HOMEWORK 8

- 作业重点在于寻找似然比统计量并给出拒绝域，解答中省略了 MLE 的验证过程。
- 方便起见，本课程约定取 $\lambda(\boldsymbol{x}) = \frac{\sup_{\Theta_0} L(\theta|\boldsymbol{x})}{\sup_{\Theta} L(\theta|\boldsymbol{x})}$, 即原假设对应的似然在分子。

[Wei] 5.30. 似然函数为

$$L(\mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{\sum_i (x_i-\mu)^2}{2\sigma^2}}.$$

对数似然为

$$l(\mu, \sigma^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{\sum_i (x_i - \mu)^2}{2\sigma^2}$$

全参数空间 Θ 上, L 在 $\hat{\theta}_{MLE} = (\bar{x}, \frac{n-1}{n}S^2)$ 达到最大值. 若 $\frac{n-1}{n}S^2 \le \sigma_0^2$, 则 $\hat{\theta}_{MLE} \in \Theta_0$ 从而 LRT 统计量 $\lambda(\mathbf{x}) = 1$. 反之，$L|_{\Theta_0}$ 在 $(\bar{x}, \sigma_0^2)$ 处达到最大, 则

$$\lambda(\mathbf{x}) = \left(\frac{\frac{n-1}{n}S^2}{\sigma_0^2}\right)^{\frac{n}{2}} e^{\frac{(n-1)}{2}S^2\left(\frac{1}{\frac{n-1}{n}S^2} - \frac{1}{\sigma_0^2}\right)} = \left(\frac{(n-1)S^2}{n\sigma_0^2}\right)^{\frac{n}{2}} e^{\frac{n}{2} - \frac{(n-1)S^2}{2\sigma_0^2}}.$$

综上，

$$\lambda(\mathbf{x}) = \begin{cases} 1, & \frac{n-1}{n}S^2 \le \sigma_0^2 \\ \left(\frac{(n-1)S^2}{n\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{n}{2}(\frac{(n-1)S^2}{2n\sigma_0^2} - 1)}, & otherwise \end{cases}.$$

考虑 $f(t) = t^{n/2} e^{-\frac{n}{2}(t-1)}, t > 0; t > 1$ 时，f 关于 t 单调递减。从而似然比检验拒绝域为 $\{\boldsymbol{x} | \frac{(n-1)S^2}{n\sigma_0^2} > \beta\}$, 其中 $\beta > 1$. □

[Wei] 5.31. 似然函数为

$$L(\mu_1, \mu_2, \sigma^2) = (2\pi\sigma^2)^{-\frac{m+n}{2}} e^{-\frac{\sum_i (x_i-\mu_1)^2 + \sum_j (y_j-\mu_2)^2}{2\sigma^2}}.$$

对数似然为

$$l(\mu_1, \mu_2, \sigma^2) = -\frac{m+n}{2}\log(2\pi) - \frac{m+n}{2}\log(\sigma^2) - \frac{\sum_i (x_i - \mu_1)^2 + \sum_j (y_j - \mu_2)^2}{2\sigma^2}$$

在 Θ 上, L 在 $(\bar{x}, \bar{y}, \frac{\sum_i (x_i - \bar{x})^2 + \sum_j (y_j - \bar{y})^2}{m+n})$ 处达到最大值.

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在 Θ_0 上, 最大化

$$f(\mu,\sigma^2) = -\frac{m+n}{2}\log(\sigma^2) - \frac{\sum_i (x_i-\mu)^2 + \sum_j (y_j-\mu)^2}{2\sigma^2}.$$

其最大值在 $(\mu_0, \sigma_0^2) = (\frac{m\bar{x}+n\bar{y}}{m+n}, \frac{\sum_i(x_i-\mu_0)^2+\sum_j(y_j-\mu_0)^2}{m+n})$ 处达到。

$$\begin{aligned}
\lambda(\mathbf{x},\mathbf{y}) =& \frac{L(\mu_0,\mu_0,\sigma_0^2)}{L(\bar{x},\bar{y},\frac{\sum_i(x_i-\bar{x})^2+\sum_j(y_j-\bar{y})^2}{m+n})} \\
=& \left(\frac{\sum_i(x_i-\bar{x})^2+\sum_j(y_j-\bar{y})^2}{(m+n)\sigma_0^2}\right)^{\frac{m+n}{2}} \cdot \\
& \exp\left\{-\frac{\sum_i(x_i-\mu_0)^2+\sum_j(y_j-\mu_0)^2}{2\sigma_0^2} + \frac{\sum_i(x_i-\bar{x})^2+\sum_j(y_j-\bar{y})^2}{2\frac{\sum_i(x_i-\bar{x})^2+\sum_j(y_j-\bar{y})^2}{m+n}}\right\} \\
=& \left(\frac{\sum_i(x_i-\bar{x})^2+\sum_j(y_j-\bar{y})^2}{\sum_i(x_i-\mu_0)^2+\sum_j(y_j-\mu_0)^2}\right)^{\frac{m+n}{2}}.
\end{aligned}$$

注意到 $\sum_i(x_i-\mu_0)^2 = \sum_i(x_i-\bar{x})^2 + m(\bar{x}-\mu_0)^2 = (m-1)S_x^2 + m(\frac{n(\bar{x}-\bar{y})}{m+n})^2$, 则

$$\lambda(\mathbf{x}) = \left(1+\frac{mn(\bar{x}-\bar{y})^2}{(m+n)\left[(m-1)S_x^2+(n-1)S_y^2\right]}\right)^{-\frac{m+n}{2}}.$$

从而似然比检验的拒绝域为 $\{(\boldsymbol{x},\boldsymbol{y}) | \frac{mn(\bar{x}-\bar{y})^2}{(m+n)\left[(m-1)S_x^2+(n-1)S_y^2\right]} > c\}$ □

5.32. 似然函数为

$$L(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2) = (2\pi\sigma_1^2)^{-\frac{m}{2}}(2\pi\sigma_2^2)^{-\frac{n}{2}} e^{-\frac{\sum_i(x_i-\mu_1)^2}{2\sigma_1^2} - \frac{\sum_j(y_j-\mu_2)^2}{2\sigma_2^2}}.$$

对数似然为

$$l(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2) = -\frac{m+n}{2}\log(2\pi) - \frac{m}{2}\log(\sigma_1^2) - \frac{n}{2}\log(\sigma_2^2) - \frac{\sum_i(x_i-\mu_1)^2}{2\sigma_1^2} - \frac{\sum_j(y_j-\mu_2)^2}{2\sigma_2^2}$$

在 Θ 中, 似然 L 的最大值在 $(\bar{x}, \bar{y}, \frac{\sum_i(x_i-\bar{x})^2}{m}, \frac{\sum_j(y_j-\bar{y})^2}{n})$ 取得.
在 Θ_0 中, 最大化

$$f(\mu_1,\mu_2,\sigma^2) = -\frac{m+n}{2}\log(\sigma^2) - \frac{\sum_i(x_i-\mu)^2+\sum_j(y_j-\mu)^2}{2\sigma^2}.$$

最大值在 $(\bar{x}, \bar{y}, \frac{\sum_i (x_i-\bar{x})^2+\sum_j (y_j-\bar{y})^2}{m+n})$ 处取得。那么,

$$\begin{aligned}
\lambda(\mathbf{x},\mathbf{y}) =& \frac{L(\bar{x}, \bar{y}, \frac{\sum_i (x_i-\bar{x})^2+\sum_j (y_j-\bar{y})^2}{m+n}, \frac{\sum_i (x_i-\bar{x})^2+\sum_j (y_j-\bar{y})^2}{m+n})}{L(\bar{x}, \bar{y}, \frac{\sum_i (x_i-\bar{x})^2}{m}, \frac{\sum_j (y_j-\bar{y})^2}{n})} \\
=& \frac{[\frac{1}{m}\sum_i (x_i-\bar{x})^2]^{\frac{m}{2}} \cdot [\frac{1}{n}\sum_j (y_j-\bar{y})^2]^{\frac{n}{2}}}{[\frac{1}{m+n}(\sum_i (x_i-\bar{x})^2+\sum_j (y_j-\bar{y})^2)]^{\frac{m+n}{2}}} \\
=& \left(\frac{m+n}{m}\right)^{m/2}\left(\frac{m+n}{n}\right)^{n/2}\left(1+\frac{\sum_j (y_j-\bar{y})^2}{\sum_i (x_i-\bar{x})^2}\right)^{-m/2}\left(1+\frac{\sum_i (x_i-\bar{x})^2}{\sum_j (y_j-\bar{y})^2}\right)^{-n/2}.
\end{aligned}$$

考虑 $f(t) = -\frac{m}{2}\log(1+t) - \frac{n}{2}\log(1+1/t), t>0$; f 关于 t 先增后减，从而检验拒绝域为

$$\left\{(\boldsymbol{x},\boldsymbol{y}) \middle| \frac{\sum_j (y_j-\bar{y})^2}{\sum_i (x_i-\bar{x})^2} > c_1 \text{ 或} \frac{\sum_j (y_j-\bar{y})^2}{\sum_i (x_i-\bar{x})^2} < c_2\right\}, c_1 > c_2 > 0.$$

□

5.37.

$$\begin{aligned}
L(\theta_1,\theta_2) &= \frac{1}{\theta_1^m}\frac{1}{\theta_2^n}\exp\{-\frac{1}{\theta_1}\sum_i x_i - \frac{1}{\theta_2}\sum_j y_j\} \quad x_{(1)}, y_{(1)} > 0, \\
l(\theta_1,\theta_2) &= -m\ln\theta_1 - n\ln\theta_2 - \frac{1}{\theta_1}m\bar{x} - \frac{1}{\theta_2}n\bar{y}.
\end{aligned}$$

在 Θ 上，最大化对数似然，有 $\widehat{(\theta_1,\theta_2)}_{MLE} = (\bar{X},\bar{Y})$, 从而 $\sup_\Theta L = (\bar{x})^{-m}(\bar{y})^{-n}e^{-m-n}$.

在 Θ_0 上，最大化 $l(\theta) = -(m+n)\ln\theta - \frac{1}{\theta}(\sum_i x_i + \sum_j y_j)$, 最大值在 $\theta_0 = \frac{m\bar{x}+n\bar{y}}{m+n}$ 处达到。$\sup_{\Theta_0} L = (\theta_0 e)^{-m-n}$. 从而

$$\lambda(\boldsymbol{x},\boldsymbol{y}) = \left(\frac{m}{m+n} + \frac{n}{m+n}\frac{\bar{y}}{\bar{x}}\right)^{-m}\left(\frac{m}{m+n}\frac{\bar{x}}{\bar{y}} + \frac{n}{m+n}\right)^{-n}$$

与 32 题类似，可知拒绝域为 $\left\{(\boldsymbol{x},\boldsymbol{y}) \middle| \frac{\bar{x}}{\bar{y}} > c_1 \text{ 或} \frac{\bar{x}}{\bar{y}} < c_2\right\}, c_1 > c_2 > 0.$ □

[Wei] 7.18. Solve the kernel of the posterior

$$\begin{aligned}
\pi(\theta|\mathbf{x},\mathbf{y}) \propto& \pi(\theta) f(\mathbf{x},\mathbf{y}|\theta) \\
\propto& \exp\left\{-\frac{(a-\mu_1)^2}{2\tau_1^2} - \frac{(b-\mu_2)^2}{2\tau_2^2} - \frac{\sum_i (x_i-a)^2}{2} - \frac{\sum_j (y_j-b)^2}{2}\right\} \\
\propto& \exp\left\{-\frac{1}{2}\left[(\frac{1}{\tau_1^2}+m)a^2 - 2(\frac{\mu_1}{\tau_1^2}+\sum_i x_i)a\right] - \frac{1}{2}\left[(\frac{1}{\tau_2^2}+n)b^2 - 2(\frac{\mu_2}{\tau_2^2}+\sum_j y_j)b\right]\right\},
\end{aligned}$$

which is the same kernel as two independent normal distribution. Explicitly,

$$\theta|\mathbf{x},\mathbf{y} \sim N\left(\begin{pmatrix} \frac{\frac{\mu_1}{\tau_1^2}+\sum_i x_i}{\frac{1}{\tau_1^2}+m} \\ \frac{\frac{\mu_2}{\tau_2^2}+\sum_j y_j}{\frac{1}{\tau_2^2}+n} \end{pmatrix}, \begin{pmatrix} \frac{1}{\frac{1}{\tau_1^2}+m} & 0 \\ 0 & \frac{1}{\frac{1}{\tau_2^2}+n} \end{pmatrix}\right).$$

Therefore,

$$\begin{aligned} a-b|\mathbf{x},\mathbf{y} \sim & N\left(\frac{\frac{\mu_1}{\tau_1^2}+\sum_i x_i}{\frac{1}{\tau_1^2}+m} - \frac{\frac{\mu_2}{\tau_2^2}+\sum_j y_j}{\frac{1}{\tau_2^2}+n}, \frac{1}{\frac{1}{\tau_1^2}+m} + \frac{1}{\frac{1}{\tau_2^2}+n}\right) := N(\mu,\sigma^2), \\ \alpha_0 = & P(a-b<0|\mathbf{x},\mathbf{y}) = \Phi(-\frac{\mu}{\sigma}). \end{aligned}$$

In Bayes test, we reject the null hypothesis H_0 if $\frac{\alpha_0}{\alpha_1} \leq 1$, or equivalently $\alpha_0 \leq \frac{1}{2}$. Notice that $\alpha_0 \leq \frac{1}{2}$ if and only if $\mu \geq 0$. We conclude that

$$H_0 \text{ is } \begin{cases} \text{rejected}, & \frac{\frac{\mu_1}{\tau_1^2}+\sum_i x_i}{\frac{1}{\tau_1^2}+m} \geq \frac{\frac{\mu_2}{\tau_2^2}+\sum_j y_j}{\frac{1}{\tau_2^2}+n} \\ \text{accepted}, & otherwise \end{cases}.$$

$\square$

Assignment 1. Use Theorem 1.1 (Find LRT by sufficient statistics) to find the LRT statistics for Example 1.1(2): $X_1,\dots,X_n \sim N(\mu,\sigma^2)$, *i.i.d.* with σ^2 unknown. Test $H_0 : \mu=\mu_0, \leftrightarrow H_1 : \mu \neq \mu_0$

Solve: Notice that $T(\mathbf{X}) = (\bar{X}, S^2)$ is sufficient for (μ,σ^2) with independent distributions $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, $\frac{n-1}{\sigma^2}S^2 \sim \chi^2_{n-1}$. The likelihood function is

$$L(\mu,\sigma^2|\bar{x},s^2) = \sqrt{\frac{n}{2\pi\sigma^2}}e^{-\frac{n(\bar{x}-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\Gamma(\frac{n-1}{2})2^{\frac{n-1}{2}}}\left(\frac{n-1}{\sigma^2}s^2\right)^{\frac{n-1}{2}-1} e^{-\frac{n-1}{2\sigma^2}s^2} \cdot \frac{n-1}{\sigma^2}.$$

Or consider the log-likelihood

$$l(\mu,\sigma^2|\bar{x},s^2) = C - \frac{n}{2}\log(\sigma^2) - \frac{n(\bar{x}-\mu)^2}{2\sigma^2} - \frac{n-1}{2\sigma^2}s^2,$$

where C is a constant independent on (μ,σ^2).

In Θ, the superior of L is reached at $(\bar{x}, \frac{(n-1)s^2}{n})$. In Θ_0 with $\mu=\mu_0$, L is maximized at $(\mu_0, \frac{n(\bar{x}-\mu_0)^2+(n-1)s^2}{n})$. Thus from Theorem 1.1,

$$\lambda(\mathbf{x}) = \frac{L(\mu_0, \frac{n(\bar{x}-\mu_0)^2+(n-1)s^2}{n})}{L(\bar{x}, \frac{(n-1)s^2}{n})} = \left(\frac{(n-1)s^2}{n(\bar{x}-\mu_0)^2+(n-1)s^2}\right)^{\frac{n}{2}} = \left(1+\frac{n(\bar{x}-\mu_0)^2}{(n-1)s^2}\right)^{-\frac{n}{2}}.$$

So the deny domain is $\{\frac{(\bar{x}-\mu_0)^2}{S^2} > c\}$. $\square$

Assignment 2. A restaurant will have a rest day after making profit for accumulated 5 days. The random variable X_i denotes whether or not the restaurant makes profit at the i-th day and is supposed to be independent with each other. $X_i = 1$ if making profit, otherwise $X_i = 0$. The probability of making profit is $P(X_i = 1) = \theta$. Suppose the prior distribution of θ is $U(0,1)$ and the restaurant had a rest day after working for 6 days. Which hypothesis is true? $H_0 : \theta > 0.5, \leftrightarrow H_1 : \theta \leq 0.5$

Solve: Notice that

$$\begin{aligned}
&P(\{\text{rest after six workdays}\}|\theta) \\
=&\sum_{i=1}^{5} P(\{\text{making profit except for the } i\text{-th day}\}|\theta) \\
=&5\theta^5(1-\theta).
\end{aligned}$$

Write the kernel of the posterior, we obtain that the posterior distribution is $Beta(6,2)$ with posterior $p.d.f.$ $f(\theta) = \frac{1}{B(6,2)}\theta^5(1-\theta)$. Thus,

$$\begin{aligned}
\alpha_1 =& \int_0^{0.5} f(\theta)d\theta \\
=&\frac{1}{B(6,2)}\left(\frac{1}{6}\theta^6 - \frac{1}{7}\theta^7\right)\Big|_{\theta=0}^{0.5} \\
=&\frac{1}{16} < \frac{1}{2},
\end{aligned}$$

which is equivalent to $\frac{\alpha_0}{\alpha_1} > \frac{1}{2}$. Therefore, we accept H_0, that is, H_0 is true. $\square$

[Wei] 5.2.

(1) Notice that $\bar{X} \sim N(\mu, \frac{9}{n})$, or equivalently $Y := \sqrt{\frac{n}{9}}(\bar{X}-\mu) \sim N(0,1)$, thus

$$\begin{aligned}
0.05 = \alpha =& P\left(|\bar{X}-\mu_0| \geq c|H_0\right) \\
=&P\left(|Y| \geq \sqrt{\frac{n}{9}}c|H_0\right) \\
=&2\Phi\left(-\sqrt{\frac{n}{9}}c\right).
\end{aligned}$$

The solution of the equation is $c = -\sqrt{\frac{9}{n}}\Phi^{-1}(0.025) = \frac{5.88}{\sqrt{n}}$.

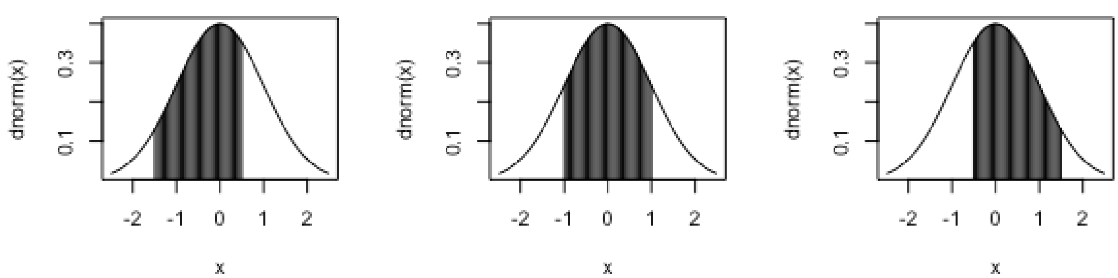
(2)

$$\begin{aligned}\beta(\mu) =& P_\mu\left(|\bar{X}-\mu_0| \geq c\right)\\ =& P_\mu\left(\bar{X} \geq \mu_0 + c \text{ or } \bar{X} \leq \mu_0 - c\right)\\ =& P\left(Y \geq \sqrt{\frac{n}{9}}(\mu_0 + c - \mu)\right) + P\left(Y \leq \sqrt{\frac{n}{9}}(\mu_0 - c - \mu)\right)\\ =& 1 - \Phi\left(\frac{\sqrt{n}}{3}(\mu_0 - \mu + c)\right) + \Phi\left(\frac{\sqrt{n}}{3}(\mu_0 - \mu - c)\right).\end{aligned}$$

(3) From the result above, we have

$$\begin{aligned}\alpha =& \beta(\mu_0) = 1 - \Phi\left(\frac{5}{3}c\right) + \Phi\left(-\frac{5}{3}c\right) = 2\Phi\left(-\frac{5}{3}c\right),\\ \beta =& 1 - \beta(\mu) = \Phi\left(\frac{5}{3}(\mu_0 - \mu + c)\right) - \Phi\left(\frac{5}{3}(\mu_0 - \mu - c)\right).\end{aligned}$$

Notice that β is increasing when $\mu \in (-\infty, \mu_0)$ and decreasing otherwise, and is centered at μ_0. (See the following plots for an illustration. The shaded area stands for β.)



Therefore, we compare α with β in different cases that

$$\begin{cases}\alpha \geq \beta, & |\mu_0 - \mu| \geq C\\ \alpha < \beta, & |\mu_0 - \mu| < C\end{cases},$$

where $C \geq 0$ is the smallest constant such that $\Phi\left(\frac{5}{3}c\right) - \Phi\left(-\frac{5}{3}c\right) + \Phi\left(\frac{5}{3}(C+c)\right) - \Phi\left(\frac{5}{3}(C-c)\right) \leq 1$. $\square$

Remark: because of the continuity of the left-hand side and the decreasing property, the choice of C can be expressed as: if $\Phi\left(\frac{5}{3}c\right) < 0.25$, then

$$\begin{cases} \alpha \geq \beta, & |\mu_0 - \mu| \geq C \\ \alpha < \beta, & |\mu_0 - \mu| < C \end{cases},$$

where $C > 0$ satisfies $\Phi\left(\frac{5}{3}c\right) - \Phi\left(-\frac{5}{3}c\right) + \Phi\left(\frac{5}{3}(C+c)\right) - \Phi\left(\frac{5}{3}(C-c)\right) = 1$. Otherwise, $\alpha > \beta$.

[Wei] 5.4. By factorization theorem, $X_{(n)}$ is sufficient for θ.

The probability of type I errors is

$$P(X_{(n)} \leq 1.5|\theta \in \Theta_0) = P(X_1, \ldots, X_n \leq 1.5|\theta \geq 2) = \left(\frac{1.5}{\theta}\right)^n, \quad \theta \geq 2.$$

Its maximum is $\left(\frac{3}{4}\right)^n$. $\square$