Q5

Two factor interest rate model: $dr = u(r,t)dt + w(r,t)\,dW_1(t)$

$$dl = p(r,t)dt + q(r,t)\,dW_2(t)$$

$$\mathbb{E}[dW_1 dW_2] = \rho dt$$

Bond with maturity T: $V(r,l,t;T)$

a) $\Pi = V(r,l,t;T) - \Delta_1 V_1(r,l,t;T_1) - \Delta_2 V_2(r,l,t;T_2)$ ⇒ portfolio hedged by two maturities

Now: $d\Pi = dV(r,l,t;T) - \Delta_1 dV_1(r,l,t;T_1) - \Delta_2 dV_2(r,l,t;T_2)$ (1)

BUT: $dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}dr^2 + \frac{1}{2}\frac{\partial^2 V}{\partial l^2}dl^2 + \frac{\partial^2 V}{\partial r \partial l}dr\,dl$ (2)

Eq (2) is by Itô, dt^2 and cross terms from $dt \approx 0$

Now, $dr^2 = \underbrace{u(r,t)^2 dt^2}_{\approx 0} + w(r,t)^2 \underbrace{dW_1^2}_{dt} + 2u(r,t)w(r,t)\underbrace{dt\,dW_1}_{O(dt^{3/2}) \approx 0}$

$dr^2 = w(r,t)^2 dt$ (3)

Now $dl^2 = \underbrace{p(r,t)^2 dt^2}_{\approx 0} + q(r,t)^2 \underbrace{dW_2^2}_{dt} + 2p(r,t)q(r,t)\underbrace{dt\,dW_2}_{\approx 0}$

$dl^2 = q(r,t)^2 dt$ (4)

Now $dr\,dl = \underbrace{u(r,t)p(r,t)dt^2}_{\approx 0} + u(r,t)q(r,t)\underbrace{dt\,dW_2}_{O(dt^{3/2}) \approx 0} + \underbrace{w(r,t)dW_1\,p(r,t)dt}_{O(dt^{3/2}) \approx 0} + w(r,t)q(r,t)\,dW_1 dW_2$

$dr\,dl = w(r,t)q(r,t)\,dW_1 dW_2$

But: $\mathbb{E}[dW_1 dW_2] = \rho dt$

So: $dr\,dl = w(r,t)q(r,t)\rho dt$ (5)

Now, replacing in (2)

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}w(r,t)^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial l^2}q(r,t)^2 dt + \frac{\partial^2 V}{\partial r \partial l}w(r,t)q(r,t)\rho dt$$

So:

$$dV = \underbrace{\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}w(r,t)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial l^2}q(r,t)^2 + \frac{\partial^2 V}{\partial r \partial l}w(r,t)q(r,t)\rho\right)dt}_{\mathcal{L}(V)dt} + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl$$

$$dV = \mathcal{L}(V)dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl \quad (6)$$

Now, applying (6) for V_1 and V_2, then replacing in (1), we have:

$$d\Pi = \underbrace{\mathcal{L}(V)dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl}_{dV} - \Delta_1\underbrace{\left(\mathcal{L}(V_1)dt + \frac{\partial V_1}{\partial r}dr + \frac{\partial V_1}{\partial l}dl\right)}_{dV_1} - \Delta_2\underbrace{\left(\mathcal{L}(V_2)dt + \frac{\partial V_2}{\partial r}dr + \frac{\partial V_2}{\partial l}dl\right)}_{dV_2}$$

(b)

Now, grouping previous equation $d\Pi$

$$d\Pi = \underbrace{\left(\mathcal{L}(V) - \Delta_1\mathcal{L}(V_1) - \Delta_2\mathcal{L}(V_2)\right)}_{A}dt + \underbrace{\left(\frac{\partial V}{\partial r} - \Delta_1\frac{\partial V_1}{\partial r} - \Delta_2\frac{\partial V_2}{\partial r}\right)}_{B}dr + \underbrace{\left(\frac{\partial V}{\partial l} - \Delta_1\frac{\partial V_1}{\partial l} - \Delta_2\frac{\partial V_2}{\partial l}\right)}_{C}dl$$

By NO-ARBITRAGE PRINCIPLE, a RISK FREE ASSET MUST EARN the RISK FREE RATE.

So, FIRST we eliminate risk (randomness) by doing $B=0$ and $C=0$:

$$\frac{\partial V}{\partial r} - \Delta_1\frac{\partial V_1}{\partial r} - \Delta_2\frac{\partial V_2}{\partial r} = 0 \quad (7)$$

$$\frac{\partial V}{\partial l} - \Delta_1\frac{\partial V_1}{\partial l} - \Delta_2\frac{\partial V_2}{\partial l} = 0 \quad (8)$$

then, the portfolio must earn the risk FREE RATE, so $Adt = r\Pi dt$

BUT $\Pi = V - \Delta_1 V_1 - \Delta_2 V_2$, so:

$$\mathcal{L}(V) - \Delta_1\mathcal{L}(V_1) - \Delta_2\mathcal{L}(V_2) = r(V - \Delta_1 V_1 - \Delta_2 V_2)$$

Rearranging: $\mathcal{L}(V) - rV = \Delta_1\left(\mathcal{L}(V_1) - V_1 r\right) + \Delta_2\left(\mathcal{L}(V_2) - V_2 r\right)$

defining: $\mathcal{L}(V) - rV = \mathcal{L}'(V)$

we have: $\mathcal{L}'(V) = \Delta_1 \mathcal{L}'(V_1) + \Delta_2 \mathcal{L}'(V_2)$

so finally: $\mathcal{L}'(V) - \Delta_1 \mathcal{L}'(V_1) - \Delta_2 \mathcal{L}'(V_2) = 0 \quad (9)$

Now, we have 3 equations (7, 8 and 9), and two unknowns Δ_1 and Δ_2, defining an inconsistent system (over-prescribed)

Now, the system can be written as:

$$\underbrace{\begin{pmatrix} \mathcal{L}'(V) & \mathcal{L}'(V_1) & \mathcal{L}'(V_2) \\ \partial V/\partial r & \partial V_1/\partial r & \partial V_2/\partial r \\ \partial V/\partial l & \partial V_1/\partial l & \partial V_2/\partial l \end{pmatrix}}_{M} \begin{pmatrix} 1 \\ -\Delta_1 \\ -\Delta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So we now set $\det(M) = 0$, this means that the first row will be a linear combination of the other two:

$$\mathcal{L}'(V) = \alpha_r \frac{\partial V}{\partial r} + \alpha_l \frac{\partial V}{\partial l}$$

Now, defining α_r and α_l in terms of market price of risk $\lambda_r(r,l,t)$ and $\lambda_l(r,l,t)$ associated with r and l:

$$\alpha_r = \lambda_r w - u \qquad \alpha_l = \lambda_l q - p$$

We have:

$$\mathcal{L}'(V) = (\lambda_r w - u)\frac{\partial V}{\partial r} + (\lambda_l q - p)\frac{\partial V}{\partial l}$$

Replacing $\mathcal{L}'(V)$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2} w^2 + \frac{1}{2}\frac{\partial^2 V}{\partial l^2} q + \frac{\partial^2 V}{\partial r \partial l} w q \rho - rV = (\lambda_r w - u)\frac{\partial V}{\partial r} + (\lambda_l q - p)\frac{\partial V}{\partial l}$$

re arranging:

$$\frac{dV}{dt} + \frac{1}{2}\frac{d^2V}{dr^2}w^2 + \frac{1}{2}\frac{d^2V}{dl^2}q + \frac{d^2V}{dr\,dl}wqp - rV + (u-\lambda_r w)\frac{dV}{dr} + (p-\lambda_l q)\frac{dV}{dl} = 0 \qquad (10)$$

(c)

So, given: $u-\lambda_r w = 0 = p-\lambda_l q$

and: $w = q = \sqrt{a+br+cl}$ with a,b,c constants.

Replacing on (10)

$$\frac{dV}{dt} + \frac{1}{2}\frac{d^2V}{dr^2}(a+br+cl) + \frac{1}{2}\frac{d^2V}{dl^2}(a+br+cl) + \frac{d^2V}{dr\,dl}(a+br+cl)\cdot p - rV + 0\cdot\frac{dV}{dr} + 0\cdot\frac{dV}{dl} = 0$$

$$\frac{dV}{dt} + \frac{1}{2}\frac{d^2V}{dr^2}(a+br+cl) + \frac{1}{2}\frac{d^2V}{dl^2}(a+br+cl) + \frac{d^2V}{dr\,dl}(a+br+cl) - rV = 0 \quad (11)$$

Solution of the form: $V = \exp\left(A(t;T) - rB(t;T) - l\,C(t;T)\right)$ (12)

Redemption Value: $V(r,l,T;T) = 1$ $\Rightarrow$ So, $A(T;T) = 0$, $B(T;T) = 0$, $C(T;T) = 0$ $\Big\}$ $\Rightarrow$ for r and l $\neq 0$

Now, let's replace (12) in (11):

$$\frac{dV}{dt} = \left(\frac{dA}{dt} - r\frac{dB}{dt} - l\frac{dC}{dt}\right)e^{A-rB-lC} = \left(\frac{dA}{dt} - r\frac{dB}{dt} - l\frac{dC}{dt}\right)\cdot V$$

$$\frac{d^2V}{dr^2} = B^2 e^{A-rB-lC} = B^2 V$$

$$\frac{d^2V}{dl^2} = C^2 e^{A-rB-lC} = C^2 V$$

$$\frac{dV}{dr\,dl} = BC e^{A-rB-lC} = BCV$$

Now in (1)

$$\left(\frac{dA}{dt} - r\frac{dB}{dt} - l\frac{dC}{dt}\right)\cdot V + \frac{1}{2}(B^2V)(a+br+cl) + \frac{1}{2}(C^2V)(a+br+cl) + BCV(a+br+cl) - rV = 0$$

Now dividing by V and grouping r and l terms:

$$r\left(-\frac{dB}{dt} + \frac{B^2b}{2} + \frac{C^2b}{2} + BCb - 1\right) + l\left(-\frac{dC}{dt} + \frac{B^2c}{2} + \frac{C^2c}{2} + BCc\right) + \frac{dA}{dt} + \frac{B^2a}{2} + \frac{C^2a}{2} + BCa = 0 \quad (B)$$

Now, with $r \neq 0$, $l \neq 0$, expressions between parentheses must be zero to fulfill the equation (B).
So now we have:

$$-\frac{dB}{dt} + \frac{B^2b}{2} + \frac{C^2b}{2} + BCb - 1 = 0 \implies \boxed{\frac{dB}{dt} = \frac{B^2b}{2} + \frac{C^2b}{2} + BCb - 1}$$

$$-\frac{dC}{dt} + \frac{B^2c}{2} + \frac{C^2c}{2} + BCc = 0 \implies \boxed{\frac{dC}{dt} = \frac{B^2c}{2} + \frac{C^2c}{2} + BCc}$$

$$\frac{dA}{dt} + \frac{B^2a}{2} + \frac{C^2a}{2} + BCa = 0 \implies \boxed{\frac{dA}{dt} = -\frac{B^2a}{2} - \frac{C^1a}{2} - BCa}$$

with boundary conditions: $A(T;T) = B(T;T) = C(T;T) = 0$ (maturity condition: $V(r,l,T;T) = 1$)