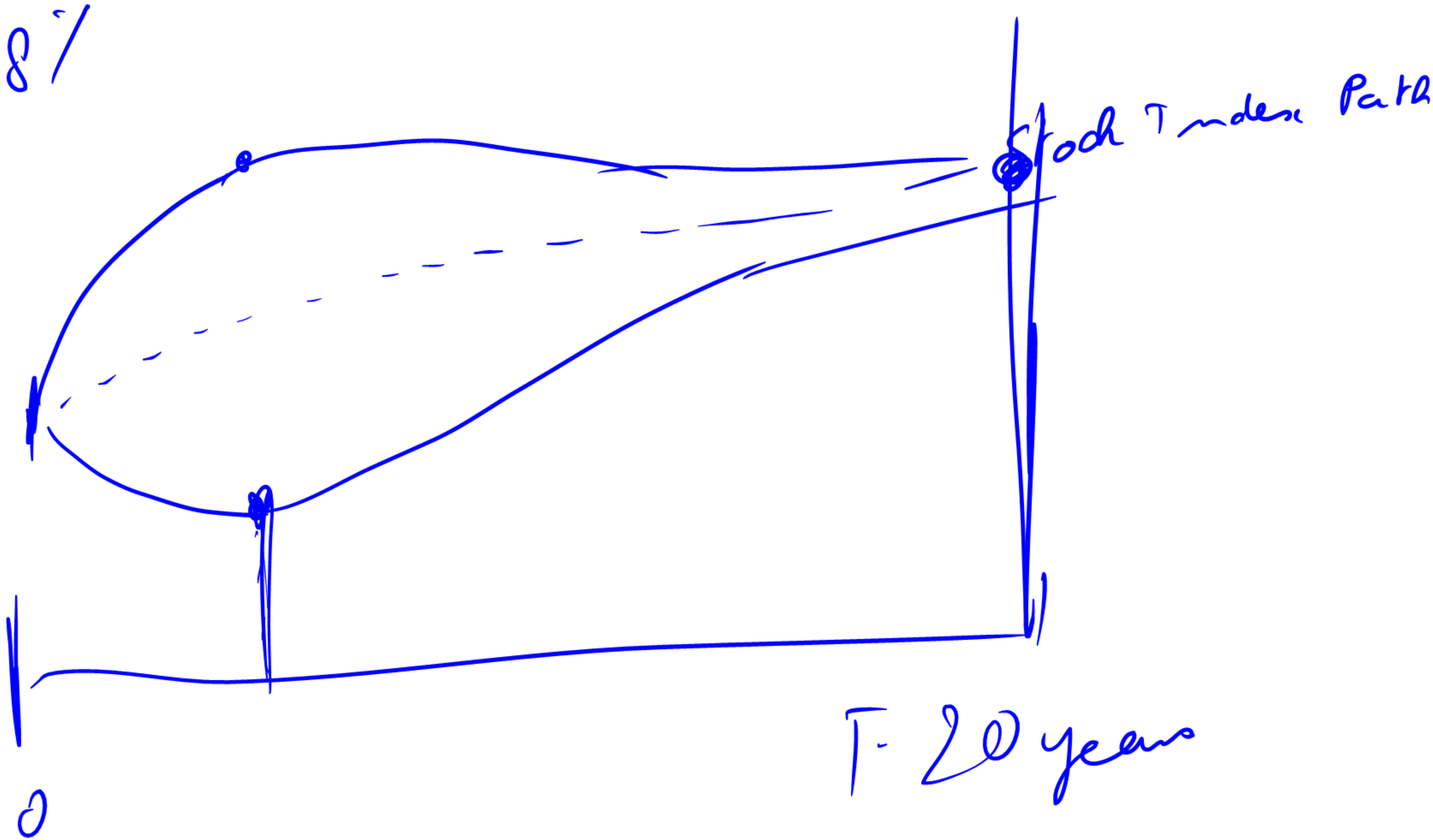

8%
Stock Index Path
0
T=20 years

Wealth / asset under management dynamics

To start, take a simple case with

- $n = 1$ risky asset
- BSM model μ;

Denote by $\begin{cases} n_t^S & \text{the number of shares in the risky asset} \\ n_t^0 & \text{the " " units in the money market asset.} \end{cases}$

Then

$$n_t^S \times S(t) = R(t) \times V(t) \qquad \Big| \qquad n_t^0 \times S_0(t) = R^0(t) \times V(t) = (1 - R(t))\, V(t)$$

$$V(t) = m_t^S S_t + m_t^0 S_0(t)$$ ← your total assets.

$$dV(t) = m_t^S \, dS_t + m_t^0 \, dS_0(t)$$

$$dV(t) = \underbrace{m_t^S S(t)}_{R(t)V(t)} (\mu dt + \sigma dW(t)) + \underbrace{m_t^0 S_0(t)}_{R^0(t)V(t)} r dt$$

$$R(t)V(t)(\mu dt + \sigma dW(t)) \qquad R^0(t)V(t) \, r dt$$

$$\boxed{\frac{dV(t)}{V(t)} = \Big(\underbrace{R^0(t)}_{1-R(t)} r + R(t)\mu\Big) dt + R(t)\sigma dW(t)}$$

$$\frac{dV(t)}{V(t)} = \Big(r + \underbrace{R(t)(\mu - r)}\Big) dt + R(t)\sigma \, dW(t)$$

$$\frac{1}{r}\mathbb{E}\left[e^{r\ln V(T)}\right]$$

$$\frac{dV(t)}{V(t)} = \left\{r + R'(t)(\mu - r\mathbb{1})\right\}dt + R'(t)\Sigma' dW(t)$$

$$\frac{\partial F}{\partial x} = \frac{1}{x} \qquad \frac{\partial^2 F}{\partial x^2} = -\frac{1}{x^2}$$

Define $Y(t) = F(V(t)) = \ln V(t)$ By Itô

$$dY(t) = \left\{r + R'(t)(\mu - r\mathbb{1})\right\}V(t)\frac{\partial F}{\partial x} + \frac{1}{2}R'(t)\Sigma'\Sigma'R(t)\,V^2(t)\frac{\partial^2 F}{\partial x^2}\Big\}dt$$

$$+ R'(t)\Sigma' V(t)\frac{\partial F}{\partial x}dW(t)$$

$$\int_t^T d\ln V(t) = \int_t^T\left\{r + R'(t)(\mu - r\mathbb{1}) - \frac{1}{2}R'(t)\Sigma'\Sigma'R(t)\right\}dt$$

$$+\int_t^T R'(t)\Sigma' dW(t)$$

$$\ln V(T) = \ln v + \int_t^T\left\{r + R'(t)(\mu - r\mathbb{1}) - \frac{1}{2}R'(t)\Sigma'\Sigma'R(t)\right\}dt \qquad V(t) = v$$

$$+\int_t^T R'(t)\Sigma\, dW(t)$$

$$e^{\gamma \ln V(T)} = \exp\left\{ \gamma \ln v + \gamma \int_0^T \left\{ r + h'(t)(\mu - r\mathbb{1}) - \frac{1}{2} h'(t)\Sigma\Sigma' h(t) \right\} dt + \gamma \int_0^T h'(t)\Sigma\, dW(t) \right\}$$

$$= v^\gamma \exp\left\{ \gamma \int_0^T \left\{ r + h'(t)(\mu - r\mathbb{1}) - \frac{1}{2} h'(t)\Sigma\Sigma' h(t) \right\} dt + \gamma \int_0^T h'(t)\Sigma\, dW(t) \underbrace{- \frac{1}{2}\gamma^2 \int_0^T h'(t)\Sigma\Sigma' h(t)\, dt} + \frac{1}{2}\gamma^2 \int_0^T h'(t)\Sigma\Sigma' h(t)\, dt \right\}$$

$$= v^\gamma \exp\left\{ \gamma \int_0^T \left\{ r + h'(t)(\mu - r\mathbb{1}) - \frac{1}{2}(1-\gamma) h'(t)\Sigma\Sigma' h(t) \right\} dt \right\}$$

$$\times \underbrace{\exp\left\{ \gamma \int_0^T h'(t)\Sigma\, dW(t) - \frac{1}{2}\gamma^2 \int_0^T h'(t)\Sigma\Sigma' h(t)\, dt \right\}}_{\chi_t^h}$$

$\to$ exponential martingale!!!

$$g(h) = -\frac{1}{2}(1-\gamma)\cancel{h'}\underbrace{\Sigma!\Sigma!' h}_{} + \boxed{h'(\mu - r\mathbb{1})} + r$$

$$h^2\sigma^2 \to 2h\sigma^2 \to 2\sigma^2 h$$

$$\left.\frac{dg}{dh}\right|_{\hat{h}} = -(1-\gamma)\Sigma!\Sigma!'\hat{h} + (\mu - r\mathbb{1}) + 0 = 0$$

$$\Sigma!\Sigma!'\hat{h} = (\mu - r\mathbb{1})$$

$$\boxed{\hat{h} = \frac{1}{1-\gamma}(\Sigma!\Sigma!')^{-1}(\mu - r\mathbb{1})}$$

$$\frac{1}{1-\gamma} \quad \underbrace{(\sigma^2)^{-1}(\mu - r)}_{\frac{\mu - r}{\sigma^2}}$$

$$\frac{d^2 g}{dh^2} < 0$$

$$\hat{h}(t) = \frac{1}{1-\gamma} \underbrace{(\Sigma \Sigma')^{-1}}_{\substack{m \times m \\ \text{matrix}}} \times \underbrace{(\mu - r \mathbb{1})}_{\text{Vector of risk premia (m-element)}}$$

-> Covariance matrix

$\mu_i - r \qquad i = 1, \dots, m$

$$\begin{pmatrix} \sigma_1^2 & \rho \sigma_i \sigma_j \\ & \ddots & \\ \rho \sigma_i \sigma_j & \sigma_m^2 \end{pmatrix}$$

$$\frac{dP_h}{d\mathbb{P}} = \chi_t^h$$

$$W_t^h = \underbrace{W_t}_{\text{under } \mathbb{P}} + \theta \int_0^t \Sigma' h(s)\, ds$$

BM under $\mathbb{P}_h$

$$dW_t^h = \underbrace{dW_t + \theta\, \Sigma' h(t)\, dt}$$

$$\Leftrightarrow dW_t = \boxed{dW_t^h - \theta \Sigma' h(t) dt}$$

We know the dynamics of the state process under $\mathbb{P}$,

under $\mathbb{P}$

$$dX(t) = (b + BX(t))dt + \Lambda \underbrace{dW(t)}_{dW_t^h - \underbrace{\theta \Sigma' h(t) dt}}$$

$$dX(t) = \left(b + BX(t) - \theta \Lambda \Sigma' h(t)\right) dt + \Lambda\, dW_t^h \quad \text{under } \mathbb{P}_h$$

Controlled factor process!

$$\phi(t,x) = \max \boxed{-\frac{1}{\theta} \ln} \, \mathbb{E}\left[\underbrace{e^{-\theta \int_0^T g(\cdot)\, dt}} \right]$$

HJB PDE for ϕ

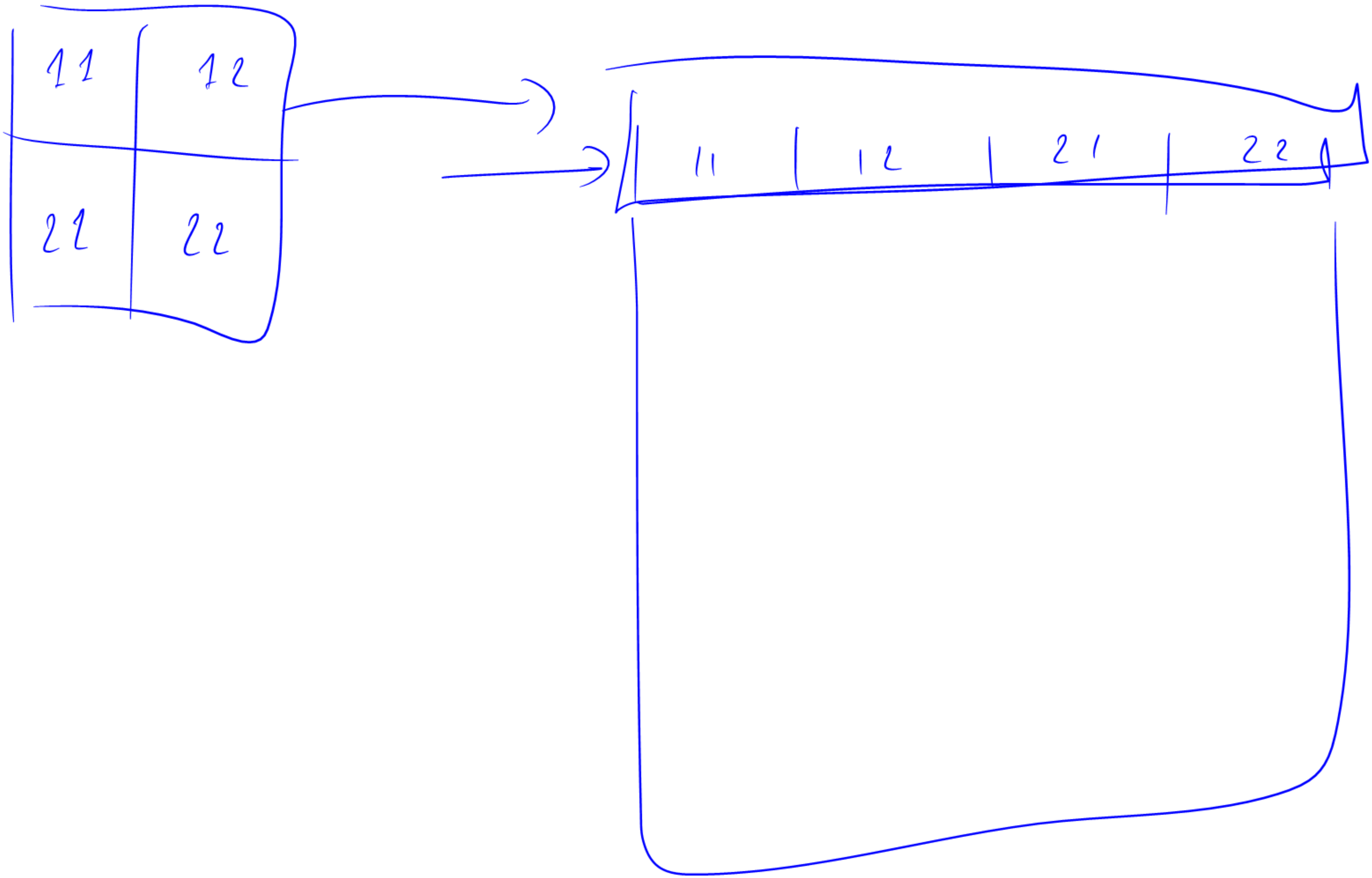
$$\phi = -\frac{1}{\theta} \ln \tilde{\phi}$$

$$\tilde{\phi}(t,x) = \underset{\sim}{\min} \; \underbrace{\mathbb{E}\left[\underbrace{e^{-\theta \int_0^T g(\cdot)\, dt}}_{\times 1} \right]}$$

$$\tilde{\phi} = e^{-\theta \phi} \iff \phi = -\frac{1}{\theta} \ln \tilde{\phi}$$

HJB PDE :

1. dynamics of $\tilde{\phi}(t, X_t)$, a function of t and X_t
   → Backward evolution operator. $D\tilde{\phi}$
2. discounting → $-g\tilde{\phi}$
3. ~~Running Rewards~~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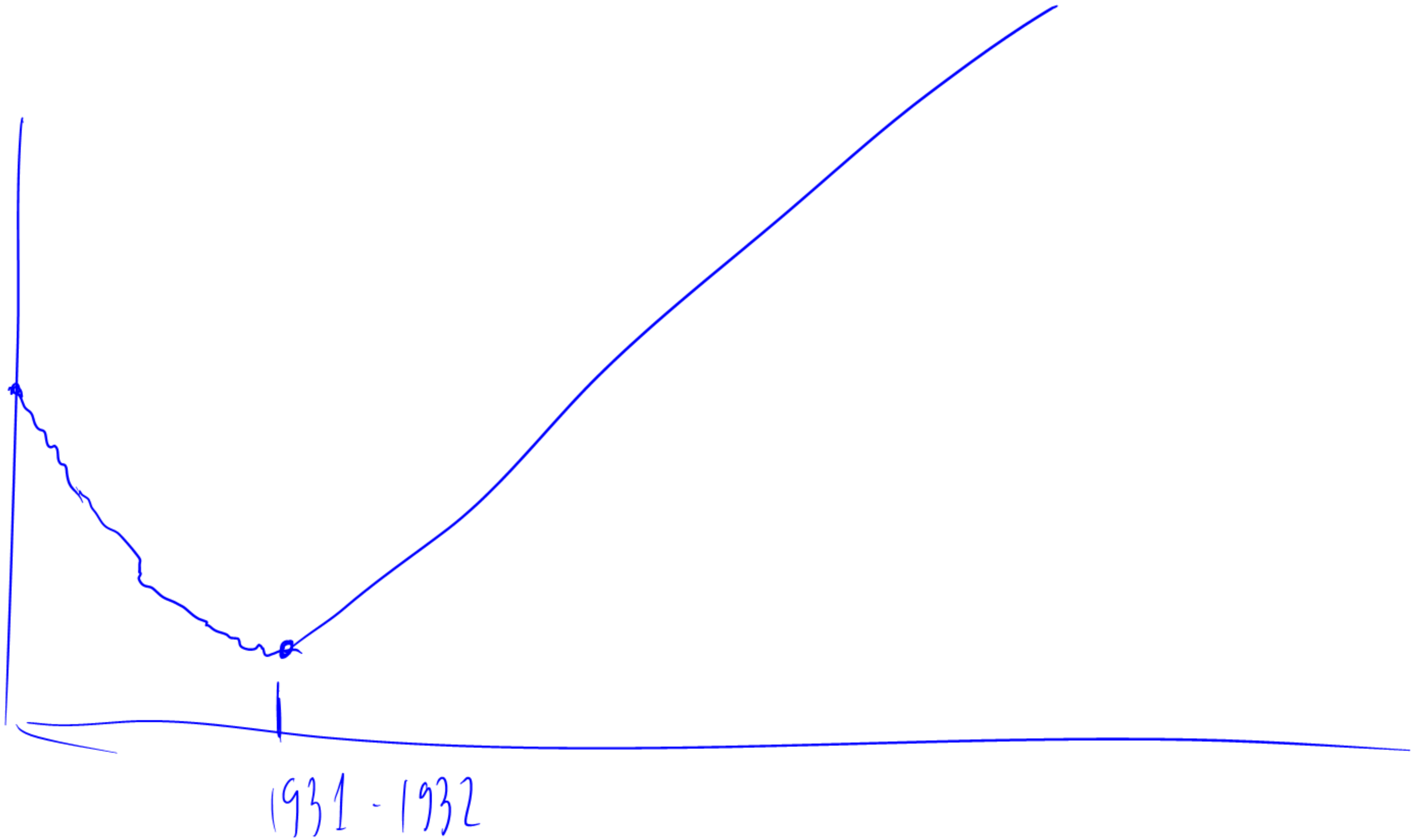
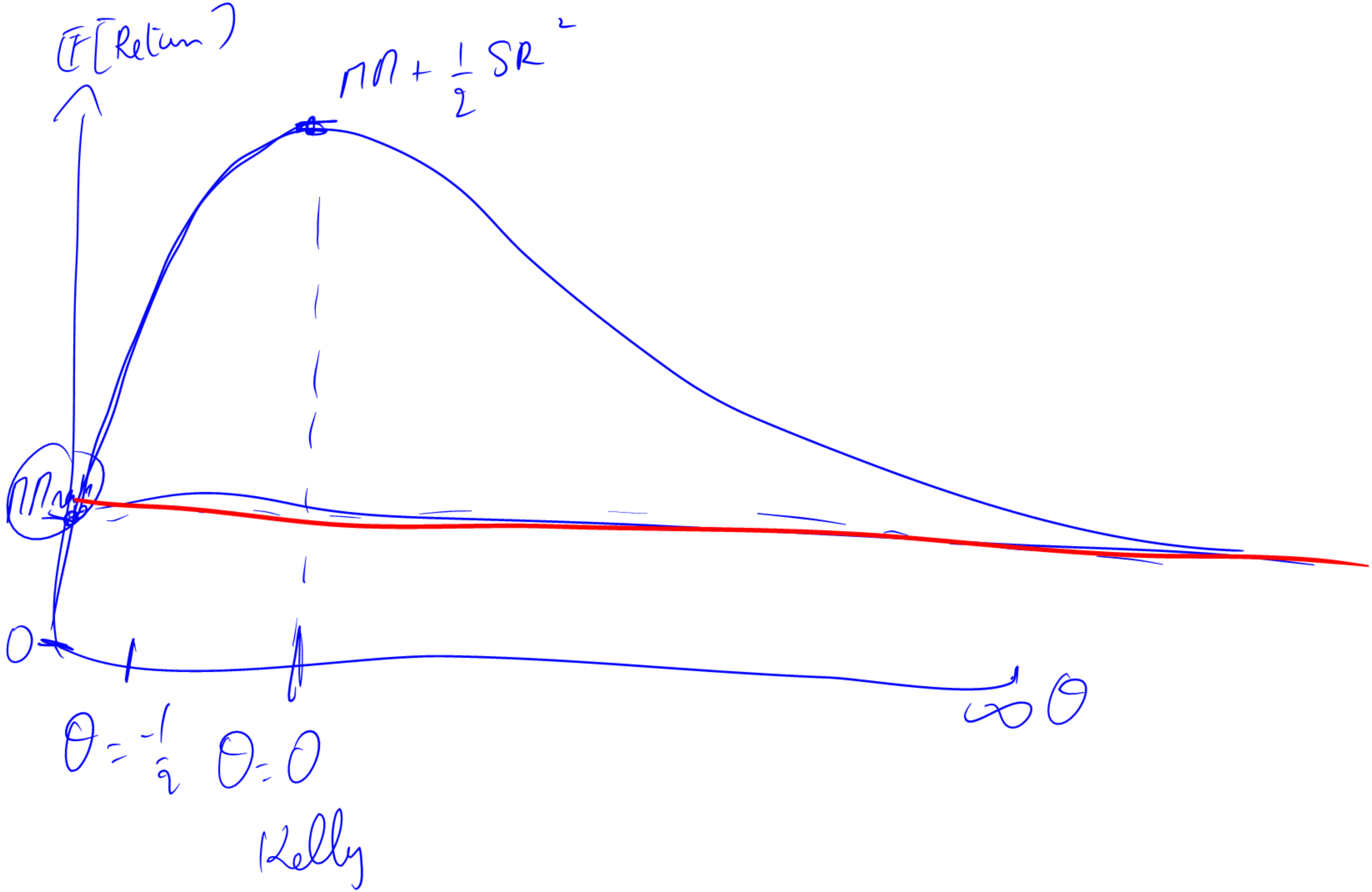

1931 - 1932



E[Return]
$\mu + \frac{1}{2} SR^2$
μ
0
$\theta = -\frac{1}{2}$
$\theta = 0$
Kelly
$\infty\theta$

Quadratic variation of a Brownian motion

$$\langle W, W \rangle_t = \lim_{K \to \infty} \sum_{k=0}^{K-1} \left(W_{t_{k+1}} - W_{t_k} \right)^2 \longrightarrow t$$

Some stochastic process Y_t

$$\langle Y, Y \rangle_t = \lim_{K \to \infty} \sum_{k=0}^{K-1} \left(Y_{t_{k+1}} - Y_{t_k} \right)^2 \longrightarrow \sigma^2 t$$

$$Y(t) = \mu \, dt + \sigma \, dW(t)$$