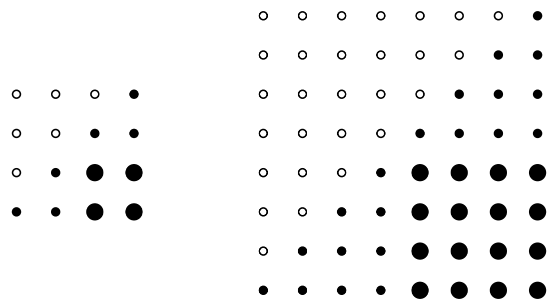
Algorithms: Some asymptotic sums

Model 1: Three proofs



$$
\begin{array}{ccccccccccc}
1 & + & 2 & + & 3 & + & \dots & + & (n-1) & + & n \\
n & + & (n-1) & + & (n-2) & + & \dots & + & 2 & + & 1 \\
\hline
(n+1) & + & (n+1) & + & (n+1) & + & \dots & + & (n+1) & + & (n+1)
\end{array}
$$

$$
\begin{aligned}
1+2+\cdots+n &< n+n+\cdots+n && (1)\\
&= n^2 && (2)
\end{aligned}
$$

$$
\begin{aligned}
1+2+\cdots+n &= 1+2+\cdots+(n/2-1)+n/2+(n/2+1)+\cdots+n && (3)\\
&> n/2+(n/2+1)+\cdots+n && (4)\\
&> n/2+n/2+\cdots+n/2 && (5)\\
&= (n/2)^2 && (6)\\
&= n^2/4 && (7)
\end{aligned}
$$

The first row of Model 1 actually shows two similar diagrams at different sizes, one 4×4 and one 8×8. Each diagram consists of a bunch of dots—some hollow and some filled; and the filled dots come in two varieties, big and small.

Learning objective: Students will understand and prove the asymptotic behavior of $1 + 2 + 3 + \cdots + n$.

Learning objective: Students will apply geometric, algebraic, and inequational reasoning to asymptotic behavior.

1 How many dots are there in total in the first diagram? In the second diagram?

2 How many big dots are there (*i.e.* the lower-right square) in the first diagram? How many are in the second?

3 How many filled dots are there in total (both big and small filled dots, *i.e.* the lower-right triangle) in the first diagram? In the second?

Now suppose that we abstract away the specific sizes of the diagrams and imagine a generic $n \times n$ version of the same diagram. To make things slightly simpler, assume that n is even.

4 In terms of n, how many dots would there be in total?

5 In terms of n, how many big dots would there be in the lower right?

6 Explain why the number of filled-in dots is equal to

$$1 + 2 + 3 + \cdots + n.$$

7 Based on the diagrams, write an inequality relating these three quantities.

8 What does this prove about the sum $1 + 2 + 3 + \cdots + n$ in terms of Θ? Justify your answer based on your answer to the previous question.

Now consider the second proof.

9 Notice that the top row is our friend $1 + 2 + 3 + \cdots + n$. What is the second row?

10 Why does the bottom row consist of copies of $(n + 1)$?

11 What is the sum of the bottom row?

12 Use this to derive a formula for $1 + 2 + \cdots + n$ in terms of n.

13 What does this formula imply about the asymptotic behavior of $1 + 2 + \cdots + n$ in terms of Θ? Justify your answer.

Finally, consider the third proof. Surprise!—once again it has to do with the sum $1 + 2 + 3 + \cdots + n$. For this proof we will again assume n is even.[1]

[1] It is not hard to fix the proof to work for odd n as well, but the details would end up obscuring the main idea somewhat.

14 Why is step (1) true?

15 Why is the right-hand side of (1) equal to (2) ?

16 What does this prove about $1 + 2 + \cdots + n$?

17 Now, what is happening in step (3)?

18 Why is (3) greater than (4)?

19 Why is (4) greater than (5)?

20 Why is (5) equal to (6)?

21 Explain why we have now shown that $n^2/4 < 1 + 2 + \cdots + n < n^2$.

22 What does this prove about $1 + 2 + \cdots + n$ in terms of Θ?

23 Two of these three proofs are in some sense the same. Which two?

24 Use one of the proof methods from this activity to derive the big-Theta asymptotic behavior of $1 + 2 + 4 + 8 + 16 + \cdots + 2^n$.