Q2: Find approximate cashflow for floorlet on one month LIBOR, using Vasicek Model

We start, as instructed, with the yield curve power series expansion:

① We know that zero-coupon bonds satisfy:

(1) $\frac{dZ}{dt} + \frac{1}{2}w^2\frac{d^2Z}{dr^2} + (u-\lambda w)\frac{dZ}{dr} - rZ = 0$ where w, u, λ are functions of (t,r)

We look for a solution of equation above of the form (Taylor series about $t=T$)

$$Z \sim 1 + a(r)(T-t) + b(r)(T-t)^2 + \ldots \quad (2)$$

Finding derivatives: $\frac{dZ}{dt} = -a - 2b(T-t)$; $\frac{dZ}{dr} = a'(T-t) + b'(T-t)^2$

$$\frac{d^2Z}{dr^2} = a''(T-t) + b''(T-t)^2$$

Now, replacing in (1)

$-a - 2b(T-t) + \frac{1}{2}w^2(a''(T-t) + b''(T-t)^2) + (u-\lambda w)(a'(T-t) + b'(T-t)^2) - r(1 + a(T-t) + b(T-t)^2) = 0$

re-arranging (grouping on $(T-t)$ and $(T-t)^2$)

$\underline{= -a - (T-t)\left[2b - \frac{1}{2}w^2a'' - a'(u-\lambda w) + ra\right] + (T-t)^2\left[\frac{1}{2}w^2b'' + (u-\lambda w)b' - rb\right]} = r$

$\qquad\qquad\qquad\underbrace{\qquad\qquad\qquad\qquad\qquad}_{0} \qquad\qquad\qquad \underbrace{\qquad\qquad\qquad\qquad}_{0}$

Now equating coefficients:

(a) $-a = r \Rightarrow \boxed{a = -r} \Rightarrow a' = -1 \Rightarrow a'' = 0$

(b) $2b - \frac{1}{2}w^2 a'' - a'(u-\lambda w) + ra = 0$

replacing $\Rightarrow 2b - 0 - (-1)(u-\lambda w) + r\cdot(-r) = 0$

$$\boxed{b = \frac{r^2 - u + \lambda w}{2}}$$

Now replacing in equation (2):

$$Z \sim 1 + (-r)(T-t) + \frac{(r^2 - u + \lambda w)}{2}\cdot(T-t)^2 + \ldots$$

So $z \sim 1 - r(T-t) + \frac{1}{2}(T-t^2)(r^2 - u + \lambda w) + \dots$ (3)

this is a Taylor series approximation for a zero-coupon bond solution

Now, as we are evaluating 1 month LIBOR, we want to find the short term rate.

So, the relation between zero-coupon and rate: $\quad z(r,t;T) = e^{-r(T-t)}$

(z)   (r)

finding r: $\quad \ln(z) = -r(T-t)$

$$r = -\frac{\ln(z)}{(T-t)} \quad (4)$$

So, we need to find (4) in eq. (3). Applying $\ln()$ in (3):

$$\ln(z) \sim \ln\left(\underbrace{1 - r(T-t) + \frac{1}{2}(T-t)^2(r^2 - u + \lambda w)}\right) + \dots$$

near the short end, is small, so:

$$\underline{\ln(1+x) \approx x} \quad \text{near } x = 0$$

$\Downarrow$ then

$$\ln(z) \sim -r(T-t) + \frac{1}{2}(T-t)^2(r^2 - u + \lambda w) + \dots$$

arranging $\Rightarrow \ln(z) \sim -r(T-t) + \underbrace{\frac{1}{2}(T-t)^2 \cdot r^2}_{\text{very small} \approx 0} - \frac{1}{2}(T-t)^2(u - \lambda w) + \dots$

$$\ln(z) \sim -r(T-t) - \frac{1}{2}(T-t)^2(u - \lambda w) + \dots$$

$$\underbrace{-\frac{\ln(z)}{(T-t)}}_{} \sim r + \frac{1}{2}(T-t)(u - \lambda w) + \dots \qquad \text{(all as } t \to T)$$
$$\text{or } (T-t) \to 0$$

by (4), this is an
approximation for the short rate (1 month LIBOR)

Then, the floorlet cashflow: $\text{MAX}[K - L, 0] = \text{MAX}[r_{floor} - r_{LIBOR}, 0]$

In this case: $r_{floorlet} = r_f$

$$r_{LIBOR} = r - \frac{1}{2}(T-t)(u - \lambda w)$$

but: $T - t = \frac{1}{12}$ (1 month)

by Vasicek: $dr = (\eta - \gamma r)dt + c\,dw$ $\Rightarrow$ $\boxed{u - \lambda w = \eta - \gamma r}$

Floorlet cash flow
$$\text{MAX}\left[r_f - \left(r - \frac{1}{2} \cdot \frac{1}{12}(\eta - \gamma r)\right), 0\right]$$

$$\text{MAX}\left[r_f - r - \frac{1}{24}(\eta - \gamma r), 0\right] \; /\!/$$

③ We wish to obtain a model of the form: $dr = u(r)dt + \nu r^{\beta} dw$ FROM DATA

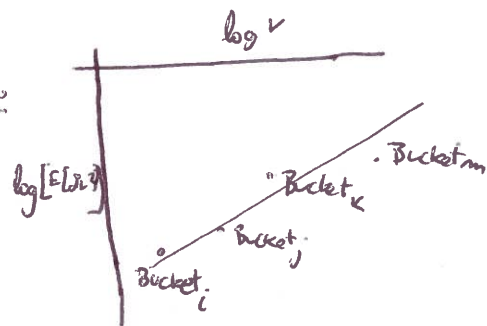ⓐ First, let's examine the **volatility structure**: $\nu r^{\beta} dw$

So: $(dr)^2 = (u(r)dt + \nu r^{\beta} dw)^2 = \underbrace{u^2(r)(dt)^2}_{\approx 0} + \nu^2 r^{2\beta} \underbrace{(dw)^2}_{dt} + 2u(r)\nu r^{\beta} \underbrace{dt\, dw}_{\approx 0}$

$$(dr)^2 = \nu^2 r^{2\beta} dt \quad (1)$$

- Now, let's divide our data in δr.

- Then, we use bucketing technique. We build buckets of δr covering a range in r values.

- Then we calculate $(\delta r)^2$ for each bucket.

- We then average the values of $(\delta r)^2$ for each bucket.

- Then, from (1) we have: $E[(\delta r)^2] = \nu^2 r^{2\beta} dt$ ⟹ from data time step δt is one day

- Applying log: $\log(E[(\delta r)^2]) = 2\log\nu + 2\beta \log r$

- Now we plot $\log(E[(\delta r)^2])$ against $\log r$:

- We estimate β with the slope and ν with the cross of the vertical axis of the estimated line.

- So volatility increases as spot rate increases

ⓑ Let's examine the **drift structure**. Is difficult to estimate drift from data, the drift term is smaller than the volatility term, and this subject to larger relative errors. Better approach is to find the drift function via empirical and analytical determination of the steady-state probability function for r

If r statistics: $dr = u(r)dt + \nu r^{\beta} dw$, then the probability density function $p(r,t)$ for r satisfies the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = \frac{1}{2}\nu^2 \frac{\partial^2}{\partial r^2}(r^{2\beta} P) - \frac{\partial}{\partial r}(u(r)P) \quad (2)$$

the steady state $P_\infty(r)$ satisfies the time-independent version of (2):

$$\frac{1}{2}\nu^2\frac{d^2}{dr^2}(r^{2\beta}P_\infty) - \frac{d}{dr}(u(r)P_\infty) = 0 \qquad (3)$$

Now, integrating once (3) $\Rightarrow$

$$\frac{1}{2}\nu^2\frac{d^2}{dr^2}(r^{2\beta}P_\infty) = \frac{d}{dr}(u(r)P_\infty) \quad \Big/ \int$$

$$\frac{1}{2}\nu^2\frac{d}{dr}(r^{2\beta}P_\infty) = u(r)P_\infty$$

$$\frac{1}{P_\infty}\frac{1}{2}\nu^2\underbrace{\frac{d}{dr}(r^{2\beta}P_\infty)}_{} = u(r)$$

Now, Product rule $\Rightarrow \dfrac{1}{P_\infty}\dfrac{1}{2}\nu^2\left[2\beta r^{2\beta-1}\cdot P_\infty + r^{2\beta}P_\infty'\right] = u(r)$

$$\nu^2\beta r^{2\beta-1} + \underbrace{\frac{P_\infty'}{P_\infty}}\cdot\frac{1}{2}\nu^2 r^{2\beta} = u(r)$$

but $\dfrac{d}{dr}(\log P_\infty) = \dfrac{1}{P_\infty}\cdot P_\infty'$ (Chain rule)

So finally: $u(r) = \nu^2\beta r^{2\beta-1} + \dfrac{1}{2}\nu^2 r^{2\beta}\dfrac{d}{dr}(\log P_\infty)$ $\quad(4)$

Now we explore the empirical data, build an histogram, and fit a density function P_∞:

So, choosing $P_\infty(r) = \dfrac{1}{ar\sqrt{2\pi}}\exp\left(\dfrac{1}{-2a^2}(\log(r/\bar{r}))^2\right)$

We find a and $\bar{r}$ fitting on data.

Now, replacing in (4) $\Rightarrow u(r) = \nu^2\beta r^{2\beta-1} + \dfrac{1}{2}\nu^2 r^{2\beta}\cdot\underbrace{\dfrac{d}{dr}\left[\log\left(\dfrac{1}{ar\sqrt{2\pi}}\cdot\exp\left(\dfrac{1}{-2a^2}(\log(r/\bar{r}))^2\right)\right)\right]}_{\text{Let's evaluate this}}$ $\quad(5)$

$$\Rightarrow \frac{d}{dr}\left[\log\left(\frac{1}{ar\sqrt{2\pi}}\right) + \log\cancel{\exp}\left(\frac{1}{-2a^2}(\log(r/\bar{r}))^2\right)\right] = \frac{d}{dr}\left[\log\left(\frac{1}{ar\sqrt{2\pi}}\right) - \frac{1}{2a^2}\left(\log\frac{r}{\bar{r}}\right)^2\right] = \frac{d}{dr}\left[\log\left(\frac{1}{r}\right) + \log\left(\frac{1}{a\sqrt{2\pi}}\right) - \frac{1}{2a^2}\left(\log\left(\frac{r}{\bar{r}}\right)\right)^2\right]$$

Evaluating $\frac{d}{dr}$ $\Rightarrow = \left[r\cdot\frac{-1}{r^2} + 0 - \frac{1}{2a^2}\left[2\log\left(\frac{r}{\bar{r}}\right)\cdot\left[\frac{1}{r} + 0\right]\cdot 1\right]\right] = \left[-\frac{1}{r} - \frac{1}{ra^2}\log\left(\frac{r}{\bar{r}}\right)\right] = +\frac{1}{r}\left[-1 - \frac{1}{a^2}\log\left(\frac{r}{\bar{r}}\right)\right]$

Now, replacing in (5) $u(r) = \nu^2\beta r^{2\beta-1} + \dfrac{1}{2}\nu^2 r^{2\beta}\cdot\left[\dfrac{1}{r}\left(-1 - \dfrac{1}{a^2}\log\left(\dfrac{r}{\bar{r}}\right)\right)\right]$

$$u(r) = \nu^2 r^{2\beta-1}\left(\beta - \frac{1}{2} - \frac{1}{2a^2}\log(r/\bar{r})\right) \quad(6)$$

So now from (6) we have a mean reverting drift $u(r)$.

Now we need to examine λ, market price of risk. There is no information in the process. We examine the short end of the curve.

So, we know that zero-coupon satisfy:
$$\frac{\partial z}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 z}{\partial r^2} + (u - \lambda w)\frac{\partial z}{\partial r} - rz = 0 \qquad (7)$$

Doing Taylor in $T = t$ $\Rightarrow$ $z \sim 1 + a(r)(T-t) + b(r)(T-t)^2 + \cdots$

<u>In question (2)</u> we do all the procedure, and show that: (replacing Taylor in (7))
$$z \sim 1 - r(T-t) + \frac{1}{2}(T-t)^2(r^2 - u + \lambda w) + \cdots$$

We also show that, to find the spot rate we need : $r = \frac{-\ln(z)}{(T-t)}$

Now, for small $(T-t)$ and assuming $\ell(1+x) \approx x$ near $x = 0$, we also show that :
$$\frac{-\ln(z)}{(T-t)} \sim r + \frac{1}{2}(T-t)(u - \lambda w) + \cdots \qquad (8)$$

Now, from the data, we can examine the time-series data to estimate $(a - \lambda w)$ empirically. Since we have u and w already, we can find λ.

Here we assume that u, w and λ only depend on r. Empirically, we can see that λ is time varying, as there are periods of fear and greed.

④ Spot rate model : $dr = (\eta - \mu r)\,dt + (\alpha r + \beta)^{\frac{1}{2}}\,dw$ (Parameters η, μ, α, β : constant)

Solution of the form : $z(r,t) = e^{(A(t;T) - r B(t;T))}$

For the following bond pricing equation : $\dfrac{dz}{dt} + \dfrac{1}{2} w^2 \dfrac{d^2 z}{dr^2} + (u - \lambda w)\dfrac{dz}{dr} - rz = 0$ (1)

In this case : $(u - \lambda w) = (\eta - \mu r)$
$w = (\alpha r + \beta)^{1/2}$

Replacing on (1) : $\dfrac{dz}{dt} + \dfrac{1}{2}\cdot(\alpha r + \beta)\dfrac{d^2 z}{dr^2} + (\eta - \mu r)\dfrac{dz}{dr} - rz = 0$ (2)   with final condition

$$z(r,T) = 1$$
(Payment at maturity)

We calculate derivatives for function $z(r,t)$:

$\dfrac{dz}{dt} = \left(\dfrac{dA}{dt} - r\dfrac{dB}{dt}\right)\cdot e^{A - rB} = \left(\dfrac{dA}{dt} - r\dfrac{dB}{dt}\right)\cdot z$

$\dfrac{dz}{dr} = -B e^{A + rB} = -B\cdot z$

$\dfrac{d^2 z}{dr^2} = B^2 e^{A - rB} = B^2 z$

Now, replacing in (2)

$\left(\dfrac{dA}{dt} - r\dfrac{dB}{dt}\right)\cdot z + \dfrac{1}{2}(\alpha r + \beta)\cdot B^2 z + (\eta - \mu r)\cdot(-B z) - rz = 0$

Dividing by z and re-arranging by grouping "r" terms :

$\left(\dfrac{dA}{dt} + \dfrac{\beta}{2}\cdot B^2 - \eta B\right) + r\left(-\dfrac{dB}{dt} + \dfrac{\alpha B^2}{2} + \mu B - 1\right) = 0$ (3)

Now, with $r \neq 0$, expressions between parentheses must be zero to fulfill the equation (3). So we now have :

$\dfrac{dA}{dt} + \dfrac{\beta \cdot B^2}{2} - \eta B = 0$ $\Rightarrow$ $\dfrac{dA}{dt} = \eta B - \dfrac{1}{2}\beta B^2$ (4)

$-\dfrac{dB}{dt} + \dfrac{\alpha B^2}{2} + \mu B - 1 = 0$ $\Rightarrow$ $\dfrac{dB}{dt} = \dfrac{1}{2}\alpha B^2 + \mu B - 1$ (5)

Now, we know that at maturity zero-coupon bond is worth 1:

$$Z(r,T,T) = 1 \quad \text{then} \quad e^{A(T,T)-rB(T,T)} = 1 \qquad /\ln()$$

(with "maturity" labeling T)

$$A(T,T) - rB(T,T) = 0 \quad (6)$$

$$\text{then} \quad (A(T,T) = 0 = B(T,T)) \qquad (\text{for } r \neq 0)$$

So we now need to solve: $\dfrac{dB}{dt} = \dfrac{1}{2}\alpha B^2 + \mu B - 1$ with condition $A(T,T) = B(T,T) = 0$

Solving: $\dfrac{2}{\alpha} \cdot \dfrac{dB}{dt} = B^2 + \dfrac{2\mu B}{\alpha} - \dfrac{2}{\alpha} \implies \dfrac{dB}{\left(B^2 + \dfrac{2\mu B}{\alpha} - \dfrac{2}{\alpha}\right)} = \dfrac{\alpha}{2}\,dt$

$$\underbrace{}_{\text{Solving roots}}$$

Roots: $B_{1,2} = \dfrac{-\dfrac{2\mu}{\alpha} \pm \sqrt{\dfrac{4\mu^2}{\alpha^2} - 4\cdot 1 \cdot \left(\dfrac{-2}{\alpha}\right)}}{2}$

$$= \dfrac{-\dfrac{2\mu}{\alpha} \pm \sqrt{\dfrac{4}{\alpha^2}\left(\mu^2 + 2\alpha\right)}}{2} = \dfrac{\dfrac{2}{\alpha}\left(-\mu \pm \sqrt{\mu^2 + 2\alpha}\right)}{2}$$

$$B_{1,2} = \dfrac{-\mu \pm \sqrt{\mu^2 + 2\alpha}}{\alpha}$$

So now we have: $\dfrac{dB}{(B-B_1)(B-B_2)} = \dfrac{\alpha}{2}\,dt$

Now, for integration purposes, let's define: $\left.\begin{array}{l} a = B_1 \\ b = -B_2 \end{array}\right\} \quad a,b = \dfrac{\mp\mu + \sqrt{\mu^2 + 2\alpha}}{\alpha}$

then: $\dfrac{dB}{(B-a)(B+b)} = \dfrac{\alpha}{2}\,dt \implies \int_0^B \dfrac{dB'}{(B'-a)(B'+b)} = \dfrac{\alpha}{2}\int dt$ where B' is auxiliary variable for integration

$$\implies \left.\dfrac{\log\left(\dfrac{B-a}{b+B}\right)}{a+b}\right|_0^B = \dfrac{\alpha}{2}(t-T) \quad (6)$$

Now we have:

$$\frac{\left(1-e^{\psi_1(t-T)}\right)(\psi_1-\mu)(\psi_1+\mu)}{\alpha\left(\mu+\psi_1-(\mu-\psi_1)e^{\psi_1(t-T)}\right)} = \frac{\left(1-e^{\psi_1(t-T)}\right)(\psi_1^2-\mu^2)}{\alpha\left(\mu+\psi_1-(\mu-\psi_1)e^{\psi_1(t-T)}\right)}$$

But, we know that: $\psi_1 = \sqrt{\mu^2+2\alpha} \Rightarrow$ So $\psi_1^2-\mu^2 = 2\alpha$ (7)

Replacing:

$$\frac{\left(1-e^{\psi_1(t-T)}\right)2\alpha}{\alpha\left(\mu+\psi_1-(\mu-\psi_1)\cdot e^{\psi_1(t-T)}\right)} = \frac{2\left(1-e^{\psi_1(t-T)}\right)}{\mu+\psi_1-\mu e^{\psi_1(t-T)}+\psi_1 e^{\psi_1(t-T)}} \qquad /\cdot -1$$

$$= \frac{2\left(e^{\psi_1(t-T)}-1\right)}{-\mu-\psi_1+\mu e^{\psi_1(t-T)}-\psi_1 e^{\psi_1(t-T)}}$$

Rearranging:

$$B = \frac{2\left(e^{\psi_1(t-T)}-1\right)}{-\mu-\psi_1+\mu e^{\psi_1(t-T)}+\psi_1 e^{\psi_1(t-T)}-2\psi_1 e^{\psi_1(t-T)}}$$

$$B = \frac{2\left(e^{\psi_1(t-T)}-1\right)}{(\mu+\psi_1)\left(e^{\psi_1(t-T)}-1\right)-2\psi_1 e^{\psi_1(t-T)}} \qquad /\cdot \frac{1}{e^{\psi_1(t-T)}}$$

$$B = \frac{2\left(1-e^{-\psi_1(t-T)}\right)}{(\mu+\psi_1)\left(1-e^{-\psi_1(t-T)}\right)-2\psi_1} \qquad /\cdot (-1)$$

$$B = \frac{2\left(e^{-\psi_1(t-T)}-1\right)}{(\mu+\psi_1)\left(e^{-\psi_1(t-T)}-1\right)+2\psi_1}$$

Then:

$$B = \frac{2\left(e^{\psi_1(T-t)}-1\right)}{(\mu+\psi_1)\left(e^{\psi_1(T-t)}-1\right)+2\psi_1}$$

Now defining: $\psi_1 = \sqrt{\mu^2 + 2\alpha}$ and evaluating (6)

$$\frac{\log\left[\dfrac{\dfrac{B - (-\mu + \psi_1)}{\alpha}}{\dfrac{(\mu + \psi_1) + B}{\alpha}}\right]}{\dfrac{2\psi_1}{\alpha}} - \frac{\log\left[\dfrac{\dfrac{0 + \mu - \psi_1}{\alpha}}{\dfrac{\mu + \psi_1}{\alpha} + 0}\right]}{\dfrac{2\psi_1}{\alpha}} = \frac{\alpha}{2}(t - T) \qquad \Big/ \cdot \frac{2\psi_1}{\alpha}$$

and rearranging

$$\log\left[\frac{\alpha B + \mu - \psi_1}{\alpha B + \mu + \psi_1}\right] - \log\left[\frac{\mu - \psi_1}{\mu + \psi_1}\right] = \psi_1(t - T)$$

$$\log\left[\frac{(\alpha B + \mu - \psi_1)}{(\alpha B + \mu + \psi_1)} \cdot \frac{(\mu + \psi_1)}{(\mu - \psi_1)}\right] = \psi_1(t - T) \qquad \Big/ e^{()}$$

$$\frac{\alpha B + \mu - \psi_1}{\alpha B + \mu + \psi_1} = \underbrace{\frac{(\mu - \psi_1)}{(\mu + \psi_1)} \cdot e^{\psi_1(t - T)}}_{\text{let's define this as } H}$$

then: $\alpha B + \mu - \psi_1 = (\alpha B + \mu + \psi_1) \cdot H$

So: $\alpha B(1 - H) = \psi_1 - \mu + H \cdot (\mu + \psi_1)$ $\qquad$ /Replacing H on the right side

$\alpha B(1 - H) = \psi_1 - \mu + (\mu - \psi_1)\left(e^{\psi_1(t - T)}\right)$

$\alpha B(1 - H) = \left(1 - e^{\psi_1(t - T)}\right) \cdot (\psi_1 - \mu)$

$$B = \frac{\left(1 - e^{\psi_1(t - T)}\right)(\psi_1 - \mu)}{\alpha(1 - H)} = \frac{\left(1 - e^{\psi_1(t - T)}\right)(\psi_1 - \mu)}{\alpha\left(1 - \dfrac{(\mu - \psi_1)}{(\mu + \psi_1)} e^{\psi_1(t - T)}\right)}$$

$$= \frac{\left(1 - e^{\psi_1(t - T)}\right)(\psi_1 - \mu)}{\dfrac{\alpha\left(\mu + \psi_1 - (\mu - \psi_1)e^{\psi_1(t - T)}\right)}{\mu + \psi_1}}$$

Q5|

Two factor interest rate model:

$$dr = u(r,t)dt + w(r,t)\,dW_1(t)$$
$$dl = p(r,t)dt + q(r,t)\,dW_2(t)$$

$$\mathbb{E}[dW_1\,dW_2] = \rho\,dt$$

Bond with maturity T: $V(r,l,t;T)$

a) $\Pi = V(r,l,t;T) - \Delta_1 V_1(r,l,t;T_1) - \Delta_2 V_2(r,l,t;T_2)$ ⟹ Portfolio hedged by two maturities

Now: $d\Pi = dV(r,l,t;T) - \Delta_1 dV_1(r,l,t;T_1) - \Delta_2 dV_2(r,l,t;T_2)$ (1)

BUT: $dV = \dfrac{\partial V}{\partial t}dt + \dfrac{\partial V}{\partial r}dr + \dfrac{\partial V}{\partial l}dl + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial r^2}dr^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial l^2}dl^2 + \dfrac{\partial^2 V}{\partial r\partial l}dr\,dl$ (2)

Eq (2) is by Itô, dt^2 and cross terms from $dt \approx 0$

Now: $dr^2 = \underbrace{u(r,t)^2 dt^2}_{\approx 0} + w(r,t)^2 \underbrace{dW_1^2}_{dt} + \underbrace{2u(r,t)w(r,t)dt\,dw_1}_{O(dt^{3/2})\approx 0}$

$dr^2 = w(r,t)^2 dt$ (3)

Now $dl^2 = \underbrace{p(r,t)^2 dt^2}_{\approx 0} + q(r,t)^2 \underbrace{dW_2^2}_{dt} + \underbrace{2p(r,t)q(r,t)dt\,dw_2}_{\approx 0}$

$dl^2 = q(r,t)^2 dt$ (4)

Now $dr\,dl = \underbrace{u(r,t)p(r,t)dt^2}_{\approx 0} + \underbrace{u(r,t)q(r,t)dt\,dw_2}_{O(dt^{3/2})\approx 0} + \underbrace{w(r,t)dW_1\,p(r,t)dt}_{O(dt^{3/2})\approx 0} + w(r,t)q(r,t)dw_1\,dw_2$

$dr\,dl = w(r,t)q(r,t)dw_1\,dw_2$

But: $\mathbb{E}[dw_1\,dw_2] = \rho\,dt$

So: $dr\,dl = w(r,t)q(r,t)\rho\,dt$ (5)

Now, replacing in (2)

$dV = \dfrac{\partial V}{\partial t}dt + \dfrac{\partial V}{\partial r}dr + \dfrac{\partial V}{\partial l}dl + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial r^2}w(r,t)^2 dt + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial l^2}q(r,t)^2 dt + \dfrac{\partial^2 V}{\partial r\partial l}w(r,t)q(r,t)\rho\,dt$

SO:

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2} w(r,t)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial l^2} q(r,t)^2 + \frac{\partial^2 V}{\partial r \partial l} w(r,t) q(r,t) \rho \right) dt + \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial l} dl$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\mathcal{L}(V)dt}$$

$$dV = \mathcal{L}(V)dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl \quad (6)$$

Now, applying (6) for V_1 and V_2, then replacing in (1), we have:

$$d\Pi = \underbrace{\mathcal{L}(V)dt + \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial l}dl}_{dV} - \Delta_1 \underbrace{\left(\mathcal{L}(V_1)dt + \frac{\partial V_1}{\partial r}dr + \frac{\partial V_1}{\partial l}dl\right)}_{dV_1} - \Delta_2 \underbrace{\left(\mathcal{L}(V_2)dt + \frac{\partial V_2}{\partial r}dr + \frac{\partial V_2}{\partial l}dl\right)}_{dV_2}$$

(b)

Now, grouping previous equation $d\Pi$

$$d\Pi = \underbrace{\left(\mathcal{L}(V) - \Delta_1 \mathcal{L}(V_1) - \Delta_2 \mathcal{L}(V_2)\right)}_{A}dt + \underbrace{\left(\frac{\partial V}{\partial r} - \Delta_1\frac{\partial V_1}{\partial r} - \Delta_2\frac{\partial V_2}{\partial r}\right)}_{B}dr + \underbrace{\left(\frac{\partial V}{\partial l} - \Delta_1\frac{\partial V_1}{\partial l} - \Delta_2\frac{\partial V_2}{\partial l}\right)}_{C}dl$$

By No-ARBITRAGE Principle, a risk FREE ASSET must earn the risk FREE RATE.

So, first we eliminate risk (randomness) by doing $B=0$ and $C=0$:

$$\frac{\partial V}{\partial r} - \Delta_1\frac{\partial V_1}{\partial r} - \Delta_2\frac{\partial V_2}{\partial r} = 0 \quad (7)$$

$$\frac{\partial V}{\partial l} - \Delta_1\frac{\partial V_1}{\partial l} - \Delta_2\frac{\partial V_2}{\partial l} = 0 \quad (8)$$

then, the portfolio must earn the risk FREE RATE, so $\Delta\Pi = r\Pi dt$
BUT $\Pi = V - \Delta_1 V_1 - \Delta_2 V_2$, so:

$$\mathcal{L}(V) - \Delta_1 \mathcal{L}(V_1) - \Delta_2 \mathcal{L}(V_2) = r(V - \Delta_1 V_1 - \Delta_2 V_2)$$

Re arranging:

$$\delta(V) - rV = \Delta_1\left(\delta(V_1) - V_1 r\right) + \Delta_2\left(\delta(V_2) - V_2 r\right)$$

defining:

$$\delta(V) - rV = \delta'(V)$$

we have:

$$\delta'(V) = \Delta_1 \delta'(V_1) + \Delta_2 \delta'(V_2)$$

So finally:

$$\delta'(V) - \Delta_1 \delta'(V_1) - \Delta_2 \delta'(V_2) = 0 \qquad (9)$$

Now, we have 3 equations (7, 8 and 9), and two unknowns Δ_1 and Δ_2, defining an inconsistent system (over-prescribed)

Now, the system can be written as:

$$\begin{vmatrix} \delta'(V) & \delta'(V_1) & \delta'(V_2) \\ \frac{\partial V}{\partial r} & \frac{\partial V_1}{\partial r} & \frac{\partial V_2}{\partial r} \\ \frac{\partial V}{\partial \ell} & \frac{\partial V_1}{\partial \ell} & \frac{\partial V_2}{\partial \ell} \end{vmatrix} \underbrace{}_{M} \begin{pmatrix} 1 \\ -\Delta_1 \\ -\Delta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So we now set $\det(M) = 0$, this means that the first row will be a linear combination of the other two:

$$\delta'(V) = \alpha_r \frac{\partial V}{\partial r} + \alpha_\ell \frac{\partial V}{\partial \ell}$$

Now, defining α_r and α_ℓ in terms of market price of risk $\lambda_r(r, \ell, t)$ and $\lambda_\ell(r, \ell, t)$ associated with r and ℓ:

$$\alpha_r = \lambda_r w - u$$

$$\alpha_\ell = \lambda_\ell q - p$$

We have:

$$\delta'(V) = (\lambda_r w - u)\frac{\partial V}{\partial r} + (\lambda_\ell q - p)\frac{\partial V}{\partial \ell}$$

Replacing $\delta'(V)$:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}w^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \ell^2}q + \frac{\partial^2 V}{\partial r \partial \ell}wq\rho - rV = (\lambda_r w - u)\frac{\partial V}{\partial r} + (\lambda_\ell q - p)\frac{\partial V}{\partial \ell}$$

re arranging:

$$\frac{dV}{dt} + \frac{1}{2}\frac{d^2V}{dr^2}w^2 + \frac{1}{2}\frac{d^2V}{dl^2}q + \frac{d^2V}{dr\,dl}wq\rho - rV + (u - \lambda_r w)\frac{dV}{dr} + (p - \lambda_l q)\frac{dV}{dl} = 0 \qquad (10)$$

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So, given: $u - \lambda_r w = 0 = p - \lambda_l q$

and : $w = q = \sqrt{a + br + cl}$ with a, b, c constants.

Replacing on (10)

$$\frac{dV}{dt} + \frac{1}{2}\frac{d^2V}{dr^2}(a + br + cl) + \frac{1}{2}\frac{d^2V}{dl^2}(a + br + cl) + \frac{d^2V}{dr\,dl}(a + br + cl)\cdot\rho - rV + 0\cdot\frac{dV}{dr} + 0\cdot\frac{dV}{dl} = 0$$

$$\frac{dV}{dt} + \frac{1}{2}\frac{d^2V}{dr^2}(a + br + cl) + \frac{1}{2}\frac{d^2V}{dl^2}(a + br + cl) + \frac{d^2V}{dr\,dl}(a + br + cl) - rV = 0 \qquad (11)$$

Solution of the form :
$$V = \exp\left(A(t;T) - rB(t;T) - lC(t;T)\right) \qquad (12)$$

Redemption Value :
$$V(r, l, T; T) = 1 \implies \text{So, } \begin{array}{l} A(T;T) = 0 \\ B(T;T) = 0 \\ C(T;T) = 0 \end{array} \Biggr\} \text{for } r \text{ and } l \neq 0$$

Now, let's replace (12) in (11) :

$$\frac{dV}{dt} = \left(\frac{dA}{dt} - r\frac{dB}{dt} - l\frac{dC}{dt}\right)e^{A - rB - lC} = \left(\frac{dA}{dt} - r\frac{dB}{dt} - l\frac{dC}{dt}\right)\cdot V \quad \checkmark$$

$$\frac{d^2V}{dr^2} = B^2 e^{A - rB - lC} = B^2 V$$

$$\frac{d^2V}{dl^2} = C^2 e^{A - rB - lC} = C^2 V$$

$$\frac{d^2V}{dr\,dl} = BC\, e^{A - rB - lC} = BC\, V$$

Now in b1)

$$\left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} - l\frac{\partial C}{\partial t}\right)\cdot V + \frac{1}{2}(B^2 V)(a+br+cl) + \frac{1}{2}(C^2 V)(a+br+cl) + BCV(a+br+cl) - rV = 0$$

Now dividing by V and grouping r and l terms:

$$r\left(-\frac{\partial B}{\partial t} + \frac{B^2 b}{2} + \frac{C^2 b}{2} + BCb - 1\right) + l\left(-\frac{\partial C}{\partial t} + \frac{B^2 c}{2} + \frac{C^2 c}{2} + BCc\right) + \frac{\partial A}{\partial t} + \frac{B^2 a}{2} + \frac{C^2 a}{2} + BCa = 0 \quad (13)$$

Now, with $r \neq 0$, $l \neq 0$, expressions between parentheses must be zero to fulfil the equation (13).
So now we have:

$$-\frac{\partial B}{\partial t} + \frac{B^2 b}{2} + \frac{C^2 b}{2} + BCb - 1 = 0 \implies \boxed{\frac{\partial B}{\partial t} = \frac{B^2 b}{2} + \frac{C^2 b}{2} + BCb - 1}$$

$$-\frac{\partial C}{\partial t} + \frac{B^2 c}{2} + \frac{C^2 c}{2} + BCc = 0 \implies \boxed{\frac{\partial C}{\partial t} = \frac{B^2 c}{2} + \frac{C^2 c}{2} + BCc}$$

$$\frac{\partial A}{\partial t} + \frac{B^2 a}{2} + \frac{C^2 a}{2} + BCa = 0 \implies \boxed{\frac{\partial A}{\partial t} = -\frac{B^2 a}{2} - \frac{C^2 a}{2} - BCa}$$

with boundary conditions: $A(T;T) = B(T;T) = C(T;t) = 0$ (maturity condition: $V(r,l,T;T) = 1$)