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NATURAL FREQUENCIES OF VIBRATION OF CANTILEVER SANDWICH BEAMS†

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Abstract—Theoretical and experimental studies were made in obtaining the natural frequencies of cantilever sandwich beams subjected to only gravity forces. The method of minimizing the total energy of the system was used for determining the frequencies. A vibration system made by Unholtz-Dickie was utilized to set the beam in vibration. Resonance occurred when the frequency of the shaker coincided with the natural frequency of the beam. The resonance frequencies were measured by transducers mounted at various locations on the beam. A total of sixteen beams of various lengths, thickness and core density were tested.

It was found that the natural frequency of a cantilever sandwich beam depends largely upon the thickness, length, core density and stiffness of the beam. In addition, the natural frequency has a nonlinear variation with the mode and for any particular mode, the value of the frequency increases as the length of the beam decreases.

Design factors were developed based upon the ratios of the theoretical frequencies of homogeneous beams having the same thicknesses and stiffnesses of that of sandwich beams and of the frequencies experimentally determined for similar sandwich beams.

NOMENCLATURE

- x, y, z rectangular coordinates
- t time coordinate
- a length of panel in direction of loading
- c thickness of core
- f, f' thickness of lower and upper facing respectively
- E modulus of elasticity of facings
- ν Poisson's ratio of facings
- E_c modulus of elasticity of core
- G_{xz} modulus of rigidity of core in xz plane
- W_c displacements of core in z direction respectively
- W displacement in z direction of any point in sandwich panel
- $\epsilon_{zc}, \gamma_{xzc}$ normal and shear strains, respectively in core
- $\epsilon_{xB}, \epsilon'_{xB}$ bending strains in lower and upper facings respectively
- τ_{xz} shear stress in core
- N_{FL}, N_{FU} normal forces of the lower and upper faces respectively
- $\epsilon_{xL}, \epsilon_{xU}$ strains in the x direction of lower and upper facings respectively
- σ_f normal stress in an arbitrary fiber at distance z
- V_c elastic strain energy per unit width of core
- V_{NF}, V'_{NF} elastic strain energy per unit width associated with normal force in lower and upper facing respectively
- V total elastic strain energy per unit width of panel
- T kinetic energy per unit width of panel
- ρ mass density of composite panel per unit length and width
- ω theoretical natural frequencies of vibration of cantilever sandwich beam, radians per unit time
- A_1, A_2, A_3, A_4 constants, configuration parameters
- ω_e experimental natural frequencies of cantilever sandwich beam cycle/sec
- ω_h natural frequencies of homogeneous beam equivalent to the sandwich beam cycle/sec
- L length of homogeneous beam
- (W'/g) mass per unit length of the homogeneous beam
- D sandwich panel stiffness lb-in^2

INTRODUCTION

The structure of sandwich plates generally consists of two relatively thin external materials called the facings separated by and bonded to a relatively thick internal structure called the core. The facings are usually of a material which has high strength and stiffness compared with that of the core which is normally of a lighter density and relatively low strength and stiffness.

Due to the nature of their construction, sandwich plates are known to have an extremely high strength-weight ratio as compared to that of a single homogeneous plate. For this reason, their applications are very favorable in the construction of airplanes, guided missiles and space ships, where the weight is a major factor in the design of such structures. Recent improved techniques of bonding and fabrication have increased their application in many other industries and especially in the domestic appliances.

In many instances, these structures are subjected to severe vibration and the designer must have a full understanding of the nature of vibration and the resulting frequencies. For the past two decades or so a great deal of theoretical and experimental effort has been made in

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the dynamic study of sandwich structures. The early attempt towards solving the natural frequencies of vibration of a simply supported sandwich beam has been made by Kimel *et al.*[1]. Later Yu[2] extended the new flexural theory developed by Mindlin[3] to include the effects of transverse-shear deformation and rotatory inertia in both the core and faces of the sandwich plates. The same author[4] used the new flexural theory to solve vibration problems of elastic sandwich plates. Hearmon[5] adopted Rayleigh method to derive closed formula for the frequencies of vibration of orthotropic plates with several combinations of clamped or supported edges. In later years Yu[6, 7] extended his earlier work further to include flexural vibration of sandwich plate in plane strain. On the other hand Raville *et al.*[8] carried out theoretical and experimental work on the natural frequencies of vibration of fixed-fixed sandwich beam. They used the energy approach in which Lagrangian multipliers were utilized to satisfy the boundary conditions of the problem. Ditaranto[9] developed the theory of vibratory bending for elastic and viscoelastic layered finite-length beam. He used the theory of elasticity to derive a sixth order differential equation for the deflection of sandwich beams. In the past decade Yu and Lai[10] presented another theory of nonlinear vibration of sandwich plates where the transverse shear and edge condition were included. They extended the theory to solve dynamic buckling of homogeneous and sandwich plates. Later, Ueng[11] derived the theory of natural frequencies of the all clamped rectangular sandwich plates using the energy method in which Lagrangian multipliers were utilized to satisfy the boundary conditions of the clamped edges. Nicholas and Heller[12] used the theory of elasticity to determine the complex shear modulus of a filled elastomer from the vibration of sandwich beams. At the same time, Ditaranto and Blasingame[13] extended the work of Ditaranto[9] for composite damping of vibrating sandwich beams by adding a dumping factor term into the equilibrium equation. On the other hand, Mead and Markus[14] again extended the work of Ditaranto[9] to include force vibration of three layer damping sandwich beams with arbitrary boundary conditions. Lately, Yan and Dowell[15] presented a governing equation for the damped vibration of a constrained-layer sandwich plates and beams where the principle of virtual work was used.

These preceding analyses do not reveal any rigorous solution for the natural frequencies of a vibrating cantilever sandwich beam subjected only to gravity forces. Therefore, the object of this study is to develop mathematical equations for the natural frequencies of a cantilever sandwich beam, clamped at one end and free at the other end. The method of minimizing the total energy of the system will be used. Furthermore, it is aimed to conduct experimental analysis for the purpose of verifying the theoretical results. An attempt will be made to develop design factors based upon the ratios of the theoretical natural frequencies of homogeneous beams having the same thicknesses and stiffnesses of that of sandwich beams and to the frequencies experimentally obtained for such sandwich beams.

THEORETICAL ANALYSIS

(a) *For the cantilever sandwich beam*

Consider the sandwich beam shown in Fig. 1 which is only subjected to gravity force. In developing the theoretical analysis for the natural frequencies of such a beam, the following assumptions have been used.

(1) The facings of the sandwich are homogeneous, isotropic and elastic thin plates;

(2) The core consists of an elastic, orthotropic continuum whose load carrying capacity in the plane of the sandwich is negligible;

(3) The modulus of elasticity of the core in the direction perpendicular to the facing is infinite and

(4) Perfect continuity exists at the interfaces where the facings and core are bonded together.

The theoretical analysis will be outlined here as follows: The deflection of the beam is taken in a polynomial form[16] as:

$$W = W_c = (A_1x^2 + A_2x^3 + A_3x^4)\sin \omega t. \tag{1}$$

The strains in the facing which occur as a result of bending of the facings about their own middle surfaces are referred to as bending strains. The bending strains, ϵ_{xB} and ϵ'_{xB} in the lower and upper facings respectively,

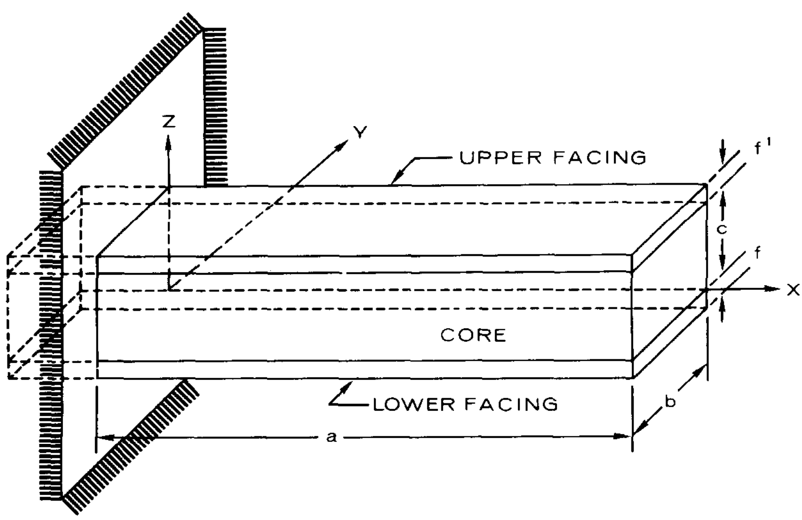

Fig. 1. Cantilever sandwich beam.

can be expressed in terms of the core displacement as;

$$\epsilon_{xB}=\left(z+\frac{f}{2}\right)\left(\frac{\partial^2 W_c}{\partial x^2}\right)_{z=0} \tag{2}$$

and

$$\epsilon'_{xB}=\left(x-c-\frac{f'}{2}\right)\left(\frac{\partial^2 W_c}{\partial x^2}\right)_{z=c} \tag{3}$$

substitution of eqn (1) into eqns (2) and (3) yields:

$$\epsilon_{xB}=\left(z+\frac{f}{2}\right)(2A_1+6A_2x+12A_3x^2)\sin\omega t \tag{4}$$

and

$$\epsilon'_{xB}=\left(z-c-\frac{f'}{2}\right)(2A_1+6A_2x+12A_3x^2)\sin\omega t. \tag{5}$$

The elastic strain energy, V_{BF} associated with the strain caused by bending of the lower facing about its middle surface can be written as:

$$V_{BF}=\frac{E}{2(1-\nu^2)}\int_{-f}^{0}\mathrm{d}z\int_0^a \epsilon_{xB}^2\,\mathrm{d}x \tag{6}$$

or

$$V_{BF}=\frac{Ef^3}{24(1-\nu^2)}[4aA_1^{\,2}+12a^2A_1A_2+16a^3A_1A_3+12a^3A_2^{\,2}+36a^4A_2A_3+28.8a^5A_3^{\,2}]\sin^2\omega t \tag{7}$$

and for the upper facing as:

$$V'_{BF}=\frac{E}{2(1-\nu^2)}\int_c^{c+f}\mathrm{d}z\int_0^a(\epsilon'_{xB})^2\,\mathrm{d}x \tag{8}$$

or

$$V'_{BF}=\frac{E(f')^3}{24(1-\nu^2)}[4aA_1^{\,2}+12a^2A_1A_2+16a^3A_1A_3+12a^3A_2^{\,2}+36a^4A_2A_3+28.8a^5A_3^{\,2}]\sin^2\omega t. \tag{9}$$

The normal stress in an arbitrary fiber of the facing may be written as:

$$\sigma_f=-E\left(z-\frac{c}{2}\right)\frac{\partial^2 W_c}{\partial x^2}. \tag{10}$$

Let N_{FL} and N_{FU} be the normal forces of the lower and upper faces. These forces may be expressed as:

$$N_{FL}=-\int_{-f}^{0}E\left(z-\frac{c}{2}\right)\frac{\partial^2 W_c}{\partial x^2}\,\mathrm{d}z \tag{11}$$

or

$$N_{FL}=E\frac{\partial^2 W_c}{\partial x^2}\frac{f}{2}(f+c) \tag{12}$$

and likewise:

$$N_{FU}=-\int_c^{c+f'}E\left(z-\frac{c}{2}\right)\frac{\partial^2 W_c}{\partial x^2}\,\mathrm{d}z \tag{13}$$

or

$$N_{FU}=-E\frac{\partial^2 W f'}{\partial x^2\ 2}(f'+c). \tag{14}$$

The elastic strain energy associated with the normal force in the lower facing can be written as:

$$V_{NF}=\frac{1}{2}\int_0^a N_{FL}\epsilon_{xL}\,\mathrm{d}x. \tag{15}$$

From beam bending theory and considering the neutral axis is in the middle of the sandwich beam, then the strains in the x direction of lower and upper facings [ϵ_{xL} and ϵ_{xU} respectively] can be expressed as:

$$\epsilon_{xL}=\frac{1}{2}(c+f)\frac{\partial^2 W_c}{\partial x^2}$$

and

$$\epsilon_{xU}=-\frac{1}{2}(c+f')\frac{\partial^2 W_c}{\partial x^2}. \tag{16}$$

Therefore,

$$V_{NF}=\frac{1}{2}\int_0^a E\frac{f}{2}(f+c)\left(\frac{\partial^2 W_c}{\partial x^2}\right)^2\left(\frac{1}{2}(c+f)\right)\mathrm{d}x \tag{17}$$

or

$$V_{NF}=\frac{Ef(f+c)^2}{8}\int_0^a[4A_1^{\,2}+24A_1A_2x+48A_1A_3x^2+36A_2^{\,2}x^2+144A_2A_3x^3+144A_3^{\,2}x^4]\sin^2\omega t\,\mathrm{d}x \tag{18}$$

or

$$V_{NF}=\frac{Ef(f+c)^2}{8}[4aA_1^{\,2}+12a^2A_1A_2+16a^3A_1A_3+12a^3A_2^{\,2}+36a^4A_2A_3+28.8a^5A_3^{\,2}]\sin^2\omega t. \tag{19}$$

In like manner the strain energy V'_{NF} associated with the normal force in the upper facing may be expressed as:

$$V'_{NF}=\frac{1}{2}\int(-N_{FU})(-\epsilon_{xU})\,\mathrm{d}x \tag{20}$$

or

$$V'_{NF}=\frac{Ef'(f'+c)^2}{8}[4aA_1^{\,2}+12a^2A_1A_2+16a^3A_1A_3+12a^3A_2^{\,2}+36a^4A_3+28.8a^5A_3^{\,2}]\sin^2\omega t. \tag{21}$$

The strain energy in the core may be written as:

$$V_c=\frac{1}{2}\int_0^c\mathrm{d}z\int_0^a G_{xz}\gamma_{xz}^2\,\mathrm{d}x \tag{22}$$

where

$$\gamma_{xz}=\frac{1}{G_{xz}}\tau_{xz} \tag{23}$$

and from Fig. 2 it follows that:

$$\tau_{xz}\,\mathrm{d}x=-\frac{\partial N_{FL}}{\partial x}\mathrm{d}x. \tag{24}$$

Substitution of eqn (12) in (24) will give:

$$\gamma_{xz}=-\frac{1}{G_{xz}}\frac{1}{2}Ef(f+c)(6A_2+24A_3x)\sin\omega t \tag{25}$$

and the expression for the strain energy in the core

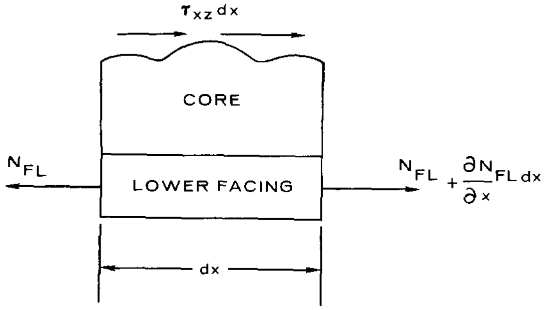

Fig. 2. Differential element of lower facing and core.

becomes:

$$V_c = \frac{cE^2f^2(f+c)^2}{8G_{xz}}[36aA_2^2 + 144a^2A_2A_3 + 192a^3A_3^2]\sin^2\omega t. \quad (26)$$

Hence, the total elastic strain energy of the cantilever beam is expressed as:

$$V = V_c + V_{BF} + V'_{BF} + V_{NF} + V'_{NF} \quad (27)$$

or

$$\begin{aligned} V = \Bigg[&\frac{cE^2f^2(f+c)^2}{8G_{xz}}(36aA_2^2 + 144a^2A_2A_3 + 192a^3A_3^2) \\ &+ \frac{Ef^3}{24(1-\nu^2)}(4aA_1^2 + 12a^2A_1A_2 + 16a^3A_1A_3 \\ &+ 12a^3A_2^2 + 36a^4A_2A_3 + 28.8a^5A_3^2) \\ &+ \frac{E(f')^3}{24(1-\nu^2)}(4aA_1^2 + 12a^2A_1A_2 + 16a^3A_1A_3 \\ &+ 12a^3A_2^2 + 36a^4A_2A_3 + 28.8a^5A_3^2) \\ &+ \frac{Ef(f+c)^2}{8}(4aA_1^2 + 12a^2A_1A_2 + 16a^3A_1A_3 \\ &+ 12a^2A_2^2 + 36a^4A_2A_3 + 28.8a^5A_3^2) \\ &+ \frac{Ef'(f'+c)^2}{8}(4aA_1^2 + 12a^2A_1A_2 + 16a^3A_1A_3 \\ &+ 12a^3A_2^2 + 36a^4A_2A_3 + 28.8a^5A_3^2)\Bigg]\sin^2\omega t. \end{aligned} \quad (28)$$

The kinetic energy of the vibrating beam may be expressed as:

$$T = \frac{1}{2}\rho\int_0^a\left(\frac{\partial W}{\partial t}\right)^2 dx \quad (29)$$

or

$$\begin{aligned} T = \frac{1}{2}\rho\omega^2\Big(&\frac{a^5}{5}A_1^2 + \frac{a^6}{3}A_1A_2 + \frac{2}{7}a^7A_1A_3 \\ &+ \frac{a^7}{7}A_2^2 + \frac{a^8}{4}A_2A_3 + \frac{a^9}{9}A_3^2\Big)\cos^2\omega t. \end{aligned} \quad (30)$$

The vibrating system is assumed to be conservative so that:

$$\frac{\partial}{\partial A_1}(V_{\max} - T_{\max}) = 0 = \frac{\partial V_{\max}}{\partial A_1} - \frac{\partial T_{\max}}{\partial A_1} \quad (31)$$

$$\frac{\partial}{\partial A_2}(V_{\max} - T_{\max}) = 0 = \frac{\partial V_{\max}}{\partial A_2} - \frac{\partial T_{\max}}{\partial A_2} \quad (32)$$

$$\frac{\partial}{\partial A_3}(V_{\max} - T_{\max}) = 0 = \frac{\partial V_{\max}}{\partial A_3} - \frac{\partial T_{\max}}{\partial A_3}. \quad (33)$$

Equations (31), (32) and (33) lead to the three equations in A_1, A_2 and A_3 as:

$$\left(X1 - \frac{1}{2}\rho\omega^2\frac{2}{5}a^5\right)A_1 + \left(X2 - \frac{1}{2}\rho\omega^2\frac{a^6}{3}\right)A_2 + \left(X3 - \frac{1}{2}\rho\omega^2\frac{2}{7}a^7\right)A_3 = 0 \quad (34)$$

$$\left(Y1 - \frac{1}{2}\rho\omega^2\frac{a^6}{3}\right)A_1 + \left(Y2 - \frac{1}{2}\rho\omega^2\frac{2}{7}a^7\right)A_2 + \left(Y3 - \frac{1}{2}\rho\omega^2\frac{a^8}{4}\right)A_3 = 0 \quad (35)$$

$$\left(Z1 - \frac{1}{2}\rho\omega^2\frac{2}{7}a^7\right)A_1 + \left(Z2 - \frac{1}{2}\rho\omega^2\frac{a^8}{4}\right)A_2 + \left(Z3 - \frac{1}{2}\rho\omega^2\frac{2}{9}a^9\right)A_3 = 0. \quad (36)$$

The solution of eqns (34), (35) and (36) other than the trivial one for which $(A_1 = A_2 = A_3 = 0)$ requires that the determinant of these coefficients must be zero, that is:

$$\begin{bmatrix} X1 - \frac{1}{2}\rho\omega^2\frac{2}{5}a^5 & X2 - \frac{1}{2}\rho\omega^2\frac{a^6}{3} & X3 - \frac{1}{2}\rho\omega^2\frac{2}{7}a^7 \\ Y1 - \frac{1}{2}\rho\omega^2\frac{a^6}{3} & Y2 - \frac{1}{2}\rho\omega^2\frac{2}{7}a^7 & Y3 - \frac{1}{2}\rho\omega^2\frac{a^8}{4} \\ Z1 - \frac{1}{2}\rho\omega^2\frac{2}{7}a^7 & Z2 - \frac{1}{2}\rho\omega^2\frac{a^8}{4} & Z3 - \frac{1}{2}\rho\omega^2\frac{2}{9}a^9 \end{bmatrix} = 0 \quad (37)$$

where

$$X1 = \frac{8aEf^3}{24(1-\nu^2)} + \frac{8aE(f')^3}{24(1-\nu^2)} + \frac{8aEf(f+c)^2}{8} + \frac{8aEf'(f'+c)^2}{8} \quad (38)$$

$$X2 = \frac{12a^2Ef^3}{24(1-\nu^2)} + \frac{12a^2E(f')^3}{24(1-\nu^2)} + \frac{12a^2Ef(f+c)^2}{8} + \frac{12a^2Ef'(f'+c)^2}{8} \quad (39)$$

$$X3 = \frac{16a^3Ef^3}{24(1-\nu^2)} + \frac{16a^3E(f')^3}{24(1-\nu^2)} + \frac{16a^3Ef(f+c)^2}{8} + \frac{16a^3Ef'(f'+c)^2}{8} \quad (40)$$

$$Y1 = \frac{12a^2Ef^3}{24(1-\nu^2)} + \frac{12a^2E(f')^3}{24(1-\nu^2)} + \frac{12a^2Ef(f+c)^2}{8} + \frac{12a^2Ef'(f'+c)^2}{8} \quad (41)$$

$$Y2 = \frac{72acE^2f^2(f+c)^2}{8G_{xz}} + \frac{24a^3Ef^3}{24(1-\nu^2)} + \frac{24a^3E(f')^3}{24(1-\nu^2)} + \frac{24a^3Ef(f+c)^2}{8} + \frac{24a^3Ef'(f'+c)^2}{8} \quad (42)$$

$$Y3 = \frac{144a^2cE^2f^2(f+c)^2}{8G_{xz}} + \frac{36a^4Ef^3}{24(1-\nu^2)} + \frac{36a^4E(f')^3}{24(1-\nu^2)} + \frac{36a^4Ef(f+c)^2}{8} + \frac{36a^4Ef'(f'+c)^2}{8} \quad (43)$$

and

$$Z1 = \frac{16a^3Ef^3}{24(1-\nu^2)} + \frac{16a^3E(f')^3}{24(1-\nu^2)} + \frac{16a^3Ef(f+c)^2}{8} + \frac{16a^3Ef'(f'+c)^2}{8} \quad (44)$$

$$Z2 = \frac{144a^2cE^2f^2(f+c)^2}{8G_{xz}} + \frac{36a^4Ef^3}{24(1-\nu^2)} + \frac{36a^4E(f')^3}{24(1-\nu^2)} + \frac{36a^4Ef(f+c)^2}{8} + \frac{36a^4Ef'(f'+c)^2}{8} \quad (45)$$

$$Z3 = \frac{384a^3cE^2f^2(f+c)^2}{8G_{xz}} + \frac{57.6a^5Ef^3}{24(1-\nu^2)} + \frac{57.6a^5E(f')^3}{24(1-\nu^2)} + \frac{57.6a^5Ef(f+c)^2}{8} + \frac{57.6a^5Ef'(f'+c)^2}{8}. \quad (46)$$

(b) *For the cantilever homogeneous beam*

The governing differential equation for the vibration of a cantilever homogeneous beam was derived from the state of equilibrium as:

$$EI\frac{d^4Y}{dX^4} - \left(\frac{W'}{g}\right)\omega_h^2 \cdot Y = 0 \quad (47)$$

where (W'/g) is the mass per unit length of the beam. Solution of eqn (47) gives the natural frequency as

$$\omega_h = \frac{n_m^2}{2\pi}\left[\frac{EIg}{W'}\right]^{1/2} \quad (48)$$

where

$$n_m^2 = \frac{(2m-1)^2\pi^2}{4L^2} \quad (49)$$

and $m = 1, 2, 3, 4, \ldots$

In assuming the stiffness of the homogeneous beam (EI) equal to the stiffness of the sandwich beam, we may write as outlined in [17].

$$EI = D = E\frac{bf^3}{6} + E\frac{bfd^2}{2} + E_c\frac{bc^3}{12} \quad (50)$$

where E and E_c are the modulii of elasticity of facing and core respectively, b is the width of the sandwich beam and d is the distance between the center line of upper and lower faces, that is:

$$d = c + f. \quad (51)$$

where in this study the face thicknesses of the sandwich beam were the same, that is $(f' = f)$.

Hence the frequency equation can be written as:

$$\omega_h = \frac{n_m^2}{2\pi}\left[\frac{Dg}{W'}\right]^{1/2} \quad (52)$$

where $m = 1, 2, 3, 4, \ldots$

IBM computer programs were developed for the solutions of eqns (37) and (52) in order to determine the natural frequencies of the sandwich cantilever beam and of the homogenous cantilever beam respectively.

EXPERIMENTAL TECHNIQUES

The experimental set up used for the vibration of the cantilever sandwich beams is shown in Fig. 3. The vibration system made by Unholtz–Dickie consists of three major parts, the shaker, the control console and the electronic power amplifier. Four sandwich panels made

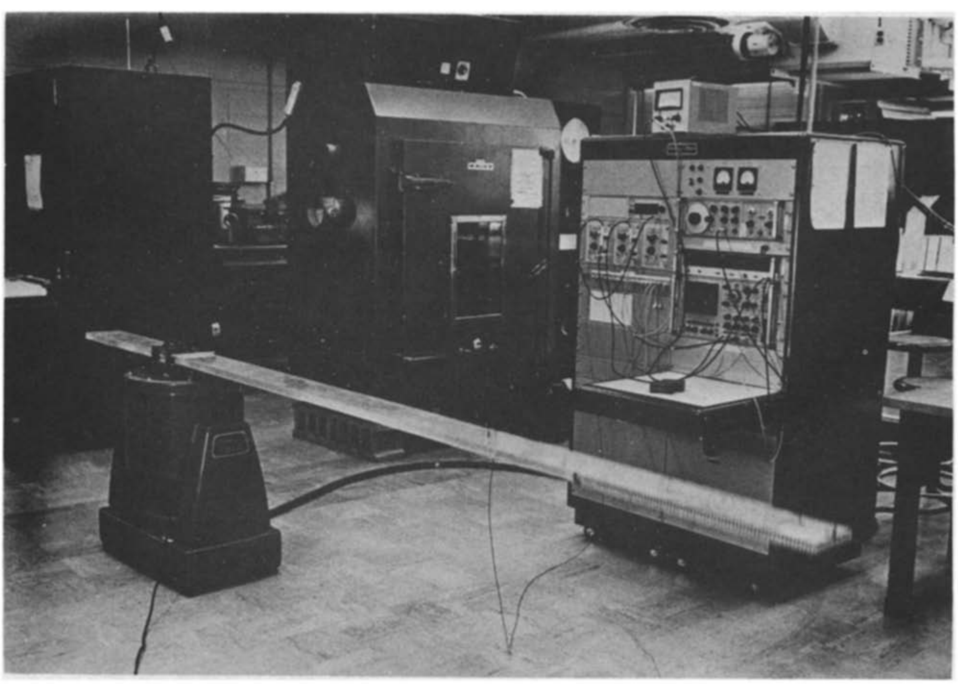

Fig. 3. Experimental set up.

by Hexel Corporation of Casa Grande, Arizona having a width of 6 in. each and of various lengths of 120 in., 100 in., 80 in. and 60 in. respectively were tested. Sixteen different tests were carried out corresponding to different length, core, stiffness and thickness. The physical properties and dimensions of the sandwich panels are given in Table 1. One end of the beam was rigidly clamped to the head of the shaker while the other end was free. The beam was set in vibration by the shaker for various frequencies and when the frequency of the shaker coincided with the natural frequency of the beam, a resonance occurred. The resonating frequencies were measured by quartz accelerometers which were mounted at various locations along the beam. The output of the accelerometers were fed simultaneously to oscilloscope, frequency counters and voltmeters.

The mode number was obtained for each resonating frequency by counting the nodel points on the beam, and the number of modes is one less than the number of nodes.

Table 1. Physical properties and dimensions of sandwich beams

Beam No.	Stiffness D (lb-in^2)	Core thickness (in.)	G_{xz} supplied by Mfg. psi	Wt. of panel lb/in.	Core style	Length of sandwich beam (in.)
1-1	3.748×10^5	1.0	4.35×10^4	0.0335	3/8-5052-0.005	120
1-2	3.748×10^5	1.0	4.35×10^4	0.0335	3/8-5052-0.005	100
1-3	3.748×10^5	1.0	4.35×10^4	0.0335	3/8-5052-0.005	80
1-4	3.748×10^5	1.0	4.35×10^4	0.0335	3/8-5052-0.005	60
2-1	2.308×10^4	0.25	4.35×10^4	0.0214	3/8-5052-0.005	120
2-2	2.308×10^4	0.25	4.35×10^4	0.0214	3/8-5052-0.005	100
2-3	2.308×10^4	0.25	4.35×10^4	0.0214	3/8-5052-0.005	80
2-4	2.308×10^4	0.25	4.35×10^4	0.0214	3/8-5052-0.005	60
3-1	4.698×10^5	1.0	2.2×10^4	0.0278	1/8-5052-0.0007	120
3-2	4.698×10^5	1.0	2.2×10^4	0.0278	1/8-5052-0.0007	100
3-3	4.698×10^5	1.0	2.2×10^4	0.0278	1/8-5052-0.0007	80
3-4	4.698×10^5	1.0	2.2×10^4	0.0278	1/8-5052-0.0007	60
4-1	2.456×10^4	0.25	2.2×10^4	0.0206	1/8-5052-0.0007	120
4-2	2.456×10^4	0.25	2.2×10^4	0.0206	1/8-5052-0.0007	100
4-3	2.456×10^4	0.25	2.2×10^4	0.0206	1/8-5052-0.0007	80
4-4	2.456×10^4	0.25	2.2×10^4	0.0206	1/8-5052-0.0007	60

Modulus of elasticity of aluminum facing $(E) = 1\times10^7$ psi.
Poisson's ratio $(\nu) = 0.33$.
Width of the sandwich panel = 6 in.
Face thickness $(f) = 0.011$ in.

Table 2. Natural frequencies of vibration of beams 1-1, 1-2, 1-3 and 1-4

Beam No. 1-1					Beam No. 1-3				
No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e	No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e
1	2.424	2.659	2.425	1.109	1	5.455	5.984	4.614	1.297
2	15.350	16.667	15.056	1.107	2	34.612	37.501	31.20	1.20
3	81.490	46.650	45.828	1.020	3	183.587	104.962	103.71	1.012
4		91.464	85.08	1.075	4		205.793	185.75	1.108
5		151.180	138.160	1.094	5		340.154	312.68	1.087
6		225.842	204.39	1.105	6		508.144	430.86	1.179
7		315.432	291.49	1.08	7		709.722	579.137	1.225
8		419.954	383.65	1.095	8		944.722	806.77	1.171
9		539.408	497.87	1.083	9		1213.668	1019.6	1.190
10		673.792	596.19	1.130	10		1516.032	1257.3	1.206
Beam No. 1-2					**Beam No. 1-4**				
No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e	No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e
1	3.491	3.830	3.450	1.10	1	9.702	10.637	8.913	1.193
2	22.124	24.001	21.590	1.112	2	61.712	66.669	60.217	1.055
3	117.428	67.176	62.645	1.072	3	327.224	186.599	164.45	1.135
4		131.707	122.140	1.078	4		365.855	339.50	1.078
5		217.699	196.29	1.109	5		604.719	539.97	1.120
6		325.213	303.31	1.072	6		903.366	774.65	1.166
7		454.222	410.49	1.107	7				
8		604.733	535.57	1.129	8				
9		776.747	691.93	1.123	9				
10		970.262	840.98	1.154	10				

DISCUSSION OF THEORETICAL AND EXPERIMENTAL RESULTS

The theoretical analysis yielded 3 real modes of natural frequencies for each sandwich panel, whereas it was experimentally possible to obtain 10 modes for each beam tested as shown in Tables 2–5. It is clear from these tables that the theoretical values for modes No. 1 and No. 2 are in closer agreement with the experimental

Table 3. Natural frequencies of vibration of beams 2-1, 2-2, 2-3 and 2-4

Beam No. 2-1					Beam No. 2-3				
No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e	No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e
1	0.798	1.493	0.76	1.960	1	1.796	3.3598	1.55	2.168
2	5.051	9.359	4.25	2.200	2	11.373	21.0575	9.492	2.218
3	26.820	26.195	12.34	2.123	3	60.388	58.938	29.22	2.017
4		51.358	23.95	2.144	4		115.556	54.137	2.135
5		84.889	40.675	2.087	5		191.001	89.26	2.140
6		126.813	60.44	2.098	6		285.330	131.57	2.169
7		177.119	83.99	2.109	7		398.518	184.36	2.162
8		235.810	110.40	2.136	8		530.517	247.00	2.148
9		302.885	142.94	2.119	9		681.491	311.14	2.19
10		378.344	178.94	2.114	10		851.274	385.14	2.210
Beam No. 2-3					Beam No. 2-4				
No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e	No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e
1	1.150	2.150	1.02	2.108	1	3.194	5.973	2.79	2.141
2	7.275	13.477	6.151	2.191	2	20.259	37.436	17.95	2.209
3	38.641	37.720	17.69	2.132	3	107.468	104.778	48.73	2.150
4		73.955	37.07	1.995	4		205.433	94.03	2.185
5		122.241	58.02	2.107	5		339.558	153.76	2.208
6		182.611	85.87	2.127	6		507.252	239.90	2.114
7		255.052	119.50	2.134	7		708.478	319.07	2.220
8		339.566	158.7	2.140	8		943.238	418.34	2.255
9		436.154	204.24	2.135	9		1211.538	539.56	2.245
10		544.815	254.87	2.138	10		1513.37	649.96	2.328

Table 4. Natural frequencies of vibration of beams 3-1, 3-2, 3-3 and 3-4

Beam No. 3-1					Beam No. 3-3				
No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e	No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e
1	2.66	2.716	2.363	1.149	1	5.991	6.110	5.120	1.193
2	16.878	17.020	13.901	1.224	2	38.134	38.297	30.795	1.244
3	89.561	47.639	37.640	1.266	3	202.074	107.188	90.240	1.188
4		93.403	80.178	1.165	4		210.156	167.150	1.257
5		154.385	128.930	1.197	5		347.365	288.57	1.204
6		230.63	188.438	1.224	6		518.917	451.93	1.148
7		322.12	228.330	1.411	7		724.768	620.57	1.168
8		428.856	368.680	1.163	8		964.927	772.67	1.249
9		550.844	400.986	1.374	9				
10		588.077	487.140	1.412	10				
Beam No. 3-2					Beam No. 3-4				
No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e	No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e
1	3.834	3.912	3.09	1.266	1	10.660	10.863	8.673	1.253
2	24.342	24.510	21.140	1.159	2	68.177	68.083	54.517	1.249
3	129.129	68.520	58.206	1.177	3	360.935	190.55	149.86	1.272
4		134.450	111.638	1.204	4		373.611	304.28	1.228
5		222.314	178.140	1.248	5		617.54	547.51	1.128
6		332.106	267.113	1.243	6		922.517	713.80	1.292
7		463.851	368.94	1.257	7		1288.48	973.45	1.324
8		617.553	530.94	1.163	8				
9		793.215	650.05	1.220	9				
10		990.832	778.55	1.273	10				

Table 5. Natural frequencies of vibration of beams 4-1, 4-2, 4-3 and 4-4

Beam No. 4-1					Beam No. 4-3				
No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e	No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e
1	0.783	1.476	0.7407	1.993	1	1.762	3.322	1.40	2.373
2	4.954	9.253	4.22	2.194	2	11.149	20.82	9.27	2.246
3	26.301	25.899	11.99	2.16	3	59.204	48.273	27.8	2.096
4		50.779	23.85	2.129	4		144.253	52.92	2.726
5		83.932	40.15	2.090	5		188.848	87.91	2.148
6		125.384	59.2	2.118	6		282.113	130.00	2.170
7		175.123	83.32	2.102	7		394.025	181.63	2.169
8		233.151	109.1	2.137	8		524.59	241.46	2.173
9		299.470	141.85	2.111	9		673.808	305.58	2.205
10		374.079	176.21	2.123	10		841.677	384.60	2.188

Beam No. 4-2					Beam No. 4-4				
No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e	No. of Mode M	ω Theory	ω_h Homo.	ω_e exp.	ω_h/ω_e
1	1.128	2.126	1.00	2.126	1	3.133	5.906	2.705	2.183
2	7.133	13.325	6.07	2.195	2	19.841	37.014	16.515	2.241
3	37.895	37.295	17.52	2.129	3	105.32	103.597	45.92	2.256
4		73.122	36.00	2.031	4		203.116	92.56	2.194
5		120.86	56.53	2.138	5		335.730	150.61	2.229
6		180.552	84.94	2.126	6		501.534	229.17	2.188
7		252.177	118.4	2.130	7		700.49	314.15	2.230
8		335.738	157.75	2.128	8		932.605	426.82	2.185
9		431.237	198.79	2.169	9		1197.88	539.0	2.222
10		538.673	253.45	2.125	10		1496.314	633.12	2.363

results than with those values obtained by the homogeneous beam approach. However for all investigated sandwich beams, the third theoretical mode was not in agreement with the value of the third experimental mode. This is a common pitfall with the energy method. Usually, as a rough rule of thumb, one may count on obtaining physical realistic results for a number of frequencies and mode shapes equal to or less than one half of the number of the unknown coefficients appearing in the assumed polynomial function of the beam deflec-

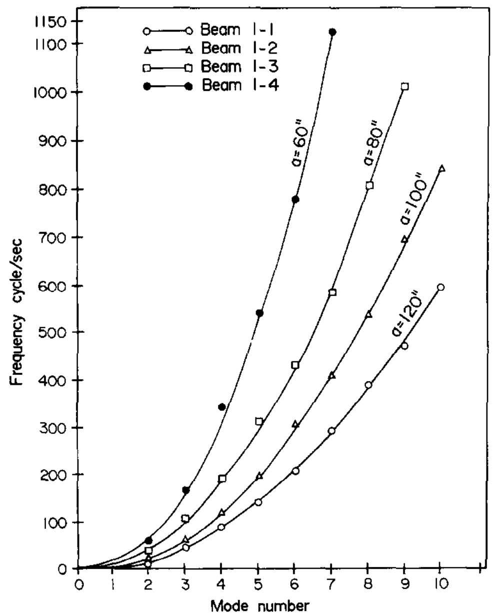

Fig. 4. Experimental variation of frequency with mode for beams 1-1, 1-2, 1-3 and 1-4.

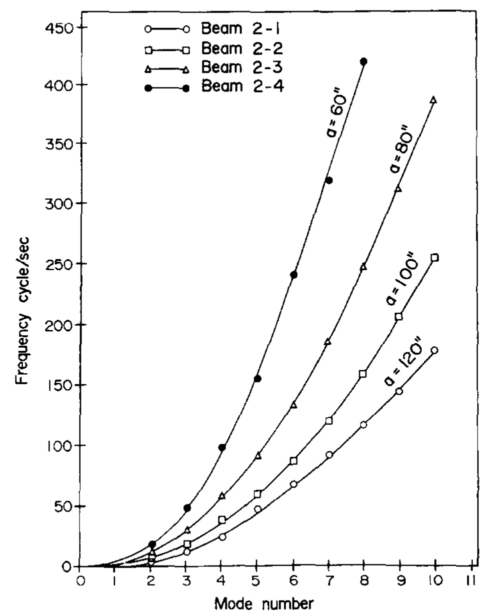

Fig. 5. Experimental variation of frequency with mode for beams 2-1, 2-2, 2-3 and 2-4.

tion. Therefore it is suggested here when higher numbers of theoretical modes are desired, then the number of coefficients in the assumed function for beam deflection should be increased by at least twice the number of the desired modes.

In general the values of the experimental results were lower than the theoretical ones for modes Nos. 1 and 2. This may be attributed to several factors such as the influence of the values of the modulus of rigidity G_{xy}, which were supplied by the manufacturer, the damping resistance of the air surrounding the vibrating sandwich beam and the dynamic rotation which were neglected in the assumption used in this theory.

The experimental results were presented in Figs. 4–7. In examining Figs. 4 and 6 for beam of 1 in. thickness it can be seen that the variational trend of the frequency with mode is nearly the same for those beams of the same length. This trend is also evident in Figs. 5 and 7 for those beams of 0.25 in. thickness. Although, the experimental results do not agree directly with the homogeneous beam theory, however, the idea of introducing the homogeneous beam theory was intended to predict the natural frequencies of sandwich beams by developing ratio factors of ω_h/ω_e as shown in Tables 2–5.

It is concluded in this study that in general the natural frequencies have a nonlinear variation with the mode and for any particular mode, the values of frequencies increase as the length of the sandwich beam decreases. Furthermore, the experimental results indicate that the natural frequencies of a cantilever beam depend largely upon the thickness, length, mass density and the stiffness of the beam. Lastly, design factors were developed based upon the ratios of the theoretical frequencies of homogeneous beams having the same thickness and stiffness of that of sandwich beams and of the frequencies experimentally obtained for similar sandwich beams. These factors will aid the designer in obtaining the natural frequencies of cantilever sandwich beams without carrying out the experimental work. In addition, the experimental results presented here should help the designer in recognizing these natural frequencies in such vibrating structures.

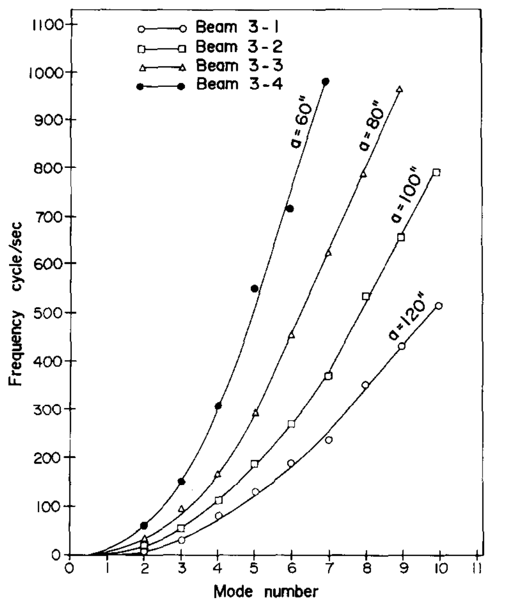

Fig. 6. Experimental variation of frequency with mode for beams 3-1, 3-2, 3-3 and 3-4.

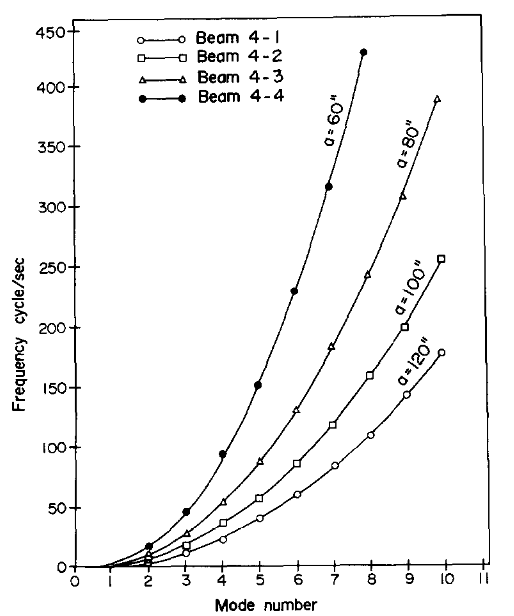

Fig. 7. Experimental variation of frequency with mode for beams 4-1, 4-2, 4-3 and 4-4.

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REFERENCES

1. W. R. Kimel, M. E. Raville, P. G. Kirmser and M. P. Patel, Natural frequencies of vibration of simply supported sandwich beams. *Proc. 4th Midwestern Conf. Solid and Fluid Mechanics*. Austin, Texas (Sept. 1959).
2. Yi-Yuan Yu, A new theory of elastic sandwich plates one dimensional case. *J. Appl. Mech.* **26**, *Trans. ASME* **81**, Series E, 415–421 (1958).
3. R. D. Mindlin, An introduction to the mathematical theory of vibration of elastic plates. A monograph prepared for U.S. Army Signal Corps Engineering Laboratories (1955).
4. Yi-Yuan Yu, Flexural vibrations of elastic sandwich plate. *J. Aero/Space Sci.* **27**, 272–282 (1960).
5. R. F. S. Hearmon, The frequency of flexural vibration of rectangular orthotropic plates with clamped or supported edges. *J. Appl. Mech.* 537–540 (1959).
6. Yi-Yuan Yu, Forced flexural vibrations of sandwich plates in plane strain. *J. Appl. Mech.* **27**, *Trans. ASME* **82**, Series E. 535–540 (1960).
7. Yi-Yuan Yu, Simplified vibration analysis of elastic sandwich plate. *J. Aero/Space Sci.* **27**, 894–900 (1960).
8. M. E. Raville, E. Ueng and M. Lei, Natural frequencies of vibration of fixed–fixed sandwich beams. *J. Appl. Mech.* 367–371 (Sept. 1961).
9. R. A. Ditaranto, Theory of vibratory bending for elastic and viscoelastic layered finite-length beams. *J. Appl. Mech.* 881–886 (Dec. 1965).
10. Yi-Yuan Yu and Jai-Lue, Influence of transverse shear and edge condition on nonlinear vibration and dynamic buckling of homogeneous and sandwich plates. *J. Appl. Mech.* 934–936 (Dec. 1966).
11. C. E. S. Ueng, Natural frequencies of vibration of an all-clamped rectangular sandwich panel. *J. Appl. Mech.* 683–684 (Sept. 1966).
12. T. Nicholas and R. A. Heller, Determination of the complex shear modulus of a filled elastomer from a vibrating sandwich beam. *Exper. Mech.* 109–116 (March 1967).
13. R. A. Ditaranto and W. Blasingame, Composite damping of vibrating sandwich beams. *J. Engng Indust.* 633–638 (Nov. 1967).
14. D. J. Mead and S. Markus, *J. Sound Vibration* **2**, 163–175 (Oct. 1969).
15. M. J. Yan and E. H. Dowell, Governing equations for vibrating constrained-layer damping sandwich plates and beams. *J. Appl. Mech.* 1041–1046 (Dec. 1972).
16. A. W. Leissa, *Vibrations of Plates*. NASA SP-160, pp. 44 (1969).
17. H. G. Allen, *Analysis and Design of Structural Sandwich Panels*. 1st Edn, pp. 9. Pergamon Press, Oxford (1969).