HOMEWORK 7

2021/11/17

[Wei] 3.30. First, $ES^2 = \frac{1}{n-1}(nEX_1^2 - nE\bar{X}^2) = \sigma^2$, which is unbiased.

Notice that $(\bar{X}, S^2)$ is sufficient for (a, σ^2) and from theorem 2.2.3 (or example 3.4.6), we only need to consider zero unbiased estimate like $U(\bar{X}, S^2)$. We can write $p.d.f$ of $(\bar{X}, S^2)$ as

$$
\begin{aligned}
f_{\bar{X},S^2}(x,y) &= \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} e^{-\frac{n(x-a)^2}{2\sigma^2}} \frac{(n-1)^{(n-1)/2}}{2^{\frac{n-1}{2}}\Gamma((n-1)/2)\sigma^{n-1}} y^{\frac{n-1}{2}-1} e^{-\frac{y}{2\sigma^2}} \\
&= \frac{\sqrt{n}(n-1)^{(n-1)/2}}{\sqrt{\pi}2^{\frac{n}{2}}\Gamma((n-1)/2)} y^{\frac{n-1}{2}-1} (\sigma^2)^{-\frac{n}{2}} e^{-\frac{n(x-a)^2+y}{2\sigma^2}}, \quad x \in \mathbb{R},\ y > 0.
\end{aligned}
$$

For any zero unbiased estimate , we have

$$0 = EU = \int_{(0,\infty)} \int_{\mathbf{R}} U(x,y) f_{\bar{X},S^2}(x,y) dxdy.$$

Taking partial derivative of σ^2 on both sides yields

$$
\begin{aligned}
0 &= \int_{(0,\infty)} \int_{\mathbf{R}} U(x,y) f_{\bar{X},S^2}(x,y) \left(-\frac{n}{2\sigma^2} + \frac{n(x-a)^2 + y}{2\sigma^4} \right) dxdy \\
&= Cov(U, \frac{n(\bar{X}-a)^2 + S^2}{2\sigma^4}).
\end{aligned}
$$

Similarly, taking twice derivative of a on both sides, we know that

$$Cov(U, \frac{n^2(\bar{X}-a)^2}{\sigma^4}) = 0.$$

Thus, $Cov(U, S^2) = 0$ and S^2 is the UMVUE for σ^2. □

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[Wei] 3.40. From the *p.d.f*

$$f(\mathbf{x}|\theta) = \frac{1}{\theta^n} e^{-\frac{\sum_i x_i}{\theta}} I_{(0,\infty)}(x_{(1)}),$$

By factorization theorem we know that $\sum_i X_i$ is sufficient. Moreover, the p.d.f. is in exponential distribution and $\Theta^* = \{-\frac{1}{\theta}\} = (-\infty, 0)$ is open in $\mathbb{R}$, so $\sum_i X_i$ is sufficient and complete.

Using Lehmann-Scheff and the fact that $E\bar{X} = EX_1 = \theta$, $\bar{X}$ is the UMVUE for θ and the variance is $Var(\bar{X}) = \frac{\theta^2}{n}$.

Since the distribution is in the exponential distribution family, we can calculate the Fisher information by

$$I(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(x,\theta)\right] = -E_\theta\left[\frac{1}{\theta^2} - \frac{2X}{\theta^3}\right] = \frac{1}{\theta^2}.$$

Thus the C-R lower bound is $\frac{(\theta')^2}{nI(\theta)} = \frac{\theta^2}{n}$.

Variance of UMVUE and C-R lower bound are the same. □

Method 2: one can first calculate the C-R lower bound, and check that $\bar{X}$ is unbiased and reaches the bound, consequently it is the UMVUE.

[Wei] 3.44. From the *p.d.f*

$$f(\mathbf{x}|\sigma^2) = (\frac{1}{2\pi\sigma^n})^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2},$$

by factorization theorem we know that $\sum_i X_i^2$ is sufficient. Moreover, the p.d.f. is in exponential distribution and $\Theta^* = \{-\frac{1}{2\sigma^2}\} = (-\infty, 0)$ is open in $\mathbb{R}$, so $\sum_i X_i 1^2$ is sufficient and complete.

Notice that $X_1^2 \sim \sigma^2\chi_1^2 = \Gamma(\frac{1}{2}, \frac{1}{2}), \sum_{i=1}^n X_i^2 \sim \Gamma(\frac{n}{2}, \frac{1}{2\sigma^2})$, checking the unbiaseness of $\hat{\sigma}$ is trivial. Moreover, $E\hat{\sigma}^2 = \frac{n\Gamma^2(\frac{n}{2})}{2\Gamma^2(\frac{n+1}{2})}\sigma^2$ and $Var(\hat{\sigma}) = \left(\frac{n\Gamma^2(\frac{n}{2})}{2\Gamma^2(\frac{n+1}{2})} - 1\right)\sigma^2$. Using L-S to prove that $\hat{\sigma}$ is UMVUE.

Next, consider the efficiency. Since the distribution is in the exponential distribution family, we can calculate the Fisher information by

$$I(\sigma) = -E_\sigma\left[\frac{\partial^2}{\partial\sigma^2}\log\left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}\right)\right] = -E_\sigma\left[\frac{1}{\sigma^2} - \frac{3X^2}{\sigma^4}\right] = \frac{2}{\sigma^2}.$$

Thus the efficiency is

$$\frac{(\sigma')^2}{nI(\sigma)Var(\hat{\sigma})} = \frac{1}{2n\left(\frac{n\Gamma^2(\frac{n}{2})}{2\Gamma^2(\frac{n+1}{2})} - 1\right)}.$$

□

One can rewrite the efficiency as $\frac{1}{n^2\frac{\Gamma^2(\frac{n}{2})}{\Gamma^2(\frac{n+1}{2})}-2n}$ *or* $\frac{1}{2n\left(\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}+1)}{\Gamma^2(\frac{n+1}{2})}-1\right)}$.

One can also consider σ^2 *as parameter, in that case* $I(\sigma^2)=\frac{1}{2\sigma^4}$.

2021/11/19

[Wei] 3.31. (1)From 3.44, $\sum_{i=1}^n X_i^2$ is sufficient and complete.

(2) Use L-S theorem.

From 3.44,

$$\sum_i X_i^2 \sim \Gamma(\frac{n}{2},\frac{1}{2\sigma^2})$$

and $\hat{\sigma}=\frac{\Gamma(\frac{n}{2})}{\sqrt{2}\Gamma(\frac{n+1}{2})}(\sum_i X_i^2)^{\frac{1}{2}}$ is the UMVUE for σ.

Since $E(\sum_i X_i^2)^2=\frac{1}{2}/(\frac{n}{2\sigma^2})^2+(\frac{n}{2}/(\frac{1}{2\sigma^2}))^2=n(n+2)\sigma^4$, $\widehat{3\sigma^4}=\frac{3}{n(n+2)}(\sum_i X_i^2)^2$ is the UMVUE for $3\sigma^4$. □

[Wei] 3.32. From Example 3.4.6, we know that $(\bar{X},S^2)$ is sufficient and complete with

$$E\bar{X}=\mu,\quad E\bar{X}^2=\frac{\sigma^2}{n}+\mu^2,\quad ES^2=\sigma^2,\quad E\frac{1}{S^2}=\frac{n-1}{\sigma^2}E\frac{1}{\chi^2_{n-1}}=\frac{n-1}{n-3}\cdot\frac{1}{\sigma^2},$$

where $\bar{X}$ and S^2 are independent with each other. Moreover,

$$E\left[\frac{n-3}{4(n-1)}\frac{\bar{X}^2}{S^2}-\frac{1}{4n}\right]=\frac{\mu^2}{4\sigma^2}$$

Use L-S theorem.

(1) $3\bar{X}+4S^2$.

(2) $\frac{n-3}{4(n-1)}\frac{\bar{X}^2}{S^2}-\frac{1}{4n}$. □

[Wei] 3.33. First check that $T:=\sum_i X_i\sim \textsf{\textit{Binomial}}(n,p)$ is sufficient and complete.

(1) From $p^s=P(X_1=\ldots=X_s=1)=E[I(X_1=\ldots=X_s=1)]$ and L-S theorem, the UMVUE for p^s is

$$\begin{aligned}\widehat{p^s}=&E[I(X_1=\ldots=X_s=1)|T]\\=&P[I(X_1=\ldots=X_s=1)|T]\\=&\begin{cases}\frac{\binom{n-s}{T-s}}{\binom{n}{T}}, & T\ge s\\ 0, & T<s\end{cases}.\end{aligned}$$

(2) Similar to (1), from $p^s + (1-p)^{n-s} = E[I(X_1 = \ldots = X_s = 1) + I(X_{s+1} = \ldots = X_n = 0)]$, its UMVUE is

$$\begin{cases} \frac{\binom{n-s}{T-s}}{\binom{n}{T}}, & T > s \\ \frac{2}{\binom{n}{T}}, & T = s \\ \frac{\binom{s}{T}}{\binom{n}{T}}, & T < s \end{cases}.$$

□

[Wei] 3.36. First check that $T_n := \sum_{i=1}^n X_i \sim \Gamma(n,\theta)$ is sufficient and complete. With the hint $e^{-\theta\tau} = P_\theta(X_1 > \tau) = E[I(X_1 > \tau)]$ and motivated by L-S theorem, calculate

$$\begin{aligned} E[I(X_1 > \tau)|T_n = t] =& P(X_1 > \tau | T_n = t) \\ =& \frac{\int_\tau^t f_{X_1}(x) f_{T_{n-1}}(t-x)dx}{f_{T_n}(t)} \\ =& \frac{\int_\tau^t \theta e^{-\theta x} \frac{\theta^{n-1}}{(n-2)!}(t-x)^{n-2} e^{-\theta(t-x)}dx}{\frac{\theta^n}{(n-1)!} t^{n-1} e^{-\theta t}} \\ =& (1 - \frac{\tau}{t})^{n-1}, \quad t > \tau. \end{aligned}$$

Therefore, the UMVUE for $e^{-\theta\tau}$ is $(1 - \frac{\tau}{T})^{n-1}$. □