T_5

$$p(x,\theta) = p(x) = \begin{cases} \dfrac{\theta - 1}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases} \qquad \theta > 1$$

$\vec{x}_n$ - выборка

а) $$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) =$$

$$= \cancel{\theta} (\theta - 1)^n \cdot \prod_{i=1} \frac{1}{\cancel{\theta} x_i^{\theta}}$$

$$\ln L(\theta) = n \ln(\theta - 1) - \theta \sum_{i=1}^{n} \cancel{\theta} \ln x_i$$

$$\frac{d \ln L}{d\theta} = \frac{n}{\theta - 1} - \sum_{i=1}^{n} \ln x_i = 0$$

$$\frac{n}{\sum_{i=1}^{n} \ln x_i} = \theta - 1$$

$$\tilde{\theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$$

точка

$$\frac{d^2 \ln L}{d\theta^2} = -\frac{n}{(\theta - 1)^2} < 0 \Rightarrow \max$$

Д-во рег.

1) перестановочность

$$\frac{\partial}{\partial\theta}\int_1^\infty \frac{\theta-1}{x^\theta}\,dx = \int_1^\infty \frac{x^\theta - \ln x\,(\theta-1)\,x^\theta}{x^{2\theta}}\,dx =$$

$$= \int_1^\infty x^{-\theta}\left(1+\ln x\cdot - \theta\ln x\right)dx =$$

$$= \int_1^\infty \frac{1}{x^\theta}\,dx + \int_1^\infty \frac{1-\theta}{x^\theta}\ln x\,dx =$$

$$= \frac{1}{1-\theta}x^{1-\theta}\Big|_1^\infty + 1-\theta\cdot\frac{1}{1-\theta}\;\underbrace{\frac{\ln x}{x^{\theta-1}}\Big|_1^\infty}_{=0} -$$

$$- \int_1^\infty \frac{1}{x^\theta}\,dx = \frac{1}{1-\theta}x^{1-\theta}\Big|_1^\infty - \frac{1}{1-\theta}x^{1-\theta}\Big|_1^\infty = 0$$

2) $I(\theta) = \frac{1}{(\theta-1)^2}$ непр. и > 0 $\forall\theta > 1$

1), 2) $\Rightarrow$ модель регулярна

~~...~~ рег.

$$J_0 = \frac{\partial^2}{\partial\theta^2}\int_1^\infty p\,dx = \underbrace{\cancel{\int_1^\infty -\frac{\theta}{x^{\theta+1}}\,dx}}_{J_1} + J_1 +$$

$$+\int_1^\infty \ln x\cdot x^{-2\theta}\left(-x^{\theta} - \ln x\cdot x^{\theta}\cdot(1-\theta)\right)dx =$$

$$= \cancel{-\theta\cdot\left(-\frac{1}{\theta}\right)x^{-\theta}\Big|_1^\infty} + J_2 + J_1$$

$$J_2 = -\int_1^\infty \ln x\, x^{-\theta}\,dx - \int_1^\infty \ln^2 x\cdot x^{-\theta}(1-\theta)\,dx =$$

$$= \underbrace{-\frac{1}{1-\theta}\ln x\cdot x^{-\theta+1}\Big|_1^\infty}_{=0} + \int_1^\infty \frac{1}{1-\theta}\cdot\frac{1}{x^{\theta}}\,dx$$

$$\underbrace{-(1-\theta)\int_1^\infty \frac{\ln^2 x}{x^{\theta}}\,dx}_{-(1-\theta)J_3} = \frac{1}{-\theta+1}\int_1^\infty \frac{1}{x^{\theta}}\,dx$$

$$J_3 = \int_1^\infty \frac{\ln^2 x}{x^{\theta}}\,dx = \underbrace{\frac{1}{1-\theta}\ln^2 x\cdot x^{-\theta+1}\Big|_1^\infty}_{=0} -$$

$$-2\cdot\frac{1}{1-\theta}\int_1^\infty \frac{\ln x}{x^{\theta}}\,dx = \frac{-2}{(1-\theta)^2}\ln x\, x^{-\theta+1}\Big|_1^\infty +$$

$$+\frac{2}{(\theta-1)^2}\int_1^{\infty}\frac{1}{x^{\theta}}\,dx=\frac{2}{(1-\theta)^3}\,x^{-\theta+1}\Big|_1^{\infty}=\frac{2}{(1-\theta)^3}$$

$$J_2=\frac{1}{(1-\theta)^2}-\frac{2}{(1-\theta)^2}$$

$$J_0=0-\frac{1}{(1-\theta)^2}+J_1$$

$$J_1=\int_1^{\infty}-\ln x\cdot\frac{1}{x^{\theta}}\,dx=-\frac{1}{1-\theta}\ln x\,x^{1-\theta}\Big|_1^{\infty}+$$

$$+\frac{1}{1-\theta}\int_1^{\infty}\frac{1}{x^{\theta}}\,dx=\frac{1}{(1-\theta)^2}$$

$J_0=0 \Rightarrow$ перестановочность $\left.\begin{array}{l}\\ \end{array}\right\}$ сильная

$\oplus$ модель регулярная $\Big\}$ регулярность б)

$$F(x) = \int_1^x \frac{\theta-1}{x^\theta}dx = \frac{1-\theta}{1-\theta}x^{-\theta+1} - 1 =$$

$$= -x^{1-\theta} + 1 =$$

$$F(x) = \frac{1}{2}$$

$$1 - x^{1-\theta} = \frac{1}{2}$$

$$x^{1-\theta} = \frac{1}{2} \qquad 2^{\frac{1}{\theta-1}} = f(\theta) = x_{med}$$

<u>Модель сильно рег.</u>

$$\sqrt{n}\,\frac{g(\tilde{\theta}) - g(\theta)}{\sigma(\theta)} \sim N(0,1)$$

ОМП асимптотическ для медианы

$$\nabla f = e^{\ln 2 \cdot \frac{1}{\theta-1}} = -\frac{\ln 2}{(\theta-1)^2}\, 2^{\frac{1}{\theta-1}}$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right] = \frac{1}{(\theta-1)^2} - \frac{2}{\theta-1}M[\ln x] + M[\ln^2 x]$$

$$\ln p = \ln(\theta-1) - \theta \ln x$$

$$\left(\frac{\partial \ln p}{\partial \theta}\right)^2 = \left(\frac{1}{\theta-1} - \ln x\right)^2 = \frac{1}{(\theta-1)^2} - \frac{2\ln x}{\theta-1} + \ln^2 x$$

$$M[\ln x] = \int_1^\infty \ln x \, \frac{\theta-1}{x^\theta} \, dx =$$

$$= \frac{(-\theta+1)}{(1-\theta)} \int_1^\infty \ln x \, d\, x^{1-\theta} =$$

$$= \ln x \, x^{1-\theta} \Big|_1^\infty - \int_1^\infty x^{-\theta} dx = \frac{1}{-\theta+1} x^{-\theta+1} \Big|_1^\infty =$$

$\downarrow$ 0

$$= \frac{1}{1-\theta}$$

$$M[\ln^2 x] = \int_1^\infty \ln^2 x \, \frac{\theta-1}{x^\theta} dx =$$

$$= \ln^2 x \cdot x^{1-\theta} \Big|_1^\infty - 2\int_1^\infty \ln x \cdot x^{-\theta} dx =$$

$$= \frac{-2}{-\theta+1} \ln x \, x^{-\theta+1} \Big|_1^\infty + 2 \frac{1}{(-\theta+1)} \int_1^\infty x^{-\theta} dx =$$

$$= \frac{2}{(1-\theta)^2}$$

$$I = \frac{1}{(\theta-1)^2}$$

$$\sigma = \sqrt{\frac{\ln^2 2}{(\theta-1)^4} 2^{\frac{2}{\theta-1}} (\theta-1)^2} = \frac{\ln 2}{\theta-1} 2^{\frac{1}{\theta-1}}$$

$$\sigma(\theta) \xrightarrow{P} \sigma(\tilde{\theta})$$

$$-1{,}96 < \frac{2^{\frac{1}{\tilde\theta-1}} - x_{med}}{\frac{\ln 2}{\tilde\theta-1} 2^{\frac{1}{\tilde\theta-1}}} \sqrt{n} < 1{,}96$$

$$\frac{-1{,}96 \cdot \frac{\ln 2}{\tilde\theta-1} 2^{\frac{1}{\tilde\theta-1}}}{\sqrt{n}} + 2^{\frac{1}{\tilde\theta-1}} < x_{med} < \frac{1{,}96 \frac{\ln 2}{\tilde\theta-1}}{\sqrt{n}} + 2^{\frac{1}{\tilde\theta-1}}$$

медиана

c) $g(\theta) = \theta \quad g(\tilde\theta) = \tilde\theta$

$$\sqrt{n}\,\frac{g(\tilde\theta) - g(\theta)}{\sigma(\theta)} \sim N(0,1)$$

$$I = \frac{1}{(\theta-1)^2}$$

$$\sigma = (\theta-1) \rightarrow \sigma(\tilde\theta) = \tilde\theta - 1$$

$$-1{,}96 < \frac{\tilde\theta - \theta}{\tilde\theta-1}\sqrt{n} < 1{,}96 \qquad \tilde\theta = \frac{n}{\sum \ln x_i} + 1$$

$$-\frac{1{,}96}{\sqrt{n}}(\tilde\theta-1) + \tilde\theta < \theta < \frac{1{,}96}{\sqrt{n}}(\tilde\theta-1) + \tilde\theta$$