4 Chaos Theory: A Brief Tutorial and Discussion

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PREFACE

The chapter begins with a quote from Sir Francis Bacon. This preface can do no better than another quote from that famous gentleman. It was a singular occasion where he departed somewhat from his customary modesty, the place being the Proem to "The Advancement of Learning," around 1600–1605.

> For myself, I found that I was fitted for nothing so well as for the study of Truth; as having a mind nimble and versatile enough to catch the resemblances of things (which is the chief point), and at the same time steady enough to fix and distinguish their subtler differences; as being gifted by nature with desire to seek, patience to doubt, fondness to meditate, slowness to assert, readiness to reconsider, carefulness to dispose and set in order; and as being a man that neither affects what is new nor admires what is old, and that hates every kind of imposture. So I thought my nature had a kind of familiarity and relationship with Truth.

This statement clothes William K.Estes like the proverbial glove. That nature has arguably led to some of the most valuable contributions to psychology in the 20th century. It has further guided several generations of experimental and theoretical psychologists through to their Ph.D. degrees and fruitful careers in research and teaching. There is little doubt that Kay C. Estes has been an inextricable part of the 'Estes equation' over much of their professional lives (see, e.g., Estes in Lindzey, 1989). Together they form a dynamic duo in psychology if there ever was one. For these reasons, the subsequent chapter is dedicated to both of them. May they continue their tradition of profound involvement and general invigoration of the science of psychology for many years to come.

"Those who aspire not to guess and divine, but to discover and to know . . . who propose to examine and dissect the nature of this very world itself, to go to facts themselves for everything." From "The Interpretation of Nature, or the Kingdom of Man", by Sir Francis Bacon, circa 1603.

This quotation from Francis Bacon emphasizes at once the preeminence of empiricism in the gaining of knowledge. It goes beyond that in hinting at the determinism to be found in observation of nature. One need not "guess" if one has gone to the root of the matter through direct inspection of natural causes. That kernel of deterministic causality has been an integral part of science now for hundreds of years. Even following the advent of mathematical probability theory in the 18th century, chance was long thought to be a manifestation of ignorance, rather than something inherent in the phenomena themselves. Quantum mechanics changed all that, but Albert Einstein, for one, refused to accept its finality of explanation, though admiring its predictive powers and consistency (e.g., see Pais, 1982, chap. 25).

With chaos theory, we have a new enigma and paradox: disorder, jumble, and confusion from certainty itself. It is apparent that fundamental issues arise in epistemology, philosophy of science, but also questions as to whether it can play a productive role in scientific theory and observation. This short essay can only touch the surface, but seeks to interest the reader in further rigorous study of the subject.

Chaos theory appears destined to be a major trend in the sciences. Perhaps it already is, to judge from its visibility in the serious specialized, the semi-serious, and the sheerly popular media. It is getting a rather slow start in the behavioral sciences, but signs abound that it is beginning to take off. When any such movement gets underway, there is some danger of faddism. If readers have any doubts that chaos theory, like many such "new items," can develop faddistic characteristics, I refer them to a recent article in one of the (usually) more responsible popular science magazines (McAuliffe, 1990).[1] The latter piece of scientific journalism saw, of course, chaos theory as being the new Nirvana, especially in biomedical and biopsychological spheres, but also managed to entangle deterministic chaotic processes with ordinary stochastic processes; in fact, it was entangled with just about anything that smacked of some degree of randomness.

Chaos theory would seem to be, on the face of it, made to order for psychology. It is deterministic and therefore seems to be strongly causal, yet often acts in a random, if not bizarre fashion. Is this not like human behavior? What is its real promise in psychology? How has it been used so far? How can it be employed to best advantage? Can we avoid the pitfalls that pandering to such vogues may augur?

[1]Considering the sorry state of science education and the dearth of graduating scientists in the United States, one can only applaud the efforts of popular science outlets to bring rather esoteric scientific information to the general public. However, there does seem to be a point at which the distortion may be so great that more harm than benefit is likely.

One imposing obstacle in attempting answers to these and similar questions lies in the area of concern itself. Chaos theory is at base a highly technical, if not abstruse subject, embedded as it is in the field of nonlinear dynamics. Even physicists and engineers well versed in many aspects of dynamics have often found themselves unprepared for the kinds of mathematics required for the serious study of chaotic processes. Thus, the content of traditional courses in differential equations and systems theory simply did not cover those techniques. Indeed, even mathematics majors typically received only a modicum of the quantitative knowledge base pertinent to that end. Engineers and physicists frequently never took a course in real analysis or point-set typology—absolutely indispensable for deep understanding of this topic. Differential geometry and topology and algebraic topology form the foundation of the most general treatment of chaos theory and nonlinear dynamics. Chaos theory is now part and parcel of nonlinear dynamics and as such is seeing day-by-day advances in a myriad of fields.

On top of all this is the fact that there is no firm accepted definition of "chaos theory." There are several aspects that can be included in the definition or omitted, although some aspects seem more inherent than others.

This chapter provides the reader with an introduction to the main concepts of chaos theory. I attempt to strike a balance between the purely metaphorical or verbal presentation and a truly rigorous foundation. The former is often all that can be found in the popular literature, whereas to completely fulfill the latter might require a year or so of concentrated study. Several sources for additional inquiry are offered along the way. A good place to begin might be the historically and epistemologically exciting book by Gleick, "Chaos: Making a New Science" (1987) and the intriguing "Order out of Chaos" (1984) by Prigogine. The reader interested in a more rigorous background could time-share Gleick with a more technical introduction such as Devaney's "An Introduction to Chaotic Dynamic Systems" (1986) and/or Barnsley's "Fractals Everywhere" (1988). This pair can be rounded out with some strong visual intuition through the visual mathematics series on dynamics by Abraham and Shaw (1985) or computer simulations through Kocak's PHASER (1989), or Devaney and Keen's edited volume on chaos and fractals (1989). Each of these has received excellent reviews although not all mathematical reviewers have been ecstatic with the mathematical part of the history offered by Gleick, (not to mention a mild disparagement of psychology; see p. 165). Some types or regions of application, especially in psychology, are mentioned and commented on. Chaos theory is compared with certain other major trends that we have seen over the past few decades, and its ultimate implications for science and psychology considered.

CHAOS AS PART OF DYNAMICS

It is fascinating, and not always appreciated in the general scientific community, that chaos can arise in extremely simple dynamic systems. To make this a little more evident we need the notion of a difference or differential equation. Difference equations are appropriate in discrete time whereas differential equations apply in continuous time. Much of our discussion can be handled in discrete time. A set of difference equations representing N inputs and N outputs, and of the first order, can be written

$$x_1(n+1) = f_1(x_1(n), x_2(n), \ldots, x_N(n), n)$$
$$x_2(n+1) = f_2(x_1(n), x_2(n)), \ldots x_N(n), n)$$
$$\vdots$$
$$x_N(n+1) = f_N(x_1(n)), x_2(n)), \ldots, x_N(n), n)$$

Note that the state of each dependent variable, say (x_2), can in general be an arbitrary function of the states of all the dependent variables at time n as well as the time itself, n. The order of a difference equation stands for the number of differences across time (akin to the kth derivative in calculus) needed to determine the dynamics of a system. Alternatively, it says that at time n, the nth state of the system can depend on the states, moving k steps in time back from the present. A single input, single output system that is of order k can be generally written

$$x(n+k) = f(x(n+k-1), x(n+k-2), \ldots, x(n), n).$$

Systems with N input and N outputs or N inputs and M outputs, and of order $k > 1$, can be similarly constructed. The sequence of states of the system across time, n, is referred to as the "trajectory." In the latter case, with a single input and output, the trajectory is denoted by the variable x(n). In the former case, it is designated by the N-dimensional vector.

Often, the number of subsystems N (corresponding to the number of equations) is referred to as the dimension of the system. A single input, single output system thus has dimension 1. As noted, if we can write $x(n + 1) = f(x(n),n)$, then it is only of the first order. The fact that n is present as a separate independent variable in the function allows the evolution to be a direct function of time itself; for instance, the system might age, warm up, or change in some other fashion. The general expression f(x(n),n) also suggests that the state transition can depend on an input, so that the state of the system at time n + 1 depends not just on the state at time n but also on the current input. For instance, perhaps $y(n + 1) = f(x(n)) = \log(x(n)) + \sin(n)$. The second term on the right hand side represents a sinusoidal input function. The first term shows that the next state is also a nonlinear (in this case), logarithmic function of the current state. In such a case, we say that the system is being "forced" by the input function sin(n). If it is possible to write $x(n + 1) = f(x(n))$ alone, then we say we are dealing with the "free response" of the system, or that the system is "autonomous." The latter case is not only important in its own right (e.g., it covers such examples as a freely swinging pendulum) but it also turns out that the free response of a system is typically required to predict what happens when an input is added.

A further move toward simplicity takes place when we require that the function "f" be a linear operation: $f(x) = a(n)x$. Note that "a" can in general still directly depend on time n. If so, we have a "time dependent system." If not, we have a "time invariant system." In a time invariant system, the next state transition depends only on the current state and input. It cannot depend on time outside of that. For instance, this constraint rules out purely time related changes to a system, such as fatigue, warm-up effects, or the like, that are not explicit functions of the state and input.

Thus, we have $x(n + 1) = ax(n) + u(n)$ as the simplest linear system with input u. It is first order, linear and time invariant. If u is always equal to 0, we have an autonomous system. Now it happens that there are chaotic systems that require very little transformation of this type of system, that is, they possess only a little bit of "nonlinearity," and therefore enjoy some entitlement to the claim of simplicity. However, we shall see that their behavior can be quite complicated indeed.

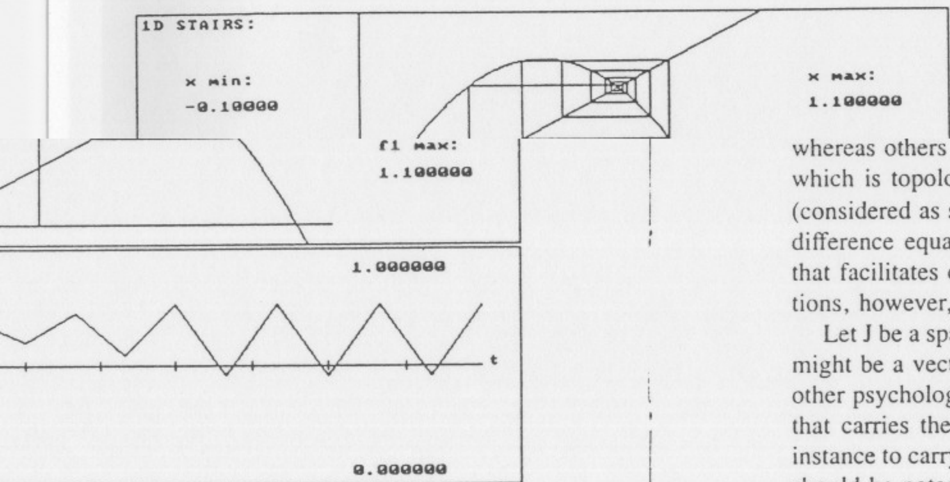
Perhaps the most elemental chaotic system is the innocent looking discrete logistic system:

$$x(n+1) = ax(n)[1 - x(n)] = ax(n) - ax^2(n)$$

Its states at time n, x(n), can take on any positive or negative value, in general. However, the interesting behavior occurs on the interval $I = [0,1]$. Note that the only nonlinearity is the squaring of the state at time n. It is even time invariant, not to mention first order and one-dimensional! Even this elemental system exhibits many of the exotic facets of more complex, multidimensional systems. It has been much employed to represent population growth. Note for instance that if at time 0, x(0) starts less than 1 but greater than 0, and the growth parameter "a" is greater than $1 + x(n)$, then growth occurs from time n to n + 1. For certain growth parameter values of "a," growth is smooth and approaches a fixed point, also known as an equilibrium point, as a limit. A fixed point is a state that remains unchanged under the dynamics of the system. Hence if a state is a fixed point, the system remains in that state forever. In this system, the fixed points are $x = 0$ and $x = (a - 1)/a$. Observe that only the second fixed point depends on the parameter "a." With two or more dependent variables, hence with dimension greater than or equal to 2, such systems can describe a wide variety of prey–predator interactions (e.g., see Beltrami, 1987).

My subsequent presentation owes much to Robert Devaney's excellent text, "Introduction to Chaotic Dynamical Systems" (1986). It focuses on discrete time systems, thereby avoiding some of the heavy topological machinery. Yet it includes analysis, in a quite rigorous fashion, of some of the best studied 1- and 2-dimensional chaotic systems. Of course, no text can do everything and this one leaves out discussion of the relationship of chaotic behavior to stochastic processes, for instance, ergodicity. A rigorous study of the latter requires some measure theory, which Devaney considered too far afield for this particular book. To be sure, this aspect is of interest to many groups, including psychologists, who might like to have a deterministic account of the randomness they regularly observe in human behavior. The present chapter contains some discussion of this side of chaotic activity.

We continue now with the logistic system already mentioned, $x(n + 1) = ax(n)[1 - x(n)]$. There is a quite nice way to plot successive states in a discrete system that illustrates exactly what is happening, at least in the short term. First we draw $y = f(x)$ as a function of x. This is, of course, the state transition function or map. The top of Fig. 4.1 shows the logistic function when $a = 3.41$



...art shows a trajectory of the logistic system with ... 3.41 and starting at x(0) = .30. The stair-step ...e trajectory is used with t = 15, that is with 15 ...art plots the evolution of the states as a direct ...tions). The abscissa in the bottom graph is ar... ...0 intervals, but the 15 interactions can be identi... ...troughs. The ordinate in the bottom graph runs

and it is clearly quadratic in appearance. This and the other illustrations in this paper were plotted using PHASER (Kocak, 1989), a simulator and animator for elementary dynamic systems. It is inexpensive and worthwhile as a teaching device and for numerical investigation of nonlinear dynamic systems (see Townsend, 1991 for a review of PHASER).

Picking a starting point, in this case $x(0) = .3$, we draw a vertical line from the x-axis to where it intersects the function. This is, of course, the next value of the system state, that is, of $x(1) = .72$. We here round off to two decimals, as a matter of course. We next move horizontally over to the diagonal that represents the new value of $x(1)$ and prepares for the next transition to $x(2)$. The value of $x(2)$ is located by again drawing a vertical line, in this case down, to the function $f(x)$, and again over to the diagonal, and so on, yielding $x(2) = .68$. We refer to this type of graph as a "stair-step" function. This figure shows the first 15 iterations from our start point. It seems to be going into a cycle; that is, toward periodic behavior, and in fact that is what is occurring. This is supported by the bottom part of Fig. 4.1, which shows the time course of the states as a function of discrete time. Observe that the last few iterations are cycling back and forth between the same two values, indicating a period of 2. These values are approximately $x = .54$ (the point on the curve at the lower right hand corner of the biggest square) and $x = .78$ (the point on the curve at the upper right hand corner of the same square). In fact, with the same parameter $a = 3.41$, the identical period will be approached from almost any starting point (initial condition).

As already mentioned, there is no universally accepted definition of chaos. Some demand only the existence of a "strange attractor" (more on that later) whereas others require more. We proceed to follow Devaney's line of thought, which is topological, that is, oriented toward behavior of states of the system (considered as sets of points in a space) under the transformation defined by the difference equation. One may alternatively take a measure theoretic approach that facilitates connections with concepts of randomness. Before giving definitions, however, we do need a modicum of notation and concepts.

Let J be a space of points, which represent states of the system. For instance, J might be a vector space representing state of knowledge, emotional status, and other psychological characteristics, expressed numerically. Let f be the function that carries the state from time n to time n + 1. If f is applied k times, for instance to carry the state at time 1 to time 1 + k, then we write $f^k[x(1)]$, where it should be noted that the superscript k is not a power, but rather the number of iterations of f. As an example, in the logistic system, $x(n + 1) = f[x(n)] = ax(n)[1 - x(n)]$, or expressed simply in terms of the position of the state, $f(x) = ax[1 - x]$.

The first criterion for chaos in our scheme is "topological transitivity," which says that if an observer looks at two sets of states, even if they do not overlap, the function f will always carry them together sometime in the future. More directly we meet the following definition.

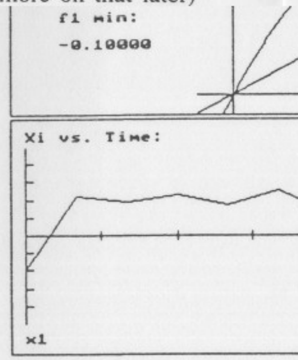


FIG. 4.1. The upper p... growth parameter a = ... method of plotting th... iterations. The lower ... function of time (itera... bitrarily divided into 1... fied by the peaks and ... from 0 to 1.

Definition 1: Topological Transitivity and Dense Trajectories

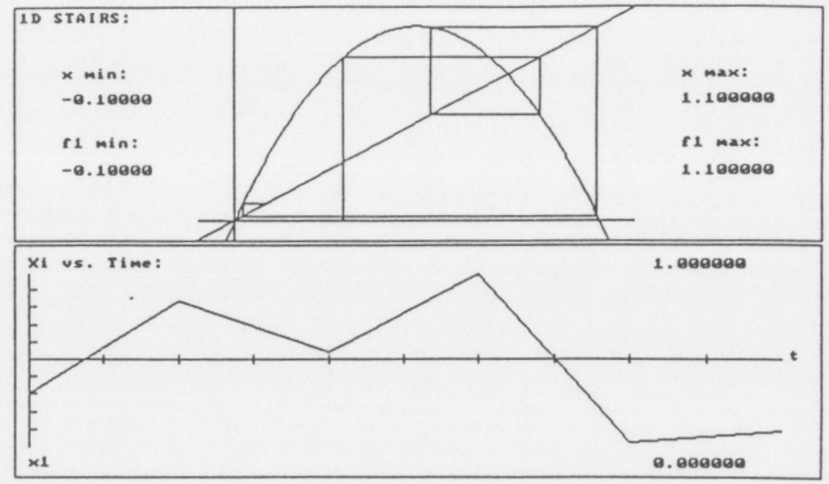
A function f mapping the space J to J is said to be topologically transitive if for any pair of open sets U,V contained in J, there exists a positive integer k such that $f^k(U) \cap V$ ‡ ϕ.#

The symbol ϕ represents the empty set. In order to be rigorous, we have retained the specification of open sets, which intuitively simply means one can always find another such set completely surrounding any point in the original set. An example of an open set is the open interval (0,1), which contains all numbers between 0 and 1 but not those endpoints themselves. All open sets composed of real numbers can be represented as unions of any number of such open intervals (a,b) and finite numbers of intersections of such unions. Expressly, Definition 1 indicates that iteration of the states represented by the set U always leads to at least one of them ending up in the set V. One outcome of this definition is that it is impossible to divide up the space into segregated regions that exhibit quite distinct kinds of behavior.

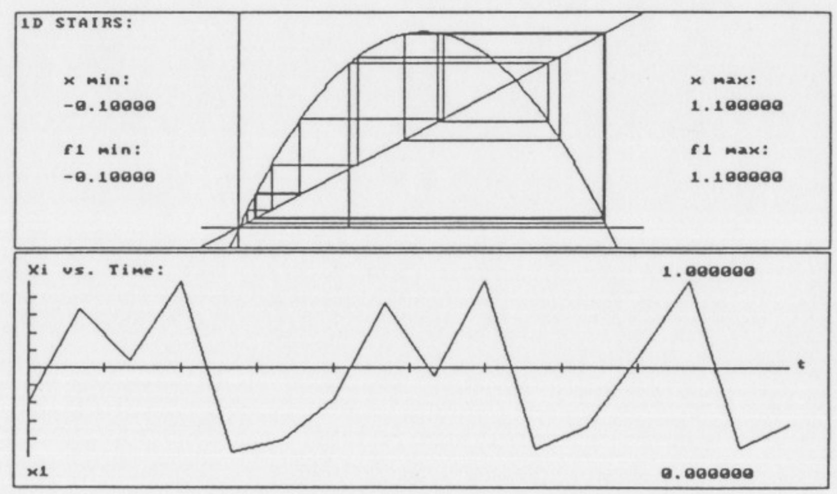
Another rather abstract notion we need is that of "density" of points in a certain set. A subset of points is dense in a set containing it, if any point in the larger set is arbitrarily close to a point in the original smaller set. For example, the set of rational numbers is dense in the set of all numbers. It turns out that in several chaotic systems of importance, obeying Definition 1 implies that there exists at least one trajectory through the space J that is dense in J. That is, there is an actual trajectory that sooner or later comes arbitrarily close to any point in the space! Of course, there may be, and usually will be, other non-overlapping trajectories in J that are not dense. In some ways, the possibility of a dense trajectory begins to give an impending sense of bizarreness—perhaps greater than the original, more general definition itself.

Let us take a new parameter value for "a" in the logistic system, a = 4, which is known to lead to chaotic behavior. Figures 4.2a, 4.2b, 4.2c show first 5 then 15 then 100 iterations, to give some idea of the evolution of the system, each time starting at the same state x(0) = .3. Both the stair-function as well as the time function as follows suggest that most areas of the interval from 0 to 1 are being visited when we allow time to become large.

The next definition is likely more familiar to readers who have exposed themselves to nontechnical but still objective articles appearing in *American Scientist, Science,* and other reputable general science magazines. It refers to a type of instability that arises in chaotic systems. Consider a system whose "initial state" we specify at time t = 0 and then observe as it evolves further over time. In chaotic systems, if the initial state is changed by any amount, no matter how small, the result in the future will be a series of states that are totally different from those associated with the first initial condition. This has come to be known as the "butterfly effect," the name suggested by meteorologist and early "chaotician" Edward Lorenz in a AAAS paper in 1979: "Predictability:



A



B

FIG. 4.2. The logistic system is now illustrated along the same lines as Fig. 4.1, but now with a chaotic growth parameter a = 4. Again we start at x(0) = .30. Part A shows 5 iterations, Part B shows 15 iterations, and Part C shows 100 iterations.

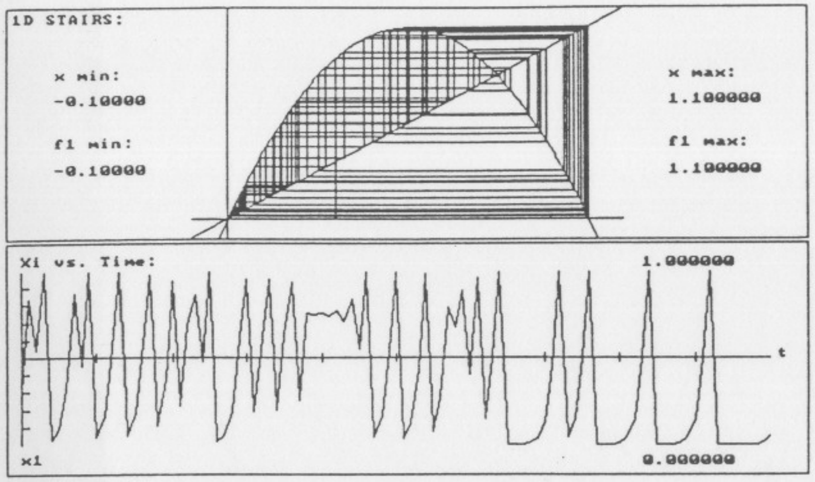


FIG. 4.2. (*Cont.*) C

Can the flap of a butterfly's wings in Brazil set off a tornado in Texas?" The idea is that the world might turn out very differently just according to whether the butterfly is flapping his or her respective wings (the initial condition) or not. The technical name is "sensitive dependence on initial conditions." In order to prepare for the definition, let "d" stand for a positive number. "N" will represent a small neighborhood (technically, this will be an open set; e.g., on the line of numbers, it might just be an open interval surrounding a point) around a state (i.e., point of J, called x). Finally let us represent the distance between the separate n-fold iterations of two points x and y, by $D[f^n(x), f^n(y)]$, where D is a metric imposed on the space J. Then we have

Definition 2: The Butterfly Effect

The function f, mapping J to J has sensitive dependence on initial conditions if there exists $d > 0$ such that for any x contained in J and any neighborhood N of x, there exists a y contained in N also, as well as an $n \geq 0$ such that $D[f^n(x), f^n(y)] > d$.#

We can translate this to mean that there is some distance greater than zero, such that no matter where we start out (i.e., x), and no matter how little we perturb it (i.e., x to some y within an arbitrarily small distance from x), if we go far enough into the future (i.e., at least n iterations), we will always find that x and y have evolved to states that are at least distance d apart. It should be noted that not all y within that small distance of x need to have trajectories that separate from x's trajectory in the future, but there must exist at least one such y for any

small neighborhood of x. Figure 4.3, computed with PHASER (Kocak, 1989) for $a = 4$ and two different starting points, $x(0) = .300$ and $y(0) = .301$, exhibits the butterfly effect because after only 8 iterations, the states of the two systems are far apart, despite their initial closeness. From then on, the two trajectories bear little similarity to one another.

One upshot of this critical aspect of chaotic behavior is that even a tiny error in measurement can lead to a vast misprediction of future behavior. There are also critical implications for computers, but these have to be bypassed in this treatment (but see, e.g., McCauley, 1986, chap. 5; or Grassberger, 1986). The "sensitive dependence" axiom is virtually always present, explicitly or implicitly, in discussions of chaos. Sometimes topological transitivity is missing, and sometimes the following characteristic is also missing, despite its profound importance for dynamic behavior.

We all know what periodic behavior is in a system: States continually repeat themselves as the system evolves. Those with sensory science backgrounds are familiar with Fourier theory and its importance in describing signals, their transmission, and decoding. Simple sine waves are the purest form of auditory stimulation. One of the primary regular types of behavior of well-behaved systems is periodic. A variation on this theme is when the amplitude of the periodic function is modified, for instance, damped, where the amplitude of the (perhaps complex)

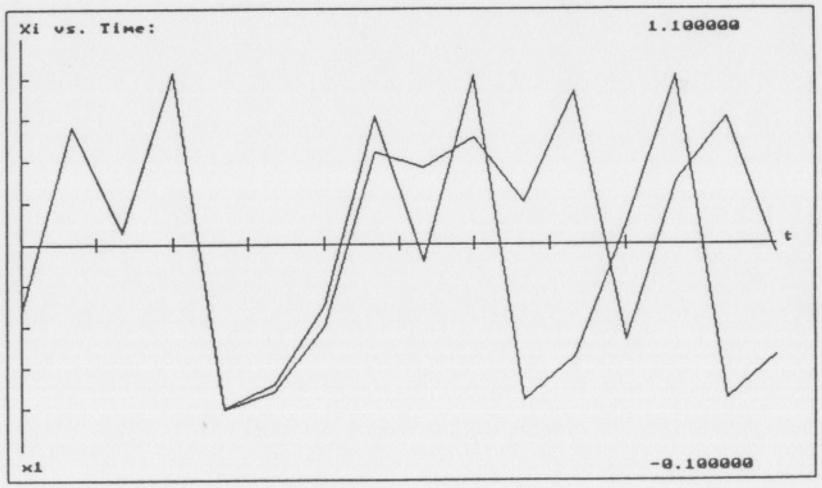


FIG. 4.3. Illustration of the Butterfly Effect in the logistic system with $a = 4$. Two start values $x(0) = .300$ and $y(0) = .301$ are so close together that they cannot be discriminated on the scale used in the figure, yet within about eight iterations their graphs deviate substantially and shortly will be totally unalike.

periodic waves goes toward zero. Unstable, but otherwise easily describable linear systems can evidence periodic behavior altered by the amplitude swinging more and more wildly out of control.

It turns out that periodic functions can emerge in chaotic systems, too, but often in a rather untoward fashion. We again require the notion of "density" of sets of points. In our case, the points are still to be thought of as states of the system. Following Devaney (1986), we nevertheless refer to this aspect as an "element of regularity."

Definition 3: Dense Periodic Trajectories

A system is said to possess an element of regularity if there exists a dense subset of periodic points in the set of states J. That is, for any point x contained in J, there is a point y that continually iterates back to itself, that is, arbitrarily close to the original point x.#

This infinite set of periodic trajectories exists for a single parameter value, say "a" in the logistic system, whenever the system is chaotic. Moreover, it has been shown that for any value of "a," there exists at most one trajectory that is attracting, in the sense that any initial state (within some set of initial states) will asymptotically approach that trajectory. Such an attracting trajectory is also known as an attracting "orbit." Some authors prefer "orbit" to "trajectory"; we hereafter use them interchangeably. When an attracting orbit or trajectory is periodic, it is referred to as a "limit cycle." In any event, it follows that at most one of the infinite set of periodic orbits in a chaotic logistic system is attracting! All the rest are repelling. This in turn implies that only one of them—the attracting trajectory—is ever likely to be observed in nature, as the probability of landing exactly on the periodic orbit to start with is zero. In fact, some chaotic systems may have no attracting periodic orbits whatsoever. This is the case in the logistic system when $a = 4$, as we learn in more detail as follows.

In Fig. 4.1, "a" ($a = 3.41$) was selected so that there exists just one periodic orbit in toto. This system is not chaotic. For "a" greater than 1 and less than 3, there exists a single equilibrium point and no periodic orbits. At $a = 3$, we see our first periodic orbit, of period 2, and this continues to hold until $a = 3.449499$ (to the sixth decimal place) at which value the system gains a period 4 limit cycle (attracting trajectory or orbit). Thus, our value of $a = 3.41$ is still (barely) within that range and does not yield chaotic behavior. As "a" continues to be increased it more and more frequently (i.e., with successively smaller increments required) hits values where more and more *even* periodic orbits appear. That is, these periods are all powers of two; this behavior on the part of "a" is referred to as a "period doubling sequence." In fact, period doubling occurs for a sequence of "a" where $a_k = a^* - c/F^k$, $F = 4.669202$, $c = 2.637$ and $a^* = 3.569946$ is the limit of this sequence. (These numbers are approximations to the unending decimal true values.) The constant "F" is known as the "Feigenbaum constant" due to its discovery by Mitchell Feigenbaum in 1975. Observe that here F^k is actually the *kth* power of F, and also that each of these periods associated with a particular value of "a" is an attracting orbit. When $a > a^*$, we first encounter chaos and $a = 4$ is usually considered the most chaotic system within this logistic family, one reason being that the system is chaotic on the entire interval $I = [0,1]$. Other interesting aspects of the system at $a = 4$ are mentioned later and note that Figs. 4.2, 4.3, and 4.5 were plotted with that parameter value. The reader is referred to Lauwerier (1986) for more detail on these phenomena in the logistic and other systems.

The three foregoing definitions establish a foundation for chaotic behavior: (a) sets of states, denoting, for example, distinct initial conditions, always eventually march into one another's province, (b) no matter how close two initial conditions are, they become unpredictably separated in the future, and (c) there are infinite numbers of periodic states as close as one likes to any other state in the system.

ALLIED NOTIONS OF CHAOS

There are a number of other concepts that are intimately related to chaotic activity. Under certain conditions, they follow from the previously described conditions, but may also themselves be used as part of the defining qualities of chaos.

Fractals, Cantor Sets, and Fractal Dimension

Loosely speaking, fractals are sets of points that are "self-similar" under magnification. An intuitive example would be a Christmas tree each of whose branches was a replica of the original tree, each of its needles was yet another replica containing its own "branches and needles," and so on. Additional sources for reading about fractals are Mandelbrot, 1983, and Barnsley, 1988. Mandelbrot has been the prophet, if not the godfather, of fractals for many years.

The mathematician Georg Cantor, in the course of establishing an axiomatic basis for the number system and the differential and integral calculus in the 19th century, invented what naturally came to be known as "Cantor sets." The most classic version starts out as the set of all numbers between zero and one. Then the (open) middle third of points is removed; that is, the points lying between $\frac{1}{3}$ and $\frac{2}{3}$, but not including the latter. Next, the middle thirds between $\frac{1}{9}$ and $\frac{2}{9}$ and between $\frac{7}{9}$ and $\frac{8}{9}$, but not including the endpoints, are removed and this continues forever. Finally, the Cantor set is all the remaining points left. This is formally constructed by taking the set intersection of the first remaining set with the second remaining set, this with the third, and so on. Figure 4.4 shows the first three removals of open intervals of powers of 3 (i.e., remove the middle $\frac{1}{3}$, then

FIG. 4.4. The construction of the "missing thirds" Cantor set is illustrated by the first three removals, of the middle third, the next middle ninths, and the next middle twenty-sevenths. The blank parts are those that have been deleted. The Cantor set is what is left after the unending sequence of continued deletions.

remove the two middle $\frac{1}{9}$'s of the remaining intervals, then the four middle $\frac{1}{27}$'s of the remaining intervals and so on).

The resulting set of points apparently rendered many mathematicians of the late nineteenth century apoplectic, due to its strange characteristics. For instance, it contains not only an infinite set of points but an uncountably infinite set; in mathematics terms, that is as many points as there are numbers on the line to start with! Yet taking the usual measure of sets of points finds that this set has measure equal to zero. In a probabilistic sense, there would be probability zero of ever landing on one of them. A related property is that the set contains no intervals; that is, no subsets of points that can be described as "all the numbers lying between the endpoint numbers "a" and "b." It is thus said to be "totally disconnected." It is furthermore "perfect," by which is meant that a region around any point in the set, however small that region, always contains other points from that set as well. The self-similarity is apparent on "blowing up" any section between, say, 0 and $(\frac{1}{3})^n$. Such a section, on magnification by 3^n, looks exactly like the original set.

Even though such odd mental creatures have been accepted by most mathematicians for the greater part of the 20th century, it is only with the increasing prominence of nonlinear dynamics and chaos that they have begun to enjoy respectability and daily employment in science and technology. Benjamin Mandelbrot (1983) is largely responsible for making the study of fractals almost a science in itself (despite his publicized antipathy toward scientific specialization), including suggestions of how nature often may employ fractal structure.

We nevertheless must be cautious in separating the use of fractals in describing static architecture, as in the structure of the brachiating alveoli in lung tissue as opposed to their presence in chaotic dynamics. It was formerly thought that the branching took place in an exponential manner, but recent evidence supports the fractal model. Fractals play a very important role in actual dynamics, too.

We now take a moment to establish the main areas of chaos of the logistic system, $x(n + 1) = ax(n)[1 - x(n)]$. We want to concentrate on the dynamics

starting in the interval [0,1], henceforth to be called "I." Obviously, for any value of "a," $x = 0$ is an equilibrium, or fixed point, because if the system is on that point it never leaves. When $a > 1$, the more interesting equilibrium point $x^* = (a - 1)/a$ does lie in I, and in fact the most intriguing dynamic occurs for even larger values of the growth parameter "a." Thus, for any value of "a" between 1 and 3, starting at any point $x(0)$ in I leads inevitably toward the equilibrium point x^*. The equilibrium point I is therefore in this case asymptotically stable as all other points converge toward it. However, this is definitely not chaos.

When $a > 3$ exciting things begin to happen, as mentioned earlier. We haven't space to develop matters in detail. But, recall that the number of periodic points increases as "a" gets larger until at $a = a^* = 3.5699$ approximately (note that this a^* is the same one mentioned earlier), one reaches an attracting periodic orbit with an infinite period! Now, as we advance beyond $a = a^*$ toward $a = 4$, we encounter small windows of "a" where there exist attracting periodic orbits. Interspersed among these windows chaos reigns.

At $a = 3.839$ approximately, and within a nonchaotic window, there are an infinite number of periodic trajectories, but only one of these, with period three, is attracting. And, it is feasible to compute how many points are associated with any particular period. For small intervals around any of the three points of the attracting orbit, the trajectory of any point in such an interval becomes ever closer to the periodic trajectory and thus mimics it. This attracting periodic trajectory constitutes an instance of a limit cycle, which is, as defined earlier, a relatively simple type of behavior that nevertheless cannot be found in linear dynamic systems. (Figure 4.1 revealed the existence of a limit cycle in our earlier discussion.) All other sets of periodic points for this value of "a" are repelling. Hence, within small intervals around such points that do not overlap the small intervals just mentioned, trajectories starting from within those intervals are repelled away from the periodic trajectory in question, and toward the attracting orbit of period three.

Now let us jump to $a > 2 + \sqrt{5}$ where we have true chaotic behavior in the sense of Definitions 1, 2, 3. Observe in beginning that here $a > a^*$. Moreover, the chaotic behavior all occurs on a Cantor set contained in I. Any point outside that Cantor set is repelled to −(infinity), so the chaotic set of points in I is repelling. Indeed, it is possible to show that the numbers in I that are eventually mapped out of this interval, and into the negative real line, correspond exactly to the open intervals removed by the construction of the Cantor set previously presented. What is left is the chaotic (Cantor) set. Correspondingly, if the initial condition $x(0)$ is within the Cantor set, then chaotic behavior is guaranteed.

There may be, and probably is, some profound yet compelling reason for the appearance of this highly regular, if bizarre set of points, outside the manifest production of it by the dynamics. However, so far I have not discovered any completely satisfying statement to this effect. Interestingly, one may achieve a bit more sense of the topology of construction of such Cantor sets by going on to view certain two-dimensional chaotic systems such as the horseshoe map (e.g., Devaney, 1986). Such an exercise would go beyond the present scope, but readers are encouraged to avail themselves of the opportunity to do so.

An additional note of interest here is that although the point of equilibrium x^* is definitely unstable in that no matter how close you start to it, points may be found there that wander off, the system satisfies "structural stability," a concept due apparently to the Soviet mathematicians Andronov and Pontriagin around 1935 (see Jackson, 1989). The concept has become an important tool in modern investigations of dynamics through the work of Stephen Smale and others (Smale, 1969). Structural stability means, roughly, that a small change in the parameters of the system lead to small changes in the overall behavior of the system: the qualitative actions are the same. So a small change to "a" around the value $2 + \sqrt{5}$ leaves the system chaotic.

A quite special value of the parameter is found at $a = 4$, the value employed before in Figs. 4.2 and 4.3. Here, the system does not obey structural stability, so changing "a" just a little to greater than or less than 4 provides drastic alterations in the system's behavior. This system, with $a = 4$, is actually chaotic on the entire interval I.

A strategic related concept is that of "fractal dimension" (see, e.g., Barnsley, 1988; Mandelbrot, 1983). We are all familiar with the usual idea of dimension as given, say, by the classical D-dimensional Euclidean space, and represented pictorially by D othogonal coordinates. It turns out that we need a refined notion to handle the "size" of fractal sets of points; for instance, states in the phase space of system trajectories. Thus, although the usual measure of the Cantor set is 0, this odd subspace has a non-zero fractal dimension. There are many generalizations of the ordinary notion of dimensions. The following has become fairly standard in fractal arithmetic.

For most of our cases of interest, we can think of our space J of points (i.e., states of the system) as being closed and bounded subsets of an ordinary Euclidean space. Because of its lack of intervals or analogous sets in higher dimensional Euclidean spaces, the dimension of J can only be approximated in a finite sense. Again let J stand for our space (i.e., set of points) and D stand for the fractal dimension, whatever it turns out to be. Suppose that $N[(\frac{1}{3})^n]$ is the smallest number of abstract boxes, each of dimension D, each side of which is of length $(\frac{1}{3})^n$ that will cover J. That is, N is the number of such boxes, but N depends on the size of the individual boxes, which in turn depend on the lengths of each side, namely $(\frac{1}{3})^n$. The particular fraction $\frac{1}{3}$ with which we represent the declining box sizes is arbitrary; here it makes the computation of dimension of the Cantor set particularly straightforward.

To paraphrase this discussion, we can loosely think of the "volume" of J as being given by these N "boxes," each of size $[(\frac{1}{3})^n]^D$. Observe that this is akin to the ordinary box size in three dimensional space with sides of length $(\frac{1}{3})^n$ being $[(\frac{1}{3})^n]^3$. That is $D = 3$ in the standard case. So in effect we are putting together N boxes, each with sides of length $(\frac{1}{3})^n$ units and dimension D (and therefore they are D-dimensional boxes) to make up the entire volume. Thus, we have $N[(\frac{1}{3})^n] \times [(\frac{1}{3})^n]^D = V$, the approximate volume. However, it is only an approximation, albeit an increasingly accurate one, as n becomes larger, and therefore the individual boxes become smaller.

Now let $V = 1$, which is simply setting an arbitrary but convenient unit of volume, and take natural logarithms of both sides to obtain approximately

$$\log\{N[(\tfrac{1}{3})^n]\} + \log[(\tfrac{1}{3})^{nD}] \doteq 0$$

Performing some algebra we find that

$$D \doteq \frac{-\mathrm{Log}\{N[(\frac{1}{3})^n]\}}{\mathrm{Log}[(\frac{1}{3})^n]} = \frac{\mathrm{Log}\{N[(\frac{1}{3})^n]\}}{\mathrm{Log}(3^n)}$$

Finally, taking the limit as n grows large yields

$$D = \lim_{n\to\infty} \frac{\mathrm{Log}\{N[(\frac{1}{3})^n]\}}{\mathrm{Log}(3^n)}$$

Applying this formula to the Cantor set finds that $D = .6309$, thus delivering a fractional or "fractal" dimension.

One way of discovering chaotic behavior is to measure the dimension and learn that it is fractional (and therefore fractal). Of course, as is the case with most of the means of pinpointing chaos, sufficient precision must be maintained that the investigator can be reasonably certain that the fraction part of the dimension is not simply measurement error or other "noise."

Lyapunov Exponents, Randomness, and Strange Attractors

Another valuable index of dynamic activity is that of the so-called "Lyapunov exponent" (see, e.g., McCaulley, 1986; Wolf, 1988). Lyapunov, a Soviet, was an early innovator in dynamics and, along with others such as Henri Poincare in France and G. D. Birkoff in the United States, was responsible for many central concepts in nonlinear systems theory. Although the Lyapunov exponent is one of those concepts that is likely to seem rather abstruse, in fact we can readily gain some pretty good intuitions about it. It is related to several critical aspects of chaotic behavior, but perhaps most immediately to the butterfly effect, that is, "small differences in initial state lead to big differences later on." Recall that the implication is that a tiny measurement error of the initial state of a system means that future predictions may be vastly wrong.

We take a general discrete difference equation as our starting point, $x(n + 1) = f[x(n)]$. Now suppose that the error in specifying the state $x(n)$ at time n, is $dx(n)$. That is, the true state is $x(n)$ but due to some type of measurement error we believe it to be $x(n) + dx(n)$. It then follows that $x(n + 1) + dx(n + 1) = f[x(n)$

+ dx(n)]; or written in terms of the error at time n + 1, dx(n + 1) = f[x(n) + dx(n)] − x(n + 1). That is, the error at time n + 1 must be the difference between what the true state should be and the propagated error from the previous time epoch. The next step is to take a linear approximation, using Taylor's series from elementary calculus to get

$$dx(n + 1) = dx(n)\cdot f'[x(n)],$$

where f′ denotes the derivative or rate of change of the function with respect to the state x at time n. But this is the simplest type of linear difference equation, easily solved to obtain, in absolute value of error,

$$|dx(n)| = |dx(0)|\cdot|f'[x(0)]\cdot f'[x(1)] \ldots f'[x(n - 1)]|.$$

Thus, the error is seen to depend on the way in which f′ changes with system state and time. For a large class of systems it happens that this error either grows exponentially fast (the chaotic systems) or decays exponentially fast (the well-behaved systems). This in turn implies that $dx(n) = dx(0)\cdot 2^{Vn}$, where V is the Lyapunov exponent. But these two equations may be set equal to one another and solved for V yielding

$$V = (1/n) \sum_{i=0}^{n-1} \mathrm{Log}_2\{f'[x(i)]\}$$

That is, V is approximately equal to the arithmetic mean of the logarithms of the derivative of f. Moreover, in the limit this operation precisely defines the Lyapunov exponent to be

$$V = \lim_{n\to\infty} \{(1/n) \sum_{i=0}^{n-1} \mathrm{Log}_2\{f'[x(i)]\}\}$$

A positive value for V implies the butterfly characteristic associated with chaos, and means that two trajectories beginning very close together can move apart very rapidly. Incidentally, it can be shown that V measures the rate at which information is lost from the system over time.

Now for the "random" aspect of chaos. In some ways, this is one of the most fascinating topics, because chance seems to emerge from determinism, an antinomy. However, most readers of this chapter have, perhaps unwittingly, used just this mechanism in the form of computer random number generators, which are deterministic means of producing numbers with some probability distribution. Even a thorough intuitive discussion would require considerably more space than we have available, but perhaps a few comments can motivate the idea.

First, observe that in any deterministic system of the kind almost always used to describe real-life mechanisms, the trajectory is unique given a starting point. Therefore any possible trajectory has probability 1 or 0 depending on the particular start point that is selected. No probability yet. Furthermore, in the absence of periodicity, single points or states will have probability 0 of ever being visited twice, so just how should we look for chance in such situations?

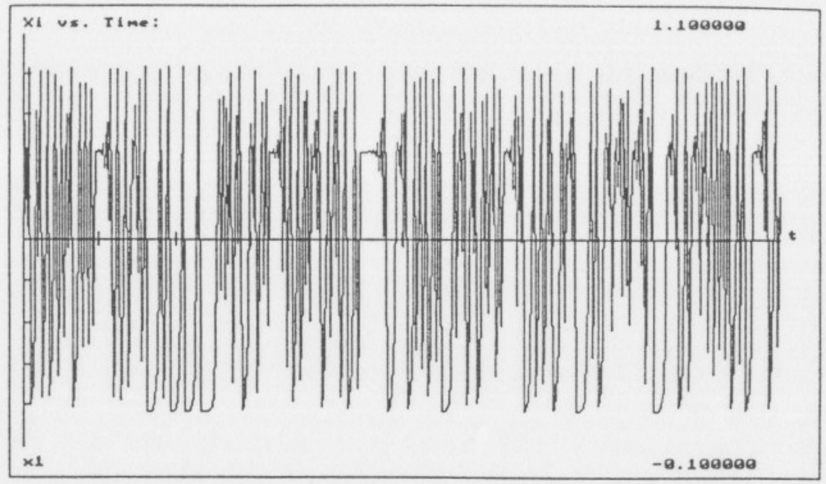


FIG. 4.5. The random appearance of the logistic system with a chaotic growth parameter a = 4 is shown with 400 iterations and starting at x(0) = .30.

A key lies in the earlier notion of topological transitivity. Recall that this assumption implies that for any open set (no matter how small) in the state space, there will be points in any other open set of states (also no matter how small that second set), such that starting at one of those points will lead in a finite time to a visit to the first open set; that is, to one of the states contained in that set. An implication is that if we are experimentally measuring the relative frequency of visiting a particular set of states, there will always be a finite measure of frequency of visitation to that set. Furthermore, the relative frequency of visitation can be taken as a measure of probability of finding the state to be contained within that set. Figure 4.5 gives some indication of randomness on the part of the logistic system when its growth parameter is in its chaotic domain. The ability of the sequence of states to span the entire available range for this parameter value (i.e., from 0 to 1), is apparent in the figure. However, the frequent and seemingly random visiting of midranges can also be appreciated from the "thicknesses" of the lines in the center parts of the graph.

In some very important cases, both physically and mathematically, the trajectory will be "randomly mixing" as well as "ergodic." Randomly mixing means that all sets of states of equal measure are visited equally often by a trajectory. A second, closely related interpretation is that if one generates a large number of trajectories simultaneously, such as placing a droplet of ink into a jar of water, each molecule of water corresponding to a separate state, then ultimately any volume of the state space will contain an amount of the original trajectories in proportion to its percentage of the total volume.

To be "ergodic" means that time averages approximate and eventually approach the space averages. This is a concept that is also useful in ordinary probability and stochastic process theory (e.g., Papoulis, 1984), but has special relevance here for deterministic processes. One intriguing sidelight is that random mixing implies ergodicity. In order to obtain an intuitive grasp of this consider any one-dimensional difference equation $x(n + 1) = f[x(n)]$, operating on a state space of size S. Observe the time average $\bar{x}(n) = (1/n)\,[x(1) + x(2) + \ldots + x(n)]$. Now segregate the visited states $x(i)$ into N bins of equal size, say, of size s. And let $n(i)$ be the number of times (iterations) that bin(i) is visited. Also, let $\hat{x}(i)$ stand for the approximate value of x within bin(i). This permits us to rewrite $\bar{x}(n) = (1/n)\,[n(1)\,\hat{x}(1) + n(2)\,\hat{x}(2) + \ldots + n(N)\,\hat{x}(N)]$. This is only the usual way that psychologists classify data when taking averages and performing other computations.

Now though, we can see that random mixing implies that visits to the ith bin occur in the ratio $n(i)/n = s/S = \Delta S$; that is, the frequency of visits to any bin is proportional to its relative size, and all of these were constructed to be equal. Therefore we can further express our average as $\bar{x}(n) = [\hat{x}(1)\,\Delta S + \hat{x}(2)\Delta S + \ldots + \hat{x}(N)\Delta S]$, and we have turned the time average into a space average, which must approach the real average by the law of large numbers. Also, if we let the bin sizes ΔS decrease toward infinitesimals, dS, then the average becomes an integral instead of a sum, $\bar{x} = \int \hat{x}\, dS$. Note too that dS takes on the properties of a density function in this case.

There are a number of other concepts that are important features of nonlinear and especially chaotic dynamics that unfortunately cannot be dealt with here. One that must be touched on because of its ubiquity in the popular press is the "strange attractor." At this point in time, there is no firmly agreed-on definition of this concept, but we now limn in something of what is available. First of all, as employed earlier in the chapter, an "attractor" of any kind is a set of points representing states of the system that trajectories approach. Typically an attractor forms an "invariant set" in the sense that if a trajectory starts within the set, it can never leave. In the special case of a single isolated point-attractor, we have the classical "asymptotically stable" state that is also an equilibrium point on account of its being an invariant set.

When an attracting set of points is itself periodic, this means that the approaching trajectories act more and more like the periodic set itself as time increases. This is the by now familiar concept of a "limit cycle," behavior that although not too complex, is never found in linear dynamic systems as noted earlier. Some authors define strange attractors by exclusion, as any attractor that is neither an ordinary asymptotically stable equilibrium point nor a limit cycle.

Usually a candidate for the status of a strange attractor is some version of a fractal set, although some writers have proselytized for yet more exotic criteria (some aspects of this debate are offered in Mandelbrot, 1983).

Interestingly, the earlier logistic system, with parameter "a" > $[2 + \sqrt{5}]$, possesses a fractal (a Cantor) set as an "invariant repellor," rather than an attractor. That is, once in the Cantor set, the trajectory can never depart, but if a trajectory is outside the repelling set, it marches away. Such a set of states thus merits the name "strange repellor." On the other hand, a number of well studied chaotic systems do possess fractal sets as attractors. And a full characterization of the attractors of some systems, though much scrutinized, are still not well understood. The Hénon map is a case in point (e.g., Devaney, 1986), and is stated as the beguilingly simple two-dimensional discrete system

$$x(n + 1) = a - by(n) - x^2(n)$$

$$y(n + 1) = x(n).$$

So far, even the employment of computer "experiments" to suggest regularities that are then proven rigorously has not fully broken the code of this system.

CHAOS, SCIENCE, PHILOSOPHY, AND PSYCHOLOGY

When even such trite-looking systems as the logistic or Hénon can be so appallingly difficult to analyze, is there any hope for a science of chaotics? The answer seems to be a tentative "yes." Already, as can perhaps be anticipated from our earlier discussion, a quite organized set of concepts, methods, and classes of behavior has evolved. Chaos theory is deeply embedded in general dynamic systems theory, its mathematical roots going back to the early 20th century. Some mathematicians actively dislike the term "chaos" because it seems to connote total disorder and an inability to ever be analyzed, understood, or dealt with. This is far from the truth as great progress has been made in understanding the phenomena, and mathematical control theory can in some cases be applied to "dechaotize" a chaotically behaving system.

Further, many scientists would claim that nonlinear dynamics in general and chaotics in particular has substantially benefitted progress in physics, chemistry, and engineering. Turbulence is a good example where chaotic dynamics has played a major role in advancing the field. Chaos theory seems to give a much better description of transitions to "noisy" behavior than does the time-honored Reynolds approach. Nevertheless, an improved ability to encompass spatial, rather than just temporal patterns in turbulence would be welcomed (e.g., Pool, 1989d).

As one would expect, there are spheres of research where the science and philosophy interact and sometimes almost merge. An instance is the study of "quantum chaos," a domain characterized by much contention and debate (see e.g., Pool, 1989c). One provocative if over-simplified question: Can chaos theory provide the answer to Einstein's long-lived disgruntlement with the absence of a deterministic or at least a causal underpinning to the probabilities of quantum theory (see e.g., Born, 1971; Pais, 1982, chap. 25)? So far, no acceptable deterministic infratheory to quantum mechanics has survived the theoretical and experimental crucible (but see the keen discussion by Suppes, in press-a).

A major enigma is that the very substance of chaos emerges from the trajectories, so far a meaningless construct in quantum mechanics. Nevertheless, there are those who seem to see quantum theory as a kind of parallel probabilistic model of chaos. For instance, the physicist Jensen (cited in pool, 1989c) suggests that instead of attempting to define "quantum chaos," it would be more profitable to determine the characteristics of a quantum system that correspond to chaos in the corresponding classical system. Others seem to feel that quantum activities should evince their own uniquely "quantum" type of chaos. In any event, presently the two theories, although agreeing on a number of predictions, diverge on several important dimensions. Thus, quantum behavior can be viewed, in most instances, as reversible (e.g., Feynman, 1965). This is a possibility unknown in chaos where the initial state is lost forever under evolution of the system.[2] Such questions could clearly be of strategic importance to future developments in quantum theory, not to mention the sheer intellectual attraction they have.

Outside of the physical sciences, the "dishes are not yet done." Certainly biology has already profited from the influx of nonlinear dynamics for many years (see, e.g., Glass & Mackey, 1988), yet the verity and usefulness of chaos theory are far from accepted by all physiologists and physicians (e.g., see Pool, 1989b). Now, there is little dispute that certain types of biological structure earlier thought to be of more mundane architecture (e.g., the aforementioned brachiation of alveoli tissue in the lung), is better described by fractal patterns, than, say, by exponential branching.

However, when one looks for indubitable evidence for chaos in the actual dynamics, it can be elusive. Even in population biology and meteorology, fields

[2]The only exception of which I am aware, of particle physics, where time is asymmetric, involves the decay of the neutral K meson. Also strictly speaking, a deterministic chaotic system is usually taken as evolving in one direction. However, because of the butterfly effect, one would have to know the initial state to an infinitesimal degree of accuracy in order to retrace the time sequence, impossible in reality. Another related point is that the mathematical form, sans physics, of Newtonian trajectories is typically time-symmetric: Differential equations are mute with regard to whether time runs from the left to right or vice versa. An engaging nontechnical discussion of the history and modern conceptions of physical time is "Arrows of Time" by Richard Morris (1984). Finally it is worth noting that the "arrow" offered by the second law of thermodynamics in terms of increasing entropy is based on statistical laws applied to closed systems. Hence, the time-symmetric character of the laws of individual particles is not violated.

boasting early pioneers in chaos research, respectively Robert May and Edward Lorenz, there are a number of respected scientists who downplay the evidence and importance of chaotic models (cf. Pool, 1989a,d).

This is true in physiology, again despite the contribution being made with nonlinear dynamics, and as forthrightly admitted by Glass and Mackey in their intriguing book "From Clocks to Chaos: The Rhythms of Life" (1988), and appears also in such fields as epidemiology (Pool, 1989a). One trend in recent years, especially salient in biological circles, is to view chaos as a "positive" rather than "negative" dynamic. Thus, it is claimed that in a number of organic systems, the heart being one of them, chaos is the preferred healthy state (e.g., Goldberger & Rigney, 1988). Others maintain that the normal brain evidences chaotic activity whereas epileptic seizures are associated with an overabundance of periodicity (e.g., Rapp, Zimmerman, Albano, de Guzman, Greenbaun, & Bashore, 1986). Here too, there is yet controversy (Glass & Mackey, 1988).

Perhaps the most frequent use of chaos theory so far in the behavioral sciences is in a directly "philosophy of science" fashion to warn of the indeterminacies of psychology. To be sure, such admonitions are requisite for any discipline studying phenomena that are reasonably complex. In fact, Suppes (1985), among others, has invoked chaos theory in dealing with such enigmas as the determinism versus probabilistic dichotomy. Are all natural phenomena, no matter how seemingly random they may appear, simply evidence of complex deterministic mechanisms? One "classic" example discussed by Suppes (1991) is that of a special-case three-body problem where any random sequence of coin tosses turns out to represent a solution to the dynamics of this "simple," and in Newtonian terms, decidedly deterministic, special case. The orbit of Pluto, impossible to predict in the long run, is another Newtonian example of chaos. As Suppes observes in another paper (1985), prediction in more complex arenas, such as real-world political scenarios, is likely to remain an "after-the-fact" reconstruction rather than true a priori rigorous anticipation. Suppes has suggested that the dichotomy between stability and instability might be a more appropriate partition than determinism versus indeterminism, given the theoretical and empirical results associated with chaos.[3]

Within psychology, Uttal (1990) has recently put forth a tractate listing cogent

[3]Although generally agreeing with Suppes' various points, I do suggest caution in dealing with the notion of "stability," because more than one definition is possible. Typically, stability can be defined in constrained but simple terms for linear systems (e.g., "nonpositive real parts of eigenvalues," "bounded inputs produce bounded outputs," and the like). Then for more general systems, new definitions often encompass the more elementary ones. For instance, Lyapunov stability is implied in linear systems by the two definitions already given. Nevertheless, in general contexts, even when the state-space is metric and the systems are autonomous (that is, they are evolving without input), a number of useful types of stability (e.g., Lyapunov, Lagrange, Poisson, Structural) can be advanced that are not necessarily equivalent (see, e.g., Jackson, 1989; Sibirsky, 1975). These diverse notions of stability may be of both philosophical as well as scientific value.

reasons that theorizing, especially reductionistic modeling, in psychology may be critically limited in its possibilities. The reasons are, briefly and not in the author's order, chaos, second law of thermodynamics, combinatorial complexity, black box identifiability problems; and, with much less emphasis, models are only abstractions of a full reality; the cumulative microcosm does not equal the macrocosm, mathematics is descriptive, not reductive, and Goedel's theorem. In my opinion, certain of the objections, such as the limits on system identification in automata (e.g., Moore, 1956), can be viewed as "the glass is half full" rather than "half empty." That is, Moore's and related theorems show that it is generally possible, from finite inputs and finite outputs, to determine the system down to a canonical class of systems within which no further identification may be possible. Given the complexity of systems and the relative poverty of input–output data, the ability to resolve the potential models down to a constrained subclass could be viewed as quite impressive.[4]

Within more standard information processing theory and methodology in psychology, the evolution of experimental technology in deciding the serial versus parallel processing issue can be interpreted in a similar optimistic way (Townsend, 1990). First, the early work, extending from the 19th century (Donders, 1869) to Sternberg's crucial experiments (1966, 1969) one hundred years later, provided much needed data, increasing experimental and theoretical sophistication and a motivating force to parallel–serial research. The second phase found that the standard experimental paradigms were primarily either testing other aspects of processing (e.g., limited vs. unlimited capacity) or quite restricted examples of the classes of models under investigation (Townsend, 1971, 1972, 1974). The third phase, overlapping with the second and still in progress, has discovered fundamental distinctions between parallel and serial processing, a number of which can be used to experimentally test the opposing concepts (Snodgrass & Townsend, 1980; Townsend, 1972, 1974, 1976a, 1976b, 1984, 1990; Townsend & Ashby, 1978, 1983).

To be sure, the case Uttal introduced for ceilings on understanding and prediction due to chaos was compelling. Nevertheless, it may be that a large portion of psychological laws might be developed that are able to skirt the tiger traps of chaos. For instance, whether random behavior emanates from chaos or pure indeterminism, averages and other statistics can be employed to reveal lawful if stochastic, for example, correlations, power spectra, and so on, systemic behavior. Perhaps an apposite example is that of reaction time. Certainly most reaction time experiments have been in the context of relatively simple cognitive operations. Yet no investigator has even found the same reaction time given on two successive trials, no matter how simple the mental task. Reaction time has been an extraordinarily useful tool for investigators of elementary perceptual and

[4]Uttal (personal communication) reminds me that he is primarily designating the limits on reductionism per se, that is, on our ability to discover the exact material entities and functions producing our behavior, rather than on modeling as a descriptive tool.

cognitive processes (e.g., Luce, 1986; Posner, 1978; Townsend & Ashby, 1983; Townsend & Kadlec, 1990; Welford, 1980). Lawfulness has emerged even though it may not be understood for a long time just where the randomness of reaction time derives, in the physiological sense; or whether the randomness is deterministic in its origin.

Another example, apt both for applied physics, engineering, and chemistry, as well as for sensory psychology, is the fact that linear systems can account so well for first order descriptions of phenomena. Much of engineering sciences is based on linear approximations to reality that have built bridges, houses—and instruments of war!—for hundreds of years, not to mention electronic equipment from around the turn of the century to the present. It is primarily with the advent of such fields as nonlinear optics (e.g., lasers, diffraction), superconductivity, and the need for more efficient air foils and turbines that nonlinear dynamics has forced itself on the engineering community. In vision and acoustic research, linearity suffices to give a very good first approximation to the functioning of the systems, although certain nonlinear phenomena have been known for some time.

Interestingly, the dependent variable "accuracy" has, alongside reaction time, been a mainstay in cognitive research. And accuracy is most frequently described in terms of probability correct; again, the phenomena and models are (sometimes implicitly) assumed to be inherently random. The origin of the probabilism is not necessarily critical for the erection of laws governing behavior, even for other indices of behavior such as rating scales, EEG recordings, and so on.

What about more direct uses of chaos, where it plays an important part in a psychological theory? There is not too much to report here so far, although Killeen (1989) argues cogently for an important post for dynamics in general, in describing human behavior. Probably the most intense development to date is the dynamic theory of olfactory discrimination advanced by Walter Freeman and his colleagues (Freeman, 1987; Freeman & Di Prisco, 1986; Freeman & Skarda, 1987). Backed up by several decades of electrophysiological data and an evolving dynamic theory of elementary olfactory processes, chaos serves a strategic purpose in the theory. Chaos is present in the state space at all phases of respiration, but not in all parts of the state space. For odor discrimination, learning leads to formation of limit cycles that attract learned odors during early inhalation. If an odor fails to "find" an attracting cycle, it falls into a chaotic well and is then identified as "novel." Further, chaotic activity is seen as ". . . a way of exercising neurons that is guaranteed not to lead to cyclic entrainment or to spatially structured activity. It also allows rapid and unbiased access to every limit cycle attractor on every inhalation, so that the entire repertoire of learned discriminanda is available to the animal at all times for instantaneous access." The last quote comes from an open commentary paper in *Behavior and Brain Sciences* (1987) by Freeman and Skarda, to which the reader is referred for an introductory account of the theory, a thorough and lively debate with the commentators, and a healthy set of references.

Notably, not all scientists believe chaos is necessary to explain such phenomena. Grossberg is one of these (see, e.g., Grossberg's commentary in the cited *Behavior and Brain Sciences* article), who has developed his own account of numerous psychological and psychobiological behaviors (Grossberg, 1982). Another intriguing exploration of nonlinear dynamics in the context of sensation psychology is found in the recent book by Gregson (1988). Here chaos does not play as integral a role as in Freeman's scheme, but may appear under various circumstances (see also, the in-depth review of Gregson's book by Heath, 1991). It remains to be seen whether chaos in fact is a strategic linchpin in psychophysical discrimination, but it appears that the Freeman and colleagues' corpus of results constitute the most impressive theoretical venture in psychobiology employing chaos to date.

My friends in the "hard" sciences and mathematics assure me that the behavioral sciences are not alone in the experience of fads and trends. However, we certainly do seem to have our share of them. I was tempted to substitute "suffering" in place of "experience" in the first sentence of this paragraph, but caught myself. When I thought of some of the various trends in experimental psychology over the past 50 years or so, it became apparent that although most of them were a bit "oversold," each in turn eventually took its place in the overall armamentarium of the science. And each added something pretty unique and helpful in doing scientific psychology.

In order to put some limits on things, suppose we start back around 1913, the time of the advent of behaviorism and, in this country, of Gestalt psychology. Later came neobehaviorism, then in the late 1940s and 1950s the trends associated with cybernetics, logic nets, game theory, automata, signal detection, and information theory. The latter, especially, were undoubtedly critical in leading inexorably to the information processing approach and artificial intelligence culminating in the field of cognitive science. But intersticed into the early 1950s was the birth of mathematical psychology as a subfield of psychology.

Some would say mathematical psychology was a fad. Mathematics goes back to the roots of experimental psychology in the 19th century, especially in psychophysics. However, Ebbinghaus and others used simple functions to describe the progress of learning as well relatively early. William Estes pioneered in the more formal development of the field with his seminal "Toward a statistical theory of learning" in 1950. This article presaged the forthcoming "stimulus sampling theory" that continues to play an important role in memory and learning theory today. The Estes paper was followed closely by a Robert Bush and Herbert Mostellar paper in 1951, introducing their linear operator approach. The early 1950s also saw the emergence of the influential signal detectability theory by the Michigan group composed of psychologists and electrical engineers (see Tanner & Swets, 1954, and their references). Another valuable cog in the mathematical machine was formal measurement theory (also called axiomatic measurement theory, foundational measurement theory, the representation approach, and so on), initiated in psychology by Patrick Suppes (1951), Duncan Luce (1956), Clyde Coombs (1950). These investigators and their colleagues began to formalize S. S. Stevens' (1946) informal schema of scale types varying in strength, and the use by Louis Thurstone (1927) and Louis Guttman (1944) of ordinal data to constrain possible psychological scales. The desire to further develop the notions of von Neuman and Morgenstern's (1944) utility theory approach to decision making was also present in some directions of this early work.

Multidimensional scaling has been another trend that stemmed from the work of investigators like Coombs and Guttman but also to some extent from a more psychometric tradition (e.g., factor analysis, and the metric scaling efforts of Young & Householder, 1938). Later under the impact of contributions by Shepard (1966) and Kruskal (1965), multidimensional scaling took on a separate life of its own and has occasionally been abused. But, when employed in a careful way, particularly in the context of a substantive theory, it has also contributed to our understanding of many phenomena.

In the ensuing years, mathematical psychology took up the cudgel in almost every area of psychology including the still active information processing approach (Atkinson & Shiffrin, 1968; Estes & Taylor, 1964; Laberge, 1962; Link & Heath, 1975; Townsend, 1971, 1972). Although mathematical psychology is not, if it ever was, viewed as the panacea to all psychology's dilemmas, it has apparently become an inseparable part of the psychological panorama. Even the Markov approach, highly voguish in the 1960s and thereafter falling into decline, continues to enrich theory in learning and memory (e.g., Riefer & Batchelder, 1988).

Artificial intelligence held sway as the dominant force in cognitive science from the early 1960s until recently, with some of the major players in the field being Feigenbaum, Minsky, Papert, Suppes, Newall, Simon, and McCarthy. John Anderson and Gordon Bower have been prolific contributors to cognitive science, in particular through their synthesis of artificial intelligence and mathematical modeling strategies.

The Johnny-come-lately, but with perhaps the broadest and greatest early inertia, is connectionism, although it has its roots in work going back many years. Its hallmark is its connotations and occasionally denotations of neural and biological realism. Its mathematical underpinnings are largely in dynamic systems theory, although stochastic processes and other aspects of applied physics play a role. Earlier pioneers were Rashevsky, Ashby, and McCulloch and Pitts. This new field intersects many disciplines, including electric engineering, computer sciences, artificial intelligence, neurosciences, psychology, and philosophy. Yet certain of the seminal resurgence papers have been published in the *Journal of Mathematical Psychology* (e.g., Grossberg, 1969; Hoffman, 1966) or other theoretical psychology journals (e.g., Anderson, 1973; McClelland, 1979), and several leaders of the movement, namely, Rumelhart and James Anderson, received doctoral or postdoctoral training in mathematical psychology laboratories.

Undoubtedly, there will be a lot of "noise" associated with this, as with the other vogues, but, just as surely, the approach will offer something that has been missing in our theoretical battery. In the present instance, in my opinion, what has been lacking in most cognitive science theory oriented around the metaphor of digital computers (or automata in general) has been the sense of a naturally functioning dynamic system. That has been more important than the "neuralism" per se. Connectionism has also upped the ante in forcing cognitive scientists to go one level deeper in their assumptions about the "process" aspects of their model; deeper, say, than the level reached by the typical "production system" routines.

Now along comes chaos theory, although as mentioned earlier, it really has a history going back to the late 19th and early 20th century. We have seen that outside some relatively narrow areas of the physical sciences, it has still not captured universal approval, despite its big splash. I suspect chaotic mechanisms will be increasingly visible in psychological models. Much of its form will be metaphorical rather than mathematical. When it assumes a mathematical formulation, it will still likely be difficult to come up with data that is sufficiently tight relative to the difficulty in estimating chaotic parameters, to test the chaotic part of the model (see, e.g., chapters by Wolf and by Grassberger in Holden, 1986; plus other references in Glass & Mackey, 1988). In some cases, its contribution may be by way of giving a cogent deterministic interpretation to an observably random phenomenon.

The metaphorical uses are not necessarily to be sneered at. The very sensationalism of chaos theory has attracted the attention of many who might not otherwise have been drawn to consider dynamic systems theory as a workable tool. Thus, it can help areas or problems previously conceived of in a static fashion to move towards a more dynamic, and therefore perhaps psychologically and biologically realistic accounting of the data. This seems to be happening in developmental psychology, for example, long treated by most psychologists as a series of static plateaus, with little concern about how the evolving individual gets from one stage to the next.

In addition, chaos will almost certainly be adopted by connectionism, because as intimated, virtually all nonlinear as well as linear dynamics can be considered "connectionistic" in some broad sense; the only necessary requirement being the allusion to neuralistic conceptions.

So far its greatest contribution has been to challenge mathematicians and other scientists to dig deeply into the general and profound aspects of nonlinear dynamics.[5] This, in turn, may offer the field of psychology a more powerful technology with which to study complex cognitive systems and behavior.

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[5]A recent integrative and readable presentation that places chaos nicely within the subjects of dynamics and neutral phenomena is "Exploring Complexity" by Nicolis and Prigogine (1989).

ACKNOWLEDGMENTS

The impetus for this chapter comes from my own longstanding interest in dynamics and systems theory (e.g., Townsend and Evans, 1983). Its specific theme emerged from a talk I gave at the 1990 Interdisciplinary Conference at Jackson Hole. I am indebted to Stephen Link, Richard Shiffrin, and William Uttal for their insightful comments on a draft of this manuscript. I thank Joseph Stampfli for inviting me to participate in his and John Ewing's excellent seminar on chaos theory and for his participation in my own seminar on nonlinear systems and dynamics theory. The writing of this chapter was partly supported by NSF Grant #8710163 from the Human Cognition and Perception Section.

REFERENCES

Abraham, R. H., & Shaw, C. D. (1985). *Dynamics—The geometry of behavior*, Parts 1 through 4. In The Visual Mathematics Library, R. H. Abraham (Ed.). Santa Cruz, Ca: Aerial Press.

Anderson, J. A. (1973). A theory for the recognition of items from short memorized lists. *Psychological Review, 80*, 417–438.

Atkinson, R. C., & Shiffrin, R. M. (1968). Human memory: A proposed system and its control processes. In K. W. & J. T. Spence (Eds.), *The psychology of learning and motivation: Advances in research and theory, Vol. 2*. New York: Harper & Row.

Barnsley, M. (1988). *Fractals everywhere*. San Diego: Academic Press.

Beltrami, E. (1987). *Mathematics for dynamic modeling*. New York: Academic Press.

Born, M. (1971). *The Born-Einstein letters: The correspondence between Albert Einstein and Max and Hedwig Born, 1916–1955*. New York: Walker & Co.

Bush, R. R., & Mosteller, F. (1951). A mathematical model for simple learning. *Psychological Review, 58*, 313–323.

Coombs, C. H. (1950). Psychological scaling without a unit of measurement. *Psychological Review, 57*, 145–158.

Devaney, R. (1986). *An introduction to chaotic dynamical systems*. New York: Addison Wesley.

Devaney, R., & Keen, L. (Eds.). (1989). Chaos and fractals: The mathematics behind the computer graphics. *Proceedings of the Symposium in Applied Mathematics*, Vol. 39. Providence, RI: American Mathematical Society.

Donders, R. C. (1869). On the speed of mental processes (W. G. Koster, Trans.). In W. G. Koster (Ed.), *Attention and performance, Vol. 2*, Amsterdam: North Holland.

Estes, William K. (1989). In G. Lindzey (Ed.). *A history of psychology in autobiography, Vol. VIII*. Stanford, CA: Stanford University Press.

Estes, W. K., & Taylor, H. A. (1964). A detection method and probabilistic models for assessing information processing from brief visual displays. *Proceedings of the National Academy of Sciences, 52*, 446–454.

Feynman, R. (1965). *The character of physical law*. Cambridge, MA: MIT Press.

Freeman, W. J. (1987). Simulation of chaotic EEG patterns with a dynamic model of the olfactory system. *Biological Cybernetics, 56*, 139–150.

Freeman, W. J., & Skarda, C. A. (1987). How brains make chaos in order to make sense of the world. *Behavioral and Brain Sciences, 10*, 161–195.

Freeman, W. J., & Viana Di Prisco, G. (1986). EEG spatial pattern differences with discriminated odors manifest chaotic and limit cycle attractors in olfactory bulb of rabbits. In G. Palm (Ed.), *Brain theory*. New York: Springer-Verlag.

Glass, L., & Mackey, M. C. (1988). *From clocks to chaos: The rhythms of life*. Princeton, NJ: Princeton University Press.

Gleick, J. (1987). *Chaos: Making a new science*. New York: Viking Penguin.

Goldberger, A. L., & Rigney, D. R. (1988). Sudden death is not chaos. In J. A. S. Kelso, M. F. Schlesinger, & A. J. Mandell (Eds.), *Dynamic patterns in complex systems ()*. Singapore: World Scientific.

Grassberger, P. (1986). Estimating the fractal dimensions and entropies of strange attractors. In A. V. Holden (Ed.), *Chaos*. Princeton, NJ: Princeton University Press.

Gregson, R. (1988). *Nonlinear psychophysical dynamics*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Grossberg, S. (1969). Embedding fields: A theory of learning with physiological implications. *Journal of Mathematical Psychology, 6*, 209–239.

Grossberg, S. (1982). *Studies of mind and brain*. Dordrecht, Holland: D. Reidel.

Guttman, L. (1944). A basis for scaling qualitative data. *American Sociological Review, 7*, 139–150.

Hoffman, W. C. (1966). The Lie algebra of visual perception. *Journal of Mathematical Psychology, 3*, 65-98.

Jackson, E. A. (1989). *Perspectives of nonlinear dynamics*. Cambridge: Cambridge University Press.

Killeen, P. R. (1989). Behavior as a trajectory through a field of attractors. In J. R. Brink & C. R. Haden (Eds.), *The computer and the brain: Perspectives on human and artificial intelligence ()*. Amsterdam: North-Holland.

Koçak, H. (1989). *Differential and difference equations through computer experiments*. New York: Springer Verlag.

Kruskal, J. B. (1965). Analysis of factorial experiments by estimating monotone transformations of the data. *Journal of the Royal Statistical Society, Series B., 27*, 251–163.

LaBerge, D. (1962). A recruitment theory of simple behavior. *Psychometrika, 27*, 375–396.

Lauwerier, H. A. (1986). One-dimensional iterative maps. In A. V. Holden (Ed.), *Chaos*. Princeton, NJ: Princeton University Press.

Link, S., & Heath, R. A. (1975). A sequential theory of psychological discrimination. *Psychometrika, 40*, 77–105.

Lorenz, E. (1979). Predictability: *Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?* Address at the Annual Meeting of the American Association for the Advancement of Science, Washington, D.C.

Luce, R. D. (1956). Semi-orders and a theory of utility discrimination. *Econometrica, 24*, 178–191.

Luce, R. D. (1986). *Response times*. New York: Oxford University Press.

Mandelbrot, B. B. (1983). *The fractal geometry of nature*. San Francisco: W. H. Freeman.

McAuliffe, K. (1990, February). Get smart: Controlling chaos. *Omni*.

McCauley, J. L. (1988). An introduction to nonlinear dynamics and chaos theory. *Physica Scripta*, Series T, Vol. T20.

McClelland, J. L. (1979). On the time relations of mental processes: An examination of systems of processes in cascade. *Psychological Review, 86*, 287–330.

Moore, E. F. (1956). Gedanken-experiments on sequential machines. In C. E. Shannon & J. McCarthy (Eds.), *Automata studies ()*. Princeton, NJ: Princeton University Press.

Morris, R. (1984). *Times arrows: Scientific attitudes toward time*. New York: Simon & Schuster.

Nicolis, G., & Prigogine, I. (1989). *Exploring complexity*. New York: W. H. Freeman.

Pais, A. (1982). *'Subtle is the Lord. . .': The science and life of Albert Einstein*. New York: Oxford University Press.

Papoulis, A. (1984). *Probability, random variables, and stochastic processes*. New York: McGraw Hill.

Pool, R. (1989a). Ecologists flirt with chaos. *Science, 243*(Jan.), 310–313.

Pool, R. (1989b). Is it healthy to be chaotic? *Science, 143*(Feb.), 604–607.

Pool, R. (1989c). Quantum chaos: Enigma wrapped in a mystery. *Science, 243*(Feb.), 893–895.

Pool, R. (1989d). Is something strange about the weather? *Science, 243*(Mar.), 1290–1293.

Posner, M. I. (1978). *Chronometric explorations of mind*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Prigogine, I. (1984). *Order out of chaos*. New York: Bantam Books.

Rapp, P. E., Zimmerman, I. D., Albano, A. M., de Gusman, G. C., Greenbaun, M. N., & Bashore, T. R. (1986). Experimental studies of chaotic neural behavior: Cellular activity and electroencephalographic signals. In H. G. Othmer (Ed.), *Nonlinear oscillations in biology and chemistry ()*. Berlin: Springer-Verlag.

Riefer, D. M., & Batchelder, W. H. (1988). Multinomial modeling and the measurement of cognitive processes. *Psychological Review, 95*, 318–339.

Shepard, R. N. (1966). Metric structures in ordinal data. *Journal of Mathematical Psychology, 3*, 287–315.

Sibirsky, K. S. (1975). *Introduction to topological dynamics* (L. F. Boron, Trans.). Leyden, Holland: Noordhoff.

Smale, S. (1969). What is global analysis? *American Mathematics Monthly, 76*, 4–9.

Snodgrass, J. G., & Townsend, J. T. (1980). Comparing parallel and serial: Theory and implementation. *Journal of Experimental Psychology: Human Perception and Performance, 6*, 330–354.

Sternberg, S. (1966). High-speed scanning in human memory. *Science, 153*, 652–654.

Sternberg, S. (1969). The discovery of processing stages: Extensions of Donders' method. In W. G. Koster (Ed.), *Attention and Performance, Vol. 2*. Amsterdam: North Holland.

Stevens, S. S. (1946). On the theory of scales of measurement. *Science, 103*, 677–680.

Suppes, P. (1951). A set of independent axioms for extensive quantities. *Portugaliae Mathematica, 10*, 163–172.

Suppes, P. (1985). Explaining the unpredictable. *Erkenntnis, 22*, 187–195.

Suppes, P. (1990). Probabilistic causality in quantum mechanics. *Journal of Statistical Planning and Inference, 25*, 293–302.

Suppes, P. (1991). *Indeterminism or instability, does it matter?* In G. G. Brittan, Jr. (Ed.), Causality, Method and Modality (pp. 5–22). Kluwer Academic Publishers.

Tanner, W. P., & Swets, J. A. (1954). A decision-making theory of visual detection. *Psychological Review, 61*, 401-409.

Thurstone, L. L. (1927). A law of comparative judgement. *Psychological Review, 34*, 273–286.

Townsend, J. T. (1971). A note on the identifiability of parallel and serial processes. *Perception & Psychophysics, 10*, 161–163.

Townsend, J. T. (1972). Some results concerning the identifiability of parallel and serial processes. *British Journal of Mathematical and Statistical Psychology, 25*, 168–199.

Townsend, J. T. (1974). Issues and models concerning the processing of a finite number of inputs. In B. H. Kantowitz (Ed.), *Human information processing: Tutorials in performance and cognition*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Townsend, J. T. (1976a). A stochastic theory of matching processes. *Journal of Mathematical Psychology, 14*, 1–52.

Townsend, J. T. (1976b). Serial and within-stage independent parallel model equivalence on the minimum processing time. *Journal of Mathematical Psychology, 14*, 219–238.

Townsend, J. T. (1984). Uncovering mental processes with factorial experiments. *Journal of Mathematical Psychology, 28*, 363–400.

Townsend, J. T. (1990). Serial versus parallel processing: Sometimes they look like Tweedledum and Tweedledee but they can (and should) be distinguished. *Psychological Science, 1*, 46–54.

Townsend, J. T., & Ashby, F. G. (1978). Methods of modeling capacity in simple processing

Lawrence Erlbaum Associates, Inc., Publishers
365 Broadway
Hillsdale, New Jersey 07642

Library of Congress Cataloging-in-Publication Data

Essays in honor of William K. Estes / edited by Alice F. Healy, Stephen M. Kosslyn, Richard M. Shiffrin.
p. cm.
Includes bibliographical references and indexes.
Contents: v. 1. From learning theory to connectionist theory — v. 2. From learning processes to cognitive processes.
ISBN 0-8058-1097-8 (hard : v. 1). — ISBN 0-8058-0759-4 (hard : v. 2). — ISBN 0-8058-1098-6 (pbk. : v. 1). — ISBN 0-8058-0760-8 (pbk. : v. 2)
1. Learning, Psychology of—Mathematical models. 2. Cognition. I. Estes, William Kaye. II. Healy, Alice F. III. Kosslyn, Stephen Michael, 1948– . IV. Shiffrin, Richard M.
BF318.E86 1992
153.1'5—dc20 91-39697
CIP

Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

FROM LEARNING THEORY TO CONNECTIONIST THEORY:

Essays in Honor of William K. Estes

Volume 1

Edited by
ALICE F. HEALY
University of Colorado
STEPHEN M. KOSSLYN
Harvard University
RICHARD M. SHIFFRIN
Indiana University

LEA LAWRENCE ERLBAUM ASSOCIATES, PUBLISHERS
1992 Hillsdale, New Jersey Hove and London