Firm Commonality and Inference in Corporate Finance

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1 Introduction

[Empirical literature has found that, in a wide range of firm policies, from operation performance to corporate governance, from investment decisions to executive payment policies, firm decisions should not be seen as an isolated island.]

[references: casual peer effect under certain circumstances.] There are at least three channels by which one firm's decisions can be influenced by other firms. Firstly, competition. For example, in a classical Cournot competition model, a firm's output depends not only on its own characteristics but also on the actions of its rivals.

Secondly, peer effects. A firm's decisions are rarely autonomous, on the opposite, they are often influenced by the strategies and actions of peer entities within their business ecosystem.

Dougal, Parsons, and Titman (2015) finds that a firm's investment decision is highly sensitive to the investments of other firms headquartered nearby, even those in very different industries, suggesting that local agglomeration economies are important determinants of firm investment and growth.

Leary and Roberts (2014) shows that, in a large part, firms' financing decisions are responses to the financing decisions and the characteristics of peer firms. These peer effects are more important for capital structure determination than most previously identified determinants.

Gao, Ng, and Wang (2011) shows that firms exhibit conformity in their financing policies to those of geographically proximate firms and the location of corporate headquarters helps explain the cross-sectional variation of capital structure in the United States. The location effect is robust to changes in the local economic environment. The results suggest that non-economic factors, such as local culture and social interactions among corporate executives, play a significant role in influencing the corporate financial policies of firms headquartered in the same metropolitan area.

Bustamante and Frésard (2021) find a sizeable complementarity of investment among product market peers, holding across a large majority of sectors. Peer effects are stronger in concentrated markets, featuring more heterogeneous firms, and for smaller firms with less precise information. It challenges the notion that firms operate in a vacuum, emphasizing the interconnectedness of corporate decisions. It illustrates that This interdependence signifies that investment behaviors are part of a broader, interconnected financial landscape, where decisions ripple through networks of firms, underscoring the necessity of considering these relational dynamics in corporate finance.

Thirdly,

[Conclusion and start for next topic: Firms are not alone, instead, they operate in a virtual space where each firm is connected with other firms nearer or further. One can think of this as a geographic space where many lands live inside and each land has different distances from other lands. All together, these firms make up the corporate landscape just as the lands make up the geographic landscape.]

[To address the interconnectedness stemming from geographic proximity is nothing strange in the literature.] [references about spatial regressions, preferably chronically]

[The contribution of this paper: How are we different than previous work? We study a broader landscape and the comovements of firm characteristics and its consequent impact on statistical inference.]

2 Data Description

The dataset consists of two parts. The first part is the firm characteristic variables and the second part is the measurement of commonality. For firm characteristics, I use the variables that have been widely used in the existing finance literature. For commonality measurement, I follow Hoberg and Phillips (2010, 2016) because their method is sufficient as well as simple and their dataset is publicly accessible.

2.1 Data of Firm Characteristics

In this section, I provide details about the firm characteristic variables. Firm fundamentals are from Compustat from 1962 to 2022. The definition of firm characteristics can be found in [Appendix]. [Table] shows the summary statistics.

Later in section **??**, I extract a balanced panel subsample from the whole dataset. During this process, xxx firms are dropped.

2.2 Measuring Commonality

As mentioned above, the measurement of firm commonality follows Hoberg and Phillips (2010, 2016). In their papers, firm commonality is measured by the cosine similarity of firms' business descriptions in their 10K filings, hereafter referred to as the commonality score. The idea is that when firms pick similar words to describe themselves, they are likely to have similar business models and thus higher commonality. This method puts every firm at a specific virtual location in the corporate landscape and tells us the interconnectedness among firms.

For a given firm i at year t, its virtual location can be represented by a vocabulary vector P_{it}, with each element equal to 1 if firm i uses the given word in its business description at year t, and zero if it does not[1]. The vocabulary vectors are normalized to have a unit length as follows:

$$V_{it} = \frac{P_{it}}{\sqrt{P_{it} \cdot P_{it}}}$$

V_{it} can be interpreted as the virtual location of firm i at year t.

The commonality score $w_{ij,t}$ between firm i and firm j at year t is therefore defined as

$$w_{ij,t} = V_{it} \cdot V_{jt} \tag{1}$$

where $\cdot$ represents the inner product of two vectors. In fact, we can see that

$$w_{ij,t} = \frac{P_{it} \cdot P_{jt}}{\sqrt{\|P_{it}\| \times \|P_{jt}\|}} \tag{2}$$

which is the definition of cosine similarity of vocabulary vector P_{it} and P_{jt}.

[1] The whole dictionary of vocabularies is constructed by all words appear in at least one business descriptions at year t. The dictionary excludes words other than nouns or proper nouns, and also excludes the words that appear in more than 25% of all business descriptions in the given year.

[why this method is valid] The formula assigns each firm a virtual location based on its business description. Each firm has a unique location and its own neighbors in the corporate landscape based on commonality score. [Higher commonality score implies higher Comovement]

The commonality between two firms can come from different channels. Firstly, [First, competitors. Second, learning. Third, supply chain. Fourth, common factor loadings. Literature reviews.]

For simplicity, in the following analysis, I use a static version of the commonality score. The static score between firm i and firm j (w_{ij}) is taken as the median value of commonality scores $w_{ij,t}$ across all year t[2]. The simplification is appropriate since $w_{ij,t}$ do not vary a lot across t. Table 1 shows the descriptive statistics of commonality scores and within-firm-pair variance contributes to only 15% to 25%.

Table 1: Describe Statistics of Commonality Scores

This table presents the descriptive statistics of commonality scores. The results illustrate three different samples: the whole Compustat universe, constituents of Russell 3000 index, and constituents of S&P 1500 index. Column 2 to 6 display the number of firm-pair × year observations, the number of firms, the mean value of the commonality scores, the total variance of the commonality scores, the between-firm-pair variance of the commonality scores, and the percentage of variance explained by between variance, respectively.

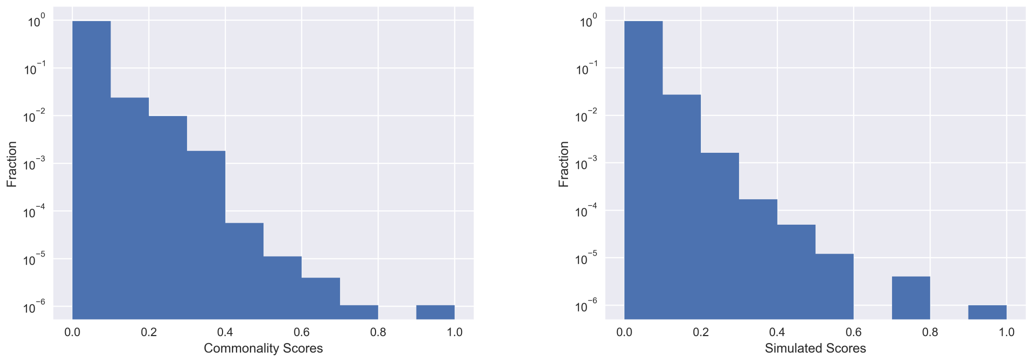
Sample range	Observations	Firms	Mean	Total Var.	Btw. Var.	Between %
Compustat	983,570,310	18633	0.0178	0.0014	0.0012	86.19%
Russell 3000	79,009,590	2766	0.0178	0.0013	0.0011	84.39%
S&P 1500	66,746,114	1929	0.0167	0.0010	0.0008	77.48%

Figure 1 displays the distribution of pair-wise commonality scores of Russell 3000 constituents. 49.30% of the firm pairs have a median commonality score of 0 and 96.44% of the firm pairs' median commonality score is under 0.1. The distribution of commonality scores decreases exponentially with the exception of 4 firm pairs with commonality scores higher than 0.9.

[Simulation gives a similar results. 2000 Firms, 1000 length dictionary, firm randomly choose 1-30 words]

A natural question is how the commonality score relates to the SIC code, which also reflects partly how firms are related. Figure 2 compares the two-digit SIC code and the

[2]Taking the mean value of commonality scores does not lead to a result change.



(a) Real Commonality Scores (b) Simulated Commonality Scores

Figure 1: Histogram of Commonality Scores

commonality score.

As one would expect, along the diagonal of the graph, firms within the same SIC industry have higher commonality scores. Particularly, some industries such as Chemicals and allied products (SIC code 28) and Depository Institutions (SIC 60) have high within-industry commonality scores at around 0.4. This does not come as a surprise, since these industries are highly specialized and firms are likely tightly connected.[reference quote]

Across SIC industries, we can see that firms with SIC code 60 to 67 have high commonality scores with each other. This is also aligned with our expectation. Finance, Insurance, Real Estate are deeply linked industries. (In fact, they are so tightly linked that we even have an abbreviation to describe them – FIRE emoji emoji)

[Last but not least, the heatmap shows that the two-digit SIC code is not a reliable measurement of firm commonality.]

[Descprition of heatmap]

3 Firm commonality reflects comovement of firm characteristics

3.1 Pair-wise correlation of firm characteristics

In this section, I illustrate that the clustering of firms' characteristics indeed can be reflected by the commonality score between them. Firms with high commonality scores with each other show similar development with respect to their characteristics.

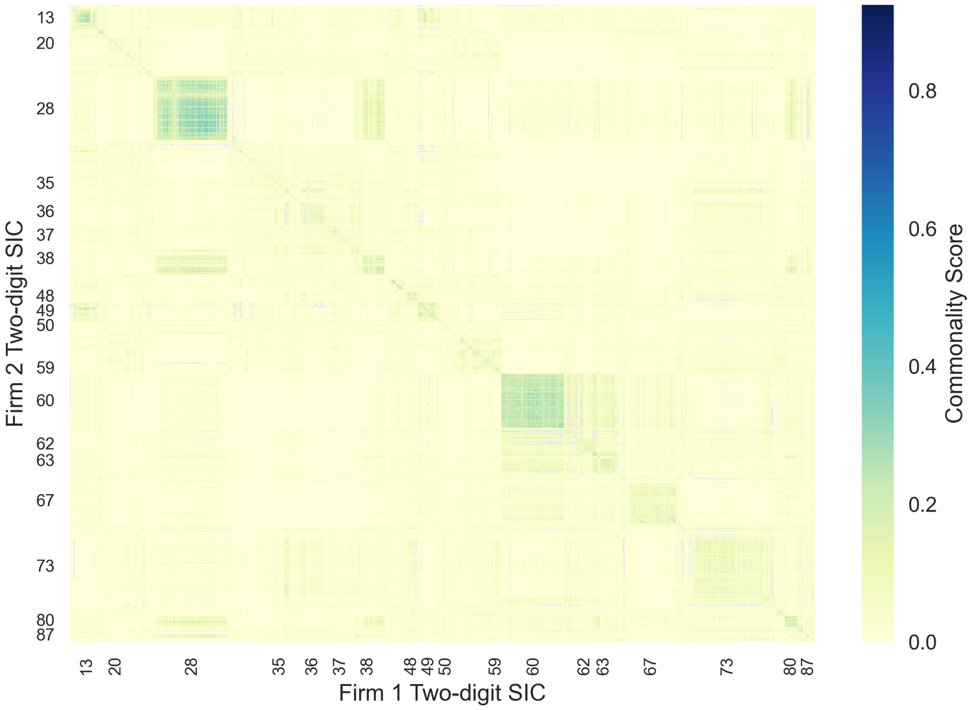
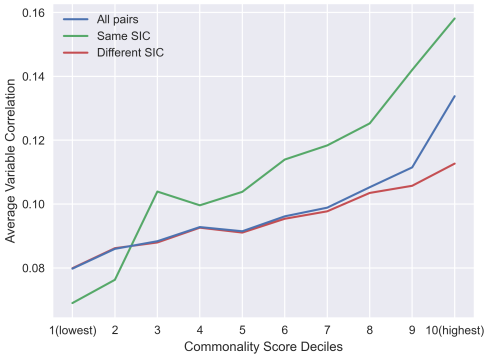


Figure 2: Heatmap of Commonality Scores

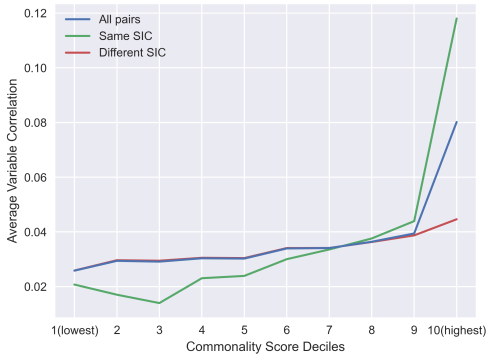
Empirically, I divide firm pairs into 10 decile groups based on their commonality scores. Then I calculate the average pair-wise correlation of firm characteristics in each decile group. Figure 3 plots the average correlation of different firm characteristics for each commonality score group.

As we can see from the blue line, the correlation of firm characteristics increases as their commonality score becomes higher. Especially, for variables that closely reflect firm decisions, such as capital expenditure, the correlation between firms is always higher than 0.2 and increases steadily with firms' commonality scores.

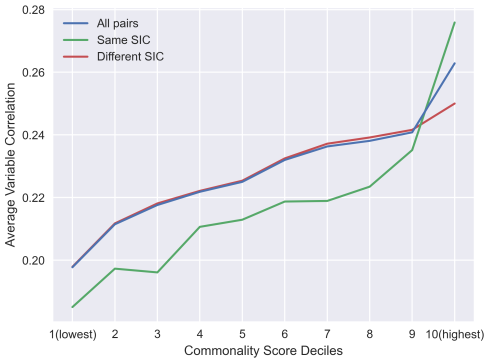
For each firm pair, I categorize it into the "same SIC" group and the "different SIC" group and then I divide them into commonality score decile groups and perform the same exercise as described in the last paragraph. This analysis yields two findings. On one hand, the SIC code is not a reliable measurement of firm commonality since firm pairs within the same industry do not always have a higher correlation of firm characteristics. On the other hand, even when firms are from different SIC industries, their commonality score mirrors their correlation in important firm characteristic variables.



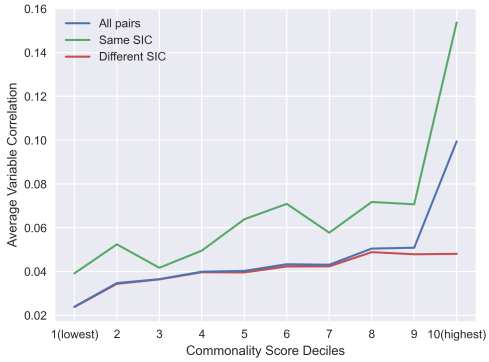
(a) Book-to-Market Ratio



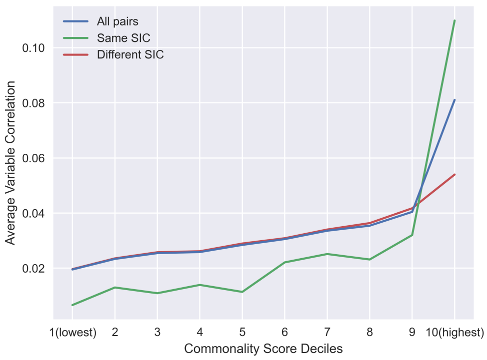
(b) Cash Holding



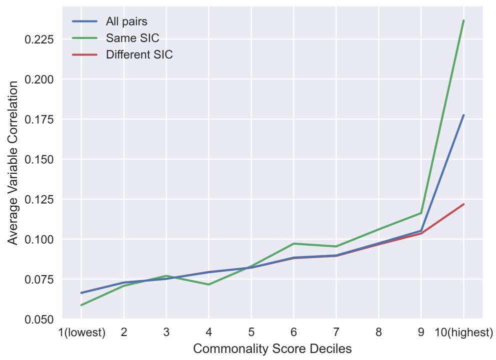
(c) Capital Expenditure



(d) Book Leverage



(e) Return of Asset



(f) Tobin's Q

Figure 3: Firm Characteristics Correlation Increases with Commonality Score

3.2 Interpretation of pair-wise correlation

So far, we have seen correlations among firm characteristics, but it remains a question of how to interpret these correlations. In this section, I illustrate this question with two methods, the standardized spatial autoregressive model, and spatial autoregressive model in the corporate landscape.

The standardized spatial autoregressive model assumes variable X_i follows a stationary Gaussian process. Specifically, the vector of variable $\boldsymbol{X}$ follows the data generating process:

$$\boldsymbol{X} \sim \mathcal{N}\left(0, \boldsymbol{\Sigma}\right)$$
$$\Sigma_{ij} = \mathrm{e}^{-\delta|s_i - s_j|}$$
$$s_i \overset{\text{iid}}{\sim} \mathcal{U}\left(0, 1\right)$$

where $\delta > 0$ measures the strength of spatial correlation.

A standardized spatial autoregressive model is a spatial equivalent of a time-series autoregressive model. Recall time-series AR(1) process follows $y_t = (1-\psi)\mu + \psi y_{t-1} + \varepsilon_t$, and its autocovariance $K_h = \mathrm{cov}(y_t, y_{t-h}) = \psi^h \sigma_y^2$ decreases exponentially with the temporal distance h.

Analogous to the time-series autoregressive model, I assume the covariance matrix of the standardized spatial autoregressive model also follows an exponential function whose base is $\mathrm{e}^{-\delta}$ and whose index is the spatial distance $|s_i - s_j|$. Figure 4(4a displays the relationship between the strength of spatial correlation δ and average pair-wise correlation. A spatial strength $\delta = 20$ implies an average pair-wise correlation of 0.1, which is equivalent to an average autocovariance of 0.1 in a time-series AR(1) process with $T = 100$ periods and $\rho = \mathrm{e}^{-\delta/(T-1)} = 0.82$

An alternative way to interpret the pair-wise correlation is to directly link the spatial autoregression coefficient ρ with the average pair-wise correlation. Specifically, we assume variable X_i follows a spatial autoregressive model:

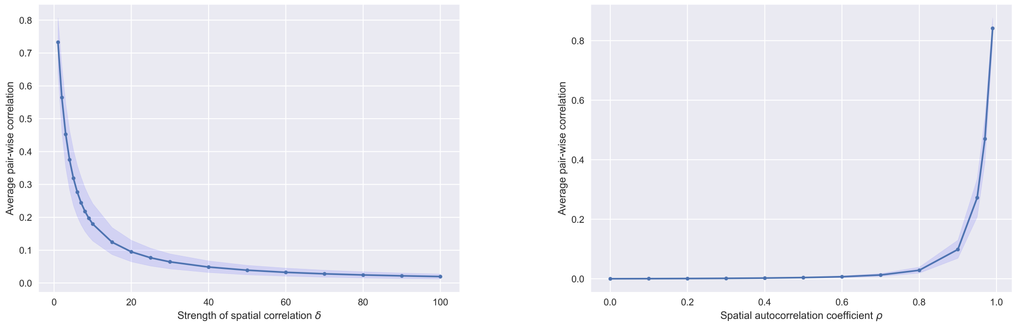
$$X_i = \sum_j \rho W_{ij} X_j + Z_i$$
$$Z_i \overset{\text{iid}}{\sim} \mathcal{N}(0, 1)$$

where ρ is the spatial autoregression coefficient with respect to commonality metrics W.

[Some explanation here]

3.3 Moran's I statistics

[Now that we have an intuitive understanding of spatial autoregression, we examine the coefficient ρ formally using Moran's I statistics[citation]. The null hypothesis is $\rho = 0$.



(a) Standardized model

(b) Corporate landscape

Figure 4: Average pair-wise correlation versus the strength of autoregression

Table 2: Moran's I statistics

This table presents Moran's I statistics of commonality scores. The mean, standard deviation, Z-score, and P-value of Moran's I statistics are calculated by conducting a random permutation procedure 1000 times.

Variable	Moran's I	Mean	Std dev	Z-score	P-value
Book-to-Market Ratio	0.0550	-0.0004	0.0012	44.6896	0.0000
Cash Holding	0.6301	-0.0004	0.0010	612.8756	0.0000
Capital Expenditure	0.1907	-0.0004	0.0010	185.1099	0.0000
Book Leverage	0.0628	-0.0004	0.0011	58.5502	0.0000
Return of Asset	0.3022	-0.0004	0.0010	294.0209	0.0000
Tobin's Q	0.2495	-0.0004	0.0011	242.9228	0.0000

Conduct permutation procedure to obtain the distribution of Moran's I statistics under the null hypothesis.]

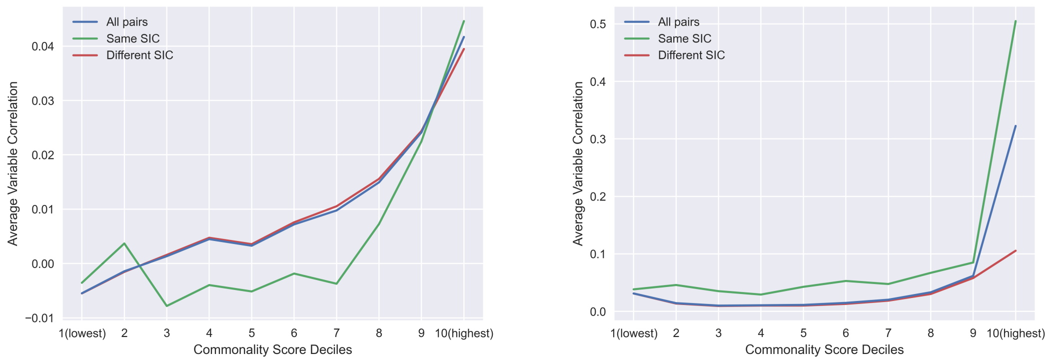
[Reprahse the following paragraph: "Global indices of spatial autocorrelation have been used to evaluate the degree to which similar observations tend to occur near each other [1–4]. Spatial autocorrelation among disease counts or incidence proportions may reflect real association between cases due to infection, or perceived association based on a spatial aggregation of similar values. Moran's I [5] is a widely used global index that measures the similarity for values in neighboring places from an overall mean value and reflects a spatially weighted form of Pearson's correlation coefficient."]

[So far, we have established the correlation between firm characteristics and commonality scores. Now we show that neglecting this relationship in the regression leads to biased results.]

4 TWFE Residual Correlates with Firm Commonality

In this section, I provide evidence that a two-way fixed effect estimator without accounting for firm commonality leads to over-rejection.

First, Figure 5 shows that the residual of a Two-Way-Fixed-Effect(TWFE) regression does not have a pattern of independent and identically distributed variables. To be more specific, had the residuals been independent and identically distributed, the lines in the graph should be horizontal and should be at the 0 level.



(a) Cash Holding (b) Capital Expenditure

Figure 5: Residuals of TWFE Estimator Correlates with Commonality Score

However, the residual shows a strong positive correlation with the commonality score. In other words, when a firm pair has a higher commonality score, these two firms' residuals also have a higher correlation. This correlation should not be overlooked. Take the capital expenditure graph as an example, as the firm commonality score increases to its highest decile, the correlation between firms' residuals increases notably to 0.5.

A vast stream of finance literature has been taking care of the correlations in residuals by using firm-clustered standard errors. However, firm-clustered standard errors only allow residuals to be correlated within a firm, not across firms. Therefore, the previous problem still exists and leads to an over-rejection. An alternative is to cluster on time dimension, however, since we only have a few decades of data, this is normally not feasible.[reference]

In the later section, Table **??** shows the over-rejection rate in detail. We will see that neither way of standard error clustering can mitigate the problem.

[If we fail to take the graph into account, then we have a bias in our estimator.]

5 Empirical Example on Capital Expenditure

6 Potential solution

6.1 Spatial regression models

6.2 Bootstrapping standard errors

7 Consistency of the OLS estimator in Panel Setting

In this section, I extend the idea from Rüttenauer (2022) and Pace and LeSage (2010) to the panel data framework. I suppose the $N \times N$ commonality score matrix $\boldsymbol{W}$ is exogenously determined, observed, and time-invariant. For simplicity of notation, I assume the dataset is already demeaned on the corresponding fixed effects. The data-generating process of $\{\boldsymbol{Y_i}\}$ follows

$$\begin{aligned}
\boldsymbol{Y_i} &= \rho \sum_j W_{ij} \boldsymbol{Y_j} + \boldsymbol{X_i}\beta + \sum_j W_{ij} \boldsymbol{X_i}\theta + \boldsymbol{\varepsilon_i} \\
\boldsymbol{\varepsilon_i} &= \lambda \sum_j W_{ij} \boldsymbol{\varepsilon_j} + \boldsymbol{u_i} \\
\boldsymbol{u_i} | \boldsymbol{X_i} &\overset{\text{iid}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma_i})
\end{aligned} \tag{3}$$

where $\boldsymbol{\Sigma_i}$ is a $T \times T$ covariance matrix of error terms u_{it} on time dimension.

Define a $NT \times 1$ vector $\boldsymbol{Y} = \text{vec}(\boldsymbol{Y_i})$ in which the first T elements $(Y_{0*N+1}, \ldots, Y_{0*N+T})$ denote the values of Firm 1 from period 1 to T, the next T elements $(Y_{1*N+1}, \ldots, Y_{1*N+T})$ denote the values of Firm 2 from period 1 to T, so on. Then equation 13 can be rewritten as

$$\boldsymbol{Y} = \left(\boldsymbol{I_{NT}} - \rho\widetilde{\boldsymbol{W}}\right)^{-1} \left(\boldsymbol{X}\beta + \widetilde{\boldsymbol{W}}\boldsymbol{X}\theta + \left(\boldsymbol{I_{NT}} - \lambda\widetilde{\boldsymbol{W}}\right)^{-1} \boldsymbol{u}\right) \tag{4}$$

where $\boldsymbol{X} = \text{vec}(\boldsymbol{X_i})$, $\boldsymbol{u} = \text{vec}(\boldsymbol{u_i})$, and $\widetilde{\boldsymbol{W}} = \boldsymbol{I_T} \otimes \boldsymbol{W}$.

Now we look at the OLS estimator. Since the data is already demeaned, the OLS estimator here is equivalent to the TWFE estimator of the undemeaned dataset.

$$
\begin{aligned}
\hat{\beta}^{\text{OLS}} &= \left(\boldsymbol{X}^{\top}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\top}\boldsymbol{Y} \\
&= \left(\boldsymbol{X}^{\top}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\top}\left(\boldsymbol{I_N}-\rho\widetilde{\boldsymbol{W}}\right)^{-1}\left(\boldsymbol{X}\beta+\widetilde{\boldsymbol{W}}\boldsymbol{X}\theta+\left(\boldsymbol{I_N}-\lambda\widetilde{\boldsymbol{W}}\right)^{-1}\boldsymbol{u}\right) \\
&= \left(\boldsymbol{X}^{\top}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\top}\widetilde{\boldsymbol{R}}_{\rho}\boldsymbol{X}\beta+\left(\boldsymbol{X}^{\top}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\top}\widetilde{\boldsymbol{R}}_{\rho}\widetilde{\boldsymbol{W}}\boldsymbol{X}\theta+\left(\boldsymbol{X}^{\top}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\top}\widetilde{\boldsymbol{R}}_{\rho}\widetilde{\boldsymbol{R}}_{\lambda}\boldsymbol{u}
\end{aligned}
\tag{5}
$$

where $\widetilde{\boldsymbol{R}}_{\rho}=\left(\boldsymbol{I_{NT}}-\rho\widetilde{\boldsymbol{W}}\right)^{-1}$ and $\widetilde{\boldsymbol{R}}_{\lambda}=\left(\boldsymbol{I_{NT}}-\lambda\widetilde{\boldsymbol{W}}\right)^{-1}$.

Because $\boldsymbol{u_i}|\boldsymbol{X_i}$ and $\boldsymbol{u_j}|\boldsymbol{X_j}$ are independently distributed and $\mathbb{E}\left[\boldsymbol{u_i}|\boldsymbol{X_i}\right]=0$, we can find that

$$
\lim_{NT\to\infty}\left(\boldsymbol{X}^{\top}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\top}\widetilde{\boldsymbol{R}}_{\rho}\widetilde{\boldsymbol{R}}_{\lambda}\boldsymbol{u}=0
$$

7.1 Independent variables are spatial independent

Assume $\boldsymbol{X_i}$ identically and independently follow standard normal distribution, according to Barry and Pace (1999) and Girard (1989), we can obtain the expectation and the variance of $\left(\boldsymbol{X}^{\top}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\top}\boldsymbol{A}\boldsymbol{X}$ for any real symmetric matrix $\boldsymbol{A}$:

$$
\begin{aligned}
\mathbb{E}\left[\boldsymbol{X}^{\top}\boldsymbol{A}\boldsymbol{X}\right] &= \frac{\operatorname{tr}(\boldsymbol{A})}{NT}\equiv\mu_A \\
\sigma^2\left[\boldsymbol{X}^{\top}\boldsymbol{A}\boldsymbol{X}\right] &= \frac{2\operatorname{tr}(\boldsymbol{A}^2)}{(NT)^2} \\
\mathbb{E}\left[\left(\boldsymbol{X}^{\top}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\top}\boldsymbol{A}\boldsymbol{X}\right] &= \mu_A \\
\sigma^2\left[\left(\boldsymbol{X}^{\top}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{\top}\boldsymbol{A}\boldsymbol{X}\right] &= \frac{2\operatorname{Var}(\lambda_A)}{NT+2}\equiv d_A^2
\end{aligned}
\tag{6}
$$

where $\operatorname{Var}(\lambda_A)$ is the "population variance" of eigenvalues of matrix $\boldsymbol{A}$, i.e. $\operatorname{Var}(\lambda_A)=\sum_i\sum_t(\lambda_{it}-\mu_A)^2/NT$ and λ_i are eigenvalues of matrix $\boldsymbol{A}$.

Recall $\widetilde{\boldsymbol{W}}=\boldsymbol{I_T}\otimes\boldsymbol{W}$. Therefore

$$
\begin{aligned}
\operatorname{tr}\left(\widetilde{\boldsymbol{R}}_{\rho}\right) &= \operatorname{tr}\left(\boldsymbol{I_T}\otimes\left(\boldsymbol{I_N}-\rho\boldsymbol{W}\right)^{-1}\right) \\
&= T\cdot\operatorname{tr}\left(\boldsymbol{I_T}\otimes\left(\boldsymbol{I_N}-\rho\boldsymbol{W}\right)^{-1}\right)\equiv T\cdot\operatorname{tr}\left(\boldsymbol{R}_{\rho}\right) \\
\operatorname{Var}\left(\lambda_{\widetilde{W}}\right) &= \frac{\sum_i\sum_t(\lambda_{it}-\mu_{\widetilde{W}})^2}{NT} \\
&= \frac{\sum_i(\lambda_i-\mu_W)^2}{N}=\operatorname{Var}(\lambda_W)
\end{aligned}
\tag{7}
$$

When N is fixed but T goes to infinite, we can immediately find that $\lim_{T\to\infty} d_A^2 \to 0$ for any real symmetric matrix $\boldsymbol{A}$. Therefore,

$$\lim_{T\to\infty} \hat{\beta}^{\text{OLS}} \xrightarrow{p} \frac{\operatorname{tr}(\boldsymbol{R_\rho})}{N}\beta + \frac{\operatorname{tr}(\boldsymbol{R_\rho W})}{N}\theta \tag{8}$$

When N goes to infinite, $\sigma^2[\boldsymbol{X_i}^\top \boldsymbol{X_i}] \to 0$ because $\boldsymbol{X_i}$ are independently and identically distributed. According to Slutsky's theorem,

$$\lim_{N\to\infty} \left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1} \boldsymbol{X}^\top \widetilde{\boldsymbol{R}}_\rho \boldsymbol{X} = \frac{\operatorname{tr}(\boldsymbol{R_\rho})}{N}$$

Then

$$\lim_{N\to\infty} \hat{\beta}^{\text{OLS}} \xrightarrow{p} \frac{\operatorname{tr}(\boldsymbol{R_\rho})}{N}\beta + \frac{\operatorname{tr}(\boldsymbol{R_\rho W})}{N}\theta \tag{9}$$

7.2 Independent variables are spatial dependent

If independent variables $\boldsymbol{X_i}$ are spatially correlated, the OLS estimator is no longer consistent in estimating the direct effects. When $\boldsymbol{X_i}|\boldsymbol{W}$ and $\boldsymbol{\varepsilon}|\boldsymbol{W}$ are not independent, there exists omitted variable bias which leads to the inconsistency of the OLS estimator.

Specifically, I assume $\boldsymbol{X_i}$ follows the data-generating process:

$$\boldsymbol{X_i} = \phi \sum_j W_{ij} \boldsymbol{X_j} + \boldsymbol{Z_i} \tag{10}$$

where $\boldsymbol{Z_i}$ is i.i.d. normal distribution. Similarly, we stack $\boldsymbol{X_i}$ and $\boldsymbol{Z_i}$ and denote $\boldsymbol{X} = \operatorname{vec}(X_i)$ and $\boldsymbol{Z} = \operatorname{vec}(Z_i)$. We can rewrite it as $\boldsymbol{X} = \widetilde{\boldsymbol{R}}_\phi \boldsymbol{Z}$ where $\widetilde{\boldsymbol{R}}_\phi = \left(\boldsymbol{I_{NT}} - \phi \widetilde{\boldsymbol{W}}\right)^{-1}$.

Following Pace and LeSage (2010), we can find that for any real symmetric matrix $\boldsymbol{A}$

$$\begin{aligned}
&\left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1} \boldsymbol{X}^\top \boldsymbol{A} \boldsymbol{X} \\
=&\left(\boldsymbol{Z}^\top \widetilde{\boldsymbol{R}}_\phi^\top \widetilde{\boldsymbol{R}}_\phi \boldsymbol{Z}\right)^{-1} \boldsymbol{Z}^\top \widetilde{\boldsymbol{R}}_\phi^\top \boldsymbol{A} \widetilde{\boldsymbol{R}}_\phi \boldsymbol{Z} \\
=&\frac{\left(\boldsymbol{Z}^\top \boldsymbol{Z}\right)^{-1} \boldsymbol{Z}^\top \widetilde{\boldsymbol{R}}_\phi^\top \boldsymbol{A} \widetilde{\boldsymbol{R}}_\phi \boldsymbol{Z}}{\left(\boldsymbol{Z}^\top \boldsymbol{Z}\right)^{-1} \boldsymbol{Z}^\top \widetilde{\boldsymbol{R}}_\phi^\top \widetilde{\boldsymbol{R}}_\phi \boldsymbol{Z}}
\end{aligned}$$

When N is fixed but T goes to infinity, equation 6 and 7 guarantee that both the variance of the nominator and denominator converges to zero. Using Slutsky's theorem, we have

$$\lim_{T\to\infty} \left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1} \boldsymbol{X}^\top \boldsymbol{A}\boldsymbol{X} = \frac{\mathrm{tr}(\widetilde{\boldsymbol{R}}_\phi^\top \boldsymbol{A}\widetilde{\boldsymbol{R}}_\phi)}{\mathrm{tr}(\widetilde{\boldsymbol{R}}_\phi^\top \widetilde{\boldsymbol{R}}_\phi)} = \frac{\mathrm{tr}(\boldsymbol{R}_\phi^\top \boldsymbol{A}\boldsymbol{R}_\phi)}{\mathrm{tr}(\boldsymbol{R}_\phi^\top \boldsymbol{R}_\phi)}$$

which implies

$$\lim_{T\to\infty} \hat{\beta}^{\mathrm{OLS}} = \frac{\mathrm{tr}(\boldsymbol{R}_\phi^\top \boldsymbol{R}_\rho \boldsymbol{R}_\phi)}{\mathrm{tr}(\boldsymbol{R}_\phi^\top \boldsymbol{R}_\phi)}\beta + \frac{\mathrm{tr}(\boldsymbol{R}_\phi^\top \boldsymbol{R}_\rho W \boldsymbol{R}_\phi)}{\mathrm{tr}(\boldsymbol{R}_\phi^\top \boldsymbol{R}_\phi)}\theta \tag{11}$$

When N goes to infinity, according to equation 6

$$\mathrm{Var}\left(\boldsymbol{Z}^\top \widetilde{\boldsymbol{R}}_\phi^\top \widetilde{\boldsymbol{R}}_\phi \boldsymbol{Z}\right) = \frac{2\mathrm{tr}\left(\widetilde{\boldsymbol{R}}_\phi^\top \widetilde{\boldsymbol{R}}_\phi \widetilde{\boldsymbol{R}}_\phi^\top \widetilde{\boldsymbol{R}}_\phi\right)}{(NT)^2} = \frac{2\mathrm{tr}\left(\boldsymbol{R}_\phi^\top \boldsymbol{R}_\phi \boldsymbol{R}_\phi^\top \boldsymbol{R}_\phi\right)}{N^2} \longrightarrow 0$$

Therefore we can apply Slutsky's theorem and obtain that

$$\lim_{N\to\infty} \hat{\beta}^{\mathrm{OLS}} = \frac{\mathrm{tr}(\boldsymbol{R}_\phi^\top \boldsymbol{R}_\rho \boldsymbol{R}_\phi)}{\mathrm{tr}(\boldsymbol{R}_\phi^\top \boldsymbol{R}_\phi)}\beta + \frac{\mathrm{tr}(\boldsymbol{R}_\phi^\top \boldsymbol{R}_\rho W \boldsymbol{R}_\phi)}{\mathrm{tr}(\boldsymbol{R}_\phi^\top \boldsymbol{R}_\phi)}\theta \tag{12}$$

7.3 Firm-clustered standard error

In this subsection, I calculate the firm-clustered standard error when error terms are spatially correlated. I assume the following data-generating process,

$$\begin{aligned} \boldsymbol{Y_i} &= \boldsymbol{X_i}\beta + \boldsymbol{\varepsilon_i} \\ \boldsymbol{\varepsilon_i} &= \lambda \sum_j W_{ij}\boldsymbol{\varepsilon_j} + \boldsymbol{u_i} \\ \boldsymbol{u_i}|\boldsymbol{X_i} &\overset{\text{iid}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma_i}) \end{aligned} \tag{13}$$

The OLS estimator here is an unbiased and consistent estimator of β.

$$\hat{\beta}^{\mathrm{OLS}} = \beta + \left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1} \boldsymbol{X}^\top \boldsymbol{\varepsilon}$$

Then we rearrange the formula as

$$\hat{\beta}^{\mathrm{OLS}} - \beta = \left(\sum_i \boldsymbol{X_i}^\top \boldsymbol{X_i}\right)^{-1} \sum_i \boldsymbol{X_i}^\top \boldsymbol{\varepsilon_i}$$

Define $\boldsymbol{\varepsilon} = \mathrm{vec}(\boldsymbol{\varepsilon_i})$, $\boldsymbol{u} = \mathrm{vec}(\boldsymbol{u_i})$, and $\boldsymbol{\Sigma} = \mathrm{diag}(\boldsymbol{\Sigma_i})$,

$$\boldsymbol{\varepsilon} = \left(\boldsymbol{I}_{NT} - \lambda\widetilde{\boldsymbol{W}}\right)^{-1}\boldsymbol{u} = \widetilde{\boldsymbol{R}}_\lambda \boldsymbol{u} \tag{14}$$

where $\widetilde{\boldsymbol{R}}_\lambda = \left(\boldsymbol{I_{NT}} - \lambda\widetilde{\boldsymbol{W}}\right)^{-1}$.

Since $\boldsymbol{u_i}$ and $\boldsymbol{u_j}$ are independent and $\boldsymbol{u_i}$ follows a normal distribution with $\boldsymbol{\Sigma_i}$ as the covariance matrix, we can obtain the variance of $\hat{\beta}^{\mathrm{OLS}}$ by

$$\mathrm{Var}(\hat{\beta}^{\mathrm{OLS}}) = \mathbb{E}\left[\left(\boldsymbol{X}^\top\boldsymbol{X}\right)^{-1}\boldsymbol{X}^\top\widetilde{\boldsymbol{R}}_\lambda^\top\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^\top\widetilde{\boldsymbol{R}}_\lambda\boldsymbol{X}\left(\boldsymbol{X}^\top\boldsymbol{X}\right)^{-1}\right]$$

However, the estimated firm-clustered standard error will be

$$\widehat{\mathrm{Var}}^{\mathrm{Firm}}(\hat{\beta}^{\mathrm{OLS}}) = \frac{N-1}{N}\left(\boldsymbol{X}^\top\boldsymbol{X}\right)^{-1}\sum_i\left(\boldsymbol{X_i}^\top\hat{\boldsymbol{\varepsilon}}_{\boldsymbol{i}}\hat{\boldsymbol{\varepsilon}}_{\boldsymbol{i}}^\top\boldsymbol{X_i}\right)\left(\boldsymbol{X}^\top\boldsymbol{X}\right)^{-1}$$

In empirical research, the autocorrelation of error terms is most likely positive but smaller than 1. I assume that all elements in $\boldsymbol{W}$ are non-negative, the maximum and principle eigenvalue of $\boldsymbol{W}$ equals 1 (if not, we normalize $\boldsymbol{W}$ by its principle eigenvalue) and $\lambda \in (0,1)$. This condition implies

$$\widetilde{\boldsymbol{R}}_\lambda^\top\boldsymbol{\Sigma}\widetilde{\boldsymbol{R}}_\lambda - \mathrm{diag}\left(\mathbb{E}\left[\hat{\boldsymbol{\varepsilon}}_{\boldsymbol{i}}\hat{\boldsymbol{\varepsilon}}_{\boldsymbol{i}}^\top\right]\right) \text{ is a positive definite matrix}$$

where $\hat{\boldsymbol{\varepsilon}}_{\boldsymbol{i}} = \boldsymbol{R}_{\lambda,ii}\boldsymbol{u_i}$.

Every elements on the main diagonal of $\mathrm{Var}(\hat{\beta}^{\mathrm{OLS}})$ is larger than $\widehat{\mathrm{Var}}^{\mathrm{Firm}}(\hat{\beta}^{\mathrm{OLS}})$. It suggests that firm-clustering standard error will underestimate the variance if spatially correlated error terms exist.

References

Barry, Ronald Paul and R Kelley Pace. 1999. "Monte Carlo estimates of the log determinant of large sparse matrices." *Linear Algebra and its applications* 289 (1-3):41–54.

Bustamante, M Cecilia and Laurent Frésard. 2021. "Does firm investment respond to peers' investment?" *Management Science* 67 (8):4703–4724.

Dougal, Casey, Christopher A Parsons, and Sheridan Titman. 2015. "Urban vibrancy and corporate growth." *The Journal of Finance* 70 (1):163–210.

Gao, Wenlian, Lilian Ng, and Qinghai Wang. 2011. "Does corporate headquarters location matter for firm capital structure?" *Financial Management* 40 (1):113–138.

Girard, A. 1989. "A fast 'Monte-Carlo cross-validation'procedure for large least squares problems with noisy data." *Numerische Mathematik* 56:1–23.

Hoberg, Gerard and Gordon Phillips. 2010. "Product market synergies and competition in mergers and acquisitions: A text-based analysis." *The Review of Financial Studies* 23 (10):3773–3811.

———. 2016. "Text-based network industries and endogenous product differentiation." *Journal of Political Economy* 124 (5):1423–1465.

Leary, Mark T and Michael R Roberts. 2014. "Do peer firms affect corporate financial policy?" *The Journal of Finance* 69 (1):139–178.

Pace, R Kelley and James P LeSage. 2010. "Omitted variable biases of OLS and spatial lag models." *Progress in spatial analysis: Methods and applications* :17–28.

Rüttenauer, Tobias. 2022. "Spatial regression models: a systematic comparison of different model specifications using Monte Carlo experiments." *Sociological Methods & Research* 51 (2):728–759.