

Model 1: Arithmetic

								1	0	1	1	0	1	0		
								×	1	1	1	0	0	1	1	
								<hr/>								
									1	0	1	1	0	1	0	
									1	0	1	1	0	1	0	0
			1	0	1	1	0	1	0	0	0	0	0	0	0	0
		1	0	1	1	0	1	0	0	0	0	0	0	0	0	0
+	1	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0
								<hr/>								
1	0	1	0	0	0	0	0	1	1	0	1	1	1	1	0	0

Learning objective: Students will analyze arithmetic operations on n -bit integers.

- 1 Approximately how many single bit operations (*e.g.* adding or comparing two bits) are needed to compute the addition $1011010_2 + 1110011_2$ shown in the model?
- 2 In general, suppose we want to add two integers of n bits each. In terms of Θ , how many bit operations are needed?

- 3 Explain why it is not possible to do any better than this.
- 4 If we have a list of $\Theta(n)$ integers, each with $\Theta(n)$ bits, how long will it take (in terms of single bit operations) to add all of them?

Now consider the multiplication shown in the model.

- 5 Why are there five rows in between the two horizontal lines?
- 6 How many operations are needed to produce each such row?
(*Hint:* you may assume that multiplying by a power of two takes constant time.)
- 7 If we are multiplying two integers with n bits each, how many intermediate rows could there be in the worst case?
- 8 How long will it take to add them all?



Model 2: Arithmetic by pieces

$$X = 01101001_2$$

$$Y = 11100100_2$$

$$X_1 = 0110_2$$

$$Y_1 = 1110_2$$

$$X_2 = 1001_2$$

$$Y_2 = 0100_2$$

Let's now consider whether it is possible to multiply two n -bit integers any faster.

- 9 What is the relationship between X , X_1 and X_2 in the model?
What about Y , Y_1 , and Y_2 ?
- 10 Suppose $Z = 1011100101_2$. What would Z_1 and Z_2 be?
- 11 What is $2^4 \cdot X_1$ in binary?
- 12 In general, if b is some number expressed in binary, what is $2^4 \cdot b$?
- 13 Write two equations expressing X in terms of X_1 and X_2 and Y in terms of Y_1 and Y_2 .
- 14 In general, suppose A is an n -bit integer, and we split it into A_1 and A_2 . Generalize your previous answer to express A in terms of A_1 and A_2 .



- 15 Suppose A and B are n -bit integers, and consider the product AB . Expand both A and B using your previous answer, and distribute the resulting product. You should end up with an expression involving only A_1 , A_2 , B_1 , and B_2 .
- 16 How many multiplications are required to compute the expression from Question 15? (Remember that multiplying by a power of two takes constant time and does not need to be counted.)
- 17 How big (how many bits) are the inputs to each multiplication in Question 15?
- 18 Explain how we can use the equation from Question 15 as a recursive algorithm to compute AB .
- 19 Let $M(n)$ denote the time taken to multiply two n -bit integers, and write a recurrence relation for $M(n)$ corresponding to this recursive algorithm.

