T3

$$p(x) = \begin{cases} e^{-\frac{x}{\theta}}, & x \geq 0 \\ 0, & x < 0 \end{cases} \cdot (\theta)^{-1} \quad \theta > 0$$

$F(x) = (1 - e^{-\frac{x}{\theta}}) \{[0, +\infty)\}$

$n = 3$

1) $\tilde{\theta}_1 = \bar{x}$ 2) $\tilde{\theta}_2 = x_{(2)}$

а) несмещенность

$M[\tilde{\theta}_1] = \theta$?

$M[\tilde{\theta}_1] = M[\frac{1}{n} \sum x_i] =$

$= \frac{1}{n} \cdot n \, M\xi = M\xi = \theta^2 \cdot \frac{1}{\theta} = \theta$ $\boxed{\text{несмещ}}$

$$M\xi = \int_{-\infty}^{\infty} x p(x) dx = \frac{1}{\theta} \int_0^{\infty} x e^{-\frac{x}{\theta}} dx =$$

$$= \frac{1}{\theta} \int_0^{\infty} x e^{-\frac{x}{\theta}} (-\theta) d\left(-\frac{x}{\theta}\right) =$$

$$= -\theta x e^{-\frac{x}{\theta}} \Big|_0^{\infty} + \int_0^{\infty} \frac{\theta}{\theta} e^{-\frac{x}{\theta}} dx = -\frac{\theta^2}{\theta} e^{-\frac{x}{\theta}} \Big|_0^{\infty} = \theta$$

$\tilde{\theta}_2 = x_{(2)}$

$M[\tilde{\theta}_2] = \theta$?

$\cancel{M[x_{(2)}] =}$

$x_{(2)} \sim h(t) = n\,p(t)\,C_{n-1}^{k-1} F(t)^{k-1} (1-F(t))^{n-k} =$

$= 3\frac{e^{-\frac{x}{\theta}}}{\theta} \cdot 2\left(1-e^{-\frac{x}{\theta}}\right) e^{-\frac{x}{\theta}} =$

$= \frac{6}{\theta} e^{-\frac{2x}{\theta}} \left(1-e^{-\frac{x}{\theta}}\right)$

$M[\tilde{\theta}_2] = \int_{-\infty}^{\infty} \cancel{x\,p(x)\,d}\; x\,h(x)\,dx =$

$= \int_0^{\infty} x \frac{6}{\theta} e^{-\frac{2x}{\theta}} \left(1-e^{-\frac{x}{\theta}}\right) dx =$

$= \frac{6}{\theta}\int_0^{\infty} x e^{-\frac{2x}{\theta}} - \frac{6}{\theta}\int_0^{\infty} x e^{-\frac{3x}{\theta}} dx = \frac{3}{2}\theta - \frac{2}{3}\theta = \frac{5}{6}\theta$ $\boxed{\text{смещ}}$

$\tilde{\theta}_2' = \frac{6}{5}\tilde{\theta}_2$ $\boxed{\text{исправление оценки}}$

б) сравнение эффективности

$D[\tilde{\theta}_1] = D[\frac{1}{n}\sum x_i] = \frac{1}{n^2} D[\sum x_i] =$

$= \frac{1}{n} D[\xi_i] = \boxed{\frac{1}{n}\theta^2}$

$D\xi = M[\xi^2] - M\xi^2 = 2\theta^2 - \theta^2 = \theta^2$

$M[\xi^2] = \int_0^\infty \frac{x^2}{\theta} e^{-\frac{x}{\theta}} dx = -x^2 e^{-\frac{x}{\theta}} \Big|_0^\infty + 2\int_0^\infty x e^{-\frac{x}{\theta}} dx =$

$= 2\theta^2$

$D[\tilde{\theta}'_2] = M[\tilde{\theta}'^2_2] - (M\tilde{\theta}'_2)^2 = \boxed{\frac{13}{25}\theta^2}$

$M[\tilde{\theta}'^2_2] = \int_0^\infty x^2 \frac{36}{25} \cdot 6 e^{-\frac{2x}{\theta}} (1 - e^{-\frac{x}{\theta}}) dx =$

$= \frac{54}{25}\theta^2 - \frac{16}{25}\theta^2 = \frac{38}{25}\theta^2$

$D[\tilde{\theta}_1]$ $\quad$ $D[\tilde{\theta}_2']$

$\frac{1}{3}\theta^2 < \frac{13}{25}\theta^2$

$\frac{25}{75} < \frac{39}{25\cdot 3}$ $\quad$ $\tilde{\theta}_1$ эффективнее $\tilde{\theta}_2'$

в) доказательство регулярности модели

$\cancel{\frac{\partial}{\partial\theta}\int\limits_{-\infty} p(x)dx}$

1) перестановочность

$$\frac{\partial}{\partial\theta}\int\limits_0^\infty e^{-\frac{x}{\theta}}\cdot\frac{1}{\theta}\,dx = \int\limits_0^\infty\left(-\frac{1}{\theta^2}e^{-\frac{x}{\theta}} + \frac{x}{\theta^3}e^{-\frac{x}{\theta}}\right)dx =$$

$$= \int\limits_0^\infty \frac{x-\theta}{\theta^3}e^{-\frac{x}{\theta}}\,dx = \frac{1}{\theta^3}\left(\int\limits_0^\infty x e^{-\frac{x}{\theta}}dx - \theta\int\limits_0^\infty e^{-\frac{x}{\theta}}dx\right) =$$

$$= \frac{1}{\theta^3}\left(\underbrace{-\theta x e^{-\frac{x}{\theta}}\Big|_0^\infty}_{0} + \theta\int\limits_0^\infty e^{-\frac{x}{\theta}}dx - \theta\int\limits_0^\infty e^{-\frac{x}{\theta}}dx\right) = 0 \ \checkmark$$

$$\ln p = -\frac{x}{\theta} - \ln\theta$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial\theta}\right)^2\right] = \int\limits_0^\infty\left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2 e^{-\frac{x}{\theta}}\cdot\frac{1}{\theta}\,dx =$$

$$= \int_0^\infty \frac{(x-\theta)^2}{\theta^4 \cdot \theta} e^{-\frac{x}{\theta}} dx =$$

$$= \frac{1}{\theta^5} \int_0^\infty (\underset{\uparrow J_1}{x^2} - \underset{\uparrow J_2}{2x\theta} + \underset{\uparrow J_3}{\theta^2}) e^{-\frac{x}{\theta}} dx = + \frac{\theta^3}{\theta^5} = + \frac{1}{\theta^2}$$

$\boxed{I}$ непр и >0

$\Downarrow$

модель регулярна

$$J_1 = \int_0^\infty x^2 e^{-\frac{x}{\theta}} dx = 2\theta^3$$

$$J_2 = -2\theta^3$$

$$J_3 = +\theta^3$$

1) $\tilde{\theta}_1 = \overline{x}$

$D[\tilde{\theta}_1] = \frac{\theta^2}{3}$ огранич. на $\forall$ компакте

$M[\tilde{\theta}_1] = \theta$ несмещ.

$\underbrace{\text{модель регулярная}}$

$\Downarrow$

$\tilde{\theta}_1$ регулярная оценка

(по дост. условию)

2) $\tilde{\theta}_2^1 = \frac{6}{5} x_{(2)}$

$D[\tilde{\theta}_2^1] = \frac{4}{15}\theta_2$ огранич на $\forall$ компакте

$M[\tilde{\theta}_2^1] = \theta$ несмещенная

модель регулярная

$\underbrace{\qquad}$

$\tilde{\theta}_2^1$ - регулярная оценка (по дост. услов.)

$I(\theta) = \frac{1}{\theta^2}$

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$D[\tilde{\theta}_1] \geq \frac{1}{nI(\theta)}$

$\frac{\theta^2}{3} = \frac{\theta^2}{3} \Rightarrow \tilde{\theta}_1$ - эффективная оценка

$D[\tilde{\theta}_2^1] = \frac{13}{25}\theta^2$

$\frac{13}{25}\theta^2 \geq \frac{\theta^2}{3}$ ничего нельзя сказать