$\Rightarrow$ $f^{op}: Y \to X$ is iso $\iff$ $f^*: \mathcal{C}(Y, c) \to \mathcal{C}(X, c)$ bij. $\forall c \in \mathcal{C}$.

$\Updownarrow$

$f: X \to Y$ is iso.

done.

Def: 1.2.7. $f: X \to Y$ morphism is

(1) monomorphism if $\forall h, k: W \rightrightarrows X$ for some W,
mono (noun)
monic (adj)
$fh = fk \Rightarrow h = k$
(left cancellable)

(2) epimorphism if $\forall h, k: Y \rightrightarrows W$ for any W,
epi
epic
$hf = kf \Rightarrow h = k$
(right cancellable)

---

f is mono $\iff$ $f_*: \mathcal{C}(c, X) \to \mathcal{C}(c, Y)$ $\forall c \in \mathcal{C}$
injective

" epi $\iff$ " "
surjective

Ex 1.2.8 $f: X \to Y$ mono in **Set**. take $x \in X$

Let $\{x\} \underset{h}{\overset{g}{\rightrightarrows}} X$, suppose $fg = fh$. $\Rightarrow$ $g = h$

Thus f is injective.

Ex 1.2.9. (split epi / split mono)

$X \xrightarrow{s} Y \xrightarrow{r} X$ s.t. $rs = 1_X$. Then

s: section (right inverse)
r: retraction (retract) (left inverse)

(s is mono
r is epic.)

In this case,

s : split mono
r : split epi.

1.2.ii) (i) If f is split epimorphism, then $\exists\, s: Y \to X$ s.t. $fs = 1_Y$. Fix $c \in C$,

let $g \in C(c, Y)$. $\Rightarrow$ $sg \in C(c, X)$

$\Rightarrow f_*(sg) = fsg = 1_Y g = g$. surjective.

Conversely, take $c = Y$. $\exists\, s \in C(Y, X)$ s.t. $fs = 1_Y$. $\therefore$ f is split epi.

1.2. ii) ii) In C^{op}, apply (i); we get

$f^{op}: Y \to X$ is split epi $\Leftrightarrow$ $\forall c \in C^{op}$, $f^{op}_*: C^{op}(c, Y) \to C^{op}(c, X)$ is surj.

$\Updownarrow$ $\Updownarrow$.

$f: X \to Y$ is split mono. $\forall c \in C$, $f^*: C(Y, c) \to C(X, c)$ is surj.

1.2. iii) done.

1.2. iv) What are monomorphism of Field?

ans) Every morphism is monomorphism.

$E \overset{h}{\underset{k}{\rightrightarrows}} F \xrightarrow{g} G$. $\Rightarrow$ g is injective function, thus, $g(h(x)) = g(k(x)) \Rightarrow h(x) = k(x)$.

But maybe not all epi is surjection. (e.g. $G \to E \rightrightarrows E$ sending distinct roots of f but ~~not~~ fixing G.)

Ex 1.2. vi) Let $f: X \rightarrowtail Y$. $\exists g: Y \to X$ s.t. $fg = 1_X$. By Lemma 1.2.11 ii) f is epic.

Apply it on C^{op}. Then $f^{op}: Y \rightarrowtail X$, $g^{op}: X \to Y$ s.t. $f^{op} g^{op} = 1_X$ By above, f^{op} is iso.

$\Rightarrow$ in C, $f: X \twoheadrightarrow Y$, $g: Y \to X$ s.t. $gf = 1_X$.

Then f is iso.

Ex 1.2.11.

By def, $^{c}/C^{op}$: $Obj = \bigcup_{x \in Obj\, C} Hom_{op}(c, x)$

$$Mor_{op}(f^{op} : c \to x,\ g^{op} : c \to y)$$

$$= \left\{ h^{op} : x \to y \quad \text{s.t.} \quad \begin{tikzcd} & c \arrow[dl, "f^{op}"'] \arrow[dr, "g^{op}"] \arrow[d, phantom, "\circlearrowright"] & \\ x \arrow[rr, "h^{op}"'] & & y \end{tikzcd} \right\}$$

By duality, its opposite category is

$(^{c}/C^{op})^{op}$: $Obj = \bigcup_{x \in Obj\, C} Hom_{op}(c, x)$

$$Mor_{op,op}(f^{op} : c \to x,\ g^{op} : c \to y)$$

$$= \{ (h^{op})^{op} : (h^{op}) \in Mor(g^{op} : c \to y, f^{op} : c \to x) \}$$

(Forget about commuting diagram; it is just abstract opposite.)

Thus, $(h^{op})^{op} \in Mor_{op,op}(f^{op} : c \to x,\ g^{op} : c \to y)$

$$\Longleftrightarrow \quad h^{op} \in Mor_{op}(g^{op} : c \to y,\ f^{op} : c \to x)$$

$$\Longleftrightarrow \quad \begin{tikzcd} & c \arrow[dl, "f^{op}"'] \arrow[dr, "g^{op}"] \arrow[d, phantom, "\circlearrowright"] & \\ x & & y \arrow[ll, "h^{op}"] \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} & c & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & \arrow[u, phantom, "\circlearrowright"] & y \arrow[ul, "g"'] \end{tikzcd}$$

Thus by sending $g \longmapsto g^{op}$ for object,

$$h \longmapsto (h^{op})^{op}$$

we can identify C/c by $(^{c}/C^{op})^{op}$.

Ex 1.2.10 Not all epi or mono is surj or inj.

ex) $\boxed{1.2.v}$ $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ canonical inclusion.

f is monic since f is injective ($fh(r) = fk(r) \Rightarrow h(r) = k(r)$)

f is epic since ($h(1) = k(1) \Rightarrow h(b) \cdot h(\frac{1}{b}) = k(b) \cdot k(\frac{1}{b})$

and $h(\frac{1}{b}) = h(b)^{-1}$, $k(\frac{1}{b}) = k(b)^{-1}$

since $h(b) = k(b)$ by f, $h(b)^{-1} = k(b)^{-1}$

$\Rightarrow h(\frac{1}{b}) = k(\frac{1}{b})$)

But f is not ring isomorphism! it is not bijective

Lemma 1.2.11. Use $\rightarrowtail$ monic $\twoheadrightarrow$ epic.

(i) $f : x \rightarrowtail y$ $g : y \rightarrowtail z$ $\Rightarrow$ $gf \; x \rightarrowtail z$.

(ii) $f : x \rightarrow y$ $g : y \rightarrow z$ s.t. $gf : x \rightarrowtail z$

$\Longrightarrow f : x \rightarrowtail y$.

pf) (i) : obvious (ii). suppose $w \overset{h}{\underset{k}{\rightrightarrows}} x$ s.t. $fh = fk$.

Then $gfh = gfk \Rightarrow h = k$.

Dually,

(i) $f^{op} : y \twoheadrightarrow x$. $g^{op} : z \twoheadrightarrow y$ $\Rightarrow$ $f^{op} g^{op} : z \twoheadrightarrow x$.

(ii) $f^{op} : y \rightarrow x$ $g^{op} : z \rightarrow y$ and $f^{op} g^{op} : z \twoheadrightarrow x$.

$\Rightarrow f^{op} : y \twoheadrightarrow$.

Delete op.

Exercise 1.2 vii) Define sup, inf on Poset $(P, \leq)$ Categorical sense, i.e. dual statement define inf.

Def : $x \in P$ is sup if $\forall y \in P$, $y \rightarrow x$ exists.

(" inf " $x \rightarrow y$ ")

If x, y are sup, $x \rightleftarrows y$ $\Rightarrow$ by anti symmetry $x = y$.

functoriality.

~~Def~~. 1.3.1. $F: C \longrightarrow D$ functor (morphism of category)

① $Fc \in D \quad \forall c \in C$

② $Ff : Fc \longrightarrow Fc' \in D. \quad \forall f: c \to c' \in C.$

Satisfying "functoriality axioms"

a) f, g composable pair, $Fg\,Ff = F(gf)$

b) $\forall c \in C$, $F(1_c) = 1_{Fc}$.

Ex 1.3.2

(1) $P: Set \longrightarrow Set \qquad A \longmapsto PA \qquad (f: A \to B) \longmapsto f_*: \underset{\subset A}{A'} \longmapsto \underset{\subset B}{f(A')}$

(2) Forgetful functor for concrete categories

$U: Group \to Set \qquad \underset{\text{as gp}}{G} \longmapsto \underset{\text{as a set}}{G}. \qquad \underset{\text{as hom}}{f} \longmapsto \underset{\text{as function}}{f}$

Similar for Ring, Field, Top, ...

$U, E: Graph \to Set \qquad G \longmapsto V(G), \ E(G) \qquad f: G \to H \longmapsto \begin{pmatrix} V(G) \to V(H) \\ E(G) \to E(H) \end{pmatrix}$

$V \sqcup E$: " $\qquad V(G) \underset{\text{dis}}{\sqcup} E(G) \qquad V(G) \sqcup E(G) \to V(H) \sqcup E(H)$

(3) Another forgetful functors

$$\begin{array}{ccccc} Mod_R & \longrightarrow & Ab & \hookleftarrow & Group \\ & \nearrow & & & \\ Field \hookrightarrow Ring & & & & \end{array}$$

(4) Ring $\longrightarrow Set_*$

Group $\longrightarrow$

$R \longmapsto (R, e) \qquad e$: identity

functorial because homomorphism preserves identity.

(5). $Top \longrightarrow Htpy$ ($Top_* \longrightarrow Htpy_*$)

$X \longmapsto X$ base pt preserving...

$f \longmapsto \tilde{f}$ homotopy class of cts function.

(6) $\pi_1 : Top_* \longrightarrow Group$ $X \longmapsto \pi_1(X)$

$f : (X, x) \longrightarrow (Y, y) \longmapsto f_* : \pi_1(X, x) \to \pi_1(Y, y)$

"Fundamental group is homotopy invariants"

$\Rightarrow \exists$ functor $Htpy_* \longrightarrow Group$ s.t.

$\pi_1 = Top_* \longrightarrow Htpy_* \longrightarrow Group$.

(7). $\Pi_1 : Top \longrightarrow Groupoid$. $X \mapsto \Pi_1(X)$, $f \longmapsto f_*$

since cts preserves path and path homotopy.

(8). $\forall n \in \mathbb{Z}$, $\begin{matrix} Z_n \\ B_n \\ H_n \end{matrix} : Ch_R \rightrightarrows Mod_R$

n-cycle $Z_n C_\cdot = \ker(d : C_n \longrightarrow C_{n-1})$

n-bdy $B_n C_\cdot = \ker(d : C_{n+1} \to C_n)$

n-homology $H_n C_\cdot = Z_n C_\cdot / B_n C_\cdot$

If you collect ~~for~~ for all $n \in \mathbb{Z}$,

$\begin{matrix} Z_* \\ B_* \\ H_* \end{matrix} : Ch_R \rightrightarrows Gr\,Mod_R$ graded R-module

ex) $Z_* : C_\cdot \longmapsto \bigoplus_{i \in \mathbb{Z}} Z_i C_\cdot$

Singular homology of Top space

$Top \longrightarrow Ch_R \xrightarrow{H_*} Gr\,Mod_R$

(ix) F : Set $\longrightarrow$ Group.

$X \longmapsto$ Free gp gen by X. $\quad f \longmapsto$ induced gp homo.

example of "free functor".

(x) Euclid_* Obj: $(\mathbb{R}^n, a)$

Mor: $f: (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$ differentiable.

Consists of $f: \mathbb{R}^n \to \mathbb{R}^m$ "

and $f(a) = b$.

Let $D: \text{Euclid}_* \longrightarrow \text{Mat}_{\mathbb{R}}$

$(\mathbb{R}^n, a) \longmapsto n$.

$f: (\mathbb{R}^n, a) \to (\mathbb{R}^m, b) \longmapsto n \xrightarrow{df_a} m$

df_a: Jacobian $m \times n$ matrix

(Say correction on Mat_R :)

It satisfy functoriality axiom due to chain rule.

(xi). Fin_* : Obj : $(\{x_1, \cdots x_n, x_{n+1}\}, x_i)$

Mor: function preserving basept.

$M^- : \text{Fin}_* \longrightarrow \text{Set}$

Let $n_+ := ([n] \cup \{a\}, \{a\})$ (M : Commutative monoid)

$M^{n_+} = M^{\times n}$ cartesian product (M^{0_+} : Singleton)

$+, \quad M^f : M^m \longrightarrow M^n$

$$(a_1, \dots, a_m) \longmapsto (b_1 \dots b_n)$$

where $b_i = \begin{cases} \prod_{j \in f^{-1}(i)} a_j & \text{if } f^{-1}(i) \neq \phi \\ 1 & \text{if empty} \end{cases}$

Then, M^f preserves unit. So,

$$m_+ \to n_+ \quad n_+ \quad Fin_* \xrightarrow{M^-} Set$$

M^n

Set_*

$M^m \longrightarrow M^n$

$\longmapsto (b_i)_i^n \quad (M^n, (1, \dots, 1))$

$= \left(\prod_{j \in f^{-1}(i)} a_j \right)$

Segal ?4

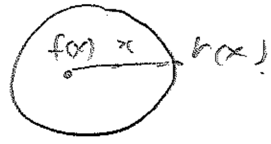
: Cohomology on some suitable category

(Alg. K-theory from Quillen).

3.3 (Brouwer fixed pt Theorem)

Any cts endo $f : D^2 \longrightarrow D^2$ has a fixed pt.

) Let $r : D^2 \longrightarrow S^1$ by



r is cts and $i : S^1 \longrightarrow D^2$ is inclusion

$\Longrightarrow \; r i = 1_{S^1}$ $\quad$ r: split epi (retract) $\quad$ i: split mono (section)

Since π_1 is functor, $Top_* \longrightarrow Group$,

$$\pi_1(S_1, x) \xrightarrow{\pi_1(i)} \pi_1(D^2, x) \xrightarrow{\pi_1(r)} \pi_1(S^1, x)$$