BLAH

0.1 TODO:

Title page Introduction WRITE ALL THE SOLUTIONS FAK Somehow synchronize with the book tex so things don't blow up when Evan permutes chapters

Implement solutions with tl;dr and actual solution?????

1 Chapter 1

1A. A finite group clearly does not have a subgroup isomorphic to itself, because the two groups have different orders. Therefore, only infinite groups may have subgroups isomorphic to themselves. □

1B. Let $g \in G$ be any element of the group, and let $g_1, g_2, \dots, g_{|G|}$ be an enumeration of the elements of G. Recall that left-multiplication by g permutes the elements of G. Therefore,

$$g_1 g_2 \cdots g_{|G|} = (gg_1)(gg_2)\cdots(gg_{|G|}).$$

Since G is abelian, we can rearrange the right-hand side and cancel $g_1 g_2 \cdots g_{|G|}$, leaving $g^{|G|} = 1$. □

1C. Observe that D_6 and S_3 both represent the symmetries of a triangle, and are therefore isomorphic.

The group D_{24}, which represents the symmetries of a regular dodecagon, has an element of order 12. Since S_4 does not have an element of order 12, the groups cannot be isomorphic. □

1D. By Lagrange's Theorem, all elements have order dividing p; thus, all elements have order 1 or p. Since exactly one element, the identity, has order 1, there is at least one element $g \in G$ with order p. Then, the elements $1, g, g^2, \dots, g^{p-1}$ are distinct, so G is isomorphic to $\mathbb{Z}_p$. □

1E. a) Let $G = \{g_1, \dots, g_{|G|}\}$. By the cancellation law, left-multiplication by any element $g_i \in G$ permutes the set $\{g_1, \dots, g_{|G|}\}$, and we may identify g_i with the corresponding permutation of $\{1, \dots, |G|\}$. Left-multiplying the set $\{g_1, \dots, g_{|G|}\}$ by $g_i, g_j \in G$ is the same as left-multiplying by g_j, then by g_i, so multiplication in G corresponds to composition in $S_{|G|}$. This is an isomorphism.

b) By (a), G is isomorphic to some subgroup of $S_{|G|}$. So, G is also isomorphic to the group of corresponding permutation matrices in $GL_{|G|}(\mathbb{R})$.

1F. Recall that S_3 is defined as the group generated by x, y under the relations $x^3 = y^2 = 1$, $yx = x^2 y$.

We identify each white chip with x and each black chip with y, so the row of chips, read left to right, is a product of x's and y's. Observe that

$$\begin{aligned} xxx &= yy \\ xxy &= yx \\ yxx &= xy \\ yxy &= xx. \end{aligned}$$

Hence, the product of the row of chips is invariant under the given operation.

If $n \equiv 1 \pmod 3$, the product of the chips is $x^n = x$. So, if it is possible to reach a state with two chips, the product of those chips must be x.

Observe that the number of black chips is always even. Thus, if two chips remain in the end, they must be both black or both white.

But, $xx = x^2 \neq x$ and $yy = 1 \neq x$. This is a contradiction. □

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12 Chapter 12

TODO FIX THIS IDIOTIC FORMATTING

12A.

Let R be a finite integral domain, and let $r \in R$ be nonzero.

We claim that the map $R \to R$ given by $x \mapsto rx$ is a permutation of R. Indeed, if $rx_1 = rx_2$ for distinct $x_1, x_2 \in R$, then $r(x_1 - x_2) = 0$, contradicting that R is an integral domain.

This means there exists x such that $rx = 1$, so r is a unit. Since this is true for all r, R is a field. ☐

12B.

a) If J is maximal, we're obviously done. Otherwise, J is contained in a proper ideal J_1. Then, if J_1 is maximal, we're done. Else, J_1 is contained in a proper ideal J_2. Continuing in this manner we get a chain $J \subsetneq J_1 \subsetneq J_2 \subsetneq \dots$. Since R is Noetherian, this chain has to stop - and it can only stop at a maximal ideal. ☐

b) Consider the poset whose elements are proper ideals of R containing J. Every chain $J_0 \subset J_1 \subset \dots$ has an upper bound $\bigcup J_i$. By Zorn's Lemma, there is a local maximum, which is a maximal ideal. ☐

12C.

12D.

Let R be a Noetherian ring. Suppose for contradiction that some ideal $J \subset R[x]$ requires infinitely many generators. We can then find an infinite sequence of non-redundant generators $\{p_i\}_i$ (i.e. p_k is not in the ideal generated by $p_1, \dots, p_{k-1}$) of J.

We initialize the ideal $I \subset R[x]$ as the zero ideal and add generators from $\{p_i\}_i$ one by one. We will show that eventually we can't add any more non-redundant generators.

For $n \in \mathbb{Z}_{\geq 0}$, define

$$C_{I,n} = \{k \in \mathbb{Z} | I \text{ contains a polynomial of degree } n \text{ and leading coefficient } k.\}$$

Observe that:

- $C_{I,n}$ is an ideal of R;
- $C_{I,n} \subset C_{I,n+1}$ for all n.

The two remaining observations we need are:

Proposition 12.1. *Adding a non-redundant generator to I will grow at least one ideal $C_{I,n}$*

Proof. Suppose we add p_k to $I = \langle p_1, \dots, p_{k-1} \rangle$. Since $p_k \notin I$, we can mod out p_k by I until we reach a polynomial p_k' such that no polynomial in I matches p_k's degree and leading coefficient. Adding p_k to I is equivalent to adding p_k'; thus when we add p_k to I we grow the ideal $C_{I,\deg p_k'}$. ☐

Proposition 12.2. *The collection of ideals $\{C_{I,n}\}_n$ can only grow a finite number of times.*

Proof. This follows from the fact that R is Noetherian and the condition $C_{I,n} \subset C_{I,n+1}$. ☐

So, after adding finitely many generators, we get stuck. ☐

12E.

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