Natural Transformation

Ex). $(-)^* \circ (-)^*$ where $(-)^* : \mathrm{Vect}_k^{op} \to \mathrm{Vect}_k$
$V \mapsto V^*$

!Induces! $V \cong V^{**}$

but this iso comes from ev_v $\forall v \in V$.
No unnatural choice of basis is needed.

Def 1.4.1. (Natural Transformation)

C, D: categories $F, G : C \rightrightarrows D$

α: Natural Transformation $F \Rightarrow G = \{\alpha_c : c \in C\}$

s.t. $\alpha_c : F_c \to G_c \in \mathrm{Mor}\, D$. $\forall c \in C$

↑ component of α.

$$\begin{tikzcd} F_c \arrow[r] \arrow[d, "Ff"'] \arrow[dr, phantom, "\circlearrowright"] & G_c \arrow[d, "Gf"] \\ F_{c'} \arrow[r] & G_{c'} \end{tikzcd}$$

s.t. $\forall f : c \to c' \in \mathrm{Mor}\, C$

commutes

Natural isomorphism: α: natural transformation
s.t. α_c is iso $\forall c \in C$.

Then $\alpha : F \cong G$.

"The arrows are natural" $\Rightarrow$ collection of arrows define natural transf.

$$\begin{tikzcd}
C \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & D
\end{tikzcd}$$

Ex 1.4.3

(i) $1_{Vect_k} \overset{ev}{\Longrightarrow} (-)^* \cdot (-)^*$

since $\forall \phi : V \to W$ morphism in $Vect_k$

$$\begin{tikzcd}
v \arrow[d, mapsto] \arrow[rrr, mapsto, bend left] & V \arrow[r, "ev"] \arrow[d, "\phi"'] & V^{**} \arrow[d, "\phi^{**}"] & f_v : g \in V^* \mapsto g(v) \arrow[d, mapsto, "\phi^{**}(W^* \to k) = V^* \xrightarrow{\phi^*} W^* \to k"] \\
\phi(v) \arrow[rrr, mapsto, bend right] & W \arrow[r, "ev"'] & W^{**} & \phi^{**}(f_v)
\end{tikzcd}$$

$f_{\phi(v)} : g \in W^* \mapsto g(\phi(v))$

$$\left.\begin{array}{lll}
\phi^{**}(f_v) & := & W^* \xrightarrow{\phi^*} V^* \xrightarrow{f_v} k \\
 & & g \longmapsto g \circ \phi \longmapsto g \circ \phi(v) \\
f_{\phi(v)} & : & g \longmapsto g(\phi(v))
\end{array}\right\} \text{Same.}$$

(ii) $1_{Vect_k} \not\Rightarrow (-)^*$.

Since I_{Vect_k} : Covariant

$(-)^*$: Contravariant.

More significantly $V \cong V^*$ cannot be defined without choice of basis (...) which is not preserved by any nonidentity linear endomorphism.

$$A \longmapsto P(A)$$

iii) $1_{Set} \Rightarrow P$. $\quad P, 1_{Set} : Set \rightrightarrows Set$.

$$A \longmapsto A$$

s.t.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & P(A) \arrow[d, "f"] \\ B \arrow[r, "\eta_B"'] & P(B) \end{tikzcd}$$

by $\eta_A : A \to P(A)$
$a \longmapsto \{a\}$

$$\bullet \longmapsto X$$

(iv). $X, Y : BG \rightrightarrows C$ $\quad$ What is natural transf? α.

$$\bullet \longmapsto Y$$

$\Rightarrow \alpha : X \to Y$ a single morphism in C.

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "g_*"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g_*"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

s.t. left diagram commutes

We say α is "G-equivariant".

(V) $\mathcal{O} : \mathrm{Top}^{op} \Longrightarrow \mathrm{Set}$ by $(\ )^c$, taking complement
$\mathcal{C}$

i.e., $\mathcal{O} \overset{(\ )^c}{\Longrightarrow} \mathcal{C}$

To see this, any cts $f^{-1}: Y \to X \in \mathrm{Top}^{op}$.

$$\begin{tikzcd}
U \arrow[dd, mapsto] & \mathcal{O}(Y) \arrow[r, "(\ )^c"] \arrow[d, "f^{-1}"'] & C(Y) \arrow[d, "f^{-1}"] & U^c \arrow[dd, mapsto] \\
& \mathcal{O}(X) \arrow[r] & C(X) & \\
f^{-1}(U) \arrow[rr, mapsto, "(\ )^c"] & & (f^{-1}(U))^c & f^{-1}(U^c)
\end{tikzcd}$$

Then, $(f^{-1}(U))^c = X - f^{-1}(U) = f^{-1}(U^c)$.

Also, It is natural iso since $(\ )^c$ is bijection.

(VI) $(-)^{op}: \mathrm{Group} \to \mathrm{Group}$

$G \longmapsto G^{op} : \left(\begin{array}{l} \text{obj} : \bullet \\ \text{Mor} := G \text{ s.t.} \\ g^{op} \cdot h^{op} = (hg)^{op} \end{array} \right.$

Thus $\phi : G \to H$ induces $\phi^{op} : G^{op} \to H^{op}$ by $g \mapsto \phi(g)$.

Let $\eta_G : G \longrightarrow G^{op}$
$g \longmapsto g^{-1}$

it is not automorphism $G \to G$
but it is homo $G \to G^{op}$

since $gh \longmapsto (gh)^{-1} = (h^{-1} \cdot g^{-1})^{op} = \eta_G(g) \cdot \eta_G(h)$
(since multiple order is deversed.

Thus

$$\begin{tikzcd}
g \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & g^{-1} \arrow[ddd, mapsto] \\
& G \arrow[r, "\eta_G"] \arrow[d, "\phi"'] & G^{op} \arrow[d, "\phi^{op}"] & \\
& H \arrow[r, "\eta_H"'] & H^{op} & \\
\phi(g) \arrow[rrr, mapsto] & & & \phi(g)^{-1} \quad \phi(g^{-1})
\end{tikzcd}$$

and $\phi(g^{-1}) = \phi(g)^{-1}$ in H.
$\Rightarrow \phi(g^{-1}) = \phi(g)^{-1}$ in H^{op}.

(VII). $F : Vect_k \longrightarrow Vect_k$
$V \longmapsto V \otimes V$.

Then,

$$\begin{tikzcd}
V \arrow[r, "\eta_V"] \arrow[d, "\phi"'] & V \otimes V \arrow[d, "\phi \otimes \phi"] \\
W \arrow[r, "\eta_W"'] & W \otimes W
\end{tikzcd}$$

commutes, when $\eta_V = 0$ $\forall V \in Vect_k$.

However, there is no basis independent definable lin. map $V \to V \otimes V$.

Ex) Ab_{fg} : Category of fin. gen. ab gp.

for given ab gp A, let TA : torsion subgp of A.

By classification Thm of ab gp,

$$A \cong TA \oplus (A/TA)$$

Prop 1.44. $A \cong TA \oplus (A/TA)$ are not natural in $A \in Ab_{fg}$.

pf) If it were natural,

$$\alpha : A \twoheadrightarrow A/TA \rightarrowtail TA \oplus (A/TA) \cong A$$

acting as a natural endomorphism of the $1_{Ab_{fg}}$, i.e. $1_{Ab_{fg}} \xrightarrow{\alpha} 1_{Ab_{fg}}$.

Claim 1 : If η : natural endomorphism of $1_{Ab_{fg}}$

$\Rightarrow \eta : x \longmapsto nx$ for some $n \in \mathbb{Z}$

pf). Since $\eta_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ is multiple by $n \in \mathbb{Z}$.

and $\forall A \in Ab_{gp}$ with map $\mathbb{Z} \xrightarrow{a} A$, $1 \longmapsto a$.

$$\begin{tikzcd}
x \arrow[r, mapsto] & nx \\
\mathbb{Z} \arrow[r, "\eta_{\mathbb{Z}}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z} \arrow[d] \\
A \arrow[r, "\eta_A"'] & A \\
ax \arrow[r, mapsto] & anx
\end{tikzcd}$$

$\eta_A(ax) = x\eta_A(a)$

By choosing $x=1$, $\eta_A(a) = na$.

Thus, $\eta_A : A \longrightarrow A$
$a \longmapsto na$. $\square$

Now, if $A = \mathbb{Z}$, $\alpha_{\mathbb{Z}} : a \longmapsto na$ by claim. Then $\alpha : A \twoheadrightarrow A/TA = A \to A \xrightarrow{\cong} A$

is iso, thus, $n \neq 0$.

If $A = \mathbb{Z}/2n\mathbb{Z}$, $TA = A$, thus

$$\alpha_{\mathbb{Z}/2n\mathbb{Z}} : A \longrightarrow 0 \longrightarrow A \longrightarrow A$$

$\therefore$ zero map.

But $n \neq 0$ in $\mathbb{Z}/2n\mathbb{Z}$ thus

$$1 \longmapsto n$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\alpha_{\mathbb{Z}}"] \arrow[d, "\pi"'] & \mathbb{Z} \arrow[d, "\pi"] \\ \mathbb{Z}/2n\mathbb{Z} \arrow[r, "\alpha_{\mathbb{Z}/2n\mathbb{Z}}"'] & \mathbb{Z}/2n\mathbb{Z} \end{tikzcd}$$

$$1 \longmapsto 0.$$

$n \neq 0$,

doesn't commute, contradiction.

Ex 1.4.6. (Riesz Representation Thm)

$\mathcal{I} : \mathrm{cHaus} \longrightarrow \mathrm{Ban}$

cpt Hausdorff   Banach space

$$\begin{array}{ccccc} X & \longmapsto & \mathcal{I}(X) := \{\text{signed Baire measure}\} & & \mu \\ f \downarrow & \longmapsto & \downarrow f^{-1} & & \downarrow \\ Y & \longmapsto & \mathcal{I}(Y) & & \mu \circ f^{-1} \end{array}$$

is a functor.

$C^* : \mathrm{cHaus} \longrightarrow \mathrm{Ban}$

$X \longmapsto C(X)^*$: dual of the Banach space $C(X)$ of cts realvalued function on X.

For each $\mu \in \mathcal{I}(X)$, $\phi_\mu : C(X) \longrightarrow \mathbb{R} \in C(X)^*$.

$$g \longmapsto \int_X g \, d\mu$$

Let $\eta : \mathcal{I} \Rightarrow C^*$ as

$$\begin{tikzcd}
\mu \arrow[rrr, mapsto] \arrow[ddd] & & & \phi_\mu : \arrow[ddd] \\
& \Sigma(X) \arrow[r, "\eta_X"] \arrow[d, "(-\circ f^{-1})"'] & C(X)^* \arrow[d, "(-\circ f)"] & \\
& \Sigma(Y) \arrow[r, "\eta_Y"'] & C(Y)^* & \\
\mu\circ f^{-1} \arrow[rr, mapsto] & & \phi_{\mu\circ f^{-1}} & \phi_\mu \circ f
\end{tikzcd}$$

s.t. $\phi_{\mu\circ f^{-1}}(g) = \int_Y g \, d(\mu\circ f^{-1})$, $\phi_\mu \circ f = \int_X g\circ f \, d\mu$.

But from real analysis, they are equal.

thus $\eta : \mu \longmapsto \phi_\mu$ is natural transf of $\Sigma \Rightarrow C^*$.

RRT: $\boxed{\text{This is natural iso.}}$

Ex 1.4.7. $\mathcal{C}$: locally small

$f : w \to z$, $h : y \to z$.

$$\Rightarrow \quad \begin{tikzcd}
C(x,y) \arrow[r, "h\cdot -"] \arrow[d, "-\cdot f"'] \arrow[dr, phantom, "\circlearrowright"] & C(x,z) \arrow[d, "-\cdot f"] \\
C(w,y) \arrow[r, "h\cdot -"'] & C(w,z)
\end{tikzcd}$$

where $h\cdot - = h_*$
$- \cdot f = f^*$
(post and pre composition)

Since $C(-,y)$, $C(-,z)$ are functors,

$$h_x : C(-,y) \Rightarrow C(-,z)$$

is natural transformation (n.t.)

Similarly,

$$f^* : C(x,-) \Rightarrow C(w,-) \quad \text{n.t.}$$

Ex 1.4.9. For $A, B \in$ Sets. $A+B :=$ disjunion of A, B.

$A^B := \{B \to A\}$

$$\Rightarrow \quad A \times (B+C) \cong (A \times B) + (A \times C), \quad (A \times B)^C \cong A^C \times B^C$$

$$A^{B+C} \cong A^B \times A^C \qquad (A^B)^C \cong A^{B \times C}$$

Actually, this is natural iso between

Set × Set × Set → Set functors

or (contravariant)

Now, restrict this natural iso in $\mathrm{Fin}_{\mathrm{iso}}$,

(Obj : finite set Mor : bijection)

Then let $|-| : \mathrm{Fin}_{\mathrm{iso}} \longrightarrow \mathbb{N}$ be functor.

(ℕ : discret category of natural #)

$$\begin{array}{ccc} A & \longmapsto & |A| \\ f \downarrow & & \downarrow 1_{|A|} \\ B & \longmapsto & |B| = |A| \end{array}$$

Then this induces

$$a(b+c) = ab + ac, \quad (ab)^c = a^c b^c$$

$$a^{b+c} = a^b a^c \qquad (a^b)^c = a^{bc}$$

a laws of mult and add in $\mathbb{N}$.

decategorification of Fin_{iso} $\Rightarrow$ $\mathbb{N}$ and these laws.

Categorification of $\mathbb{N}$ = Fin_{iso}