



המחלקה להנדסת חשמל ואלקטרוניקה

תאריך הבחינה : 16.02.18

שעות הבחינה : 8:30-11:30

מבוא לאותות אקראיים

מועד ב'

ד"ר דימה בחובסקי, אביתר רימון

תשע"ח סמסטר א'

חומר עזר - עד 2 דפי נוסחאות אישיים (משני צדדים), מחשבון

הוראות מיוחדות :

- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.
- אם לא מצויין אחרת, הסעיפים הם בעלי ניקוד זהה.

השאלון כולל 11 דפים (כולל דף זה)

בהצלחה !



1 PSD (33 נק') – מתוך Kay, 17.34, 17.37, 19.2

(א) נתון תהליך אקראי בדיד, $x[n]$, בעל צפיפות הפסק ספקטראלית (PSD) מהצורה $P_x(f) = 1 + \cos(2\pi f)$. האם הדגימות של התהליך הן חסרי קורלציה?

(ב) נתון תהליך אקראי בדיד, $x[n]$, מהצורה $x[n] = w[n] - \mu$, כאשר $w[n]$ הינו רעש לבן עם תוחלת 0 ושונות σ_w^2 ו- μ זה קבוע ממשי. מהי פוקציה אוטו-קורלציה, $R_w[n, n+k]$ האם מדובר בתהליך סטציאונרי?

(ג) נתונים תהליכים

$$(1) \quad y(t) = a_1 x_1(t) + a_2 x_2(t)$$

$$(2) \quad z(t) = b_1 x_1(t) + b_2 x_2(t),$$

כאשר $x_1(t), x_2(t)$ הם WSS במשותף. האם $y(t), z(t)$ הם גם WSS במשותף? הוכח/נמק!

2 חיזוי (22 נק') – מתוך Kay, 18.25

נתון תהליך אקראי $x[n] = \cos(2\pi f_0 n + \theta)$ כאשר $\theta \sim U[0, 2\pi]$. ידוע, שפונקציית אוטו-קורלציה של התהליך נתונה ע"י

$$(3) \quad R_x[k] = \cos(2\pi f_0 k)$$

נדרש לעשות חיזוי לינארי עבור התהליך מהצורה $\hat{x}[n+1] = ax[n] + bx[n-1]$. מצא ערכים אופטימליים של a, b במובן שגיאה ריבועית מינימלית (MMSE), והוכח ששגיאת חיזוי היא 0.

3 Poisson (22 נק') – מתוך Shaum, 5.56

קצב הגעת הלקוחות לפקיד מסויים הינו $\lambda = 1/10$ לדקה, החל מפתחת משרדו. הפקיד מתחיל לקבל לקוחות רק כאשר כבר יש 3 לקוחות בחדר המתנה.

(א) מהו זמן המתנה ממוצע להתחלת קבלת הלקוחות?

(ב) מהו הסיכוי, שאחרי 30 דקות מפתחת המשרד לא התקבל אף לקוח?



4 שאלה Markov (33 נק')

במקום מסויים, סיכוי לגשם אחרי יום שמש הוא $1/20$, וסיכוי לגשם אחרי יום גשום הוא $3/5$.

(א) רשום את מטריצת המעברים, P .

(ב) אם היום הוא יום שמש, מהו הסיכוי לגשם עוד יומיים?

(ג) מהו סיכוי ממוצע ליום גשם/שמש?

Random Processes – Formulas

1 Random Variables

1.1 Distributions

	Notation	PDF/PMF	CDF	$E[X]$	$\text{Var}[X]$
Bernoulli	$\text{Ber}(p)$	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$	$\begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$	p	$p(1-p)$
Uniform	$U[a,b]$	$\frac{1}{b-a}, a \leq x \leq b$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	$N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$\Phi(x)$	μ	σ^2
Exponential	$\text{Exp}(\lambda)$	$\lambda \exp(-\lambda x), x \geq 0$	$1 - \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$
Poisson	$\mathcal{P}(\lambda)$	$p(X=k) = \exp(-\lambda) \frac{\lambda^k}{k!}$	$\exp(-\lambda) \sum_{i=0}^k \frac{\lambda^i}{i!}$	λ	λ
Erlang	$\text{Erlang}(k, \lambda t)$	$\lambda \frac{(\lambda t)^{k-1}}{(k-1)!} \exp(-\lambda t)$	$1 - \exp(-\lambda t) \sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!}$	$\frac{k}{\lambda t}$	$\frac{k}{(\lambda t)^2}$

1.2 Properties

Definitions:

$$F_X(x) = p(X \leq x) \quad (1a)$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \quad (1b)$$

$$F_X(x) = \int_{-\infty}^x f_X(p) dp \quad (1c)$$

$$p(a < X \leq b) = F_X(b) - F_X(a) \quad (1d)$$

$$p_X[x_k] = p(X = x_k) \quad (2a)$$

$$F_X(x) = \sum_{k: x_k \leq x} p_X[x_k] \quad (2b)$$

Expectation:

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx \\ \sum_i x_i p_X[x_i] \end{cases} \quad (3a)$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_i g(x_i) p_X[x_i] \end{cases} \quad (3b)$$

$$E[aX + b] = aE[X] + b \quad (3c)$$

Variance:

$$\begin{aligned} \text{Var}[X] &= E[(X - E[X])^2] \\ &= E[X^2] - E^2[X] \end{aligned} \quad (4a)$$

$$\text{Var}[aX + b] = a^2 \text{Var}[X] \quad (4b)$$

2 Two Random Variables

2.1 Joint Distributions

Definitions:

$$F_{XY}(x, y) = p(X \leq x, Y \leq y) \quad (5a)$$

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} \geq 0 \quad (5b)$$

$$F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(s, p) dp ds \quad (5c)$$

$$p[x_j, y_k] = p(X = x_j, Y = y_k) \quad (6a)$$

$$F_{XY}(x, y) = p(X \leq x_j, Y \leq y_k) \quad (6b)$$

Conditional distribution (Bayes)
 $(f_X(x), f_Y(y), p_X[x_k], p_Y[y_k] > 0)$:

$$f_{Y|X}(y|x)f_X(x) = f_{X|Y}(x|y)f_Y(y) = f_{XY}(x, y) \quad (7a)$$

$$p_{Y|X}[y_j|x_k]p_X[x_k] = p_{X|Y}[x_k|y_j]p_Y[y_j] = p_{XY}[x_k, y_j] \quad (7b)$$

$$F_{Y|X}(y|x) = p(Y \leq y|X = x) = \int_{-\infty}^y f_{Y|X}(s|x)ds \quad (7c)$$

$$F_{Y|X}[y|x_k] = \frac{p[Y \leq y_j, X = x_k]}{p_X[x_k]} \quad (7d)$$

Expectation:

$$E[XY] = \iint xy f_{XY}(x, y) dxdy \quad (8a)$$

$$E[g(X)] = \left\{ \iint g(x, y) f_{XY}(x, y) dxdy \right. \\ \left. \sum_i \sum_k g(x_i, y_k) p_x[x_i, y_k] \right\} \quad (8b)$$

$$E[aX + bY] = aE[X] + bE[Y] \quad (8c)$$

Conditional expectation & Variance:

$$E[Y|X] = \left\{ \int y f_{Y|X}(y|x) dy \right. \\ \left. \sum_j y_j p[y_j|x_k] \right\} \quad (9a)$$

$$E[X] = E[E[X|Y]] = \iint y f_{Y|X}(y|x) f_X(x) dxdy \quad (9b)$$

$$\text{Var}[Y|X] = E[Y^2|X] - E^2[Y|X] \quad (9c)$$

$$\text{Var}[Y] = \text{Var}[E[Y|X]] + E[\text{Var}[Y|X]] \quad (9d)$$

Independent random variables:

$$f_{XY}(x, y) = f_X(x)f_Y(y) \quad (10a)$$

$$p_{XY}[x_k, y_j] = p_X[x_k]p_Y[y_j] \quad (10b)$$

$$F_{XY}(x, y) = F_X(x)F_Y(y) \quad (10c)$$

$$E[XY] = E[X]E[Y] \quad (10d)$$

$$E[g_1(X)g_2(Y)] = E[g_1(X)]E[g_2(Y)] \quad (10e)$$

$$\text{Var}[aX + bY] = a^2 \text{Var}[X] + b^2 \text{Var}[Y] \quad (10f)$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy \quad (11a)$$

$$p_X[x_k] = \sum_j p_{XY}[x_k, y_j] \quad (11b)$$

$$F_X(x) = F_{XY}(x, \infty) \quad (11c)$$

$$F_Y(y) = F_{XY}(\infty, y) \quad (11d)$$

3 Random Processes – General Properties

- PDF & CDF

$$F_{\mathbf{x}}(x; t) = p(\mathbf{x}(t) \leq x) \quad (15a)$$

$$f_{\mathbf{x}}(x; t) = \frac{\partial}{\partial x} F_{\mathbf{x}}(x; t) \quad (15b)$$

2.2 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y , covariance is given by

$$\begin{aligned} \text{Cov}[X, Y] &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y]. \end{aligned} \quad (12)$$

Main covariance properties are:

$$\text{Cov}[X, X] = \text{Var}[X] \quad (13a)$$

$$\text{Cov}[X, Y] = \text{Cov}[Y, X] \quad (13b)$$

$$\text{Cov}[X, a] = 0 \quad (13c)$$

$$\text{Cov}[aX, bY] = ab \text{Cov}[X, Y] \quad (13d)$$

$$\text{Cov}[X, Y] = \text{Cov}[X + a, Y + b] \quad (13e)$$

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2 \text{Cov}[X, Y] \quad (13f)$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X] \text{Var}[Y]}} \quad (14)$$

such that $|\rho_{XY}| \leq 1$.

2.3 Relations

- When X and Y are *orthogonal*, $E[XY] = 0$.
- When X and Y are *uncorrelated*, $\text{Cov}[X, Y] = \rho_{XY} = 0$.
- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 10).
- When X and Y are *jointly* Gaussian and uncorrelated $\Rightarrow X$ and Y are independent.
- Joint \Rightarrow marginal, marginal \nRightarrow joint

- Variance:

$$\text{Var}[\mathbf{x}(t)] = E[\mathbf{x}^2(t)] - E^2[\mathbf{x}(t)] \quad (17a)$$

$$\text{Var}[\mathbf{x}[n]] = E[\mathbf{x}^2[n]] - E^2[\mathbf{x}[n]] \quad (17b)$$

- Auto-correlation

$$R_{\mathbf{x}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{x}(t_2)] \quad (18a)$$

$$R_{\mathbf{x}}(t, t + \tau) = E[\mathbf{x}(t)\mathbf{x}(t + \tau)] \quad (18b)$$

$$R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(t_2, t_1) \quad (18c)$$

$$R_{\mathbf{x}}(t, t) = E[\mathbf{x}^2(t)] \quad (18d)$$

$$R_{\mathbf{x}}[n_1, n_2] = E[\mathbf{x}[n_1]\mathbf{x}[n_2]] \quad (18e)$$

- Auto-covariance

$$\begin{aligned} C_{\mathbf{x}}(t_1, t_2) &= R_{\mathbf{x}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)] \\ &= E[\{\mathbf{x}(t_1) - E[\mathbf{x}(t_1)]\}\{\mathbf{x}(t_2) - E[\mathbf{x}(t_2)]\}] \end{aligned} \quad (19)$$

$$C_{\mathbf{x}}(t, t) = \text{Var}[\mathbf{x}(t)] \quad (20)$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(t_1, t_2) = \frac{C_{\mathbf{x}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{x}}(t_2, t_2)}} \quad (21a)$$

$$|\rho_{\mathbf{x}}(t_1, t_2)| \leq 1 \quad (21b)$$

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *orthogonal*, $R_{\mathbf{x}}(t_1, t_2) = 0$.
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *uncorrelated*, $C_{\mathbf{x}}(t_1, t_2) = \rho_{\mathbf{x}}(t_1, t_2) = 0$.
- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *independent*, $R_{\mathbf{x}}(t_1, t_2) = E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]$.

4 Wide-Sense Stationary (WSS) Process

Definition:

$$E[\mathbf{x}(t)] = E[\mathbf{x}(0)] = \mu_{\mathbf{x}} = \text{const} \quad (22a)$$

$$R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(\tau = |t_2 - t_1|), \quad \forall t_1, t_2 \quad (22b)$$

$$E[\mathbf{x}[n]] = E[\mathbf{x}[0]] = \mu_{\mathbf{x}} = \text{const} \quad (22c)$$

$$R_{\mathbf{x}}[n_1, n_2] = R_{\mathbf{x}}(k = |n_2 - n_1|), \quad \forall n_1, n_2 \quad (22d)$$

- Auto-correlation

$$R_{\mathbf{x}}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t + \tau)] \quad (23a)$$

$$R_{\mathbf{x}}[k] = E[\mathbf{x}[n]\mathbf{x}[n + k]] \quad (23b)$$

Properties:

$$R_{\mathbf{x}}(-\tau) = R_{\mathbf{x}}(\tau) \quad (24a)$$

$$R_{\mathbf{x}}(0) = E[|\mathbf{x}(0)|^2] = E[|\mathbf{x}(t)|^2] \quad (24b)$$

$$\text{Var}[\mathbf{x}(t)] = C_{\mathbf{x}}(0) = \sigma_{\mathbf{x}}^2 \quad (24c)$$

$$|R_{\mathbf{x}}(0)| \geq R_{\mathbf{x}}(\tau) \quad (24d)$$

Deterministic definition ($x[n]$ is not random)

$$R_x(\tau) = x(\tau) * x(-\tau) \quad (25a)$$

$$R_x[k] = x[k] * x[-k] \quad (25b)$$

- Auto-covariance

$$C_{\mathbf{x}}(\tau) = R_{\mathbf{x}}(\tau) - \mu_{\mathbf{x}}^2 \quad (26a)$$

$$C_{\mathbf{x}}[k] = R_{\mathbf{x}}[k] - \mu_{\mathbf{x}}^2 \quad (26b)$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(\tau) = \frac{C_{\mathbf{x}}(\tau)}{C_{\mathbf{x}}(0)} \quad (27a)$$

$$\rho_{\mathbf{x}}[k] = \frac{C_{\mathbf{x}}[k]}{C_{\mathbf{x}}[0]} \quad (27b)$$

- Power spectral density (PSD)

$$\begin{aligned} S_{\mathbf{x}}(f) &= \mathcal{F}\{R_{\mathbf{x}}(\tau)\} = \\ &= \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau) \exp(j2\pi f\tau) d\tau \end{aligned} \quad (28a)$$

$$= 2 \int_0^{\infty} R_{\mathbf{x}}(\tau) \cos(2\pi f\tau) d\tau \quad (28b)$$

$$P_{\mathbf{x}}(f) = \mathcal{F}\{R_{\mathbf{x}}[k]\} = \quad -1/2 \leq f \leq 1/2$$

$$= \sum_{k=-\infty}^{\infty} R_{\mathbf{x}}[k] \exp(-j2\pi fk) \quad (28c)$$

$$= 2 \sum_{k=0}^{\infty} R_{\mathbf{x}}[k] \cos(-2\pi fk) \quad (28d)$$

Properties:

$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(-f) \quad (29a)$$

$$S_{\mathbf{x}}(f) \geq 0, \quad \forall f \quad (29b)$$

$$S_{\mathbf{x}}(f) \in \mathbb{R} \quad (\text{real numbers}) \quad (29c)$$

$$P_{\mathbf{x}}(f) = P_{\mathbf{x}}(-f) \quad (29d)$$

$$P_{\mathbf{x}}(f) \geq 0, \quad \forall f \quad (29e)$$

$$P_{\mathbf{x}}(f) \in \mathbb{R} \quad (\text{real numbers}) \quad (29f)$$

$$P_{\mathbf{x}}(f) = P_{\mathbf{x}}(f + 1) \quad (29g)$$

Average power

$$P_{\mathbf{x}} = E[\mathbf{x}^2(t)] = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f) df \quad (30a)$$

$$= E[\mathbf{x}^2[n]] = R_{\mathbf{x}}[0] = \int_{-1/2}^{1/2} P_{\mathbf{x}}(f) df \quad (30b)$$

Deterministic definition:

$$S_x(f) = X(f)X^*(f) = |X(f)|^2 \quad (31)$$

4.1 White Noise & White Gaussian Noise (WGN) Process

White noise process is SSS (WSS) process that is characterized by

$$R_{\mathbf{n}}(\tau) = \sigma^2 \delta(\tau) \quad (32a)$$

$$R_{\mathbf{n}}[k] = \sigma^2 \delta[k] \quad (32b)$$

$$S_{\mathbf{n}}(f) = \sigma^2 \quad \forall f \quad (32c)$$

For WGN process, $\mathbf{n}(t) \sim N(0, \sigma^2)$,

$$R_{\mathbf{n}}(\tau) = \frac{N_0}{2} \delta(\tau) \quad (33a)$$

$$S_{\mathbf{n}}(f) = \frac{N_0}{2} \quad \forall f \quad (33b)$$

5 Cross-Signal

- Cross-correlation

$$R_{\mathbf{xy}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{y}(t_2)] \quad (34)$$

- Cross-covariance

$$C_{\mathbf{xy}}(t_1, t_2) = R_{\mathbf{xy}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{y}(t_2)] \quad (35)$$

- Correlation Coefficient

$$\rho_{\mathbf{xy}}(t_1, t_2) = \frac{C_{\mathbf{xy}}(t_1, t_2)}{\sqrt{C_{\mathbf{xy}}(t_1, t_1)C_{\mathbf{xy}}(t_2, t_2)}} \quad (36)$$

5.1 WSS Cross-signal

- $\mathbf{x}(t), \mathbf{y}(t)$ are jointly WSS, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ each of them is WSS and

$$R_{\mathbf{xy}}(\tau) = E[\mathbf{x}(t)\mathbf{y}(t + \tau)] \quad (37)$$

6 LTI and WSS Random Process

Output of LTI system with impulse response $h(t)$ and random process $x(t)$,

$$y(t) = x(t) * h(t) \quad (45)$$

Average

$$m_{\mathbf{y}} = m_{\mathbf{x}} \int_{-\infty}^{\infty} h(s) ds = m_{\mathbf{x}} H(f=0) \quad (46)$$

Cross-correlation & cross-covariance:

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) \quad (47a)$$

$$C_{\mathbf{xy}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) \quad (47b)$$

$$R_{\mathbf{yx}}(\tau) = R_{\mathbf{x}}(\tau) * h(-\tau) \quad (47c)$$

$$C_{\mathbf{yx}}(\tau) = C_{\mathbf{x}}(\tau) * h(-\tau) \quad (47d)$$

$$R_{\mathbf{y}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \quad (47e)$$

$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \quad (47f)$$

- When $\mathbf{x}(t)$ and $\mathbf{y}(t + \tau)$ are *uncorrelated jointly WSS*, $C_{\mathbf{xy}}(\tau) = 0$.

Properties

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{yx}}(-\tau) \quad (38a)$$

$$|R_{\mathbf{xy}}(\tau)| \leq \sqrt{R_{\mathbf{x}}(0)R_{\mathbf{y}}(0)} \quad (38b)$$

$$|R_{\mathbf{xy}}(\tau)| \leq \frac{1}{2} [R_{\mathbf{x}}(0) + R_{\mathbf{y}}(0)] \quad (38c)$$

Deterministic definition

$$R_{xy}(\tau) = x(\tau) * y(-\tau) \quad (39)$$

- Cross-covariance

$$C_{\mathbf{xy}}(\tau) = R_{\mathbf{xy}}(\tau) - \mu_{\mathbf{x}}\mu_{\mathbf{y}} \quad (40)$$

- Cross-PSD

$$S_{\mathbf{xy}}(f) = \mathcal{F}\{R_{\mathbf{xy}}(\tau)\} \quad (41)$$

Properties

$$S_{\mathbf{xy}}(f) = S_{\mathbf{yx}}(-f) = S_{\mathbf{xy}}^*(-f) \quad (42)$$

Deterministic definition

$$S_{xy}(f) = X(f)Y^*(f) \quad (43)$$

- Coherence

$$\gamma_{\mathbf{xy}}(f) = \frac{S_{\mathbf{xy}}(f)}{\sqrt{S_{\mathbf{x}}(f)S_{\mathbf{y}}(f)}} \quad (44)$$

Power-Spectral Density (PSD) & Cross-PSD: Given frequency response $H(f) = \mathcal{F}\{h(\tau)\}$, $H^*(f) = \mathcal{F}\{h(-\tau)\}$

$$S_{\mathbf{xy}}(f) = S_{\mathbf{x}}(f)H(f) \quad (48a)$$

$$S_{\mathbf{yx}}(f) = S_{\mathbf{x}}(f)H^*(f) \quad (48b)$$

$$S_{\mathbf{y}}(f) = S_{\mathbf{x}}(f)H(f)H^*(f) = S_{\mathbf{x}}(f)|H(f)|^2 \quad (48c)$$

Power of the process:

$$P_x = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f) df \quad (49a)$$

$$P_y = R_{\mathbf{y}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f)|H(f)|^2 df \quad (49b)$$

6.1 SNR

Given input signal

$$\mathbf{x}(t) = \mathbf{s}(t) + \mathbf{n}(t), \quad (50)$$

where $\mathbf{s}(t), \mathbf{n}(t)$ are independent and $E[\mathbf{n}(t)] = 0$, the PSD of output $\mathbf{y}(t)$ is given by

$$\begin{aligned} S_{\mathbf{y}}(f) &= S_{\mathbf{x}}(f) |H(f)|^2 \\ &= S_{\mathbf{s}}(f) |H(f)|^2 + S_{\mathbf{n}}(f) |H(f)|^2, \end{aligned} \quad (51)$$

where $S_{\mathbf{s}}(f) |H(f)|^2$ is signal output PSD and $S_{\mathbf{n}}(f) |H(f)|^2$ is noise PSD.

The input and output SNRs is given by

$$\text{SNR}_x = \frac{E[\mathbf{s}^2(t)]}{E[\mathbf{n}^2(t)]} = \frac{R_{\mathbf{s}}(0)}{R_{\mathbf{n}}(0)} = \frac{\int_{-\infty}^{\infty} S_{\mathbf{s}}(f) df}{\int_{-\infty}^{\infty} S_{\mathbf{n}}(f) df} \quad (52a)$$

$$\text{SNR}_y = \frac{\int_{-\infty}^{\infty} S_{\mathbf{s}}(f) |H(f)|^2 df}{\int_{-\infty}^{\infty} S_{\mathbf{n}}(f) |H(f)|^2 df}. \quad (52b)$$

7 Multi-dimensional processes

7.1 Covariance matrix

Given random vector $\mathbf{X} = (X_1, X_2, \dots, X_N)^T$,

$$\begin{aligned} C_{\mathbf{X}} &= \text{Cov}[\mathbf{X}, \mathbf{X}] = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T] \\ &= E[\mathbf{X}\mathbf{X}^T] - E[\mathbf{X}]E[\mathbf{X}]^T \\ &= \begin{bmatrix} \text{Var}[X_1] & \text{Cov}[X_1, X_2] & \cdots & \text{Cov}[X_1, X_N] \\ \text{Cov}[X_2, X_1] & \text{Var}[X_2] & \cdots & \text{Cov}[X_2, X_N] \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}[X_N, X_1] & \text{Cov}[X_N, X_2] & \cdots & \text{Var}[X_N] \end{bmatrix} \end{aligned} \quad (53)$$

Properties:

- Symmetry

$$C_{\mathbf{X}} = C_{\mathbf{X}}^T \quad \text{Cov}[X_i, X_j] = \text{Cov}[X_j, X_i] \quad (54)$$

- Variance of linear combination: Given vector $\mathbf{a} = (a_1, a_2, \dots, a_N)^T$,

$$\text{Var}[\mathbf{a}^T \mathbf{X}] = \mathbf{a}^T C_{\mathbf{X}} \mathbf{a} \quad (55)$$

- Linear transformation: Given linear transformation $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$,

$$E[\mathbf{Y}] = \mathbf{A}E[\mathbf{X}] + \mathbf{b} \quad (56a)$$

$$C_{\mathbf{Y}} = \mathbf{A}C_{\mathbf{X}}\mathbf{A}^T \quad (56b)$$

- Uncorrelated variables

$$C_{\mathbf{X}} = \text{diag}[\text{Var}[X_1], \text{Var}[X_2], \dots, \text{Var}[X_N]] \quad (57)$$

- Cross-covariance: For two random vectors $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$, the resulting $m \times n$ cross-covariance matrix is given by

$$\begin{aligned} \text{Cov}[\mathbf{X}, \mathbf{Y}] &= C_{\mathbf{XY}} \\ &= E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])^T] \\ &= E[\mathbf{XY}^T] - E[\mathbf{X}]E[\mathbf{Y}]^T \end{aligned} \quad (58)$$

$$C_{\mathbf{YX}} = C_{\mathbf{XY}}^T \quad (59)$$

7.2 Relation Between Covariance Matrix & Auto-covariance

Given WSS process $\mathbf{x}(t)$, the corresponding correlation matrix of $\mathbf{X} = [\mathbf{x}(t_1), \dots, \mathbf{x}(t_N)]^T$ is given by

$$R_{\mathbf{X}} = E[\mathbf{XX}^T] \quad (60)$$

$$R_{\mathbf{X}}(i, j) = E[X_i X_j] = R_{\mathbf{x}}(|t_i - t_j|) \quad (61)$$

7.3 MMSE Linear Prediction

Mean square error (MSE) of predictor \hat{Y} is given by

$$mse = E[(Y - \hat{Y})^2] \quad (62)$$

- Given two random variables, X, Y , and predictor

$$\hat{Y} = aX + b, \quad (63)$$

minimum MSE (MMSE) predictor given $X = x$ is

$$\hat{Y} = E[Y] + \frac{\text{Cov}[X, Y]}{\text{Var}[X]} (x - E[X]) \quad (64)$$

- Given N samples of process $\mathbf{x}[n]$, and predictor

$$\hat{\mathbf{x}}[n+1] = \sum_{i=1}^N a_i \mathbf{x}[n-i+1], \quad (65)$$

the values of a_i are given by solution of

$$\begin{bmatrix} R_{\mathbf{x}}[0] & R_{\mathbf{x}}[1] & \cdots & R_{\mathbf{x}}[N-1] \\ R_{\mathbf{x}}[1] & R_{\mathbf{x}}[0] & \cdots & R_{\mathbf{x}}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{x}}[N-1] & R_{\mathbf{x}}[N-2] & \cdots & R_{\mathbf{x}}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} R_{\mathbf{x}}[1] \\ R_{\mathbf{x}}[2] \\ \vdots \\ R_{\mathbf{x}}[N] \end{bmatrix} \quad (66)$$

and the resulting MMSE is

$$mmse = R_{\mathbf{x}}[0] - \sum_{i=1}^N a_i R_{\mathbf{x}}[i] \quad (67)$$

8 Gaussian Variables & Processes

8.1 Bi-variate & Multivariate Normal Distribution

Joint Gaussian distribution of X_1 and X_2

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} \right]\right) \quad (68)$$

Multivariate Gaussian distribution of $\mathbf{X} = (X_1, X_2, \dots, X_N)^T$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2} \det[\mathbf{C}_{\mathbf{X}}]} \exp\left\{-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \mathbf{C}_{\mathbf{X}}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right\}, \quad (69)$$

Properties:

- Linear combination of Gaussian variables is Gaussian variable,
- Linear transformation – follows Eqs. (56a).
- If jointly distributed Gaussian random variables are *uncorrelated*, they are also *independent*

8.2 Gaussian Process

A Gaussian process $\mathbf{x}(t)$ a random process that for $\forall k > 0$ and for all times t_1, \dots, t_k , the set of random variable $\mathbf{x}(t_1), \dots, \mathbf{x}(t_k)$ is jointly Gaussian (i.e. described by Eq. (69)).

Properties:

- WSS Gaussian process is SSS.
- Gaussian process $\mathbf{x}(t)$ that passes through LTI system, $\mathbf{y}(t) = h(t) * \mathbf{x}(t)$, is also Gaussian process but with corresponding change in expectation and auto-covariance function.

9 Poisson Process

- The Poisson process, $N(t)$, is described by

$$p(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, \dots \quad (70a)$$

$$p(N(0) = 0) = 0 \quad (70b)$$

$$E[N(t)] = \lambda t \quad (70c)$$

$$\text{Var}[N(t)] = \lambda t \quad (70d)$$

$$p(N(t) \leq k) = \sum_{i=0}^k p(N(t) = i) \quad (70e)$$

- Independent & stationary increments:

For any $t_4 > t_3 \geq t_2 > t_1$ and random variables I_1, I_2 defined by

$$I_1 = N(t_2) - N(t_1) \quad (71a)$$

$$I_2 = N(t_4) - N(t_3) \quad (71b)$$

(a) I_1 and I_2 are *independent*

(b) $t_2 - t_1 = t_4 - t_3 \Rightarrow I_1, I_2$ has the same distribution (*stationary* property)

- Time increment property

$$p(N(t_2) - N(t_1) = k) = p(N(t_2 - t_1) = k) \quad (72)$$

- Joint PMF ($t_2 > t_1$)

$$\begin{aligned} p(N(t_1) = i, N(t_2) = j) &= \\ &= p(N(t_1) = i) \cdot p(N(t_2 - t_1) = j - i) \end{aligned} \quad (73)$$

- Conditional probability

$$\begin{aligned} p(N(t_1) = i | N(t_2) = j) &= \\ &= \frac{p(N(t_1) = i, N(t_2) = j)}{p(N(t_2) = j)} \end{aligned} \quad (74)$$

- Special properties:

- * Given sum of two independent distributions $X \sim \mathcal{P}(\lambda_1)$ and $Y \sim \mathcal{P}(\lambda_2)$, the resulting distribution is given by $X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2)$.
- Sub-group of Poisson process is Poisson process.
- Sum of two Poisson processes λ_1 and λ_2 is Poisson process $\lambda_1 + \lambda_2$ (but not a subtraction).

- Erlang: If $X_i \sim \text{Exp}(\lambda)$ is time difference between events, then

$$T_k = \sum_{i=1}^k X_i \sim \text{Erlang}(k, \lambda) \quad (75a)$$

$$E[T_k] = \frac{k}{\lambda} \quad (75b)$$

$$\text{Var}[T_k] = \frac{k}{\lambda^2} \quad (75c)$$

9.1 Campbell Theorem

Given

$$z(t) = \sum_{k=1}^{\infty} \delta(t - T_k) \quad (76)$$

and casual system impulse response, $h(t)$, the resulting process is given by

$$y(t) = z(t) * h(t) = \sum_{k=1}^{\infty} h(t - T_k) \quad (77)$$

and the resulting statistics is given by

$$E[y(t)] = \lambda \int_0^t h(s) ds \quad (78a)$$

$$\text{Var}[y(t)] = \lambda \int_0^t h^2(s) ds \quad (78b)$$

10 Markov Chain

- Transition matrix

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}, \quad (79)$$

where $p_{ij} = p(\mathbf{x}[n] = j | \mathbf{x}[n-1] = i)$

- Chapman-Kolmogorov equation

$$\mathbf{p}^T[n_1 + n] = \mathbf{p}^T[n_1] \mathbf{P}^n, \quad (80)$$

where $\mathbf{p}[n]$ is state probability vector

$$\mathbf{p}[n] = \begin{bmatrix} p_0[n] \\ p_1[n] \end{bmatrix} \quad (81)$$

and $p_i[n] = p(\mathbf{x}[n] = i)$, $i = 0, 1$.

- For general transition matrix of the form

$$\mathbf{P} = \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix} \quad (82)$$

$$\begin{aligned} \mathbf{P}^n &= \begin{bmatrix} \frac{\beta}{\alpha + \beta} & \frac{\alpha}{\alpha + \beta} \\ \frac{\beta}{\alpha + \beta} & \frac{\alpha}{\alpha + \beta} \end{bmatrix} + \\ &+ (1 - \alpha - \beta)^n \begin{bmatrix} \frac{\alpha}{\alpha + \beta} & -\frac{\alpha}{\alpha + \beta} \\ -\frac{\beta}{\alpha + \beta} & \frac{\beta}{\alpha + \beta} \end{bmatrix} \end{aligned} \quad (83)$$

- Steady-state probability vector

$$\boldsymbol{\pi}^T = [\pi_0 \quad \pi_1], \quad (84)$$

where $\pi_i = \lim_{n \rightarrow \infty} p(\mathbf{x}[n] = i)$, $i = 0, 1$

10.1 Ergodic Markov chain

For $\mathbf{P}^n > 0$,

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \mathbf{P} \quad (85)$$

Average number of time-steps to return to state i from the last occurrence of state i is $1/\pi_i$.

11 Different Supplementary Formulas

11.1 Derivatives

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\exp[f(x)] = \exp[f(x)] \frac{d}{dx}f(x)$$

11.2 Integrals

11.2.1 Indefinite

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$$

$$\int \exp(ax) dx = \frac{1}{a} \exp(ax)$$

$$\int x \exp(ax) dx = \exp(ax) \left[\frac{x}{a} - \frac{1}{a^2} \right]$$

$$\int x^2 \exp(ax) dx = \exp(ax) \left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right]$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

11.2.2 Definite

$$\int_0^\infty \exp(-a^2 x^2) dx = \frac{\sqrt{\pi}}{2a}$$

$$\int_0^\infty x^2 \exp(-a^2 x^2) dx = \frac{\sqrt{\pi}}{4a^3}$$

$$\int_{-\infty}^\infty \delta(x) dx = 1$$

$$\int_{-\infty}^\infty f(x) \delta(x-a) dx = f(a)$$

11.3 Fourier Transform

11.3.1 Properties

$$\frac{d^n}{dt^n} f(t) \xleftrightarrow{\mathcal{F}} (j2\pi f)^n F(f)$$

$$f(-t) \xleftrightarrow{\mathcal{F}} F^*(f)$$

$$f(t-t_0) \xleftrightarrow{\mathcal{F}} F(f) e^{-j2\pi f t_0}$$

$$f(t) e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} F(f-f_0)$$

11.3.2 Transform

$$u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} \left(\frac{1}{j\pi f} + \delta(f) \right)$$

$$\exp(-at) u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a + j2\pi f}$$

$$t \exp(-at) u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{(a + j2\pi f)^2}$$

$$\exp(-a|t|) \xleftrightarrow{\mathcal{F}} \frac{2a}{a^2 + 4\pi^2 f^2}$$

$$\exp(-at^2) \xleftrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{a}} \exp\left(-\frac{(\pi f)^2}{a}\right)$$

$$\cos(2\pi f_a t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} [\delta(f-f_a) + \delta(f+f_a)]$$

$$\sin(2\pi f_a t) \xleftrightarrow{\mathcal{F}} \frac{1}{2j} [\delta(f-f_a) - \delta(f+f_a)]$$

11.4 Trigonometry

$$\sin^2(\alpha) = \frac{1}{2} (1 - \cos(2\alpha))$$

$$\cos^2(\alpha) = \frac{1}{2} (1 + \cos(2\alpha))$$

$$\cos(\alpha) \cos(\beta) = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin(\alpha) \sin(\beta) = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\sin(\alpha) \cos(\beta) = \frac{1}{2} [\sin(\alpha - \beta) + \sin((\alpha + \beta))]$$

11.5 Matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det[A] = ad - bc$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$