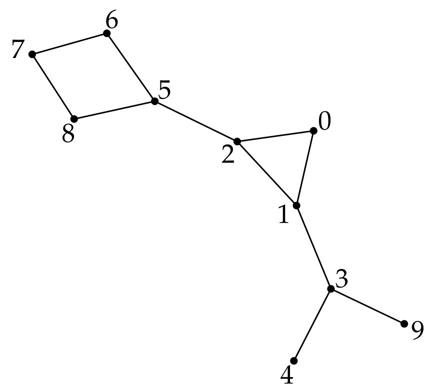
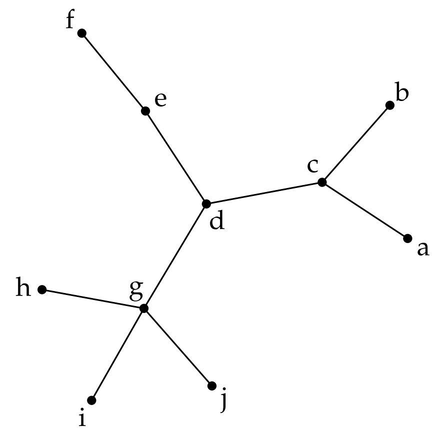
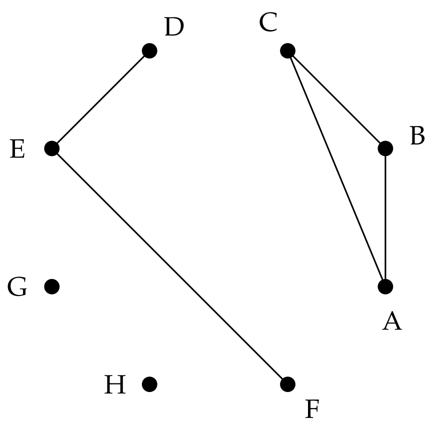
Algorithms: Graphs

Model 1: Graphs and graph terms







$G = (V, E)$

$V = \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta\}$

$E = \{\{\alpha, \delta\}, \{\theta, \eta\}, \{\beta, \alpha\}, \{\zeta, \delta\}, \{\epsilon, \eta\}, \{\gamma, \alpha\}\}$

Key: α = alpha, β = beta, γ = gamma, δ = delta, ϵ = epsilon,
ζ = zeta, η = eta, θ = theta

Definition 1. A (simple) *graph* $G = (V, E)$ is a set of *vertices* V together with a set E of *edges*, where each edge consists of a set of two vertices.

Above are shown four example graphs. In clockwise order from the upper-left, we will refer to them as the numeric graph, the lowercase graph, the Greek graph, and the uppercase graph. Here are some facts about the graphs:

- C and B are *adjacent* (aka *neighbors*). So are 6 and 5. C and F are not adjacent. f and d are not adjacent. H is not adjacent to itself.
- The *degree* of vertex C is 2. The degree of vertex H is 0. The degree of vertex g is 4.
- Vertices D, F, 4, 9, a, b, f, h, i, j are *leaves*. No other vertex is a leaf.

Don't worry if you don't remember all these graph terms! In fact you should probably try to forget what you think you might remember and just focus on the information in the model. Part of the point of this exercise is to help you either recall these terms, or learn them for the first time.

Learning objective: Students will understand and apply graph terms *edge*, *vertex*, *adjacent/neighbor*, *degree*, *leaf*, *path*, *connected*, *connected component*, *cycle*, *cyclic*, *acyclic*, and *tree*.

1 What does it mean for two vertices to be *adjacent*?

2 Are α and β adjacent? What about α and ζ?

3 What is the *degree* of a vertex? Use the term *adjacent* in your answer.

4 What is the degree of α?

5 What is the definition of a *leaf*? Use the term *degree* in your answer.

6 Which vertices are leaves in the Greek graph?

7 In the space below, draw a graph with at least three leaves, one vertex with degree five, and at least one pair of vertices that are not adjacent.

Model 2: More graph terms

Here are a few more facts about the graphs from Model 1:

- Vertices 7–8–5–2–0 are a *path*. 2–1–0–2–1 is also a path. H by itself is a path. H–F–B is not a path. 8–2–1 is also not a path.
- Vertices f and g are *connected*. So are 7 and 8. C and F are not connected. H is connected to itself.
- The graph with numbers is a *connected graph*. So is the graph with lowercase letters. The graph with uppercase letters is not a connected graph (it is *disconnected*).
- The numeric and lowercase graphs have one *connected component* each. The uppercase graph has four connected components.
- 8–7–6–5–8 is a *cycle*. So is C–A–B–C. h–g–i–h is not a cycle. h–g–h is not a cycle either. Nor is H–H.
- The numeric graph and uppercase graph are *cyclic* graphs. The lowercase graph is *acyclic*.
- The lowercase graph is a *tree*. None of the other graphs are trees (not even the Greek one).

8 How many vertices can be in a *path*?

9 What do you think is the definition of a *path*?

10 Give an example of a path in the Greek graph.

11 Can two vertices be connected but not adjacent? If so, give an example.

12 Can two vertices be adjacent but not connected? If so, give an example.

13 What do you think it means for two vertices to be connected? Be sure to use the term *path* in your answer.

14 Is the Greek graph connected?

15 What do you think is the definition of a *connected graph*?

16 How many vertices can be in a connected component?

17 How many *connected components* does the Greek graph have?

18 Is the set of vertices $\{E, F\}$ a connected component? Why or why not?

19 Write a definition for *connected component*.

20 Write an "if and only if" statement using the terms *connected graph* and *connected component*.

21 What is a *cycle*? Use the term *path* in your answer.

22 Does your definition for *cycle* correctly explain why h–g–h is not a cycle? If not, revise it so it does.

23 Does the Greek graph have a cycle?

24 Is the Greek graph cyclic or acyclic?

25 What do you think is the definition of a tree? You should use two of the other graph terms in your definition.

Warning—a tree graph is not quite the same thing as a tree data structure!

Some proofs about graphs

Learning objective: Students will write proofs about graphs.

Theorem 2 (Trees). *Let $G = (V, E)$ be a graph with $|V| = n \geq 1$. Any two of the following imply the third:*

1. *G is connected.*
2. *G is acyclic.*
3. *G has $n - 1$ edges.*

We will take each pair of statements in turn and show that they imply the third. Fill in the blanks to complete the following proofs! Note that the size of a blank does not necessarily correspond to the amount of stuff you should write in it.

Lemma 3. $(1), (2) \implies (3)$. *That is: let $G = (V, E)$ be a graph with*

$|V| = n \geq 1$. If ____

and ____,

then ____.

Proof. Let $P(n)$ denote the statement "Any graph G with n vertices

which is ____________________ and ____________________

must have ____." We wish to show that $P(n)$ holds for all $n \geq 1$.

The proof is by ____________________________________.

- The base case is when ________________________________.

  In this case, G must be ________________________________

  which indeed ______________________________________.

- For the induction step, suppose $P(k)$ holds for some $k \geq 1$. That is,

  suppose that any graph with _____________________ vertices

  which is ____

  must have _______________________________________.

Then we wish to show $P(k+1)$, that is, any graph with ________

vertices which is connected and acyclic must have ____________.

So, let G be a graph with ________________ vertices which is

____________________ and ____________________.
We claim that G must have some vertex which is a leaf, that is, a

vertex of degree ________________________________,
which we can show as follows:

- G cannot have any vertices of degree ____________________

  because ____________________________________.

- It also cannot be the case that every vertex of G has degree $\geq$ __.

  If they did, then we could find a __________ by starting at any

  vertex and walking along edges randomly until __________;

  we would never get stuck because ____________________.

  However, this is impossible because we assumed __________.

Hence, G must have some vertex which ____________________.
If we delete this vertex along with the edge adjacent to it, it results

in a graph G' with only ______________________ vertices;

we note that G' is still ______________________________

because ____

and also ____

because ____.
Hence we may apply the inductive hypothesis to conclude that G'

____________________. Adding the deleted vertex and edge

back to G' shows that G __________________________________,
which is what we wanted to show.

$\square$

Let's do one more! (You will do the third on your HW.)

Lemma 4. $(2),(3) \implies (1)$, *that is,* ____________________________

____.

Proof. This proof uses a *counting argument*: we will show what we wish to show by counting things in multiple ways.

Let c denote the number of connected components of G. We want to show that ____.

Number the components of G from $1 \dots c$, and say that component i has n_i vertices. Then

$$\sum_{i=1}^{c} n_i = \underline{\hspace{4cm}}$$

because ____.

Each connected component is by definition a ______________ graph;

each component must also be ______________________________ since we assumed that G is. Hence we may apply Lemma 3 to conclude that component i ______________________________________.
Adding these up, the total number of edges in G is

$$|E| = \sum_{i=1}^{c} \underline{\hspace{2.5cm}} = \underline{\hspace{5cm}}$$

But we already assumed the number of edges in G is _____________,

and hence __________________ as desired. $\square$