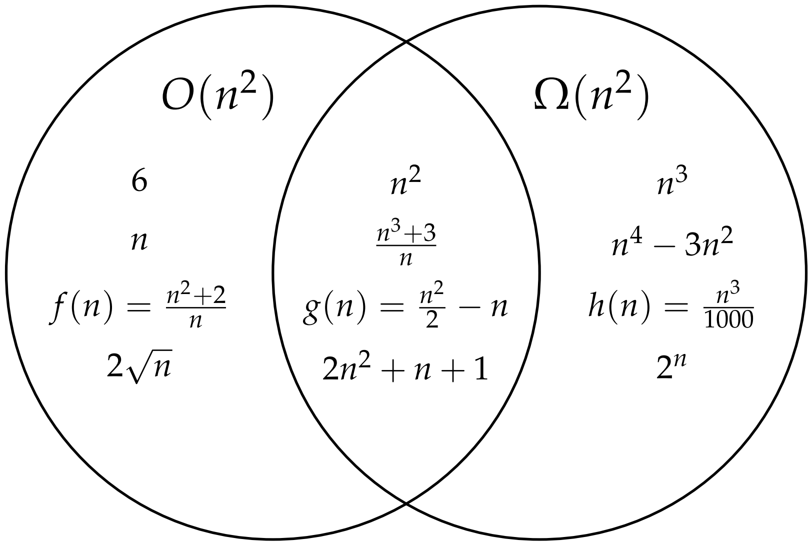
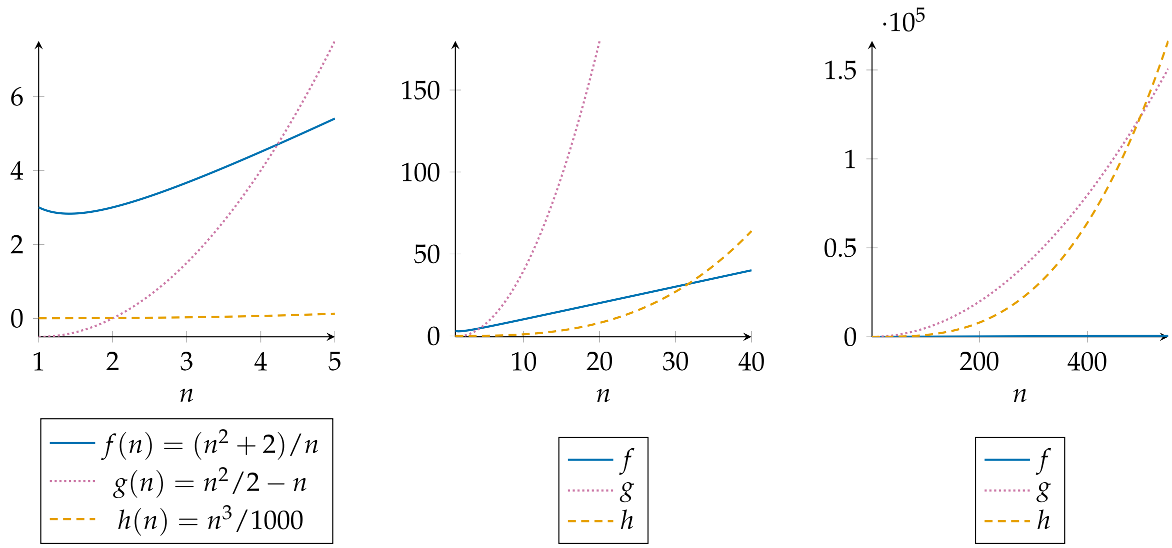
Algorithms: Introduction to Asymptotic Analysis

Model 1: Big-O and Big-Ω





Critical Thinking Questions I (20 minutes)

Important note: although any previous experience you have with big-O notation may be helpful, it is **not** assumed that you remember anything in particular! When answering the following questions, as much as possible, try to rely on the information provided in Model 1 rather than on your memory.

1 **Working together**, based on the **Venn diagram** in the model, say whether each function is $O(n^2)$, $\Omega(n^2)$, or both.

(a) $\frac{n^2+2}{n}$

According to the Venn diagram, $\frac{n^2+2}{n}$ *is* $O(n^2)$.

(b) $\frac{n^3}{1000}$

$\Omega(n^2)$

(c) $\frac{n^2}{2} - n$

Both $O(n^2)$ *and* $\Omega(n^2)$.

(d) 2^n

$\Omega(n^2)$.

For Questions 2–5, consider the functions

$$f(n) = (n^2+2)/n,$$
$$g(n) = n^2/2 - n, \text{and}$$
$$h(n) = n^3/1000.$$

Graphs of these functions are shown in the model (or rather, *one* graph is shown three times at different zoom levels).

2 Look at the graphs to determine which function is biggest when $2 \le n \le 4$.

The left-hand graph shows that $f(n)$ *(the blue line) is biggest on this interval.*

3 The following table has four columns representing different intervals for n. For each interval, the table is supposed to show which function is smallest, which is biggest, and which is in between. A couple entries have already been filled in for you. Using

Learning objective: Extrapolating from examples, students will develop and apply informal definitions to classify functions as $O(n^2)$, $\Omega(n^2)$, or $\Theta(n^2)$.

Process objective: Students will process information from a model to explore the meaning of big-O and big-Omega notation.

Process objective: Students will think critically to discover counterexamples and assemble evidence.

Ω is pronounced “big omega” (amusingly, “o-mega” is itself Greek for “big O”, although they meant “big” in the sense of a long vowel, not uppercase).

the graphs in the model, fill in the rest of the table. Note that the graphs do not quite show what happens at $n = 600$; when filling in the last column of the table, simply use your best judgment to predict what will happen.

Make sure your group agrees on the best way to fill in the table.

biggest	f			
mediumest		f		
smallest				
	$2 \leq n \leq 4$	$5 \leq n \leq 30$	$35 \leq n \leq 450$	$n = 600$

f	g	g	h
g	f	h	g
h	h	f	f

4 Does the same relative order continue for all $n \geq 600$, or do the functions ever change places again? Justify your answer.

The functions never change places again. Since h is proportional to n^3 it will continue to grow faster than g, which will in turn continue to grow faster than f.

5 Based on the model and your answers to the previous questions, match each statement on the left with an appropriate informal definition on the right. $q(n)$ represents an arbitrary function.

You will see more formal definitions on the next activity!

A function $q(n)$ is $O(n^2)$

A function $q(n)$ is $\Omega(n^2)$

$q(n)$ is greater than n^2 for all $n \geq 0$

Eventually, for big enough values of n, $q(n)$ grows at the same rate or more slowly than n^2

$q(n)$ grows more slowly than n^2

$q(n) \geq n^2$ for big enough values of n

The definition of $q(n)$ has n^2 in it

$q(n)$ eventually grows at a similar rate or more quickly than n^2

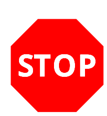
- *"$q(n)$ is $O(n^2)$" goes with "Eventually, for big enough values of n, $q(n)$ grows at the same rate or more slowly than n^2". The function $g(n)$ grows at the same rate as n^2, and $f(n)$ grows more slowly than n^2; both are $O(n^2)$. We also know that it only matters what happens for big enough n, because both $f(n)$ and $g(n)$ start out larger than $h(n)$, but eventually $h(n)$ becomes larger, and $h(n)$ is not $O(n^2)$.*
- *"$q(n)$ is $\Omega(n^2)$" goes with "$q(n)$ eventually grows at a similar rate or more quickly than n^2". The reasoning is similar to that for $O(n^2)$.*

6 In what way(s) do you think the definition of $\Omega(n^2)$ is similar to and different from that of $O(n^2)$?

Answers may vary. For example, the definitions both have something to do with n being "sufficiently large", and they both involve comparing something to n^2; one involves something being $\leq$ something else, and the other involves $\geq$.

7 Choose two *incorrect* definitions from the previous question. For each one, write one or two sentences explaining why it is incorrect. Be sure to mention evidence from your previous answers.

- *"$q(n)$ is greater than n^2 for all $n \geq 0$": this is an incorrect definition of $\Omega(n^2)$ since it requires something to be true for all $n \geq 0$, whereas the real definition only cares what happens once n gets big enough. For example, we know from the model that $h(n) = n^3/1000$ is $\Omega(n^2)$, but it is only greater than n^2 for $n > 1000$.*
- *"$q(n)$ grows more slowly than n^2": a function which is $O(n^2)$ may grow more slowly* or at the same rate *as n^2. For example, $2n^2 + n + 1$ is $O(n^2)$ but does not grow more slowly than n^2.*
- *"$q(n) \geq n^2$ for big enough values of n": for $q(n)$ to be $\Omega(n^2)$, we only require that it grows at a similar or greater rate as n^2, not that it is literally greater than n^2. For example, from the model, $g(n) = n^2/2 - n$ is $\Omega(n^2)$, but it is* never *greater than n^2.*
- *"The definition of $q(n)$ has n^2 in it": there are clearly functions in the model which are $O(n^2)$ or $\Omega(n^2)$ that do not mention n^2; for example, 2^n is $\Omega(n^2)$ and n is $O(n^2)$.*



Critical Thinking Questions II (10 minutes)

8 If a function is both $O(n^2)$ and $\Omega(n^2)$, we say it is $\Theta(n^2)$. For each of the below functions, say whether you think it is $\Theta(n^2)$, or only $O(n^2)$ or $\Omega(n^2)$. Justify your answers.

Θ is pronounced "big theta".

(a) $3n^2 + 2n - 10$

> *This is $\Theta(n^2)$. It is very similar to $2n^2 + n + 1$ which we know is $\Theta(n^2)$. It grows at a similar rate to n^2.*

(b) $\dfrac{n^3 - 5}{n}$

> *This is also $\Theta(n^2)$. It is similar to $(n^3 + 3)/n$. If we rewrite it as $n^2 + 3/n$ we can see that it will also grow at a similar rate to n^2.*

(c) $\dfrac{n^3 - 5}{\sqrt{n}}$

> *This is $\Omega(n^2)$ but not $O(n^2)$. Divding n^3 by $\sqrt{n}$ produces something that still grows faster than n^2.*

(d) $(n+1)(n-2)$

> *This is $\Theta(n^2)$. It is equal to $n^2 - n - 2$.*

(e) $n + n\sqrt{n}$

> *This is $O(n^2)$ but not $\Omega(n^2)$. $n\sqrt{n} = n \cdot n^{1/2} = n^{1.5}$ which grows more slowly than n^2.*

(f) $n^2 \cdot \log_2 n$

> *This is $\Omega(n^2)$ but not $O(n^2)$; multiplying by $\log_2 n$ means it grows strictly faster than n^2.*

9 In your answers to Question 8, in which cases did you make use of evidence from the model (the Venn diagram or graphs) to justify your answers? In which cases did you make use of team members' previous knowledge?

> *Student answers will vary.*

Facilitation plan

Learning Objectives

Content objectives

- Extrapolating from examples, students will develop and apply informal definitions to classify functions as $O(n^2)$, $\Omega(n^2)$, or $\Theta(n^2)$.

Process objectives

- Students will process information from a model to explore the meaning of big-O and big-Omega notation.
- Students will think critically to discover counterexamples and assemble evidence.

Announcements (2 minutes)

- Remember HW 1 due Friday. Start early, come ask for help if you need it.
- Today, take a role you haven't had. Review duties.
- Remind managers to look at the time limits on the activities, make sure you stay on track!

CTQs I (Big-O) (30 mins: 20 activity + 10 discussion)

(Up to 3 minutes to get started, look at role cards, etc.)

- Share and discuss answers to 5, 6, and 7. Note that the next activity will present the real definitions, so it is not critical that students converge on an exactly correct definitions; the goal is to get them to think about the important issues.

CTQs II (Big-Theta, classification) (15 mins: 10 activity + wrap-up)

- Discuss answers as necessary.
- Wrap-up: today was about building intuition and examples. Promise we will see the real definitions next time!

Author notes

In the past when I have used a previous version of the activity in a 50-minute class, I only made it through CTQ I and never made it to CTQ II. I hope that

- this version is more streamlined, and
- encouraging managers to keep track of time will help so that we can get to the application questions.