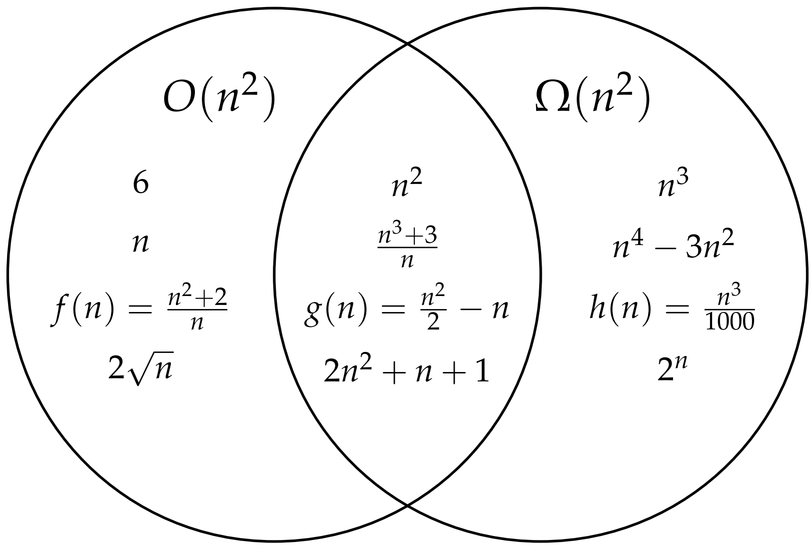
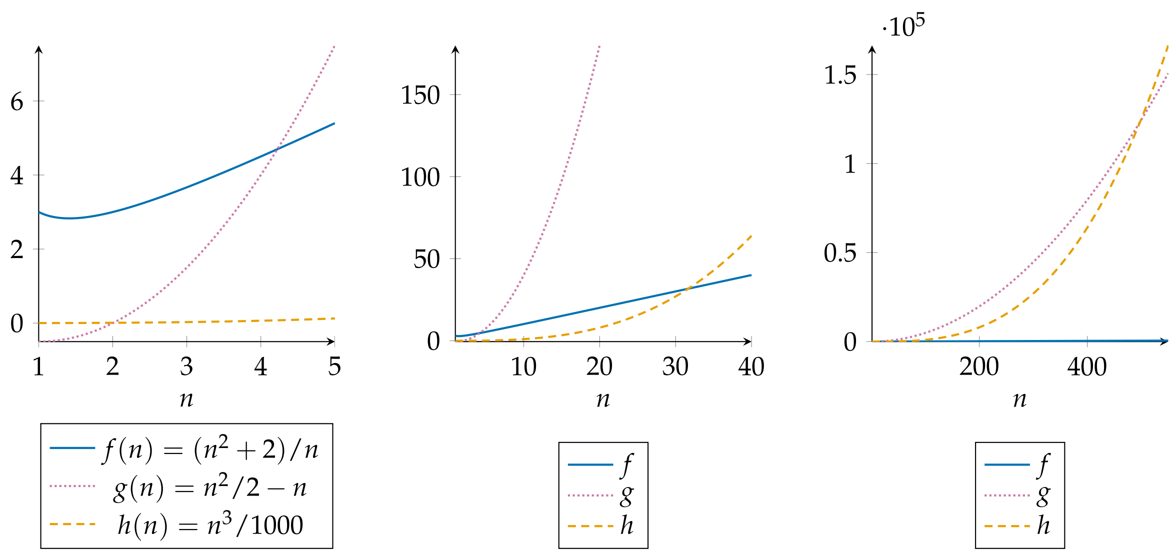
Algorithms: Introduction to Asymptotic Analysis

Model 1: Big-O and Big-Ω





Critical Thinking Questions I (20 minutes)

Learning objective: Extrapolating from examples, students will develop and apply informal definitions to classify functions as $O(n^2)$, $\Omega(n^2)$, or $\Theta(n^2)$.

Important note: although any previous experience you have with big-O notation may be helpful, it is **not** assumed that you remember anything in particular! When answering the following questions, as much as possible, try to rely on the information provided in Model 1 rather than on your memory.

Process objective: Students will process information from a model to explore the meaning of big-O and big-Omega notation.

1 **Working together**, based on the **Venn diagram** in the model, say whether each function is $O(n^2)$, $\Omega(n^2)$, or both.

Process objective: Students will think critically to discover counterexamples and assemble evidence.

(a) $\frac{n^2+2}{n}$

(b) $\frac{n^3}{1000}$

Ω is pronounced "big omega" (amusingly, "o-mega" is itself Greek for "big O", although they meant "big" in the sense of a long vowel, not uppercase).

(c) $\frac{n^2}{2} - n$

(d) 2^n

For Questions 2–8, consider the functions

$$f(n) = (n^2 + 2)/n,$$
$$g(n) = n^2/2 - n, \text{and}$$
$$h(n) = n^3/1000.$$

Graphs of these functions are shown in the model (or rather, *one* graph is shown three times at different zoom levels).

2 Look at the graphs to determine which function is biggest when $2 \le n \le 4$.

3 The following table has four columns representing different intervals for n. For each interval, the table is supposed to show which function is smallest, which is biggest, and which is in between. A couple entries have already been filled in for you. Using

the graphs in the model, fill in the rest of the table. Note that the graphs do not quite show what happens at $n = 600$; when filling in the last column of the table, simply use your best judgment to predict what will happen.

Make sure your group agrees on the best way to fill in the table.

biggest	f			
mediumest		f		
smallest				
	$2 \le n \le 4$	$5 \le n \le 30$	$35 \le n \le 450$	$n = 600$

4 Does the same relative order continue for all $n \ge 600$, or do the functions ever change places again? Justify your answer.

5 Look at all the functions in the Venn diagram which are *both* $O(n^2)$ *and* $\Omega(n^2)$. What do they have in common?

6 Now look at the functions which are $O(n^2)$ but *not* $\Omega(n^2)$. What do they have in common?

7 Suppose we have three algorithms to solve a particular problem:

- Algorithm F takes $f(n) = (n^2 + 2)/n$ seconds to solve the problem on a particular computer when given an input of size n.
- Algorithm G takes $g(n) = n^2/2 - n$ seconds to solve the problem on a particular computer when given an input of size n.
- Algorithm H takes $h(n) = n^3/1000$ seconds to solve the problem on a particular computer when given an input of size n.

Label each of the following statements as True or False, and write a sentence or phrase explaining your reasoning.

(a) Algorithm H would be the best choice if we only ever need to solve the problem for small ($n \leq 10$) inputs.

(b) Algorithm G would be the best choice if we need to solve the problem for very large ($n \geq 10^6$) inputs.

(c) Algorithm G will take exactly $g(n) = n^2/2 - n$ seconds to solve a problem of size n on *any* computer.

(d) If we run Algorithm G on a different computer, there is some constant k such that Algorithm G will take $k \cdot g(n) = k(n^2/2 - n)$ seconds to solve a problem of size n on that computer.

(e) If we need to solve inputs of any size $n \geq 10$ in at most $n^2/4 - n/2$ seconds, we could run Algorithm G on a sufficiently fast computer.

(f) If we need to solve inputs of any size $n \geq 10$ in at most $n^2/4 - n/2$ seconds, we could run Algorithm F on a sufficiently fast computer.

(g) If we need to solve inputs of any size $n \geq 10$ in at most $n^2/4 - n/2$ seconds, we could run Algorithm H on a sufficiently fast computer.

(h) Suppose Algorithm J solves the problem for inputs of size n in some amount of time that is $O(n^2)$. In general, assuming n may be large, we would prefer Algorithm J over Algorithm H.

8 Based on the model and your answers to the previous questions, match each statement on the left with an appropriate informal definition on the right. $q(n)$ represents an arbitrary function.

You will see more formal definitions on the next activity!

A function $q(n)$ is $O(n^2)$

A function $q(n)$ is $\Omega(n^2)$

$q(n)$ is greater than n^2 for all $n \geq 0$

Eventually, for big enough values of n, $q(n)$ grows at a similar rate or more slowly than n^2

$q(n)$ grows more slowly than n^2

$q(n) \geq n^2$ for big enough values of n

The definition of $q(n)$ has n^2 in it

$q(n)$ eventually grows at a similar rate or more quickly than n^2

9 Choose two *incorrect* definitions from the previous question. For each one, write one or two sentences explaining why it is incorrect. Be sure to mention evidence from your previous answers.

STOP

Critical Thinking Questions II (10 minutes)

10 If a function is both $O(n^2)$ and $\Omega(n^2)$, we say it is $\Theta(n^2)$. For each of the below functions, say whether you think it is $\Theta(n^2)$, only $O(n^2)$, or only $\Omega(n^2)$. Justify your answers.

Θ is pronounced "big theta".

(a) $3n^2 + 2n - 10$

(b) $\dfrac{n^3 - 5}{n}$

(c) $\dfrac{n^3 - 5}{\sqrt{n}}$

(d) $(n+1)(n-2)$

(e) $n + n\sqrt{n}$

(f) $n^2 \cdot \log_2 n$

11 In your answers to Question 10, in which cases did you make use of evidence from the model (the Venn diagram or graphs) to justify your answers? In which cases did you make use of team members' previous knowledge?

Facilitation plan

Learning Objectives

Content objectives

- Extrapolating from examples, students will develop and apply informal definitions to classify functions as $O(n^2)$, $\Omega(n^2)$, or $\Theta(n^2)$.

Process objectives

- Students will process information from a model to explore the meaning of big-O and big-Omega notation.
- Students will think critically to discover counterexamples and assemble evidence.

Announcements (2 minutes)

- Remember HW 1 due Friday. Start early, come ask for help if you need it.
- Today, take a role you haven't had. Review duties.
- Remind managers to look at the time limits on the activities, make sure you stay on track!

CTQs I (Big-O) (30 mins: 20 activity + 10 discussion)

(Up to 3 minutes to get started, look at role cards, etc.)

- Share and discuss answers to 8, **??**, and 9. Note that the next activity will present the real definitions, so it is not critical that students converge on an exactly correct definitions; the goal is to get them to think about the important issues.

CTQs II (Big-Theta, classification) (15 mins: 10 activity + wrap-up)

- Discuss answers as necessary.
- Wrap-up: today was about building intuition and examples. Promise we will see the real definitions next time!

Author notes

In the past when I have used a previous version of the activity in a 50-minute class, I only made it through CTQ I and never made it to CTQ II. I hope that

- this version is more streamlined, and
- encouraging managers to keep track of time will help so that we can get to the application questions.