AXIS, ANGLE & Rotation Matrix

BYIII

1 Composition of Rotation Matrix

Problem: Given rotation axis, a rotation angle, generate the corresponding rotation atrix.

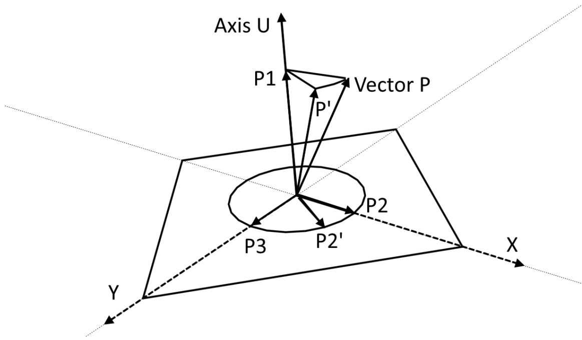

Figure 1: Rotating a vector P about the axis U

Solution

As show in Figure 1, the axis $\mathbf{U}$ goes through the origin, a vector $\mathbf{P}$ rotates about $\mathbf{U}$ to $\mathbf{P'}$. Let θ denotes the rotation angle, and let ϕ denotes the angle between $\mathbf{U}$ and $\mathbf{P}$. From $\mathbf{U}$ we can define a plane S(whose normal is $\mathbf{U}$). Now we have:

$$\begin{aligned}\mathbf{U} &= [u_x, u_y, u_z]^T,\\ u_x^2 + u_y^2 + u_z^2 &= 1,\\ \mathbf{P} &= [x, y, z]^T,\\ \mathbf{P'} &= [x', y', z']^T.\end{aligned}$$

Project $\mathbf{P}$ to $\mathbf{U}$ and S, we can get $\mathbf{P}_1$ and $\mathbf{P}_2$:

$$\begin{aligned}
\mathbf{P} &= \mathbf{P}_1 + \mathbf{P}_2, \\
\mathbf{P}_1 &= (\mathbf{P} \cdot \mathbf{U})\mathbf{U}, \\
|\mathbf{P}_1| &= |\mathbf{P}| \cos \phi, \\
\mathbf{P}_2 &= \mathbf{P} - \mathbf{P}_1, \\
|\mathbf{P}_2| &= |\mathbf{P}| \sin \phi.
\end{aligned}$$

Notice that rotating $\mathbf{P}$ is equivalent to firstly rotating $\mathbf{P}_2$ a θ angle to $\mathbf{P}_2'$ and then combining $\mathbf{P}_1$ and $\mathbf{P}_2'$ to form $\mathbf{P}'$. So, next step is to determine $\mathbf{P}_2'$.

To represent $\mathbf{P}_2'$, we first to establish some frame for plane S. It is convenient to choose vector $\mathbf{P}_2$ be the *unit vector*(though, maybe $|\mathbf{P}_2| \neq 1$) of the X axis, and some other vector $\mathbf{P}_3$ with the same length of $\mathbf{P}_2$ as the Y axis of plane S, which satisfies

$$\begin{aligned}
\mathbf{P}_2 \cdot \mathbf{P}_3 &= 0, \\
|\mathbf{P}_2| = |\mathbf{P}_2'| &= |\mathbf{P}_3|.
\end{aligned}$$

Now, according to the rotation matrix in 2D plane, we can write:

$$\mathbf{P}_2' = \mathbf{P}_2 \cos \theta + \mathbf{P}_3 \sin \theta.$$

Then, we determine $\mathbf{P}_3$. It is true that $\mathbf{P}_3$ is parallel to $\mathbf{P} \times \mathbf{U}$, and

$$|\mathbf{P} \times \mathbf{U}| = |\mathbf{P}| \cdot 1 \cdot \sin \phi = |\mathbf{P}_2| = |\mathbf{P}_3|.$$

So we can get:

$$\mathbf{P}_3 = \mathbf{P} \times \mathbf{U}.$$

Combining above:

$$\begin{aligned}
\mathbf{P}' &= \mathbf{P}_1 + \mathbf{P}_2' \\
&= (\mathbf{P} \cdot \mathbf{U})\mathbf{U} + [\mathbf{P} - (\mathbf{P} \cdot \mathbf{U})\mathbf{U}] \cos \theta + \mathbf{P} \times \mathbf{U} \sin \theta \\
&= \mathbf{P} \cos \theta + (\mathbf{P} \cdot \mathbf{U})\mathbf{U}(1 - \cos \theta) + \mathbf{P} \times \mathbf{U} \sin \theta
\end{aligned}$$

2 Decomposition