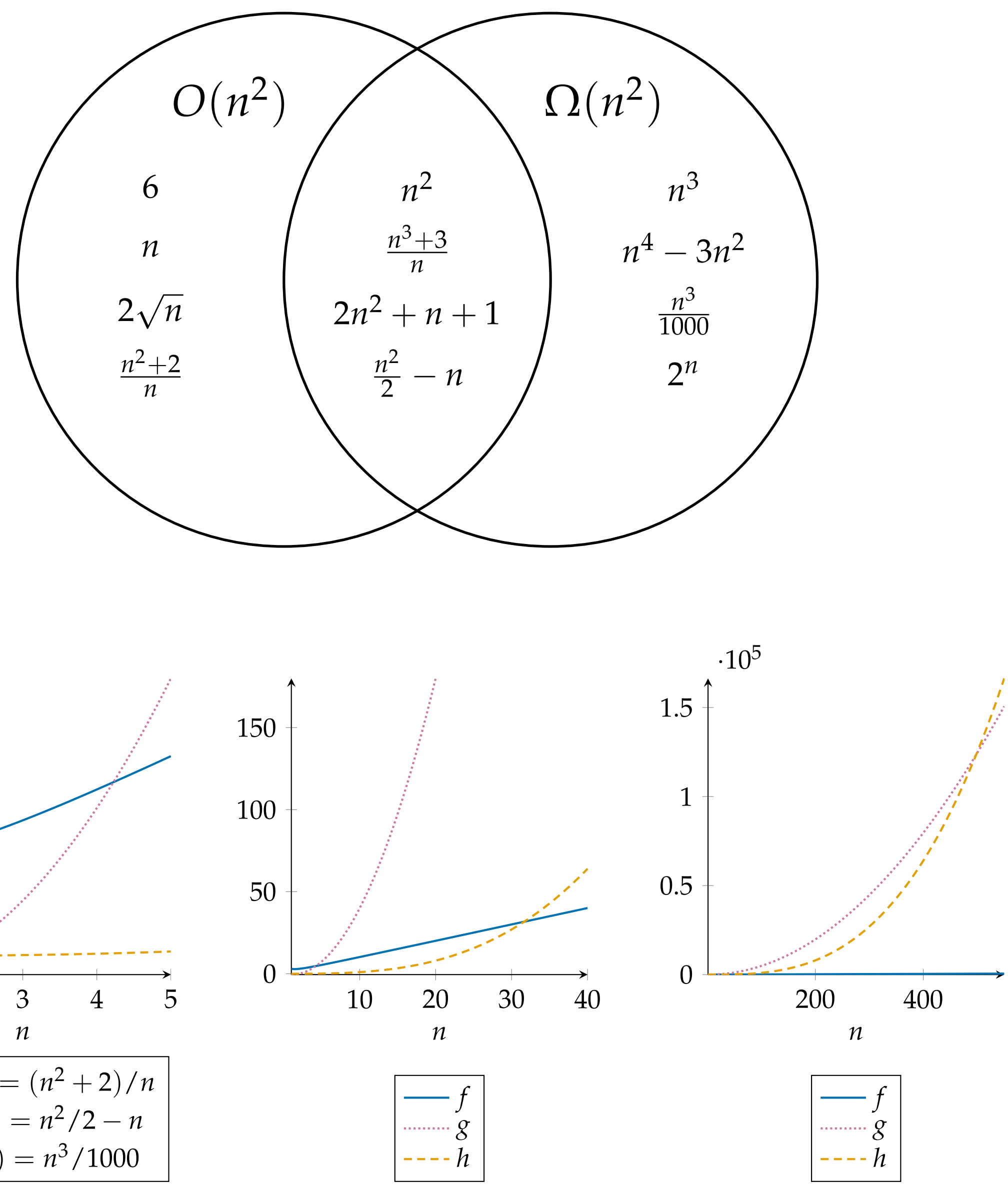
Algorithms: Introduction to Asymptotic Analysis

Model 1: Big-O and Big-Ω



Critical Thinking Questions I (20 minutes)

Learning objective: Extrapolating from examples, students will develop an intuitive mental model to classify functions as $O(n^2)$, $\Omega(n^2)$, or $\Theta(n^2)$.

Process objective: Students will process information from a model to explore the meaning of big-O and big-Omega notation.

Process objective: Students will think critically to discover counterexamples and assemble evidence.

Ω is pronounced "big omega" (amusingly, "o-mega" is itself Greek for "big O", although they meant "big" in the sense of a long vowel, not uppercase).

Important note: although any previous experience you have with big-O notation may be helpful, it is **not** assumed that you remember anything in particular! When answering the following questions, as much as possible, try to rely on the information provided in Model 1 rather than on your memory.

1 **Working together**, based on the **Venn diagram** in the model, say whether each function is $O(n^2)$, $\Omega(n^2)$, or both.

(a) $2\sqrt{n}$

According to the Venn diagram, $2\sqrt{n}$ is $O(n^2)$.

(b) n^3

$\Omega(n^2)$

(c) $2n^2 + n + 1$

Both $O(n^2)$ and $\Omega(n^2)$.

(d) 2^n

$\Omega(n^2)$.

For Questions 2–6, consider the functions

$$f(n) = (n^2 + 2)/n,$$
$$g(n) = n^2/2 - n, \text{and}$$
$$h(n) = n^3/1000.$$

Graphs of these functions are shown in the model (or rather, *one* graph is shown three times at different zoom levels).

2 Look at the graphs to determine which function is biggest when $2 \le n \le 4$.

The left-hand graph shows that $f(n)$ (the blue line) is biggest on this interval.

3 The following table has four columns representing different intervals for n. For each interval, the table is supposed to show which function is smallest, which is biggest, and which is in between. A couple entries have already been filled in for you. Using

the graphs in the model, fill in the rest of the table. Note that the graphs do not quite show what happens at $n = 600$; when filling in the last column of the table, simply use your best judgment to predict what will happen.

Make sure your group agrees on the best way to fill in the table.

biggest	f			
middle		f		
smallest				
	$2 \leq n \leq 4$	$5 \leq n \leq 30$	$35 \leq n \leq 450$	$n = 600$

f	g	g	h
g	f	h	g
h	h	f	f

4 Does the same relative order continue for all $n \geq 600$, or do the functions ever change places again? Justify your answer.

The functions never change places again. Since h is proportional to n^3 it will continue to grow faster than g, which will in turn continue to grow faster than f.

5 Again using the Venn diagram, for each function, say whether it is $O(n^2)$, $\Omega(n^2)$, or both.

(a) $f(n) = (n^2 + 2)/n$

$f(n)$ is $O(n^2)$.

(b) $g(n) = n^2/2 - n$

$g(n)$ is both $O(n^2)$ and $\Omega(n^2)$.

(c) $h(n) = n^3/1000$

h is $\Omega(n^2)$.

6 Based on your answers to the previous three questions, which grow more quickly in general, functions which are $O(n^2)$, or functions which are $\Omega(n^2)$? Write one or two sentences explaining your reasoning. Be sure to mention evidence from your answers to each of the previous three questions.

f, which we know is $O(n^2)$ from question 5, is the smallest function for $n \geq 600$ (based on the table in question 3 and the answer to question 4); h, which is $\Omega(n^2)$, is the biggest. Eventually, $\Omega(n^2)$ functions grow more quickly than $O(n^2)$ functions.

Using evidence from the model, come to a consensus within your group as to whether each of the following statements is true or false. Write a short justification for each answer.

7 If $f(n)$ is $O(n^2)$, then it has n^2 in its definition.

False; e.g. 6 *is* $O(n^2)$ *but does not have* n^2 *in its definition.*

8 If $f(n)$ has n^2 in its definition, then $f(n)$ is $O(n^2)$.

False; e.g. $n^4 - 3n^2$ *has* n^2 *in its definition but it is not* $O(n^2)$.

9 If $f(n)$ is both $O(n^2)$ and $\Omega(n^2)$, then it has n^2 in its definition.

False; e.g. $\frac{n^3+3}{n}$.

10 If $f(n) \leq n^2$ for all $n \geq 0$, then $f(n)$ is $O(n^2)$.

This is true; although we can't know for sure without knowing the definition of $O(n^2)$, *all the functions in the model with this property are in fact* $O(n^2)$.

11 If $f(n)$ is $O(n^2)$, then $f(n) \leq n^2$ for all $n \geq 0$.

False. For example, $2n^2 + n + 1$ *is* $O(n^2)$ *but it is never* $\leq n^2$.

12 If $f(n) \leq n^2$ for all n that are sufficiently large, then $f(n)$ is $O(n^2)$.

True. For example, $\frac{n^2+2}{n}$ *is only* $\leq n^2$ *for* $n > 1$.

13 If $f(n)$ is $O(n^2)$ and $g(n)$ is $\Omega(n^2)$, then $f(n) \leq g(n)$ for all n.

False. For example, consider $(n^2+2)/n$ *(which is* $O(n^2)$*) and* $n^3/1000$ *(which is* $\Omega(n^2)$*). From the graphs we know that the first function is actually greater than the second until around* $n = 30$.

14 Using one or more complete English sentences, propose a definition of $O(n^2)$ by completing the following statement.

A function $f(n)$ *is* $O(n^2)$ *if and only if...*

Answers may vary. Students may talk about how it only matters what happens to $f(n)$ *when* n *is big enough; how* $f(n)$ *should "grow at a similar rate" to* n^2, *or always be* $\leq kn^2$ *for some constant* k*; they may note that only the "biggest term" of* $f(n)$ *matters.*


STOP

Critical Thinking Questions II (10 minutes)

15 In what way(s) do you think the definition of $\Omega(n^2)$ is similar to and different from that of $O(n^2)$?

> *Answers may vary. For example, the definitions both have something to do with n being "sufficiently large", and they both involve comparing something to n^2; one involves something being $\leq$ something else, and the other involves $\geq$.*

16 If a function is both $O(n^2)$ and $\Omega(n^2)$, we say it is $\Theta(n^2)$. For each of the below functions, say whether you think it is $\Theta(n^2)$, $O(n^2)$, or $\Omega(n^2)$. Justify your answers.

Θ is pronounced "big theta".

(a) $3n^2 + 2n - 10$

> *This is $\Theta(n)$. It is very similar to $2n^2 + n + 1$ which we know is $\Theta(n)$.*

(b) $\dfrac{n^3 - 5}{n}$

> *This is also $\Theta(n)$. It is similar to $(n^3 + 3)/n$.*

(c) $\dfrac{n^3 - 5}{\sqrt{n}}$

> *This is $\Omega(n^2)$ but not $O(n^2)$. Divding n^3 by $\sqrt{n}$ produces something that still grows faster than n^2.*

(d) $(n + 1)(n - 2)$

> *This is $\Theta(n^2)$. It is equal to $n^2 - n - 2$.*

(e) $n + n\sqrt{n}$

> *This is $O(n^2)$ but not $\Omega(n^2)$. $n\sqrt{n} = n \cdot n^{1/2} = n^{1.5}$ which grows more slowly than n^2.*

17 In your answers to Question 16, in which cases did you make use of evidence from the model (the Venn diagram or graphs) to justify your answers? In which cases did you make use of team members' previous knowledge?

Facilitation plan

Learning Objectives

Content objectives

- Students will develop intuitive mental models to classify functions as $O(n^2)$, $\Omega(n^2)$, and/or $\Theta(n^2)$.

Process objectives

- Information processing (interpreting Venn diagram and graph models)
- Critical thinking (finding counterexamples; synthesizing examples to come up with intuitive models for $O(n^2)$ and $\Omega(n^2)$)

Announcements (2 minutes)

- Remember HW 1 due Friday. Start early, come ask for help if you need it.
- Today, take a role you haven't had. Review duties.
- Remind managers to look at the time limits on the activities, make sure you stay on track!

CTQs I (Big-O) (30 mins: 20 activity + 10 discussion)

(Up to 3 minutes to get started, look at role cards, etc.)

- Make grid, go around and tell them put up answers to T/F questions once they get there. Discuss as necessary.
- Share and discuss proposed definitions of $O(n^2)$. Note that the next activity will present the real definition, so it is not critical that students converge on an exactly correct definition; the goal is to get them to think about the important issues.

CTQs II (Big-Theta, classification) (15 mins: 10 activity + wrap-up)

- Discuss answers as necessary.
- Wrap-up: today was about building intuition and examples. Promise we will see the real definition next time!

Author notes

In the past when I have used a previous version of the activity in a 50-minute class, I only made it through CTQ I and never made it to CTQ II. I hope that

- This version is more streamlined
- Encouraging managers to keep track of time will help

so that we can get to the application questions.

Some unused questions:

- $f(n)$ being $O(n^2)$ and/or $\Omega(n^2)$ has nothing to do with whether it literally has n^2 in its definition.

  > *True.*

- Every function $f(n)$ is either $O(n^2)$ or $\Omega(n^2)$ (or both).

  > *This is true, though there is no particular way to answer this from the model alone; student answers may vary.*

- Do you think $n^2 \cdot \log_2 n$ is $O(n^2)$, $\Omega(n^2)$, or $\Theta(n^2)$? Why?

  > *Answers may vary. In fact, it is $\Omega(n^2)$ but not $O(n^2)$; multiplying by $\log_2 n$ means it grows strictly faster than n^2.*