HOMEWORK 11

[Wei] 4.1. Notice that $Q(\mathbf{X},\mu)=\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}\sim N(0,1)$ is a pivot. Then a 0.95 confidence interval is $\{-z_{0.05/2}\le Q\le z_{0.05/2}\}$. With $\bar{X}=4.7832$, $\sigma=0.01$, $n=5$, the confidence interval is

$$[\bar{X}-\frac{\sigma}{\sqrt{n}}z_{0.025},\bar{X}+\frac{\sigma}{\sqrt{n}}z_{0.025}]=[4.774,4.792].$$

□

[Wei] 4.2. Notice that $Q(\mathbf{X},\mu)=\frac{\sqrt{n}(\bar{X}-\mu)}{S}\sim t_{n-1}$ is a pivot. Then a $1-\alpha$ confidence interval is $\{t_{n-1}(\alpha/2)\le Q\le t_{n-1}(\alpha/2)\}$, that is,

$$[\bar{X}-\frac{S}{\sqrt{n}}t_{n-1}(\alpha/2),\bar{X}+\frac{S}{\sqrt{n}}t_{n-1}(\alpha/2)].$$

With $\bar{X}=4.7832$, $S=0.0105$, $n=5$, the confidence interval is $[4.770,4.796]$ for $\alpha=0.05$ and $[4.762,4.805]$ for $\alpha=0.01$. □

[Wei] 4.3. Similar to Exercise 4.2, a 0.05 confidence interval is $[\bar{X}-\frac{S}{\sqrt{n}}t_{n-1}(\alpha/2),\bar{X}+\frac{S}{\sqrt{n}}t_{n-1}(\alpha/2)]=[1784,2116]$. □

[Wei] 4.4. Here we have

$$\frac{\sqrt{n}(\bar{X}-\mu)}{4}\sim N(0,1).$$

and we may transform the probability as

$$P(\bar{X}-1<\mu<\bar{X}+1)=P(-\frac{\sqrt{n}}{4}<Z<\frac{\sqrt{n}}{4})\ge 0.9.$$

where $Z\sim N(0,1)$. We also note that $z_{\alpha/2}=1.65$, which implies that $\frac{\sqrt{n}}{4}\ge 1.65$, thus we have $n\ge 44$.

[Wei] 4.11. Notice that $Q=\frac{\bar{A}-\bar{B}-(a_1-a_2)}{S}\sqrt{\frac{mn}{m+n}}\sim t_{m+n-2}$ is a pivot, where $S^2=\frac{(m-1)S_A^2+(n-1)S_B^2}{m+n-2}$ (See Page 137 for more details). Then a $1-\alpha$ confidence interval is

$$[\bar{A}-\bar{B}-S\sqrt{\frac{m+n}{mn}}t_{n-1}(\alpha/2),\bar{A}-\bar{B}+S\sqrt{\frac{m+n}{mn}}t_{n-1}(\alpha/2)].$$

With $\bar{A}=0.14125$, $\bar{B}=0.1392$, $S=0.00255$, $m=4$, $n=5$, the 0.05 confidence interval is $[-0.002,0.006]$ for a_1-a_2. □

[Wei] 4.12. From Page 138, $Q=\frac{\bar{Y}-\bar{X}-(b-a)}{\sqrt{S_1^2/m+S_2^2/n}}\xrightarrow{\mathscr{L}}N(0,1)$ is a pivot. Then an asymptotic $1-\alpha$ confidence interval for $b-a$ is

$$[\bar{Y}-\bar{X}-\sqrt{S_1^2/m+S_2^2/n}z_{\alpha/2},\bar{Y}-\bar{X}+\sqrt{S_1^2/m+S_2^2/n}z_{\alpha/2}].$$

With the given data, the confidence interval is $[-155,-95]$. □

[Wei] 4.14. Facts:

(1) $\frac{\bar{Y}-\bar{X}-(b-a)}{\sqrt{\sigma_1^2/m+\sigma_2^2/n}} \sim N(0,1)$.

(2) $\frac{(m-1)S_1^2}{\sigma_1^2} + \frac{(n-1)S_2^2}{\sigma_2^2} \sim \chi^2_{m+n-2}$.

Since $\sigma_2^2 = \lambda\sigma_1^2$, $Q = \frac{\frac{\bar{Y}-\bar{X}-(b-a)}{\sqrt{\sigma_1^2/m+\sigma_2^2/n}}}{\sqrt{\frac{1}{m+n-2}\left(\frac{(m-1)S_1^2}{\sigma_1^2}+\frac{(n-1)S_2^2}{\sigma_2^2}\right)}} = \frac{\frac{\bar{Y}-\bar{X}-(b-a)}{\sqrt{1/m+\lambda/n}}}{\sqrt{\frac{1}{m+n-2}\left((m-1)S_1^2+(n-1)S_2^2/\lambda\right)}} \sim t_{m+n-2}$ is a pivot. Then a $1-\alpha$ confidence interval for $b-a$ is

$$\bar{Y} - \bar{X} \pm \sqrt{\frac{(1/m+\lambda/n)\big((m-1)S_1^2 + (n-1)S_2^2/\lambda\big)}{(m+n-2)}}\, t_{m+n-2}(\alpha/2).$$

(I abbreviate the interval expression since it's too long.)

[Wei] 4.10. The problem should be revised as "how many should n be at most?" Notice that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, and from the definition of lower confidence limit,

$$0.95 \leq P\left(\sigma \geq \frac{\sqrt{(n-1)S^2}}{4}\right) = P(\chi^2_{n-1} \leq 16).$$

Thus we have $n \leq 9$. $\square$

[Wei] 4.15.

(1) According to [Wei] Page 138, in the case that m and n are not sufficiently large, an approximate $1-\alpha$ confidence interval for $\mu_2 - \mu_1$ is

$$[\bar{Y} - \bar{X} - S_* t_r(\alpha/2), \bar{Y} - \bar{X} + S_* t_r(\alpha/2)],$$

where $S_*^2 = \frac{S_1^2}{m} + \frac{S_2^2}{n} = 177.73$, $r = S_*^4/\left[\frac{S_1^4}{m^2(m-1)} + \frac{S_2^4}{n^2(n-1)}\right] = 7.77 \approx 8$. Therefore, the confidence interval should be $[-39.72, 21.76]$. $\square$

(2) According to [Wei] Page 139 (4.2.13), a $1-\alpha$ confidence interval for σ_1^2/σ_2^2 is

$$\left[\frac{S_1^2}{S_2^2}\frac{1}{F_{m-1,n-1}(\alpha/2)}, \frac{S_1^2}{S_2^2}\frac{1}{F_{m-1,n-1}(1-\alpha/2)}\right].$$

With the given data, it should be $[4.88, 79.36]$. $\square$

[Wei] 4.19. We only consider the case $n > 1$.

(1) Let $Y_i = \frac{X_i-\theta_1}{\theta_2-\theta_1} \sim U(0,1)$, $Z_i = \frac{\theta_2-X_i}{\theta_2-\theta_1} \sim U(0,1)$, $Q = \frac{(\theta_2-\theta_1)-(X_{(n)}-X_{(1)})}{\theta_2-\theta_1}$ then $Q = Y_{(1)} + Z_{(1)}$ and

$$f_{Y_{(1)},Z_{(1)}}(y,z) = f_{Y_{(1)},Y_{(n)}}(y, 1-z) = n(n-1)(1-z-y)^{n-2} I_{(0,1)}(y,z,y+z).$$

Therefore, Q is a pivot with $p.d.f.$

$$f_Q(x) = \int f_{Y_{(1)},Z_{(1)}}(y, x-y)dy = n(n-1)x(1-x)^{n-2} I_{(0,1)}(x).$$

We can observe that $Q \sim Beta(2, n-1)$.

Suppose $P(Q \in (1-c, 1)) = 1-\alpha$ (this assumption is convenient for following analysis, you can generally suppose a confidence interval (a, b) instead), then $c = Beta_{n-1,2}(\alpha)$. Also notice that $Q = 1 - \frac{(X_{(n)} - X_{(1)})}{\theta_2 - \theta_1}$,

$$Q \in (1-c, 1) \Leftrightarrow \theta_2 - \theta_1 \in \big((X_{(n)} - X_{(1)})/c, \infty\big).$$

As a result, a $1-\alpha$ confidence interval for $\theta_2 - \theta_1$ is

$$\big((X_{(n)} - X_{(1)})/Beta_{n-1,2}(\alpha), \infty\big).$$

□

(2) Hint: the objective statistic can be rewritten as $R := \frac{Y_{(1)} + Y_{(n)} - 1}{Y_{(n)} - Y_{(1)}}$, which is auxiliary. Notice that $(X_{(1)}, X_{(n)})$ is sufficient and complete for (θ_1, θ_2). Basu's theorem tolds us $S := Y_{(n)} - Y_{(1)}$ is independent with R. Now write the $p.d.f$ of $(Y_{(1)}, Y_{(n)})$. With the relationship

$$\begin{cases} Y_{(1)} = \frac{1}{2}(RS - S + 1) \\ Y_{(n)} = \frac{1}{2}(RS + S + 1) \end{cases},$$

write the $p.d.f$ of (R, S). Consequently, $f_R = f_{(R,S)}/f_S$. The explicit form is very complicated, and should be discussed with nodes $0, \pm 1$. □

(3) Using R as the pivot. □

[Wei] 4.23.

(1) $X_{(1)} - \theta \sim Exp(n)$. □

(2) Using $Q := X_{(1)} - \theta$ in (1) as a pivot, suppose the interval as $(0, c)$ for Q. After calculation, the interval for θ is $(X_{(1)} + \log \alpha/n, X_{(1)})$. □

[Wei] 5.56, 5.57. See [Wei] Page 135 (4.2.4), Page 141 (4.2.13) for the answer. But take exercise on using inverting hypothesis method!

	5.56	5.57
CI	$[\frac{(n-1)S^2}{\chi^2_{n-1}(\alpha/2)}, \frac{(n-1)S^2}{\chi^2_{n-1}(1-\alpha/2)}]$	$[\frac{S_1^2}{S_2^2}\frac{1}{F_{m-1,n-1}(\alpha/2)}, \frac{S_1^2}{S_2^2}\frac{1}{F_{m-1,n-1}(1-\alpha/2)}]$
CL	$\frac{(n-1)S^2}{\chi^2_{n-1}(\alpha)}$	$\frac{S_1^2}{S_2^2}\frac{1}{F_{m-1,n-1}(\alpha)}$
CU	$\frac{(n-1)S^2}{\chi^2_{n-1}(1-\alpha)}$	$\frac{S_1^2}{S_2^2}F_{n-1,m-1}(\alpha)$

□