תאריך הבחינה : 28.01.20
שעות הבחינה : 13:30-16:30

מבוא לאותות אקראיים
מועד א׳
ד״ר דימה בחובסקי

תשע״ט סמסטר א׳

חומר עזר - דף נוסחאות אישי (משני צדדים), מחשבון

הוראות מיוחדות :

- השאלון כולל 7 סעיפים ללא בחירה, סך הכל של 105 נקודות.

- סעיפים הם בעלי ניקוד זהה, אלא אם צוין אחרת.

- יש לציין באופן מלא וברור את שלבי הפתרון. תשובה ללא הסבר לא תתקבלנה.

- במקום בו נדרש חישוב מספרי, יש קודם לרשום את הנוסחא, ורק אח׳׳כ להציב!

- יש לציין יחידות למספרים, ובמידה וקיימות!

- כל השרטוטים יהיו גדולים, ברורים, עם סימון צירים!

- אין חובה להגיע לערך מספרי של הפונקציה $Q(x)$, במידה ומופיעה בתשובה.

השאלון כולל 11 דפים (כולל דף זה)

בהצלחה !

1   תהליכים בזמן בדיד (60 נק')

נתונים תהליכים אקראיים

$$\mathbf{x}[n] = \mathbf{w}[n] + \mathbf{w}[n-1]$$

$$\mathbf{y}[n] = \mathbf{w}[n] - \mathbf{w}[n-1],$$

כאשר $\mathbf{w}[n]$ הוא רעש לבן גאוסי בעל שונות 1.

(א) חשב $R_{\mathbf{x}}[n, n+k], C_{\mathbf{x}}[n, n+k], R_{\mathbf{y}}[n, n+k], C_{\mathbf{y}}[n, n+k]$.

האם מדובר בתהליכים סטציואנריים?

מהו הספק הממוצע של $\mathbf{x}[n], \mathbf{y}[n]$?

(ב) חשב $R_{\mathbf{xy}}[n, n+k], C_{\mathbf{xy}}[n, n+k]$.

האם $\mathbf{x}[n], \mathbf{y}[n]$ סטציאונריים במשותף?

האם $\mathbf{x}[n], \mathbf{y}[n]$ האם אורטוגונליים ו/או חסרי קורלציה ו/או בלתי תלויים? הוכח/נמק!

(ג) מהי התפלגות המשותפת של $\mathbf{x}[2], \mathbf{x}[3], \mathbf{y}[4]$?

הערה : ניתן להשתמש בתשובות לסעיפים הקודמים בצורה פרמטרית, כגון שימוש ב-

$R_{\mathbf{x}}[n, n+k], R_{\mathbf{y}}[n, n+k], R_{\mathbf{xy}}[n, n+k]$ ללא חישוב ערך מספרי.

(ד) יש לבנות חזאי אופטימלי במובן שגיאה ריבוית מינימלית, של $\mathbf{y}[n]$ מתוך $\mathbf{x}[n]$, מהצורה

$$\hat{\mathbf{y}}[n] = a_1 x[n] + a_2 x[n-1].$$

מהם הערכים של a_1, a_2?

הערה : ניתן להשתמש בתשובות לסעיפים הקודמים בצורה פרמטרית, כגון שימוש ב-

$R_{\mathbf{x}}[n, n+k], R_{\mathbf{y}}[n, n+k], R_{\mathbf{xy}}[n, n+k]$ ללא חישוב ערך מספרי.

2   קשר בין תהליכים וחיזוי לא לינארי (45 נק')

נתון חזאי ממשתנה אקראי X למשתנה אקראי Y, מהצורה $\hat{Y} = ax^2$.

(א) חשב מקדם a אופטימלי במובן שגיאה ריבועית ממוצעת מינימלית. חשב ביטוי לשגיאה ריבועית ממוצעת מינימלית המתקבלת עבור מקדם a אופטימלי.

עבור סעיפים הבאים, נתונים משתנים אקראיים X, Y מהצורה

$$X = 0.9W_1 + 0.1W_2$$
$$Y = W_1^2$$
,

כאשר $W_1, W_2 \sim U[0, 1]$ ובלתי תלויים.

(ב) חשב ערך מספרי של שגיאת חיזוי המתקבלת.

נוסחאות עזר :

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

(ג) חשב את הערך של שגיאת חיזוי לינארי מהצורה $\hat{Y} = ax + b$. השוואה את התוצאה המתקבלת לתוצאה של הסעיף הקודם ונמק.

Random Processes – Formulas

1 Distributions

1.1 Continuous

	Notation	PDF	CDF	$E[X]$	$\text{Var}[X]$
Uniform	U[a,b]	$\begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \end{cases}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
Normal	$N(\mu, \sigma^2)$	$\dfrac{1}{\sigma\sqrt{2\pi}} \exp\left[-\dfrac{(x-\mu)^2}{2\sigma^2} \right]$	$\Phi(x)$	μ	σ^2
Exponential	$Exp(\lambda)$	$\lambda \exp(-\lambda x), x \geq 0$	$1 - \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$

1.1.1 Q-function

Given $Y \sim N(\mu, \sigma^2)$

$$p(Y > y) = Q\left(\frac{y-\mu}{\sigma} \right) \tag{1a}$$

$$Q(x) = 1 - \Phi(x) \tag{1b}$$

$$Q(-x) = 1 - Q(x) \tag{1c}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{s^2}{2} \right) ds. \tag{2}$$

1.2 Discrete

	Notation	PMF	CDF	$E[X]$	$\text{Var}[X]$
Bernoulli	Ber(p)	$\begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$	$\begin{cases} 0 & x < 0 \\ 1-p & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$	p	$p(1-p)$

2 Random Variables

Definitions:

$$F_X(x) = p(X \leq x) \tag{3a}$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} \geq 0 \tag{3b}$$

$$F_X(x) = \int_{-\infty}^x f_X(p)\, dp \tag{3c}$$

$$p(a < X \leq b) = F_X(b) - F_X(a) \tag{3d}$$

$$f_X(x) \geq 0 \tag{3e}$$

$$\int_{-\infty}^\infty f_X(x)\, dx = 1 \tag{3f}$$

$$p_X[x_k] = \Pr[X = x_k] \tag{4a}$$

$$0 \leq p_X[x_i] \leq 1 \ \forall i \tag{4b}$$

$$\sum_i p_X[x_i] = 1 \tag{4c}$$

$$F_X(x) = \Pr(X \leq x), \ x \in \mathbb{R} \tag{4d}$$

$$F_X(x) = \sum_{k:x_k \leq x} p_X[x_k] \tag{4e}$$

Expectation:

$$E[X] = \begin{cases} \int_{-\infty}^{\infty} x f_X(x)\,dx \\ \sum_i x_i\, p_X[x_i] \end{cases} \tag{5a}$$

$$E[g(X)] = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \\ \sum_i g(x_i)\, p_X[x_i] \end{cases} \tag{5b}$$

$$E[aX + b] = aE[x] + b \tag{5c}$$

Variance:

$$\text{Var}[X] = E\big[(X - E[X])^2\big]$$
$$= E[X^2] - E^2[X] \tag{6a}$$
$$\text{Var}[aX + b] = a^2\,\text{Var}[X] \tag{6b}$$
$$\text{Var}[b] = 0 \tag{6c}$$

3 Two Random Variables

3.1 Joint Distributions

Definitions:

$$F_{XY}(x, y) = p(X \le x, Y \le y) \tag{7a}$$

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} \ge 0 \tag{7b}$$

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(s, p)\,dp\,ds \tag{7c}$$

$$p[x_j, y_k] = p(X = x_j, Y = y_k) \tag{8a}$$
$$F_{XY}(x, y) = p(X \le x_j, Y \le y_k) \tag{8b}$$

Expectation:

$$E[g(X, Y)] = \begin{cases} \iint g(x, y) f_{XY}(x, y)\,dx\,dy \\ \sum_i \sum_k g(x_i, y_k)\, p_X[x_i, y_k] \end{cases} \tag{9a}$$

$$E[aX + bY] = aE[X] + bE[Y] \tag{9b}$$

For **independent** random variables:

$$f_{XY}(x, y) = f_X(x) f_Y(y) \tag{10a}$$
$$p_{XY}[x_k, y_j] = p_X[x_k] p_Y[y_j] \tag{10b}$$
$$F_{XY}(x, y) = F_X(x) F_Y(y) \tag{10c}$$
$$E[XY] = E[X]E[Y] \tag{10d}$$
$$E[g_1(X)g_2(Y)] = E[g_1(X)]E[g_2(Y)] \tag{10e}$$
$$\text{Var}[aX + bY] = a^2\,\text{Var}[X] + b^2\,\text{Var}[Y] \tag{10f}$$

Marginal distribution:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dy \tag{11a}$$

$$p_X[x_k] = \sum_j p_{XY}[x_k, y_j] \tag{11b}$$

$$F_X(x) = F_{XY}(x, \infty) \tag{11c}$$
$$F_Y(y) = F_{XY}(\infty, y) \tag{11d}$$

3.2 Conditional Relations

Conditional distribution (Bayes),
for $f_X(x), f_Y(y), p_X[x_k], p_Y[y_k] > 0$:

$$f_{Y|X}(y|x) f_X(x) = f_{X|Y}(x|y) f_Y(x) = f_{XY}(x, y) \tag{12a}$$

$$p_{Y|X}[y_j|x_k] p_X[x_k] = p_{X|Y}[x_k|y_j] p_Y[y_j] = p_{XY}[x_k, y_j] \tag{12b}$$

$$F_{Y|X}(y|x) = p(Y \le y|X = x) \tag{12c}$$

$$= \int_{-\infty}^{y} f_{Y|X}(s|x)\,ds \tag{12d}$$

$$F_{Y|X}[y|x_k] = \frac{p[Y \le y_j, X = x_k]}{p_X[x_k]} \tag{12e}$$

Conditional expectation & Variance:

$$E[Y|X] = \begin{cases} \int y f_{Y|X}(y|x)\,dy \\ \sum_j y_j\, p[y_j|x_k] \end{cases} \tag{13a}$$

$$E[X] = E[E[X|Y]] = \iint y f_{Y|X}(y|x) f_X(x)\,dx\,dy \tag{13b}$$

$$\text{Var}[Y|X] = E[Y^2|X] - E^2[Y|X] \tag{13c}$$

$$\text{Var}[Y] = \text{Var}\big[E[Y|X]\big] + E\big[\text{Var}[Y|X]\big] \tag{13d}$$

3.3 Correlation, Covariance & Correlation Coefficient

- For two jointly-distributed random variables X and Y, covariance is given by

$$\text{Cov}[X, Y] = E\big[(X - E[X])(Y - E[Y])\big]$$
$$= E[XY] - E[X]E[Y]. \tag{14}$$

Main covariance properties are:

$$\text{Cov}[X, X] = Var[X] \tag{15a}$$
$$\text{Cov}[X, Y] = \text{Cov}[Y, X] \tag{15b}$$
$$\text{Cov}[X, a] = 0 \tag{15c}$$
$$\text{Cov}[aX, bY] = ab\,\text{Cov}[X, Y] \tag{15d}$$
$$\text{Cov}[X, Y] = \text{Cov}[X + a, Y + b] \tag{15e}$$
$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X, Y] \tag{15f}$$

$$\big|E[XY]\big| \le \sqrt{E[X^2]E[Y^2]} \text{ Cauchy-Schwatz} \tag{15g}$$

- Correlation coefficient (also termed as Pearson product-moment correlation coefficient) is given by

$$\rho_{XY} = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}} \qquad (16)$$

such that $|\rho_{XY}| \le 1$.

3.4   MMSE Linear Prediction

Mean square error (MSE) of predictor $\hat{Y}$ is given by

$$mse = E[(Y - \hat{Y})^2] \qquad (17)$$

Linear prediction of $\hat{Y} = ax + b$ for $X = x$ is

$$\hat{Y} = E[Y] + \frac{\text{Cov}[X, Y]}{\text{Var}[X]}(x - E[X]) \qquad (18)$$

and

$$mse_{min} = E\left[(Y - (aX + b))^2\right] = \text{Var}(Y)(1 - \rho_{XY}^2) \qquad (19)$$

When X, Y are jointly Gaussian, this prediction is optimal among **all** possible predictors

3.5   Relations

- When X and Y are *orthogonal*, $E[XY] = 0$.

- When X and Y are *uncorrelated*, $\text{Cov}[X, Y] = \rho_{XY} = 0$.

- When X and Y are *independent*, they are also uncorrelated (see also Eqs. 10).

- When X and Y are *jointly* Gaussian and uncorrelated $\Rightarrow X$ and Y are independent.

4   Multi-dimensional Random Variables

4.1   Covariance matrix

Given random vector $\mathbf{X} = (X_1, X_2, \ldots, X_N)^T$,

$$\begin{aligned}
\mathbf{C_X} &= \text{Cov}[\mathbf{X}, \mathbf{X}] = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T] \\
&= E[\mathbf{X}\mathbf{X}^T] - E[\mathbf{X}]E[\mathbf{X}]^T \\
&= \begin{bmatrix}
\text{Var}[X_1] & \text{Cov}[X_1, X_2] & \cdots & \text{Cov}[X_1, X_N] \\
\text{Cov}[X_2, X_1] & \text{Var}[X_2] & \cdots & \text{Cov}[X_2, X_N] \\
\vdots & \vdots & \ddots & \vdots \\
\text{Cov}[X_N, X_1] & \text{Cov}[X_N, X_2] & \cdots & \text{Var}[X_N]
\end{bmatrix}
\end{aligned} \qquad (20)$$

Properties:

- Symmetry

$$\mathbf{C_X} = \mathbf{C_X}^T \quad \text{Cov}[X_i, X_j] = \text{Cov}[X_j, X_i] \qquad (21)$$

- Variance of linear combination: Given vector $\mathbf{a} = (a_1, a_2, \ldots, a_N)^T$,

$$\text{Var}[\mathbf{a}^T\mathbf{X}] = \mathbf{a}^T C_\mathbf{X} \mathbf{a} \qquad (22)$$

- Linear transformation: Given linear tranformation $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$,

$$E[\mathbf{Y}] = \mathbf{A}E[\mathbf{X}] + \mathbf{b} \qquad (23a)$$
$$C_\mathbf{Y} = \mathbf{A}C_\mathbf{X}\mathbf{A}^T \qquad (23b)$$

- Uncorrelated variables

$$\mathbf{C_X} = \text{diag}\left[\text{Var}[X_1], \text{Var}[X_2], \ldots, \text{Var}[X_N]\right] \qquad (24)$$

- Cross-covariance: For two random vectors $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$, the resulting $m \times n$ cross-covariance matrix is given by

$$\begin{aligned}
\text{Cov}[\mathbf{X}, \mathbf{Y}] &= C_\mathbf{XY} \\
&= E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])^T] \\
&= E[\mathbf{X}\mathbf{Y}^T] - E[\mathbf{X}]E[\mathbf{Y}]^T \qquad (25)
\end{aligned}$$
$$C_\mathbf{YX} = C_\mathbf{XY}^T \qquad (26)$$

4.2   Bi-variate & Multivariate Normal Distribution

Joint Gaussian distribution of X_1 and X_2 with expectation $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and covariance matrix $\mathbf{C_X} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ is

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2}\right]\right) \qquad (27)$$

Multivariate Gaussian distribution of $\mathbf{X} = (X_1, X_2, \ldots, X_N)^T$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{N/2} \det[C_{\mathbf{X}}]} \exp\left\{-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T C_{\mathbf{X}}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right\}, \tag{28}$$

Properties:

- Random vector $\mathbf{X}$ is **jointly** Gaussian distributed, iff (if and only if) for all possible real vectors $\mathbf{a} = (a_1, \ldots, a_n)^T$ linear combination $Y = \mathbf{a}^T \mathbf{X}$ is Gaussian,
$$Y \sim N(\mathbf{a}^T \boldsymbol{\mu}, \mathbf{a}^T C_{\mathbf{X}} \mathbf{a}). \tag{29}$$

- If $X_1, X_2, \ldots, X_N$, $X_k \sim N(0,1), 1 \leq k \leq n$ are identically and independently distributed (IID) normal Gaussian random variables, it is termed as *normalized Gaussian random vector*. Its joint PDF is given by

$$f_{\mathbf{X}}(\mathbf{x}) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_N}(x_N) = \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{x_1^2 + x_2^2 + \ldots + x_N^2}{2}\right) = \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{\mathbf{x}^T \mathbf{x}}{2}\right) \tag{30}$$

The covariance matrix of such vector is given by identity matrix of size $N \times N, C_{\mathbf{X}} = I_n$ and its expectation is $\boldsymbol{\mu} = \mathbf{0}_{N \times 1}$.

- Linear combination of **independent** Gaussian variables, $X_i \sim N(\mu_i, \sigma_i^2)$ is Gaussian

$$\sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} (a_i \sigma_i)^2\right). \tag{31}$$

- Linear transformation – follows Eqs. (23a).

- If jointly distributed Gaussian random variables are *uncorrelated*, they are also *independent*

5 Random Processes – General Properties

- PDF & CDF

$$F_{\mathbf{x}}(x; t) = p(\mathbf{x}(t) \leq x) \tag{32a}$$

$$f_{\mathbf{x}}(x; t) = \frac{\partial}{\partial x} F_{\mathbf{x}}(x; t) \tag{32b}$$

$$p_{\mathbf{x}}[x_k; n] = p(\mathbf{x}[n] = x_k) \tag{32c}$$

- Average:

$$E[\mathbf{x}(t)] = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x; t) dx \tag{33a}$$

$$E[\mathbf{x}[n]] = \sum_i x_i p_{\mathbf{x}}[x_k; n] \tag{33b}$$

- Variance:

$$\text{Var}[\mathbf{x}(t)] = E[\mathbf{x}^2(t)] - E^2[\mathbf{x}(t)] = \sigma_{\mathbf{x}}(t) \tag{34a}$$

$$\text{Var}[\mathbf{x}[n]] = E[\mathbf{x}^2[n]] - E^2[\mathbf{x}[n]] = \sigma_{\mathbf{x}}[n] \tag{34b}$$

- Auto-correlation

$$R_{\mathbf{x}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{x}(t_2)] \tag{35a}$$

$$R_{\mathbf{x}}(t, t + \tau) = E[\mathbf{x}(t)\mathbf{x}(t + \tau)] \tag{35b}$$

$$R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(t_2, t_1) \tag{35c}$$

$$R_{\mathbf{x}}(t, t) = E[\mathbf{x}^2(t)] \tag{35d}$$

$$R_{\mathbf{x}}[n_1, n_2] = E[\mathbf{x}[n_1]\mathbf{x}[n_2]] \tag{35e}$$

$$R_{\mathbf{x}}[n, n] = E[\mathbf{x}^2[n]] \tag{35f}$$

- Auto-covariance

$$C_{\mathbf{x}}(t_1, t_2) = E\left[\{\mathbf{x}(t_1) - E[\mathbf{x}(t_1)]\}\{\mathbf{x}(t_2) - E[\mathbf{x}(t_2)]\}\right] \tag{36}$$

$$= R_{\mathbf{x}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)] \tag{37}$$

$$C_{\mathbf{x}}[n_1, n_2] = E\left[\{\mathbf{x}[n_1] - E[\mathbf{x}[n_1]]\}\{\mathbf{x}[n_2] - E[\mathbf{x}[n_2]]\}\right] \tag{38}$$

$$= R_{\mathbf{x}}[n_1, n_2] - E[\mathbf{x}[n_1]]E[\mathbf{x}[n_2]] \tag{39}$$

$$C_{\mathbf{x}}(t, t) = Var[\mathbf{x}(t)] \tag{40a}$$

$$C_{\mathbf{x}}[n, n] = Var[\mathbf{x}[n]] \tag{40b}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(t_1, t_2) = \frac{C_{\mathbf{x}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{x}}(t_2, t_2)}} \tag{41a}$$

$$|\rho_{\mathbf{x}}(t_1, t_2)| \leq 1 \tag{41b}$$

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *orthogonal*, $R_{\mathbf{x}}(t_1, t_2) = 0$.

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *uncorrelated*, $C_{\mathbf{x}}(t_1, t_2) = \rho_{\mathbf{x}}(t_1, t_2) = 0$.

- When $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ are *independent*, $R_{\mathbf{x}}(t_1, t_2) = E[\mathbf{x}(t_1)]E[\mathbf{x}(t_2)]$.

6 Wide-Sense Stationary (WSS) Process

Definition:

$$E[\mathbf{x}(t)] = E[\mathbf{x}(0)] = \mu_{\mathbf{x}} = \text{const} \tag{42a}$$

$$R_{\mathbf{x}}(t_1, t_2) = R_{\mathbf{x}}(\tau = |t_2 - t_1|), \quad \forall t_1, t_2 \tag{42b}$$

$$E[\mathbf{x}[n]] = E[\mathbf{x}[0]] = \mu_{\mathbf{x}} = \text{const} \tag{42c}$$

$$R_{\mathbf{x}}[n_1, n_2] = R_{\mathbf{x}}(k = |n_2 - n_1|), \quad \forall n_1, n_2 \tag{42d}$$

- Auto-correlation

$$R_{\mathbf{x}}(\tau) = E[\mathbf{x}(t)\mathbf{x}(t + \tau)] \tag{43a}$$

$$R_{\mathbf{x}}[k] = E\left[\mathbf{x}[n]\mathbf{x}[n + k]\right] \tag{43b}$$

Properties:

$$R_{\mathbf{x}}(-\tau) = R_{\mathbf{x}}(\tau) \tag{44a}$$

$$R_{\mathbf{x}}(0) = E[|\mathbf{x}(0)|^2] = E[|\mathbf{x}(t)|^2] \tag{44b}$$

$$Var[\mathbf{x}(t)] = C_{\mathbf{x}}(0) = \sigma_{\mathbf{x}}^2 \tag{44c}$$

$$R_{\mathbf{x}}(0) \geqslant |R_{\mathbf{x}}(\tau)| \tag{44d}$$

- Auto-covariance

$$C_{\mathbf{x}}(\tau) = R_{\mathbf{x}}(\tau) - \mu_{\mathbf{x}}^2 \tag{45a}$$

$$C_{\mathbf{x}}[k] = R_{\mathbf{x}}[k] - \mu_{\mathbf{x}}^2 \tag{45b}$$

- Correlation Coefficient

$$\rho_{\mathbf{x}}(\tau) = \frac{C_{\mathbf{x}}(\tau)}{C_{\mathbf{x}}(0)} \tag{46a}$$

$$\rho_{\mathbf{x}}[k] = \frac{C_{\mathbf{x}}[k]}{C_{\mathbf{x}}[0]} \tag{46b}$$

6.1 Power Spectral Density (PSD)

$$S_{\mathbf{x}}(F) = \mathscr{F}\left\{R_{\mathbf{x}}(\tau)\right\} = \quad -\infty \leq f \leq \infty$$
$$= \int_{-\infty}^{\infty} R_{\mathbf{x}}(\tau) \exp\left(-j2\pi F\tau\right) d\tau \tag{47a}$$

$$R_{\mathbf{x}}(\tau) = \mathscr{F}^{-1}\left\{S_{\mathbf{x}}(F)\right\} =$$
$$= \int_{-\infty}^{\infty} S_{\mathbf{x}}(f) \exp\left(j2\pi F\tau\right) dF \tag{47b}$$

$$S_{\mathbf{x}}(f) = \text{DTFT}\{R_{\mathbf{x}}[k]\} = \sum_{k=-\infty}^{\infty} R_{\mathbf{x}}[k]e^{-j2\pi fk} \tag{47c}$$

Properties:

$$S_{\mathbf{x}}(F) = S_{\mathbf{x}}(-F) \tag{48a}$$

$$S_{\mathbf{x}}(F) \geqslant 0, \ \forall F \tag{48b}$$

$$S_{\mathbf{x}}(F) \in \mathbb{R} \tag{48c}$$

$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(-f) \tag{48d}$$

$$S_{\mathbf{x}}(f) \geqslant 0, \ \forall f \tag{48e}$$

$$S_{\mathbf{x}}(f) \in \mathbb{R} \tag{48f}$$

$$S_{\mathbf{x}}(f) = S_{\mathbf{x}}(f + 1) \tag{48g}$$

Average power

$$P_{\mathbf{x}} = E\left[\mathbf{x}^2(t)\right] = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F) dF \tag{49a}$$

$$P_{\mathbf{x}} = E\left[\mathbf{x}^2[n]\right] = R_{\mathbf{x}}[0] = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{\mathbf{x}}(f) df \tag{49b}$$

6.2 White Noise & White Gaussian Noise (WGN) Process

White noise process is SSS (WSS) process that is characterized by

$$R_{\mathbf{n}}(\tau) = \sigma^2 \delta(\tau) \tag{50a}$$

$$S_{\mathbf{n}}(F) = \sigma^2 \quad \forall F \tag{50b}$$

For WGN process, $\mathbf{n}(t) \sim N(0, \sigma^2)$,

$$R_{\mathbf{n}}(\tau) = \frac{N_0}{2}\delta(\tau) \tag{51a}$$

$$S_{\mathbf{n}}(F) = \frac{N_0}{2} \quad \forall F \tag{51b}$$

6.3 Relation Between Covariance Matrix & Auto-covariance

Given WSS process $\mathbf{x}(t)$, the corresponding correlation matrix of $\mathbf{X} = [\mathbf{x}(t_1), \ldots, \mathbf{x}(t_N)]^T$ is given by

$$R_{\mathbf{X}} = E\left[\mathbf{X}\mathbf{X}^T\right] \tag{52}$$

$$R_{\mathbf{X}}(i, j) = E\left[X_i X_j\right] = R_{\mathbf{x}}\left(|t_i - t_j|\right) \tag{53}$$

7 Cross-Signal

- Cross-correlation

$$R_{\mathbf{xy}}(t_1, t_2) = E[\mathbf{x}(t_1)\mathbf{y}(t_2)] \tag{54}$$

- Cross-covariance

$$C_{\mathbf{xy}}(t_1, t_2) = R_{\mathbf{xy}}(t_1, t_2) - E[\mathbf{x}(t_1)]E[\mathbf{y}(t_2)] \tag{55}$$

- Correlation Coefficient

$$\rho_{\mathbf{xy}}(t_1, t_2) = \frac{C_{\mathbf{xy}}(t_1, t_2)}{\sqrt{C_{\mathbf{x}}(t_1, t_1)C_{\mathbf{y}}(t_2, t_2)}} \tag{56}$$

7.1 WSS Cross-signal

- $\mathbf{x}(t), \mathbf{y}(t)$ are jointly WSS, if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ each of them is WSS and

$$R_{\mathbf{xy}}(\tau) = E[\mathbf{x}(t)\mathbf{y}(t + \tau)] \tag{57}$$

- When $\mathbf{x}(t)$ and $\mathbf{y}(t + \tau)$ are *uncorrelated jointly WSS*, $C_{\mathbf{xy}}(\tau) = 0$.

  Properties

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{yx}}(-\tau) \tag{58a}$$

$$\left|R_{\mathbf{xy}}(\tau)\right| \leqslant \sqrt{R_{\mathbf{x}}(0)R_{\mathbf{y}}(0)} \tag{58b}$$

$$\left|R_{\mathbf{xy}}(\tau)\right| \leqslant \frac{1}{2}\left[R_{\mathbf{x}}(0) + R_{\mathbf{y}}(0)\right] \tag{58c}$$

- Cross-covariance

$$C_{\mathbf{xy}}(\tau) = R_{\mathbf{xy}}(\tau) - \mu_{\mathbf{x}}\mu_{\mathbf{y}} \qquad (59)$$

- Cross-PSD

$$S_{\mathbf{xy}}(f) = \mathscr{F}\left\{R_{\mathbf{xy}}(\tau)\right\} \qquad (60)$$

Properties

$$S_{\mathbf{xy}}(f) = S_{\mathbf{yx}}(-f) = S_{\mathbf{xy}}^{*}(-f) \qquad (61)$$

Correlation coefficient

$$\rho_{\mathbf{xy}}(\tau) = \frac{C_{\mathbf{xy}}(\tau)}{C_{\mathbf{xy}}(0)} \qquad (62)$$

- Coherence

$$\gamma_{\mathbf{xy}}(f) = \frac{S_{\mathbf{xy}}(f)}{\sqrt{S_{\mathbf{x}}(f)S_{\mathbf{y}}(f)}} \qquad (63)$$

8   LTI and WSS Random Process

Output of LTI system with impulse response $h(t)$ and random process $x(t)$,

$$y(t) = x(t) * h(t) \qquad (64)$$

Average

$$m_{\mathbf{y}} = m_{\mathbf{x}}\int_{-\infty}^{\infty} h(s)\,ds = m_{\mathbf{x}}H(f=0) \qquad (65)$$

Cross-correlation & cross-covariance:

$$R_{\mathbf{xy}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) \qquad (66a)$$
$$C_{\mathbf{xy}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) \qquad (66b)$$
$$R_{\mathbf{yx}}(\tau) = R_{\mathbf{x}}(\tau) * h(-\tau) \qquad (66c)$$
$$C_{\mathbf{yx}}(\tau) = C_{\mathbf{x}}(\tau) * h(-\tau) \qquad (66d)$$
$$R_{\mathbf{y}}(\tau) = R_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \qquad (66e)$$
$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \qquad (66f)$$

Power-Spectral Density (PSD) & Cross-PSD: Given frequency response $H(F) = \mathscr{F}\{h(\tau)\}$, $H^{*}(F) = \mathscr{F}\{h(-\tau)\}$

$$S_{\mathbf{xy}}(F) = S_{\mathbf{x}}(F)H(F) \qquad (67a)$$
$$S_{\mathbf{yx}}(F) = S_{\mathbf{x}}(F)H^{*}(F) \qquad (67b)$$
$$S_{\mathbf{y}}(F) = S_{\mathbf{x}}(F)H(F)H^{*}(F) = S_{\mathbf{x}}(f)\left|H(F)\right|^{2} \qquad (67c)$$

Power of the process:

$$P_x = R_{\mathbf{x}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F)\,dF \qquad (68a)$$

$$P_y = R_{\mathbf{y}}(0) = \int_{-\infty}^{\infty} S_{\mathbf{x}}(F)\left|H(F)\right|^{2}\,dF \qquad (68b)$$

$$P_x = R_{\mathbf{x}}[0] = \int_{-1/2}^{1/2} S_{\mathbf{x}}(f)\,df \qquad (68c)$$

$$P_y = R_{\mathbf{y}}[0] = \int_{-1/2}^{1/2} S_{\mathbf{x}}(f)\left|H(f)\right|^{2}\,df \qquad (68d)$$

Same process passes two different systems

$$R_{\mathbf{yz}}(\tau) = R_{\mathbf{x}}(\tau) * h_1(-\tau) * h_2(\tau) \qquad (69)$$
$$S_{\mathbf{yz}}(F) = S_{\mathbf{x}}(F)H_1^{*}(F)H_2(F) \qquad (70)$$

8.1   Z-Transform

Auto-correlation

$$H(z) = \mathscr{Z}\left\{h[n]\right\} = \frac{B(z)}{A(z)}$$

$$\mathscr{Z}\left\{h[n] * h[-n]\right\} = \frac{B(z)B(z^{-1})}{A(z)A(z^{-1})}$$

$$S_{\mathbf{x}}(z) = \mathscr{Z}\left\{R_{\mathbf{x}}[n]\right\}$$

PSD

$$S_{\mathbf{xy}}(z) = S_{\mathbf{x}}(z)H(z) \qquad (71a)$$
$$S_{\mathbf{yx}}(z) = S_{\mathbf{x}}(z)H(z^{-1}) \qquad (71b)$$
$$S_{\mathbf{y}}(z) = S_{\mathbf{x}}(z)H(z)H(z^{-1}) \qquad (71c)$$

Two different systems

$$R_{\mathbf{yz}}[k] = R_{\mathbf{x}}[k] * h_1[-k] * h_2[k] \qquad (72a)$$
$$S_{\mathbf{yz}}(f) = S_{\mathbf{x}}(f)H_1^{*}(f)H_2(f) \qquad (72b)$$
$$S_{\mathbf{yz}}(z) = S_{\mathbf{x}}(z)H_1(1/z)H_2(z) \qquad (72c)$$

8.2   Gaussian Process

A Gaussian process $\mathbf{x}(t)$ a random process that for $\forall k > 0$ and for all times $t_1,\ldots,t_k$, the set of random variable $\mathbf{x}(t_1),\ldots,\mathbf{x}(t_k)$ is jointly Gaussian (i.e. described by Eq. (28)).

Properties:

- WSS Gaussian process is SSS.

- Gaussian process $\mathbf{x}(t)$ that passes through LTI system, $\mathbf{y}(t) = h(t) * \mathbf{x}(t)$, is also Gaussian process that may be described by the change of expectation and auto-correlation,

$$E[\mathbf{y}(t)] = E[\mathbf{x}(t)]\int_{-\infty}^{\infty} h(s)\,ds \qquad (73a)$$

$$= E[\mathbf{x}(t)]H(0), \quad H(F) = \mathscr{F}\{h(t)\}$$
$$C_{\mathbf{y}}(\tau) = C_{\mathbf{x}}(\tau) * h(\tau) * h(-\tau) \qquad (73b)$$

- The resulting autocorrelation may be used for producing the correspondent covariance matrix $C_{\mathbf{Y}}$ of a multivariate Gaussian $\mathbf{Y} = [\mathbf{y}(t_1),\ldots,\mathbf{y}(t_N)]^{T}$

8.3   Linear Prediction

Given N samples of process $\mathbf{x}[n]$, and predictor

$$\hat{\mathbf{x}}[n+1] = \sum_{i=1}^{N} a_i \mathbf{x}[n-i+1], \tag{74}$$

the values of a_i are given by a solution of

$$\begin{bmatrix} R_{\mathbf{x}}[0] & R_{\mathbf{x}}[1] & \cdots & R_{\mathbf{x}}[N-1] \\ R_{\mathbf{x}}[1] & R_{\mathbf{x}}[0] & \cdots & R_{\mathbf{x}}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{x}}[N-1] & R_{\mathbf{x}}[N-2] & \cdots & R_{\mathbf{x}}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} R_{\mathbf{x}}[1] \\ R_{\mathbf{x}}[2] \\ \vdots \\ R_{\mathbf{x}}[N] \end{bmatrix} \tag{75}$$

and the resulting minimum MSE is

$$mse_{min} = R_{\mathbf{x}}[0] - \sum_{i=1}^{N} a_i R_{\mathbf{x}}[i] \tag{76}$$

9   Different Supplementary Formulas

9.1   Derivatives

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} \exp[f(x)] = \exp[f(x)] \frac{d}{dx} f(x)$$

9.2   Integrals

9.2.1   Indefinite

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1$$

$$\int \exp(ax) dx = \frac{1}{a} \exp(ax)$$

$$\int x \exp(ax) dx = \exp(ax) \left[\frac{x}{a} - \frac{1}{a^2} \right]$$

$$\int x^2 \exp(ax) dx = \exp(ax) \left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right]$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

9.2.2   Definite

$$\int_0^\infty \exp\left(-a^2 x^2\right) dx = \frac{\sqrt{\pi}}{2a}$$

$$\int_0^\infty x^2 \exp\left(-a^2 x^2\right) dx = \frac{\sqrt{\pi}}{4a^3}$$

$$\int_{-\infty}^\infty \delta(x) dx = 1$$

$$\int_{-\infty}^\infty f(x) \delta(x-a) dx = f(a)$$

9.3   Fourier Transform

9.3.1   Properties

$$\frac{d^n}{dt^n} f(t) \overset{\mathscr{F}}{\Longleftrightarrow} (j2\pi f)^n F(f)$$

$$f(-t) \overset{\mathscr{F}}{\Longleftrightarrow} F^*(f)$$

$$f(t-t_0) \overset{\mathscr{F}}{\Longleftrightarrow} F(f) e^{-j2\pi f t_0}$$

$$f(t) e^{j2\pi f_0 t} \overset{\mathscr{F}}{\Longleftrightarrow} F(f-f_0)$$

9.3.2   Transform pairs

$$u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2} \left(\frac{1}{j\pi f} + \delta(f) \right)$$

$$\exp(-at) u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{a + j2\pi f}$$

$$t \exp(-at) u(t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{(a + j2\pi f)^2}$$

$$\exp(-a|t|) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{2a}{a^2 + 4\pi^2 f^2}$$

$$\exp\left(-at^2\right) \overset{\mathscr{F}}{\Longleftrightarrow} \sqrt{\frac{\pi}{a}} \exp\left(-\frac{(\pi f)^2}{a} \right)$$

$$\cos(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2} \left[\delta(f - f_a) + \delta(f + f_a) \right]$$

$$\sin(2\pi f_a t) \overset{\mathscr{F}}{\Longleftrightarrow} \frac{1}{2j} \left[\delta(f - f_a) - \delta(f + f_a) \right]$$

9.4   Convolution

$$x(t) * y(t) = \int_{-\infty}^\infty f(s) g(t-s) ds$$

$$x(t) * y(t) \overset{\mathscr{F}}{\Longleftrightarrow} X(f) Y(f)$$

$$\delta(t) * y(t) = y(t)$$

9.5 Trigonometry

$$\sin^2(\alpha) = \frac{1}{2}\left(1 - \cos(2\alpha)\right)$$

$$\cos^2(\alpha) = \frac{1}{2}\left(1 + \cos(2\alpha)\right)$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\left[\cos(\alpha + \beta) + \cos(\alpha - \beta)\right]$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left[\sin(\alpha - \beta) + \sin\left((\alpha + \beta)\right)\right]$$

9.6 Matrices

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$\det[\mathbf{A}] = ad - bc$$

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

10 Z-transforms

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}$$

10.1 Usual Transforms

Signal	Z transform	ROC
$\delta[n]$	1	$\mathbb{C}$
$u[n]$	$\dfrac{1}{1 - z^{-1}}$	$|z| > 1$
$-u[-n-1]$	$\dfrac{1}{1 - z^{-1}}$	$|z| < 1$
$\delta[n - m]$	z^{-m}	$\mathbb{C} - \{0\}$ if $m > 0$, $\mathbb{C} - \{\infty\}$ if $m < 0$
$a^n u[n]$	$\dfrac{1}{1 - az^{-1}}$	$|z| > a$
$-a^n u[-n-1]$	$\dfrac{1}{1 - az^{-1}}$	$|z| < a$

10.2 Properties

Property	Discrete Signal	Z transform	ROC
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(z) + a_2 X_2(z)$	includes $R_1 \cap R_2$
Time shift	$x[n - n_0]$	$z^{-n_0} X(z)$	R
Frequency scaling	$z_0^n x[n]$	$X\left(\dfrac{z}{z_0}\right)$	$|z_0|R$
Time reversal	$x[-n]$	$X(z^{-1})$	R^{-1} if $m < 0$
Convolution	$(x_1 * x_2)[n]$	$X_1(z) X_2(z)$	$R_1 \cap R_2$ (or possibly more)
Time differentiation	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	$R \cap \{|z| > 0\}$
Accumulation	$\displaystyle\sum_{k=-\infty}^{n} x[k]$	$\dfrac{X(z)}{1 - z^{-1}}$	$R \cap \{|z| > 1\}$