Q2- Find approximate cashflow for floorlet on one month LIBOR, using Vasicek Model

We start, as instructed, with the yield curve power series expansion:

① We know that zero-coupon bonds satisfy:

$$(1) \quad \frac{dZ}{dt} + \frac{1}{2} w^2 \frac{d^2 Z}{dr^2} + (u - \lambda w)\frac{dZ}{dr} - rZ = 0$$

where $w, \mu, \lambda$ are functions of $(t, r)$

We look for a solution of equation above of the form (Taylor series about $t=T$)

$$Z \sim 1 + a(r)(T-t) + b(r)(T-t)^2 + \ldots \quad (2)$$

Finding derivatives: $\frac{dZ}{dt} = -a - 2b(T-t)$ ; $\frac{dZ}{dr} = a'(T-t) + b'(T-t)^2$

$$\frac{d^2 Z}{dr^2} = a''(T-t) + b''(T-t)^2$$

Now, replacing in (1)

$$-a - 2b(T-t) + \frac{1}{2}w^2\left(a''(T-t) + b''(T-t)^2\right) + (u - \lambda w)\left(a'(T-t) + b'(T-t)^2\right) - r\left(1 + a(T-t) + b(T-t)^2\right) = 0$$

re-arranging (grouping on $(T-t)$ and $(T-t)^2$)

$$\underline{-a} - (T-t)\underbrace{\left[2b - \frac{1}{2}w^2 a'' - a'(u - \lambda w) + ra\right]}_{0} + (T-t)^2 \underbrace{\left[\frac{1}{2}w^2 b'' + (u - \lambda w)b' - rb\right]}_{0} = \underline{r}$$

Now equating coefficients:

(a) $-a = r \Rightarrow \boxed{a = -r} \Rightarrow a' = -1 \Rightarrow a'' = 0$

(b) $2b - \frac{1}{2}w^2 a'' - a'(u - \lambda w) + ra = 0$

replacing $\Rightarrow 2b - 0 - (-1)(u - \lambda w) + r \cdot (-r) = 0$

$$\boxed{b = \frac{r^2 - u + \lambda w}{2}}$$

Now replacing in equation (2):

$$Z \sim 1 + (-r)(T-t) + \frac{(r^2 - u + \lambda w)}{2} \cdot (T-t)^2 + \ldots$$

So $z \sim 1 - r(T-t) + \frac{1}{2}(T-t^2)(r^2 - u + \lambda w) + \dots$ (3)

this is a Taylor series approximation for a zero-coupon bond solution

New, as we are evaluating 1 month LIBOR, we want to find the short term rate.

So, the relation between zero-coupon $(z)$ and rate $(r)$: $z(r,t;T) = e^{-r(T-t)}$

finding $r$: $\ln(z) = -r(T-t)$

$$r = -\frac{\ln(z)}{(T-t)} \quad (4)$$

So, we need to find (4) in eq. (3); Applying $\ln()$ in (3):

$$\ln(z) \sim \ln(\underbrace{1 - r(T-t) + \frac{1}{2}(T-t)^2(r^2 - u + \lambda w)}_{\text{near the short end, is small, so:}}) + \dots$$

$$\ln(1+x) \approx x \quad \text{near } x = 0$$

$\Downarrow$ then

$$\ln(z) \sim -r(T-t) + \frac{1}{2}(T-t)^2(r^2 - u + \lambda w) + \dots$$

arranging $\Rightarrow$ $\ln(z) \sim -r(T-t) + \underbrace{\frac{1}{2}(T-t)^2 \cdot r^2}_{\text{very small} \approx 0} - \frac{1}{2}(T-t)^2(u - \lambda w) + \dots$

$$\ln(z) \sim -r(T-t) - \frac{1}{2}(T-t)^2(u - \lambda w) + \dots$$

$$\underbrace{-\frac{\ln(z)}{(T-t)}}_{\text{by (4), this is an approximation for the short rate (1 month LIBOR)}} \sim r + \frac{1}{2}(T-t)(u - \lambda w) + \dots \quad (\text{all as } t \to T \text{ or } (T-t) \to 0)$$

Then, the floorlet cashflow: $\text{MAX}[K - L, 0] = \text{MAX}[r_{floorlet} - r_{LIBOR}, 0]$

In this case: $r_{floorlet} = r_f$

$$r_{LIBOR} = r - \frac{1}{2}(T-t)(u - \lambda w)$$

but: $T - t = \frac{1}{12}$ (1 month)

by Vasicek: $dr = (\eta - \gamma r)dt + c\,dw$ $\Rightarrow$ $\boxed{u - \lambda w = \eta - \gamma r}$

Floorlet cashflow

$$\text{MAX}\left[r_f - \left(r - \frac{1}{2} \cdot \frac{1}{12} \cdot (\eta - \gamma r)\right), 0\right]$$

$$\text{MAX}\left[r_f - r - \frac{1}{24}(\eta - \gamma r), 0\right] //$$

③ We wish to obtain a model of the form: $dr = u(r)dt + \nu r^{\beta} dW$ FROM DATA

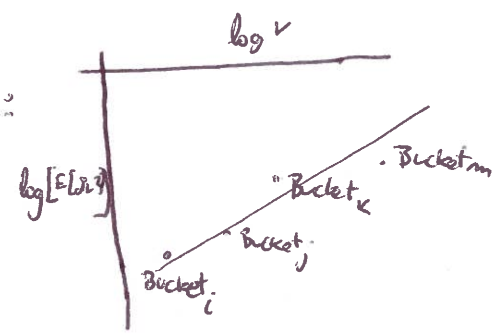
ⓐ First, let's examine the volatility structure: $\nu r^{\beta} dW$

For so: $(dr)^2 = (u(r)dt + \nu r^{\beta} dw)^2 = \underbrace{u^2(r)(dt)^2}_{\approx 0} + \nu^2 r^{2\beta} \underbrace{(dw)^2}_{dt} + 2u(r)\nu r^{\beta} \underbrace{dt\,dw}_{\approx 0}$

$$(dr)^2 = \nu^2 r^{2\beta} dt \quad (1)$$

- Now, let's divide our data in $\delta r$.
- Then, we use bucketing technique. We build buckets of $\delta r$ covering a range in $r$ values.
- Then we calculate $(\delta r)^2$ for each bucket.
- We then Avenge the values of $(\delta r)^2$ for each bucket.
- then, from (1) we have: $E[(\delta r)^2] = \nu^2 r^{2\beta} dt$ ⇒ from data time step $\delta t$ is one day
- Applying log: $\log(E[(\delta r)^2]) = 2\log \nu + 2\beta \log r$

- Now we plot $\log(E[(\delta r)^2])$ against $\log r$:




- We stimate $\beta$ with the slope and $\nu$ with the cross of the vertical axis of the stimated line.

- So volatility increses as spot rate increases

ⓑ Let's examine the drift structure. Is difficult to estimate drift from data, the drift term is smaller than the volatility term, and thus subject to larger relative errors.
Better approach is to find the drift function via empirical and analytical determination of the steady-state probability function for $r$

If $r$ statistics: $dr = u(r)dt + \nu r^{\beta} dw$, then the probability density function $p(r,t)$ for $r$ satisfies the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \frac{1}{2}\nu^2 \frac{\partial^2}{\partial r^2}(r^{2\beta} p) - \frac{\partial}{\partial r}(u(r)p) \quad (2)$$

the steady state $p_\infty(r)$ satisfies the time-independent version of (2):

$$\frac{1}{2}\nu^2\frac{\partial^2}{\partial r^2}\left(r^{2\beta}p_\infty\right) - \frac{\partial}{\partial r}\left(u(r)p_\infty\right) = 0 \qquad (3)$$

Now, integrating once (3) $\Rightarrow$ $\frac{1}{2}\nu^2\frac{d^2}{dr^2}\left(r^{2\beta}p_\infty\right) = \frac{d}{dr}\left(u(r)p_\infty\right)$ // $\int$

$$\frac{1}{2}\nu^2\frac{d}{dr}\left(r^{2\beta}p_\infty\right) = u(r)p_\infty$$

$$\frac{1}{p_\infty}\frac{1}{2}\nu^2\underbrace{\frac{d}{dr}\left(r^{2\beta}p_\infty\right)} = u(r)$$

Now, Product rule $\Rightarrow$ $\frac{1}{p_\infty}\frac{1}{2}\nu^2\left[2\beta r^{2\beta-1}\cdot p_\infty + r^{2\beta}p'_\infty\right] = u(r)$

$$\nu^2\beta r^{2\beta-1} + \underbrace{\frac{p'_\infty}{p_\infty}}\cdot\frac{1}{2}\nu^2 r^{2\beta} = u(r)$$

but $\frac{d}{dr}(\log p_\infty) = \frac{1}{p_\infty}\cdot p'_\infty$ (Chain rule)

So finally: $u(r) = \nu^2\beta r^{2\beta-1} + \frac{1}{2}\nu^2 r^{2\beta}\frac{d}{dr}(\log p_\infty)$ (4)

Now we explore the empirical data, build an histogram, and fit a density function $p_\infty$:


So, choosing $p_\infty(r) = \frac{1}{ar\sqrt{2\pi}}\exp\left(\frac{1}{-2a^2}\left(\log(r/\bar{r})\right)^2\right)$

We find $a$ and $\bar{r}$ fitting on data.

Now, replacing in (4) $\Rightarrow$ $u(r) = \nu^2\beta r^{2\beta-1} + \frac{1}{2}\nu^2 r^{2\beta}\underbrace{\frac{d}{dr}\left[\log\left(\frac{1}{ar\sqrt{2\pi}}\exp\left(\frac{1}{-2a^2}\left(\log(r/\bar{r})\right)^2\right)\right)\right]}_{\text{Let's evaluate this}}$ (5)

$\Rightarrow \frac{d}{dr}\left[\log\left(\frac{1}{ar\sqrt{2\pi}}\right) + \cancel{\log}\,\cancel{\exp}\left(\frac{1}{-2a^2}\left(\log(r/\bar{r})\right)^2\right)\right] = \frac{d}{dr}\left[\log\left(\frac{1}{ar\sqrt{2\pi}}\right) - \frac{1}{2a^2}\left(\log\frac{r}{\bar{r}}\right)^2\right] = \frac{d}{dr}\left[\log\left(\frac{1}{r}\right) + \log\left(\frac{1}{a\sqrt{2\pi}}\right) - \frac{1}{2a^2}\left(\log\left(\frac{r}{\bar{r}}\right)\right)^2\right]$

Evaluating $\frac{d}{dr}$ $\Rightarrow$ $\left[r\cdot\frac{-1}{r^2} + 0 - \frac{1}{2a^2}\left[2\log\left(\frac{r}{\bar{r}}\right)\cdot\left[\frac{1}{r}+0\right]\cdot 1\right]\right] = \left[-\frac{1}{r} - \frac{1}{ra^2}\log\left(\frac{r}{\bar{r}}\right)\right] = +\frac{1}{r}\left[-1 - \frac{1}{a^2}\log\left(\frac{r}{\bar{r}}\right)\right]$

Now, replacing in (5) $u(r) = \nu^2\beta r^{2\beta-1} + \frac{1}{2}\nu^2 r^{2\beta}\left[\frac{1}{r}\left(-1 - \frac{1}{a^2}\log\left(\frac{r}{\bar{r}}\right)\right)\right]$

$$u(r) = \nu^2 r^{2\beta-1}\left(\beta - \frac{1}{2} - \frac{1}{2a^2}\log(r/\bar{r})\right) \qquad (6)$$

So now from (6) we have a mean reverting drift $u(r)$.

Now we need to examine $\lambda$, market price of risk. There is no information in the process. We examine the short end of the curve.

So, we know that zero-coupon satisfy: $\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} + (u - \lambda w) \frac{\partial Z}{\partial r} - rZ = 0$ (7)

Doing Taylor in $T=t$ $\Rightarrow Z \sim 1 + a(r)(T-t) + b(r)(T-t)^2 + \dots\dots$

In question (2) we do all the procedure, and show that: (replacing Taylor in (7))

$$Z \sim 1 - r(T-t) + \frac{1}{2}(T-t)^2(r^2 - u + \lambda w) + \dots$$

We also show that, to find the spot rate we need: $r = \frac{-\ln(Z)}{(T-t)}$

Now, for small $(T-t)$ and assuming $\ln(1+x) \approx x$ near $x=0$, we also show that:

$$\frac{-\ln(Z)}{(T-t)} \sim r + \frac{1}{2}(T-t)(u - \lambda w) + \dots\dots \quad (8)$$

Now, from the data, we can examine the time-series data to estimate $(u - \lambda w)$ empirically. Since we have $u$ and $w$ already, we can find $\lambda$.

Here we assume that $u$, $w$ and $\lambda$ only depend on $r$. Empirically, we can see that $\lambda$ is time varying, as there are periods of fear and greed.

4) Spot rate model: $dr = (\eta - \gamma r)\,dt + (\alpha r + \beta)^{1/2}\,dW$ (Parameters $\eta, \gamma, \alpha, \beta$, constants)

Solution of the form: $Z(r,t) = e^{(A(t;T) - rB(t;T))}$

For the following bond pricing equation: $\frac{\partial Z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 Z}{\partial r^2} + (u - \lambda w)\frac{\partial Z}{\partial r} - rZ = 0$ (1)

In this case: $(u - \lambda w) = (\eta - \gamma r)$

$w = (\alpha r + \beta)^{1/2}$

Replacing on (1): $\frac{\partial Z}{\partial t} + \frac{1}{2}\cdot(\alpha r + \beta)\frac{\partial^2 Z}{\partial r^2} + (\eta - \gamma r)\frac{\partial Z}{\partial r} - rZ = 0$ (2) with final condition $Z(r,T) = 1$ (payment at maturity)

We calculate derivatives for function $Z(r,t)$:

$$\frac{\partial Z}{\partial t} = \left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)\cdot e^{A-rB} = \left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)\cdot Z$$

$$\frac{\partial Z}{\partial r} = -Be^{A-rB} = -B\cdot Z$$

$$\frac{\partial^2 Z}{\partial r^2} = B^2 e^{A-rB} = B^2 Z$$

Now, replacing in (2)

$$\left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)\cdot Z + \frac{1}{2}(\alpha r + \beta)\cdot B^2 Z + (\eta - \gamma r)\cdot(-B\cdot Z) - r\cdot Z = 0$$

Dividing by Z and re-arranging by grouping "r" terms:

$$\left(\frac{\partial A}{\partial t} + \frac{\beta}{2}\cdot B^2 - \eta B\right) + r\left(-\frac{\partial B}{\partial t} + \frac{\alpha B^2}{2} + \gamma B - 1\right) = 0 \quad (3)$$

Now, with $r \neq 0$, expressions between parentheses must be zero to fulfill the equation (3). So we now have:

$$\frac{\partial A}{\partial t} + \frac{\beta\cdot B^2}{2} - \eta B = 0 \quad \Rightarrow \quad \frac{\partial A}{\partial t} = \eta B - \frac{1}{2}\beta B^2 \quad (4)$$

$$-\frac{\partial B}{\partial t} + \frac{\alpha B^2}{2} + \gamma B - 1 = 0 \quad \Rightarrow \quad \frac{\partial B}{\partial t} = \frac{1}{2}\alpha B^2 + \gamma B - 1 \quad (5)$$

Now, we know that at maturity zero-coupon bond is worth 1:

$$\underbrace{Z(r,T;T)}_{\text{maturity}}=1 \quad \text{then} \quad e^{A(T;T)-rB(T;T)}=1 \qquad / \ln()$$

$$A(T;T)-rB(T;T)=0 \qquad (6)$$

$$\text{then } (A(T;T)=0=B(T;T)) \qquad (\text{for } r\neq 0)$$

So we now need to solve: $\frac{dB}{dt}=\frac{1}{2}\alpha B^2+\mu B-1$ with condition $A(T;T)=B(T;T)=0$

Solving: $\frac{2}{\alpha}\cdot\frac{dB}{dt}=B^2+\frac{2\mu B}{\alpha}-\frac{2}{\alpha} \Rightarrow \frac{dB}{\underbrace{\left(B^2+\frac{2\mu B}{\alpha}-\frac{2}{\alpha}\right)}_{\text{Solving roots}}}=\frac{\alpha}{2}dt$

Roots: $B_{1,2}=\dfrac{-\frac{2\mu}{\alpha}\pm\sqrt{\frac{4\mu^2}{\alpha^2}-4\cdot 1\cdot\left(\frac{-2}{\alpha}\right)}}{2}$

$$=\frac{-\frac{2\mu}{\alpha}\pm\sqrt{\frac{4}{\alpha^2}(\mu^2+2\alpha)}}{2}=\frac{\frac{2}{\alpha}\left(-\mu\pm\sqrt{\mu^2+2\alpha}\right)}{2}$$

$$B_{1,2}=\frac{-\mu\pm\sqrt{\mu^2+2\alpha}}{\alpha}$$

So now we have: $\dfrac{dB}{(B-B_1)(B-B_2)}=\dfrac{\alpha}{2}dt$

Now, for integration purposes, let's define: $\left.\begin{array}{l}a=B_1\\ b=-B_2\end{array}\right\}\ a,b=\dfrac{\mp\mu+\sqrt{\mu^2+2\alpha}}{\alpha}$

Then: $\dfrac{dB}{(B-a)(B+b)}=\dfrac{\alpha}{2}dt \Longrightarrow \displaystyle\int_0^B\frac{dB'}{(B'-a)(B'+b)}=\frac{\alpha}{2}\int dt$ where $B'$ is auxiliary variable for integration

$$\Longrightarrow \left.\frac{\log\left(\frac{B-a}{b+B}\right)}{a+b}\right|_0^B=\frac{\alpha}{2}(t-T) \qquad (6)$$

Now we have: $$\frac{(1-e^{\varphi_1(t-T)})(\varphi_1-\mu)(\varphi_1+\mu)}{\alpha(\mu+\varphi_1-(\mu-\varphi_1)e^{\varphi_1(t-T)})} = \frac{(1-e^{\varphi_1(t-T)})(\varphi_1^2-\mu^2)}{\alpha(\mu+\varphi_1-(\mu-\varphi_1)e^{\varphi_1(t-T)})}$$

But, we now that: $\varphi_1 = \sqrt{\mu^2+2\alpha} \Rightarrow$ So $\varphi_1^2-\mu^2 = 2\alpha \quad (7)$

Replacing: $$\frac{(1-e^{\varphi_1(t-T)})\,2\cancel{\alpha}}{\cancel{\alpha}(\mu+\varphi_1-(\mu-\varphi_1)\cdot e^{\varphi_1(t-T)})} = \frac{2(1-e^{\varphi_1(t-T)})}{\mu+\varphi_1-\mu e^{\varphi_1(t-T)}+\varphi_1 e^{\varphi_1(t-T)}} \quad /\cdot -1$$

$$= \frac{2(e^{\varphi_1(t-T)}-1)}{-\mu-\varphi_1+\mu e^{\varphi_1(t-T)}-\varphi_1 e^{\varphi_1(t-T)}}$$

Rearranging: $$B = \frac{2(e^{\varphi_1(t-T)}-1)}{-\mu-\varphi_1+\mu e^{\varphi_1(t-T)}+\varphi_1 e^{\varphi_1(t-T)}-2\varphi_1 e^{\varphi_1(t-T)}}$$

$$B = \frac{2(e^{\varphi_1(t-T)}-1)}{(\mu+\varphi_1)(e^{\varphi_1(t-T)}-1)-2\varphi_1 e^{\varphi_1(t-T)}} \quad /\cdot \frac{1}{e^{\varphi_1(t-T)}}$$

$$B = \frac{2(1-e^{-\varphi_1(t-T)})}{(\mu+\varphi_1)(1-e^{-\varphi_1(t-T)})-2\varphi_1} \quad /\cdot(-1)$$

$$B = \frac{2(e^{-\varphi_1(t-T)}-1)}{(\mu+\varphi_1)(e^{-\varphi_1(t-T)}-1)+2\varphi_1}$$

Then: $$B = \frac{2(e^{\varphi_1(T-t)}-1)}{(\mu+\varphi_1)(e^{\varphi_1(T-t)}-1)+2\varphi_1}$$