Question 1

Ordering: shortest weighted processing time first.

Proof:
Let $i(1)$ be the first job processed, $i(2)$ the second etc
With shortest weighted processing time we have:

$$w_1 p_{i(1)} \leq w_2 p_{i(2)} \leq \cdots \leq w_n p_{i(n)}$$

Now consider some order with

$$w_k p_{i(k)} \geq w_{k+1} p_{i(k+1)} \tag{1}$$

Let job $i(k-1)$ be completed at time t, then:

$$\begin{aligned}
C_{i(k)} &= t + p_{i(k)} \\
C_{i(k+1)} &= t + p_{i(k)} + p_{i(k+1)} \\
\sum_{i}^{n} w_i C_i &= \sum_{\substack{j=1 \\ j \neq k, j \neq k+1}}^{n} w_j C_{i(j)} + w_k C_{i(k)} + w_{k+1} C_{i(k+1)}
\end{aligned}$$

The objective function becomes:

$$z_1 = \sum_{\substack{j=1 \\ j \neq k, j \neq k+1}}^{n} w_j C_{i(j)} + w_k(t + p_{i(k)}) + w_{k+1}(t + p_{i(k)} + p_{i(k+1)})$$

Now swap job order of $i(k), i(k+1)$:

$$z_2 = \sum_{\substack{j=1 \\ j \neq k, j \neq k+1}}^{n} w_j C_{i(j)} + w_{k+1}(t + p_{i(k+1)}) + w_k(t + p_{i(k)} + p_{i(k+1)})$$

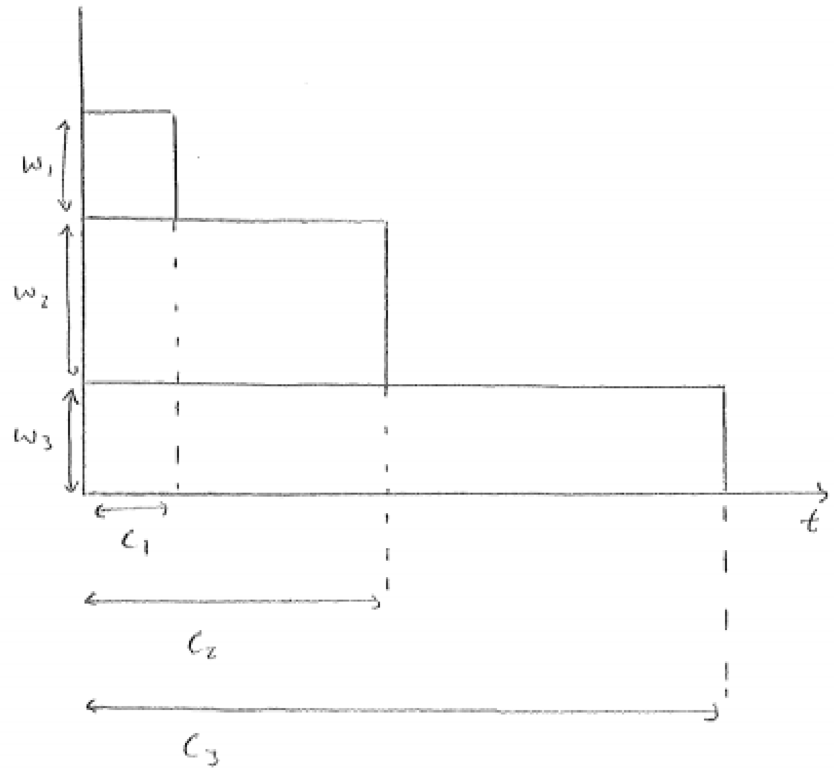
The change in objective function becomes:

$$\begin{aligned}
z_2 - z_1 &= w_{k+1}(t + p_{i(k+1)}) + w_k(t + p_{i(k)} + p_{i(k+1)}) - w_k(t + p_{i(k)}) - w_{k+1}(t + p_{i(k)} + p_{i(k+1)}) \\
&= w_k p_{i(k+1)} - w_{k+1} p_{i(k)} \\
&= w_{k+1} p_{i(k+1)} \left[\frac{w_k}{w_{k+1}} - \frac{p_{i(k)}}{p_{i(k+1)}} \right]
\end{aligned}$$

From (1), knowing weights and processing times are positive, we can state:

$$\frac{p_{i(k+1)}}{p_{i(k)}} \geq \frac{w_k}{w_{k+1}}$$

Therefore $z_2 - z_1 \leq 0$, meaning switching to shortest weighted processing time order can only improve or maintain the solution. This proves shortest weighted processing time order is optimal.

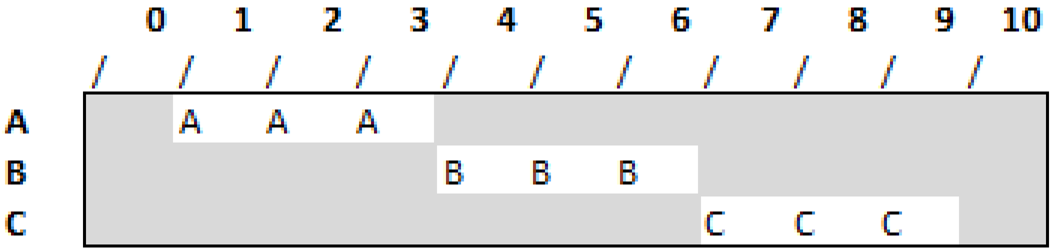

w_1
w_2
w_3
t
C_1
C_2
C_3

Question 2

a)

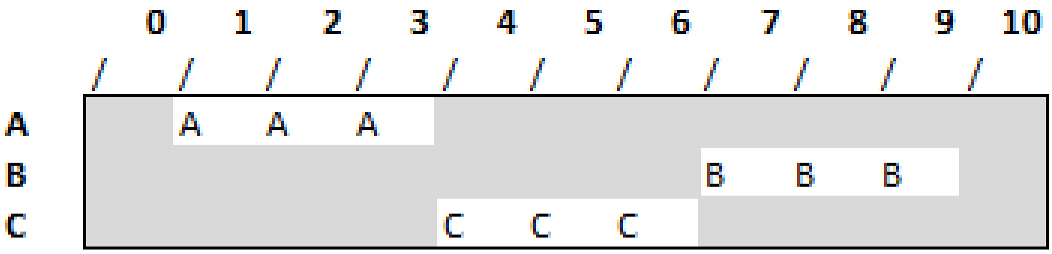
Jobs	A	B	C
p_j	3	3	3
d_j	0	0	0
r_j	1	2	3

Optimal solution via rule



L_max = 10

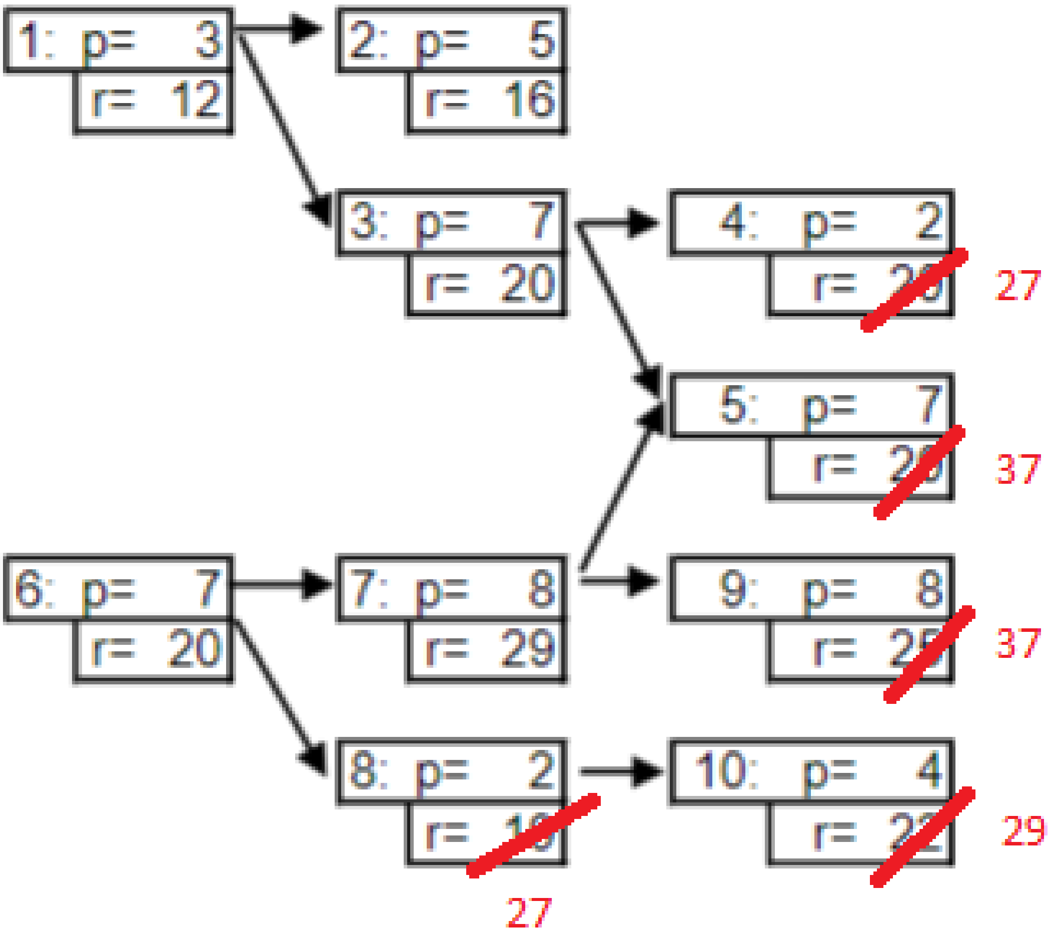
Optimal solution alternative



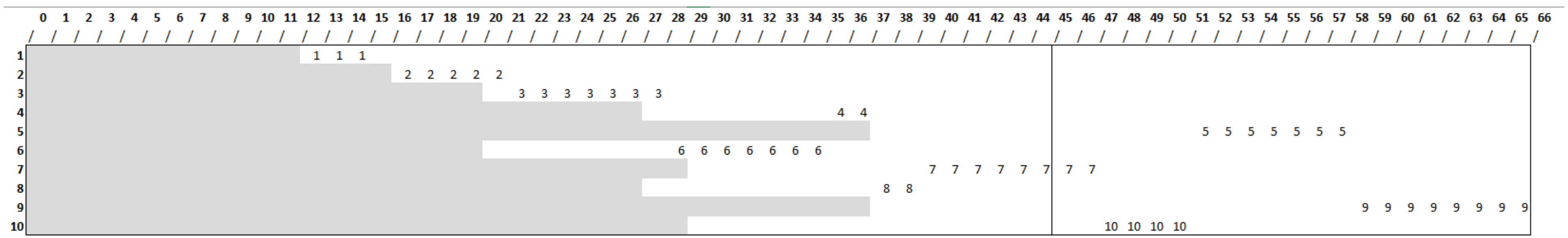
L_max = 10

b) $1/r_j/C_{\max}$

Question 3



Maximum lateness = 21
Order: 1, 2, 3, 6, 4, 8, 7, 10, 5, 9



L_max = 21

Order:
1 2 3 6 4 8 7 10 5 9

Question 4

Let t_j be the time job j starts.
Let x_{ij} be 1 if job i precedes job j, 0 otherwise.
Let T_j be the tardiness of job j.
Let other variables be as defined as in the notes/assignment.

$$
\begin{aligned}
\text{minimize} \quad & \sum T_i \\
\text{subject to} \quad & x_{ij} + x_{ji} = 1 && \forall i, j, i \neq j \\
& t_i - t_j + M x_{ij} \geq p_j && \forall i, j \\
& t_j - t_i \geq p_i && \forall j \in S_i, \forall i \\
& t_i \geq r_i && \forall i \\
& T_i - t_i \geq p_i - d_i && \forall i \\
\\
& t_i \geq 0, \ \text{integer} && \forall i \\
& T_i \geq 0, \ \text{integer} && \forall i \\
& x_{ij} \ \text{binary} && \forall i, j
\end{aligned}
$$

Question 5

Jobs	A	B	C	D	E
p_j	13	9	13	10	8
d_j	6	18	10	11	13
w_j	2	4	3	5	4

K=	2

t=	0	p_avg=	10.6
job	w_j/p_j	slack	I_j(t)
A	0.153846	0	0.153846
B	0.444444	9	0.290701
C	0.230769	0	0.230769
D	0.5	1	0.476963
E	0.5	5	0.39495

Choose jobs D, E
Job D complete at time 10
Job E complete at time 8

Machine 1: D
Machine 2: E

Job E complete.
Machine 1: Job D. Machine 2: Free. Jobs A, B, C left.

t=	8	p_avg=	11.66667
job	w_j/p_j	slack	I_j(t)
A	0.153846	0	0.153846
B	0.444444	1	0.425799
C	0.230769	0	0.230769
D	0.5	0	0.5
E	0.5	0	0.5

Choose job B
Job D complete at time 10
Job B complete at time 17

Machine 1: D
Machine 2: E, B

Job D complete
Machine 1: Free. Machine 2: Job B. Jobs A, C left.

t=	10	p_avg=	13
job	w_j/p_j	slack	I_j(t)
A	0.153846	0	0.153846
B	0.444444	0	0.444444
C	0.230769	0	0.230769
D	0.5	0	0.5
E	0.5	0	0.5

Choose job C
Job C complete at time 23
Job B complete at time 17

Machine 1: D, C
Machine 2: E, B

Job B complete
Machine 1: Job C. Machine 2: Free. Job A left.

t=	17	p_avg=	13
job	w_j/p_j	slack	I_j(t)
A	0.153846	0	0.153846
B	0.444444	0	0.444444
C	0.230769	0	0.230769
D	0.5	0	0.5
E	0.5	0	0.5

Choose job A
Job C complete at time 23
Job A complete at time 30

Machine 1: D, C
Machine 2: E, B, A

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	D	D	D	D	D	D	D	D	D	D	C	C	C	C	C	C	C	C	C	C	C	C	C								
2	E	E	E	E	E	E	E	E	B	B	B	B	B	B	B	B	B	A	A	A	A	A	A	A	A	A	A	A	A	A	

	T	w	wT
A	24	2	48
B	0	4	0
C	13	3	39
D	0	5	0
E	0	4	0

obj= 87

Objective = 87

Question 6

Score:

$$\left[\frac{1}{r_j} + \frac{1}{\max(\text{SETUP, } 0.5)}\right] * \frac{1}{\max(d_j - p_j - t, 3)} * w_j$$

where SETUP is equal to the setup cost between the current job and job j.
Final score: 176

			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
		Jobs	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
a	1	**p_j**	11	12	7	2	11	14	8	16	14	10	7	10	15	3	2	10	13	6
		r_j	4	8	7	47	31	11	13	55	45	19	71	28	9	11	22	14	28	16
b	1	**d_j**	66	123	36	166	114	70	189	122	120	58	152	182	213	70	199	34	86	171
c	3	**Colour**	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
d	0.5	**wj**	9	7	1	3	4	7	5	9	4	2	8	6	6	4	10	1	10	1

Colour ('Col') Setup

From\To	1	2	3
1	0	5	7
2	7	0	9
3	2	5	0

Time=	0	Col=	1	Score=	0.37	0.02	0.01	0.04	0.01	0.03	0.06	0.02	0.01	0.09	0.01	0.01	0.06	0.02	0.01	0.09	0.03	0
Time=	15	Col=	1	Score=	0.51	0.02	0.02	0.04	0.01	0.04	0.06	0.02	0.01	0.12	0.01	0.01	0.07	0.02	0.01	0.23	0.04	0
Time=	25	Col=	1	Score=	0.68	0.03	0.07	0.04	0.01	0.05	0.07	0.02	0.01	0.18	0.01	0.01	0.07	0.03	0.01	0.69	0.05	0
Time=	35	Col=	1	Score=	1.01	0.03	0.1	0.05	0.01	0.08	0.07	0.03	0.01	0.32	0.02	0.01	0.08	0.04	0.01	0.69	0.06	0
Time=	49	Col=	3	Score=	1.13	0.04	0.71	0.01	0.02	2.09	0.02	0.03	0.14	0.37	0.02	0.1	0.02	0.06	0.14	0.19	0.1	0.02
Time=	63	Col=	3	Score=	2.25	0.05	0.71	0.02	0.02	4.88	0.02	0.05	0.19	0.37	0.02	0.11	0.03	0.29	0.15	0.19	0.24	0.02
Time=	71	Col=	2	Score=	1.18	0.37	0.08	0.01	0.25	0.47	0.01	0.52	0.02	0.13	0.22	0.01	0.01	2.79	0.01	0.07	6.79	0
Time=	84	Col=	2	Score=	1.18	0.55	0.08	0.01	0.43	0.47	0.01	0.83	0.02	0.13	0.26	0.01	0.01	2.79	0.01	0.07	6.79	0
Time=	100	Col=	2	Score=	1.18	1.35	0.08	0.01	2.71	0.47	0.01	3.03	0.09	0.13	0.36	0.01	0.02	2.79	0.02	0.07	6.79	0
Time=	111	Col=	2	Score=	1.18	4.96	0.08	0.01	2.71	0.47	0.02	6.05	0.18	0.13	0.47	0.01	0.02	2.79	0.02	0.07	6.79	0
Time=	123	Col=	2	Score=	1.18	4.96	0.08	0.01	2.71	0.47	0.02	6.05	0.18	0.13	0.73	0.02	0.02	2.79	0.02	0.07	6.79	0
Time=	130	Col=	2	Score=	1.18	4.96	0.08	0.01	2.71	0.47	0.02	6.05	0.18	0.13	1.07	0.02	0.02	2.79	0.02	0.07	6.79	0
Time=	153	Col=	3	Score=	2.25	0.76	0.71	0.14	0.31	4.88	0.1	0.65	2.7	0.37	0.57	0.64	0.08	0.39	0.46	0.19	0.79	0.17
Time=	163	Col=	3	Score=	2.25	0.76	0.71	0.52	0.31	4.88	0.16	0.65	2.7	0.37	0.57	1.36	0.1	0.39	0.6	0.19	0.79	0.69
Time=	169	Col=	3	Score=	2.25	0.76	0.71	0.52	0.31	4.88	0.24	0.65	2.7	0.37	0.57	4.07	0.13	0.39	0.73	0.19	0.79	0.69
Time=	171	Col=	3	Score=	2.25	0.76	0.71	0.52	0.31	4.88	0.29	0.65	2.7	0.37	0.57	4.07	0.14	0.39	0.79	0.19	0.79	0.69
Time=	175	Col=	1	Score=	6.75	0.76	0.1	2.02	0.31	0.55	1.73	0.65	0.22	1.37	0.57	0.36	0.55	0.39	0.09	0.69	0.79	0.07
Time=	183	Col=	1	Score=	6.75	0.76	0.1	2.02	0.31	0.55	3.46	0.65	0.22	1.37	0.57	0.36	0.84	0.39	0.13	0.69	0.79	0.07

BestScore=	0.37	Best Job=	1	pj=	11	rj=	4	wj=	9	dj=	66	col=	1	Setup=	0	Start=	4	Cj=	15	Tj=	0	Fj=	11
BestScore=	0.23	Best Job=	16	pj=	10	rj=	14	wj=	1	dj=	34	col=	1	Setup=	0	Start=	15	Cj=	25	Tj=	0	Fj=	11
BestScore=	0.18	Best Job=	10	pj=	10	rj=	19	wj=	2	dj=	58	col=	1	Setup=	0	Start=	25	Cj=	35	Tj=	0	Fj=	16
BestScore=	0.1	Best Job=	3	pj=	7	rj=	7	wj=	1	dj=	36	col=	3	Setup=	7	Start=	42	Cj=	49	Tj=	13	Fj=	42
BestScore=	2.09	Best Job=	6	pj=	14	rj=	11	wj=	7	dj=	70	col=	3	Setup=	0	Start=	49	Cj=	63	Tj=	0	Fj=	52
BestScore=	0.29	Best Job=	14	pj=	3	rj=	11	wj=	4	dj=	70	col=	2	Setup=	5	Start=	68	Cj=	71	Tj=	1	Fj=	60
BestScore=	6.79	Best Job=	17	pj=	13	rj=	28	wj=	#	dj=	86	col=	2	Setup=	0	Start=	71	Cj=	84	Tj=	0	Fj=	56
BestScore=	0.83	Best Job=	8	pj=	16	rj=	55	wj=	9	dj=	122	col=	2	Setup=	0	Start=	84	Cj=	100	Tj=	0	Fj=	45
BestScore=	2.71	Best Job=	5	pj=	11	rj=	31	wj=	4	dj=	114	col=	2	Setup=	0	Start=	100	Cj=	111	Tj=	0	Fj=	80
BestScore=	4.96	Best Job=	2	pj=	12	rj=	8	wj=	7	dj=	123	col=	2	Setup=	0	Start=	111	Cj=	123	Tj=	0	Fj=	115
BestScore=	0.73	Best Job=	11	pj=	7	rj=	71	wj=	8	dj=	152	col=	2	Setup=	0	Start=	123	Cj=	130	Tj=	0	Fj=	59
BestScore=	0.18	Best Job=	9	pj=	14	rj=	45	wj=	4	dj=	120	col=	3	Setup=	9	Start=	139	Cj=	153	Tj=	33	Fj=	108
BestScore=	0.64	Best Job=	12	pj=	10	rj=	28	wj=	6	dj=	182	col=	3	Setup=	0	Start=	153	Cj=	163	Tj=	0	Fj=	135
BestScore=	0.69	Best Job=	18	pj=	6	rj=	16	wj=	1	dj=	171	col=	3	Setup=	0	Start=	163	Cj=	169	Tj=	0	Fj=	153
BestScore=	0.73	Best Job=	15	pj=	2	rj=	22	wj=	#	dj=	199	col=	3	Setup=	0	Start=	169	Cj=	171	Tj=	0	Fj=	149
BestScore=	0.52	Best Job=	4	pj=	2	rj=	47	wj=	3	dj=	166	col=	1	Setup=	2	Start=	173	Cj=	175	Tj=	9	Fj=	128
BestScore=	1.73	Best Job=	7	pj=	8	rj=	13	wj=	5	dj=	189	col=	1	Setup=	0	Start=	175	Cj=	183	Tj=	0	Fj=	170
BestScore=	0.84	Best Job=	13	pj=	15	rj=	9	wj=	6	dj=	213	col=	1	Setup=	0	Start=	183	Cj=	198	Tj=	0	Fj=	189
																					176		1579
																				Score=	176		

Plot