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Natural Frequencies and
Damping Capabilities of
Laminated Beams

by

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ABSTRACT

Further analytical investigations are made into the damping capability and determination of natural frequencies of laminated beams, consisting of elastic-viscoelastic-elastic layers, as a means for reducing the vibratory energy transmitted through machine foundation supports in naval vessels.

An exact analytical solution is obtained for determining the natural frequencies of simply-supported sandwich beams having no rivets at the ends. Three possible modes of vibration are shown to exist. The case of the simply-supported sandwich beam having rivets at each end is considered and the equations reduced to the solution of 12 x 12 determinant for calculation on a digital computer.

An approximate method is suggested for determining the natural frequencies of sandwich beams having any end conditions. The procedure is simple to use and is exact for simply-supported beams.

A simple but approximate expression is also developed for determining the composite loss factors of sandwich beams. The procedure yields good engineering results.

Administrative Information

This study was conducted under contract No. N161-26236 and this report is submitted as part of the requirements thereof in connection with the Structural Damping Program of the Ships Silencing Division of the U. S. Navy Marine Engineering Laboratory.

References:

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4. DiTaranto, R. A. "Theory of Vibratory Bending for Elastic and Viscoelastic Layered Finite-Length Beams" Journal of Applied Mechanics December 1965.

5. DiTaranto, R. A. "Analytical Investigation of the Damping Capability of Laminated Beams" MEL R&D Report 235/65 June 18, 1965.

6. DiTaranto, R. A. "A Short Survey of Viscoelastic Theory" MEL R&D Report 37/66.

Nomenclature:

ζ_1 = mid-plane extension

$R = R_1(1+i\beta)$

β = material loss factor

$R_1 = \frac{G_1 b}{2H_2 K_1}$

b = width of beam

$G = G_1(1+i\beta)$ - complex shear modulus

G_1 = real part of complex modulus

$2H_1$ = thickness of elastic layer

$2H_2$ = thickness of viscoelastic layer

$K = EA$

E = elastic modulus

A = cross-sectional area

$\delta = H_1 + H_3 + 2H_2 \doteq H_1 + H_3$

ρ = mass per unit length

$B = EI$

$I = \frac{1}{12} b(2H_1)^3$

$S = \frac{K_1+K_3}{K_1}$

$\omega^2 = \omega_1^2\,(1+i\eta)$

ω_1 = natural circular frequency

γ = mass density of elastic material

$h_1 \doteq 48\,\frac{\gamma E}{G_1^2}\,H_2^2\,\omega_1{}^2$

η = composite loss factor

L = length of beam

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Natural Frequencies and Damping Capability of Laminated Beams

I. Synopsis

1.0 Introduction

1.1 Overall Program: As part of the overall Navy Program to reduce noise emanating from vessels, a 'Structural Damping Program' is in progress at the U. S. Navy MEL. The aim of this program is to investigate methods for attenuating vibratory energy in the structure between the machines and the hull.

1.2 Specific Program: The U.S.N.M.E.L. has taken an approach to accomplish the above goal through the use of laminated material as structural members. This material is made of steel and viscoelastic layers, so that structural rigidity is maintained concurrent with damping (or dissipation of vibratory energy).

1.3 Previous Results. Tests have shown that reasonable amounts of damping are possible by using laminated material. In order to better understand, use and design with the material analytical investigations have been in progress. The results of References 1,2,3 and 4 indicate that mathematical expressions are available for determining the composite loss-factor of a laminated beam as a function of the natural frequency. These results also show that the composite loss-factor versus frequency curve is independent of the boundary conditions and solely dependent upon the physical and geometric properties of the cross-section of the beam. The natural frequencies themselves are dependent upon the boundary conditions. In reference 5 the free and forced vibrations of sandwich beams were investigated. The eigenvalues n_0, s_1, s_2, t_1 and t_2 were obtained as a function of h_1 and plotted. The mathematical expression for ζ, ζ', y, ϕ, M and V were explicitely written in terms of the eigenvalues. A generalized plot is given of the composite loss factor versus h_1 and β from which the composite loss-factor η can be readily found. Reference 6 is a short survey of the theory of viscoelasticity especially as it pertains to this program.

2.0 Scope of this Study: Details of the free vibrations of sandwich beams have been investigated. The 'exact' natural frequencies have been obtained for a simply-supported sandwich beam. In view of the complexity of the 'exact' solution, approximate solutions have been evolved which can be used to calculate the natural frequencies of sandwhich beams having any end conditions. These approximate procedures allow one to design beams for optimum damping and/or measure the properties of the viscoelastic material.

3.0 Results

3.1 Natural Frequencies - Exact: The two cases considered are the simply-supported sandwich beam with and without rivets at each end. The unriveted case yielded the analytical result that a natural frequency exists when

$$\sin n_0^{\frac{1}{2}} R_1^{\frac{1}{2}} L = 0$$

or

$$n_0^{\frac{1}{2}} R^{\frac{1}{2}} L = n\pi \; .$$

This result is to be expected and is the result for a sinusoidal deflection $y = y_0 \sin \frac{n\pi x}{L}$. An unexpected result is that natural frequencies exist when

$$2t_1'' R_1^{\frac{1}{2}} L = n\pi$$

and

$$2t_2'' R_1^{\frac{1}{2}} L = m\pi \; .$$

This result appears to be new and indicates that natural frequencies can occur in simply-supported sandwich beams for other than sinusoidal mode shapes. The t_1' vs h_1 curves show that only the lower values of n yield natural frequencies and a cut-off value of n exists above which none of this type of mode of vibration occurs. The t_2' vs h_1 curve displays a similar characteristic; i.e. only the first few values of n yield natural frequencies and a cut-off value of n exists above which no natural frequencies exist.

The case having riveted ends did not yield to a closed-form solution and required the calculation of a 12 by 12 determinant. The necessary elements of the determinant are presented in a form ready for use in a digital computer.

3.2 Approximate Relations for determining the Natural Frequencies of Sandwich Beams.

(a) The natural frequency is written in the form

$$\omega_i^2 = \frac{a_n}{\rho L^2} [B_1 + B_2 + K_i \delta^2\alpha]$$

in which

a_n is a constant dependent upon end-conditions (see page 35).

ρ is the mass/unit length

B_i is the stiffness of the i^{th} elastic layer.

$K_1 = EA_1$

A_1 is the cross-sectional area of the elastic layer.

δ is the distance between the neutral axes of layer 1 and 3.

α is shear parameter given by equation III-10

For the case in which $\beta \ll 1$, it is found that

$$\alpha = [a_n(\frac{2H_1 2H_2}{L^2}) \frac{E}{G} + 2]^{-1}$$

and

$$\omega_1^2 = \frac{4EH_1{}^3 a_n{}^2}{3(4\gamma_1 H_1 + 2\gamma_2 H_2)L^4} [1+6\alpha].$$

3.3 A simplified method for determining the shear moduli and loss-factor of a viscoelastic for $\beta \ll 1$ is proposed in section III in which the composite loss factor η and natural frequency ω_1 are measured and G_1 and β calculated. This procedure may be used as a procurement test-method for the viscoelastic material.

3.4 Optimum damping for a sandwich beam.

(a) Design - Given E, H_2; γ, ω_1 and the end conditions, one can determine the optimum length of the beam and elastic layer thickness to yield maximum damping for a given viscoelastic material.

(b) Design of the shear properties of the viscoelastic may be performed so that optimum damping may be obtained over a wide frequency range. By letting $\frac{G_1}{\omega_1}$ be constant for the maximum value of η, maximum damping is attainable over the frequency range. Thus, if the real part of the shear modulus is specifically designed into a material to increase ω_1, then an optimum material can be obtained. In all cases the tangent loss factor, β, should be as large as possible.

3.5 An approximate equation has been found for the composite loss factor of a sandwich beam, which is within 10% of the correct value. The relation is

$$\eta \doteq .66\beta \frac{h_1^{.8}}{[(2+.475\ h_1^{.6})^2+(2\beta)^2]} .$$

3.6 It is further shown in section III, that the natural frequency of a sandwich beam increases with damping. This is in agreement with a result derived in reference 3.

4.0 Continued Investigations

It is recommended that the following be considered for further investigations of sandwich beams.

4.1 Computer solutions be performed to find the natural frequencies for the following end-condition

a. Simply-supported riveted ends
b. Free-free
c. Cantilever
d. Fixed-fixed
e. Fixed-pinned

Correlation with the approximate methods should be investigated.

4.2 Consider the mode shapes and meaning of the natural frequencies associated with

$t_1L = n\pi$ and $t_2L = m\pi$.

Consider correlation with tests if necessary

II. Determination of the natural frequencies of sandwich beams - exact solution.

1.0 As discussed in reference 5, the determination of the natural frequencies of three layer laminated beams may be found by satisfying six boundary conditions, i.e., the usual four of deflection, and/or slope, and/or moment and/or shear; plus an extensional effect at each end, i.e. zero stress, and/or zero deflection and/or zero shear.

The equations for extensional effect

$$\zeta = 0$$

$$\zeta'' = 0$$

$$\zeta''' = 0$$

and for 'lateral' effects

$$y = 0$$

$$\phi = 0$$

$$M = 0$$

$$V = 0$$

are given in reference 5, as equations II-1C, II-2, II-3, II-5, II-7, II-8, and II-10. Equations for M and V are rewritten in the appendix as errata to II-8 and II-10. The general case of different elastic layers can be solved on the basis of individual cases. In general, the solution requires the evaluation of a 12 by 12 determinant and the results cannot be easily generalized. The special but important results of the sandwich beam can be generalized and this case has been solved herein for the simply-supported beam with and without rivets at each end. The simply-supported-no-rivets case requires that

$$M = y = \zeta = 0.$$

since the moment deflection and axial stress is zero at each end. These six boundary conditions yield 12 linear homogeneous equations. By selecting, the coordinate system at one end, the twelve equations become two sets of six equations each having its own set of six undetermined coefficients. Since one set is independent of the length of the beam (the only remaining variable in the determinant of coefficients) then this set can only be satisfied, in general, if the six undetermined coefficients in that determinantal set are all zero. The determinant of coefficients of the second set of six equations can be made equal to zero for three cases

$$\sin a_1 L = 0$$

$$\sin 2t_1 L = 0$$

and

$$\sin 2t_2 L = 0 \ .$$

The first case is the expected one in which the mode shape is sinusoidal ($y = y_0 \sin \frac{n\pi x}{L}$) and the natural frequencies are obtained when

$$a_1 = \frac{n\pi}{L}$$

or

$$n_0 = (\frac{n\pi}{L})^2 \cdot \frac{1}{R_1}$$

where n is an integer 1, 2, 3, etc. For a given L, R_1 and n the value of n_0 may be computed. With this value of n_0 and a value of β, one may find h_1 from the curve on page II-34 of reference 5. These natural frequencies correspond to the usual relation

$$\omega^2 = n^2 \pi^2 \frac{EI}{\rho L^4} \ .$$

The second and third cases were unexpected results. These imply that natural frequencies also occur for a simply-supported no-rivets case at other frequencies and mode-shapes than sinusoidal. The mode shapes were not investigated but the frequencies were. For a natural frequency to exist

$$2t_1 L = n\pi$$

or

$$2t_2 L = m\pi$$

where m and n are integers. A given value of n or m and L yields a value for t_1 and t_2 which can be used in curves II-37 and II-38 of reference 5. Recalling that

$$t_1 = R_1 t_1''$$

and

$$t_2 = R_1 t_2''$$

the values of t_1'' and t_2'' may be calculated and used in the curves to obtain h_1 from which ω_1 may be obtained.

It is seen that the t_1'' type of vibration could yield an h_1 for a given L and n = 1 and for n = 2 it is possible for lower value of h_1 to be obtained. At some value of n and β the curves indicate a cut-off of any of these type of vibrations. Thus the lower mode is associated with a high natural frequency of vibration in this kind of vibration. This is contrary to the usual vibration phenomenon and merits further academic investigation. The t_2'' type of vibration can yield a value of h_1 for m = 1 and a given L. For higher values of m, larger values of h_1 may be found until a cut-off frequency is reached. This too merits further academic study. The simply-supported beam with riveted ends requires that

$$M = 0$$

$$y = 0$$

and

$$\zeta'' = 0$$

at each end.

Since each equation is complex, the six boundary conditions yield twelve linear homogeneous algebraic equations. Unlike the unriveted case, the resulting 12 x 12 determinant does not simplify by factoring. In this case, the origin of the coordinate system was placed at the center of the beam and the equations written accordingly. In order to retain some generalization to the results, a factor of R_1 was eliminated from each of the equations by using

$$a_1 = n_0^{\frac{1}{2}} R_1^{\frac{1}{2}} \qquad t_1 = t_1' R_1$$

$$s_1 = s_1' R_1 \qquad t_2 = t_2' R_1$$

$$s_2 = s_2' R_1$$

and using $F = R_1^{\frac{1}{2}} L$ as a parameter. The specific equations and the 144 elements of the resulting determinant have been written and are to be used to find the natural frequencies of this type of beam. Computations are in progress on the IBM 360 computer for finding the first five natural frequencies. Using the results of reference 5 for n_0, s_1', s_2', t_1' and t_2' and values of h_1 equal to 6, 20, 50, 150, 500, 2000 and 5000 for β equal to .1 and 1, values of F will be found which satisfy the 12 x 12 determinant. These results will be reported in the future as a plot F vs h_1 for the first five modes of vibration.

2.0 Simply-supported - 3 layer beam - ends unconstrained

The boundary conditions involve the Moment, the deflection y and the axial strain.

Now

$$M = K_1\delta\zeta_1' + (B_1 + B_3)\,\phi' \qquad -1$$

and

$$\phi' = \frac{s}{\delta}\zeta_1' - \frac{1}{R\delta}\zeta_1''' \qquad -2$$

so that

$$M = [K_1\delta + (B_1+B_1)\,\frac{s}{\delta}]\zeta_1' - (\frac{B_1 + B_3}{R\delta})\,\zeta_1''' \qquad -3$$

also

$$\zeta' = 0 \quad @\ x = 0\ \&\ x = L$$

and in addition

$$y = \frac{s}{\delta}\int \zeta dx - \frac{\zeta'}{R\delta}\,. \qquad -4$$

In view of the fact that ζ' must be zero at each end, then this term in M and y may be dropped so that for these boundary conditions, there remains

$$\left.\begin{aligned} \zeta_1''' &= 0 \\ \int\zeta dx &= 0 \\ \zeta' &= 0 \end{aligned}\right\} \quad @\ x = 0 \text{ and } x = L.$$

In order to evaluate the above expressions the values of the coefficients R_{ij}, S_{ij}, T_{ij}, U_{ij} and $\int R_{ij}\,dx$ are evaluated at $x = 0$. They are:

$$
\left.\begin{array}{ll}
R_{31} = 1 & R_{32} = 0 \\
R_{41} = 0 & R_{42} = 0 \\
R_{51} = 1 & R_{52} = 0 \\
R_{61} = 0 & R_{62} = 0 \\
 & \\
S_{31} = 0 & S_{51} = 0 \\
S_{32} = 0 & S_{52} = 0 \\
S_{41} = s_1 & S_{61} = s_2 \\
S_{42} = t_1 & S_{62} = t_2 \\
T_{31} = (s_1^2 - t_1^2) & T_{51} = (s_2^2 - t_2^2) \\
T_{32} = 2s_1t_1 & T_{52} = 2s_2t_2 \\
T_{41} = 0 & T_{61} = 0 \\
T_{42} = 0 & T_{62} = 0 \\
 & \\
U_{31} = 0 & U_{51} = 0 \\
U_{32} = 0 & U_{52} = 0 \\
U_{41} = s_1[s_1^2 - 3t_1^2] & U_{61} = s_2[s_2^2 - 3t_2^2] \\
U_{42} = t_1[3s_1^2 - t_1^2] & U_{62} = t_2[3s_2^2 - t_2^2] \\
 & \\
\int R_{31}dx = 0 & \int R_{51}dx = 0 \\
\int R_{32}dx = 0 & \int R_{52}dx = 0 \\
\int R_{41}dx = \dfrac{s_1}{s_1^2 + t_1^2} & \int R_{61}dx = \dfrac{s_2}{s_2^2 + t_2^2} \\
\int R_{42}dx = -\dfrac{t_1}{s_1^2 + t_1^2} & \int R_{62}dx = -\dfrac{t_2}{s_2^2 + t_2^2}
\end{array}\right\} 5
$$

(1) at $x = 0$

$\zeta_1' = 0$

so that

(1-1) $E_{21}a_1 + E_{41}s_1 - E_{42}t_1 + E_{61}s_2 - E_{62}t_2 = 0$

and

(1-2) $E_{22}a_1 + E_{41}t_1 + E_{42}s_1 + E_{61}t_2 + E_{62}s_2 = 0$

(2) at $x = 0$

$\zeta''' = 0$

so that

(2-1) $$-E_{21}a_1^3 + E_{41}s_1(s_1^2-3t_1^2) - E_{42}t_1(3s_1^2-t_1^2) + E_{61}s_2[s_2^2-3t_2^2] - E_{62}t_2(3s_2^2-t_2^2) = 0$$

and

(2-2) $$-E_{22}a_1^3 + E_{41}t_1(3s_1^2-t_1^2) + E_{42}s_1(s_1^2-3t_1^2) + E_{62}s_2(s_2^2-3t_2^2) + E_{61}t_2(3s_2^2-t_2^2) = 0$$

(3) at $x = 0$

$\int\zeta dx = 0$

so that

(3-1) $$-E_{21}\frac{1}{a_1} + E_{41}\frac{s_1}{s_1^2+t_1^2} + E_{42}\frac{t_1}{s_1^2+t_1^2} + E_{61}\frac{s_2}{s_2^2+t_2^2} + E_{62}\frac{t_2}{s_2^2+t_2^2} = 0$$

$$(3\text{-}2) \quad -E_{22}\frac{1}{a_1} - E_{41}\frac{t_1}{s_1^2 + t_1^2} + E_{42}\frac{s_1}{s_1^2 + t_1^2} - E_{61}\frac{t_2}{s_2^2 + t_2^2} + E_{62}\frac{s_2}{s_2^2 + t_2^2} = 0$$

These six equations are all independent of the length L, indicating that the only non-trivial solution is for:

$$E_{21} = 0$$

$$E_{22} = 0$$

$$E_{41} = 0$$

$$E_{42} = 0$$

$$E_{61} = 0$$

$$E_{62} = 0$$

@ x = L

$R_{31} = \cos t_1L \cosh s_1L$ | $S_{31} = s_1R_{41} - t_1R_{42}$

$R_{32} = \sin t_1L \sinh s_1L$ | $S_{32} = s_1R_{42} + t_1R_{41}$

$R_{41} = \sinh s_1L \cos t_1L$ | $S_{41} = s_1R_{31} - t_1R_{32}$

$R_{42} = \sin t_1L \cosh s_1L$ | $S_{42} = s_1R_{32} + t_1R_{31}$

$R_{51} = \cos t_2L \cosh s_2L$ | $S_{51} = s_2R_{61} - t_2R_{62}$

$R_{52} = \sin t_2L \sinh s_2L$ | $S_{52} = s_2R_{62} + t_2R_{61}$

$R_{61} = \sinh s_2L \cos t_2L$ | $S_{61} = s_2R_{51} - t_2R_{52}$

$R_{62} = \sin t_2L \cosh s_2L$ | $S_{62} = s_2R_{52} + t_2R_{51}$

$$U_{31} = s_1[s_1^2 - 3t_1^2]\, R_{41} - t_1\, [3s_1^2 - t_1^2]R_{42}$$
$$U_{32} = s_1[s_1^2 - 3t_1^2]\, R_{42} + t_1\, [3s_1^2 - t_1^2]R_{41}$$
$$U_{41} = s_1[s_1^2 - 3t_1^2]\, R_{31} - t_1\, [3s_1^2 - t_1^2]R_{32}$$
$$U_{42} = s_1[s_1^2 - 3t_1^2]\, R_{32} + t_1\, [3s_1^2 - t_1^2]R_{31}$$
$$U_{51} = s_2[s_2^2 - 3t_2^2]\, R_{61} - t_2\, [3s_2^2 - t_2^2]R_{62}$$
$$U_{52} = s_2[s_2^2 - 3t_2^2]\, R_{62} + t_2\, [3s_2^2 - t_2^2]R_{61}$$
$$U_{61} = s_2[s_2^2 - 3t_2^2]\, R_{51} - t_2\, [3s_2^2 - t_2^2]R_{52}$$
$$U_{62} = s_2[s_2^2 - 3t_2^2]\, R_{52} + t_2\, [3s_2^2 - t_2^2]R_{51}$$

Letting

$$P_{ij} = \int R_{ij}\, dx$$

We have

$$P_{31} = \frac{s_1}{s_1^2+t_1^2} R_{41} + \frac{t_1}{s_1^2+t_1^2} R_{42}; \quad P_{52} = \frac{s_2}{s_2^2+t_2^2} R_{62} - \frac{t_2}{s_2^2+t_2^2} R_{61}$$

$$P_{32} = \frac{s_1}{s_1^2+t_1^2} R_{42} - \frac{t_1}{s_1^2+t_1^2} R_{41}; \quad P_{61} = \frac{s_2}{s_2^2+t_2^2} R_{51} + \frac{t_2}{s_2^2+t_2^2} R_{52}$$

$$P_{41} = \frac{s_1}{s_1^2+t_1^2} R_{31} + \frac{t_1}{s_1^2+t_1^2} R_{32}; \quad P_{62} = \frac{s_2}{s_2^2+t_2^2} R_{52} - \frac{t_2}{s_2^2+t_2^2} R_{51}$$

$$P_{42} = \frac{s_1}{s_1^2+t_1^2} R_{32} - \frac{t_1}{s_1^2+t_1^2} R_{31}$$

$$P_{51} = \frac{s_2}{s_2^2+t^2} R_{61} + \frac{t_2}{s_2^2+t_2^2} R_{62}$$

Considering the boundary conditions at the other end.

(4) at $\chi = L$

$\zeta' = 0$

(4-1) $-E_{11}\, a_1 \sin a_1 L + E_{31}S_{31} - E_{32}S_{32} + E_{51}S_{51} - E_{52}S_{52} = 0$

and

(4-2) $-E_{12}\, a_1 \sin a_1 L + E_{31}S_{32} + E_{32}S_{31} + E_{51}S_{52} + E_{52}S_{51} = 0$

(5) at $\chi = L$

$\zeta''' = 0$

so that

(5-1) $E_{11}\, a_1^3 \sin a_1 L + E_{31}U_{31} - E_{32}U_{32} + E_{51}U_{51} - E_{52}U_{52} = 0$

and

(5-2) $E_{12}\, a_1^3 \sin a_1 L + E_{31}U_{32} + E_{32}U_{31} + E_{51}U_{52} + E_{52}U_{51} = 0$

(6) at $\chi = L$

$\int \zeta\, dx = 0$

so that

(6-1) $$E_{11}\left(\frac{\sin a_1 L}{a_1}\right) + E_{31}\left[\frac{s_1}{s_1^2 + t_1^2} R_{41} + \frac{t_1}{s_1^2 + t_1^2} R_{42}\right]$$

$$-E_{32}\left[\frac{s_1}{s_1^2 + t_1^2} R_{42} - \frac{t_1}{s_1^2 + t_1^2} R_{41}\right] + E_{51}\left[\frac{s_2}{s_2^2 + t_2^2} R_{61} + \frac{t_2}{s_2^2 + t_2^2} R_{62}\right]$$

$$-E_{52}\left[\frac{s_2}{s_2^2 + t_2^2} R_{62} - \frac{t_2}{s_2^2 + t_2^2} R_{61}\right] = 0$$

and

(6-2) $$E_{12}\left(\frac{\sin a_1L}{a_1}\right) + E_{31}\left[\frac{s_1}{s_1^2 + t_1^2} R_{42} - \frac{t_1}{s_1^2 + t_1^2} R_{41}\right]$$

$$-E_{32}\left[\frac{s_1}{s_1^2 + t_1^2} R_{41} + \frac{t_1}{s_1^2 + t_1^2} R_{42}\right]$$

$$-E_{51}\left[\frac{s_2}{s_2^2 + t_2^2} R_{62} - \frac{t_2}{s_2^2 + t_2^2} R_{61}\right]$$

$$-E_{52}\left[\frac{s_2}{s_2^2 + t_2^2} R_{61} + \frac{t_2}{s_2^2 + t_2^2} R_{62}\right] = 0$$

It is seen that when the six above equations are written in matrix form

(6-3) $$|a_{ij}|\ |E_{ik}| = 0$$

that the first two columns are made up of a the common factor $\sin a_1L$, so that the matrix equation may be written as

(6-4) $$\sin^2 aL\ |b_{ij}||E_{ik}| = 0$$

The matrix $|b_{ij}|$ is the same as $|a_{ij}|$ except for the first two columns in which

(6-5) $$a_{i1} = b_{i1} \sin a_1L$$

and

(6-6) $$a_{i2} = b_{i2} \sin a_1L.$$

Equation (   ) shows that a solution is obtained when

(6-7) $$\sin a_1L = 0$$

or

(6-8) $$a_1L = n\pi$$

or

(6-9) $$a_1 = \frac{n\pi}{L}.$$

The b_{ij} determinant has the following elements

$b_{11} = -a_1$ | $b_{21} = 0$
$b_{12} = 0$ | $b_{22} = -a_1$
$b_{13} = s_1R_{41}-t_1R_{42}$ | $b_{23} = s_1R_{42}+t_1R_{41}$
$b_{14} = -s_1R_{42}-t_1R_{41}$ | $b_{24} = s_1R_{41}-t_1R_{42}$
$b_{15} = s_2R_{61}-t_2R_{62}$ | $b_{25} = s_2R_{62}+t_2R_{61}$
$b_{16} = -s_2R_{62}-t_2R_{61}$ | $b_{26} = s_2R_{61}-t_2R_{62}$

$b_{31} = a_1^{\,3}$ | $b_{41} = 0$
$b_{32} = 0$ | $b_{42} = a_1^{\,3}$
$b_{33} = C_1R_{41}-C_2R_{42}$ | $b_{43} = C_1R_{42}+C_2R_{41}$
$b_{34} = -C_1R_{42}-C_2R_{41}$ | $b_{44} = C_1R_{41}-C_2R_{42}$
$b_{35} = C_3R_{61}-C_4R_{62}$ | $b_{45} = C_3R_{61}-C_4R_{62}$
$b_{36} = -C_3R_{62}+C_4R_{61}$ | $b_{46} = C_3R_{62}+C_4R_{61}$

where
$C_1 = s_1(s_1^{\,2}-3t_1^{\,2})$ and $C_2 = t_1(3s_1^{\,2}-t_1^{\,2})$
$C_3 \quad s_2(s_2^{\,2}-3t_2^{\,2})$ and $C_4 = t_2(3s_2^{\,2}-t_2^{\,2})$

$b_{51} = \frac{1}{a_1}$ | $b_{61} = 0$
$b_{52} = 0$ | $b_{62} = \frac{1}{a_1}$
$b_{53} = d_1R_{41}+d_2R_{42}$ | $b_{63} = d_1R_{42}-d_2R_{41}$
$b_{54} = -d_1R_{42}+d_2R_{41}$ | $b_{64} = -d_1R_{41}-d_2R_{42}$
$b_{55} = d_3R_{61}+d_4R_{62}$ | $b_{65} = -d_3R_{62}+d_4R_{61}$
$b_{56} = -d_3R_{62}+d_4R_{61}$ | $b_{66} = -d_3R_{61}-d_4R_{62}$

where $d_1 = \frac{s_1}{s_1^{\,2}+t_1^{\,2}}$ and $d_2 = \frac{t_1}{s_1^{\,2}+t_1^{\,2}}$

$d_3 = \frac{s_2}{s_2^{\,2}+t_2^{\,2}}$ and $d_4 = \frac{t_2}{s_2^{\,2}+t_2^{\,2}}$

Equation (6-4) may be written as the sum of two matrices or

$$b'_{ij}\, E_{kj} + b''_{ij}\, E_{kj} = 0.$$

$$\begin{vmatrix} -a_1 & 0 & s_1R_{41} & -s_1R_{42} & s_2R_{61} & -s_2R_{62} \\ 0 & -a_1 & t_1R_{41} & -t_1R_{42} & t_2R_{61} & -t_2R_{62} \\ a_1{}^3 & 0 & C_1R_{41} & -C_1R_{42} & C_3R_{61} & -C_3R_{62} \\ 0 & a_1{}^3 & C_2R_{41} & -C_2R_{42} & C_3R_{61} & C_3R_{62} \\ 0 & 0 & d_1R_{41} & -d_1R_{42} & d_3R_{61} & -d_3R_{62} \\ 0 & 0 & -d_2R_{41} & -d_2R_{42} & d_4R_{61} & -d_4R_{62} \end{vmatrix} \begin{vmatrix} E_{11} \\ E_{12} \\ E_{13} \\ E_{14} \\ E_{15} \\ E_{16} \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & 0 & -t_1R_{42} & -t_1R_{41} & -t_2R_{62} & -t_2R_{61} \\ 0 & 0 & s_1R_{42} & s_1R_{41} & s_2R_{62} & -s_2R_{61} \\ 0 & 0 & -C_2R_{42} & -C_2R_{41} & -C_4R_{62} & C_4R_{61} \\ 0 & 0 & C_1R_{42} & C_1R_{41} & -C_4R_{62} & C_4R_{61} \\ \frac{1}{a_1} & 0 & d_2R_{42} & d_2R_{41} & d_4R_{62} & d_4R_{61} \\ 0 & \frac{1}{a_1} & d_1R_{42} & -d_1R_{41} & -d_3R_{62} & -d_3R_{61} \end{vmatrix} \begin{vmatrix} E_{11} \\ E_{12} \\ E_{13} \\ E_{14} \\ E_{15} \\ E_{16} \end{vmatrix} = 0 \qquad (6\text{-}10)$$

It is seen that the above equation may be written in the form

$$R_{41}R_{42}R_{61}R_{62} \quad C_{ij} \quad E_{kj} \quad + \quad R_{42}R_{41}R_{62}R_{61} \quad D_{ij} \quad E_{kj} \quad = \quad 0 \tag{6-11}$$

Now

$$|C_{ij}| \nless 0$$

and $|D_{ij}| \nless 0$

therefore

$$R_{41}R_{42}R_{61}R_{62} = 0 \tag{6-12}$$

This becomes

$$\sinh s_1L \cos t_1L \sin t_1L \cosh s_1L \sinh s_2L \cos t_2L \sin t_2L \cosh s_2L = 0$$

or

$$(\tfrac{1}{2} \sin 2t_1L)(\tfrac{1}{2} \sin 2t_2L)(\tfrac{1}{2} \sinh 2s_1L)(\tfrac{1}{2}\sinh s_2L) = 0 \tag{6-13}$$

therefore

$$2t_1L = n\pi \tag{6-14}$$

and

$$2t_2L = m\pi. \tag{5-15}$$

Using the relation developed in ref (5) we have

$$t_1' \ R_1^{\frac{1}{2}} = t_1 \tag{6-16}$$

$$t_2' \ R_1^{\frac{1}{2}} = t_2 \tag{6-17}$$

so that a natural frequency exists when

$$2t_1'R_1^{\frac{1}{2}}L = 2t_1'F = n\pi \tag{6-18}$$

and

$$2t_2'R_1^{\frac{1}{2}}L = 2t_2'F = m\pi \ . \tag{6-19}$$

This may be written as

$$t_1' = \frac{n\pi}{2F} \tag{6-20}$$

and

$$t_2' = \frac{m\pi}{2F} \ . \tag{6-21}$$

This may be interpreted in the following manner. Given a beam in which R_1 and L are known then assume n = 1; a value for t_1' is calculated. (recall $R_1 = \frac{G_1 b}{2H_2K_1}$). This value of t_1' is used to find h_1 for a given β using the chart on pg II - 37 ref (5). Knowing h_1, the corresponding composite loss-factor η may be found from pg II - 33 of the same reference and the natural frequency ω_1 may be calculated using

$$h_1 \doteq 48 \frac{\gamma E}{G_1^{\,2}} H_2^{\,2} \omega_1^{\,2} .$$

Integer values of n (2, 3, 4 etc) may be substituted to obtain larger values of t_1' and by use of the charts obtain the corresponding natural frequency and composite loss-factor. For a given β the t_1' vs h_1 curves indicate lower natural frequencies for increasing n with a cut-off frequency for which no natural frequencies occur above a given value of n.

A similar procedure may be followed to obtain the natural frequencies and associated composite loss-factor for t_2' using the curves of t_2' vs h_1 on pg II - 38. In this case the natural frequencies increase with m and reach a cut-off frequency for a given value of β.

3.0 Simply-supported-3 layer beam-ends constrained (riveted)

The boundary conditions at both ends are

$$M = 0$$
$$y = 0$$
$$\text{and} \quad \zeta'' = 0, \text{ (ends riveted).}$$

The moment and deflection equations do not simplify as for the previous case with no rivets. The moment equation of reference 5 is corrected and shown as errata at the end of this report. For ease of writing, the three equations are written as

$$M = \sum\sum[p_{ji}x_i + i\alpha_{ji}x_i] = 0$$

$$y = \sum\sum[d_{ji}x_i + ig_{ji}x_i] = 0$$

and

$$\zeta'' = \sum\sum[u_{ji}x_i + il_{ji}x_j] = 0 \ .$$

The origin of the coordinate system is selected at the middle of the beam, so that the above three relations hold at x equal to plus and minus L/2. For a solution to exist the resulting 12 x 12 determinant of coefficients must be zero. All elements of the determinant are written in terms of R_{ij}. The R_{ij} terms are trigonometric or hyperbolic functions of

$$n_0 \frac{R_1^{\frac{1}{2}}L}{2}, \ s_1' R_1^{\frac{1}{2}} \frac{L}{2}, \ s_2' R_1^{\frac{1}{2}} \frac{L}{2}, \ t_1' R_1^{\frac{1}{2}} \frac{L}{2} \text{ and/or } t_2' R_1^{\frac{1}{2}} \frac{L}{2}.$$

The factor $R^{\frac{1}{2}}L$ is made a parameter F and for a given geometric cross section and physical properties of the elastic and visco-elastic layers a variation of F with h_1 can be obtained by solving the 12 x 12 determinant on a digital computer. This will be done in the near future.

$$m_2' = m_2 R_1 = \frac{B_1+B_3}{1+\beta^2}$$

Moment

Real part of $M = \sum \sum p_{ji} X_i$

$$p_{11} = [-m_1 n_o^{1/2} - m_2' n_o^{3/2}] \sin a_1 X = q_{11} \sin a_1 X$$

$$p_{12} = [m_2' \beta n_o^{3/2}] \sin a_1 X = q_{12} \sin a_1 X$$

$$p_{13} = [m_1 a_1 + m_2' n_o^{3/2}] \cos a_1 X = q_{13} \cos a_1 X$$

$$p_{14} = [-m_2' \beta n_o^{3/2}] \cos a_1 X = q_{14} \cos a_1 X$$

$$p_{15} = [m_1 s_1' - m_2'(s_1'^3 - 3s_1' t_1'^2) + m_2' \beta(3s_1'^2 t_1' - t_1'^3)] R_{41}$$

$$+ [-m_1 t_1' + m_2'(3s_1'^2 t_1' - t_1'^3) + m_2' \beta(s_1'^3 - 3s_1' t_1'^2)] R_{42}$$

$$= q_{15} R_{41} + r_{15} R_{42}$$

$$p_{16} = [-m_1 t_1' + m_2'(3s_1'^2 t_1' - t_1'^3) + m_2' \beta(s_1'^3 - 3s_1' t_1'^2)] R_{41}$$

$$+ [-m_1 s_1' + m_2'(s_1'^3 - 3s_1' t_1'^2) - m_2' \beta(3s_1'^2 t_1' - t_1'^3)] R_{42}$$

$$= q_{16} R_{41} + r_{16} R_{42}$$

$$p_{17} = q_{15} R_{31} + r_{15} R_{32}$$

$$p_{18} = q_{16} R_{31} + r_{16} R_{32}$$

$$p_{19} = [m_1 s_2' - m_2'(s_2'^3 - 3s_2' t_2'^2) + m_2'\beta(3s_2'^2 t_2' - t_2'^3)]R_{61}$$

$$+ [-m_1 t_2' + m_2'(3s_2'^2 t_2' - t_2'^3) + m_2'\beta \quad (s_2'^3 - 3s_2' t_2'^2)]R_{62}$$

$$= q_{19}R_{61} + r_{19}R_{62}$$

$$p_{1,10} = [-m_1 t_2' + m_2'(s_2'^3 t_2' - t_2'^3) + m_2'\beta(s_2'^3 - 3s_2' t_2'^2)]R_{61}$$

$$+ [-m_1 s_2' + m_2'(s_2'^3 - 3s_2' t_2'^2) - m_2'\beta(3s_2'^2 t_2' - t_2'^3)]R_{62}$$

$$= q_{1,10}R_{61} + r_{1,10}R_{62}$$

$$p_{1,11} = q_{19}R_{51} + r_{19}R_{52}$$

$$p_{1,12} = q_{1,10}R_{51} + r_{1,10}R_{52}$$

Imag. part of $M = \sum \sum \alpha_{ji} X_i$

$$\alpha_{11} = [m_2 \beta n_o^{3/2}] \sin a_1 X = \beta_{11} \sin a_1 X$$

$$\alpha_{12} = [-m_1 n_o^{\frac{1}{2}} - m_2' n_o^{3/2} + m_2' \beta n_o^{3/2}] \sin a_1 X = \beta_{12} \sin a_1 X$$

$$\alpha_{13} = [-m_2 \beta n_o^{3/2}] \cos a_1 X = \beta_{13} \cos a_1 X$$

$$\alpha_{14} = [m_1 n_o^{\frac{1}{2}} + m_2'(1-\beta^2) n_o^{3/2}] \cos a_1 X = \beta_{14} \cos a_1 X$$

$$\alpha_{15} = [m_1 t_1' - m_2'(3s_1'^2 t_1' - t_1'^3) + m_2' \beta (s_1'^3 - 3s_1' t_1'^2)] R_{41}$$

$$+ [m_1 s_1' - m_2'(s_1'^3 - 3s_1' t_1'^2) - m_2' \beta (3s_1'^2 t_1' - t_1'^3)] R_{42}$$

$$= \beta_{15} R_{41} + \gamma_{15} R_{42}$$

$$\alpha_{16} = q_{15} R_{41} + \gamma_{15} R_{42}$$

$$\alpha_{17} = \beta_{15} R_{31} + \gamma_{15} R_{32}$$

$$\alpha_{18} = q_{15} R_{31} + \gamma_{15} R_{32}$$

$$\alpha_{19} = [m_1 t_2' - m_2(3s_2'^2 t_2' - t_2'^3) + m_2' \beta (s_2'^3 - 3s_2' t_2'^2)] R_{61}$$

$$+ [m_1 s_2' - m_2'(s_2'^3 - 3s_2' t_2'^2) - m_2' \beta (3s_2'^2 t_2' - t_2'^3)] R_{62}$$

$$\alpha_{19} = \beta_{19} R_{61} + \gamma_{19} R_{62}$$

$$\alpha_{1,10} = q_{19} R_{61} + \gamma_{19} R_{62}$$

$$\alpha_{1,11} = \beta_{19} R_{51} + \gamma_{19} R_{52}$$

$$\alpha_{1,12} = q_{19} R_{51} + \gamma_{19} R_{52}$$

$$\sum \sum d_{ji}X_i = \text{Real part of } y$$

Using $a_1 = n_o^{\frac{1}{2}} R_1^{\frac{1}{2}}$

$$d_{11} = [\frac{S}{n_o^{\frac{1}{2}}} + \frac{n_o^{\frac{1}{2}}}{(1+\beta^2)}] \sin a_1X = \ell_{11} \sin a_1X$$

$$d_{12} = [\frac{-\beta n_o^{\frac{1}{2}}}{(1+\beta^2)}] \sin a_1X = \ell_{12}\sin a_1X$$

$$d_{13} = -[\frac{S}{n_o^{\frac{1}{2}}} + \frac{n_o^{\frac{1}{2}}}{1+\beta^2}] \cos a_1X = \ell_{13} \cos a_1X$$

$$d_{14} = [\frac{n_o^{\frac{1}{2}}\beta}{1+\beta^2}] \cos a_1X = \ell_{14}\cos a_1X$$

$$d_{15} = [\frac{Ss_1'}{s_1'^2+t_1'^2} - \frac{s_1'}{(1+\beta^2)} + \frac{\beta t_1'}{(1+\beta^2)}]R_{41} + [\frac{St_1'}{s_1'^2+t_1'^2} + \frac{t_1'}{1+\beta^2} + \frac{\beta s_1'}{1+\beta^2}] R_{42}$$

$$= \ell_{15}R_{41}+f_{15}R_{42}$$

$$d_{16} = [\frac{St_1'}{s_1'^2+t_1'^2} + \frac{t_1'}{1+\beta^2} + \frac{\beta s_1'}{1+\beta^2}]R_{41}+[\frac{-Ss_1'}{s_1'^2+t_1'^2} + \frac{s_1'}{1+\beta^2} - \frac{\beta t_1'}{1+\beta^2}]R_{42}$$

$$= \ell_{16}R_{41}+f_{16}R_{42}$$

$$d_{17} = [\frac{Ss_1'}{s_1'^2+t_1'^2} - \frac{s_1'}{1+\beta^2} + \frac{\beta t_1'}{1+\beta^2}]R_{31}+[\frac{St_1'}{s_1'^2+t_1'^2} + \frac{t_1'}{1+\beta^2} + \frac{\beta s_1'}{1+\beta^2}]R_{32}$$

$$= \ell_{15}R_{31}+f_{15}R_{32}$$

$$d_{18} = \ell_{16}R_{31}+f_{16}R_{32}$$

$$d_{19} = [\frac{Ss_2'}{s_2'^2+t_2'^2} - \frac{s_2'}{1+\beta^2} + \frac{\beta t_2'}{1+\beta^2}]R_{61}+[\frac{St_2'}{s_2'^2+t_2'^2} + \frac{t_2'}{1+\beta^2} + \frac{\beta s_2'}{1+\beta^2}]R_{62}$$

$$= \ell_{19}R_{61}+f_{19}R_{62}$$

$$d_{1,10} = [\frac{St_2'}{s_2'^2+t_2'^2} + \frac{t_2'}{1+\beta^2} + \frac{\beta s_2'}{1+\beta^2}]R_{61}+[\frac{-Ss_2'}{s_2'^2+t_2'^2} + \frac{s_2'}{1+\beta^2} - \frac{\beta t_2'}{1+\beta^2}]R_{62}$$

$$d_{1,10} = \ell_{1,10}R_{61} + f_{1,10}R_{62}$$

$$d_{1,11} = \ell_{19}R_{51}+f_{1,9}R_{52}$$

$$d_{1,12} = \ell_{1,10}R_{51}+f_{1,10}R_{52}$$

Imaginary part of y

$$\text{Im } y = \sum \sum g_{ji} X_i$$

factor out $\frac{1}{R_1^{1/2}}$

$$g_{11} = \left[\frac{n_o^{1/2}}{1+\beta^2}\right] \sin a_1 X = h_{11} \sin a_1 X$$

$$g_{12} = \left[\frac{S}{n_o^{1/2}} - \frac{\beta n_o^{1/2}}{1+\beta^2}\right] \sin a_1 X = h_{12} \sin a_1 X$$

$$g_{13} = \left(\frac{-n_o^{1/2}}{1+\beta^2}\right) \cos a_1 X = h_{13} \cos a_1 X$$

$$g_{14} = \left[- \frac{S}{n_o^{1/2}} + \frac{n_o^{1/2}\beta}{1+\beta^2}\right] \cos a_1 X = h_{14} \cos a_1 X$$

$$g_{15} = \left[\frac{St_1'}{s_1'^2+t_1'^2} - \frac{t_1'}{1+\beta^2} + \frac{\beta s_1'}{1+\beta^2}\right]R_{41} + \left[\frac{Ss_1'}{s_1'^2+t_1'^2} - \frac{s_1'}{1+\beta^2} - \frac{\beta t_1'}{1+\beta^2}\right]R_{42}$$

$$= h_{15}R_{41} + k_{15}R_{42}$$

$$g_{16} = \left[\frac{Ss_1'}{s_1'^2+t_1'^2} - \frac{s_1'}{1+\beta^2} - \frac{\beta t_1'}{1+\beta^2}\right]R_{41} + \left[\frac{St_1'}{s_1'^2+t_1'^2} + \frac{t_1'}{1+\beta^2} - \frac{s_1'}{1+\beta^2}\right]R_{42}$$

$$= h_{16}R_{41} + k_{16}R_{42}$$

$$g_{17} = h_{15}R_{31} + k_{15}R_{32}$$

$$g_{18} = h_{16}R_{31} + k_{16}R_{32}$$

$$g_{19} = \left[\frac{St_2'}{s_2'^2+t_2'^2} - \frac{t_2'}{1+\beta^2} + \frac{\beta s_2'}{1+\beta^2}\right]R_{61} + \left[\frac{Ss_2'}{s_2'^2+t_2'^2} - \frac{s_2'}{1+\beta^2} - \frac{\beta t_2'}{1+\beta^2}\right]R_{62}$$

$$g_{19} = h_{19}R_{61} + k_{19}R_{62}$$

$$g_{1,10} = \left[\frac{Ss_2'}{s_2'^2+t_2'^2} - \frac{s_2'}{1+\beta^2} - \frac{\beta t_2'}{1+\beta^2}\right]R_{61} + \left[\frac{St_2'}{s_2'^2+t_2'^2} + \frac{t_2'}{1+\beta^2} - \frac{s_2'}{1+\beta^2}\right]R_{62}$$

$$= h_{1,10}R_{61} + k_{1,10}R_{62}$$

$$g_{1,11} = h_{19}R_{51} + k_{19}R_{52}$$

$$g_{1,12} = h_{1,10}R_{51} + k_{1,10}R_{52}$$

Real part of $\zeta'' \doteq \sum u_{ij} X_j$

$$u_{11} = -n_o \cos a_1 X = v_{11} \cos a_1 X$$

$$u_{12} = 0$$

$$u_{13} = -n_o \sin a_1 X = v_{11} \sin a_1 X$$

$$u_{14} = 0$$

$$u_{15} = (s_1'^2 - t_1'^2) R_{31} - 2 s_1' t_1' R_{32} = v_{15} R_{31} + w_{15} R_{32}$$

$$u_{16} = 2 s_1' t_1' R_{31} + (s_1'^2 - t_1'^2) R_{32} = - w_{15} R_{31} + v_{15} R_{32}$$

$$u_{17} = (s_1'^2 - t_1'^2) R_{41} - 2 s_1' t_1' R_{42} = v_{15} R_{41} + w_{15} R_{42}$$

$$u_{18} = 2 s_1' t_1' R_{41} + (s_1'^2 - t_1'^2) R_{42} = -w_{15} R_{41} + v_{15} R_{42}$$

$$u_{19} = (s_2'^2 - t_2'^2) R_{51} - 2 s_2' t_2' R_{52} = v_{19} R_{51} + w_{19} R_{52}$$

$$u_{1,10} = 2 s_2' t_2' R_{51} + (s_2'^2 - t_2'^2) R_{52} = -w_{19} R_{51} + v_{19} R_{52}$$

$$u_{1,11} = (s_2'^2 - t_2'^2) R_{61} - 2 s_2' t_2' R_{62} = v_{19} R_{61} + w_{19} R_{62}$$

$$u_{1,12} = 2 s_2' t_2' R_{61} + (s_2'^2 - t_2'^2) R_{62} = -w_{19} R_{61} + v_{19} R_{62}$$

Imag. part of $\zeta'' = \sum l_{ij} X_j$

$l_{11} = 0$

$l_{12} = -n_o \cos a_1 X$

$l_{13} = 0$

$l_{14} = -n_o \sin a_1 X$

$l_{15} = w_{15} R_{31} - v_{15} R_{32}$

$l_{16} = v_{15} R_{31} + w_{15} R_{32}$

$l_{17} = w_{15} R_{41} - v_{15} R_{42}$

$l_{18} = v_{15} R_{41} + w_{15} R_{42}$

$l_{19} = w_{19} R_{51} - v_{19} R_{52}$

$l_{1,10} = v_{19} R_{51} + w_{19} R_{52}$

$l_{1,11} = w_{19} R_{61} - v_{19} R_{62}$

$l_{1,12} = v_{19} R_{61} + w_{19} R_{62}$

III. Determination of Natural Frequencies - Approximate Method

1.0 In view of the complexity in solving the 12 by 12 determinants for various boundary conditions, an approximate procedure was considered. The characteristic equation of the sixth order homogeneous differential equation developed in reference 5 is re-examined. This is re-written as equation (III-1). When solved for the natural frequency, equation III-4 is obtained. An investigation of equation III-4 reveals that it is in the usual form for homogeneous beams

$$\omega_1^2 = \frac{a_n^{\ 2}}{L^4}\left[\frac{EI}{\rho}\right]$$

in which the eigenvalue λ_0 is comparable to $\frac{a_n}{L^2}$, and EI, the stiffness in the homogeneous beam, is comparable to $(B_1+B_3+K_i\delta^2\alpha)$. The values of a_n are determined by the boundary conditions and a table of such values is given on page 35. The quantity $(B_1+B_3+K_i\delta^2\alpha)$ represents the effective stiffness on the vibrating beam, in which B_1 and B_3 are stiffnesses of the individual steel layers about their own neutral axes and $K_i\delta^2$ is the portion of the stiffness due to transferring the area moments of inertia of the elastic layers to the composite neutral axis. The factor α is a factor which indicates the shear carrying capacity of the viscoelastic. When α is equal to one-half then there is no shear strain in the viscoelastic, whereas when α is equal to zero, the viscoelastic cannot transmit any shear stress and each elastic layer bends independently except for being restricted to moving laterally the same amount. For the case of the simply-supported beam having no axial constraints at each end, the value for the eigenvalue λ_0 is exact; i.e.; $\lambda_0 = (\frac{n\pi}{L})^2$, so that substituting this value for λ_0 into the frequency equation yields the exact natural frequencies. The factor α is exact for this case and truly represents the effect of frequency on the stiffness of the beam. Guided by this form of the frequency equation and by the exactness of using this form for a simply-supported beam, it is postulated that this form of the equation for the natural frequency may be used as an approximation to the natural frequency of beams having other end conditions. Thus it is assumed that $\lambda_0 = \frac{a_n}{L^2}$ where a_n is the usual factor determined by the boundary conditions. Thus, given the geometry of the cross-section, length of the beam, the physical properties of the materials and the boundary conditions, the natural frequency may be approximated by equation III-4 in which α is given by equation III-5. For ease of calculation the curves of α versus G are plotted for the first five modes of the cantilever, simply-supported, free-free, fixed-fixed, and fixed-pinned beams.

It is recognized that this procedure, suggested for finding the natural frequencies of laminated beams, is approximate and that its accuracy can only be checked by exact solutions of the kind performed in section II, and/or by tests of actual beams. It is felt that the procedure should yield good engineering results.

1.1 Approximation of Natural Frequencies

Equation (36) of reference 3 , is

$$\lambda_o^4 + \lambda_o^3 \left[\frac{R_1(2d_1S+1)}{d_1}\right] + \lambda_o^2 \left[\frac{SR_1^2(1+\beta^2)(1+Sd_1)}{d_1} - \frac{d_2}{d_1}\omega_1^2\right]$$

$$-\lambda_o\left[\frac{2SR_1d_2\omega_1^2}{d_1}\right] - \frac{d_2}{d_1}\omega_1^2S^2R_1^2(1+\beta^2) = 0 \qquad \text{III-1}$$

Solving for ω_1^2, we obtain,

$$\omega_1^2 = \frac{d_1}{d_2}\lambda_o^2 \left[\frac{\lambda_o^2 + \lambda_o[2SR_1+\frac{R_1}{d_1}] + R_1^2S(1+\beta^2)(\frac{1}{d_1}+S)}{\lambda_o^2 + 2\lambda_oSR_1+S^2R_1^2(1+\beta^2)}\right] \qquad \text{III-2}$$

or,

$$\omega_1^2 = \frac{B_1+B_3}{\rho}\lambda_o^2 \left[1+\frac{K_1\delta^2R_1[\lambda_o+R_1S(1+\beta^2)]}{(B_1+B_3)[\lambda_o^2 + 2\lambda_oSR_1+S^2R_1^2(1+\beta^2)]}\right] \qquad \text{III-3}$$

where $d_1 = \dfrac{B_1+B_3}{K_1\delta^2}$

$d_2 = \dfrac{\rho}{K_1\delta^2}$.

The last equation may be written as,

$$\omega_1^2 = \lambda_o^2\left[\frac{B_1+B_3}{\rho} + \frac{K_1\delta^2}{\rho}\alpha\right] \qquad \text{III-4}$$

where

$$\alpha = \frac{R_1[\lambda_o+R_1S(1+\beta^2)]}{\lambda_o^2 + 2\lambda_oSR_1+S^2R_1^2(1+\beta^2)} \qquad \text{III-5}$$

If $\lambda_o = n_oR_1$

then

$$\alpha = \frac{n_o+S(1+\beta^2)}{n_o^2 + 2n_oS+S^2(1+\beta^2)} \qquad \text{III-6}$$

For a sandwich beam having a thin viscoelastic layer $S \doteq 2$, so that

$$\alpha \doteq \frac{n_o+2(1+\beta^2)}{n_o^2 + 4n_o+4(1+\beta^2)} . \qquad \text{III-7}$$

Reconsidering equation III-5, for α, it is seen that, for a sandwich beam, if we let

$$\gamma_o = \frac{a_n}{L^2}$$

then

$$\alpha = \frac{a_n R_1 L^2 + 2R_1^2 L^4 (1+\beta^2)}{a_n^2 + 4a_n R_1 L^2 + 4R_1^2 L^4 (1+\beta^2)} \qquad \text{III-8}$$

or letting

$$G = F^2 = R_1 L^2 \qquad \text{III-9}$$

$$\alpha = \frac{a_n G + 2G^2 (1+\beta^2)}{a_n^2 + 4a_n G + 4G^2 (1+\beta^2)} . \qquad \text{III-10}$$

For the special case in which $\beta \ll 1$

$$\alpha \doteq \frac{G}{a_n + 2G} . \qquad \text{III-11}$$

Using the relations III-10 and III-11, plots of α versus G are obtained and are given for the following end conditions

1. Cantilever
2. Simply-supported
3. Free-free
4. Fixed-fixed
5. Fixed-pinned

For a simply supported beam with no axial constraints on the layers at each support it is found that

$$\lambda_o = a_1^2 = (\frac{n\pi}{L})^2 \qquad \text{(See page 35)} \qquad \text{III-12}$$

so that

$$\omega_1^2 = (\frac{n\pi}{L})^4 \left\{\frac{B_1+B_3+K_1\delta^2\alpha}{\rho}\right\}. \qquad \text{III-13}$$

It can be seen that the term $[B_1+B_3+K_1\delta^2\alpha]$ represents the stiffness of the laminated beam. When R_1 is infinite (no shear strain in the V.E. layer) then $\alpha \rightarrow \frac{1}{S}$ (or for a sandwich beam in which $S \doteq 2$ then $\alpha \rightarrow \frac{1}{2}$) so that the stiffness approaches a value of

$$B_1 + B_3 + \frac{K_1\delta 2}{2}.$$

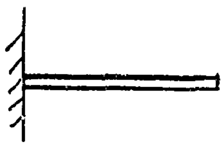
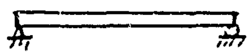
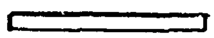
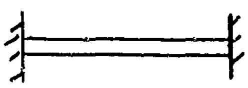
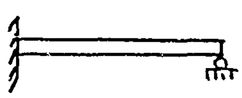
This is the stiffness of the composite cross-section in which the shear is carried directly through the V.E. layer without shear deformation. The other extreme case occurs when $R_1 \rightarrow 0$. In this case α becomes zero, and the stiffness is (B_1+B_3) and each beam contributes solely the stiffness about its own axis; no shear stress is transmitted through the viscoelastic layer although the theory imposed the condition that both beams move laterally by the same amount.

The above description indicates that the parameter α is an inverse measure of the shear strain occurring in the Viscoelastic layer. We note that for small values of β

$$\alpha \doteq \frac{1}{n_o+S}$$

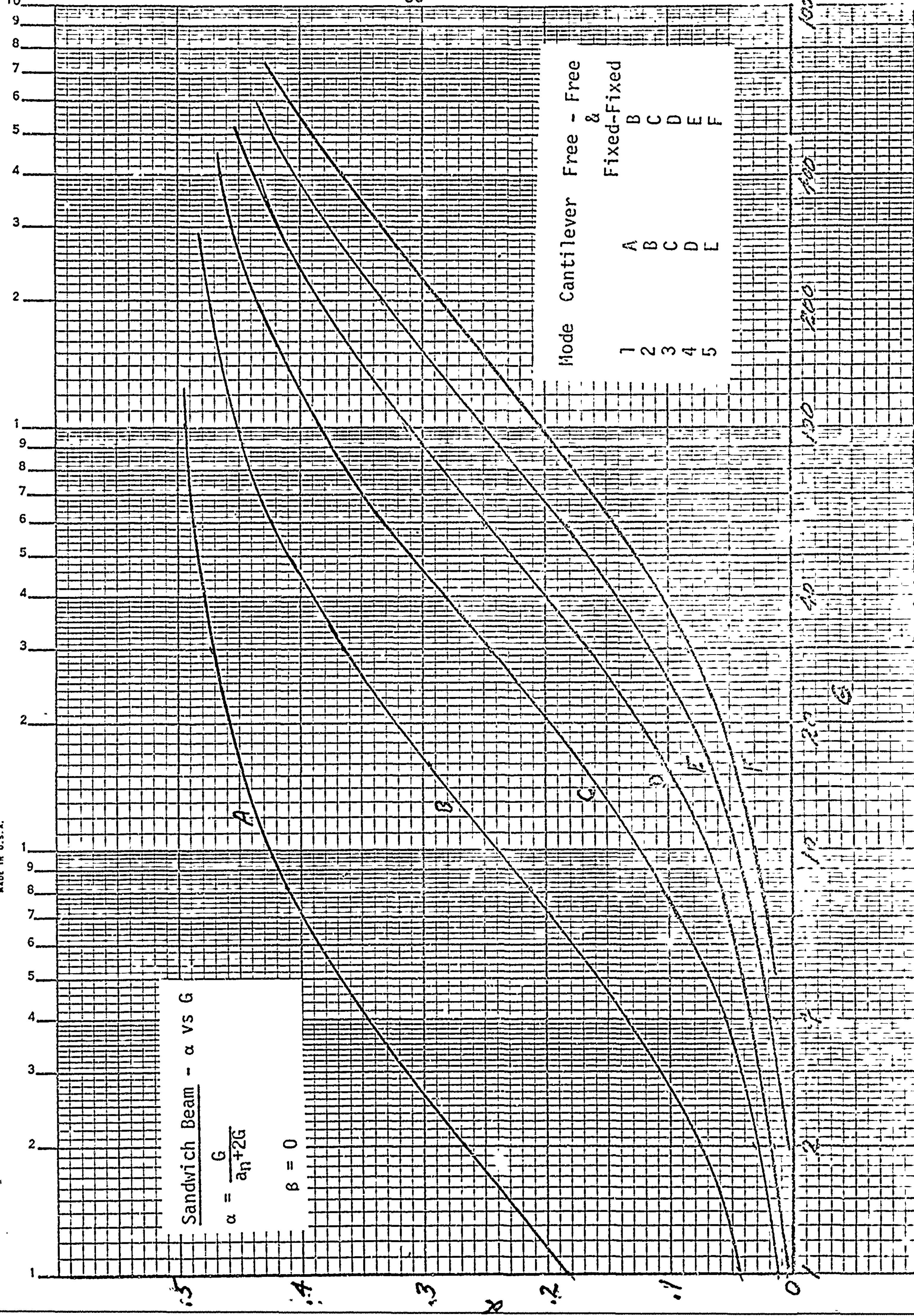
so that a maximum value of α is $\frac{1}{S}$ for n_o equal to zero; and α approaches zero as n_o becomes large compared to S. The quantity n_o is always positive.

It is seen that for the condition of simply-supported, unrestrained ends, using $\lambda_o = (\frac{n\pi}{L})^2$, the value obtained by the usual homogeneous beam, an exact solution is obtained by using equation(III-13).The factor α is exact in this case. This procedure suggests a means for obtaining the natural frequencies of laminated beams having other restraints, i.e.; use the a_n constants obtained for homogeneous beams in the relation $\lambda_o = \frac{a_n}{L^2}$. A table of values for a_n is given on page 35 , for several boundary conditions.

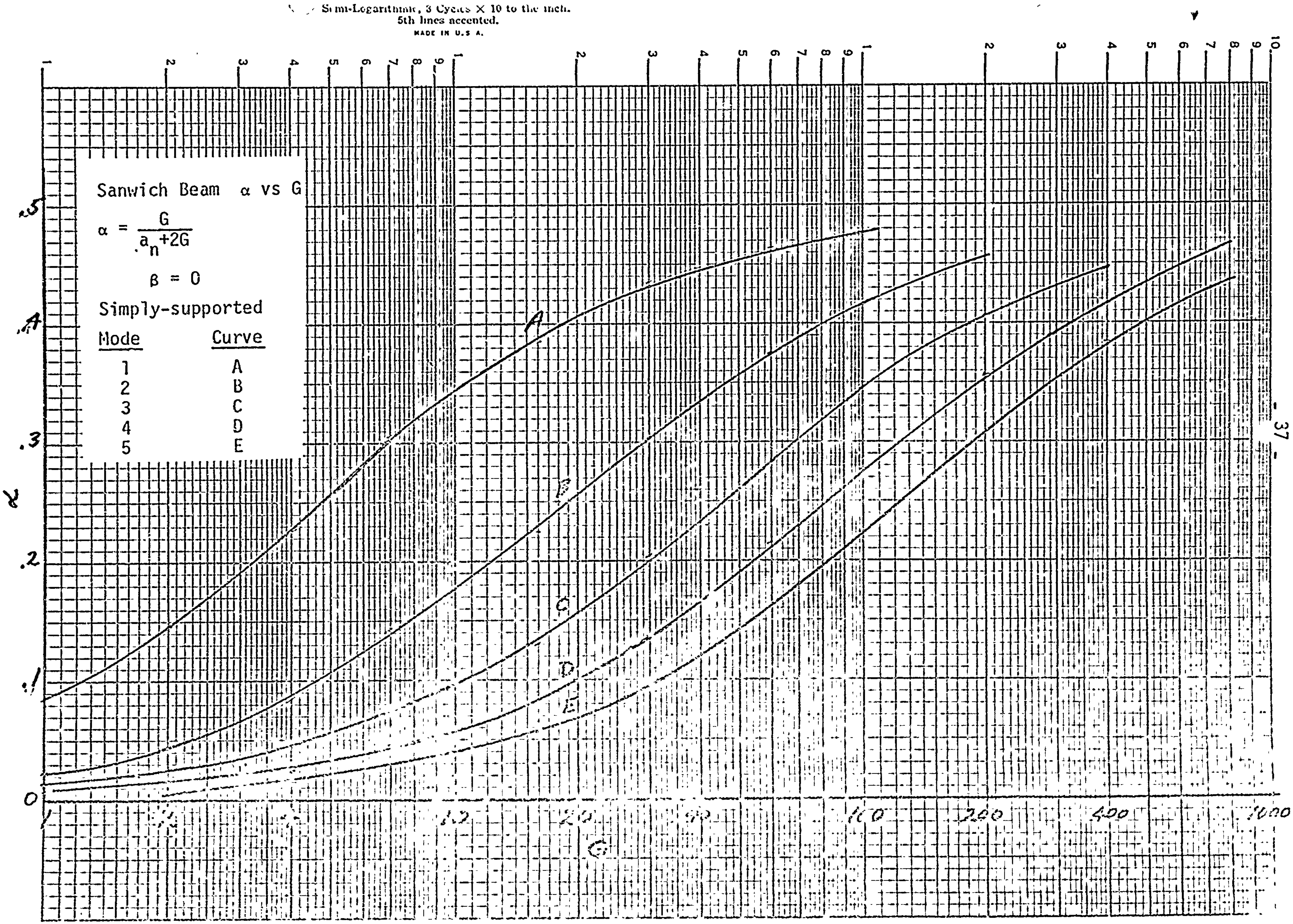
Beam Condition	a_n	
	General Relation	Specific Relations
1. Cantilever 	$a_n \approx (n-\frac{1}{2})^2\pi^2$ for $n \geq 2$	$a_1 = 3.52$ $a_2 = 22.0$ $a_3 = 61.7$ $a_4 = 121.0$ $a_5 = 200.0$
2. Simply - supported 	$a_n = (n\pi)^2$	$a_1 = 9.87$ $a_2 = 39.5$ $a_3 = 88.9$ $a_4 = 158.$ $a_5 = 247.$
3. Free - Free  4. Fixed - Fixed 	$a_n \approx (n+\frac{1}{2})^2\pi^2$ [same a_n for Free - Free and Fixed - Fixed]	$a_1 = 22.0$ $a_2 = 61.7$ $a_3 = 121.0$ $a_4 = 200.0$ $a_5 = 298.2$
5. Fixed - pinned 	$a_n \approx (n+\frac{1}{4})^2\pi^2$	$a_1 = 15.4$ $a_2 = 50.0$ $a_3 = 104.0$ $a_4 = 178.0$ $a_5 = 272.$

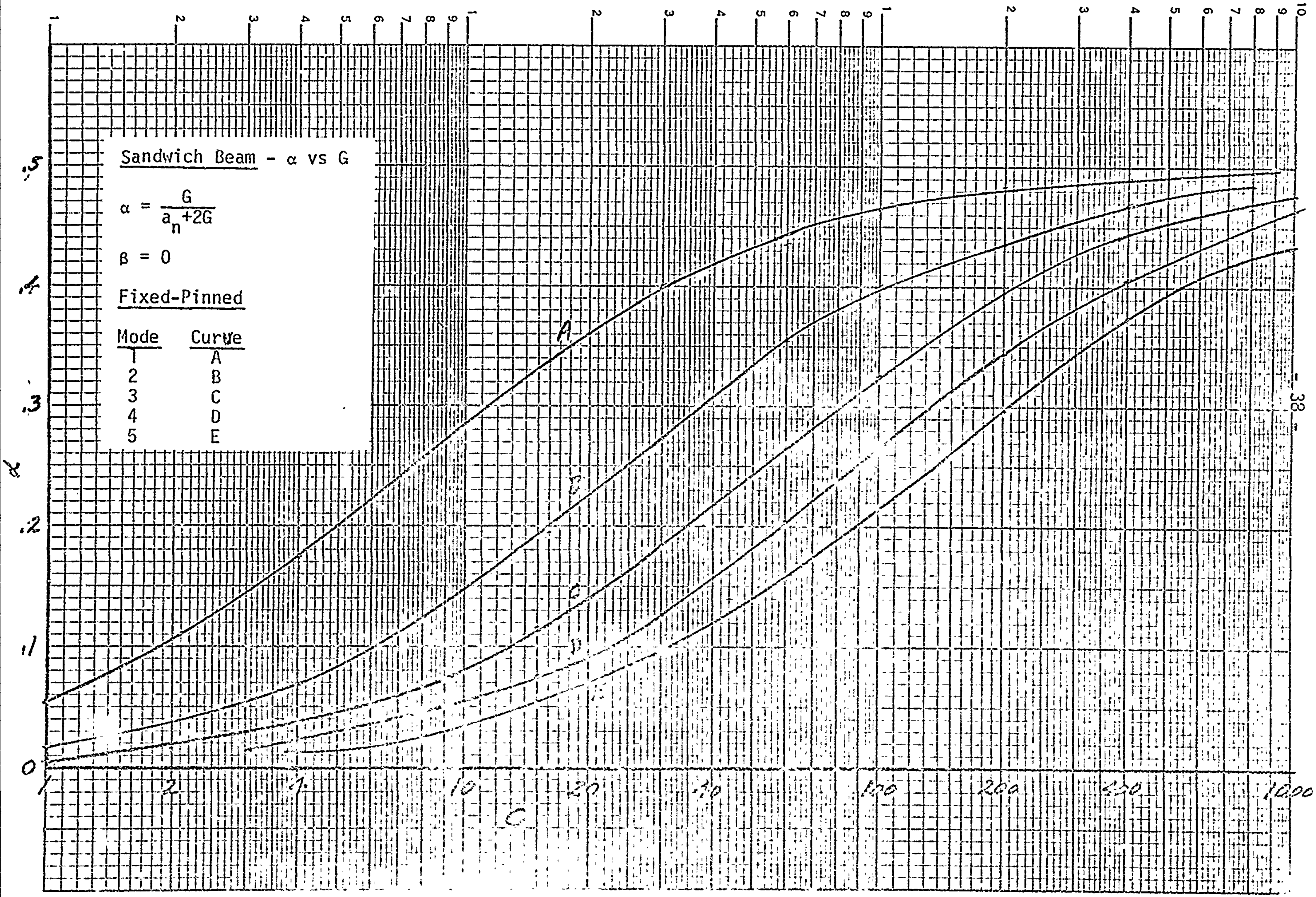
$$\lambda_o = \frac{a_n}{L^2}$$

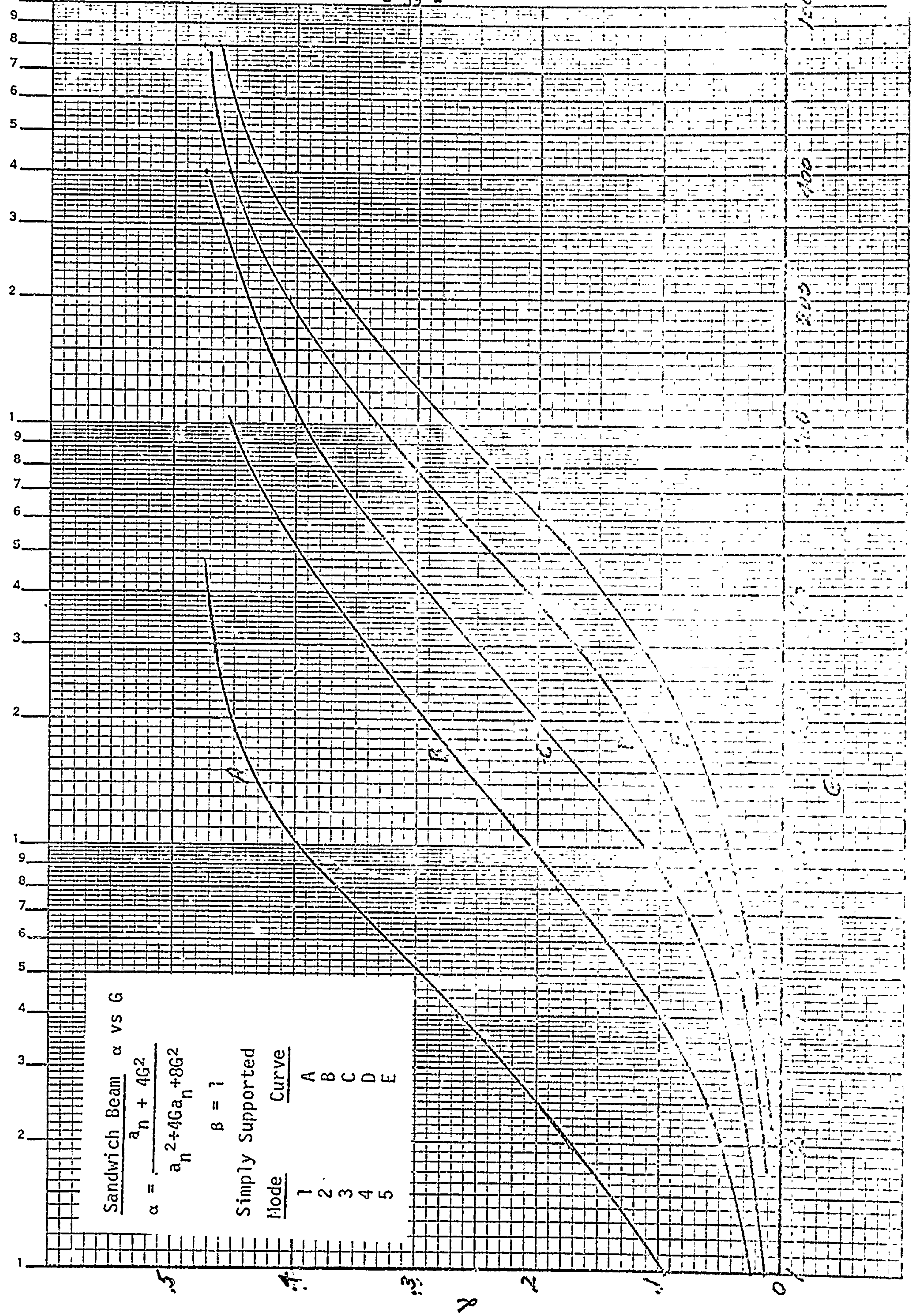
Table 1

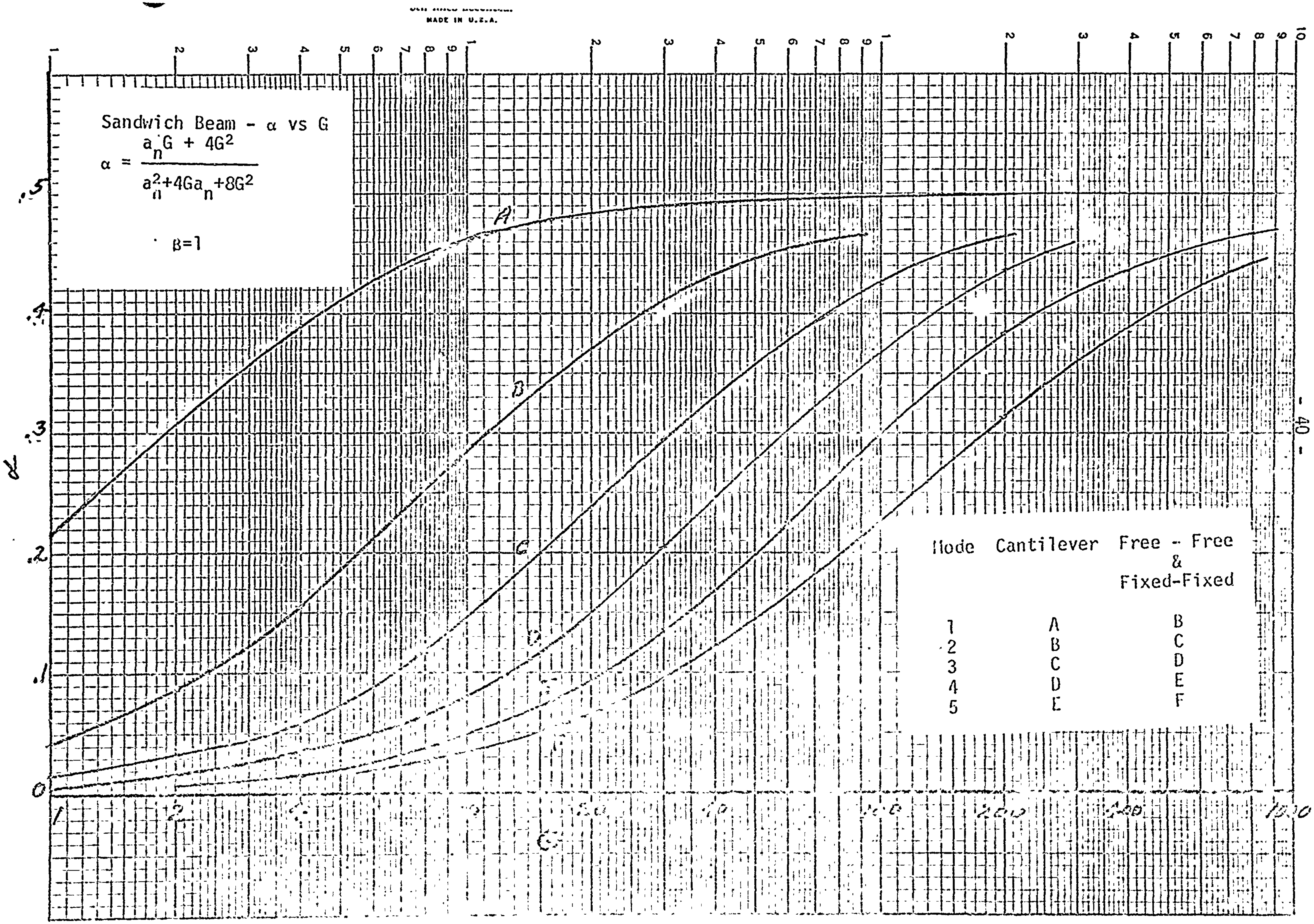


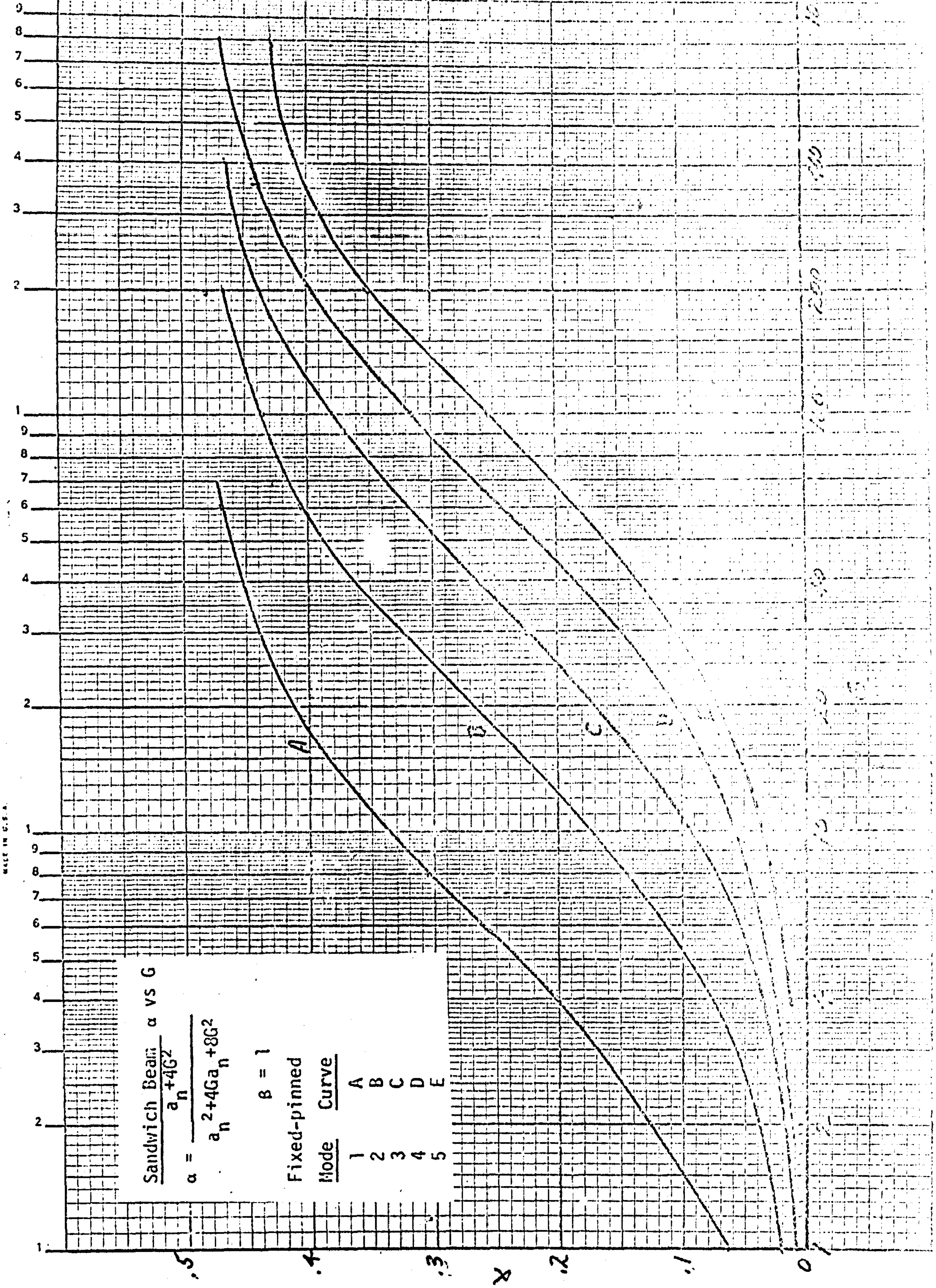
Sandwich Beam - α vs G
$\alpha = \frac{G}{a_n+2G}$
β = 0
Mode Cantilever Free - Free & Fixed-Fixed
1 A
2 B B
3 C C
4 D D
5 E E
F
A
B
C
D
E
F
G
1000
400
200
100
40
20
10
4
2
1
.5
.4
.3
.2
.1
0
α


Semi-Logarithmic, 3 Cycles × 10 to the inch.
5th lines accented.
MADE IN U.S.A.
Sanwich Beam α vs G
$\alpha = \frac{G}{a_n + 2G}$
β = 0
Simply-supported
Mode Curve
1 A
2 B
3 C
4 D
5 E
A
B
C
D
E
α
G
0
.1
.2
.3
.4
.5
100
200
400
1000


Sandwich Beam - α vs G
$\alpha = \frac{G}{a_n+2G}$
β = 0
Fixed-Pinned
Mode Curve
1 A
2 B
3 C
4 D
5 E
A
B
C
D
E
G
α
.5
.4
.3
.2
.1
0
1
2
4
10
20
40
100
200
400
1000


Sandwich Beam α vs G
$\alpha = \dfrac{a_n + 4G^2}{a_n^2 + 4Ga_n + 8G^2}$
β = 1
Simply Supported
Mode Curve
1 A
2 B
3 C
4 D
5 E
α
G


Sandwich Beam - α vs G
$\alpha = \dfrac{a_n G + 4G^2}{a_n^2 + 4Ga_n + 8G^2}$
β=1
Mode | Cantilever | Free - Free & Fixed-Fixed
1 | A | B
2 | B | C
3 | C | D
4 | D | E
5 | E | F
A
B
C
D
α
G
.5
.4
.3
.2
.1
0
1
2
10
20
40
100
200
400
1000


Sandwich Beam α vs G
α = $\frac{a_n + 4G^2}{a_n^2 + 4Ga_n + 8G^2}$
B = 1
Fixed-pinned
Mode Curve
1 A
2 B
3 C
4 D
5 E
A
B
C
D
.5
.4
.3
γ
.2
.1
0

IV. Miscellaneous Considerations

1.0 The approximate expression for the natural frequency of a sandwich beam further approximated for the case in which $\beta \ll 1$. The expression for α simplifies to

$$\alpha \doteq \frac{1}{n_0+2} .$$

The expression for ω_1 is written explicitely in terms of the geometric and physical properties of the elastic and visco-elastic material.

2.0 Consideration is given to designing a beam for optimum damping if the physical properties of the materials and the natural frequency of the beam is selected. The relation is based on the case in which $\beta \ll 1$. Considering ω_{10} as the natural frequency obtained when $\alpha = 0$, a relation, equation (V-13), is obtained for n_0. For optimum damping ref. 5 shows that $n_0 = 5$. Using this fact a relation, equation (V-17), is obtained for the necessary half thickness, H_1, which would yield optimum damping. The optimum length is obtained from

$$\lambda_0 = n_0 R_1 = \frac{a_n}{L^2}$$

yielding

$$L = [\frac{a_n}{5R_1}]^{\frac{1}{2}} .$$

3.0 Based on the results of section III, for $\beta \ll 1$, expressions are derived for G_1 and β which allow one to solve these properties of the viscoelastic material if the geometric properties, elastic properties, and the end-conditions are known and if η and ω_1 are measured. These equations may be used to find the shear properties of a viscoelastic material if $\beta \ll 1$. This procedure may be considered as a means for testing material to insure adherance to procurement specification of a viscoelastic material.

4.0 Using the results of ref. 5, an approximate expression for the composite loss factor, equation (V-23), is obtained in terms of h_1 and β. It is shown that this expression yields results which are within 10% accuracy.

5.0 It was shown in ref. 3 that the natural frequency of a sandwich beam increases as the loss factor, β, is increased. It is shown herein that the same situation occurs using the approximate expressions for the natural frequency developed in section III, thus reinforcing the analytical result that viscoelastic damping causes an increase in natural frequency for a laminated beam.

1.1 Determination of the natural frequencies for small values of material loss factor - ($\beta < 1$)

Considering equation (III-4) for small values of β we note in the equation

$$\omega_1^2 = \lambda_0^2 \left[\frac{B_1+B_3}{\rho} + \frac{K_1\delta^2}{\rho}\alpha\right] \qquad \text{IV-1}$$

that for a sandwich beam having a thin viscoelastic layer

$$\alpha = \frac{n_0 + 2(1+\beta^2)}{n_0^2 + 4n_0 + 4(1+\beta^2)}, \qquad \text{IV-2}$$

and that for $\beta<1$

$$\alpha \doteq \frac{1}{n_0+2}. \qquad \text{IV-3}$$

We note that since n_0 can have values from zero to infinity then α has values between one half and zero.

Using the relation

$$\lambda_0 = n_0 R_1 \doteq \frac{a_n}{L^2} \qquad \text{IV-4}$$

in equation (IV-3)

then

$$\alpha \doteq \frac{R_1 L^2}{a_n + 2R_1 L^2} \qquad \text{IV-5}$$

or since

$$R_1 = \frac{G_1 b}{2H_2 K_1} = \frac{G_1 b}{2H_2 Eb2H_1} \qquad \text{IV-6}$$

then

$$\alpha \doteq \frac{G_1 L^2}{4a_n H_2 H_1 E + 2G_1 L^2}. \qquad \text{IV-7}$$

or

$$\alpha \doteq \frac{1}{[a_n(\frac{2H_1 2H_2}{L^2}) \frac{E}{G_1} + 2]} \quad . \qquad \text{IV-8}$$

Now

$$B_1 = \frac{1}{12}(2H_1)^3 b = \frac{2}{3} H_1^3 \; b \; E \qquad \text{IV-9}$$

and for the sandwich beam

$$B_1 = B_3$$

therfore

$$B_1 + B_3 = \frac{4}{3} H_1^3 bE.$$

also

$$K_1 \delta^2 \doteq Eb2H_1[2H_1]^2 = 8EbH_1^3$$

for

$$H_2 << H_1.$$

The value for ρ is

$$\gamma_1[4H_1 b] + \gamma_2[2H_2 b]$$

where

γ_1 is the mass density of the elastic material and γ_2 is the mass density of the viscoelastic material. The natural frequency may be written as

$$\omega_1^2 = \frac{a_n^2}{L^4}\left[\frac{4H_1^3 E}{3[4\gamma_1 H_1 + 2\gamma_2 H_2]} + \alpha \frac{8EH_1^3}{4\gamma_1 H_1 + 2\gamma_2 H_2}\right] \qquad \text{IV-10}$$

or

$$\omega_1^2 = \frac{4EH_1^3 \; a_n^2}{3(4\gamma_1 H_1 + 2\gamma_2 H_2)L^4}[1+6\alpha] \qquad \text{IV-11}$$

in which α is found from equation (IV-8).

Thus, in order to calculate the natural frequency of a sandwich beam, one would need to know

1. a_n - This is determined from the end conditions and the mode of vibration. A list of these values is given on page 35.

2. $(2H_1)$ - This is the thickness of the elastic layers.

3. $(2H_2)$ - This is the thickness of the viscoelastic layer.

4. L - This is the length of the beam.

5. E - Young's Modulus of the elastic material.

6. G_1 - This is the real part of the shear modulus (storage modulus) of the viscoelastic material.

7. γ_1 - This is the mass density of the elastic material.

8. γ_2 - This is the mass density of the viscoelastic material (Note that the mass per unit length and per unit width is the quantity $4\gamma_1 H_1 + 2\gamma_2 H_2$).

2.1 Design of a Sandwich Beam for Optimum Composite - Loss Factor

Using the suggested approximate procedure for finding the natural frequency of a sandwich beam having a thin viscoelastic layer, we have

$$\lambda_o = n_o R_1 \doteq \frac{a_n}{L^2} .$$

But using eq. ( ).

$$\omega_{10}^2 = \frac{a_n^2}{3L^4} \left[\frac{4E\, H_1^3}{4\gamma_1 H_1 + 2\gamma_2 H_2}\right] \qquad \text{IV-12}$$

we can obtain from (IV-12)

$$(n_o R_1)^2 \doteq \frac{3(4\gamma_1 H_1 + 2\gamma_2 H_2)\, \omega_{10}^2}{4\, EH_1^3} \qquad \text{IV-13}$$

For optimum damping,

$h_1 \doteq 50$ (see pg. II-33 ref. 5).

Also for $h_1 = 50$ we find from the n_o vs h_1 plot that $n_o \doteq 5$.

for $n_o = 5$, we find for $\beta \ll 1$ that

$$\omega_1^2 = \omega_{10}^2 \left[\frac{13}{7}\right] \qquad \text{IV-14}$$

or

$$\omega_{10}^2 = \frac{7}{13}\, \omega_1^2 . \qquad \text{IV-15}$$

Since $R_1 = \dfrac{G_1}{4EH_1 H_2}$

then

$$25\left[\frac{G_1^2}{16E^2 H_1^2\, H_2^2}\right] = \frac{21(4\gamma_1 H_1 + 2\gamma_2 H_2)\omega_1^2}{52\, EH_1^3} \qquad \text{IV-16}$$

Solving for H_1 we find

$$H_1 = \frac{2\gamma_2 H_2^3 E\omega_1^2}{3.86\ G_1^2 - 4\gamma_1 E\omega_1^2 H_2^2} \,. \qquad \text{IV-17}$$

Thus for given elastic material (E and γ_1) and a given viscoelastic material (G_1 and γ_2) one may select an ω_1 with its associated G_1 and a viscoelastic thickness layer ($2H_2$), then solve for the half thickness of the elastic layer (H_1) for optimum damping. The length of the beam may be found from the expression.

$$n_o R_1 = \frac{a_n}{L^2}$$

so that for $n_o = 5$

$$L = [\frac{a_n}{5R_1}]^{\frac{1}{2}} = [\frac{4a_n EH_1H_2}{5G_1}]^{\frac{1}{2}} \qquad \text{IV-18}$$

in which a_n is found on page 35.

Equations (IV-17) and (IV-18) yield the half thickness of the elastic layer and the associated length of the beam to achieve optimum damping.

3.1 A Suggested Method for Determining the Shear Moduli and Loss-Factor of a Viscoelastic Material - $\beta<<1$.

A method for determining the shear moduli and loss-factor of a viscoelastic material is suggested, based on the analysis on pages 44 and 45 If in some manner, as for example impedance measurements of a free-free beam, the composite loss factor η and the natural frequency ω_1 and its associated mode number are measured for a sandwich beam having a thin viscoelastic layer, then the shear modulus G_1 and material loss factor β may be calculated.

For the case in which β is small G_1 and η can be obtained in the following manner. Knowing the mode number and the end-conditions, the value of a_n may be obtained from table 1. This may be used in equation (IV-11) to solve for α, i. e.,

$$\alpha = \frac{3(4\gamma_1H_1+2\gamma_2H_2)L^4}{24E\, H_1^3\, a_n^2}\,\omega_1^2 - \frac{1}{6} \qquad \text{IV-19}$$

Using equation (IV-7) and solving for G_1, one obtains

$$G_1 = \frac{4\alpha a_n H_2 H_1 E}{L^2[1-2\alpha]}, \qquad \text{IV-20}$$

in which α is obtained by solving eq. (IV-19). The real part of the shear modulus is thus calculated.

The relation for the composite loss factor η for a sandwich beam is shown in ref. 5 , to be eq. II-15.

$$\eta = \frac{R_1K_1\lambda_0^3\,\beta\delta^2}{\rho\omega_1^2\,[(R_1S+\lambda_0)^2+(R_1S\beta)^2]} \qquad \text{IV-21}$$

which for the assumptions being considered i.e. ($\beta<1$), becomes,

$$\beta = \eta\,\frac{(4\gamma_1H_1+2\gamma_2H_2)\,\omega_1^2[G_1+2\lambda_0\,H_2H_1E]^2}{4EG_1\lambda_0^3\,H_1^3} \qquad \text{IV-22}$$

This last equation allows one to calculate the material loss factor β. Having found G_1 and β, the loss modulus G_2 may be found using

$$G_2 = \beta G_1.$$

4.1 An Approximate Equation for the Composite Loss-Factor

It was found in ref. 5, that for a sandwich beam having a thin viscoelastic layer

$$\eta = \frac{6\, n_o^3 \beta}{h_1[(2+n_o)^2+(2\beta)^2]}$$

An inspection of the n_o vs h_1 plot for values of β between .1 and 1 (pg II-34 ref. 5) indicates that the log n_o vs log h_1, curve is approximately a straight line and rather independent of β for values of β between .1 and 1. The relation can be assumed to be

$$n_o = c\, h_1^k \qquad \text{IV-23}$$

in which c and k must be evaluated from the curve. We see that two corresponding points on the n_o-h_1 plot are

$$n_o = 20 \quad \text{and } h_1 = 500$$

$$n_o = 2 \quad \text{and } h_1 = 11$$

Solving for c and k by using the above values we find

$$k = .6$$

and $$c = .475$$

This then yields the relation

$$\eta = .66\beta \frac{h_1^{.8}}{[(2+.475h_1^{.6})^2+(2\beta)^2]} \qquad \text{IV-24}$$

A check of three values of h_1 indicates a good correlation with the exact results. These are shown below.

h_1	η Approx.	η Exact	η % error	β
4	.022	.0238	-7.5	.1
4	.153	.150	+1.95	1
40	.031	.033	-6.05	.1
40	.282	.280	+.7	1
600	.0195	.0185	+5.4	.1
600	.195	.185	+5.4	1

The approx. value of η uses equation (IV-24) whereas the exact value of η is taken from the η vs h_1 plot of ref. 5..

As was shown previously in ref. 5 , the approximate equations for small values of h_1 i.e. h_1 less than 1, and the approximate equation for larger values of h_1, i.e., $h_1 > 2000$ are given as

$$\eta \doteq \frac{3h_1^{\frac{1}{2}}}{16} \quad \frac{\beta}{1+\beta^2} \qquad \text{for } h_1 < 1$$

$$\eta \doteq \frac{6\beta}{h_1^{\frac{1}{2}}} \qquad \text{for } h_1 >> 1.$$

5.1 Comments on the apparent increase in ω_1 when damping is present

In ref. 2 , it was shown that

$$\omega_1 \doteq \omega_o \left[1 + \frac{3\eta^2}{8}\right]$$

where

ω_1 is the natural frequency including damping effects

$\omega_o = p^2 \left[\frac{EI}{\rho}\right]^{\frac{1}{2}}$, the undamped natural frequency

η is the composite loss-factor

p is associated with the mode number

$$p^2 = \frac{a_n}{L^2}$$

EI - stiffness

ρ - mass/unit length.

In the above EI is the stiffness and should take into account the decrease in stiffness due to the shearing effect occuring in the viscoelastic layer. Thus if used properly I is the total I of the cross-section about the composite neutral axis, at the low frequencies, and decreases to the sum of the individual I's about their own neutral axis as the vibration frequency increases. But for a given EI, indeed the relation above indicates the natural frequency would increase with the addition of damping. As a further substantiation of this increase we look at the expression for the sandwich beam having a thin visco-elastic layer,

$$\omega_1^2 = \omega_{10}^2 \left[1 + 6\alpha\right].$$

The factor α accounts for the shear effect of the viscoelastic layer as was pointed out previously since , for $\beta = 0$, it varies from 0 (no shear carrying capacity by V.E.) to ½ (all shear carried). Now α also contains β, the material loss factor. In particular we have

$$\alpha = \frac{n_o + 2(1 + \beta^2)}{n_o^2 + 4n_o + 4(1 + \beta^2)}$$

If we compare α containing β and that for which $\beta = 0$ i.e. α_o, and if this ratio $\frac{\alpha}{\alpha}$ is greater than one, then the natural frequency of a beam would tend to be higher with damping than it would be without damping. Looking at this ratio we see,

$$\frac{\alpha}{\alpha_o} = \frac{\dfrac{n_o + 2(1 + \beta^2)}{n_o^2 + 4n_o + 4(1 + \beta^2)}}{\dfrac{n_o + 2}{(n_o + 2)^2}} = \frac{(n_o+2)[n_o+2(1+\beta^2)]}{n_o^2+4\,n_o+4\,(1+\beta^2)}$$

$$\frac{\alpha}{\alpha_o} = 1 + \frac{2\, n_o\, \beta^2}{n_o^{\,2} + 4\, n_o + 4(1+\beta^2)}$$

For β small compared to n_o, we have

$$\frac{\alpha}{\alpha_o} \doteq 1 + \frac{2\, n_o}{(n_o + 2)^2}\, \beta^2$$

This shows $\frac{\alpha}{\alpha_o} > 1$, therefore this indicates that the natural frequency will increase with damping, from the value it would have with no damping.

V. Errata

The following two equations are the errata for equations II-10 and II-11 (Moment and shear) of reference 5.

Let

$$\text{II-9}\quad \begin{cases} m_1 = K_1\delta^2 + (B_1+B_3)S \\ m_2 = \dfrac{B_1+B_3}{R_1(1+\beta^2)} \end{cases}$$

Then,

$$\begin{aligned}
\text{(II-10)}\quad \delta M = {} & E_{11}[-m_1a_1\sin a_1x - m_2a_1{}^3\sin a_1x] \\
& + E_{12}[m_2\beta a_1{}^3\sin a_1x] - E_{22}[m_2\beta a_1{}^3\cos a_1x] \\
& + E_{21}[m_1a_1\cos a_1x + m_2a_1{}^3\cos a_1x] \\
& + E_{31}[m_1S_{31} - m_2U_{31} + m_2\beta U_{32}] + E_{32}[-m_1S_{32} + m_2U_{32} + m_2\beta U_{31}] \\
& + E_{41}[m_1S_{41} - m_2U_{41} + m_2\beta U_{42}] + E_{42}[-m_1S_{42} + m_2U_{42} + m_2\beta U_{41}] \\
& + E_{51}[m_1S_{51} - m_2U_{51} + m_2\beta U_{52}] + E_{52}[-m_1S_{52} + m_2U_{52} + m_2\beta U_{51}] \\
& + E_{61}[m_1S_{51} - m_2U_{61} + m_2\beta U_{62}] + E_{62}[-m_1S_{62} + m_2U_{62} + m_2\beta U_{61}] \\
& + i\{E_{11}[m_2\beta a_1{}^3\sin a_1x] - E_{21}[m_2\beta a_1{}^3\cos a_1x] \\
& + E_{12}[-m_1a_1\sin a_1x - m_2a_1{}^3\sin a_1x + m_2\beta a_1{}^3\sin a_1x] \\
& + E_{22}[m_1a_1\cos a_1x + m_2a_1{}^3\cos a_1x - m_2\beta a_1{}^3\cos a_1x] \\
& + E_{31}[m_1S_{32} - m_2U_{32} + m_2\beta U_{31}] + E_{32}[m_1S_{31} - m_2U_{31} + m_2\beta U_{32}] \\
& + E_{41}[m_1S_{42} - m_2U_{42} + m_2\beta U_{41}] + E_{42}[m_1S_{41} - m_2U_{41} + m_2\beta U_{42}] \\
& + E_{51}[m_1S_{52} - m_2U_{52} + m_2\beta U_{51}] + E_{52}[m_1S_{51} - m_2U_{51} + m_2\beta U_{52}] \\
& + E_{61}[m_1S_{62} - m_2U_{62} + m_2\beta U_{61}] + E_{62}[m_1S_{61} - m_2U_{61} + m_2\beta U_{62}]\}. \qquad \text{II-15}
\end{aligned}$$

Evaluation of Shear - V

$$V = -\frac{dm}{dx}.$$

$$(\text{II-11}) \quad -\delta V = E_{11}[-m_1 a_1{}^2 \cos a_1 x - m_2 a_1{}^4 \cos a_1 x]$$

$$+ E_{12}[m_2 \beta a_1{}^4 \cos a_1 x] + E_{22}[m_2 \beta a_1{}^4 \sin a_1 x]$$

$$+ E_{21}[-m_1 a_1{}^2 \sin a_1 x - m_2 a_1{}^4 \sin a_1 x]$$

$$+ E_{31}[m_1 T_{31} - m_2 V_{31} + m_2 \beta V_{32}] + E_{32}[-m_1 T_{32} + m_2 V_{32} + m_2 \beta V_{31}]$$

$$+ E_{41}[m_1 T_{41} - m_2 V_{41} + m_2 \beta V_{42}] + E_{42}[-m_1 T_{42} + m_2 V_{42} + m_2 \beta V_{41}]$$

$$+ E_{51}[m_1 T_{51} - m_2 V_{51} + m_2 \beta V_{52}] + E_{52}[-m_1 T_{52} + m_2 V_{52} + m_2 \beta V_{51}]$$

$$+ E_{61}[m_1 T_{61} - m_2 V_{61} + m_2 \beta V_{62}] + E_{62}[-m_1 T_{62} + m_2 V_{62} + m_2 \beta V_{61}]$$

$$+ i\{E_{11}[m_2 \beta a_1{}^4 \cos a_1 x] + E_{21}[m_2 \beta a_1{}^4 \sin a_1 x]$$

$$+ E_{12}[-m_1 a_1{}^2 \cos a_1 x - m_2 a_1{}^4 \cos a_1 x + m_2 \beta a_1{}^4 \cos a_1 x]$$

$$+ E_{22}[-m_1 a_1{}^2 \sin a_1 x - m_2 a_1{}^4 \sin a_1 x + m_2 \beta a_1^4 \sin a_1 x]$$

$$+ E_{31}[m_1 T_{32} - m_2 V_{32} + m_2 \beta V_{31}] + E_{32}[m_1 S_{31} - m_2 V_{31} + m_2 \beta V_{32}]$$

$$+ E_{41}[m_1 T_{42} - m_2 V_{42} + m_2 \beta V_{41}] + E_{42}[m_1 S_{41} - m_2 V_{41} + m_2 \beta V_{42}]$$

$$+ E_{51}[m_1 T_{52} - m_2 V_{52} + m_2 \beta V_{51}] + E_{52}[m_1 S_{51} - m_2 V_{51} + m_2 \beta V_{52}]$$

$$+ E_{61}[m_1 T_{62} - m_2 V_{62} + m_2 \beta V_{61}] + E_{62}[m_1 S_{61} - m_2 V_{61} + m_2 \beta V_{62}]\},$$

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Further analytical investigations are made into the damping capability and determination of natural frequencies of laminated beams, consisting of elastic-viscoelastic-elastic layers, as a means for reducing the vibratory energy transmitted through machine foundation supports in naval vessels.

An exact analytical solution is obtained for determining the natural frequencies of simply-supported sandwich beams having no rivets at the ends. Three possible modes of vibration are shown to exist. The case of the simply-supported sandwich beam having rivets at each end is considered and the equations reduced to the solution of 12 x 12 determinant for calculation on a digital computer.

An approximate method is suggested for determining the natural frequencies of sandwich beams having any end conditions. The procedure is simple to use and is exact for simply-supported beams.

A simple but approximate expression is also developed for determining the composite loss factors of sandwich beams. The procedure yields good engineering results.

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