Then the maximum of f reaches at $(\bar{x}, \bar{y}, \frac{\sum_i (x_i-\bar{x})^2+\sum_j (y_j-\bar{y})^2}{m+n})$. Thus,

$$\begin{aligned}\lambda(\mathbf{x},\mathbf{y}) =& \frac{L(\bar{x}, \bar{y}, \frac{\sum_i (x_i-\bar{x})^2+\sum_j (y_j-\bar{y})^2}{m+n}, \frac{\sum_i (x_i-\bar{x})^2+\sum_j (y_j-\bar{y})^2}{m+n})}{L(\bar{x}, \bar{y}, \frac{\sum_i (x_i-\bar{x})^2}{m}, \frac{\sum_j (y_j-\bar{y})^2}{n})} \\ =& \frac{[\sum_i (x_i-\bar{x})^2]^{\frac{m}{2}} \cdot [\sum_j (y_j-\bar{y})^2]^{\frac{n}{2}}}{[\sum_i (x_i-\bar{x})^2+\sum_j (y_j-\bar{y})^2]^{\frac{m+n}{2}}}.\end{aligned}$$

□

[Wei] 5.35.

(1) The likelihood function is

$$L(\lambda) = \lambda^n \exp(-\lambda \sum_{i=1}^n X_i)$$

In Θ, the superior of L is reached at $\frac{1}{\bar{X}}$ and $L = \bar{X}^{-n} \exp(-n)$.

In Θ_0, $\sup_{\lambda\in\{\lambda_0\}} = \lambda^0 \exp(-\lambda_0 \sum_{i=1}^n X_i)$.

Thus,

$$\lambda(\mathbf{x}) = (\lambda_0 \bar{X} e)^n \exp(-\lambda_0 n \bar{X})$$

(2) If $(\bar{X})^{-1}$ is reachable for $\lambda \le \lambda_0$, then the likelihood ratio is 1. Otherwise, λ_0 maximizes the likelihood function in Θ_0 and thus we obtain the above likelihood ration again.

□

[Wei] 5.36. The likelihood function is

$$L(\mu) = e^{n\mu - \sum_i x_i} I_{(-\infty, x_{(1)}]}(\mu).$$

In $\Theta = \mathbb{R}$, the superior of L is reached at $x_{(1)}$ and $L = e^{nx_{(1)} - \sum_i x_i}$.

In $\Theta_0 = \{\mu_0\}$, $L = e^{n\mu_0 - \sum_i x_i} I_{(-\infty, x_{(1)}]}(\mu_0)$. Thus,

$$\lambda(\mathbf{x}) = \begin{cases} 0, & x_{(1)} < \mu_0 \\ e^{n(\mu_0 - x_{(1)})}, & otherwise \end{cases}.$$

□

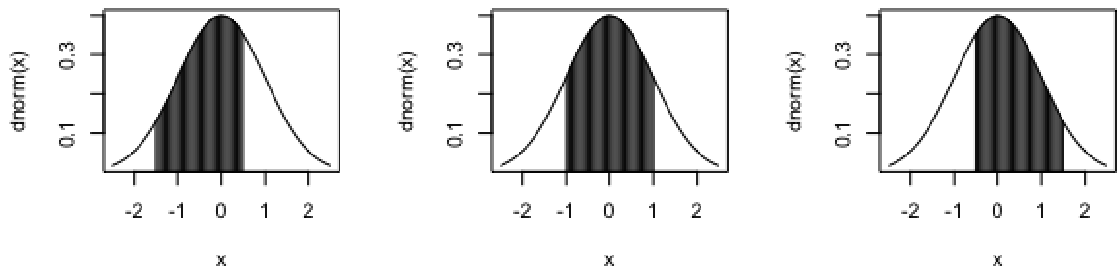
[Wei] 5.37. The likelihood function is

$$L(\theta_1, \theta_2 | n.m) = \frac{1}{\theta_1^m \theta_2^n} \exp(-\frac{1}{\theta_1} m\bar{X}) \exp(-\frac{1}{\theta_2} n\bar{Y}).$$

In $\Theta = \mathbb{R}^2$, the superior of L is reached at $\theta_1 = \bar{X}, \theta_2 = \bar{Y}$ and $L = \frac{1}{\bar{X}^m \bar{Y}^n} \exp(-m-n)$.

In $\Theta_0 = \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \theta_1 = \theta_2\}$, the superior of L is reached at $\theta_1 = \theta_2 = \frac{m\bar{X}+n\bar{Y}}{m+n}$ and $L = \frac{1}{(\frac{m\bar{X}+n\bar{Y}}{m+n})^{m+n}} \exp(-m-n)$. Thus, we have

$$\lambda(\mathbf{x},\mathbf{y}) = \frac{\bar{X}^m \bar{Y}^n}{(\frac{m\bar{X}+n\bar{Y}}{m+n})^{m+n}}$$



Therefore, we compare α with β in different cases that

$$\begin{cases} \alpha \geq \beta, & |\mu_0 - \mu| \geq C \\ \alpha < \beta, & |\mu_0 - \mu| < C \end{cases},$$

where $C \geq 0$ is the smallest constant such that $\Phi\left(\frac{5}{3}c\right) - \Phi\left(-\frac{5}{3}c\right) + \Phi\left(\frac{5}{3}(C+c)\right) - \Phi\left(\frac{5}{3}(C-c)\right) \leq 1$. □

Remark: because of the continuity of the left-hand side and the decreasing property, the choice of C can be expressed as: if $\Phi\left(\frac{5}{3}c\right) < 0.25$, then

$$\begin{cases} \alpha \geq \beta, & |\mu_0 - \mu| \geq C \\ \alpha < \beta, & |\mu_0 - \mu| < C \end{cases},$$

where $C > 0$ satisfies $\Phi\left(\frac{5}{3}c\right) - \Phi\left(-\frac{5}{3}c\right) + \Phi\left(\frac{5}{3}(C+c)\right) - \Phi\left(\frac{5}{3}(C-c)\right) = 1$. Otherwise, $\alpha > \beta$.

[Wei] 5.4. The probability of type I errors is

$$P(X_{(n)} \leq 1.5 | \theta \in \Theta_0) = P(X_1, \ldots, X_n \leq 1.5 | \theta \geq 2) = \left(\frac{1.5}{\theta}\right)^n, \quad \theta \geq 2.$$

Its maximum is $\left(\frac{3}{4}\right)^n$. □

[Wei] 5.7. After discussion, three TAs agree that we are supposed to test for both expectation and variance and each test is supposed to be conducted under the assumption that the other parameter is unknown. However, for the final examination, the null hypothesis shall be clear. After calculation, we have

$$\bar{X} = 10.48, \quad S^2 = 0.056$$

For the test of μ, we shall use the statistic $T = \frac{\sqrt{n}(\bar{X}-\mu_0)}{S} \sim t_{n-1}$. Since $|T| = 0.3273268 < t_9(0.025) = 2.144787$, thus we accept the null hypothesis that $\mu = \mu_0$.

For the test of σ^2, we shall use the statistic $T = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$. Since $T = 34.84444 > \chi_{14}(0.025) = 26.11895$, we reject the null hypothesis that $\sigma^2 = \sigma_0^2$ and conclude that the machine does not work well.

Remark: If we accept that $\mu = \mu_0$ first, we are supposed to test for the variance with the knowledge of μ_0. In this case, the degree of freedom of chi-square distribution shall be n. $\square$

[Wei] 5.8. The hypothesis is

$$H_0 : \mu \leq 1600 \leftrightarrow H_1 : \mu > 1600.$$

Suppose that the reject region is $\{\bar{X} > c\}$, then from $\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu) \sim N(0, 1)$, we have

$$\begin{aligned}
\alpha \geq& \sup_{\mu \leq 1600} P(\bar{X} > c) \\
=& \sup_{\mu \leq 1600} P\left(\frac{\sqrt{n}}{\sigma}(\bar{X} - \mu) > \frac{\sqrt{n}}{\sigma}(c - \mu)\right) \\
=& \sup_{\mu \leq 1600} 1 - \Phi\left(\frac{\sqrt{n}}{\sigma}(c - \mu)\right) \\
=& 1 - \Phi\left(\frac{\sqrt{26}}{150}(c - 1600)\right).
\end{aligned}$$

Therefore, $c \geq 1648.39$, and the reject region is $\{\bar{X} \geq 1649\}$. We accept the null hypothesis that $\mu \leq 1600$. $\square$

[Wei] 5.10. We first obtain that $\bar{X} = 31.12667$ and $S^2 = 1.260067$. Since $|T| = |\frac{\sqrt{n}(\bar{X} - \mu_0)}{S}| = 2.99678 < t_5(0.01/2) = 4.032143$, we accept the null hypothesis that $\mu = \mu_0$. $\square$

[Wei] 5.11.

(1) Since $|T| = |\frac{\sqrt{n}(\bar{X} - \mu_0)}{S}| = -4.102414 < t_9(0.05) = 1.833113$, we accept the null hypothesis that $\mu \leq \mu_0$.
(2) Since $T = \frac{(n-1)S^2}{\sigma_0^2} = 7.700625 > \chi_9^2(1 - 0.05) = 3.325113$, we accept the null hypothesis that $\sigma^2 \geq \sigma_0^2$.

$\square$

[Wei] 5.15 (Using LRT method). Notice that $T(\mathbf{X}) = \sum_i X_i$ is a sufficient statistic and $X_i \overset{i.i.d}{\sim} \Gamma(1, \frac{1}{\theta})$, then $T \sim \Gamma(n, \frac{1}{\theta})$ with *p.d.f.*

$$f_T(t) = \frac{1}{(n-1)!\theta^n} t^{n-1} e^{-\frac{t}{\theta}}, \quad t > 0$$

and log likelihood

$$l(\theta) = c(n, t) - n \log(\theta) - \frac{t}{\theta}.$$

Taking derivative of $l(\theta)$ yields that its maximum is reached at $\theta = \frac{t}{n}$. Therefore, the LRT statistic

$$\begin{aligned}\lambda(\mathbf{X}) = \lambda^*(T) =& \frac{f_T(T)|_{\theta=2000}}{\sup_{\theta\in\mathbb{R}} f_T(T)} \\ =& \frac{2000^{-n}e^{-\frac{T}{2000}}}{\left(\frac{T}{n}\right)^{-n} e^{-n}} \\ =& \left(\frac{e}{2000n}\right)^n T^n e^{-\frac{T}{2000}},\end{aligned}$$

where $T = \sum_i X_i$.

The reject region can be expressed as $\{\mathbf{X} : \lambda^*(T) \leq c\}$. Since λ is increasing when $T < 2000n$ and decreasing otherwise, we set the reject region as $\{\mathbf{X} : T \leq c_1 \text{ or } T \geq c_2\}$, where $c_1 \leq 2000n \leq c_2$ are constants such that

$$P(T \leq c_1) = P(T \geq c_2) = \frac{\alpha}{2}.$$

(This region is different from $\{\lambda^*(T) \leq c\}$, but is computationally attractive.) Observe that under H_0, $\frac{2}{2000}T \sim \Gamma(n, \frac{1}{2}) = \chi^2_{2n}$, the two constants can be solved as

$$\begin{aligned}c_1 =& \frac{2000}{2}\chi^2_{2n}(1 - \frac{\alpha}{2}) = 9591, \\ c_2 =& \frac{2000}{2}\chi^2_{2n}(\frac{\alpha}{2}) = 34170.\end{aligned}$$

Since $c_1 < 30200 < c_2$, we accept the null hypothesis H_0. □

Remark: if we write the equivalent form of $\{\lambda^*(T) \leq c\}$, that is, $\{T \leq c_1 \text{ or } T \geq c_2\}$, where $c_1 \leq 2000n \leq c_2$ are constants such that $\lambda^*(c_1) = \lambda^*(c_2)$ and $P(T \leq c_1) + P(T \geq c_2) = \alpha$, it is not easy to write an explicit solution or calculate $\tilde{c_1}$ such that $\lambda^*(\tilde{c_1}) = \lambda^*(30200)$. However, my computer tells me that $\tilde{c_1} \approx 12405$, and $P(T \leq 12405) + P(T \geq 30200) \approx 0.1652 > 0.05$, thus we accept H_0 with this exact choice of reject region.

[Wei] 5.36. Notice that $T(\mathbf{X}) = X_{(1)}$ is a sufficient statistic with *p.d.f.*

$$f_T(t) = ne^{-n(t-\mu)}, \quad t \geq \mu$$

and the likelihood function

$$L(\mu) = ne^{-n(t-\mu)}I_{(-\infty,t]}(\mu).$$

Then the maximal likelihood is reached at $\mu = t$. Therefore, the LRT statistic

$$\lambda(\mathbf{X}) = \lambda^*(T) = \begin{cases} 0, & X_{(1)} < \mu_0 \\ e^{-n(X_{(1)}-\mu_0)}, & X_{(1)} \geq \mu_0 \end{cases}.$$

The reject region can be expressed as

$$R = \{\mathbf{X} : \lambda^*(T) \leq c\} = \{X_{(1)} < \mu_0 \text{ or } X_{(1)} \geq C\}, \quad C \in [\mu_0, +\infty].$$

With level α, we can calculate C, which satisfies

$$\alpha = P(\mu_0 \in R) = P_{\mu_0}(X_{(1)} \geq C) = e^{-n(C-\mu_0)}.$$

Thus, $C = \mu_0 - \frac{\log(\alpha)}{n}$.

In summary, we reject the null hypothesis H_0 if

$$X_{(1)} < \mu_0, \text{ or } X_{(1)} \geq \mu_0 - \frac{\log(\alpha)}{n}.$$

Otherwise, H_0 is accepted. □

HOMEWORK 11

WEIYU LI

2018/12/07

Solve the UMP level α test for:

(1) $X_1, \ldots, X_n \sim N(0, \sigma^2)$**, i.i.d. with unknown** $\sigma^2 > 0$**.**
$H_0 : \sigma^2 = \sigma_0^2 \leftrightarrow H_1 : \sigma^2 = \sigma_1^2$, **where** $\sigma_0^2 < \sigma_1^2$**.**

Solve: Notice that $T = \sum X_i^2 \sim \sigma^2 \chi_n^2$ is sufficient, and its $p.d.f$ is

$$f(t|\sigma) = \frac{1}{\sigma^2} f_{\chi_n^2}\left(\frac{t}{\sigma^2}\right) \propto \frac{1}{\sigma^n} e^{-\frac{t}{2\sigma^2}}.$$

From Neyman-Pearson lemma, the reject region should be $\{T : f(T|\sigma_1) > kf(T|\sigma_0)\}$. Since

$$\begin{aligned} f(T|\sigma_1) &> kf(T|\sigma_0) \\ \Leftrightarrow -n\log(\sigma_1) - \frac{T}{2\sigma_1^2} &> \log(k) - n\log(\sigma_2) - \frac{T}{2\sigma_2^2} \\ \Leftrightarrow \qquad T &> c\ , \end{aligned}$$

where c is a constant, the reject region can be rewritten as $\{T : T > c\}$.

With level $\alpha = P(T > c|H_0) = P(\chi_n^2 > \frac{c}{\sigma_0^2})$, we have $c = \sigma_0^2 \chi_n^2(\alpha)$. Thus, we reject H_0 if

$$T = \sum X_i^2 > \sigma_0^2 \chi_n^2(\alpha).$$

□

(2) $X_1, \ldots, X_n \sim Poisson(\lambda)$**, i.i.d. with unknown** $\lambda > 0$**.**
$H_0 : \lambda = \lambda_0 \leftrightarrow H_1 : \lambda = \lambda_1$, **where** $\lambda_0 < \lambda_1$**.**

Solve: Notice that $T = \sum X_i \sim Poisson(n\lambda)$ is sufficient, and its $p.m.f$ is

$$P(t|\lambda) = e^{-n\lambda} \frac{(n\lambda)^t}{t!} \propto e^{-n\lambda} \lambda^t.$$

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From Neyman-Pearson lemma, the reject region should be $\{T : f(T|\lambda_1) > kf(T|\lambda_0)\}$. Since

$$\begin{aligned} & f(T|\lambda_1) > kf(T|\lambda_0) \\ \Leftrightarrow & -n\lambda_1 + T\log(\lambda_1) > \log(k) - n\lambda_0 + T\log(\lambda_0) \\ \Leftrightarrow & \quad T > c\ , \end{aligned}$$

where c is a constant, the reject region can be rewritten as $\{T : T > c\}$.

With level $\alpha \geq P(T > c|H_0) = \sum_{t=\lfloor c\rfloor+1}^{\infty} e^{-n\lambda}\frac{(n\lambda)^t}{t!}$, we reject H_0 if

$$T = \sum X_i^2 > c,$$

where c is the smallest integer such that $\sum_{t=c+1}^{\infty} e^{-n\lambda}\frac{(n\lambda)^t}{t!} \leq \alpha$. □