T_4

$\xi \sim R(\theta, 2\theta) \qquad p(x) = \frac{1}{\theta}\{(\theta, 2\theta)\}$

а) $\vec{x}_n$ - выборка

ОММ

$$\alpha_1 = M[\xi] = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_{\theta}^{2\theta} x \frac{1}{\theta}\, dx =$$

$$= \frac{3}{2}\theta = \bar{x}$$

$$\tilde{\theta} = \frac{2}{3}\bar{x}$$

1) несмещ.

$$M[\tilde{\theta}] = \frac{2}{3} M[\xi] = \theta$$

2) состоят

$$\cancel{= \frac{4}{9} D[\xi] = \frac{4}{9}\left(\frac{7}{3}\theta^2 - \frac{9}{4}\theta^2\right) = \frac{1}{27}\theta^2}$$

$$\cancel{M[\xi^2] = \int_{\theta}^{2\theta} x^2 \cdot \frac{1}{\theta}\, dx = \frac{7}{3}\theta^2}$$

$$D[\tilde{\theta}] = \frac{4}{9} D[\bar{x}] = \frac{4}{9} \cdot \frac{1}{n^2} \cdot n\, D[\xi] =$$

$$= \frac{4}{9n}\left(\frac{7}{3}\theta^2 - \frac{9}{4}\theta^2\right) = \boxed{\frac{1}{27n}\theta^2} \xrightarrow[n \to \infty]{} 0$$

+ несмещ.

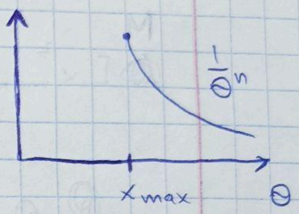
$\boxed{\text{состоят}}$ по дост. условию

ОМП

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = \frac{1}{\theta^n}\{(\theta \le x_i \le 2\theta)\}$$

$L(\theta) \to \sup$ $\qquad 2\theta > x_{max}$

$$\tilde{\theta} = \frac{x_{max}}{2}$$



1) несмещ.

$$M[\tilde{\theta}] = \frac{1}{2} M[x_{max}]$$

$$g(y) = n(F(y))^{n-1} \cdot F'(y) = \qquad F(x) = \int_{\theta}^{x} \frac{1}{\theta}\, dt =$$

$$= n\left(\frac{x-\theta}{\theta}\right)^{n-1} \frac{1}{\theta} \qquad = \frac{x-\theta}{\theta}\{(\theta, 2\theta)\}$$

$$M[x_{max}] = \int_{\theta}^{2\theta} x\, n\left(\frac{x-\theta}{\theta}\right)^{n-1} \frac{1}{\theta}\, dx =$$

$$= \frac{n}{\theta}\int_{\theta}^{2\theta} (\theta + x - \theta)\left(\frac{x-\theta}{\theta}\right)^{n-1} dx =$$

$$= \frac{n}{\theta}\int_{\theta}^{2\theta} \theta \left(\frac{x-\theta}{\theta}\right)^{n} dx + \theta n \int_{\theta}^{2\theta} \left(\frac{x-\theta}{\theta}\right)^{n-1} d\left(\frac{\theta - x}{\theta}\right) =$$

$$= \frac{n\theta}{n+1}\left(\frac{x-\theta}{\theta}\right)^{n+1}\Bigg|_{\theta}^{2\theta} + \frac{n\theta}{n}\left(\frac{x-\theta}{\theta}\right)^{n}\Bigg|_{\theta}^{2\theta} =$$

$$= \frac{n\theta}{n+1} + \theta =$$

$$= \frac{2n\theta + \theta}{n+1} = \theta\,\frac{(2n+1)}{n+1} \quad \boxed{\text{смещ.}}$$

$$\tilde{\theta}' = \frac{n+1}{(2n+1)}\,\theta = \frac{n+1}{(2n+1)}\,x_{max}$$

$$D\left[\frac{n+1}{2n}\,x_{max}\right] = \frac{(n+1)^2}{4}\cdot\frac{1}{n}\,D[x_{max}]$$

$$M[x_{max}^2] = \int_{\theta}^{2\theta} x^2\,\frac{n}{\theta}\left(\frac{x-\theta}{\theta}\right)^{n-1} dx =$$

$$= \frac{\theta x^2}{n}\,\frac{n}{\theta}\left(\frac{x-\theta}{\theta}\right)^{n}\Bigg|_{\theta}^{2\theta} - 2\int_{\theta}^{2\theta} \theta x\,\frac{1}{\theta}\left(\frac{x-\theta}{\theta}\right)^{n} dx =$$

$$= 4\theta^2 - \theta$$

$$- 2x\theta\cdot\frac{1}{n+1}\left(\frac{x-\theta}{\theta}\right)^{n+1}\Bigg|_{\theta}^{2\theta} + 2\int_{\theta}^{2\theta} \frac{\theta}{\theta}\cdot\frac{\theta^2}{(n+1)}\left(\frac{x-\theta}{\theta}\right)^{n+1} dx$$

$$= 4\theta^2 - \frac{4\theta^2}{n+1} + 2\theta^2\,\frac{1}{(n+1)(n+2)} =$$

$$= \theta^2\left(\frac{4(n+1)(n+2) - 4(n+2) + 2}{(n+1)(n+2)}\right)$$

$$D[\tilde{\theta}'] = \frac{(n+1)^2}{(2n+1)^2}\left[\theta^2\left(\frac{4n^2+12n+8-4n-8+2}{(n+1)(n+2)}\right) + \theta^2\,\frac{4n^2}{(n+1)^2}\right] \xrightarrow[n\to\infty]{} 0$$

$$D[\tilde{\theta}'] = \frac{n\theta^2}{(2n+1)^2(n+2)} \quad + \text{несм}$$

$\tilde{\theta}'$ $\boxed{\text{состоят}}$ по дост. усл.

c) $I = \frac{1}{\theta^2}$

$\Delta[\tilde\theta_{ОММ}] = \frac{1}{27n}\theta^2$

$\Delta[\tilde\theta'_{ОМП}] = \frac{n}{(2n+1)^2(n+2)}\theta^2$

$\frac{1}{27}\theta^2 > 0$

$\Rightarrow \tilde\theta'_{ОМП}$ ассимп. эффективнее

d) доверительный интервал

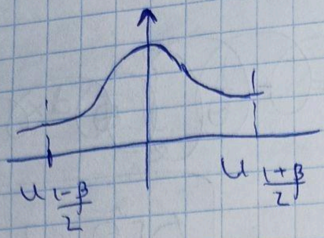
Точный

$f(\theta, \vec{x}_n)$

$$\frac{\bar{x} - \frac{3}{2}\theta}{\theta \frac{1}{\sqrt{12}}}\sqrt{n} \sim N(0,1)$$

$$f(\vec{x}_n, \theta) = \frac{\bar{x} - \frac{3}{2}\theta}{\sqrt{\frac{1}{12}}\,\theta}\sqrt{n} \sim N(0,1)$$

не зависит от θ



$\parallel$

$u_{0,025}$ $\qquad$ $u_{0,975} = 1,96$

~~$-1 \cdot 0,728 < \theta < \bar{x} \cdot 0,615$~~

~~б) асимптотический дов интервал~~

$$-1,96 < \left(\frac{\bar{x}}{\theta}\sqrt{12} - 3\sqrt{3}\right)\sqrt{n} < 1,96$$

$$3\sqrt{3} - \frac{1,96}{\sqrt{n}} < \frac{\bar{x}\sqrt{12}}{\theta} < 3\sqrt{3} + \frac{1,96}{\sqrt{n}}$$

$$\frac{x\sqrt{12}}{3\sqrt{3} + \frac{1,96}{\sqrt{n}}} < \theta < \frac{\bar{x}\sqrt{12}}{3\sqrt{3} - \frac{1,96}{\sqrt{n}}}$$

точный дов. интервал

е) асимптотический дов интервал по ОММ

$$\tilde{\theta} = \frac{2}{3}\bar{x} = g(\tilde{\alpha}) = \frac{2}{3}\tilde{\alpha}_1$$

$$\theta = g(\alpha) = \frac{2}{3}\alpha_1$$

$$\sigma(\alpha) = \sqrt{\frac{2}{3}(\alpha_2 - \alpha_1^2)\frac{2}{3}} = \frac{2}{3}\sqrt{\alpha_2 - \alpha_1^2}$$

$$\cancel{\sqrt{n}\,\frac{g(\tilde{\alpha}) - g(\alpha)}{\frac{2}{3}}} \qquad \sqrt{n}\,\frac{\frac{2}{3}\tilde{\alpha}_1 - \frac{2}{3}\alpha_1}{\frac{2}{3}\sqrt{\alpha_2 - \alpha_1^2}} \rightsquigarrow N(0,1)$$

$$\underbrace{\sqrt{n}\,\frac{\tilde{\theta} - \theta}{\frac{3}{2}\sqrt{\alpha_2 - \alpha_1^2}}}_{\substack{\| \\ f(\vec{x}_n, \theta) \\ \xrightarrow{F} N(0,1)}} \cdot \underbrace{\frac{\sqrt{\alpha_2 - \alpha_1^2}}{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}}_{\xrightarrow{P} 1} \rightsquigarrow N(0,1)$$

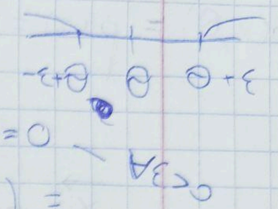
по лемме Слуцкого

$$-1{,}96 < \sqrt{n}\,\frac{\tilde{\theta} - \theta}{\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}\,\frac{3}{2}} < 1{,}96$$

$$-\frac{1{,}96\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}{\sqrt{n}\,\frac{2}{3}} + \tilde{\theta} < \theta < \frac{1{,}96\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}{\sqrt{n}\cdot\frac{2}{3}} + \tilde{\theta}$$

Исследование $\frac{x_{max}}{2} = \tilde{\Theta}_2$ на несмещ

$\forall \varepsilon > 0 \quad P(|\tilde{\Theta}_2 - \Theta| \geq \varepsilon) \to 0, \ n \to \infty$



$P(x_{max}/2 \geq \Theta + \varepsilon) = P(x_{max} \geq 2\Theta + \varepsilon \cdot 2) = 0$

$P(x_{max}/2 \leq \Theta - \varepsilon) = P(x_{max} < 2\Theta - 2\varepsilon) =$

$= \Phi(2(\Theta - \varepsilon)) = \left(\frac{2\Theta - 2\varepsilon - \Theta}{\Theta}\right)^n = \left(\frac{\Theta - 2\varepsilon}{\Theta}\right)^n =$

$= \left(1 - 2\frac{\varepsilon}{\Theta}\right)^n \underset{n \to \infty}{\longrightarrow} 0$

$\Theta < 2\Theta - 2\varepsilon < 2\Theta$

$0 < \varepsilon < \frac{\Theta}{2}$

Если $\varepsilon \geq \frac{\Theta}{2}$, то $2(\Theta - \varepsilon) \leq \Theta \Rightarrow$

$\Rightarrow \Phi(2(\Theta - \varepsilon)) = 0 \to 0 \quad n \to \infty$