AXIS, ANGLE & Rotation Matrix

BYIII

1 Composition of Rotation Matrix

Problem: Given rotation axis, a rotation angle, generate the corresponding rotation atrix.

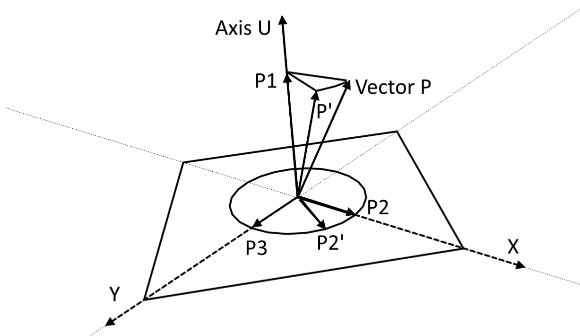

Figure 1: Rotating a vector P about the axis U

Solution

As show in Figure 1, the axis $\mathbf{U}$ goes through the origin, a vector $\mathbf{P}$ rotates about $\mathbf{U}$ to $\mathbf{P}'$. Let θ denote the rotation angle, and let ϕ denote the angle between $\mathbf{U}$ and $\mathbf{P}$. From $\mathbf{U}$ we can define a plane S(whose normal is $\mathbf{U}$). Now we have:

$$\begin{aligned}
\mathbf{U} &= [u_x, u_y, u_z]^T, \\
u_x^2 &+ u_y^2 + u_z^2 = 1, \\
\mathbf{P} &= [x, y, z]^T, \\
\mathbf{P}' &= [x', y', z']^T.
\end{aligned}$$

Project $\mathbf{P}$ to $\mathbf{U}$ and S, we can get $\mathbf{P}_1$ and $\mathbf{P}_2$:

$$\begin{aligned}
\mathbf{P} &= \mathbf{P}_1 + \mathbf{P}_2, \\
|\mathbf{P}_1| &= \mathbf{P} \cdot \mathbf{U} = |\mathbf{P}| \cos\phi, \\
\mathbf{P}_1 &= (\mathbf{P} \cdot \mathbf{U})\mathbf{U}, \\
\mathbf{P}_2 &= \mathbf{P} - \mathbf{P}_1, \\
|\mathbf{P}_2| &= |\mathbf{P}| \sin\phi.
\end{aligned}$$

Notice that rotating $\mathbf{P}$ is equivalent to firstly rotate $\mathbf{P}_2$ a θ angle to $\mathbf{P}_2'$ and then combining $\mathbf{P}_1$ and $\mathbf{P}_2'$ to form $\mathbf{P}'$. So, next step is to determine $\mathbf{P}_2'$.

To represent $\mathbf{P}_2'$, we first to establish some frame for plane S. It is convenient to choose vector $\mathbf{P}_2$ be the *unit vector*(though, maybe $|\mathbf{P}_2| \neq 1$) of the X axis, and some other vector

$\mathbf{P}_3$ with the same length of $\mathbf{P}_2$ as the Y axis of plane S, which satisfies

$$\mathbf{P}_2 \cdot \mathbf{P}_3 = 0,$$
$$|\mathbf{P}_2| = |\mathbf{P}'_2| = |\mathbf{P}_3|.$$

Now, according to the rotation matrix in 2D plane, we can write:

$$\mathbf{P}'_2 = \mathbf{P}_2 \cos\theta + \mathbf{P}_3 \sin\theta.$$

Then, we determine $\mathbf{P}_3$. It is true that $\mathbf{P}_3$ is parallel to $\mathbf{P} \times \mathbf{U}$, and

$$|\mathbf{P} \times \mathbf{U}| = |\mathbf{P}| \cdot 1 \cdot \sin\phi = |\mathbf{P}_2| = |\mathbf{P}_3|.$$

So we can get:

$$\mathbf{P}_3 = \mathbf{P} \times \mathbf{U}.$$

Combining above:

$$\begin{aligned}
\mathbf{P}' &= \mathbf{P}_1 + \mathbf{P}'_2 \\
&= (\mathbf{P} \cdot \mathbf{U})\mathbf{U} + [\mathbf{P} - (\mathbf{P} \cdot \mathbf{U})\mathbf{U}]\cos\theta + \mathbf{P} \times \mathbf{U} \sin\theta. \\
&= \mathbf{P}\cos\theta + (\mathbf{P} \cdot \mathbf{U})\mathbf{U}(1 - \cos\theta) + \mathbf{P} \times \mathbf{U} \sin\theta
\end{aligned}$$

Therefore, the rotatio matrix is

$$\mathbf{R} = \begin{bmatrix} u_x^2(1-\cos\theta) + \cos\theta & u_x u_y(1-\cos\theta) - u_z \sin\theta & u_x u_z(1-\cos\theta) + u_y \sin\theta \\ u_x u_y(1-\cos\theta) + u_z \sin\theta & u_y^2(1-\cos\theta) + \cos\theta & u_y u_z(1-\cos\theta) - u_x \sin\theta \\ u_x u_z(1-\cos\theta) - u_y \sin\theta & u_y u_z(1-\cos\theta) + u_x \sin\theta & u_z^2(1-\cos\theta) + \cos\theta \end{bmatrix}.$$

Reference: Rodrigues' rotation formula

2 Decomposition

Problem: Given the rotation matrix, to extract rotation axis and the rotation angle.
Solution

Denotes the rotation matrix as $\mathbf{R}$:

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

If from the matrix $\mathbf{R}$ we find rotation axis $\mathbf{U}$ and rotation angle θ, then $\mathbf{R}$ can be rewrotten as:

$$\mathbf{R} = \begin{bmatrix} u_x^2(1-\cos\theta) + \cos\theta & u_x u_y(1-\cos\theta) - u_z \sin\theta & u_x u_z(1-\cos\theta) + u_y \sin\theta \\ u_x u_y(1-\cos\theta) + u_z \sin\theta & u_y^2(1-\cos\theta) + \cos\theta & u_y u_z(1-\cos\theta) - u_x \sin\theta \\ u_x u_z(1-\cos\theta) - u_y \sin\theta & u_y u_z(1-\cos\theta) + u_x \sin\theta & u_z^2(1-\cos\theta) + \cos\theta \end{bmatrix}.$$

Thinking matrix $\mathbf{R}$ as a linear transformation, it just changes a vector's orientation and has nothing to do with the vector's length. So, 1 must be an eigen value of $\mathbf{R}$:

$$\mathbf{R}\mathbf{v} = 1 \cdot \mathbf{v}.$$

And the eigen vector corresponding to eigen value 1 must be the rotation axis, because $\mathbf{R}$ does not change its orientation. Therefore, let's consider the eigen value decomposition of matrix $\mathbf{R}$.

Consider a very simple case, **A** being the rotation matrix of rotating about the X axis by a θ angle:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

The eigen values of **A** are: 1, $e^{i\theta}$, $e^{-i\theta}$. Then the trace of matrix **A** is:

$$trA = 1 + 2\cos\theta.$$

So the rotation angle can be calculated from the trace of the rotation matrix:

$$\cos\theta = \frac{1}{2}(tr\mathbf{A} - 1).$$

For the complex formation of rotation matrix **R**, it is the same that

$$\begin{aligned} tr\mathbf{R} &= (u_x^2 + u_y^2 + u_z^2)(1 - \cos\theta) + 3\cos\theta \\ &= 1 + 2\cos\theta. \end{aligned}$$

Since the geometry multiplicity of eigen value 1 of rotation matrix is 1, the rotation axis can be calculated from the null space of matrix $(\mathbf{R} - \mathbf{I})$, or the eigen vector of 1.