



המחלקה להנדסת חשמל ואלקטרוניקה

תאריך הבחינה: 01.05.2023

שעות הבחינה: 9:00-12:00

מבוא לאותות אקראיים

מועד א'

ד"ר דימה בחובסקי, מר טל פאר

תשפ"ד סמסטר א'

חומר עזר - דף נוסחאות אישי (עמוד אחד), מחשבון

הוראות מיוחדות:

- השאלון כולל שאלות ללא בחירה, סך הכל של 103 נקודות.
- סעיפים הם בעלי ניקוד זהה, אלא אם צוין אחרת.
- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.
- במקום בו נדרש חישוב מספרי, יש קודם לרשום את הנוסחה, ורק אח"כ להציב!
- יש לציין יחידות למספרים, ובמידה וקיימות!
- כל השרטוטים יהיו גדולים, ברורים, עם סימון צירים!
- אין חובה להגיע לערך מספרי של הפונקציה $Q(x)$, במידה ומופיעה בתשובה.

השאלון כולל 12 דפים (כולל דף זה)

בהצלחה !



1 היסטוגרמה (39 נק')

נתונה פונקציית PDF של משתנה אקראי X , מהצורה

$$f_X(x) = \frac{1}{2}e^{2x}; \quad -\infty < x < \ln(2) \approx 0.69314718$$

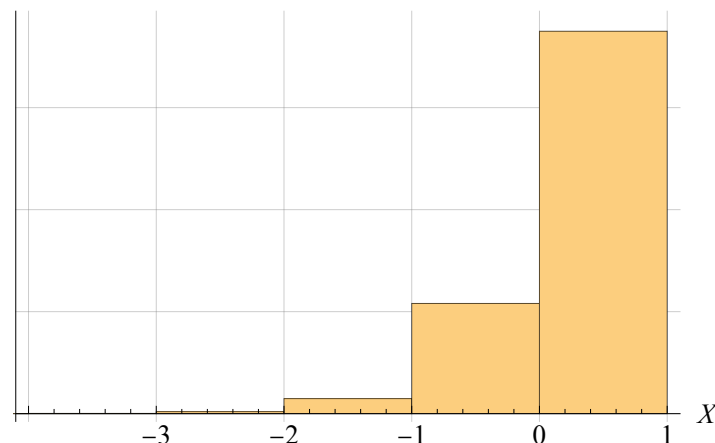
יש למצוא:

1. $E[X]$

2. $E[X^2]$

3. $\Pr(X < 0)$

4. (15 נק') נתונה היסטוגרמה (איור 1) עבור ערכים אקראיים המתפלגים לפי X . ציר x מחושב עם מרווחים של $\Delta x = 1$, כמתואר שאיור.



איור 1: היסטוגרמה של משתנה X .

עבור ההיסטוגרמה, יש לחשב את הערכים הבאים:

(א) עבור ציר y , שהוא מספר הערכים של היסטוגרמה בכל מרווח: מהו מספר הערכים התאורטי במרווח בין 0 ל-1 בתוך ההיסטוגרמה, מתוך סה"כ 100,000 הערכים?

(ב) עבור ציר y , שהוא ההסתברות של הערך במרווח: מהי הסתברות לערכים במרווח בין 0 ל-1, בתוך ההיסטוגרמה?

(ג) עבור ציר y , שהוא חישוב מקורב של $f_X(x)$: מהו הערך התאורטי של $f_X(0.5)$ עבור ההיסטוגרמה?

תזכורת:

$$\lim_{x \rightarrow -\infty} x^n e^x \rightarrow 0, \quad e^{\ln(a)} = a$$

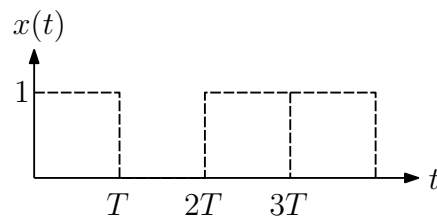
2 אותות אקראיים - תקשורת baseband (24 נק')

נתון אות ספרתי בינארי מהצורה

$$x(t) = \sum_n a_n g(t - nT)$$

כאשר $a_n \in \{0, 1\}$, $\Pr(a_n = 0) = \Pr(a_n = 1) = \frac{1}{2}$

והאות $g(t) = \begin{cases} 1 & 0 \leq t < T \\ 0 & \text{אחרת} \end{cases}$ הינו פולס בגובה 1 ברוחב T . דוגמא של האות מופיעה באיור להלן.



יש לחשב:

איור 2: דוגמא לאות $x(t)$.

(א) $E[x(t)]$

(ב) $P_x, \text{Var}[x(t)]$

3 משתנים אקראיים וקשר בין משתנים (40 נק')

נתונות מדידות רדיוס של מעגלים. ידועה שרדיוסים מתפלגים $r \sim U[a = 1, b = 3]$ ושגיאת מדידה היא $X = r + n$ $n \sim N(\mu = 0, \sigma^2 = 0.01)$. תוצאת המדידה בלתי תלויות. תוצאת המדידה היא $X = r + n$. היקף הוא $C = 2\pi X$, ושטח הוא $A = \pi X^2$. יש לחשב:

(א) מהו היקף הממוצע של המעגל הנמדד, $E[C]$?

(ב) מהי שונות של היקף של המעגל הנמדד, $\text{Var}[C]$?

(ג) מהו שטח הממוצע של המעגל הנמדד, $E[A]$?

(ד) יש לחשב $\text{Cov}[X, r]$.

Random Processes – Formulas

1 Distributions

1.1 Continuous

	Notation	PDF	CDF	$E[X]$	$\text{Var}[X]$
Uniform	$U[a,b]$	$\begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \end{cases}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	$N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$	$\Phi(x)$	μ	σ^2
Exponential	$Exp(\lambda)$	$\lambda \exp(-\lambda x), x \geq 0$	$1 - \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$

1.1.1 Q-function

$$\begin{array}{lcl} \text{Given } Y \sim N(\mu, \sigma^2) & & Q(x) = 1 - \Phi(x) \quad (3) \\ \frac{Y - \mu}{\sigma} \sim N(0, 1) & (1) & Q(-x) = 1 - Q(x) \quad (4) \\ p(Y > y) = Q\left(\frac{y - \mu}{\sigma}\right) & (2) & Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{s^2}{2}\right) ds \quad (5) \end{array}$$

1.2 Discrete

	Notation	PDF	CDF	$E[X]$	$\text{Var}[X]$
Bernoulli	$\text{Ber}(p)$	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$	$\begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$	p	$p(1-p)$
Binomial	$\text{Bin}(n, p)$	$\binom{n}{k} p^k (1-p)^{n-k}$		np	$np(1-p)$
Geometric	$\text{Geo}(p)$	$p(1-p)^{k-1}$	$1 - (1-p)^k$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

$\binom{n}{k} = \frac{n!}{(n-k)!k!}$

2 Random Variables

$$\begin{array}{lcl} \text{Definitions:} & & f_X(x) \geq 0 \quad (10) \\ F_X(x) = p(X \leq x) & (6) & \int_{-\infty}^{\infty} f_X(x) dx = 1 \quad (11) \\ f_X(x) = \frac{\partial F_X(x)}{\partial x} \geq 0 & (7) & p_X[x_k] = \Pr[X = x_k] \quad (12) \\ F_X(x) = \int_{-\infty}^x f_X(p) dp & (8) & 0 \leq p_X[x_i] \leq 1 \quad \forall i \quad (13) \\ p(a < X \leq b) = F_X(b) - F_X(a) & (9) & \sum_i p_X[x_i] = 1 \quad (14) \end{array}$$

$$F_X(x) = \Pr(X \leq x), \quad x \in \mathbb{R} \quad (15)$$

$$F_X(x) = \sum_{k: x_k \leq x} p_X[x_k] \quad (16)$$

Expectation:

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x) dx \\ \sum_i x_i p_X[x_i] \end{cases} \quad (17a)$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx \\ \sum_i g(x_i) p_X[x_i] \end{cases} \quad (17b)$$

$$E[aX + b] = aE[X] + b \quad (17c)$$

Variance:

$$\begin{aligned} \text{Var}[X] &= E[(X - E[X])^2] \\ &= E[X^2] - E^2[X] \end{aligned} \quad (18a)$$

$$\text{Var}[aX + b] = a^2 \text{Var}[X] \quad (18b)$$

$$\text{Var}[b] = 0 \quad (18c)$$

2.1 Numerical calculation

$$E[X] = \frac{1}{N} \sum_{i=1}^N x_i \quad (19)$$

$$\text{Var}[X] = \frac{1}{N} \sum_{i=1}^N (x_i - E[X])^2 \quad (20)$$

2.2 Histogram

$$p_X[x_i] \approx \frac{n_i}{N} \quad i = 1, \dots, k \quad (21)$$

$$f_X(x_i) \approx \frac{n_i}{N} \cdot \frac{1}{\Delta x} \quad i = 1, \dots, k \quad (22)$$

3 Two Random Variables

3.1 Joint Distributions

Definitions:

$$F_{XY}(x, y) = p(X \leq x, Y \leq y) \quad (23a)$$

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} \geq 0 \quad (23b)$$

$$F_{XY}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{XY}(s, p) dp ds \quad (23c)$$

$$p[x_j, y_k] = p(X = x_j, Y = y_k) \quad (24a)$$

$$F_{XY}(x, y) = p(X \leq x, Y \leq y) \quad (24b)$$

Expectation:

$$E[g(X, Y)] = \begin{cases} \iint g(x, y) f_{XY}(x, y) dx dy \\ \sum_i \sum_k g(x_i, y_k) p_{XY}[x_i, y_k] \end{cases} \quad (25a)$$

$$E[aX + bY] = aE[X] + bE[Y] \quad (25b)$$

For **independent** random variables:

$$f_{XY}(x, y) = f_X(x) f_Y(y) \quad (26a)$$

$$p_{XY}[x_k, y_j] = p_X[x_k] p_Y[y_j] \quad (26b)$$

$$F_{XY}(x, y) = F_X(x) F_Y(y) \quad (26c)$$

$$(26d)$$

Independent random variables properties:

$$E[XY] = E[X]E[Y] \quad (27a)$$

$$E[g_1(X)g_2(Y)] = E[g_1(X)]E[g_2(Y)] \quad (27b)$$

$$\text{Var}[aX + bY] = a^2 \text{Var}[X] + b^2 \text{Var}[Y] \quad (27c)$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy \quad (28a)$$

$$p_X[x_k] = \sum_j p_{XY}[x_k, y_j] \quad (28b)$$

$$F_X(x) = F_{XY}(x, \infty) \quad (28c)$$

$$F_Y(y) = F_{XY}(\infty, y) \quad (28d)$$

3.2 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y , covariance is given by

$$\begin{aligned} \text{Cov}[X, Y] &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y] \end{aligned} \quad (29)$$

Main covariance properties are:

$$\text{Cov}[X, X] = \text{Var}[X] \quad (30a)$$

$$\text{Cov}[X, Y] = \text{Cov}[Y, X] \quad (30b)$$

$$\text{Cov}[X, a] = 0 \quad (30c)$$

$$\text{Cov}[aX, bY] = ab \text{Cov}[X, Y] \quad (30d)$$

$$\text{Cov}[X, Y] = \text{Cov}[X + a, Y + b] \quad (30e)$$

$$\text{Var}[X \pm Y] = \text{Var}[X] + \text{Var}[Y] \pm 2 \text{Cov}[X, Y] \quad (30f)$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X] \text{Var}[Y]}} \quad (31)$$

such that $|\rho_{XY}| \leq 1$.

3.3 MMSE Linear Prediction

Mean square error (MSE) of predictor \hat{Y} is given by

$$mse = E[(Y - \hat{Y})^2] \quad (32)$$

Linear prediction of $\hat{Y} = ax + b$ for $X = x$ is

$$\hat{Y} = E[Y] + \frac{\text{Cov}[X, Y]}{\text{Var}[X]} (x - E[X]) \quad (33)$$

and

$$mse_{min} = E \left[\left(Y - (aX + b) \right)^2 \right] = \text{Var}[Y] (1 - \rho_{XY}^2) \quad (34)$$

When X, Y are jointly Gaussian, this prediction is optimal among **all** possible predictors

3.4 Relations

- When X and Y are *orthogonal*, $E[XY] = 0$.

- When X and Y are *uncorrelated*, $\text{Cov}[X, Y] = \rho_{XY} = 0$.

- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 26).

- When X and Y are *jointly* Gaussian and uncorrelated $\Rightarrow X$ and Y are independent.

3.5 Bi-variate Normal Distribution

Joint Gaussian distribution of X_1 and X_2

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N \left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \mathbf{C}_\mathbf{x} \right) \quad (35)$$

with covariance matrix

$$\mathbf{C}_\mathbf{x} = \begin{bmatrix} \text{Cov}[X_1, X_1] & \text{Cov}[X_1, X_2] \\ \text{Cov}[X_2, X_1] & \text{Cov}[X_2, X_2] \end{bmatrix} \quad (36)$$

Important properties:

- Sum of independent Gaussian variables is a Gaussian variable.
- Random vector $[X_1, \dots, X_n]$ is **jointly** Gaussian distributed, iff (if and only if) for all possible real vectors $\mathbf{a} = (a_1, \dots, a_n)^T$ linear combination $Y = a_1 X_1 + \dots + a_n X_n$ is Gaussian distributed.
- If jointly distributed Gaussian random variables are *uncorrelated*, they are also *independent*

4 Random Processes – General Properties

- PDF & CDF

$$F_\mathbf{x}(x; t) = p(\mathbf{x}(t) \leq x) \quad (37a)$$

$$f_\mathbf{x}(x; t) = \frac{\partial}{\partial x} F_\mathbf{x}(x; t) \quad (37b)$$

$$p_\mathbf{x}[x_k; n] = p(\mathbf{x}[n] = x_k) \quad (37c)$$

- Average:

$$E[\mathbf{x}(t)] = \int_{-\infty}^{\infty} x f_\mathbf{x}(x; t) dx \quad (38a)$$

$$E[\mathbf{x}[n]] = \sum_i x_i p_\mathbf{x}[x_k; n] \quad (38b)$$

- Variance:

$$\text{Var}[\mathbf{x}(t)] = E[\mathbf{x}^2(t)] - E^2[\mathbf{x}(t)] = \sigma_\mathbf{x}^2(t) \quad (39a)$$

$$\text{Var}[\mathbf{x}[n]] = E[\mathbf{x}^2[n]] - E^2[\mathbf{x}[n]] = \sigma_\mathbf{x}^2[n] \quad (39b)$$

- Auto-correlation

$$R_\mathbf{x}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{x}(t_2)] \quad (40a)$$

$$R_\mathbf{x}(t, t + \tau) = E[\mathbf{x}(t)\mathbf{x}(t + \tau)] \quad (40b)$$

$$R_\mathbf{x}(t_1, t_2) = R_\mathbf{x}(t_2, t_1) \quad (40c)$$

$$R_\mathbf{x}(t, t) = E[\mathbf{x}^2(t)] \quad (40d)$$

$$R_{\mathbf{x}}[n_1, n_2] = E[\mathbf{x}[n_1]\mathbf{x}[n_2]] \quad (40e)$$

$$R_{\mathbf{x}}[n, n] = E[\mathbf{x}^2[n]] \quad (40f)$$

- Auto-covariance

$$C_{\mathbf{x}}(t_1, t_2) = E\left[\{\mathbf{x}(t_1) - E[\mathbf{x}(t_1)]\}\{\mathbf{x}(t_2) - E[\mathbf{x}(t_2)]\}\right] \quad (41)$$

$$= R_{\mathbf{x}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)] \quad (42)$$

$$C_{\mathbf{x}}[n_1, n_2] = E\left[\{\mathbf{x}[n_1] - E[\mathbf{x}[n_1]]\}\{\mathbf{x}[n_2] - E[\mathbf{x}[n_2]]\}\right] \quad (43)$$

$$= R_{\mathbf{x}}[n_1, n_2] - E[\mathbf{x}[n_1]]E[\mathbf{x}[n_2]] \quad (44)$$

$$C_{\mathbf{x}}(t, t) = \text{Var}[\mathbf{x}(t)] \quad (45a)$$

$$C_{\mathbf{x}}[n, n] = \text{Var}[\mathbf{x}[n]] \quad (45b)$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(t_1, t_2) = \frac{C_{\mathbf{x}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{x}}(t_2, t_2)}} \quad (46a)$$

$$|\rho_{\mathbf{x}}(t_1, t_2)| \leq 1 \quad (46b)$$

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *orthogonal*, $R_{\mathbf{x}}(t_1, t_2) = 0$.

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *uncorrelated*, $C_{\mathbf{x}}(t_1, t_2) = \rho_{\mathbf{x}}(t_1, t_2) = 0$.

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *independent*, $R_{\mathbf{x}}(t_1, t_2) = E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]$.

5 Wide-Sense Stationary (WSS) Process

Definition:

$$E[\mathbf{x}(t)] = E[\mathbf{x}(0)] = \mu_{\mathbf{x}} = \text{const} \quad (47a)$$

$$R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(\tau = |t_2 - t_1|), \quad \forall t_1, t_2 \quad (47b)$$

$$E[\mathbf{x}[n]] = E[\mathbf{x}[0]] = \mu_{\mathbf{x}} = \text{const} \quad (47c)$$

$$R_{\mathbf{x}}[n_1, n_2] = R_{\mathbf{x}}(k = |n_2 - n_1|), \quad \forall n_1, n_2 \quad (47d)$$

- Auto-correlation

$$R_{\mathbf{x}}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t + \tau)] \quad (48a)$$

$$R_{\mathbf{x}}[k] = E[\mathbf{x}[n]\mathbf{x}[n + k]] \quad (48b)$$

Properties:

$$R_{\mathbf{x}}(-\tau) = R_{\mathbf{x}}(\tau) \quad (49a)$$

$$R_{\mathbf{x}}(0) = E[|\mathbf{x}(0)|^2] = E[|\mathbf{x}(t)|^2] \quad (49b)$$

$$\text{Var}[\mathbf{x}(t)] = C_{\mathbf{x}}(0) = \sigma_{\mathbf{x}}^2 \quad (49c)$$

$$R_{\mathbf{x}}(0) \geq |R_{\mathbf{x}}(\tau)| \quad (49d)$$

- Auto-covariance

$$C_{\mathbf{x}}(\tau) = R_{\mathbf{x}}(\tau) - \mu_{\mathbf{x}}^2 \quad (50a)$$

$$C_{\mathbf{x}}[k] = R_{\mathbf{x}}[k] - \mu_{\mathbf{x}}^2 \quad (50b)$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(\tau) = \frac{C_{\mathbf{x}}(\tau)}{C_{\mathbf{x}}(0)} \quad (51a)$$

$$\rho_{\mathbf{x}}[k] = \frac{C_{\mathbf{x}}[k]}{C_{\mathbf{x}}[0]} \quad (51b)$$

5.1 Power Spectral Density (PSD)

$$S_{\mathbf{x}}(F) = \mathcal{F}\{R_{\mathbf{x}}(\tau)\} = \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau) \exp(-j2\pi F\tau) d\tau \quad -\infty \leq F \leq \infty \quad (52a)$$

$$R_{\mathbf{x}}(\tau) = \mathcal{F}^{-1}\{S_{\mathbf{x}}(F)\} = \int_{-\infty}^{\infty} S_{\mathbf{x}}(f) \exp(j2\pi F\tau) dF \quad (52b)$$

$$S_{\mathbf{x}}(f) = \text{DTFT}\{R_{\mathbf{x}}[k]\} = \sum_{k=-\infty}^{\infty} R_{\mathbf{x}}[k] e^{-j2\pi f k} \quad (52c)$$

Properties:

$$S_{\mathbf{x}}(F) = S_{\mathbf{x}}(-F) \quad (53a)$$

$$S_{\mathbf{x}}(F) \geq 0, \quad \forall F \quad (53b)$$

$$S_{\mathbf{x}}(F) \in \mathbb{R} \quad (53c)$$

$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(-f) \quad (53d)$$

$$S_{\mathbf{x}}(f) \geq 0, \quad \forall f \quad (53e)$$

$$S_{\mathbf{x}}(f) \in \mathbb{R} \quad (53f)$$

$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(f + 1) \quad (53g)$$

Average power

$$P_{\mathbf{x}} = E[\mathbf{x}^2(t)] = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F) dF \quad (54a)$$

$$P_{\mathbf{x}} = E[\mathbf{x}^2[n]] = R_{\mathbf{x}}[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{\mathbf{x}}(f) df \quad (54b)$$

5.2 White Noise & White Gaussian Noise (WGN) Process

White noise process is SSS (WSS) process that is characterized by

$$R_{\mathbf{n}}(\tau) = \sigma^2 \delta(\tau) \quad (55a)$$

$$S_{\mathbf{n}}(F) = \sigma^2 \quad \forall F \quad (55b)$$

For WGN process, $\mathbf{n}(t) \sim N(0, \sigma^2)$,

$$R_{\mathbf{n}}(\tau) = \frac{N_0}{2} \delta(\tau) \quad (56a)$$

$$S_{\mathbf{n}}(F) = \frac{N_0}{2} \quad \forall F \quad (56b)$$

5.3 Relation Between Covariance Matrix & Auto-covariance

Given WSS process $\mathbf{x}(t)$, the corresponding correlation matrix of $\mathbf{X} = [\mathbf{x}(t_1), \dots, \mathbf{x}(t_N)]^T$ is given by

$$R_{\mathbf{X}} = E[\mathbf{X}\mathbf{X}^T] \quad (57)$$

$$R_{\mathbf{X}}(i, j) = E[X_i X_j] = R_{\mathbf{x}}(|t_i - t_j|) \quad (58)$$

6 Cross-Signal

- Cross-correlation

$$R_{\mathbf{xy}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{y}(t_2)] \quad (59)$$

- Cross-covariance

$$C_{\mathbf{xy}}(t_1, t_2) = R_{\mathbf{xy}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{y}(t_2)] \quad (60)$$

- Correlation Coefficient

$$\rho_{\mathbf{xy}}(t_1, t_2) = \frac{C_{\mathbf{xy}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{y}}(t_2, t_2)}} \quad (61)$$

6.1 WSS Cross-signal

- $\mathbf{x}(t), \mathbf{y}(t)$ are jointly WSS, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ each of them is WSS and

$$R_{\mathbf{xy}}(\tau) = E[\mathbf{x}(t)\mathbf{y}(t + \tau)] \quad (62)$$

- When $\mathbf{x}(t)$ and $\mathbf{y}(t + \tau)$ are *uncorrelated jointly WSS*, $C_{\mathbf{xy}}(\tau) = 0$.

Properties

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{yx}}(-\tau) \quad (63a)$$

$$|R_{\mathbf{xy}}(\tau)| \leq \sqrt{R_{\mathbf{x}}(0)R_{\mathbf{y}}(0)} \quad (63b)$$

$$|R_{\mathbf{xy}}(\tau)| \leq \frac{1}{2} [R_{\mathbf{x}}(0) + R_{\mathbf{y}}(0)] \quad (63c)$$

- Cross-covariance

$$C_{\mathbf{xy}}(\tau) = R_{\mathbf{xy}}(\tau) - \mu_{\mathbf{x}}\mu_{\mathbf{y}} \quad (64)$$

- Cross-PSD

$$S_{\mathbf{xy}}(f) = \mathcal{F}\{R_{\mathbf{xy}}(\tau)\} \quad (65)$$

Properties

$$S_{\mathbf{xy}}(f) = S_{\mathbf{yx}}(-f) = S_{\mathbf{xy}}^*(-f) \quad (66)$$

Correlation coefficient

$$\rho_{\mathbf{xy}}(\tau) = \frac{C_{\mathbf{xy}}(\tau)}{C_{\mathbf{xy}}(0)} \quad (67)$$

- Coherence

$$\gamma_{\mathbf{xy}}(f) = \frac{S_{\mathbf{xy}}(f)}{\sqrt{S_{\mathbf{x}}(f)S_{\mathbf{y}}(f)}} \quad (68)$$

7 LTI and WSS Random Process

Output of LTI system with impulse response $h(t)$ and random process $x(t)$,

$$y(t) = x(t) * h(t) \quad (69)$$

Average

$$m_{\mathbf{y}} = m_{\mathbf{x}} \int_{-\infty}^{\infty} h(s)ds = m_{\mathbf{x}}H(F=0) \quad (70)$$

Cross-correlation & cross-covariance:

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) \quad (71a)$$

$$C_{\mathbf{xy}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) \quad (71b)$$

$$R_{\mathbf{yx}}(\tau) = R_{\mathbf{x}}(\tau) * h(-\tau) \quad (71c)$$

$$C_{\mathbf{yx}}(\tau) = C_{\mathbf{x}}(\tau) * h(-\tau) \quad (71d)$$

$$R_{\mathbf{y}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \quad (71e)$$

$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \quad (71f)$$

Power-Spectral Density (PSD) & Cross-PSD:
Given frequency response

$$H(F) = \mathcal{F}\{h(\tau)\}, H^*(F) = \mathcal{F}\{h(-\tau)\}$$

$$S_{\mathbf{x}\mathbf{y}}(F) = S_{\mathbf{x}}(F) H(F) \quad (72a)$$

$$S_{\mathbf{y}\mathbf{x}}(F) = S_{\mathbf{x}}(F) H^*(F) \quad (72b)$$

$$S_{\mathbf{y}}(F) = S_{\mathbf{x}}(F) |H(F)|^2 \quad (72c)$$

Power of the process:

$$P_x = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F) dF \quad (73a)$$

$$P_y = R_{\mathbf{y}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F) |H(F)|^2 dF \quad (73b)$$

Same process passes two different systems

$$R_{\mathbf{y}\mathbf{z}}(\tau) = R_{\mathbf{x}}(\tau) * h_1(-\tau) * h_2(\tau) \quad (74)$$

$$S_{\mathbf{y}\mathbf{z}}(F) = S_{\mathbf{x}}(F) H_1^*(F) H_2(F) \quad (75)$$

7.1 Discrete-Time

Auto-correlation

$$H(z) = \mathcal{Z}\{h[n]\} = \frac{B(z)}{A(z)}$$

$$\mathcal{Z}\{h[n] * h[-n]\} = \frac{B(z)B(z^{-1})}{A(z)A(z^{-1})}$$

$$S_{\mathbf{x}}(z) = \mathcal{Z}\{R_{\mathbf{x}}[n]\}$$

$$h[k] * h[-k] = \sum_m h[m]h[m+k]$$

PSD

$$S_{\mathbf{x}\mathbf{y}}(z) = S_{\mathbf{x}}(z)H(z) \quad (76a)$$

$$S_{\mathbf{y}\mathbf{x}}(z) = S_{\mathbf{x}}(z)H(z^{-1}) \quad (76b)$$

$$S_{\mathbf{y}}(z) = S_{\mathbf{x}}(z)H(z)H(z^{-1}) \quad (76c)$$

Two different systems

$$R_{\mathbf{y}\mathbf{z}}[k] = R_{\mathbf{x}}[k] * h_1[-k] * h_2[k] \quad (77a)$$

$$S_{\mathbf{y}\mathbf{z}}(f) = S_{\mathbf{x}}(f)H_1^*(f)H_2(f) \quad (77b)$$

7.3 Linear Prediction

Given N samples of process $\mathbf{x}[n]$, and predictor

$$\hat{\mathbf{x}}[n+1] = \sum_{i=1}^N a_i \mathbf{x}[n-i+1], \quad (82)$$

the mean-square error is given by

$$\begin{aligned} mse &= E \left[(\mathbf{x}[n+1] - \hat{\mathbf{x}}[n+1])^2 \right] \\ &= E \left[(\mathbf{x}[n+1] - a_0 \mathbf{x}[n] - a_1 \mathbf{x}[n-1] - \dots - a_N \mathbf{x}[n-N])^2 \right] \end{aligned} \quad (83)$$

$$S_{\mathbf{y}\mathbf{z}}(z) = S_{\mathbf{x}}(z)H_1(1/z)H_2(z) \quad (77c)$$

Power of the process:

$$P_x = R_{\mathbf{x}}[0] = \int_{-1/2}^{1/2} S_{\mathbf{x}}(f) df \quad (78a)$$

$$P_y = R_{\mathbf{y}}[0] = \int_{-1/2}^{1/2} S_{\mathbf{x}}(f) |H(f)|^2 df \quad (78b)$$

Average

$$\mu_{\mathbf{y}} = \mu_{\mathbf{x}} \sum_m h[m] \quad (79)$$

For white Gaussian noise input

$$\text{Var}[\mathbf{y}[n]] = \text{Var}[\mathbf{x}[n]] \sum_m h^2[m] \quad (80)$$

7.2 Gaussian Process

A Gaussian process $\mathbf{x}(t)$ a random process that for $\forall k > 0$ and for all times t_1, \dots, t_k , the set of random variable $\mathbf{x}(t_1), \dots, \mathbf{x}(t_k)$ is jointly Gaussian.

Properties:

- WSS Gaussian process is SSS.
- Gaussian process $\mathbf{x}(t)$ that passes through LTI system, $\mathbf{y}(t) = h(t) * \mathbf{x}(t)$, is also Gaussian process that may be described by the change of expectation and auto-correlation,

$$E[\mathbf{y}(t)] = E[\mathbf{x}(t)] \int_{-\infty}^{\infty} h(s) ds \quad (81a)$$

$$= E[\mathbf{x}(t)]H(0), \quad H(F) = \mathcal{F}\{h(t)\}$$

$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \quad (81b)$$

- The resulting autocorrelation may be used for producing the correspondent covariance matrix $C_{\mathbf{Y}}$ of a multivariate Gaussian $\mathbf{Y} = [\mathbf{y}(t_1), \dots, \mathbf{y}(t_N)]^T$

and the values of a_i are given by a solution of

$$\begin{bmatrix} R_{\mathbf{x}}[0] & R_{\mathbf{x}}[1] & \cdots & R_{\mathbf{x}}[N-1] \\ R_{\mathbf{x}}[1] & R_{\mathbf{x}}[0] & \cdots & R_{\mathbf{x}}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{x}}[N-1] & R_{\mathbf{x}}[N-2] & \cdots & R_{\mathbf{x}}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} R_{\mathbf{x}}[1] \\ R_{\mathbf{x}}[2] \\ \vdots \\ R_{\mathbf{x}}[N] \end{bmatrix} \quad (84)$$

and the resulting minimum MSE is

$$mse_{min} = R_{\mathbf{x}}[0] - \sum_{i=1}^N a_i R_{\mathbf{x}}[i] \quad (85)$$

7.4 Match Filter

The goal of filter $h(t)$ is to provide maximum SNR at time $t = t_0$ for *deterministic* signal $x(t)$ and noise $n(t)$.

$$H(f) = \alpha \frac{X^*(f)}{S_n(f)} e^{-j2\pi f t_0} \quad (86)$$

$$y(t) = \frac{1}{\alpha} R_v(t - t_0) \quad (87)$$

For white noise, $n(t)$, with $S_N(f) = N_0/2$, the filter

is given by

$$H_{mf}(f) = X^*(f) e^{-j2\pi f t_0} \longleftrightarrow h_{mf}(t) = x(t_0 - t) \quad (88)$$

and the resulting maximum SNR is given by

$$SNR_{max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |X(f)|^2 df = \frac{2E_x}{N_0} \quad (89)$$

8 Different Supplementary Formulas

8.1 Derivatives

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{d}{dx} \exp[f(x)] = \exp[f(x)] \frac{d}{dx} f(x)$$

8.2 Integrals

8.2.1 Indefinite

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1$$

$$\int \exp(ax) dx = \frac{1}{a} \exp(ax)$$

$$\int x \exp(ax) dx = \exp(ax) \left[\frac{x}{a} - \frac{1}{a^2} \right]$$

$$\int x^2 \exp(ax) dx = \exp(ax) \left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right]$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \cos(ax + b) dx = \frac{\sin(ax + b)}{a}$$

8.2.2 Definite

$$\int_0^{\infty} \exp(-a^2 x^2) dx = \frac{\sqrt{\pi}}{2a}$$

$$\int_0^{\infty} x^2 \exp(-a^2 x^2) dx = \frac{\sqrt{\pi}}{4a^3}$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$\int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a)$$

8.3 Fourier Transform

8.3.1 Properties

$$\frac{d^n}{dt^n} g(t) \xleftrightarrow{\mathcal{F}} (j2\pi F)^n G(F)$$

$$g(-t) \xleftrightarrow{\mathcal{F}} G^*(F)$$

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(F) e^{-j2\pi F t_0}$$

$$g(t) e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} G(F - F_0)$$

8.3.2 Transform pairs

$$\begin{aligned}
 u(t) &\xleftrightarrow{\mathcal{F}} \frac{1}{2} \left(\frac{1}{j\pi F} + \delta(F) \right) \\
 \exp(-at)u(t) &\xleftrightarrow{\mathcal{F}} \frac{1}{a + j2\pi F} \\
 t \exp(-at)u(t) &\xleftrightarrow{\mathcal{F}} \frac{1}{(a + j2\pi F)^2} \\
 \exp(-a|t|) &\xleftrightarrow{\mathcal{F}} \frac{2a}{a^2 + 4\pi^2 F^2} \\
 \exp(-at^2) &\xleftrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{a}} \exp\left(-\frac{(\pi F)^2}{a}\right) \\
 \cos(2\pi f_a t) &\xleftrightarrow{\mathcal{F}} \frac{1}{2} [\delta(F - F_a) + \delta(F + F_a)] \\
 \sin(2\pi f_a t) &\xleftrightarrow{\mathcal{F}} \frac{1}{2j} [\delta(F - F_a) - \delta(F + F_a)] \\
 u(t+a) - u(t-a) &\xleftrightarrow{\mathcal{F}} \text{sinc}(2\pi F a) \quad \text{pulse in time} \\
 \text{sinc}(2\pi F a) &\xleftrightarrow{\mathcal{F}} u(F+a) - u(F-a)
 \end{aligned}$$

8.4 Convolution

$$x(t) * y(t) = \int_{-\infty}^{\infty} f(s)g(t-s)ds$$

$$\begin{aligned}
 x(t) * y(t) &\xleftrightarrow{\mathcal{F}} X(F)Y(F) \\
 \delta(t) * y(t) &= y(t)
 \end{aligned}$$

8.5 Trigonometry

$$\begin{aligned}
 \sin^2(\alpha) &= \frac{1}{2} (1 - \cos(2\alpha)) \\
 \cos^2(\alpha) &= \frac{1}{2} (1 + \cos(2\alpha)) \\
 \cos(\alpha) \cos(\beta) &= \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)] \\
 \sin(\alpha) \sin(\beta) &= \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)] \\
 \sin(\alpha) \cos(\beta) &= \frac{1}{2} [\sin(\alpha - \beta) + \sin((\alpha + \beta))]
 \end{aligned}$$

8.6 Matrices

$$\begin{aligned}
 \mathbf{A} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \\
 \det[\mathbf{A}] &= ad - bc \\
 \mathbf{A}^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
 \end{aligned}$$

9 Discrete-Time

Series sum

$$\begin{aligned}
 \sum_{n=0}^{N-1} r^n &= \frac{1 - r^N}{1 - r} \\
 \sum_{n=N_1}^{N_2-1} r^n &= \frac{r^{N_1} - r^{N_2}}{1 - r} \quad N_1 \leq N_2 \\
 \sum_{n=0}^{\infty} r^n &= \frac{1}{1 - r} \quad |r| < 1 \\
 \sum_{n=N_1}^{\infty} r^n &= \frac{1}{1 - r^{N_1}} \quad |r| < 1 \\
 \sum_{n=0}^{\infty} nr^n &= \frac{r}{(1 - r)^2} \quad |r| < 1
 \end{aligned}$$

9.1 Z-transforms

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

Signal	Z transform	ROC
$\delta[n]$	1	\mathbb{C}
$u[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
$-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z < 1$
$\delta[n - m]$	z^{-m}	$\mathbb{C} - \{0\}$ if $m > 0$, $\mathbb{C} - \{\infty\}$ if $m < 0$
$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z > a$
$-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z < a$

Property	Discrete Signal	Z transform	ROC
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(z) + a_2 X_2(z)$	includes $R_1 \cap R_2$
Time shift	$x[n - n_0]$	$z^{-n_0} X(z)$	R
Frequency scaling	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$ z_0 R$
Time reversal	$x[-n]$	$X(z^{-1})$	R^{-1} if $m < 0$
Convolution	$(x_1 * x_2)[n]$	$X_1(z)X_2(z)$	$R_1 \cap R_2$ (or possibly more)
Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{X(z)}{1 - z^{-1}}$	$R \cap \{ z > 1\}$

9.2 DTFT

$$X(f) = X(z) \Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\omega}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{jn\omega} d\omega.$$