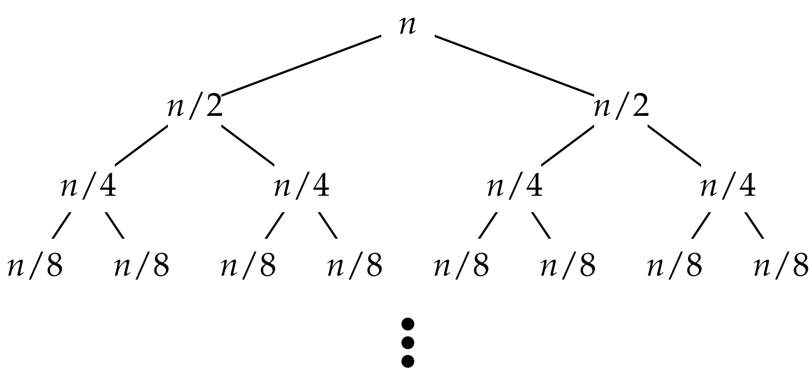
Algorithms: Introduction to Divide & Conquer

Model 1: Merge sort

mergesort(xs) =
    if $len(xs) \leq 1$ **then return** xs
    split xs into halves (xs_1, xs_2)
    $xs'_1 \leftarrow$ mergesort(xs_1)
    $xs'_2 \leftarrow$ mergesort(xs_2)
    $xs' \leftarrow$ merge(xs'_1, xs'_2)
    return xs'

$$T(1) = \Theta(1)$$
$$T(n) = 2T(n/2) + \Theta(n)$$



Recall the *merge sort* algorithm, which works by splitting the input list into halves, recursively sorting the two halves, and then merging the two sorted halves back together.

Learning objective: Students will use recurrence relations and recursion trees to describe and analyze divide and conquer algorithms.

1 How long does mergesort take on a list of length 1?

2 *Just by looking at the code*, how many recursive calls does mergesort make at each step?

Don't overthink this one. Yes, it's really that easy.

3 If xs has size n, what is the size of the inputs to the recursive calls to mergesort?

4 (Review) How long does it take (in big-Θ terms) to merge xs_1 and xs_2 after they are sorted?

5 Let $T(n)$ denote the total amount of time taken by mergesort on an input list of length n. Use your answers to the previous questions to explain the equations for $T(n)$ given in the model. This is called a *recurrence relation* because it defines $T(n)$ via recursion.

6 Suppose algorithm X takes an input of size n, splits it into three equal-sized pieces, and makes a recursive call on each piece. Deciding how to split up the input into pieces takes $\Theta(n^2)$ time; combining the results of the recursive calls takes additional $\Theta(n)$ time. In the base case, algorithm X takes constant time on an input of size 1. Write a recurrence relation $X(n)$ describing the time taken by algorithm X, similar to the one given in the model.

7 Now suppose algorithm X makes only two recursive calls instead of three, but each recursive call is still on an input one-third the size of the original input. How does your recurrence relation for X change?

8 Write a recurrence relation for binary search.

Now let's return to considering merge sort. The tree shown in the model represents the call tree of merge sort on an input of size n, that is, each node in the tree represents one recursive call to merge sort. The expression at each node shows how much work happens at that node (from merging), that is, how much time is spent processing data in that specific recursive call.

9 Notice that the entire tree is not shown; the dots indicate that the tree continues further with the same pattern. What is the depth (number of levels) of the tree, in terms of n?

10 How much total work happens on each individual level of the tree?

11 How much total work happens in the entire tree?

12 Draw a similar tree for the second version of algorithm X. Be careful to distinguish between the *size of the input* and the *amount of work* done at each node.