HOMEWORK 10

[Wei] 5.42.

(1) Notice that $X_{(n)}$ is sufficient with $p.d.f.$

$$f(x|\theta) = n\frac{x^{n-1}}{\theta^n} I_{(0,\theta)}(x).$$

For any $\theta_2 > \theta_1$,

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = \begin{cases} \frac{\theta_1^n}{\theta_2^n}, & 0 < x < \theta_1 \\ +\infty, & \theta_1 \le x < \theta_2 \end{cases},$$

which is monotone non-decreasing with respect to x. Therefore, its family of $p.d.f.$s has MLR and by Karlin-Rubin theorem, the reject region for a level α UMPT should be $\{X_{(n)} > x_0\}$, where x_0 can be solved by

$$\alpha = P(X_{(n)} > x_0|H_0) = \int_{x_0}^{1} nx^{n-1}dx = 1 - x_0^n.$$

In conclusion, the level α UMPT rejects H_0 if $X_{(n)} > (1-\alpha)^{\frac{1}{n}}$. □

(2) Analogously, using K-R theorem, the reject region is $\{X_{(n)} < x_0\}$. Solving the equation

$$\alpha = P(X_{(n)} < x_0|H_0) = \int_{0}^{x_0} nx^{n-1}dx = x_0^n$$

yields that the level α UMPT rejects H_0 if $X_{(n)} < \alpha^{\frac{1}{n}}$. □

[Wei] 5.43. Notice that $\bar{X} \sim N(\mu, \frac{1}{n})$ is sufficient with $p.d.f.$

$$f(x|\mu) = \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n}{2}(x-\mu)^2}.$$

For any $\mu_2 > \mu_1$,

$$\frac{f(x|\mu_2)}{f(x|\mu_1)} = e^{-\frac{n}{2}(x-\mu_2)^2 + \frac{n}{2}(x-\mu_1)^2} = e^{\frac{n}{2}(\mu_2-\mu_1)(2x-\mu_1-\mu_2)},$$

which is monotone non-decreasing with respect to x. Therefore, its family of $p.d.f.$s has MLR and by K-R theorem, the reject region for a level α UMPT should be $\{\bar{X} < x_0\}$, where x_0 can be solved by

$$\alpha = P(X_{(n)} < x_0|\mu_0) = P\big(\sqrt{n}(X_{(n)} - \mu_0) < \sqrt{n}(x_0 - \mu_0)|\mu_0\big) = \Phi\big(\sqrt{n}(x_0 - \mu_0)\big).$$

In conclusion, the UMPT rejects H_0 if $\bar{X} < \mu_0 - \frac{z_\alpha}{\sqrt{n}}$, where $\Phi(z_\alpha) = 1 - \alpha$. □

Date: 2018/12/21.

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[Wei] 5.44. Notice that $\sum_i X_i \sim Poisson(n\lambda)$ is sufficient with $p.m.f.$

$$P(x|\lambda) = e^{-n\lambda}\frac{(n\lambda)^x}{x!}, \quad x \in \mathbb{Z}_{\geq 0}.$$

For any $\lambda_2 > \lambda_1$,

$$\frac{P(x|\lambda_2)}{P(x|\lambda_1)} = e^{n(\lambda_1-\lambda_2)}\left(\frac{\lambda_2}{\lambda_1}\right)^x,$$

which is monotone non-decreasing with respect to x. Therefore, its family of $p.d.f.$s has MLR and by K-R theorem, the reject region for a level α UMPT should be $\{\sum_i X_i < x_0\}$, where x_0 satisfies

$$\begin{cases} P(\sum_i X_i < x_0|\lambda_0) = \sum_{x=0}^{\lceil x_0\rceil-1} e^{-n\lambda}\frac{(n\lambda)^x}{x!} \leq \alpha \\ \sum_{x=0}^{\lceil x_0\rceil} e^{-n\lambda}\frac{(n\lambda)^x}{x!} > \alpha \end{cases}.$$

In conclusion, the UMPT rejects H_0 if $\sum_i X_i \leq x_0$, where $x_0 \in \mathbf{Z}_{\geq 0}$ satisfies

$$\begin{cases} \sum_{x=0}^{x_0} e^{-n\lambda}\frac{(n\lambda)^x}{x!} \leq \alpha \\ \sum_{x=0}^{x_0+1} e^{-n\lambda}\frac{(n\lambda)^x}{x!} > \alpha \end{cases}.$$

Besides, if $e^{-n\lambda} > \alpha$, we always accept H_0. □

[Wei] 5.45. Notice that $\sum_i X_i^2 \sim \sigma^2\chi_n^2$ is sufficient with $p.d.f.$

$$f(x|\sigma) = \frac{1}{\Gamma(n/2)2^{n/2}}x^{n/2-1}e^{-\frac{x}{2\sigma^2}}(\sigma^2)^{-n/2}.$$

For any $\sigma_2^2 > \sigma_1^2$,

$$\frac{f(x|\sigma_2)}{f(x|\sigma_1)} = e^{\frac{x}{2}(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_2^2})}\left(\frac{\sigma_1^2}{\sigma_2^2}\right)^{n/2},$$

which is monotone non-decreasing with respect to x. Therefore, its family of $p.d.f.$s has MLR and by K-R theorem, the reject region for a level α UMPT should be $\{\sum_i X_i^2 > x_0\}$, where x_0 can be solved by

$$\alpha = P(\sum_i X_i^2 > x_0|\sigma_0^2) = P\big(\chi_n^2 > \frac{x_0}{\sigma_0^2}\big).$$

Thus, the UMPT rejects H_0 if $\sum_i X_i^2 > \sigma_0^2\chi_n^2(\alpha)$, where $P\big(\chi_n^2 > \chi_n^2(\alpha)\big) = \alpha$. □

[Wei] 5.46. Notice that $\sum_i X_i \sim b(n,p)$ is sufficient with $p.m.f.$

$$P(x|p) = \binom{n}{x}p^x(1-p)^{n-x} = \binom{n}{x}(1-p)^n\left(\frac{p}{1-p}\right)^x, \quad x = 0,1,\ldots,n.$$

For any $p_2 > p_1$,

$$\frac{P(x|p_2)}{P(x|p_1)} = \left(\frac{1-p_2}{1-p_1}\right)^n\left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)^x,$$

which is monotone non-decreasing with respect to x. Therefore, its family of $p.d.f.$s has MLR and by K-R theorem, the reject region for a level α UMPT should be $\{\sum_i X_i < x_0\}$, where x_0 satisfies

$$\begin{cases} P(\sum_i X_i < x_0 | p = \frac{1}{2}) = \frac{\sum_{x=0}^{\lceil x_0 \rceil - 1} \binom{n}{x}}{2^n} \leq \alpha \\ \frac{\sum_{x=0}^{\lceil x_0 \rceil} \binom{n}{x}}{2^n} > \alpha \end{cases}.$$

In conclusion, the UMPT rejects H_0 if $\sum_i X_i \leq x_0$, where $x_0 \in \mathbf{Z}_{\geq 0}$ satisfies

$$\begin{cases} \sum_{x=0}^{x_0} \binom{n}{x} \leq \alpha \cdot 2^n \\ \sum_{x=0}^{x_0+1} \binom{n}{x} > \alpha \cdot 2^n \end{cases}.$$

Besides, if $\alpha < 2^{-n}$, we always accept H_0. □

[Wei] 5.48. Notice that $\sum_i X_i \sim \Gamma(n, \lambda)$ is sufficient with $p.d.f.$

$$f(x|\lambda) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}, \quad x > 0.$$

Also from Ex. 2.26, $2\lambda \sum_i X_i \sim \chi^2_{2n}$.

A trick we use here is to consider an equivalent test:

$$H_0 : -\lambda \leq -\lambda_0 \leftrightarrow H_1 : -\lambda > -\lambda_0,$$

where we regard $-\lambda$ as the parameter. For any $-\lambda_2 > -\lambda_1$ ($i.e.$, $\lambda_2 < \lambda_1$),

$$\frac{f(x| - \lambda_2)}{f(x| - \lambda_1)} = \left(\frac{\lambda_2}{\lambda_1}\right)^n e^{(\lambda_1 - \lambda_2)x},$$

which is monotone non-decreasing with respect to x. Therefore, its family of $p.d.f.$s has MLR and by K-R theorem, the reject region for a level α UMPT should be $\{\sum_i X_i > x_0\}$, where x_0 can be solved by

$$\alpha = P(\sum_i X_i > x_0 | -\lambda_0) = P\big(\chi^2_{2n} > 2\lambda_0 x_0\big).$$

Thus, the UMPT rejects H_0 if $\sum_i X_i^2 > \frac{\chi^2_{2n}(\alpha)}{2\lambda_0}$, where $P\big(\chi^2_{2n} > \chi^2_{2n}(\alpha)\big) = \alpha$. □

Assignment: $X_1, \ldots, X_n$ *i.i.d.* $\sim$ *Exp*(λ). **Solve the level** $\alpha = 0.1$ **UMPT for** $H_0 :\ \lambda \leq 1$ **or** $\lambda \geq 2$**,** $H_1 :\ 1 < \lambda < 2$**.**

Solve: Notice that the joint $p.d.f.$ is

$$f(\mathbf{x}|\lambda) = \lambda^n e^{-\lambda \sum_i x_i} I_{(0,\infty)}(x_{(1)}),$$

where λ is strictly increasing with respect to λ and $T(\mathbf{X}) := -\sum_i X_i$. Then the reject region for a level α UMPT should be $\{t_1 < T < t_2\}$. Similar to Ex. 5.48, we know that $-2\lambda T \sim \chi^2_{2n}$, thus t_1, t_2 can be solved by

$$\begin{cases} \alpha = P(t_1 < T < t_2 | \lambda = 1) = P\big(-2t_1 > \chi^2_{2n} > -2t_2\big) = \chi^2_{2n}(-2t_1) - \chi^2_{2n}(-2t_2) \\ \alpha = P(t_1 < T < t_2 | \lambda = 2) = P\big(-4t_1 > \chi^2_{2n} > -4t_2\big) = \chi^2_{2n}(-4t_1) - \chi^2_{2n}(-4t_2) \end{cases},$$

where $P\big(\chi^2_{2n} > \chi^2_{2n}(\alpha)\big) = \alpha$.

Thus, the UMPT rejects H_0 if $x_1 < \sum_i X_i < x_2$, where x_1, x_2 satisfy

$$\begin{cases} 0.1 = \chi^2_{2n}(2x_2) - \chi^2_{2n}(2x_1) \\ 0.1 = \chi^2_{2n}(4x_2) - \chi^2_{2n}(4x_1) \end{cases}$$

. □

[Wei] 5.8: find the p value. The hypothesis is that

$$H_0 : \mu \le 1600 \quad \text{versus} \quad H_1 : \mu > 1600.$$

Let $\mu_0 = 1637$, $Z = \frac{\sqrt{n}}{\sigma}(\bar{X} - \mu)$, then $Z \sim N(0, 1)$, we also have

$$\begin{aligned} p &= \sup_{\mu \in H_0} P(\bar{X} > \mu_0) \\ &= \sup_{\mu \in H_0} P(Z > \frac{\sqrt{n}}{\sigma}(\mu_0 - \mu)) \\ &= P(Z \ge 1.258) \\ &= \Phi(-1.258). \end{aligned}$$

□

[Wei] 5.13: find the p value. The hypothesis is that

$$H_0 : \mu \le 23.8 \quad \text{versus} \quad H_1 : \mu > 23.8.$$

Let $T = \frac{\sqrt{n}}{S}(\bar{X} - \mu)$, where $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$. Then $T \sim t_{n-1}$. Here the mean of the samples is $\mu_0 = 24.2$ and $n = 7$, $S = 2.296$. We also have

$$\begin{aligned} p &= \sup_{\mu \in H_0} P(\bar{X} > \mu_0) \\ &= \sup_{\mu \in H_0} P(T > \frac{\sqrt{n}}{S}(\mu_0 - \mu)) \\ &= P(T > 0.461) \end{aligned}$$

where $T \sim t_6$ here. □

Remark: It also make sense to set $H_0 : \mu = 23.8$ versus $H_1 : \mu \ne 23.8$. However, it's wrong to set $H_0 : \mu > 23.8$ versus $H_1 : \mu \le 23.8$. H_0 is designed as a case which we want to "protect", and in this case we only think the new hypnotic meet our demand if we have a "strong" reason to say H_0 is not right.

[Wei] 5.16: find the p value. We notice that

$$Z = \frac{(\bar{Y} - \bar{X}) - (\mu_2 - \mu_1)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} \sim N(0, 1).$$

Therefore, the p value can be calculated as

$$\begin{aligned}
p &= \sup_{(\mu_1,\mu_2)\in H_0} P(\bar{Y} - \bar{X} > \bar{y} - \bar{x}) \\
&= \sup_{(\mu_1,\mu_2)\in H_0} P(Z > \frac{(\bar{y} - \bar{x}) - (\mu_2 - \mu_1)}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}) \\
&= P(Z > \frac{\bar{y} - \bar{x}}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}) \\
&= \Phi(4.200).
\end{aligned}$$

$\square$