)_d \quad \forall d \in D$

Hence $F(f'f,-) = F(f',-) \cdot F(f,-)$ as natural iso.

In case of UF, applyi it on.

$$(c,d) \xrightarrow{(f,g)} (c',d') \xrightarrow{(f',g')} (c'',d'')$$

we get

$$F'(c)(d) \xrightarrow{F'(f)(g)} (c',d') \xrightarrow{F'(f')(g')} (c'',d'')$$

WTS $\quad F'(f')(g') \cdot F'(f)(g) = F(f'f)(g'g)$

```latex
\begin{tikzcd}
F'(c)(d) \arrow[r] \arrow[d] \arrow[rr, bend left, "{F(f'f),(1_d)}"] \arrow[dr, bend left, "F'(f)(g)"] \arrow[dr, bend right, "F'(f)(g)"'] & F'(c')(d) \arrow[r] \arrow[d] \arrow[dr, bend left, "F'(f')(g)"] \arrow[dr, bend right] & F'(c'')(d) \arrow[d] \arrow[dd, bend left=60, "{F(1_{c''},g'g)}"] \\
F'(c)(d') \arrow[r] \arrow[d] \arrow[dr, bend left, "F'(f)(g')"] \arrow[dr, bend right] & F'(c')(d') \arrow[r] \arrow[d] \arrow[dr, bend left, "F'(f')(g')"] \arrow[dr, bend right, "F'(f')(g')"'] & F'(c'')(d') \arrow[d] \\
F'(c)(d'') \arrow[r] & F(c')(d'') \arrow[r] & F'(c'')(d'')
\end{tikzcd}
```

Notes that each rectangular commutes by def of $F'(f)(g)$, $F'(f')(g)$, $F'(f')(g')$, $F'(f')(g)$.

Hence outer rectangular commutes.

So it suffices to show that each outer edge is $F(f'f)(1_d)$ and $F(1_{c''})(g',g)$.

But

$$\begin{tikzcd}
F'(C)(d) \arrow[d, "F'(f)(1_d)"'] \arrow[dd, bend left=80, "F(f'f, 1_d) = F'(f'f)_d \circ F'(C)(1_d)"] \\
F'(C')(d) \arrow[d, "F'(f')(1_d)"'] \\
F'(C'')(d)
\end{tikzcd}$$

gives

$$F'(f')(1_d) \cdot F'(f)(1_d)$$

$$= F'(f')_d \circ F'(C')(1_d) \cdot F'(C')(1_d) \circ F'(f)_d$$

$$= \underline{F'(f')_d \quad F'(C')(1_d)} \; F'(f)_d$$

$$= F'(f')_d \quad \underline{1_{F'(C')(d)}} \quad F'(f)_d \qquad \text{by Functoriality of } F'(C')$$

$$= F'(f')_d \cdot F'(f)_d \qquad \text{by functoriality}$$

$$= F'(f'f)_d \qquad \text{of } F'(\;\;).$$

done. (Vertical one is similar)

Hence given map is well-def, thus bijection.