



המחלקה להנדסת חשמל ואלקטרוניקה

תאריך הבחינה : 14.01.19

שעות הבחינה : 09:00-12:00

מבוא לאותות אקראיים

מועד א'

ד"ר דימה בחובסקי, אלן פריד

תשע"ט סמסטר א'

חומר עזר - דף נוסחאות אישי (משני צדדים), מחשבון

הוראות מיוחדות :

- סעיפים הם בעלי ניקוד זהה, אלא אם צוין אחרת.
- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.
- במקום בו נדרש חישוב מספרי, יש קודם לרשום את הנוסחא, ורק אח"כ להציב!
- יש לציין יחידות למספרים, ובמידה וקיימות!
- כל השרטוטים יהיו גדולים, ברורים, עם סימון צירים!
- אין חובה להגיע לערך מספרי של הפונקציה $Q(x)$, במידה ומופיעה בתשובה.

השאלון כולל 12 דפים (כולל דף זה)

בהצלחה !



1 שאלה (20 נק') 7.27, 7.31 Kay

(א) חשב ביטוי עבור $\text{Var}[X - Y]$, כאשר X, Y הם משתנים אקראיים כלשהם.

(ב) נתון שמשתנים X, Y הם גאוסיים במשותף,
$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N \left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \rho \\ \sigma_1 \sigma_2 \rho & \sigma_2^2 \end{bmatrix} \right)$$

חשב $E[X - Y]$, $\text{Var}[X - Y]$ כפונקציה של פרמטרי ההתפלגות.

2 שאלה (30 נק') 14.22 Kay

(א) חשב ערך אופטימלי של a עבור חזאי לינארי של Y מתוך $X = x$ מהצורה $\hat{Y} = ax$, במובן של שגיאה ריבועית מינימלית.

(ב) חשב שגיאה ממוצעות המתקבלת, $E[Y - \hat{Y}]$.

(ג) חשב שגיאה ריבועית ממוצעות המתקבלת, $E[(Y - \hat{Y})^2]$.

3 שאלה (10 נק')

הסבר, איך בעזרת משתנה אקראי בעל התפלגות אחידה, אפשר לקבל משתנה אקראי בעל התפלגות Cauchy מהצורה

$$f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$

$$F_X(x) = \frac{1}{\pi} \arctan(x) + \frac{1}{2}$$

4 שאלה (10 נק')

נתון אות דגום של 7 דגימות,

$$\mathbf{x}[n] = \{+1, +1, +1, -1, -1, +1, -1\}$$

חשב $R_x[k]$ עבור $k = 0, 1, 2, 3$.



5 שאלה (30 נק')

נתון תהליך אקראי $x(t)$ בעל תכונות הבאות :

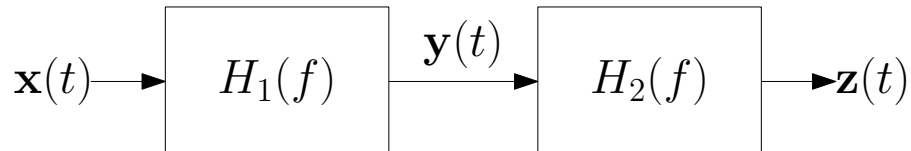
• WSS

• Gaussian

• $E[x(t)] = 0$

• $R_x(\tau) = 10\delta(\tau)$

התהליך עובר דרך 2 מערכות הבאות :



(1)
$$H_1(f) = \begin{cases} 1, & |f| < 3 \\ 0, & \text{אחרת} \end{cases}$$

(2)
$$H_2(f) = \exp\left(-\frac{f^2}{4}\right)$$

(א) חשב הספק P_y ?

הבהרה עבור סעיפים הבאים : אין חובה להגיע לערך מספרי של הפונקציה $Q(x)$, במידה ומופיעה בתשובה.

(ב) חשב הספק P_z ?

(ג) חשב הסתברות $p(y(t) > 3)$

Random Processes – Formulas

1 Distributions

1.1 Continuous

	Notation	PDF	CDF	$E[X]$	$\text{Var}[X]$
Uniform	$U[a, b]$	$\begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	$N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$\Phi(x)$	μ	σ^2
Exponential	$Exp(\lambda)$	$\lambda \exp(-\lambda x), x \geq 0$	$1 - \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$

1.1.1 Q-function

Given $Y \sim N(\mu, \sigma^2)$

$$p(Y > y) = Q\left(\frac{y - \mu}{\sigma}\right) \quad (1a)$$

$$Q(x) = 1 - \Phi(x) \quad (1b)$$

$$Q(-x) = 1 - Q(x) \quad (1c)$$

1.2 Discrete

	Notation	PMF	CDF	$E[X]$	$\text{Var}[X]$
Bernoulli	$\text{Ber}(p)$	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$	$\begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$	p	$p(1-p)$
Poisson	$\mathcal{P}(\lambda)$	$p(X=k) = \exp(-\lambda) \frac{\lambda^k}{k!}$	$\exp(-\lambda) \sum_{i=0}^k \frac{\lambda^i}{i!}$	λ	λ
Erlang	$Erlang(k, \lambda t)$	$\lambda \frac{(\lambda t)^{k-1}}{(k-1)!} \exp(-\lambda t)$	$1 - \exp(-\lambda t) \sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!}$	$\frac{k}{\lambda t}$	$\frac{k}{(\lambda t)^2}$

2 Random Variables

Definitions:

$$F_X(x) = p(X \leq x) \quad (2a)$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \quad (2b)$$

$$F_X(x) = \int_{-\infty}^x f_X(p) dp \quad (2c)$$

$$p(a < X \leq b) = F_X(b) - F_X(a) \quad (2d)$$

$$p_X[x_k] = p(X = x_k) \quad (3a)$$

$$F_X(x) = \sum_{k: x_k \leq x} p_X[x_k] \quad (3b)$$

Expectation:

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx \\ \sum_i x_i p_x[x_i] \end{cases} \quad (4a)$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_i g(x_i) p_x[x_i] \end{cases} \quad (4b)$$

$$E[aX + b] = aE[X] + b \quad (4c)$$

Variance:

$$\begin{aligned} \text{Var}[X] &= E[(X - E[X])^2] \\ &= E[X^2] - E^2[X] \end{aligned} \quad (5a)$$

$$\text{Var}[aX + b] = a^2 \text{Var}[X] \quad (5b)$$

$$\text{Var}[b] = 0 \quad (5c)$$

2.1 Simulation

Given CDF of the required random variable, $F_X(x)$, it may be generated from $Z \sim U(0, 1)$ by $X = F_X^{-1}(Z)$.

Example: Given $X \sim U[0, 1]$, $Y = a + (b - a)X \sim U[a, b]$

3 Two Random Variables

3.1 Joint Distributions

Definitions:

$$F_{XY}(x, y) = p(X \leq x, Y \leq y) \quad (6a)$$

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} \geq 0 \quad (6b)$$

$$F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(s, p) dp ds \quad (6c)$$

$$p[x_j, y_k] = p(X = x_j, Y = y_k) \quad (7a)$$

$$F_{XY}(x, y) = p(X \leq x_j, Y \leq y_k) \quad (7b)$$

Expectation:

$$E[XY] = \iint xy f_{XY}(x, y) dx dy \quad (8a)$$

$$E[g(X)] = \begin{cases} \iint g(x, y) f_{XY}(x, y) dx dy \\ \sum_i \sum_k g(x_i, y_k) p_x[x_i, y_k] \end{cases} \quad (8b)$$

$$E[aX + bY] = aE[X] + bE[Y] \quad (8c)$$

For **independent** random variables:

$$f_{XY}(x, y) = f_X(x) f_Y(y) \quad (9a)$$

$$p_{XY}[x_k, y_j] = p_X[x_k] p_Y[y_j] \quad (9b)$$

$$F_{XY}(x, y) = F_X(x) F_Y(y) \quad (9c)$$

$$E[XY] = E[X] E[Y] \quad (9d)$$

$$E[g_1(X) g_2(Y)] = E[g_1(X)] E[g_2(Y)] \quad (9e)$$

$$\text{Var}[aX + bY] = a^2 \text{Var}[X] + b^2 \text{Var}[Y] \quad (9f)$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy \quad (10a)$$

$$p_X[x_k] = \sum_j p_{XY}[x_k, y_j] \quad (10b)$$

$$F_X(x) = F_{XY}(x, \infty) \quad (10c)$$

$$F_Y(y) = F_{XY}(\infty, y) \quad (10d)$$

3.2 Conditional Relations

Conditional distribution (Bayes),

for $f_X(x), f_Y(y), p_X[x_k], p_Y[y_k] > 0$:

$$f_{Y|X}(y|x) f_X(x) = f_{X|Y}(x|y) f_Y(y) = f_{XY}(x, y) \quad (11a)$$

$$p_{Y|X}[y_j|x_k] p_X[x_k] = p_{X|Y}[x_k|y_j] p_Y[y_j] = p_{XY}[x_k, y_j] \quad (11b)$$

$$F_{Y|X}(y|x) = p(Y \leq y | X = x) \quad (11c)$$

$$= \int_{-\infty}^y f_{Y|X}(s|x) ds \quad (11d)$$

$$F_{Y|X}[y|x_k] = \frac{p[Y \leq y_j, X = x_k]}{p_X[x_k]} \quad (11e)$$

Conditional expectation & Variance:

$$E[Y|X] = \frac{\int y f_{Y|X}(y|x) dy}{\sum_j y_j p[y_j|x_k]} \quad (12a)$$

$$E[X] = E[E[X|Y]] = \iint y f_{Y|X}(y|x) f_X(x) dx dy \quad (12b)$$

$$\text{Var}[Y|X] = E[Y^2|X] - E^2[Y|X] \quad (12c)$$

$$\text{Var}[Y] = \text{Var}[E[Y|X]] + E[\text{Var}[Y|X]] \quad (12d)$$

3.3 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y , covariance is given by

$$\begin{aligned} \text{Cov}[X, Y] &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y]. \end{aligned} \quad (13)$$

Main covariance properties are:

$$\text{Cov}[X, X] = \text{Var}[X] \quad (14a)$$

$$\text{Cov}[X, Y] = \text{Cov}[Y, X] \quad (14b)$$

$$\text{Cov}[X, a] = 0 \quad (14c)$$

$$\text{Cov}[aX, bY] = ab \text{Cov}[X, Y] \quad (14d)$$

$$\text{Cov}[X, Y] = \text{Cov}[X + a, Y + b] \quad (14e)$$

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2 \text{Cov}[X, Y] \quad (14f)$$

$$|E[XY]| \leq \sqrt{E[X^2]E[Y^2]} \text{ Cauchy-Schwatz} \quad (14g)$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X] \text{Var}[Y]}} \quad (15)$$

such that $|\rho_{XY}| \leq 1$.

3.4 MMSE Linear Prediction

Mean square error (MSE) of predictor \hat{Y} is given by

$$mse = E[(Y - \hat{Y})^2] \quad (16)$$

Linear prediction of $\hat{Y} = ax + b$ for $X = x$ is

$$\hat{Y} = E[Y] + \frac{\text{Cov}[X, Y]}{\text{Var}[X]} (x - E[X]) \quad (17)$$

and

$$mse_{min} = E[(Y - (aX + b))^2] = \text{Var}(Y)(1 - \rho_{XY}^2) \quad (18)$$

When X, Y are jointly Gaussian, this prediction is optimal among **all** possible predictors

3.5 Relations

- When X and Y are *orthogonal*, $E[XY] = 0$.
- When X and Y are *uncorrelated*, $\text{Cov}[X, Y] = \rho_{XY} = 0$.
- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 9).
- When X and Y are *jointly* Gaussian and uncorrelated $\Rightarrow X$ and Y are independent.
- Joint \Rightarrow marginal, marginal \nRightarrow joint

4 Multi-dimensional Random Variables

4.1 Covariance matrix

Given random vector $\mathbf{X} = (X_1, X_2, \dots, X_N)^T$,

$$\begin{aligned} \mathbf{C}_\mathbf{X} &= \text{Cov}[\mathbf{X}, \mathbf{X}] = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T] \\ &= E[\mathbf{X}\mathbf{X}^T] - E[\mathbf{X}]E[\mathbf{X}]^T \\ &= \begin{bmatrix} \text{Var}[X_1] & \text{Cov}[X_1, X_2] & \cdots & \text{Cov}[X_1, X_N] \\ \text{Cov}[X_2, X_1] & \text{Var}[X_2] & \cdots & \text{Cov}[X_2, X_N] \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}[X_N, X_1] & \text{Cov}[X_N, X_2] & \cdots & \text{Var}[X_N] \end{bmatrix} \end{aligned} \quad (19)$$

Properties:

- Symmetry

$$\mathbf{C}_\mathbf{X} = \mathbf{C}_\mathbf{X}^T \quad \text{Cov}[X_i, X_j] = \text{Cov}[X_j, X_i] \quad (20)$$

- Variance of linear combination: Given vector $\mathbf{a} = (a_1, a_2, \dots, a_N)^T$,

$$\text{Var}[\mathbf{a}^T \mathbf{X}] = \mathbf{a}^T \mathbf{C}_\mathbf{X} \mathbf{a} \quad (21)$$

- Linear transformation: Given linear transformation $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$,

$$E[\mathbf{Y}] = \mathbf{A}E[\mathbf{X}] + \mathbf{b} \quad (22a)$$

$$\mathbf{C}_\mathbf{Y} = \mathbf{A}\mathbf{C}_\mathbf{X}\mathbf{A}^T \quad (22b)$$

- Uncorrelated variables

$$\mathbf{C}_\mathbf{X} = \text{diag}[\text{Var}[X_1], \text{Var}[X_2], \dots, \text{Var}[X_N]] \quad (23)$$

- Cross-covariance: For two random vectors $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$, the resulting $m \times n$ cross-covariance matrix is given by

$$\begin{aligned} \text{Cov}[\mathbf{X}, \mathbf{Y}] &= \mathbf{C}_{\mathbf{X}\mathbf{Y}} \\ &= E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])^T] \\ &= E[\mathbf{X}\mathbf{Y}^T] - E[\mathbf{X}]E[\mathbf{Y}]^T \end{aligned} \quad (24)$$

$$\mathbf{C}_{\mathbf{Y}\mathbf{X}} = \mathbf{C}_{\mathbf{X}\mathbf{Y}}^T \quad (25)$$

4.2 Decorrelation

Given random vector \mathbf{X} with covariance matrix $\mathbf{C}_\mathbf{X}$, and linear transformation $\mathbf{Y} = \mathbf{V}^T \mathbf{X}$, where \mathbf{V} is eigenvectors matrix of $\mathbf{C}_\mathbf{X}$, the resulting covariance matrix $\mathbf{C}_\mathbf{Y}$ is of the form $\mathbf{C}_\mathbf{Y} = \text{diag}[\lambda_1, \dots, \lambda_N]$, where λ_i are eigenvalues of $\mathbf{C}_\mathbf{X}$.

4.3 Bi-variate & Multivariate Normal Distribution

Joint Gaussian distribution of X_1 and X_2 with expectation $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and covariance matrix $C_{\mathbf{X}} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ is

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} \right]\right) \quad (26)$$

Multivariate Gaussian distribution of $\mathbf{X} = (X_1, X_2, \dots, X_N)^T$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2} \det[C_{\mathbf{X}}]} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T C_{\mathbf{X}}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right\}, \quad (27)$$

Properties:

- Random vector \mathbf{X} is **jointly** Gaussian distributed, iff (if and only if) for all possible real vectors $\mathbf{a} = (a_1, \dots, a_n)^T$ linear combination $Y = \mathbf{a}^T \mathbf{X}$ is Gaussian,

$$Y \sim N(\mathbf{a}^T \boldsymbol{\mu}, \mathbf{a}^T C_{\mathbf{X}} \mathbf{a}). \quad (28)$$

- If $X_1, X_2, \dots, X_N, X_k \sim N(0, 1), 1 \leq k \leq n$ are identically and independently distributed (IID) normal Gaussian random variables, it is termed as *normalized Gaussian random vector*. Its joint PDF is given by

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1}(x_1)f_{X_2}(x_2) \cdots f_{X_N}(x_N) = \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{x_1^2 + x_2^2 + \dots + x_N^2}{2}\right) = \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{\mathbf{x}^T \mathbf{x}}{2}\right) \quad (29)$$

The covariance matrix of such vector is given by identity matrix of size $N \times N$, $C_{\mathbf{X}} = I_n$ and its expectation is $\boldsymbol{\mu} = \mathbf{0}_{N \times 1}$.

- Linear combination of **independent** Gaussian variables, $X_i \sim N(\mu_i, \sigma_i^2)$ is Gaussian

$$\sum_{i=1}^n a_i X_i \sim N\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n (a_i \sigma_i)^2\right). \quad (30)$$

- Linear transformation – follows Eqs. (22a).
- If jointly distributed Gaussian random variables are *uncorrelated*, they are also *independent*

5 Random Processes – General Properties

- PDF & CDF

$$F_{\mathbf{X}}(x; t) = p(\mathbf{x}(t) \leq x) \quad (31a)$$

$$f_{\mathbf{X}}(x; t) = \frac{\partial}{\partial x} F_{\mathbf{X}}(x; t) \quad (31b)$$

$$p_{\mathbf{X}}[x_k; n] = p(\mathbf{x}[n] = x_k) \quad (31c)$$

- Average:

$$E[\mathbf{x}(t)] = \int_{-\infty}^{\infty} x f_{\mathbf{X}}(x; t) dx \quad (32a)$$

$$E[\mathbf{x}[n]] = \sum_i x_i p_{\mathbf{X}}[x_k; n] \quad (32b)$$

- Variance:

$$\text{Var}[\mathbf{x}(t)] = E[\mathbf{x}^2(t)] - E^2[\mathbf{x}(t)] \quad (33a)$$

$$\text{Var}[\mathbf{x}[n]] = E[\mathbf{x}^2[n]] - E^2[\mathbf{x}[n]] \quad (33b)$$

- Auto-correlation

$$R_{\mathbf{X}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{x}(t_2)] \quad (34a)$$

$$R_{\mathbf{X}}(t, t + \tau) = E[\mathbf{x}(t)\mathbf{x}(t + \tau)] \quad (34b)$$

$$R_{\mathbf{X}}(t_1, t_2) = R_{\mathbf{X}}(t_2, t_1) \quad (34c)$$

$$R_{\mathbf{X}}(t, t) = E[\mathbf{x}^2(t)] \quad (34d)$$

$$R_{\mathbf{X}}[n_1, n_2] = E[\mathbf{x}[n_1]\mathbf{x}[n_2]] \quad (34e)$$

- Auto-covariance

$$\begin{aligned} C_{\mathbf{X}}(t_1, t_2) &= R_{\mathbf{X}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)] \\ &= E[\{\mathbf{x}(t_1) - E[\mathbf{x}(t_1)]\} \{\mathbf{x}(t_2) - E[\mathbf{x}(t_2)]\}] \end{aligned} \quad (35)$$

$$C_{\mathbf{X}}(t, t) = \text{Var}[\mathbf{x}(t)] \quad (36)$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(t_1, t_2) = \frac{C_{\mathbf{x}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{x}}(t_2, t_2)}} \quad (37a)$$

$$|\rho_{\mathbf{x}}(t_1, t_2)| \leq 1 \quad (37b)$$

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *orthogonal*, $R_{\mathbf{x}}(t_1, t_2) = 0$.

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *uncorrelated*, $C_{\mathbf{x}}(t_1, t_2) = \rho_{\mathbf{x}}(t_1, t_2) = 0$.

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *independent*, $R_{\mathbf{x}}(t_1, t_2) = E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]$.

6 Wide-Sense Stationary (WSS) Process

Definition:

$$E[\mathbf{x}(t)] = E[\mathbf{x}(0)] = \mu_{\mathbf{x}} = \text{const} \quad (38a)$$

$$R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(\tau = |t_2 - t_1|), \quad \forall t_1, t_2 \quad (38b)$$

$$E[\mathbf{x}[n]] = E[\mathbf{x}[0]] = \mu_{\mathbf{x}} = \text{const} \quad (38c)$$

$$R_{\mathbf{x}}[n_1, n_2] = R_{\mathbf{x}}(k = |n_2 - n_1|), \quad \forall n_1, n_2 \quad (38d)$$

- Auto-correlation

$$R_{\mathbf{x}}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t + \tau)] \quad (39a)$$

$$R_{\mathbf{x}}[k] = E[\mathbf{x}[n]\mathbf{x}(n + k)] \quad (39b)$$

Properties:

$$R_{\mathbf{x}}(-\tau) = R_{\mathbf{x}}(\tau) \quad (40a)$$

$$R_{\mathbf{x}}(0) = E[|\mathbf{x}(0)|^2] = E[|\mathbf{x}(t)|^2] \quad (40b)$$

$$\text{Var}[\mathbf{x}(t)] = C_{\mathbf{x}}(0) = \sigma_{\mathbf{x}}^2 \quad (40c)$$

$$R_{\mathbf{x}}(0) \geq |R_{\mathbf{x}}(\tau)| \quad (40d)$$

Deterministic definition ($x[n], x(t)$ not random)

$$R_x(\tau) = x(\tau) * x(-\tau) \quad (41a)$$

$$= \int_{-\infty}^{\infty} x(s)x(s + \tau)ds$$

$$R_x[k] = x[k] * x[-k] \quad (41b)$$

$$= \sum_{n=-\infty}^{\infty} x[n]x[n + k]$$

- Auto-covariance

$$C_{\mathbf{x}}(\tau) = R_{\mathbf{x}}(\tau) - \mu_{\mathbf{x}}^2 \quad (42a)$$

$$C_{\mathbf{x}}[k] = R_{\mathbf{x}}[k] - \mu_{\mathbf{x}}^2 \quad (42b)$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(\tau) = \frac{C_{\mathbf{x}}(\tau)}{C_{\mathbf{x}}(0)} \quad (43a)$$

$$\rho_{\mathbf{x}}[k] = \frac{C_{\mathbf{x}}[k]}{C_{\mathbf{x}}[0]} \quad (43b)$$

6.1 Linear Prediction

Given N samples of process $\mathbf{x}[n]$, and predictor

$$\hat{\mathbf{x}}[n + 1] = \sum_{i=1}^N a_i \mathbf{x}[n - i + 1], \quad (44)$$

the values of a_i are given by a solution of

$$\begin{bmatrix} R_{\mathbf{x}}[0] & R_{\mathbf{x}}[1] & \cdots & R_{\mathbf{x}}[N-1] \\ R_{\mathbf{x}}[1] & R_{\mathbf{x}}[0] & \cdots & R_{\mathbf{x}}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{x}}[N-1] & R_{\mathbf{x}}[N-2] & \cdots & R_{\mathbf{x}}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} R_{\mathbf{x}}[1] \\ R_{\mathbf{x}}[2] \\ \vdots \\ R_{\mathbf{x}}[N] \end{bmatrix} \quad (45)$$

and the resulting minimum MSE is

$$mse_{min} = R_{\mathbf{x}}[0] - \sum_{i=1}^N a_i R_{\mathbf{x}}[i] \quad (46)$$

6.2 Power Spectral Density (PSD)

$$S_{\mathbf{x}}(f) = \mathcal{F}\{R_{\mathbf{x}}(\tau)\} = \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau) \exp(-j2\pi f\tau) d\tau \quad (47a)$$

$$= 2 \int_0^{\infty} R_{\mathbf{x}}(\tau) \cos(2\pi f\tau) d\tau \quad (47b)$$

$$R_{\mathbf{x}}(\tau) = \mathcal{F}^{-1}\{S_{\mathbf{x}}(f)\} = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f) \exp(j2\pi f\tau) df \quad (47c)$$

$$(47d)$$

Properties:

$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(-f) \quad (48a)$$

$$S_{\mathbf{x}}(f) \geq 0, \forall f \quad (48b)$$

$$S_{\mathbf{x}}(f) \in \mathbb{R} \quad (\text{real numbers}) \quad (48c)$$

$$(48d)$$

Average power

$$P_{\mathbf{x}} = E[\mathbf{x}^2(t)] = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f) df \quad (49)$$

Deterministic definition:

$$S_x(f) = X(f)X^*(f) = |X(f)|^2 \quad (50)$$

6.3 White Noise & White Gaussian Noise (WGN) Process

White noise process is SSS (WSS) process that is characterized by

$$R_{\mathbf{n}}(\tau) = \sigma^2 \delta(\tau) \quad (51a)$$

$$S_{\mathbf{n}}(f) = \sigma^2 \quad \forall f \quad (51b)$$

For WGN process, $\mathbf{n}(t) \sim N(0, \sigma^2)$,

$$R_{\mathbf{n}}(\tau) = \frac{N_0}{2} \delta(\tau) \quad (52a)$$

$$S_{\mathbf{n}}(f) = \frac{N_0}{2} \quad \forall f \quad (52b)$$

6.4 Relation Between Covariance Matrix & Auto-covariance

Given WSS process $\mathbf{x}(t)$, the corresponding correlation matrix of $\mathbf{X} = [\mathbf{x}(t_1), \dots, \mathbf{x}(t_N)]^T$ is given by

$$R_{\mathbf{X}} = E[\mathbf{X}\mathbf{X}^T] \quad (53)$$

$$R_{\mathbf{X}}(i, j) = E[X_i X_j] = R_{\mathbf{x}}(|t_i - t_j|) \quad (54)$$

7 Cross-Signal

- Cross-correlation

$$R_{\mathbf{xy}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{y}(t_2)] \quad (55)$$

- Cross-covariance

$$C_{\mathbf{xy}}(t_1, t_2) = R_{\mathbf{xy}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{y}(t_2)] \quad (56)$$

- Correlation Coefficient

$$\rho_{\mathbf{xy}}(t_1, t_2) = \frac{C_{\mathbf{xy}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{y}}(t_2, t_2)}} \quad (57)$$

7.1 WSS Cross-signal

- $\mathbf{x}(t), \mathbf{y}(t)$ are jointly WSS, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ each of them is WSS and

$$R_{\mathbf{xy}}(\tau) = E[\mathbf{x}(t)\mathbf{y}(t+\tau)] \quad (58)$$

- When $\mathbf{x}(t)$ and $\mathbf{y}(t+\tau)$ are *uncorrelated jointly WSS*, $C_{\mathbf{xy}}(\tau) = 0$.

Properties

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{yx}}(-\tau) \quad (59a)$$

$$|R_{\mathbf{xy}}(\tau)| \leq \sqrt{R_{\mathbf{x}}(0)R_{\mathbf{y}}(0)} \quad (59b)$$

$$|R_{\mathbf{xy}}(\tau)| \leq \frac{1}{2} [R_{\mathbf{x}}(0) + R_{\mathbf{y}}(0)] \quad (59c)$$

Deterministic definition

$$R_{xy}(\tau) = x(\tau) * y(-\tau) \quad (60)$$

- Cross-covariance

$$C_{\mathbf{xy}}(\tau) = R_{\mathbf{xy}}(\tau) - \mu_{\mathbf{x}}\mu_{\mathbf{y}} \quad (61)$$

- Cross-PSD

$$S_{\mathbf{xy}}(f) = \mathcal{F}\{R_{\mathbf{xy}}(\tau)\} \quad (62)$$

Properties

$$S_{\mathbf{xy}}(f) = S_{\mathbf{yx}}(-f) = S_{\mathbf{xy}}^*(-f) \quad (63)$$

Deterministic definition

$$S_{xy}(f) = X(f)Y^*(f) \quad (64)$$

- Coherence

$$\gamma_{\mathbf{xy}}(f) = \frac{S_{\mathbf{xy}}(f)}{\sqrt{S_{\mathbf{x}}(f)S_{\mathbf{y}}(f)}} \quad (65)$$

8 LTI and WSS Random Process

Output of LTI system with impulse response $h(t)$ and random process $x(t)$,

$$y(t) = x(t) * h(t) \quad (66)$$

Average

$$m_y = m_x \int_{-\infty}^{\infty} h(s) ds = m_x H(f=0) \quad (67)$$

Cross-correlation & cross-covariance:

$$R_{xy}(\tau) = R_x(\tau) * h(\tau) \quad (68a)$$

$$C_{xy}(\tau) = C_x(\tau) * h(\tau) \quad (68b)$$

$$R_{yx}(\tau) = R_x(\tau) * h(-\tau) \quad (68c)$$

$$C_{yx}(\tau) = C_x(\tau) * h(-\tau) \quad (68d)$$

$$R_y(\tau) = R_x(\tau) * h(\tau) * h(-\tau) \quad (68e)$$

$$C_y(\tau) = C_x(\tau) * h(\tau) * h(-\tau) \quad (68f)$$

Power-Spectral Density (PSD) & Cross-PSD: Given frequency response $H(f) = \mathcal{F}\{h(\tau)\}$, $H^*(f) = \mathcal{F}\{h(-\tau)\}$

$$S_{xy}(f) = S_x(f) H(f) \quad (69a)$$

$$S_{yx}(f) = S_x(f) H^*(f) \quad (69b)$$

$$S_y(f) = S_x(f) H(f) H^*(f) = S_x(f) |H(f)|^2 \quad (69c)$$

Power of the process:

$$P_x = R_x(0) = \int_{-\infty}^{\infty} S_x(f) df \quad (70a)$$

$$P_y = R_y(0) = \int_{-\infty}^{\infty} S_x(f) |H(f)|^2 df \quad (70b)$$

8.1 SNR

Given input signal

$$\mathbf{x}(t) = \mathbf{s}(t) + \mathbf{n}(t), \quad (71)$$

where $\mathbf{s}(t), \mathbf{n}(t)$ are independent and $E[\mathbf{n}(t)] = 0$, the PSD of output $\mathbf{y}(t)$ is given by

$$\begin{aligned} S_y(f) &= S_x(f) |H(f)|^2 \\ &= S_s(f) |H(f)|^2 + S_n(f) |H(f)|^2, \end{aligned} \quad (72)$$

where $S_s(f) |H(f)|^2$ is signal output PSD and $S_n(f) |H(f)|^2$ is noise PSD.

The input and output SNRs is given by

$$\text{SNR}_x = \frac{E[\mathbf{s}^2(t)]}{E[\mathbf{n}^2(t)]} = \frac{R_s(0)}{R_n(0)} = \frac{\int_{-\infty}^{\infty} S_s(f) df}{\int_{-\infty}^{\infty} S_n(f) df} \quad (73a)$$

$$\text{SNR}_y = \frac{\int_{-\infty}^{\infty} S_s(f) |H(f)|^2 df}{\int_{-\infty}^{\infty} S_n(f) |H(f)|^2 df}. \quad (73b)$$

8.2 Gaussian Process

A Gaussian process $\mathbf{x}(t)$ a random process that for $\forall k > 0$ and for all times t_1, \dots, t_k , the set of random variable $\mathbf{x}(t_1), \dots, \mathbf{x}(t_k)$ is jointly Gaussian (i.e. described by Eq. (27)).

Properties:

- WSS Gaussian process is SSS.
- Gaussian process $\mathbf{x}(t)$ that passes through LTI system, $\mathbf{y}(t) = h(t) * \mathbf{x}(t)$, is also Gaussian process that may be described by the change of expectation and auto-correlation,

$$E[\mathbf{y}(t)] = E[\mathbf{x}(t)] \int_{-\infty}^{\infty} h(s) ds \quad (74a)$$

$$= E[\mathbf{x}(t)] H(0), \quad H(f) = \mathcal{F}\{h(t)\}$$

$$C_y(\tau) = C_x(\tau) * h(\tau) * h(-\tau) \quad (74b)$$

- The resulting autocorrelation may be used for producing the correspondent covariance matrix C_Y of a multivariate Gaussian $\mathbf{Y} = [\mathbf{y}(t_1), \dots, \mathbf{y}(t_N)]^T$

9 Poisson Process

- The Poisson process, $N(t)$, is described by

$$p(N(t) = k) = \exp(-\lambda t) \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, \dots \quad (75a)$$

$$p(N(0) = 0) = 0 \quad (75b)$$

$$E[N(t)] = \lambda t \quad (75c)$$

$$\text{Var}[N(t)] = \lambda t \quad (75d)$$

$$p(N(t) \leq k) = \sum_{i=0}^k p(N(t) = i) \quad (75e)$$

- Independent & stationary increments:

For any $t_4 > t_3 \geq t_2 > t_1$ and random variables I_1, I_2 defined by

$$I_1 = N(t_2) - N(t_1) \quad (76a)$$

$$I_2 = N(t_4) - N(t_3) \quad (76b)$$

- (a) I_1 and I_2 are *independent*
- (b) $t_2 - t_1 = t_4 - t_3 \Rightarrow I_1, I_2$ has the same distribution (*stationary* property)

- Time increment property

$$p(N(t_2) - N(t_1) = k) = p(N(t_2 - t_1) = k) \quad (77)$$

- Joint PMF ($t_2 > t_1$)

$$\begin{aligned} p(N(t_1) = i, N(t_2) = j) &= \\ &= p(N(t_1) = i) \cdot p(N(t_2 - t_1) = j - i) \end{aligned} \quad (78)$$

- Conditional probability

$$\begin{aligned} p(N(t_1) = i | N(t_2) = j) &= \\ &= \frac{p(N(t_1) = i, N(t_2) = j)}{p(N(t_2) = j)} \end{aligned} \quad (79)$$

- Special properties:

* Given sum of two independent distributions $X \sim \mathcal{P}(\lambda_1)$ and $Y \sim \mathcal{P}(\lambda_2)$, the resulting distribution is given by $X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2)$.

– Sub-group of Poisson process is Poisson process.

– Sum of two Poisson processes λ_1 and λ_2 is Poisson process $\lambda_1 + \lambda_2$ (but not a subtraction).

- Erlang: If $X_i \sim \text{Exp}(\lambda)$ is time difference between

events, then

$$T_k = \sum_{i=1}^k X_i \sim \text{Erlang}(k, \lambda) \quad (80a)$$

$$E[T_k] = \frac{k}{\lambda} \quad (80b)$$

$$\text{Var}[T_k] = \frac{k}{\lambda^2} \quad (80c)$$

9.1 Campbell Theorem

Given

$$z(t) = \sum_{k=1}^{\infty} \delta(t - T_k) \quad (81)$$

and casual system impulse response, $h(t)$, the resulting process is given by

$$y(t) = z(t) * h(t) = \sum_{k=1}^{\infty} h(t - T_k) \quad (82)$$

and the resulting statistics is given by

$$E[y(t)] = \lambda \int_0^t h(s) ds \quad (83a)$$

$$\text{Var}[y(t)] = \lambda \int_0^t h^2(s) ds \quad (83b)$$

10 Different Supplementary Formulas

10.1 Derivatives

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\exp[f(x)] = \exp[f(x)] \frac{d}{dx}f(x)$$

10.2 Integrals

10.2.1 Indefinite

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$$

$$\int \exp(ax) dx = \frac{1}{a} \exp(ax)$$

$$\int x \exp(ax) dx = \exp(ax) \left[\frac{x}{a} - \frac{1}{a^2} \right]$$

$$\int x^2 \exp(ax) dx = \exp(ax) \left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right]$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

10.2.2 Definite

$$\int_0^\infty \exp(-a^2 x^2) dx = \frac{\sqrt{\pi}}{2a}$$

$$\int_0^\infty x^2 \exp(-a^2 x^2) dx = \frac{\sqrt{\pi}}{4a^3}$$

$$\int_{-\infty}^\infty \delta(x) dx = 1$$

$$\int_{-\infty}^\infty f(x) \delta(x-a) dx = f(a)$$

10.3 Fourier Transform

10.3.1 Properties

$$\frac{d^n}{dt^n} f(t) \xLeftrightarrow{\mathcal{F}} (j2\pi f)^n F(f)$$

$$f(-t) \xLeftrightarrow{\mathcal{F}} F^*(f)$$

$$f(t-t_0) \xLeftrightarrow{\mathcal{F}} F(f) e^{-j2\pi f t_0}$$

$$f(t) e^{j2\pi f_0 t} \xLeftrightarrow{\mathcal{F}} F(f-f_0)$$

10.3.2 Transform pairs

$$u(t) \xLeftrightarrow{\mathcal{F}} \frac{1}{2} \left(\frac{1}{j\pi f} + \delta(f) \right)$$

$$\exp(-at)u(t) \xLeftrightarrow{\mathcal{F}} \frac{1}{a + j2\pi f}$$

$$t \exp(-at)u(t) \xLeftrightarrow{\mathcal{F}} \frac{1}{(a + j2\pi f)^2}$$

$$\exp(-a|t|) \xLeftrightarrow{\mathcal{F}} \frac{2a}{a^2 + 4\pi^2 f^2}$$

$$\exp(-at^2) \xLeftrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{a}} \exp\left(-\frac{(\pi f)^2}{a}\right)$$

$$\cos(2\pi f_a t) \xLeftrightarrow{\mathcal{F}} \frac{1}{2} [\delta(f-f_a) + \delta(f+f_a)]$$

$$\sin(2\pi f_a t) \xLeftrightarrow{\mathcal{F}} \frac{1}{2j} [\delta(f-f_a) - \delta(f+f_a)]$$

10.4 Trigonometry

$$\sin^2(\alpha) = \frac{1}{2} (1 - \cos(2\alpha))$$

$$\cos^2(\alpha) = \frac{1}{2} (1 + \cos(2\alpha))$$

$$\cos(\alpha) \cos(\beta) = \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$\sin(\alpha) \sin(\beta) = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\sin(\alpha) \cos(\beta) = \frac{1}{2} [\sin(\alpha - \beta) + \sin((\alpha + \beta))]$$

10.5 Matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\det[\mathbf{A}] = ad - bc$$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

10.5.1 Eigenvalues/vectors

$$\mathbf{A}\mathbf{V} = \mathbf{\Lambda}\mathbf{V}$$

$$\det[\mathbf{A} - \mathbf{\Lambda}\mathbf{V}] = 0$$