Robust Portfolio Construction with Volatility Filtering
CVA Calculation

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1 CVA

In this section, calculation of Credit Valuation Adjustment (CVA) is done for an interest rate swap. To achieve this, Monte Carlo simulations for the Libor rate are done using the HJM model that evolves the forward curve using the same-form SDE. Different sensitivity analysis are done in order to explore the robustness of the results. Python is used for implementation.

1.1 Introduction: CVA for Interest Rate Swap

Credit Valuation Adjustment (CVA) is a form of "adjustment" or "correction" that must be applied to the price of a financial instrument to take in account counterparty risk.

$$Price_{adjusted} = Price_{non-adjusted} - CVA$$

In particular, let's consider a plain vanilla interest rate swap (IRS) between two counterparties. In this case, the price or *mark to market* (MtM) for this derivative needs to consider the likelihood of the counterparty entering in default, hence there is a CVA associated to this instrument that will depend on the *creditworthiness* of the counterparties.

In this example, let's suppose counterparty A is entering in an interest rate swap contract with counterparty B, hence A will calculate the CVA that will need to take in account given counterparty B creditworthiness[1]. The first step is to calculate the MtM for the IRS not only today, but also the expected MtM for this instrument in the future. In particular, A will be particularly interested in the cases in which the derivative moves *in the money* for counterparty B, as those are the cases in which B has a liability with A. This will define the *expected exposure* (EE) A will have on B due to this instrument. Formally:

$$EE_t = \mathbb{E}[Max(MtM_t, 0)]$$

Now intuitively, CVA accounts for the expected or potential loss counterparty A has on the IRS with counterparty B. Expected loss can be written as:

$$Loss = (AmountLoss)x(Probability of Default)x(DiscountFactor)$$

Which can be written as:

$$L(t) = (1 - R)EE(t) * PD(t) * DF(t)$$

With R being the Recovery Rate, EE(t) the expected exposure at time t, PD(t) the probability of default at time t, and DF(t) the correspondent Discount Factor as the loss happens in the future. Now, CVA can be written as the expected loss:

$$CVA = \mathbb{E}[(1 - R)EE(t) * PD(t) * DF(t)]$$

$$CVA = \int_0^T (1 - R)EE(t) * DF(t) * dPD(t)$$

Where: $dPD(t) = \Delta PD(t) = PD(t_{i-1}, t_i) = P(0, t_{i-1}) - P(0, t_i)$, as for the integral, the probability of default of counterparty B in that particular period $PD(t_{i-1}, t_i)$ is needed.

So, R, EE(t), DF(t) and PD(t) are needed for CVA calculation. In particular:

- R: In this case 40% is assumed.
- PD(t): Obtained by implied market spreads or CDS.

[1] Assumption is made that the derivative is uncollateralized

- EE(t) and DF(t): Evolution for interest rates in the future is needed. For this, there are two main approaches for the calculation:
  - Static: It is assumed that the evolution for interest rates is what it is implied in the actual spot curve. So forward rates are obtained from the actual spot curve, and then EE(t) and DF(t) are calculated.
  - Dynamic: A model for interest rates is built. Then simulations are made using this model, obtaining different exposures and discount factors for each simulation. EE(t) is then calculated using different indicators such as the mean, median or 97.5th percentile from the simulations.

In this case, a Dynamic approach is used. For this, the Heath Jarrow Morton (HJM) framework is used to model the evolution of forward rates, from which Monte Carlo simulations are drawn.

1.2 HJM model

For this exercise, forward rates for tenor j are simulated using the following Heath Jarrow Morton (HJM) framework considering Musiela Parametrisation[2] for τ:

$$\bar{f}_j(t+dt) = \bar{f}_j(t) + d\bar{f}_j$$

$$d\bar{f}(t,\tau) = \bar{m}(t,\tau)dt + \sum_{i=1}^{3} \bar{v}_i(t,\tau)dX_i + \frac{\bar{f}(t,\tau)}{\partial \tau}dt$$

$$\bar{v}_i(t,\tau)dX_i = \sqrt{\lambda_i} e_\tau^i \phi_i \sqrt{\delta t}$$

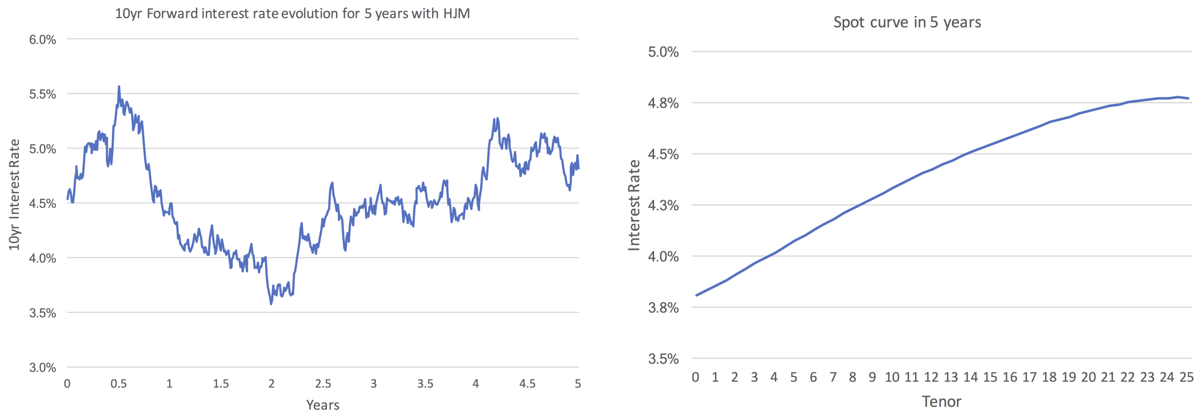
Principal Component Analysis is used to calibrate volatility functions $\bar{v}_i$, which implement a linear decomposition of changes in interest rates at each tenor. Such approach provides systematic factors that describe movement of the curve as a whole. Also, factor attribution is well established for yield curve analysis, where Parallel shifts, Steepening/Flattening, and Curvature are normally identified on the first three principal components, which normally explains more than 90% of the yield curve variation.

Drift $\bar{m}(t,\tau)$ is calculated using numerical integration over the fitted volatility functions (trapezium rule is used), where:

$$m(t,\tau) = v_i(t,\tau) \int_t^\tau v_i(t,s)ds$$

In this case, calibration was done using instantaneous forward rates data provided by BOE with data pre-January 2007 regime. Forward curve is modeled in 6 months increment, up to 25 years of maturity. The following graph shows the 5 year evolution of the 10yr forward interest rate done by the calibrated HJM model:

[2] This corrects for the evolution for the forward rate curve as simulations move forward in time: $\tau = T - t$



(a) Evolution of 10yr forward rate

(b) Spot curve in 5 years

Figure 1: Evolution of HJM model

1.3 Interest Rate Swap MtM

An interest rate swap is a derivative contract in which two counterparties agrees an exchange of payments benchmarked against an interest rate index. The most common IRS is a fixed for floating swap, in which one party will make payments to the other based on an initially agreed fixed rate of interest, to receive back payments based on a floating interest rate index. Each of these series of payments is termed a 'leg', so a typical IRS has both a fixed and a floating leg.

This type of plain vanilla swaps, like most derivative instruments, have zero value at initiation. However, this value changes over time, due to changes in factors affecting the value of the underlying rates. To calculate the MtM it's necessary to calculate the Present Value (PV) of each leg (fixed and float). The difference between them corresponds to the MtM:

- PV of Fixed Leg: Discount known future cash flows at the correspondent Discount Factors.
- PV of Floating Leg: Infer from the market term structure the expected value for the forward floating leg. Then discount this values at the correspondent Discount Factors. Also it's possible to build models to simulate the evolution of interest rates in the future, and therefore extract the expected forward floating leg form this simulations.

In this exercise, as mentioned before, Monte Carlo simulations are made with an HJM model to simulate de forward interest rate evolution. Therefore, there will be a MtM for each simulation.

1.4 Probability of Default

Probability of default may be bootstrapped from the CDS market. In particular, modeling default times as the first jump of a poisson process with parameter λ, gives as result the following survival probability between times t and T:

$$P(t,T) = exp(-\lambda(T-t))$$

Or in a more general form, it can be defined $\lambda(t)$ as a non-negative function of time, then:

$$P(t,T) = exp\left(-\int_t^T \lambda(s)ds\right)$$

Now, in particular, for a piecewise term structure of $\lambda(t)$:

$$P(T_n) = exp\left(-\sum_{1}^{n} \lambda_k \Delta t\right) \tag{1}$$

Also, by non arbitrage principle, the fair spread quote S_n for the N-period CDS is:

$$S_N = \frac{(1-R)\sum_{n=1}^{N} D(0,T_n)(P(T_{n-1}) - P(T_n))}{\sum_{n=1}^{N} D(0,T_n)P(T_n)\Delta t_n} \tag{2}$$

Where R is the recovery rate and $D(0, T_n)$ the discount factor of tenor T_n. Using equations 1 and 2 is possible to bootstrap probability of defaults using "observable" market CDS prices and discount factors.

1.5 Implementation

For CVA calculation, a 5 year IRS is modeled, with notional of 1 written on 6M Libor. It is assumed that the counterparty wants to pay the fixed leg and receive the floating leg. 6m Libor is modeled by the HJM framework. In order to obtain the proper OIS curve for discount factors, a constant LIBOR-OIS spread of 80bps is assumed. Python code for implementation is provided. Pseudo-code is as follows:

1. Function `ExpExposure` calculates expected exposure for the IRS. For this, the FRA matrix is simulated with the HJM model (defined as `SimFRA` in the code), using the starting FRAs (t_0) and volatility functions in *HJM Model MC.xlsm* (defined as functions, `Vol_1`, `Vol_2` and `Vol_3`). For the drift, numerical integration is coded (defined as `M`).

2. Once the FRA matrix is created, expected exposure is calculated. For this, the correspondent 6m forward Libor rate is obtained from the FRA matrix and calculate: $(forwardlibor - fixedrate) * Notional * dt * DF$. This is done for each tenor while moving forward in time step. Every step moving forward, means the IRS has less time to maturity. The discount factor is also calculated, substracting 80bps to the Libor rate and integrating in the forward curve.

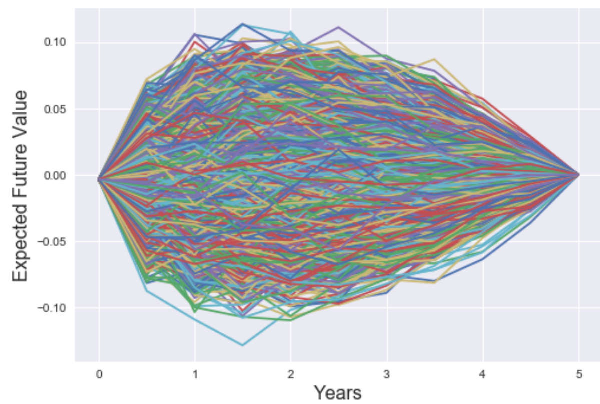
3. Points 1 and 2 are iterated N times for different draws of the Monte Carlo simulation. Therefore, N expected exposure profiles are obtained.

4. Once the expected exposure is obtained, CVA is calculated using the function `FCVA`. This calculates the area under the expected exposure curve using the center of the "Riemann sum rectangle" for each tenor. For this, the average expected exposure between two tenors is used, and discount factors are interpolated using: $ln(DF(0,\tau)) = \frac{\tau-\tau_i}{\tau_{i+1}-\tau_i} ln(DF(0,\tau_{i+1})) + \frac{\tau_{i+1}-\tau}{\tau_{i+1}-\tau_i} ln(DF(0,\tau_i))$, for $\tau_i < \tau < \tau_{i+1}$. Probability of defaults are calculated given by a term structure of hazard rates. Then $EE * DF * \Delta PD * (1 - RR)$ is calculated for each tenor. This is done for the mean, median and 97.5th percentile expected exposure profile, so three CVAs are obtained.

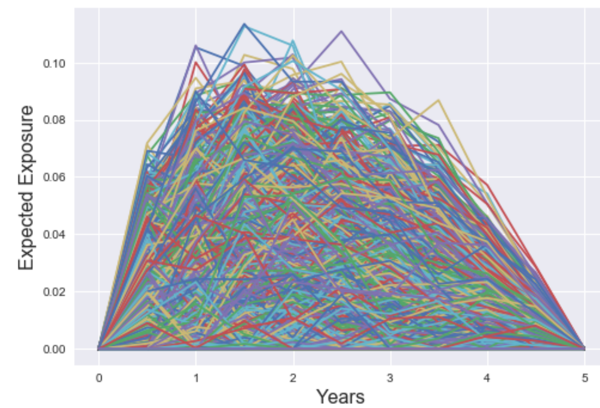
5. Several sensitivity analysis are computed, analysing the CVA value against: different fixed rate coupons, number of simulations, different recovery rates, and different lambda structures for survival probability. CVA is computed for the mean, median and 97.5th percentile of the simulated exposures.

1.5.1 Case 1: Constant hazard rate

For the first exercise, constant hazard rate of 3% is assumed. 10.000 Monte Carlo simulations are made, using a fixed rate coupon of 4.5%. Simulations for the Expected Future Value of the contract (MtM) and the Expected Exposure ($Max(MtM, 0)$), are the following:



(a) Expected Future Value, MtM



(b) Expected Exposure, $Max(MtM, 0)$

Figure 2: Simulations for IRS: EFV and EE

Then, calculating the mean, median and 97.5th percentile of exposure, we get:

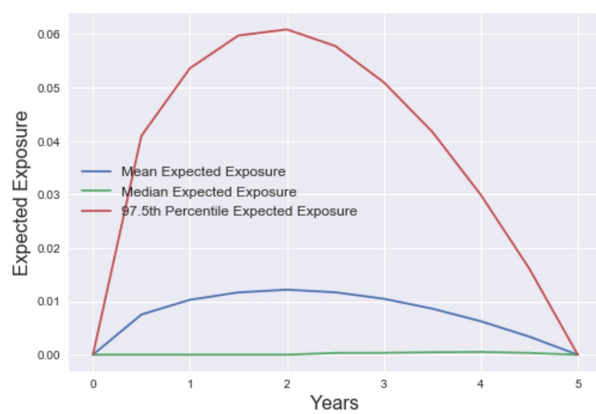


Figure 3: Expected Exposure: Mean, Median and 97.5th Percentile

It is interesting to note that, while as expected, the 97.5th case has a much higher exposure than the mean and median cases, the mean exposure is much higher than the median. This is due to the fact that expected exposure only takes in account the positive side of the MtM distribution. Looking at the frequency (or location) of the simulated paths, the majority are concentrated near '0', so the median tends to be near this number. Also is interesting to see that peak exposure is achieved at the second year for the mean and 97.th cases., while for the median case on the third year.

Table 1 shows the CVA calculated for each of the exposure cases.

CVA		
Mean	Median	97.5th %ile
6.38×10^{-4}	1.4×10^{-5}	3.208×10^{-3}

Table 1: CVA for constant hazard rate 3%, fixed rate coupon 4.5%, and 10.000 MC simulations

1.5.2 Case 2: Constant hazard rate, varying number of simulations

In this case, same conditions for case 1 are applied, but CVA is calculated for different number of Monte Carlo simulations in order to see how the convergence behaves to a stable CVA result.

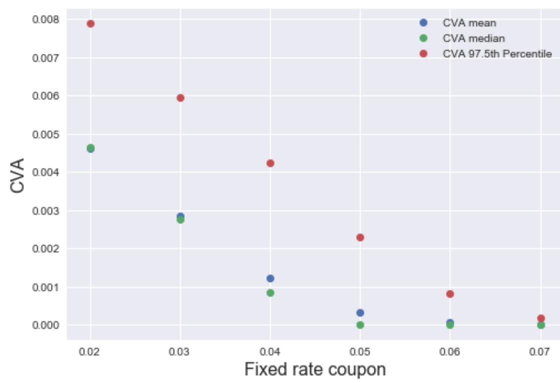
	CVA		
Simulations	Mean	Median	97.5th %ile
50	5.03×10^{-4}	6.5×10^{-5}	2.224×10^{-3}
100	6.13×10^{-4}	5.6×10^{-5}	2.864×10^{-3}
200	6.35×10^{-4}	4.7×10^{-5}	3.178×10^{-3}
500	6.68×10^{-4}	3.4×10^{-5}	3.208×10^{-3}
1.000	6.31×10^{-4}	2.9×10^{-5}	3.232×10^{-3}
2.000	6.54×10^{-4}	2.5×10^{-5}	3.230×10^{-3}
5.000	6.25×10^{-4}	1.5×10^{-5}	3.196×10^{-3}
10.000	6.38×10^{-4}	1.4×10^{-5}	3.208×10^{-3}

Table 2: CVA for constant hazard rate 3%, fixed rate coupon 4.5%, and different number of MC simulations

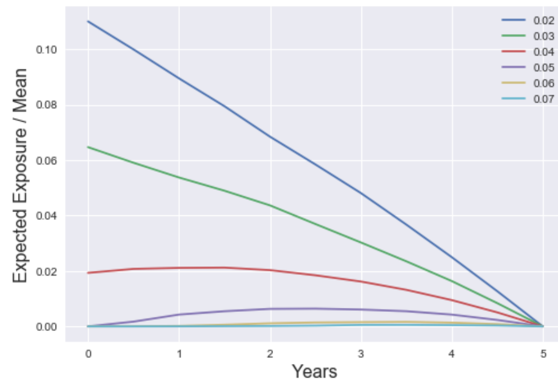
Table 2 shows that at around 1.000 Monte Carlo simulations, reasonable convergence is achieved. It is interesting to mention the behaviour of the CVA median. As pointed out in the previous case, Expected Future Value paths simulations have higher frequency near '0'. As more simulations are done, more paths near 0 are obtained, thus "pulling" the median towards '0'.

1.5.3 Case 3: Constant hazard rate, varying fixed rate coupon

For this case, different fixed rate coupons are used, keeping a constant hazard rate 3% and 1.000 Monte Carlo simulations. Results are the following:



(a) CVA mean, median and 97.5th, for different fixed rate coupons



(b) Expected Exposure Mean for different fixed rate coupons

Figure 4: Simulations for varying fixed rate coupons

	CVA		
Fixed Coupon	Mean	Median	97.5th %ile
2%	4.619×10^{-3}	4.630×10^{-3}	7.880×10^{-3}
3%	2.839×10^{-3}	2.754×10^{-3}	5.938×10^{-3}
4%	1.231×10^{-3}	8.51×10^{-4}	4.227×10^{-3}
5%	3.23×10^{-4}	0	2.304×10^{-3}
6%	6.4×10^{-5}	0	8.10×10^{-4}
7%	1.7×10^{-5}	0	1.88×10^{-4}

Table 3: CVA for constant hazard rate 3%, 1.000 MC simulations and varying fixed coupon

As expected, the lower the fixed coupon rate, higher is the CVA exposure, as the counterparty starts with a positive MtM. On the other hand, as shown in table 3, CVA median goes to 0 for coupons of 5% and higher. This happens because as the MtM starts negative, the median of the Expected Future Value simulations is on the negative side during the whole period of the IRS. Figure 2 illustrates this fact, which shows simulations for EFV using 6% fixed coupon:

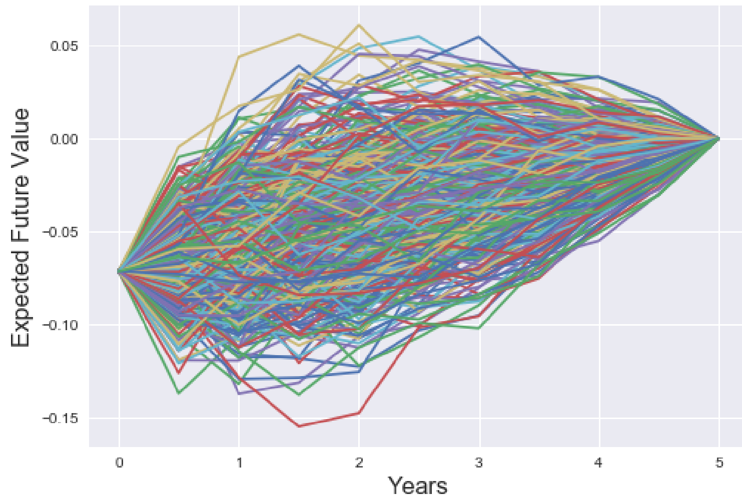
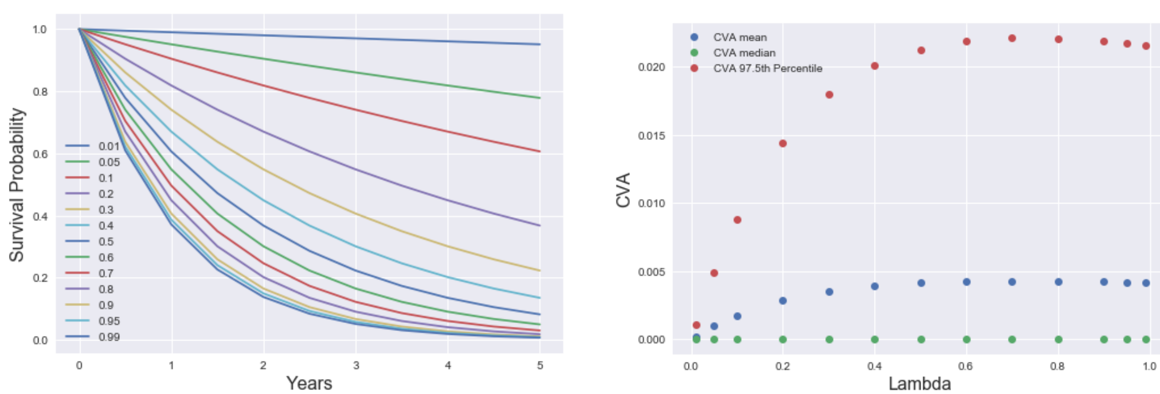


Figure 5: Expected Future Value for fixed coupon of 6%

On the other hand, while median goes below zero the whole period, the 97.5th percentile path has positive values. For the case of the mean, it just need one non negative path to be over zero.

1.5.4 Case 4: Constant hazard rate, varying fixed hazard rate

In this case, CVA results are tested for different constant hazard rates λ. Fixed rate coupon 4.5%, and 1.000 MC simulations are used.



(a) Survival Probability function for different λ

(b) CVA for different λ

Figure 6: Simulations for varying fixed rate coupons

	CVA		
λ	Mean	Median	97.5th %ile
1%	2.13×10^{-4}	4×10^{-6}	1.069×10^{-3}
5%	9.74×10^{-4}	1.8×10^{-5}	4.904×10^{-3}
10%	1.749×10^{-3}	3.0×10^{-5}	8.828×10^{-3}
20%	2.852×10^{-3}	4.2×10^{-5}	1.4446×10^{-2}
30%	3.530×10^{-3}	4.4×10^{-5}	1.7954×10^{-2}
40%	3.932×10^{-3}	4.1×10^{-5}	2.0073×10^{-2}
50%	4.154×10^{-3}	3.6×10^{-5}	2.1279×10^{-2}
60%	4.257×10^{-3}	3.0×10^{-5}	2.1885×10^{-2}
70%	4.285×10^{-3}	2.5×10^{-5}	2.2101×10^{-2}
80%	4.264×10^{-3}	2.0×10^{-5}	2.2062×10^{-2}
90%	4.214×10^{-3}	1.6×10^{-5}	2.1863×10^{-2}
95%	4.181×10^{-3}	1.5×10^{-5}	2.1723×10^{-2}
99%	4.152×10^{-3}	1.3×10^{-5}	2.1597×10^{-2}

Table 4: CVA for varying constant hazard rate λ, 1.000 MC simulations and 4.5% fixed coupon

It is interesting to observe that from λ around 70%, CVA starts to decrease. This is due to the expected exposure structure and the Δ PD structure for each survival probability function. While higher λ implies a more negative gradient for the survival probability at the beginning and therefore high Δ PD, also imply lower Δ PD in future tenors. Then, given the expected exposure for each tenor $(EE(t) * \Delta PD_t)$, higher λ end having lower CVA results. Figure 7 illustrates this behaviour.

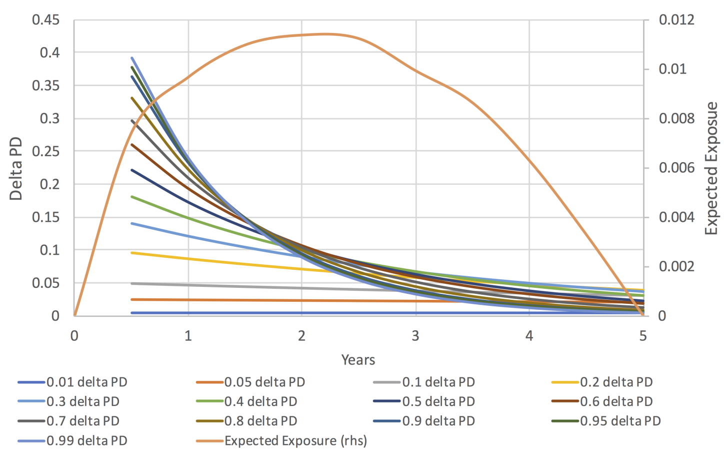


Figure 7: Δ PD for different λ vs. Expected Exposure

1.5.5 Case 5: Constant hazard rate, varying recovery rate

CVA value as a function of recovery rate is tested. Constant hazard rates λ of 3%, fixed rate coupon of 4.5%, and 1.000 MC simulations are used.

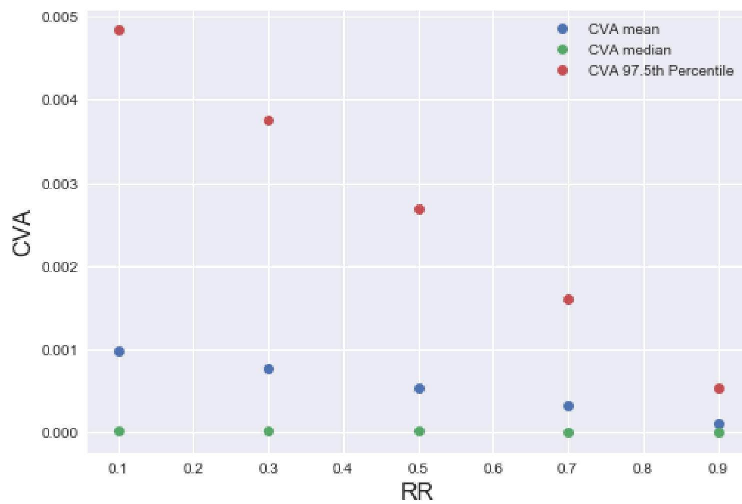


Figure 8: CVA for different recovery rates

	CVA		
RR	Mean	Median	97.5th %ile
10%	9.84×10^{-4}	2.9×10^{-5}	4.836×10^{-3}
30%	7.65×10^{-4}	2.2×10^{-5}	3.761×10^{-3}
50%	5.47×10^{-4}	1.6×10^{-5}	2.687×10^{-3}
70%	3.28×10^{-4}	1.0×10^{-5}	1.612×10^{-3}
90%	1.09×10^{-4}	3×10^{-6}	5.37×10^{-4}

Table 5: CVA for varying recovery rate, 1.000 MC simulations and 4.5% fixed coupon, 3% constant hazard rate

As expected by the CVA formula, in this case CVA behaviour is linear on the recovery rate.

1.5.6 Case 5: Varying hazard rate

In this section, three different term structures of hazard rates are modeled. Counterparty X has a downward sloping hazard rate term structure, reflecting a stress scenario in the short term but with a better scenario (in marginal terms) in the medium term. Counterparty Y has a steep upward sloping hazard rate term structure, reflecting increasing difficulties in the medium term. Counterparty Z, has a low upward sloping hazard rate term structure, showing it's good creditworthiness:

	Hazard rate λ		
Tenor period	X	Y	Z
0 - 0.5	107%	8%	0.9%
0.5 - 1	79%	12%	1.2%
1 - 1.5	41%	17%	1.5%
1.5 - 2	19%	26%	1.7%
2 - 2.5	16%	37%	1.9%
2.5 - 3	10%	52%	2.1%
3 - 3.5	8%	68%	2.2%
3.5 - 4	6%	82%	2.4%
4 - 4.5	5%	103%	2.5%
4.5 - 5	4%	150%	2.7%

Table 6: Different Hazard Rates Term Structures

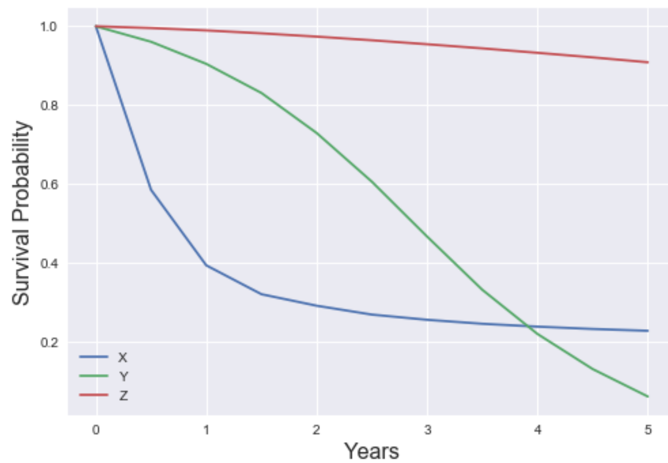


Figure 9: Survival Probabilities for X, Y and Z

Table 7 shows the results for the calculation of the CVA for each of the three counterparties (assuming 4.5% fixed coupon rate and 1.000 Monte Carlo simulations):

	CVA		
Counterparty	Mean	Median	97.5th %ile
X	2.903×10^{-3}	4.1×10^{-5}	1.4610×10^{-2}
Y	4.760×10^{-3}	3.70×10^{-4}	2.2932×10^{-2}
Z	4.17×10^{-4}	3.2×10^{-5}	2.009×10^{-3}

Table 7: CVA for different term structures

As expected, counterparty Z has the lowest CVA due to it's high survival probability relative to X and Y. What is not intuitive, is that counterparty Y has a higher CVA than X, while X's survival probability has a very negative gradient at the beginning. As pointed out in case 4, this is due to the expected exposure structure and the Δ PD structure for each counterparty. Figure 10 illustrates this relationship. Although ΔPD is very high por counterparty X at the beginning, for 1.5 year tenor onwards ΔPD for counterparty Y is higher, while ΔPD for counterparty X converges to values near 0. Then, given the expected exposure for each tenor $(EE(t) * \Delta PD_t)$, counterparty Y ends with a higher CVA than X.

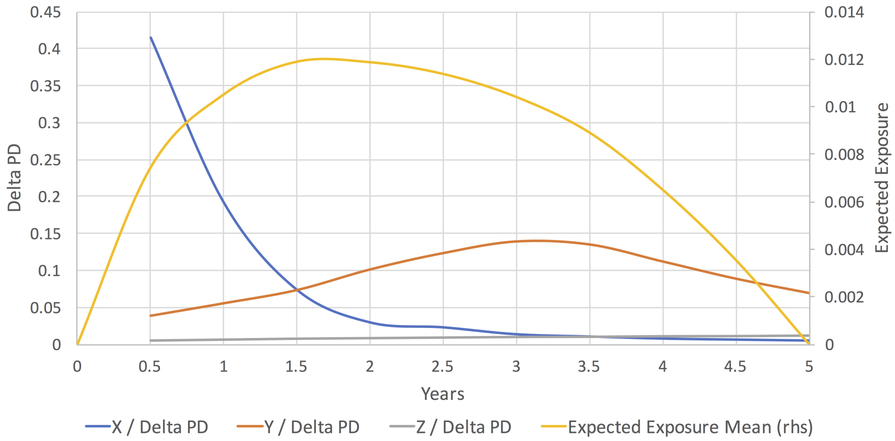


Figure 10: Δ PD vs. Expected Exposure

2 Robust Portfolio Construction with Volatility Filtering

This section seeks to construct a robust portfolio allocation using volatility filtering. To achieve this, the Black-Litterman asset allocation model with different processes for estimating volatility is used. In particular, this method is applied to the Chilean equity and fixed income market using macro quantitative and bottom up analysis to generate the views.

2.1 The Black-Litterman model

The Black-Litterman model enable investors to combine their unique views regarding the performance of various assets with the implied market equilibrium expected returns (prior distribution) to form a new, mixed estimate of expected returns (posterior distribution). This leads to intuitive, mean-efficient stable portfolios, and address the main problems of the traditional Markowitz mean-variance approach of highly concentrated and input-sensitive portfolios. As starting point, the Black-Litterman uses "equilibrium" returns as starting neutral point. This are defined as the set of returns that "clear" the market. This returns are obtained by "reverse optimization" using the following formula for excess returns:

$$\Pi_{prior} = \lambda \Sigma w_{mkt} \tag{3}$$

where, for N assets, Π_{prior} is the implied excess equilibrium return vector (Nx1), λ is the risk aversion coefficient, Σ is the covariance matrix of excess returns (NxN), and w_{mkt} is the market equilibrium weight for the assets (Nx1). In particular, λ characterizes the expected risk-return tradeoff. In other words, is the rate at which an investor will forego expected return for less variance. In this case it acts as an "scaling factor". More excess returns per unit of risk, increases the estimated excess returns:

$$\lambda = \frac{E(r) - r_f}{\sigma^2} = \frac{SharpeRatio}{\sigma} \tag{4}$$

As equation 4 shows, λ can be expressed as a function of the sharpe ratio. There is no "absolute" number for a "good" sharpe ratio, as it depends on the asset class and it's volatility, but normally will go between levels of 0.5 and 2. It is important to mention that situations in which the variance is very small with respect to the mean can lead to different levels of sharpe ratios (as it will be shown further in the fixed income application).

Now, the equation for the posterior Black-Litterman of expected returns ($\Pi_{posterior}$) is the following:

$$\Pi_{posterior} = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi_{prior} + P'\Omega^{-1}Q\right] \tag{5}$$

where, for N assets, $\Pi_{posterior}$ is the new (posterior) combined excess return vector (Nx1), τ is a scalar, Σ is the covariance matrix of excess returns (NxN), P is the matrix that identifies the assets involved in the views (KxN, for a number of "K" views). Ω is a diagonal covariance matrix of error terms from expressed views that represent the uncertainty of each view (KxK), Π_{prior} is the implied excess equilibrium return vector (Nx1), and Q is the view vector (Kx1).

As mentioned before, the Black-Litterman model allows investment managers views to be combined with the implied equilibrium return, to obtain a new "weighted" view. This views can be expressed as absolute views or relative views. For example, in a 4 asset portfolio with assets A, B, C and D., let's consider the following views:

- Asset C will have an absolute excess return of 6%

- Asset A outperforms asset B in 3%

Matrix P and Q will be identified as follows:

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 6\% \\ 3\% \end{bmatrix}$$

Matrix Ω, will express the uncertainty of each view. In particular, Ω represents the covariance matrix of error terms ϵ:

$$Q + \epsilon = \begin{bmatrix} 6\% \\ 3\% \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

Ω is a diagonal covariance matrix with 0's in all off diagonal terms. This assumes views are independent of one another.

$$\Omega = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_k \end{bmatrix}$$

Determining the individual variances of the error terms ω is normally done by obtaining the volatility of each view $\omega_k = p_k \Sigma p_k^{'}$, where p_k is a single row vector (1xN) from matrix P, and Σ is the covariance matrix of excess returns. This way, the uncertainty of the view will be linked to the volatility of the underlying "portfolio p_k" that expresses the view. In other words, views will have a higher impact in the posterior returns $\Pi_{posterior}$ when ω_k is small, as the view has a lower uncertainty or a higher conviction.

The last parameter needed for equation 5 is τ. Conceptually, τ acts as a scalar that weights matrix Σ in order to obtain the "best estimate" for the actual covariance matrix Σ. Particularly, Black-Litterman assumes that distributions of excess returns $f_u(u)$ is Normal:

$$\mu \sim N(\Pi_{prior}, \tau\Sigma) \tag{6}$$

So $\tau\Sigma$ is effectively the estimation error on u. A way τ can be defined is as proportionally inverse to the number of observations for the estimation of Σ , so $\tau = 1/N_{obs}$.

Having all the parameters, $\Pi_{posterior}$ can now be estimated from equation 5. Also, $\Sigma_{posterior}$ and $w_{posterior}$ can be calculated as:

$$\Sigma_{posterior} = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1} \tag{7}$$

$$w_{posterior} = \frac{1}{\lambda}\Sigma^{-1}\Pi_{posterior} \tag{8}$$

2.2 Volatility Filtering

In general, volatility in financial markets present several stylized facts that makes it's modeling more accurate than modeling returns themselves. Some of this facts are:

- Volatility Clustering: Volatility is not constant and tends to cluster through time. Having large volatility today is a good precursor of having large volatility in the coming days (positive autocorrelation over several days).

- Long memory: Changes in volatility (typically due to a shock) have a long-lasting impact on it's subsequent evolution. It tends to have a slow decay in it's autocorrelation.

- Leverage effect: Volatility is typically negatively correlated with the returns of it's underlying asset. On negative returns, volatility goes up, while when positive returns are observed, volatility goes down. Also it presents asymmetry, meaning that on positive returns, volatility decreases to a lesser extent compared to when volatility increases due to negative returns.

There are several models that try to capture this effects. EWMA (Exponentially Weighted Moving-Average), ARCH (Autoregressive Conditional Heteroskedasticity), GARCH (Generalized Autoregressive Conditional Heteroskedasticity) and EGARCH (Exponential Generalized Autoregressive Conditional Heteroskedasticity) processes are examples of this. There are several other models, particularly within the GARCH family such us GJR-GARCH, TGARCH, QGARCH, IGARCH, NGARCH among others, but for the purpose of this investigation the focus will be on the first three. To briefly explain each of them, let's consider a financial asset that has a vector of returns R. Return in time t, can be modeled with a drift and a stochastic part as:

$$r_t = \mu_t + \epsilon_t \tag{9}$$

$$\epsilon_t = \sigma_t Z_t \qquad Z_t \sim N(0,1) \tag{10}$$

Where μ_t is the drift or mean at time t, ϵ_t is the error term or innovation at time t, and σ_t the standard deviation at time t. So, EWMA, ARCH, GARCH and EGARCH for process σ_t are the following:

2.2.1 EWMA

The EWMA (Exponentially Weighted Moving-Average) process is the following:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda)\epsilon_{t-1}^2 \tag{11}$$

The decay factor λ sets how much "weight" to give to the last σ_{t-1}^2 estimation, versus the square of the innovation ϵ_{t-1}. Although it varies between different asset classes, λ is typically is fixed previously, with a range between 0.90 and 0.98, where 0.94 is normally used.

2.2.2 ARCH

Proposed by Engle[4], the ARCH (Autoregressive Conditional Heteroskedasticity) process is the following:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 \tag{12}$$

Where $\omega > 0$, $\alpha_i \geq 0$, and $\sum_{i=1}^{p} \alpha_i \leq 0$, to ensure positive variance and stationarity. The model name, comes from the fact that it is *autoregressive* on the squares innovations, with a number of p lags (it can be defined as ARCH(p)), σ_t is *conditional* to information up to $t-1$, and it is heteroskedastic as volatility is non constant (as opposed to homoskedastic).

2.2.3 GARCH

ARCH typically needs a large number of lags on the innovation to correctly fit the process. As a solution, Bollerslev[5] proposed the GARCH model. It is based on an "infinite" specification of the ARCH model, and significantly reduces the estimated parameters in order to fit the data. The GARCH (Generalized Autoregressive Conditional Heteroskedasticity) process is the following:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2 \tag{13}$$

Where $\omega > 0$, $\alpha_i, \beta_i \geq 0$, and $\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{p} \beta_i \leq 0$, to ensure positive variance and stationarity.

2.2.4 EGARCH

As mentioned before, another empirically noticed fact in returns, is that there is an asymmetric effect to volatility between having positive and negative returns. Nelson[6] address this and proposed a new framework called EGARCH. The EGARCH (Exponential Generalized Autoregressive Conditional Heteroskedasticity), written as an AR process, is the following:

$$\begin{aligned} \ln \sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i (|e_{t-i}| - \sqrt{\frac{2}{\pi}}) + \sum_{j=1}^{o} \gamma_j |e_{t-j}| + \sum_{k=1}^{q} \beta_k \ln \sigma_{t-k}^2 \\ e_t &= \frac{\epsilon_t}{\sigma_t} \end{aligned} \tag{14}$$

Where $\ln \sigma_t^2$ is modeled rather than σ_t in order to ensure positive variance.

2.3 Portfolio Construction: Chilean market

In this section, the Black-Litterman model with volatility filtering is applied to the Chilean equity and fixed income market. Different sensitive analysis is done, exploring different volatility estimations, risk aversions and asset allocation optimizations (mean-var, sharpe max, VaR min). Python is used for implementation. Python pseudo code is as follows:

1. Data for the prices of the assets to compute and the risk free rate are saved in the excel named BLData.xlsx. Then the data is read from the code and excess return is computed (function `getReturns`.

2. Volatility estimations, EWMA, ARCH, GARCH and EGARCH are computed. This is done with function `xgarchModel`. Covariance matrix is built.

3. Starting from a particular sharpe ratio, λ is calculated. Then, together with the investor's view (matrix P and Q), posterior Black-Litterman returns, weights and Covariance are calculated with function `BL`. Black-Litterman constrained posterior weight to $w_n > 0$ is also calculated with function `wBLConstrained`

4. Different asset allocations for different minimization techniques are done with functions `MinVariance, MinVarianceVaR, MinVarianceSharpe`. Function `effFrontier` creates the efficient frontier for the minimum variance approach. Functions `wBLVaR, wBLSharpe` calculate the optimum weights for VaR minimization and Sharpe maximization.

2.4 Equity market

The IPSA (Indice de Precio Selectivo de Acciones) is the most used equity index in the Chilean market. It is generated with the 40 most traded stocks, which are annually reviewed and weighted by market capitalisation. The 10 stocks with more weight in the index are used (they account for more than 50% of the index), since they have enough historical data, enough liquidity (daily traded) and are stable members of the local index. Weekly data since September 2006 is used, while for the risk free rate a local index for the 1 day OIS curve is used[3].

Initial market weights for the 10 stocks are the following:

Stock	w_{mkt}
Empresas COPEC	14.7%
Enel Americas	14.0%
Latam Airlines Group	12.2%
SACI Falabella	12.1%
Banco Santander Chile	9.7%
Cencosud	9.0%
Sociedad Qumica y Minera de Chile	7.4%
Banco de Crdito e Inversiones	7.2%
Banco de Chile	6.9%
Empresas CMPC	6.8%

Table 8: Market weight for 10 more important stock in IPSA (normalized to 100%)

2.4.1 Volatility

First, volatility is estimated using five different approaches: Constant historic volatility, EWMA(0.94), ARCH(1), GARCH(1,1), EGARCH(1,1,1). The following figure shows the volatility for "Cencosud" for the whole period with the different approaches:

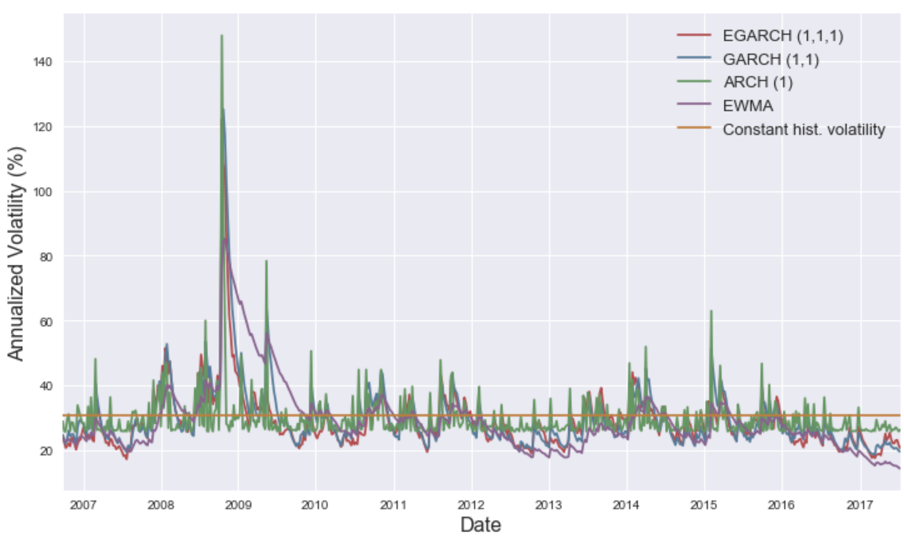


Figure 11: Volatility for Cencosud since 2006

Doing a "zoom" for 2010 onward to take a more detailed look at the behaviour post 2008 crisis:

[3]Public index constructed by RiskAmerica, www.riskamerica.com

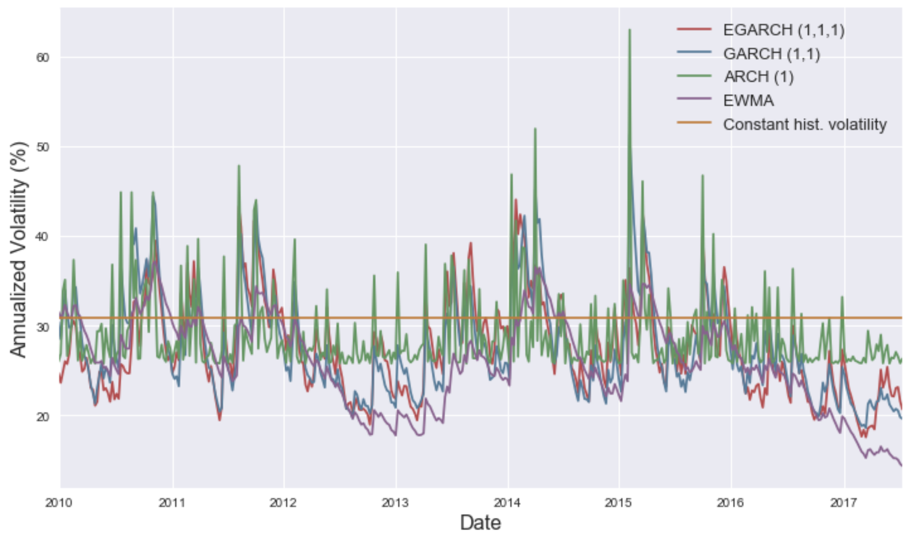


Figure 12: Volatility for Cencosud since 2010

From the results, the ARCH(1) model is much "noisier" and fails to capture the downside moves in volatility due to it's dependency to only one lagged innovation: $\omega + \alpha\epsilon_{i-1}^2$. As in this case α is positive, it behaves as a constant plus some "noise". Now, doing the same example but this time with ARCH(5):

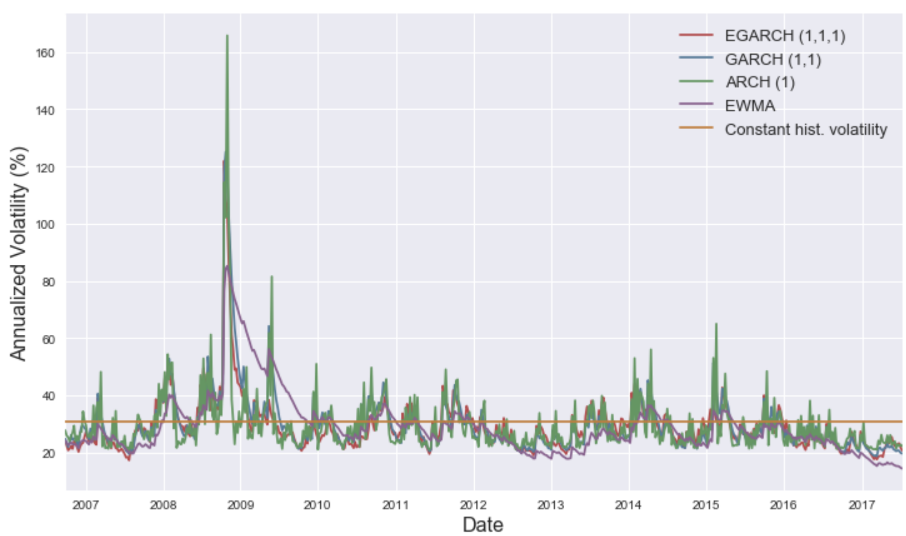


Figure 13: Volatility for Cencosud since 2006, with ARCH(5)

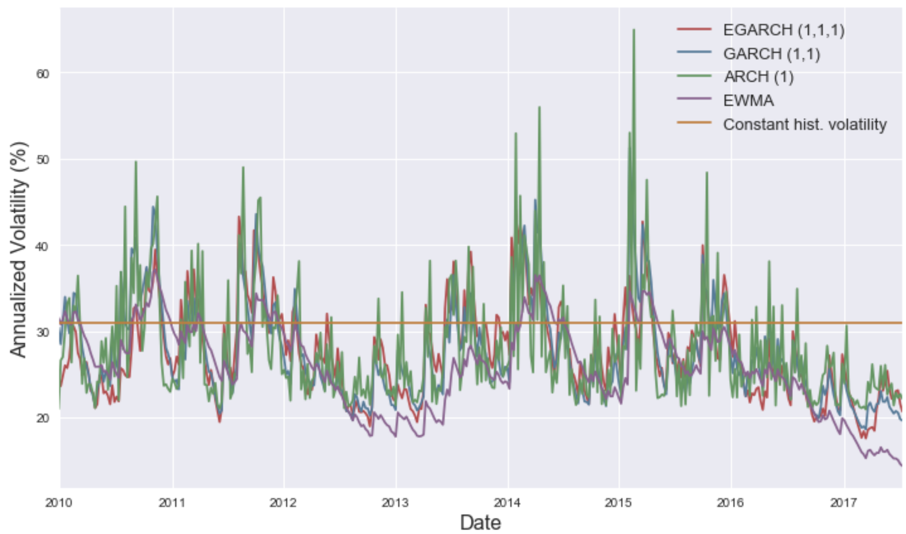


Figure 14: Volatility for Cencosud since 2010, with ARCH(5)

Behaviour of ARCH(5) model improved with respect to ARCH(1). However, EGARCH and GARCH models present a smoother series, with good sensitivity to changes in volatility. In particular, figure 2 shows volatility for Cencosud from 2010 onwards without the ARCH model:

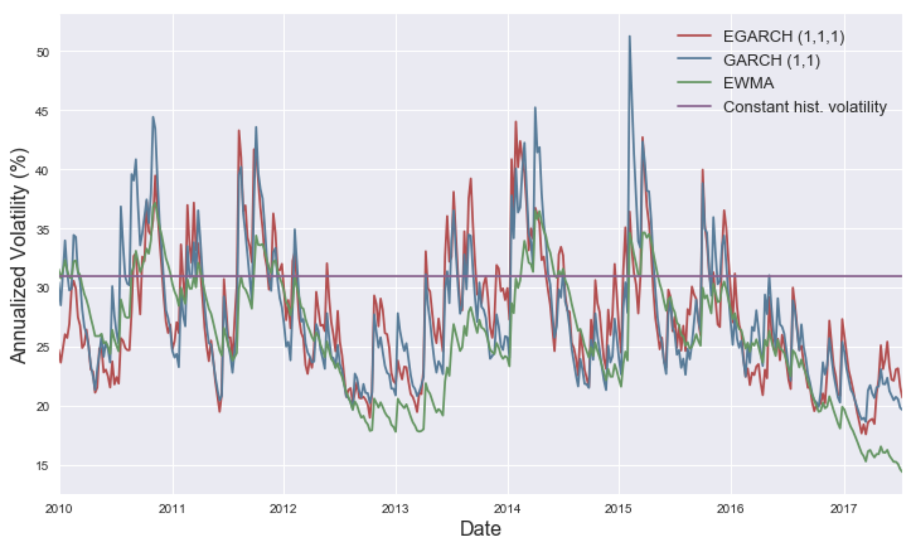


Figure 15: Volatility for Cencosud since 2010, for EGARCH, GARCH, EWMA and Historic Volatility

It is interesting to see how volatility has decreased the last two years. This is a global phenomenon, in tandem with consecutively positive returns in global equity markets. The VIX[4] for example is now in historic lows, below 10.

[4]Chicago Board Options Exchange Volatility Index for SPX

2.4.2 Black-Litterman Allocation

The Black-Litterman allocation framework is applied using the different volatility approaches to estimate the covariance matrix. For λ estimation, initially the sharpe ratio approach mentioned in equation 4 using 0.5 sharpe is used. Prior equilibrium returns are as follows:

Stock	Π_{EGARCH}	Π_{GARCH}	Π_{ARCH}	Π_{EWMA}	$\Pi_{HistoricVol}$
Empresas COPEC	7.09%	7.7%	9.8%	6.12%	9.46%
Enel Americas	8.39%	10.3%	10.76%	7.11%	9.55%
Latam Airlines Group	10.38%	12.52%	11.84%	12.52%	11.92%
SACI Falabella	4.99%	5.16%	8.66%	4.52%	9.72%
Banco Santander Chile	6.54%	7.43%	9.31%	6.28%	9.78%
Cencosud	7.52%	7.02%	9.6%	5.08%	11.48%
Sociedad Quimica y Minera de Chile	8.78%	8.74%	12.88%	7.68%	11.49%
Banco de Credito e Inversiones	5.59%	6.12%	7.94%	5.32%	7.99%
Banco de Chile	7.81%	8.39%	8.64%	4.63%	7.84%
Empresas CMPC	7.23%	7.28%	9.4%	6.23%	9.84%

Table 9: Prior returns for Black-Litterman, using different volatility approaches

In general, within the prior equilibrium return estimation, the higher the volatility of the asset, the higher the implied return. In this case, is interesting to see that as historic volatility is higher than the estimated through the other models, this implies higher expected returns.

Now, for building the views, in this case bottom up views where built using buy side analysts estimations from a local asset management firm. In particular, views are the following:

1. Cencosud will have an absolute excess return of 20%
2. Sociedad Quimica y Minera de Chile will have an absolute excess return of 17%
3. Latam Airlines Group will have an negative absolute excess return of -10%
4. BCI will outperform Banco de Chile by 5%

Posterior returns and allocations given the views are as follows:

Stock	Π_{EGARCH}	Π_{GARCH}	Π_{ARCH}	Π_{EWMA}	$\Pi_{HistoricVol}$
Empresas COPEC	7.41%	8.52%	9.11%	8.44%	8.4%
Enel Americas	6.7%	8.87%	7.95%	7.64%	6.82%
Latam Airlines Group	3.23%	5.02%	3.4%	6.72%	3.23%
SACI Falabella	6.01%	6.57%	9.1%	7.26%	9.65%
Banco Santander Chile	6.02%	7.3%	7.84%	7.87%	8.01%
Cencosud	11.94%	11.87%	12.72%	11.41%	13.62%
Sociedad Quimica y Minera de Chile	11.77%	12.03%	13.45%	12.48%	12.58%
Banco de Credito e Inversiones	7.23%	8.26%	8.78%	8.72%	8.41%
Banco de Chile	6.17%	7.28%	6.8%	5.87%	6.06%
Empresas CMPC	7.68%	8.21%	8.96%	8.87%	8.84%

Table 10: Posterior returns for Black-Litterman, using different volatility approaches

Stock	ω_{market}	ω_{EGARCH}	ω_{GARCH}	ω_{ARCH}	ω_{EWMA}	$\omega_{HistoricVol}$
Empresas COPEC	14.74%	14.74%	14.74%	14.74%	14.74%	14.74%
Enel Americas	13.98%	13.98%	13.98%	13.98%	13.98%	13.98%
Latam Airlines Group	12.22%	-41.39%	-35.24%	-41.14%	-33.34%	-39.59%
SACI Falabella	12.13%	12.13%	12.13%	12.13%	12.13%	12.13%
Banco Santander Chile	9.64%	9.64%	9.64%	9.64%	9.64%	9.64%
Cencosud	9.01%	65.4%	78.51%	50.89%	120.98%	35.65%
Sociedad Quimica y Minera de Chile	7.41%	26.38%	26.88%	15.6%	26.42%	19.78%
Banco de Credito e Inversiones	7.16%	37.48%	36.09%	28.08%	33.49%	27.11%
Banco de Chile	6.93%	-23.4%	-22%	-13.99%	-19.41%	-13.03%
Empresas CMPC	6.78%	6.78%	6.78%	6.78%	6.78%	6.78%

Table 11: Posterior allocations for Black-Litterman, using different volatility approaches

Stock	ω_{market}	ω_{EGARCH}	ω_{GARCH}	ω_{ARCH}	ω_{EWMA}	$\omega_{HistoricVol}$
Empresas COPEC	14.74%	8.18%	5.3%	10.68%	3.63%	16.07%
Enel Americas	13.98%	0%	0%	3.73%	0%	9.34%
Latam Airlines Group	12.22%	0%	0%	0%	0%	0%
SACI Falabella	12.13%	27.82%	23.71%	17.76%	11.4%	13.55%
Banco Santander Chile	9.64%	3.34%	0.44%	7.06%	0%	0.85%
Cencosud	9.01%	26.29%	36.83%	25.51%	60%	11.53%
Sociedad Quimica y Minera de Chile	7.41%	9.88%	11.08%	4.83%	10.62%	7.79%
Banco de Credito e Inversiones	7.16%	23.56%	19.59%	23.06%	12.69%	25.36%
Banco de Chile	6.93%	0%	0%	2.41%	0%	12.5%
Empresas CMPC	6.78%	0.93%	3.06%	4.97%	1.67%	3.01%

Table 12: Posterior allocations for Black-Litterman, **with constrains** $\sum \omega_n = 1, \omega_n > 0$, using different volatility approaches

First interesting thing to note, is that as table 11 shows, the stocks that aren't involved in the views, have their allocations unchanged. Also, as expected by view 3, Latam Airlines Group decreases it's weight, while Cencosud and Sociedad Quimica y Minera de Chile increases their weight (views 1 and 2). BCI also increases while Banco de Chile decreases as expected by view 4.

Now, following the optimal allocation of Markowitz mean-variance approach, a efficient frontier can be made using the prior and posterior distributions. Results are the following using constrains $\sum \omega_n = 1, \omega_n > 0$:

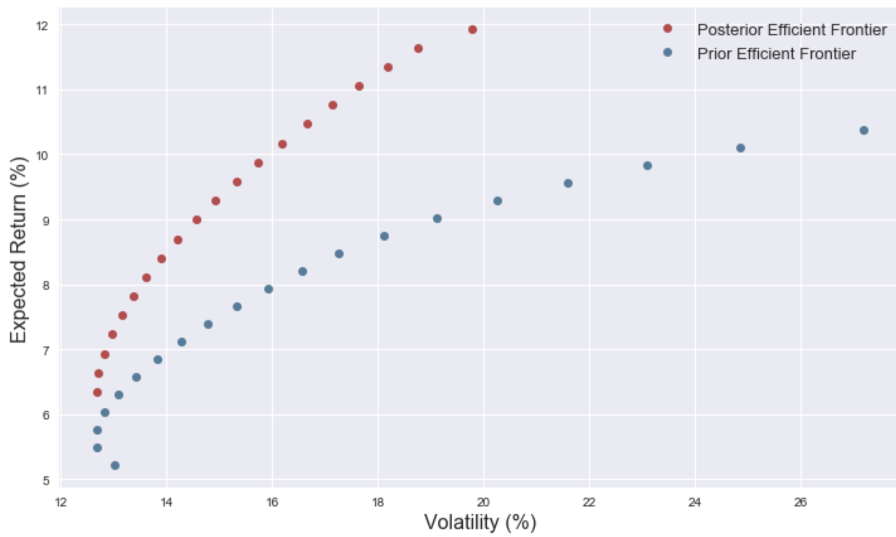
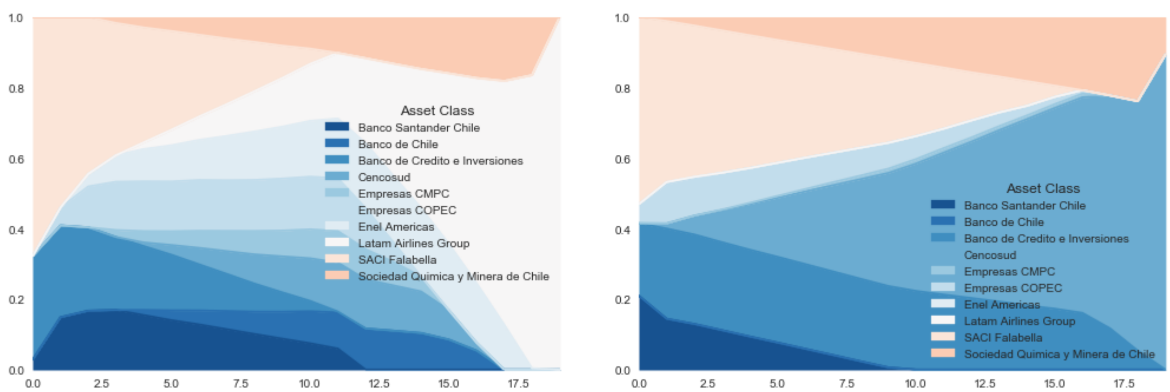


Figure 16: Efficient frontier using EGARCH



(a) Prior allocation efficient frontier

(b) Posterior allocation efficient frontier

Figure 17: Change in allocations due to Black-Litterman

It is interesting to note that Latam Airlines Group was an important part of the allocation in the prior for high volatility portfolios, but due to view 3, it's allocation decreases almost to 0, while mainly Cencosud and to a lesser part Sociedad Quimica y Minera increases due to views 1 and 2. Banco de Credito e Inversiones has a higher proportion at the expense of a lower proportion of Banco de Chile.

Approaches as maximization of Sharpe Ratio and VaR minimization subject to a certain risk budget also may be done:

$$\begin{aligned}
\text{Sharpe Ratio} \quad & argmax_w \frac{w'\mu - r_f}{\sqrt{w'\Sigma w}} \qquad \text{s.t.} \qquad w'1 = 1 \\
\text{VaR minimisation} \quad & argmax_w w'\mu - \lambda w'\Sigma w \qquad \text{s.t.} \qquad \sqrt{\lambda w'\Sigma w} = \frac{\text{constant}}{\text{factor}} \\
\text{VaR factor} =& \Phi^{-1}(1-c)
\end{aligned} \tag{15}$$

In particular, for the sharpe ratio, maximising the previous equation gives the the optimal tangency portfolio for the efficient frontier. Solving it with the constrains $\sum \omega_n = 1, \omega_n > 0$ gives:

Stock	ω_{market}	$\omega_{sharpeprior}$	$\omega_{sharpeposterior}$
Empresas COPEC	14.74%	10.88%	0%
Enel Americas	13.98%	18.42%	0%
Latam Airlines Group	12.22%	25.14%	0%
SACI Falabella	12.13%	0%	0%
Banco Santander Chile	9.64%	0%	0%
Cencosud	9.01%	13.77%	71.13%
Sociedad Quimica y Minera de Chile	7.41%	11.87%	23.97%
Banco de Credito e Inversiones	7.16%	0%	4.90%
Banco de Chile	6.93%	11.64%	0%
Empresas CMPC	6.78%	8.29%	0%

Table 13: Optimal tangency portfolio for sharpe maximization

For r_f, 2.5% is used as it is the actual 1d OIS for the Chilean market. It is interesting to see that in the prior, Latam Airlines had a biggest allocation in the tangency portfolio, which was in line with a higher expected prior equilibrium return. On the posterior, Cencosud, Sociedad Quimica y Minera and Banco de Credito e Inversiones are preferred in line with the views and posterior returns.

For VaR minimization, the following graph resumes the optimization done for different VaR factor constrains:

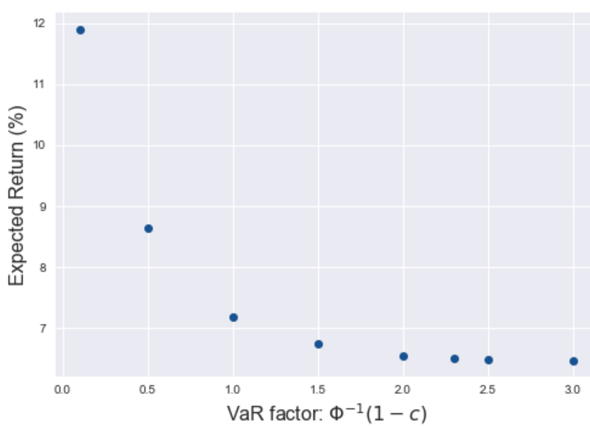


Figure 18: Optimization using VaR factor

As expected, as the constraining factor increases, expected return decreases. For example, for a factor of 2.33 equivalent to a 99% level of confidence, the expected return is approximately 6.5%, while for a factor of 0.5, that is approximately equivalent to a 70% level of confidence, the expected return is 8.6%. Another interesting thing to note is that as the factor increases, it convergence to a minimum level of return of 6.5% approximately. This is coherent with the efficient frontier shown in figure 16, where the minimum variance portfolio is approximately a that level. In fact, calculating the minimum variance portfolio by it's closed form solution, gives an expected return of 6.34%, with an annualized volatility of 12.69%.

Another interesting test to do is to see how allocation vary, while the coefficient of risk aversion λ varies. Up to now, λ was the result of equation 4 using a sharpe ratio of 0.5. This, using the

EGARCH model, sets λ at 3.34. Testing for levels of alternative levels of lambda of 0.5 and 6 give the following results for the new posterior allocation

Stock	ω_{market}	$\omega_{posterior}\lambda = 3.34$	$\omega_{posterior}\lambda = 0.5$	$\omega_{posterior}\lambda = 6$
Empresas COPEC	14.74%	14.74%	14.74%	14.74%
Enel Americas	13.98%	13.98%	13.98%	13.98%
Latam Airlines Group	12.22%	-41.39%	-257.46%	-24.54%
SACI Falabella	12.13%	12.13%	12.13%	12.13%
Banco Santander Chile	9.64%	9.64%	9.64%	9.64%
Cencosud	9.01%	65.4%	493.17%	32.03%
Sociedad Quimica y Minera de Chile	7.41%	26.38%	193.18%	13.37%
Banco de Credito e Inversiones	7.16%	37.48%	180.63%	26.32%
Banco de Chile	6.93%	-23.4%	-166.54%	-12.23%
Empresas CMPC	6.78%	6.78%	6.78%	6.78%

Table 14: Optimal Black-Littemmann allocations with different risk aversion coefficients λ

As expected, the lower the λ the less risk averse so allocations tend to be exacerbated. For $\lambda = 0.5$ it allocated near 500% in Cencosud that has one of the best posterior returns, and going short Latam Airlines and Banco de Chile with lower expected returns. On the other hand, $\lambda = 6$, it tends to moderate the results, going less aggressive in the longs and the shorts.

2.5 Fixed Income Market

The Chilean fixed income market has developed quickly in the last 15 years. An asset management industry of roughly USD 50 billion, from which about 50% is invested in the local fixed income market, and a pension fund industry of about USD 180 billion from which 35-40% is invested in local fixed income has helped the market to develop. During 2017, the sovereign fixed income market has been gaining weight in global indices such us the JP Morgan GBI-EM, and liquidity has been increasing rapidly.

In this section the Black-Litterman model is applied to the Chilean local fixed income market. For this, 11 local fixed income indices are selected[5]. Five of them associated to local government bonds, both nominal and inflation linked. Four of them associated with credit spread, and two of them with money market time deposits in nominal and inflation linked terms. For this exercise, to avoid possible biases, the initial market weight is the "naive" 1/N weighting. Market capitalization may bias the allocation towards money market and government bonds, although liquidity is reasonable for corporate bonds allowing for a equally weighted portfolio without having liquidity problems.

[5]Public indices provided by www.riskamerica.com

Index	w_{mkt}
Gov CLP 3-5	9.1%
Gov CLP 5+	9.1%
Gov UF 3-5	9.1%
Gov UF 5-7	9.1%
Gov UF 7+	9.1%
Corp AAA UF 3-5	9.1%
Corp AAA UF 5+	9.1%
Corp A UF 3-5	9.1%
Corp A UF 5+	9.1%
Dep CLP 9-12M	9.1%
Dep UF 9-12M	9.1%

Table 15: Initial market weights for the Chilean local fixed income market

The notation refers to: Government, Corporate or Time Deposits (Gov/Corp/Dep), CLP or UF (nominal or inflation linked respectively), AAA or A (local ratings on corporates), and duration bucket where the index is built (example, 3-5 years or 9-12 months). It is interesting to point out that corporate indices are in inflation linked terms (UF) as about 90% of the outstanding debt on this market is in this currency. The inflation linked market was the first market to develop as in the 90s the monetary policy rate target was in 'real' terms, fact that was changed late 90s. Up to now, the Chilean market has a very liquid inflation linked market and most corporates prefer to issue in inflation linked terms to offset inflation risks, although inflation has been very stable since late 90s (with the exception of 2008, where it reached 9.85% in YoY terms due to the depreciation of the exchange rate and the increase in oil prices).

As in the equity section, weekly data since September 2006 is used, while for the risk free rate a local index for the 1 day OIS curve is used.

2.5.1 Volatility

As in the equity market, different volatility processes are tested. The following figure shows different volatility measures for the 'Gov CLP 5+' Index.

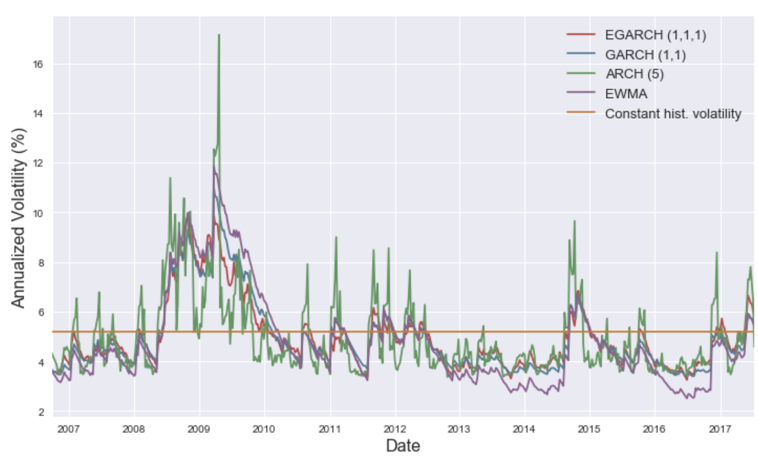


Figure 19: Volatility estimation for Gov CLP 5+ index since 2006

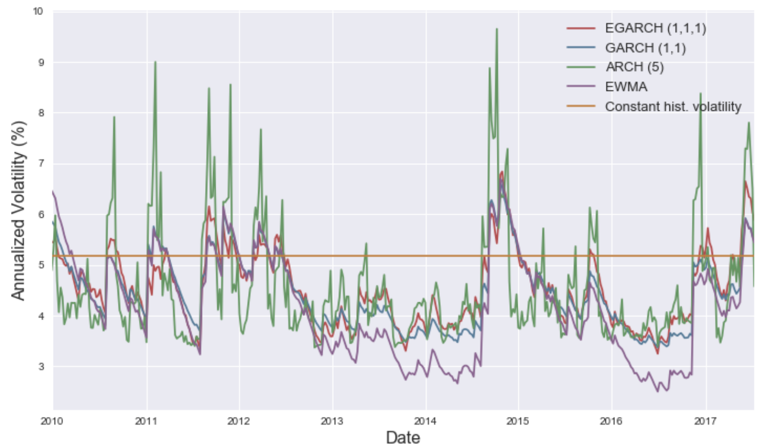


Figure 20: Volatility estimation for Gov CLP 5+ index since 2010

Is interesting to see that in spite of the low volatility in the equity world (local and global), after 2016 US elections and the start of the FED's hiking process in December 2016, volatility picked up in the fixed income market. In particular, fixed income local markets in EM are one of the most affected assets in an environment in which the US monetary policy starts to tighten.

Again, EGARCH and GARCH shows as the most smooth but at the same time reactive to changes in volatility. For further deep sensitivity analysis, EGARCH will be used.

2.5.2 Black-Litterman Allocation: Prior

As in previous section, the Black-Litterman allocation framework is applied using the different volatility approaches to estimate the covariance matrix. This time although, is necessary to find the right λ estimation. Using the sharpe ratio approach, an estimate for the sharpe ratio is needed for the chilean fixed income market. As mentioned before, sharpe ratios have to be carefully evaluated depending on the asset class in study. Also, in the fixed income world, the problem of the volatility being much smaller than the mean can be encountered. In particular, taking the ratio of excess return with respect to it's average volatility since 2006 for the Chilean market, gives a sharpe ratio of around 1.8 (2.2% annualized excess return with 1.2% annualized volatility). Using this sharpe ratio, prior equilibrium returns are as follows:

Index	Π_{EGARCH}
Gov CLP 3-5	2.8%
Gov CLP 5+	7.74%
Gov UF 3-5	3.31%
Gov UF 5-7	4.78%
Gov UF 7+	9.15%
Corp AAA UF 3-5	2.55%
Corp AAA UF 5+	6.36%
Corp A UF 3-5	1.53%
Corp A UF 5+	4.92%
Dep CLP 9-12M	0.09%
Dep UF 9-12M	0.41%

Table 16: Prior returns on Black-Litterman for EGARCH volatility

On table 16 it can be observed again that the higher the volatility of the asset, the higher the implied equilibrium return. Assuming a constant sharpe ratio, in equilibrium, higher volatility assets have higher expected returns.

2.5.3 Black-Litterman Allocation: Views

A macro-quantitative approach is built for generating the views on the fixed income market. The framework used is the following:

- Different macro scenarios are defined. For each scenario, a monetary policy path and inflation expectations are determined.
- Nominal and inflation linked zero coupon curves are built using the expected monetary policy path and inflation.
- Corporate spreads are estimated for each scenario.
- Each fixed income instrument is valued at each scenario with the correspondent zero coupon curve and spread.
- Probabilities are assigned to each scenario, to obtain a fair value for each instrument.
- Fair value is compared to market value to establish expected returns. Capital gain and carry is taken in account.
- Views are created using the expected returns.

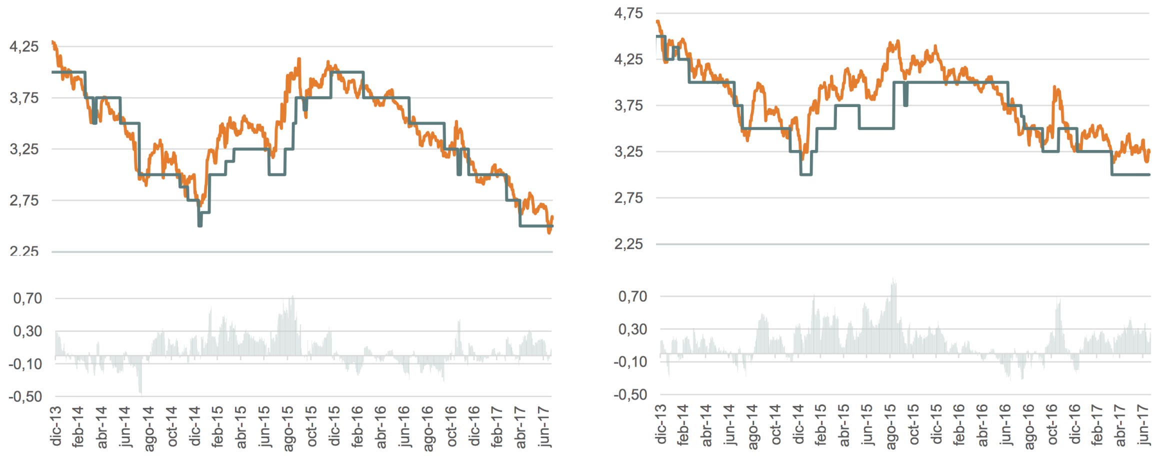
This approach has the advantage that intuitively links macro views to fair value in fixed income. Normally, investors think in a 'macro manner' and express their macro ideas in a monetary policy and inflation view. In a way, thinking in monetary policy is like thinking in the 'one day forward curve' in fixed income. The problem is that the link to the spot yield curve is not always easy, and it's even more difficult for the inflation linked curve, where not only inflation expectations are important, but also the running carry it's relevant.

Another important aspect to take in account is the term premium within the yield curve. There are different approaches in the literature, from simple to more complex ones using affine term structure models[7] [6]. Here, a simple approach is taken. The framework is the following:

- Up to the 2y tenor, the Chilean Central Bank survey to market participants is taken. Here participants are asked for their monetary policy view up to 2 years. Then, the survey is compared to what is implied in the forward curve (1 day FRA). This difference is considered as the term premium (premium expected in the market by investors on top of the survey expectations).
- For the 5y and 10y tenor, the implied forward rates are compared to the Central Bank's neutral policy rate. Each year in September, the Central Bank publishes their new estimation for the neutral policy rate. The difference between the implied forward rates (1d FRA) in the market and the neutral policy rate is considered as the term premium.

[6] NY FED on US Term Premium: https://www.newyorkfed.org/medialibrary/media/research/staff_reports/sr658.pdf

The following figures shows the difference between the survey and the implied forward rate for the 12m and 24m tenor:



(a) 12m survey (green) vs. implied forward rate expectation (orange)

(b) 24m survey (green) vs. implied forward rate expectation (orange)

Figure 21: Term premium on the short end of the Chilean fixed income market

For the long end, the following figure shows the nominal zero coupon curve (orange) and the forward curve (green). In the forward curve, the dates marked show where 'full hikes' of 25pb are completed. In the long end, the forward curve approaches 5%, while the Central Bank estimated a neutral rate between 4-4.5%. Using the center of the range, there is roughly 75pb of term premia on the forward curve.

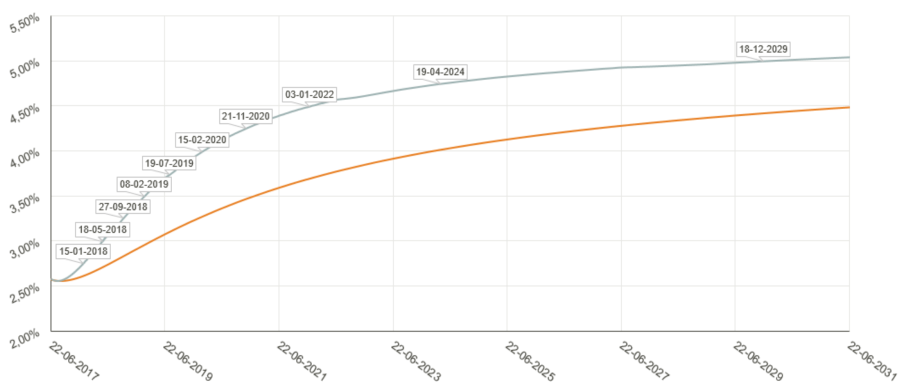


Figure 22: Implied nominal forward curve (green) vs. spot zero curve (orange). Dates mark a 'complete' 25pb of hike.

Now, in order to simplify the analysis, let's consider three macro scenarios: 'dove', 'base', 'hawk'. Each one of them implies a certain monetary policy path as follows:

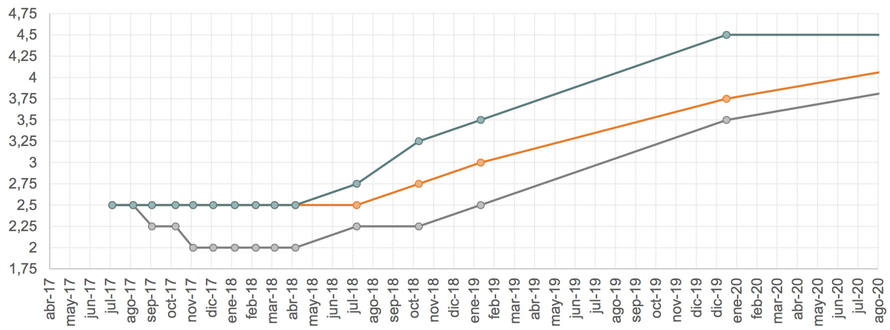


Figure 23: Monetary Policy Rate scenarios: Dove, Base, Hawk

The Central Bank of Chile monetary policy rate is now at 2.5%. In the dove scenario, the Central Bank would deliver two more rate cuts, to then start hiking on 2018 and then converge to the lower band of the neutral rate at 4%. In the base scenario, the Central Bank stays put and starts hiking in 2018 converging to the mid of the neutral policy rate band at 4.25%. In the hawk scenario, the Central Bank starts hiking earlier, converging to the upper part of the neutral rate band at 4.5%.

Adding the term premium, zero coupon curves are calculated by compounding the forward rates. Then, the yield for the different fixed income instruments are calculated discounting each of their cash flows to the zero coupon curve. The following figure shows the result:

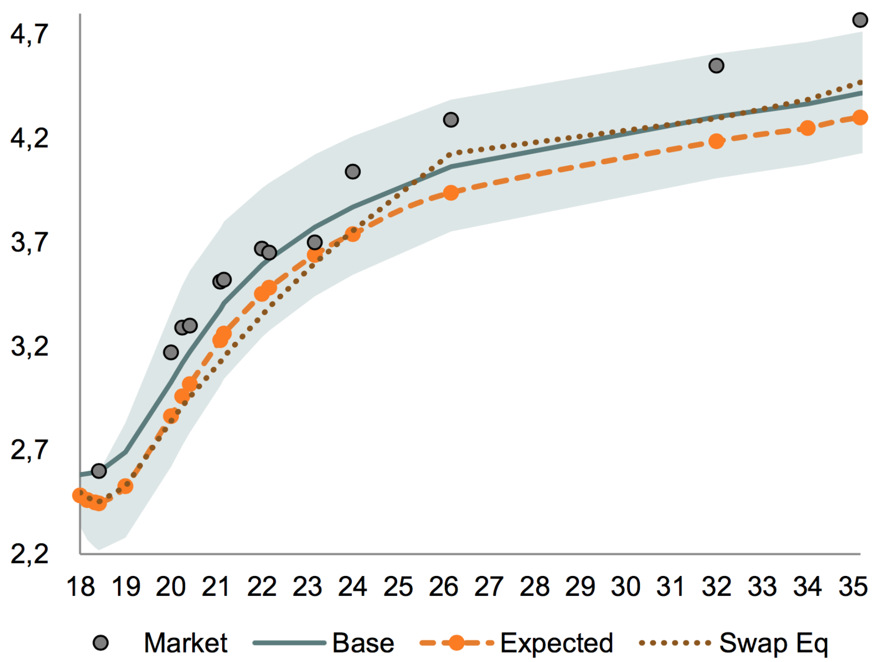


Figure 24: Nominal bond valuation using scenarios: Dove, Base, Hawk

In the figure, the area shows the difference between the hawk and dove scenario. Also the 'swap equivalent' curve is added in order to have a reference for the swap spread. Finally, probabilities for each scenario are added, in which 70%, 40% and 0% are assigned to the base, dove and hawk scenario respectively, having as result the 'expected' curve. Figure 25 shows the valuation in detail.

Instrument	Valuation	Market	Δ	Dove	Central	Hawk	Swap Eq	Swap Spread	1M Carry [bp]	3M Carry [bp]
BTP0600118	2,48			2,33	2,58	2,58	2,50			
BTP0600318	2,46			2,27	2,59	2,59	2,47			
BCP0600318	2,46			2,27	2,59	2,59	2,47			
BCP0600518	2,45			2,23	2,59	2,59	2,45			
BCP0600618	2,45	2,60	0,15	2,22	2,60	2,60	2,45	-0,15	-2,8	-8,4
BTP0600119	2,53			2,28	2,69	2,83	2,53			
BTP0600120	2,86	3,17	0,31	2,62	3,03	3,37	2,84	-0,33	1,1	3,2
BCP0450420	2,96	3,29	0,33	2,72	3,12	3,49	2,91	-0,38	1,4	4,2
BCP0450620	3,02	3,30	0,28	2,79	3,17	3,56	2,96	-0,34	1,3	4,0
BCP0600221	3,23	3,51	0,28	3,01	3,38	3,77	3,12	-0,39	1,7	5,1
BTP0450321	3,26	3,52	0,26	3,04	3,41	3,80	3,15	-0,37	1,7	5,0
BTP0600122	3,45	3,67	0,22	3,24	3,59	3,96	3,35	-0,32	1,7	5,1
BCP0600322	3,48	3,65	0,17	3,28	3,62	3,99	3,39	-0,26	1,6	4,9
BCP0600323	3,64	3,70	0,06	3,44	3,77	4,12	3,60	-0,10	1,5	4,4
BTP0600124	3,74	4,04	0,30	3,54	3,87	4,21	3,76	-0,28	1,8	5,5
BTP0450326	3,94	4,29	0,35	3,75	4,06	4,39	4,13	-0,16	1,7	5,1
BTP0600132	4,19	4,55	0,36	4,01	4,30	4,61	4,30	-0,25	1,4	4,2

Figure 25: Nominal bond valuation using scenarios: Dove, Base, Hawk

Finally, the total expected return for each instrument is computed comparing the market price to the expected price. For carry calculations, the holding period return used is 3 months. For corporate bonds is necessary to add a view for the spread in each scenario. For the inflation linked curve the same procedure is done, assigning different inflation expectations to each scenario and the compounding those expectations to the estimated nominal curve. Figure 25 shows the expected return for different asset classes.

			Overview		
Asset Class	Spot	Estimation	Carry	Capital G/L	Tot Ret Period
IF $ 120	0,23	0,23	0,69%	0,00%	0,69%
IF $ 360	0,26	0,22	0,78%	0,35%	1,13%
IF UF 120	2,00	0,23	0,89%	0,15%	1,04%
IF UF 360	1,17	0,22	0,68%	0,71%	1,39%
PESO-02	3,15	2,46	0,79%	1,21%	2,00%
PESO-05	3,70	3,48	0,93%	0,97%	1,90%
PESO-07	4,04	3,94	1,01%	0,61%	1,62%
PESO-10	4,28	4,25	1,07%	0,22%	1,29%
UF-02	0,80	-0,49	0,59%	2,25%	2,84%
UF-05	1,17	0,58	0,68%	2,64%	3,32%
UF-07	1,34	0,82	0,73%	3,14%	3,87%
UF-10	1,49	0,96	0,76%	3,74%	4,50%
Corp UF 02 / AAA-/	1,80	0,51	0,84%	2,25%	3,09%
Corp UF 05 / AAA-/	2,27	1,68	0,96%	2,64%	3,60%
Corp UF 07 / AAA-/	2,49	1,97	1,01%	3,14%	4,15%
Corp UF 10 / AAA-/	2,69	2,16	1,06%	3,74%	4,80%
Corp UF 02 / A	2,00	0,71	0,89%	2,25%	3,14%
Corp UF 05 / A	2,52	1,93	1,02%	2,64%	3,66%
Corp UF 10 / A	2,99	2,46	1,14%	3,17%	4,31%

Figure 26: Nominal bond valuation using scenarios: Dove, Base, Hawk

Then, expected returns are 'normalized' to build the P matrix, taking in account 5 main relative value views: 2 relative value trades of inflation breakeven in 5 and 10yrs, 2 relative value trade on 10yr spread for AAA and A, 1 relative value steepener trade in the nominal curve. For this exercise, the views are the following:

1. 5yr inflation breakeven will have a positive return of 1.67%
2. 10yr inflation breakeven will have a positive return of 3.52%
3. 10yr AAA spreads will have a positive returno of 0.3%
4. 10yr A spreads will have a negative return of -0.19%
5. 5/10 nominal curve will have a negative return of -0.61%

2.5.4 Black-Litterman Allocation: Posterior

Using the views established in previous section, the posterior Black-Litterman returns and weights are the following:

Index	Prior Π_{EGARCH}	Post Π_{EGARCH}	Post ω_{EGARCH}	Post Constrained ω_{EGARCH}
Gov CLP 3-5	2.8%	1.14%	27.61%	0%
Gov CLP 5+	7.74%	2.61%	-18.1%	0%
Gov UF 3-5	3.31%	2.96%	7.1%	3.94%
Gov UF 5-7	4.78%	3.94%	28.13%	13.54%
Gov UF 7+	9.15%	6.36%	9.09%	1.23%
Corp AAA UF 3-5	2.55%	2.33%	9.09%	5.53%
Corp AAA UF 5+	6.36%	4.7%	1.9%	0%
Corp A UF 3-5	1.53%	1.42%	9.09%	9.22%
Corp A UF 5+	4.92%	3.82%	7.91%	5.91%
Dep CLP 9-12M	0.09%	0.01%	9.09%	52.69%
Dep UF 9-12M	0.41%	0.52%	9.09%	7.94%

Table 17: Posterior results of Black-Litterman using the EGARCH model

Table 17 shows some interesting results. It can be noticed the overweight in Gov UF 5-7 and Gov CLP 3-5 due to view 2 and 5. Both of this views go against Gov CLP 5+, which goes underweight. Another interesting result is that although view 3 imply a positive performance of Corp AAA UF 5+ against Gov UF 5-7, the 0.3% implied is less that what the model had in the prior (6.36% - 4.78% = 1.58%). Then the model underweights AAA corporate spreads 5+. This is an important feature of Black-Litterman, as the views and their impact on posterior allocations are relative to prior equilibrium returns. For view 4 for example, although a negative relative performance is expected, this expectation is less far from what the prior was expecting (4.92% - 4.78% = 0.14%). This view is 0.27% worse than the prior, while view 3 is 1.28% worse than expected by the prior. That is why Corp A UF 5+ decreases less than Corp AAA UF 5+.

Now, for the prior and posterior distributions, the efficient frontier is computed following the optimal allocation of Markowitz mean-variance approach. Using constrains $\sum \omega_n = 1, \omega_n > 0$, results are the following:

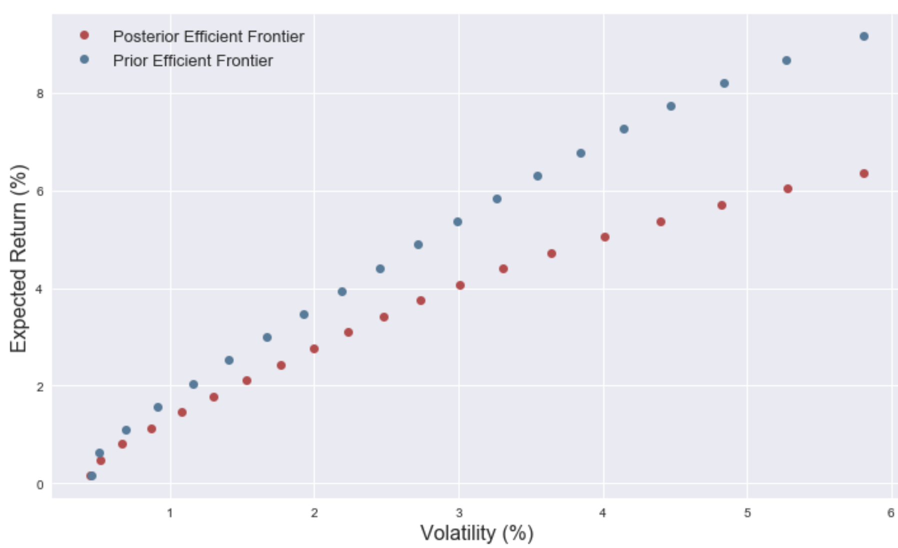
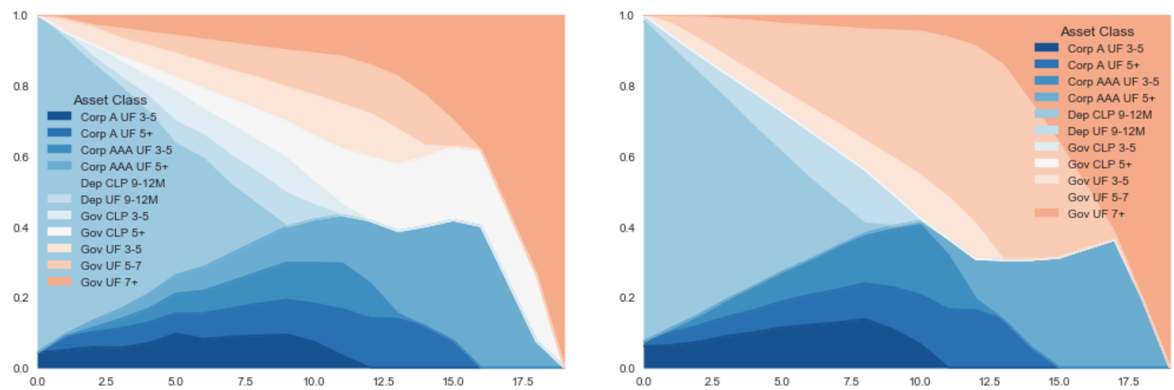


Figure 27: Efficient frontier using EGARCH



(a) Prior allocation efficient frontier

(b) Posterior allocation efficient frontier

Figure 28: Change in allocations due to Black-Litterman

The first interesting thing to note from the efficient frontier, is that in this case the result is a "worse" frontier with respect to the prior. This is because the relative views imposed where less optimistic than the prior, dragging posterior returns down. Actually all assets have less expected return in the posterior, with the exception of Dep UF 0-12M that increases 0.11%.

Figure 28 shows the changes in the allocation due to the views. As expected, Gov UF 5-7 increases it's participation in the portfolio while Gov CLP 5+ decreases. It is interesting to see that although Corp AAA UF 5+ decreases it's allocation in the posterior Black-Litterman allocation, it is still necessary for portfolios intending to achieve high returns together with Gov UF 7+.

As in the previous equity section, approaches as maximization of Sharpe Ratio and VaR minimization subject to a certain risk budget also may be done. The optimal portfolio allocation for sharpe maximization using $r_f = 2.5\%$ and constrains $\sum \omega_n = 1, \omega_n > 0$ is:

Index	$\omega_{sharpeprior}$	$\omega_{sharpeposterior}$
Gov CLP 3-5	0%	0%
Gov CLP 5+	20.19%	0%
Gov UF 3-5	0%	0%
Gov UF 5-7	0%	0%
Gov UF 7+	55.99%	77.71%
Corp AAA UF 3-5	0%	0%
Corp AAA UF 5+	23.83%	22.29%
Corp A UF 3-5	0%	0%
Corp A UF 5+	0%	0%
Dep CLP 9-12M	0%	0%
Dep UF 9-12M	0%	0%

Table 18: Optimal tangency portfolio for sharpe maximization

It is interesting to note that in the prior, the optimal tangency portfolio had 20.19% of Gov CLP 5+, while given the views, the posterior portfolio allocated all between Corp AAA UF 5+ and Gov UF 5-7. Another thing to point out, is that this two assets have the highest expected return. Looking at figure 27, the r_f level is high compared to the level of the efficient frontier, so the tangency portfolio that maximizes the slope between the risk free asset and a portfolio in the frontier (sharpe ratio), is a combination of the risk free asset with a portfolio structured with assets that have high volatility and high expected returns.

With respect to VaR optimization, the following figure shows how expected returns behave as the VaR constrain increases:

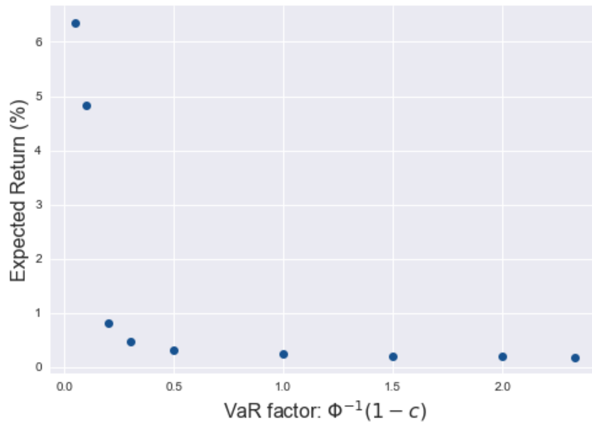


Figure 29: Optimization using VaR factor

Again, as expected, as the constraining factor increases, expected return decreases. For a factor of 2.33 equivalent to a 99% level of confidence, an expected return of 1.87% is achieved. To achieve a return of 4.76%, a factor of 0.3 is needed, meaning 61.8% level of confidence. In other words, a high level of risk budget is needed to achieve higher returns, as the fixed income assets in the portfolio have limited expected returns.

3 Implementation: Python

For the implementation of both main topics Python is used. It's a high level powerful programming language with an elaborate ecosystem of tools an libraries. Some of the advantages of this language presents [1]:

- Open source: Python and the majority of supporting libraries and tools available are open source.
- Syntax: Easy to learn, code is compact and highly readable.
- Interpreted: Is an interpreted language which makes rapid prototyping and development in general more convenient.
- Libraries: There is a wide range of powerful libraries available and supply grows steadily.
- Speed: In spite is an interpreted language, code execution speed is similar to languages like C or C++.
- Market: Python is getting popular at large financial institutions and in the hedge fund industry.

For the portfolio topic, data is provided to Python in an excel spreadsheet (BLdata.xlsx). Two spreadsheets are provided, one for the equity market and other for the fixed income market. In order for the code to read them, they must be in the same folder as the python code, and the spread sheet at use must have the name BLdata.xlsx.

4 Conclusions

This project is divided in two main topics. The first one seeks to calculate the Credit Valuation Adjustment (CVA) for an interest rate swap. To achieve this, Monte Carlo simulations for the Libor rate are done using the HJM model that evolves the forward curve using the same-form SDE. Several sensitivity analysis where done in order to explore the robustness of the results, including how the CVA evolved while changing different inputs for it's calculation. In particular the CVA was tested against the number of Monte Carlo simulations, level for the fixed rate coupon, different constant hazard rates, different recovery rates and three examples of hazard rate term structures.

The second topic seeks to construct a robust portfolio allocation using volatility filtering. To achieve this, the Black-Litterman asset allocation model was used with different volatility processes such as EGARCH, GARCH, ARCH, EWMA and constant volatility.

The method was applied to the Chilean equity and fixed income market. For the equity market, views where constructed using buy side analysts estimations from a local asset management firm. For the fixed income market, a macro-quantitative framework was presented which has the advantage that intuitively links macro views to fair value in fixed income. Mean-variance efficient frontier, Sharpe and VaR optimization where done for both asset classes, arriving to interesting and not necessarily intuitive results.

As further improvements, it would be interesting to develop a model for calculating term premium on the Chilean fixed income market in order to link the expected term premium for each macro scenario in a quantitative way. Also, it would be interesting to add quantitative data on generating the views for the equity market, such as company ratios and/or macroeconomic ratios.

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