HOMEWORK 2

2021/09/24

[Wei] 2.8.

(1)

$$
\begin{aligned}
\because \quad & P\left(X_{(n)} < 0.99\right) = P\left(X_1, \ldots, X_n < 0.99\right) = 0.99^n \leq 0.05 \\
\therefore \quad & n \geq \frac{\log(0.05)}{\log(0.99)} = 298.0729. \\
\therefore \quad & n \geq 299.
\end{aligned}
$$

□

(2) From (2.3.8) in our textbook, we have

$$
\begin{aligned}
f_{R_n, X_{(1)}}(r, x) =& \frac{n!}{(n-2)!} r^{n-2} I_{(0,1-x)}(r) I_{(0,1)}(x). \\
\therefore \quad f_{R_n}(r) =& \int f_{R_n, X_{(1)}}(r, x) dx \\
=& \int \frac{n!}{(n-2)!} r^{n-2} I_{(0,1)}(r) I_{(0,1-r)}(x) dx \\
=& n(n-1) r^{n-2}(1-r) I_{(0,1)}(r).
\end{aligned}
$$

□

(3)

$$
\begin{aligned}
\because \quad f_Z(z) &= f_{R_n}\left(1 - \frac{z}{2n}\right) \left|\frac{dR_n}{dz}\right| = n(n-1)\left(1 - \frac{z}{2n}\right)^{n-2} \left(\frac{z}{2n}\right) I_{(0,2n)}(z) \frac{1}{2n} \\
&= \frac{n-1}{4n} z \left(1 - \frac{z}{2n}\right)^{\frac{2n}{z} \frac{(n-2)z}{2n}} I_{(0,2n)}(z) \\
\therefore \quad f_Z(z) &\to \frac{1}{4} z e^{-\frac{z}{2}} I_{(0,+\infty)}(z) \quad (n \to \infty). \\
\therefore \quad Z &\to \chi_4^2 \quad (n \to \infty).
\end{aligned}
$$

□

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[Wei] 2.10. Define the survival function $S(x) = P(X \geq x) = 1 - F(x)$, then

$$S_X(x) = \begin{cases} e^{-(x/\beta)^\alpha} & , x \geq 0 \\ 1 & , x < 0 \end{cases}.$$

$$\begin{aligned} \therefore \quad S_Y(x) &= P(X_1, \ldots, X_n \geq x) = (S_X(x))^n \\ &= \begin{cases} e^{-n(x/\beta)^\alpha} = e^{-(x/(\beta/n^{1/\alpha}))^\alpha} & , x \geq 0 \\ 1 & , x < 0 \end{cases}. \end{aligned}$$

$$\therefore \quad Y \text{ is also a Weibull distribution with parameters}$$
$$\alpha_Y = \alpha, \quad \beta_Y = \frac{\beta}{n^{1/\alpha}}.$$

□

[Wei] 2.26.

$$\because \quad Exp(\lambda) = \Gamma\left(1, \frac{1}{\lambda}\right), \quad \chi_p^2 = \Gamma\left(\frac{p}{2}, \frac{1}{2}\right).$$

$$\therefore \quad \frac{2}{\lambda} Exp(\lambda) = \chi_2^2. \tag{1}$$

Let $Y_1 = X_{(1)}$, $Y_i = X_{(i)} - X_{(i-1)}$, $i = 2, \ldots, n$, then $\left|\frac{\partial(Y_1,\ldots,Y_n)}{\partial(X_{(1)},\ldots,X_{(n)})}\right| = 1$.

$$\begin{aligned} \therefore \quad f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) &= f_{X_{(1)},\ldots,X_{(n)}}(y_1, y_1 + y_2, \ldots, y_1 + \ldots + y_n) \left|\frac{\partial(Y_1, \ldots, Y_n)}{\partial(X_{(1)}, \ldots, X_{(n)})}\right| \\ &= n! \prod_{i=1}^n f_X(y_i) \\ &= \prod_{i=1}^n \left(\frac{n+1-i}{\lambda} e^{\frac{n+1-i}{\lambda} y_i} I_{(0,+\infty)}(y_i)\right). \end{aligned}$$

Separated the joint $p.d.f$, we conclude that $Y_i, i = 1, \ldots, n$ are independently distributed as $\text{Exp}(\frac{\lambda}{n+1-i})$. From (1),

$$\frac{2(n+1-i)}{\lambda} Y_i \overset{i.i.d.}{\sim} \chi_2^2. \tag{2}$$

$$\begin{aligned} \therefore \quad \frac{2T}{\lambda} &= \frac{2}{\lambda}\left(nX_{(1)} - (n-1)X_{(1)} + (n-1)X_{(2)} - (n-2)X_{(2)} + \ldots + (n+1-r)X_{(r)}\right) \\ &= \frac{2}{\lambda}\left(nY_1 + (n-1)Y_2 + \ldots + (n+1-r)Y_r\right) \\ &= \sum_{i=1}^r \frac{2(n+1-i)}{\lambda} Y_i \sim \chi_{2r}^2. \end{aligned}$$

□

Remark: Intuitively speaking, the *memoryless* property of exponential distributions makes $X_{(i)} - X_{(i-1)}$ "forget" the information before (to say, $X_{(i-1)} - X_{(i-2)}, \dots, X_{(2)} - X_{(1)}$) and therefore be independent with them.

[Wei] 2.27. Since $\frac{2(n-i+1)}{\sigma}\left(X_{(i)} - X_{(i-1)}\right) = \frac{2(n-i+1)}{\sigma}\left((X_{(i)} - \mu) - (X_{(i-1)} - \mu)\right)$, we can assume $\mu = 0$ without loss of generality. Then it's the case in 2.26 with $\lambda = \sigma$. According to (2), we have

$$\frac{2(n+1-i)}{\sigma}(X_{(i)} - X_{(i-1)}) \overset{i.i.d.}{\sim} \chi^2_2.$$

$\square$

2021/09/29, 10/08

[Wei] 2.39. Here we denote $Negbin(r, p)$ for negative binomial distribution with only one parameter $0 < p < 1$ (fix r). $Exp(\lambda)$ has the same $p.d.f.$ as in **2.26**.

- $Negbin(r, p)$:

$$\begin{aligned} f(n;p) =& \binom{n-1}{r-1} p^r (1-p)^{n-r} \overset{\theta := \log(1-p)}{=} \left(\frac{1-e^\theta}{e^\theta}\right)^r \exp\{\theta n\} \binom{n-1}{r-1} \\ :=& C(\theta) \exp\{\theta n\} h(n), \quad n \in \mathbb{Z}_{\geq r}, \end{aligned}$$

  where $C(\theta) := \left(\frac{1-e^\theta}{e^\theta}\right)^r$, $h(n) := \binom{n-1}{r-1}$.
  Its natural parametric space is $\{\theta : \theta \in (-\infty, 0)\}$.
- $Exp(\lambda)$:

$$\begin{aligned} f(x;\lambda) =& \frac{1}{\lambda} e^{-\frac{x}{\lambda}} I_{(0,+\infty)}(x) \overset{\theta := -\frac{1}{\lambda}}{=} -\theta \exp\{\theta x\} I_{(0,+\infty)}(x) \\ :=& C(\theta) \exp\{\theta x\} h(x), \end{aligned}$$

  where $C(\theta) := -\theta$, $h(x) := I_{(0,+\infty)}(x)$.
  Its natural parametric space is $\{\theta : \theta \in (-\infty, 0)\}$.

$\square$

Remarks: (i) The answer can be various.
(ii) For negative binomial, some may think that $r = \sum X_i$ which depends on the sample. In this case, construct another $\tilde{\theta} := \log\left(\frac{p}{1-p}\right)$ to obtain a natural form.

2.40 由 $\int f(x,\theta)\,dx = 1$，知

$$C(\theta) = \frac{1}{\int \exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)} := \frac{1}{p(\theta)}$$

则 $$\frac{\partial C(\theta)}{\partial\theta_j} = -\frac{1}{p^2(\theta)} \int T_j(x) \exp\{\sum_{i=1}^{k}\theta_i T_i(x)\} h(x)\,dx = -\frac{1}{C(\theta)} E_\theta T_j(x)$$

$$\Rightarrow \quad -\frac{\partial \log C(\theta)}{\partial\theta_j} = -\frac{1}{C(\theta)} \frac{\partial C(\theta)}{\partial\theta_j} = E_\theta (T_j(x)) \qquad ①$$

$$\frac{\partial^2 \log C(\theta)}{\partial\theta_j \partial\theta_s} = -\frac{1}{C^2(\theta)} \frac{\partial C}{\partial\theta_s}\frac{\partial C}{\partial\theta_j} + \frac{1}{C(\theta)} \frac{\partial^2 C}{\partial\theta_j\partial\theta_s} \qquad ②$$

$$\frac{\partial^2 C}{\partial\theta_j\partial\theta_s} = \frac{\partial}{\partial\theta_j}\left(-\frac{1}{p^2(\theta)} \int T_s(x) \exp\{\sum_{i=1}^{k}\theta_i T_i(x)\} h(x)\,dx\right)$$

$$= \frac{2}{p^3(\theta)} \int T_j(x) \exp\{\sum_{i=1}^{k}\theta_i T_i(x)\} h(x)\,dx \int T_s(x) \exp\{\sum_{i=1}^{k}\theta_i T_i(x)\} h(x)\,dx$$

$$-\frac{1}{p^2(\theta)} \int T_j(x)\, T_s(x) \exp\{\sum_{i=1}^{k}\theta_i T_i(x)\} h(x)\,dx$$

$$= 2C(\theta)\, E_\theta T_j(x)\, E_\theta T_s(x) - C(\theta)\, E_\theta T_j(x) T_s(x) \qquad ③$$

联立 ①-③ $\Rightarrow \dfrac{\partial^2 \log C(\theta)}{\partial\theta_j\partial\theta_s} = -\mathrm{Cov}_\theta(T_j(x), T_s(x))$ □

2.42 解：(1) $X_1, \cdots, X_n \overset{i.i.d.}{\sim} P(\lambda)$，则 $T(X) = \sum_{i=1}^{n} X_i \sim P(n\lambda)$

$$f(\vec{x} \mid t) = \frac{f(\vec{x}, t)}{f_T(t)} = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}(x_i!)^{-1} 1(x_i \in N, 1\le i\le n)}{e^{-n\lambda} (n\lambda)^{\sum_{i=1}^{n} x_i} [(\sum_{i=1}^{n} x_i)!]^{-1}} = \frac{(\sum_{i=1}^{n} x_i)!}{n^{\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}(x_i!)} 1(x_i \in N, 1\le i\le n)$$

与 λ 无关，由定义知 $T = \sum_{i=1}^{n} X_i$ 是充分统计量.

(2)

样本联合 p.m.f. 为

$$f(\vec{x}; \lambda) = e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}(x_i!)^{-1} 1(x_i \in N, 1\le i\le n)$$

$$g(t, \lambda) = e^{-n\lambda} \lambda^{t(\vec{x})} \qquad h(\vec{x}) = \prod_{i=1}^{n}(x_i!)^{-1} 1(x_i \in N, 1\le i\le n)$$

由因子分解定理知，$T(\vec{x}) = \sum_{i=1}^{n} X_i$ 是 λ 的充分统计量.

□

2.43 解：(1) $X_1, \cdots, X_n \overset{i.i.d.}{\sim} Nb(1,p)$，$P(X=x;p)=p(1-p)^{x-1}$，$x\in\mathbb{Z}_+$，$0<p<1$

则 $T=\sum_{i=1}^n X_i \sim Nb(n,p)$（负二项分布），$P(T=t;n,p)=\binom{t-1}{n-1}p^n(1-p)^{t-n}$，$t=n,n+1,\cdots$

则 $$f(\vec{x}\mid t)=\frac{p^n(1-p)^{\sum_{i=1}^n x_i-n}\,1(x_i\in\mathbb{Z}_+,\forall i)\,1(\sum_{i=1}^n x_i=t)}{\binom{t-1}{n-1}p^n(1-p)^{\sum_{i=1}^n x_i-n}}$$

$$=\frac{1}{\binom{t-1}{n-1}}\,1(x_i\in\mathbb{Z}_+,\forall i)\,1(\sum_{i=1}^n x_i=t)$$

与 p 无关，由定义知 T 是 p 的充分统计量.

(2) 样本联合 p.m.f.：

$$f(\vec{x};p)=p^n(1-p)^{\sum_{i=1}^n x_i-n}\,1(x_i\in\mathbb{Z}_+,\forall i)$$

则 $g(t(\vec{x});p)=p^n(1-p)^{t(\vec{x})-n}$，$h(\vec{x})=1(x_i\in\mathbb{Z}_+,\forall i)$

由因子分解定理知，$T(\vec{x})=\sum_{i=1}^n X_i$ 是 p 的充分统计量.

□

2.46 解：不是，证明如下：

样本联合 p.d.f.

$$f(\vec{x};\theta)=\frac{1}{(\sqrt{2\pi\theta^2})^n}e^{-\frac{1}{2\theta^2}\sum_{i=1}^n(x_i-\theta)^2}=\frac{1}{(\sqrt{2\pi\theta^2})^n}e^{-\frac{1}{2\theta^2}\sum_{i=1}^n x_i^2+\frac{1}{\theta}\sum_{i=1}^n x_i}\cdot e^{-\frac{n}{2}}$$

取 $T=(\sum_{i=1}^n x_i,\sum_{i=1}^n x_i^2):=(T_1,T_2)$，由因子分解定理可见 T 是 θ 的充分统计量

又 $\forall$ 两样本点 $\vec{x},\vec{y}$

$$\frac{f(\vec{x};\theta)}{f(\vec{y};\theta)}=e^{-\frac{1}{2\theta^2}(T_2(\vec{x})-T_2(\vec{y}))}\,e^{\frac{1}{\theta}(T_1(\vec{x})-T_1(\vec{y}))}\equiv const\ (与\theta无关)$$

当且仅当 $T(\vec{x})=T(\vec{y})$

故 $T(\vec{X})$ 是 θ 的极小充分统计量

若 $\bar{X}$ 是充分统计量，因 $\bar{X}=\frac{1}{n}T_1$，则 $\bar{X}$ 极小充分

但不存在 $\bar{X}$ 到 T 的 1-1 映射，矛盾.

故 $\bar{X}$ 不是充分统计量.

□

Assignment on PPT

Assignment 1: Prove that the *double exponential distribution* family with $p.d.f$ $f(x;\mu) = \frac{1}{4}\exp\left\{-\frac{|x-\mu|}{2}\right\}$**, where** $x \in \mathbb{R}$**,** $\mu \in \mathbb{R}$ **unknown, is not an exponential family.**

See Wu Yue's proof in the QQ group.

Assignment 2: Let $X_1, \ldots, X_n \sim Exp(\theta)$**,** $i.i.d.$**. Prove that** $T(\mathbf{X}) = X_{(1)}$ **is not a sufficient statistic for** θ**.**

$$\because \quad f(x) = \theta e^{-\theta x} I_{(0,+\infty)}(x) \text{ is the } p.d.f \text{ of } X_1 \sim Exp(\theta),$$

$$\therefore \quad f_{\mathbf{X},T}(\mathbf{x}, t) = \theta^n e^{-\theta \sum_{i=1}^n x_i} \delta_t(x_{(1)}) \prod_{i=1}^n I_{(0,+\infty)}(x_i).$$

$$\because \quad P(T > t) = P(X_i > t, \forall i) = e^{-n\theta t}, \forall t \geq 0.$$

$$\therefore \quad f_T(t) = n\theta e^{-n\theta t} I_{(0,+\infty)}(t).$$

$$\therefore \quad f_{\mathbf{X}|T}(\mathbf{x}|t) = \frac{f_{\mathbf{X},T}(\mathbf{x}, t)}{f_T(t)} = \frac{\theta^{n-1}}{n} e^{\theta(nt - \sum_{i=1}^n x_i)} \delta_t(x_{(1)}) \prod_{i=1}^n I_{(0,+\infty)}(x_i).$$

The conditional distribution is not constant as a function of θ unless $n = 1$. Therefore we conclude that T is not a sufficient statistic. □

Remark: One can also show that $\tilde{T(\mathbf{X})} = \sum_{i=1}^n X_i$ or equivalently $\bar{X}$ is the minimal sufficient statistic for θ. Since there doesn't exist a function ϕ, such that $\tilde{T} = \phi(T)$, from the definition of minimal sufficient statistic, T is not a sufficient statistic.

Assignment 3: Let $X_1, \ldots, X_n$ $i.i.d.$ **with density** $f(x; \theta = (a, b)) = c(a, b)\phi(x) I_{(a,b)}(x)$**, where** $-\infty < a < b < +\infty$ **unknown and** $\int_a^b \phi(x)dx < +\infty$**. Prove that** $T = (X_{(1)}, \ldots, X_{(n)})$ **is a sufficient statistic for** θ**.**

Hint: Factorization Theorem

Assignment 4: Let $X_1, \ldots, X_n \sim U(\theta, 2\theta)$**,** $i.i.d.$**,** $\theta > 0$ **unknown. Solve a minimal sufficient statistic for** θ**.**

$$\because \quad f(\mathbf{x}; \theta) = \frac{1}{\theta^n} I(\theta < x_{(1)} \leq x_{(n)} < 2\theta) = \frac{1}{\theta^n} I_{(x_{(n)}/2, x_{(1)})}(\theta),$$

$$\therefore \quad \frac{f(\mathbf{x}; \theta)}{f(\mathbf{y}; \theta)} = \frac{I_{(x_{(n)}/2, x_{(1)})}(\theta)}{I_{(y_{(n)}/2, y_{(1)})}(\theta)},$$

which is a constant with respect to θ iff $x_{(1)} = y_{(1)}$ and $x_{(n)} = y_{(n)}$. Therefore we conclude that $T(\mathbf{X}) = (X_{(1)}, X_{(n)})$ is a minimal sufficient statistic. (The answer is unique in the sense of one-to-one mapping.) □