1. The Apportionment Problem

Typical University needs to create a student government with 16 representatives from 4 groups of students. If we desire that each representative represents an equal number of students, how many representatives will each class have?

Class	Population	Representatives
Freshmen	12,000	
Sophomores	10,000	
Juniors	6,000	
Seniors	4,000	
Total		

The ______________________________, h, is the total number of representatives.
The ______________________________, s is the total population, p, divided by the house size, h. So $s = p/h$.
A group's ______________________________, q_i, is the group's population, p_i divided by the standard divisor, s. So $q_i = p_i/s$.

Example 1.1. If Typical University can only have 15 representatives, how many representatives will each class have?

$s =$

Class	Population	Representatives
Freshmen	12,000	
Sophomores	10,000	
Juniors	6,000	
Seniors	4,000	
Total		

We have to round to a whole number (because we can't split people). However, this causes an issue because that would require more representatives than we have available. This is an ______________________________because the goal is to round the set of numbers (quotas in this case) to whole numbers in a way that maintains the original sum (the number of representatives in this case). A procedure for solving an apportionment problem without making arbitrary choices is called an ______________________________. We will finish this problem after we discuss some background information.
When q is not already an integer, there are multiple ways to round.

- Round q up to the next integer, $\lceil q \rceil$.
- Round q down to the previous integer, $\lfloor q \rfloor$.
- Round to the nearest integer, $[q]$. If q is halfway to the next integer or larger, round up to the next integer. Otherwise, round down to the previous integer.
- Round according to the geometric mean. The geometric mean of $\lfloor q \rfloor$ and $\lceil q \rceil$ is $q^* = \sqrt{\lfloor q \rfloor \lceil q \rceil}$. If q is equal to or larger than q^*, round up to the next integer. Otherwise, round down to the previous integer.

Complete the following chart.

q	$[q]$	$\lfloor q \rfloor$	$\lceil q \rceil$	q^*	Round according to q^*
2.5					
2.45					
2.44					
2.1					
2					
1.9					
1.45					
1.4					
0.6					
0.1					
0.00001					

Different apportionment methods will use different rounding rules.

The U.S. constitution says the House of Representatives "shall be apportioned among the several states within this union according to their respective Numbers...." Therefore, much of the history of apportionment is related to the House of Representatives. For a historical summary, see http://www.ctl.ua.edu/math103/apportionment/apphisty.htm.

2. Hamilton Method

This was the first apportionment method approved by Congress for the U.S. House of Representatives, but it was vetoed by President Washington. It was later used for most of the years between 1852 and 1901.

(1) Compute the standard divisor.
(2) Compute the quota for each "state" (group).
(3) Round each quota down.
(4) Calculate the number of seats left to be assigned.
(5) Assign the remaining seats to the states with the *largest fractional part* of q.

Example 2.1. Let's return to Typical University's apportionment of 15 representatives.

Class	Population	q	Rounded quota	Hamilton Apportionment
Freshmen	12,000	5.6250		
Sophomores	10,000	4.6875		
Juniors	6,000	2.8125		
Seniors	4,000	1.8750		
TOTAL	**32,000**	**15**		

Example 2.2. A school district received 46 computers to distribute to 5 high schools based on the number of AP statistics students at each school. Use the Hamilton method to distribute the computers.
$s =$

High School	Population	q	Rounded quota	Hamilton Apportionment
Northside	39			
Southside	71			
Eastside	18			
Westside	223			
Central	209			
TOTAL				

Example 2.3. Several small towns were working together to reorganize their county representation. There are 8 council seats. Use the Hamilton method to distribute the seats among the towns.
$s =$

Town	Population	q	Rounded quota	Hamilton Apportionment
A	3862			
B	2818			
C	1881			
D	500			
E	21			
TOTAL				

Example 2.4. A university allocated funds for 100 student worker positions. The positions will be apportioned to four colleges according to their enrollment each Fall. Use the Hamilton method to distribute the positions.
$s =$

College	Population (Fall 2014)	q	Rounded quota	Hamilton Apportionment
J	5565			
K	3417			
L	3864			
M	197			
TOTAL				

3. (14.3-4) Divisor Methods and Which Method is Best

We have used the standard divisor, s, to represent the average district population. We will use s for all apportionment methods to calculate the quota.

The ________________will also use an ________________d, to calculate an ________________. The adjusted quota combined with the appropriate rounding rules for each method will give the final apportionment for divisor methods.

Definition 3.1 (Jefferson Method)**.** The U.S. Constitution requires a minimum population per congressional district, so Thomas Jefferson incorporated that minimum into his apportionment method. Jefferson's method was used to apportion the House of Representatives from 1791 until 1840. The Jefferson method favors larger states.

(1) Compute the standard divisor.
(2) Compute the quota for each "state" (group).
(3) Round each quota *down.*
(4) If the total number of seats is not correct, call the current apportionment N and find new divisors, $d_i = \frac{p_i}{N_i+1}$ correspond to giving each state one more seat.
(5) Assign a seat to the state with the largest d. (Notice that divisor methods look at the entire number of d rather than the fractional part of the number.)
(6) Repeat Steps 4 and 5 until the total number of seats is correct. The last d_i used is the ________________________________

Example 3.2. Let's return to the school district that received 46 computers to distribute to 5 high schools based on the number of AP statistics students at each school. Use the Jefferson method to distribute the computers.

$s = 560/46 \approx 12.174$.

High School	Pop.	q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
North	39	3204					
South	71	5832					
East	18	1479					
West	223	18318					
Cent.	209	17168					
Total	560	46					

$d =$

Example 3.3. Let's return to the small towns who were working together to reorganize their county representation. There are 8 council seats. Use the Jefferson method to distribute the seats among the towns.

$s = 9082/8 \approx 1135.25$

Town	Pop.	q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
A	3862	3.402					
B	2818	2.483					
C	1881	1.657					
D	500	0.440					
E	209	0.019					
Total	9082	8					

$d =$

Example 3.4. Let's return to the university's 100 student worker positions. Use the Jefferson method to distribute the positions.

$s = 13097/100 \approx 130.97$

College	Pop.	q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
J	5565	42.491					
K	3471	26.502					
L	3864	29.503					
M	197	1.504					
Total	13097	100					

$d =$

4. Webster Method

Definition 4.1.
The *Webster method* does not favor large or small states. The Webster method (or a method that gave the same apportionment as the Webster method) was used to apportion the House of Representatives for the majority of the years between 1842 and 1931. For a bit of political drama, research the apportionment of 1872 when the apportionment did not match any of the methods and affected the presidential election of 1876.

(1) Compute the standard divisor.
(2) Compute the quota for each "state" (group).
(3) Round each quota to the nearest integer.
(4) If the total number of seats is not correct, call the current apportionment N, and find new divisors.
   - If the number of seats needs to increase, use $d_i^+ = p_i/(N_i + 0.5)$.
   - If the number of seats needs to decrease, use $d_i^- = p_i/(N_i - 0.5)$.
(5) Adjust the seats according to d.
   - If the number of seats needs to increase, assign a seat to the state with the largest d_i^+.
   - If the number of seats needs to decrease, remove a seat from the state with the smallest d_i^-.
   - Repeat Steps 4 and 5 until the total number of seats is correct. The last d_i used is the adjusted divisor, d.

Example 4.2.
Let's return to the school district that received 46 computers to distribute to 5 high schools based on the number of AP statistics students at each school. Use the Webster method to distribute the computers.
$s = 560/46 \approx 12.174$

High School	Population	q	Rounded quota	d_i	Webster App.
Northside	39	3.2036			
Southside	71	5.8321			
Eastside	18	1.4876			
Westside	223	18.3179			
Central	209	17.1679			
TOTAL	560	46			

$d =$

Example 4.3.

Let's return to the small towns who were working together to reorganize their county representation. There are 8 council seats. Use the Webster method to distribute the seats among the towns.

$s = 9082/8 = 1135.25$

Town	Population	q	Rounded quota	d_i	Webster App.
A	3862	3.402			
B	2818	2.483			
C	1881	1.657			
D	500	0.440			
E	21	0.019			
TOTAL	9082	8			

$d =$

Example 4.4.

Let's return to the university's 100 student worker positions. Use the Webster method to distribute the positions.

$s = 13097/100 = 130.97$

College	Population	q	Rounded quota	d_i	Webster App.
J	5565	42.491			
K	3471	26.502			
L	3864	29.503			
M	197	1.504			
TOTAL	13097	100			

$d =$

5. Hill-Huntington Method

Definition 5.1.
The *Hill-Huntington method* does a great job of keeping the relative differences of representative share (i.e., $\frac{\text{“apportionment”}}{\text{“population”}}$) and district population (i.e., $\frac{\text{“population”}}{\text{“apportionment”}}$) stable between states. It also ensures that every group gets at least one representative, so it favors small states. Since 1941, the Hill-Huntington method with a house size of 435 has been used to apportion the House of Representatives.

(1) Compute the standard divisor.
(2) Compute the quota for each “state” (group).
(3) Round each quota according to the geometric mean of $\lfloor q \rfloor$ and $\lceil q \rceil$, and $q^* = \sqrt{\lfloor q \rfloor \lceil q \rceil}$
(4) If the total number of seats is not correct, call the current apportionment N, and find new divisors.
  - If the number of seats needs to increase, use $d_i^+ = \frac{p_i}{\sqrt{N_i(N_i+1)}}$.
  - If the number of seats needs to decrease, use $d_i^- = \frac{p_i}{\sqrt{N_i(N_i-1)}}$.
(5) Adjust the seats according to d.
  - If the number of seats needs to increase, assign a seat to the state with the largest d_i^+.
  - If the number of seats needs to decrease, remove a seat from the state with the smallest d_i^-.
(6) Repeat Steps 4 and 5 until the total number of seats is correct. The last d_i used is the adjusted divisor, d.

Example 5.2.
Let’s return to the school district that received 46 computers to distribute to 5 high schools based on the number of AP statistics students at each school. Use the Hill-Huntington method to distribute the computers.
$s = 560/46 \approx 12.174$

High School	Pop.	q	q^*	Rounded quota	d_i	HH App.
North	39	3.2036				
South	71	5.8321				
East	18	1.4786				
West	223	18.3179				
Cent.	209	17.1679				
Total	560	46				

$d =$

Example 5.3.

Let's return to the small towns who were working together to reorganize their county representation. There are 8 council seats. Use the Hill-Huntington method to distribute the seats among the towns.

$s = 9082/8 = 1135.25$

High School	Pop.	q	q^*	Rounded quota	d_i	HH App.
A	3862	3.402				
B	2818	2.438				
C	1881	1.657				
D	500	0.440				
E	21	0.019				
Total	9082	8				

$d =$

Example 5.4.

Let's return to the university's 100 student worker positions. Use the Hill-Huntington method to distribute the positions.

$s = 13097/100 = 130.97$

College	Pop.	q	q^*	Rounded quota	d_i	HH App.
J	5565	42.491				
K	3471	26.502				
L	3864	29.503				
M	197	1.504				
Total	13097	100				

$d =$

Example 5.5. A college has 5 departments to which it plans to apportion 12 new faculty positions based on the number of students in the department. The current number of students in each department is given below. Use the Hill-Huntington method to apportion the 12 faculty positions.
$s = 600/12 = 50$

Dept	Student.	q	q^*	Rounded quota	d_i	HH App.
A	174	3.48				
B	166	3.32				
C	121	2.42				
D	70	1.4				
E	69	1.38				
Total	600	12				

$d =$

6. Paradoxes of apportionment

6.1. **Alabama Paradox.**

Example 6.1. The high school that was distributing computers based on the number of AP statistics students had another computer donated, so there are now 47 computers to distribute. Use Hamilton's plan to distribute the computers.
$s =$

High School	Pop.	q	Rounded quota	Hamiltonian Apportionment.
North	39			
South	71			
East	18			
West	223			
Cent.	209			
Total	560			

Compare the distribution of 46 computers to the distribution of 47 computers.

High School	Pop.	Ham. App. of 46 computers	Ham. App. of 47 computers	App. Changes.
North	39	3		
South	71	6		
East	18	2		
West	223	18		
Cent.	209	17		
Total	560	46		

What seems odd about these results?

Definition 6.2.
A ________________________________ is a statement that is seemingly contradictory or opposed to common sense and yet is perhaps true.
The ________________________________ occurs when a state loses a seat as the result of an increase in the house size, with no change in any state's population.

The Alabama paradox was discovered in the apportionment based on the 1880 census. The Alabama paradox is possible with the Hamilton method, but not with the divisor methods.
What information tells you that the Alabama paradox occurred in this example?

6.2. **Population Paradox.** Consider two numbers, A and B, where $A > B$.

Definition 6.3. The ________________________________ between the two numbers is $A - B$
The ________________________________ between the two numbers is $\frac{A-B}{B} \cdot 100\%$

Example 6.4.
Compare the colleges' populations and make a case for any change in the distribution of student workers.

Colleges	Pop. Fall 2014	Pop. Fall 2015	Absolute Difference	Relative Difference
J	5565	5573		
K	3471	3481		
L	3864	3878		
M	197	198		

Use the Hamilton method to redistribute the positions for Fall 2015.
$s =$

Colleges	Pop. Fall 2015	q	Rounded quota	Hamilton App.
J	5573			
K	3481			
L	3878			
M	198			
Total	13130			

Complete this chart to document the appropriation change.

Colleges	Abs. Pop. Diff.	Rel. Pop. Diff.	Hamiltonian App. 2014	Hamiltonian App. 2015	App Changes
J	8	0.1438 %	42		
K	10	0.2881%	26		
L	14	0.3623 %	30		
M	1	0.5076 %	2		
Total			100		

Comment on the appropriation change in relation to the absolute and relative population differences.

Definition 6.5.
The ________________________________occurs when there is a fixed number of seats and a reapportionment causes a state to lose a seat to another state even though the percent increase in the population of the state that loses the seat is larger than the percent increase of the state that wins the seat.
The population paradox is possible with the Hamilton method, but not with the divisor methods.
What information tells you that the population paradox occurred in this example?

6.3. **New States Paradox.**

Definition 6.6.
The ________________________________occurs in a reapportionment in which an increase in the total number of states causes a shift in the apportionment of existing states.
The new states paradox was discovered in 1907 when Oklahoma joined the union. It is possible for this paradox to occur with the Hamilton method, but not with the divisor methods.

Example 6.7.
A pre-school received 20 picnic tables to distribute to two age level groups, the three-year olds and the four-year olds. Use the Hamilton method to distribute the tables.
$s =$

Age group	Population	q	Rounded quota	Hamiltonian App
3-year olds	71			
4-year olds	119			

Later a two-year old class was added that has 51 students. Five additional picnic tables were purchased for the additional students because 51/9.5 ≈5.38. Redistribute using the Hamilton method.
$s =$

Age group	Population	q	Rounded quota	Hamiltonian App
2-year olds	51			
3-year olds	71			
4-year olds	119			

What information tells you that the new states paradox occurred in this example?

6.4. **Quota Condition.**

Example 6.8.
A company will hire 200 new workers to work at one of the four facilities around the state. The new workers will be apportioned using the Jefferson method according to the current production levels at each facility. The location and production levels are given below.

$s =$

Facility	Production Level	q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
Q	12520						
R	4555						
S	812						
T	947						

Definition 6.9.
The ________________________________ says that the number assigned to each represented unit must be the standard quota, q, rounded up or rounded down.
What information tells you that the quota condition was violated in this example?
The quota condition can be violated by all of the divisor methods (Jefferson, Webster, and Hill-Huntington), but NOT by the Hamilton method.

6.5. **Comparing Method.**

Balinski and Young found that no apportionment method that satisfies the quota condition is free of paradoxes.

- Divisor methods are free of the paradoxes, but they can violate the quota condition.
- Hamilton's method may have paradoxes but does not violate the quota condition.

Example 6.10.

We have used each method to apportion multiple items. Let's look at the final apportionments.

High School	Population	q	Ham. App.	Jeff. App.	Web. App.	HH App.
Northside	39	3.2036	3	3	3	3
Southside	71	5.8321	6	6	6	6
Eastside	18	1.4786	2	1	1	2
Westside	223	18.3179	18	19	19	18
Central	209	17.1679	17	17	17	17
TOTAL	560	46	46	46	46	46

Town	Pop.	q	Ham. App.	Jeff. App.	Web. App.	HH App.
A	3862	3.4019	3	4	3	3
B	2818	2.4823	3	2	3	2
C	1881	1.6569	2	2	2	1
D	500	0.4404	0	0	0	1
E	21	0.0185	0	0	0	1
TOTAL	9082	8	8	8	8	8

Class	Pop.	q	Ham. App.	Jeff. App.	Web. App.	HH App.
J	5565	42.4906	42	43	42	42
K	3471	26.5023	26	26	26	27
L	3864	29.5029	30	30	30	29
M	197	1.5042	2	1	2	2
TOTAL	13097	100	100	100	100	100

Notice that although the total number of computers, seats, or student workers stayed constant, certain "states" had a vested interest in which apportionment method was used.

A town has 3 districts. The North district has a population of 98,000, the East district has a population of 26,000, and the South district has a population of 6,000. The total population is 130,000. Apportion 10 representatives using the Hamilton, Jefferson, Webster, and Hill-Huntington methods.

Hamilton	quota	
North	98,000/13,000 = 7.538	
East	26,000/13,000 = 2	
South	6000/13,000 = 0.461	

Jefferson	quota	
North	7.538	
East	2	
South	0.461	

Webster	quota	
North	7.538	
East	2	
South	0.461	

Hill-Huntington	quota	
North	7.538	
East	2	
South	0.461	