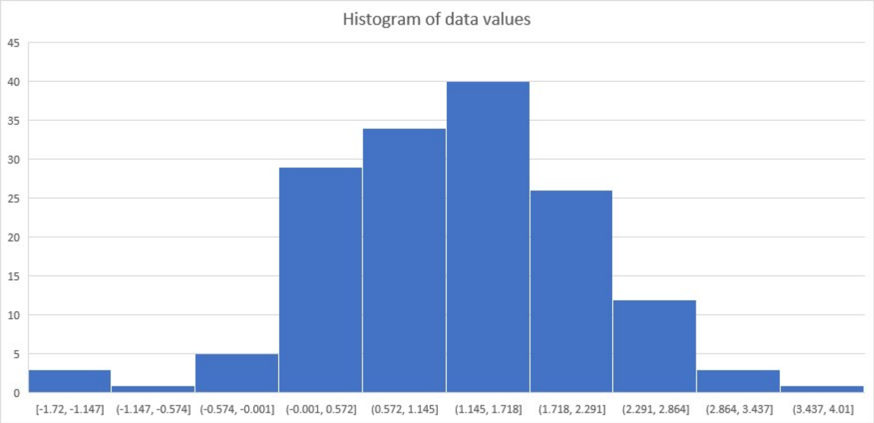
Question 1

$$
\begin{aligned}
P(Z=z) &= P(X+Y=z) \\
&= \sum_{i=0}^{z} P(X+Y=z, X=i) \\
&= \sum_{i=0}^{z} P(Y=z-i, X=i) \\
&= \sum_{i=0}^{z} P(Y=z-i) \times P(X=i) \\
&= \sum_{i=0}^{z} \left[\frac{\mu^{(z-i)}e^{-\mu}}{(z-i)!}\right] \times \left[\frac{\lambda^i e^{-\lambda}}{i!}\right] \\
&= \sum_{i=0}^{z} \frac{e^{-(\mu+\lambda)}\mu^{(z-i)}\lambda^i}{(z-i)! \times i!} \\
&= \sum_{i=0}^{z} \frac{1}{z!} \times \frac{z!}{(z-i)! \times i!} \times e^{-(\mu+\lambda)}\mu^{(z-i)}\lambda^i \\
&= \sum_{i=0}^{z} \left[\frac{z!}{(z-i!) \times i!} \times \lambda^i \times \mu^{(z-i)}\right] \times \frac{e^{-(\mu+\lambda)}}{z!} \\
&= \sum_{i=0}^{z} \left[\binom{z}{i} \times \lambda^i \times \mu^{(z-i)}\right] \times \frac{e^{-(\mu+\lambda)}}{z!} \\
&= (\lambda+\mu)^z \times \frac{e^{-(\mu+\lambda)}}{z!}
\end{aligned}
$$

The last step appears from recognizing the binomial theorem. The final expression is the equation for a Poisson process with rate $= (\lambda + \mu)$, therefore $Z(t)$ is also a Poisson process.

Question 2



Looking at the histogram of the data with number of bins = 10, I hypothesised a normal distribution for the data, with the first and last bin extended to positive and negative infinity. Using Maximum Likelihood Estimators to estimate the two parameters of a normal distribution:

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} \times \exp\left(- \frac{(x-\mu)^2}{2\sigma^2} \right) \\
L(\mu, \sigma^2) &= \prod_{i=0}^{j} \frac{1}{\sqrt{2\pi\sigma^2}} \times \exp\left(- \frac{(x_i-\mu)^2}{2\sigma^2} \right) \\
&= \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \times \prod_{i=0}^{j} \exp\left(- \frac{(x_i-\mu)^2}{2\sigma^2} \right) \\
&= \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \times \exp\left(- \frac{1}{2\sigma^2} \sum_{i=0}^{n} (x_i-\mu)^2 \right)
\end{aligned}
$$

Using log maximum likelihood:

$$
\begin{aligned}
l(\mu, \sigma^2) &= \ln(L(\mu, \sigma^2)) \\
&= \ln\left(\left(\frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \right) + \ln\left(\exp\left(- \frac{1}{2\sigma^2} \sum_{i=0}^{n} (x_i-\mu)^2 \right) \right) \\
&= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=0}^{n} (x_i-\mu)^2 \\
&= -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=0}^{n} (x_i-\mu)^2
\end{aligned}
$$

Differentiating and setting to zero to find parameters:

$$
\begin{aligned}
\frac{\partial l}{\partial \mu} &= \frac{1}{\sigma^2}\sum_{i=0}^{n}(x_i-\mu)^2 \\
0 &= \sum_{i=0}^{n}(x_i-\mu)^2 \\
0 &= \sum_{i=0}^{n}(x_i) - n\mu \\
\mu &= \frac{\sum_{i=0}^{n} x_i}{n}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=0}^{n}(x_i-\mu)^2 \\
&= \frac{1}{2\sigma^2}\left[\frac{1}{\sigma^2}\sum_{i=0}^{n}(x_i-\mu)^2 - n\right] \\
0 &= \frac{1}{\sigma^2}\sum_{i=0}^{n}(x_i-\mu)^2 - n \\
\sigma^2 &= \frac{\sum_{i=0}^{n}(x_i-\mu)^2}{n}
\end{aligned}
$$

Therefore the maximum likelihood parameters are the sample mean and sample variance.

Using the ten bins as shown previous, a χ^2 test was performed to evaluate goodness of fit.

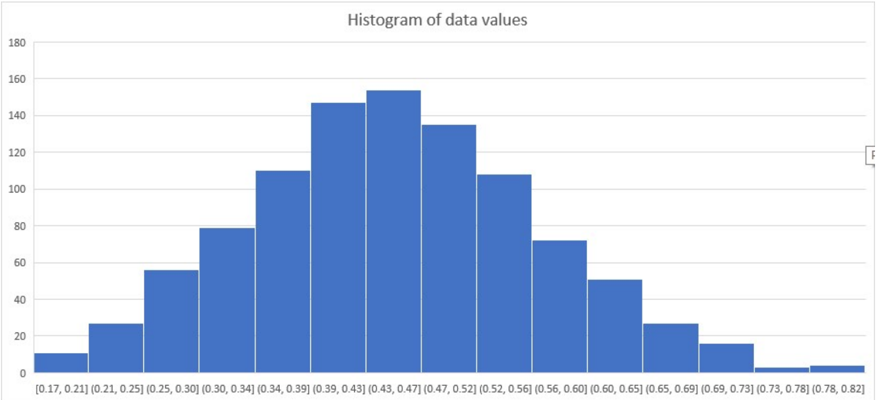
Bin	Range	Estimated n	Actual n	χ^2
1	-inf, -1.147	0.753	3	6.71
2	-1.147, -0.574	3.09	1	1.41
3	-0.574, -0.001	10.1	5	2.56
4	-0.001, 0.572	22.6	29	1.81
5	0.572, 1.145	34.8	34	0.0177
6	1.145, 1.718	36.8	40	0.281
7	1.718, 2.291	26.7	26	0.0198
8	2.291, 2.864	13.3	12	0.134
9	2.864, 3.437	4.57	3	0.540
10	3.437, +inf	1.27	1	0.0571

χ^2 sums to 13.54, which is lower than the value 16.92 needed for a 95% confidence with 9 degrees of freedom - therefore we accept the null hypothesis and are satisfied with goodness of fit with normal distribution with mean = 1.23 and variance = 0.847.

Question 3

The random observations were generated in python, with the following code:

```
import random
import numpy as np

average = []

for i in range(1000):

    random_set = [random.random() for x in range(10)]
    random_set.sort()

    average.append((random_set[2] + random_set[4] + random_set[6]) / 3.0)

np.savetxt("list.csv", average)
```



Looking at the histogram of the data with number of bins = 15, I hypothesised a beta distribution for the data. Matching moments to estimate the two parameters of a beta distribution:

$$\begin{aligned}
\mu &= \frac{\alpha}{\alpha+\beta} \\
\alpha &= \mu(\alpha+\beta) \\
\beta &= \frac{\alpha(1-\mu)}{\mu}
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
&= \alpha^2\mu^2(1-\mu)\Big/\left[\mu^2\left(\alpha+\frac{\alpha(1-\mu)}{\mu}\right)^2 \times \mu\left(\alpha+\frac{\alpha(1-\mu)}{\mu}+1\right)\right] \\
&= \alpha^2\mu^2(1-\mu)\Big/\left[\left(\alpha\mu+\alpha(1-\mu)\right)^2 \times \left(\alpha\mu+\alpha(1-\mu)+\mu\right)\right] \\
&= \alpha^2\mu^2(1-\mu)\Big/\left[\alpha^2(\alpha+\mu)\right] \\
&= \frac{\mu^2(1-\mu)}{(\alpha+\mu)}
\end{aligned}$$

Re-arrange to give:

$$\alpha = \frac{\mu^2(1-\mu)}{\sigma^2} - \mu$$
$$\beta = \frac{\alpha(1-\mu)}{\mu}$$

Using 15 bins[1] , a χ^2 test was performed to evaluate goodness of fit.

Bin	Range	Estimated n	Actual n	χ^2
1	0, 1/15	0.00354	0	0.00354
2	1/15, 2/15	0.585	0	0.585
3	2/15, 3/15	8.31	7	0.208
4	3/15, 4/15	40.5	42	0.0580
5	4/15, 5/15	106	107	0.0133
6	5/15, 6/15	181	170	0.636
7	6/15, 7/15	221	239	1.42
8	7/15, 8/15	202	194	0.347
9	8/15, 9/15	139	137	0.0368
10	9/15, 10/15	70.5	70	0.00344
11	10/15, 11/15	24.7	27	0.207
12	11/15, 12/15	5.35	5	0.0224
13	12/15, 13/15	0.561	2	3.69
14	13/15, 14/15	0.0168	0	0.0168
15	14/15, 15/15	0.0000269	0	0.0000269

χ^2 sums to 7.24, which is lower than the value 23.68 needed for a 95% confidence with 14 degrees of freedom - therefore we accept the null hypothesis and are satisfied with goodness of fit with beta distribution with $\alpha = 8.21$ and $\beta = 9.98$.

[1] Not as in figure, because Excel does not allow fine grain tuning of histogram bins

Question 4

$$\begin{aligned}
f(x) &= (1-p)^{x-1}p \\
L(p) &= \prod_{i=1}^{n}(1-p)^{x_i-1}p \\
&= p^n \prod_{i=1}^{n}(1-p)^{x_i-1} \\
&= \frac{p^n}{(1-p)^n}\prod_{i=1}^{n}(1-p)^{x_i}
\end{aligned}$$

$$\begin{aligned}
l(p) &= \ln L(p) \\
&= \ln\left(\frac{p^n}{(1-p)^n}\right) + \sum_{i=0}^{n}\ln\left[(1-p)^{x_i}\right] \\
&= n\ln\left(\frac{p}{1-p}\right) + \sum_{i=0}^{n}x_i\ln(1-p) \\
&= n\ln(p) - n\ln(1-p) + \sum_{i=0}^{n}x_i\ln(1-p) \\
&= n\ln(p) + \left[\sum_{i=0}^{n}x_i - n\right]\ln(1-p)
\end{aligned}$$

Differentiating and setting to zero to find parameter:

$$\begin{aligned}
\frac{dl}{dp} &= \frac{n}{p} + \left[\sum_{i=0}^{n}x_i - n\right]\frac{1}{1-p} \\
0 &= n(1-p) + \left[\sum_{i=0}^{n}x_i - n\right]p \\
0 &= n - np + \left[\sum_{i=0}^{n}x_i - n\right]p \\
0 &= p\left(\left[\sum_{i=0}^{n}x_i - n\right] - n\right) + n \\
p &= -n\Bigg/\left(\left[\sum_{i=0}^{n}x_i - n\right] - n\right) \\
&= \frac{n}{\sum_{i=0}^{n}x_i} \\
&= \frac{1}{\mu}
\end{aligned}$$

The maximum likelihood estimator for p is the reciprocal of the mean.

This estimator is not unbiased. To be unbiased, $E[\hat{p}] = p$ for all n. Consider n = 1:

$$\begin{aligned} E\left[\frac{1}{X_1}\right] &= \sum_{x=1}^{\infty} \frac{1}{x} P(X_1 = x) \\ &= \sum_{x=1}^{\infty} \frac{1}{x} (1-p)^{x-1} p \\ &= p + \sum_{x=2}^{\infty} \frac{1}{x} (1-p)^{x-1} \quad > p \end{aligned}$$

The condition is not met so the estimator is biased.

Question 5

a) The variances were found using python, with the following code:

```
import random
import numpy as np

sim = []
simanti = []
for i in range(1000):

    randlist = [random.random() for x in range(10000)]
    fx = [100 if (x > 0.9 and x <= 0.91) else 0 for x in randlist]

    sim.append(sum(fx)/float(len(fx)))

    randlistanti = [random.random() for x in range(5000)]
    randlistanti = randlistanti + [1 - x for x in randlistanti]
    fxanti = [100 if (x > 0.9 and x <= 0.91) else 0 for x in
        ↪ randlistanti]

    simanti.append(sum(fxanti)/float(len(fxanti)))

print(np.mean(sim), np.mean(simanti))
print(np.var(sim), np.var(simanti))
```

Variance under iid and under antithetic sampling were very similar (0.01009 vs 0.00944 respectively). Therefore antithetic sampling is neutral for the example.

b) Antithetic sampling is useful when the problem is mostly odd, as it removes the odd part of the variance and doubles the even part of the variance. Because antithetic sampling was neutral for this example, it means the problem was equally even and odd (or close enough that the simulation could not find a difference in 1000 runs).

Question 6

a) It is clear that to maintain feasibility and minimise cost, x must equal 0.5 for the first stage. Then:

$$y = \begin{cases} \omega - x & \text{if } \omega \geq x \text{ with } p = \frac{2}{3} \\ 0 & \text{otherwise with } p = \frac{1}{3} \end{cases}$$

$$\begin{aligned} E[z] &= 5 \times 0.5 + \left(E[\omega|\omega \geq x] - 0.5\right)\left(\frac{2}{3}\right) + 0 \times \left(\frac{1}{3}\right) \\ &= 2.5 + \left(1 - 0.5\right)\left(\frac{2}{3}\right) \\ &= 2.83 \end{aligned}$$

The expected cost is 2.83

b)

$$\begin{aligned} E[\omega] &= 0.75 \\ x &= 0 \\ y &= \omega \end{aligned}$$

$$\begin{aligned} E[z] &= 0 + E[\omega|\omega \leq 1]\left(\frac{2}{3}\right) + \infty \times \left(\frac{1}{3}\right) \\ &= \infty \end{aligned}$$

The infinity arises due to potential infeasibility as in stage one, we set x = 0. Then in stage 2, there is a 1/3 chance ω takes upon a value greater than 1, and the constraints cannot be met. This gives a cost of infinity.

c) As the cost of expected value problem is infinity, the VSS is also infinity ($\infty - 2.83 = \infty$).