1 Preliminaries

Let $\mathcal{S}_n$ denote the symmetric group of order n, that is, the group of permutations on $\{1, \ldots, n\}$ under composition. It is a well-known fact that every permutation $\sigma \in \mathcal{S}_n$ can be uniquely decomposed as a product of cycles. The *cycle type* of σ is the sequence of natural numbers $\sigma_1, \sigma_2, \sigma_3, \ldots$ where σ_i is the number of i-cycles in the cycle decomposition of σ. For example, the permutation $(132)(45)(78)(6)$ has cycle type $1, 2, 1, 0, 0, 0, \ldots$

The *exponential generating function* (egf) associated to a combinatorial species F is defined by

$$F(x) = \sum_{n \geq 0} |F[n]| \frac{x^n}{n!}.$$

That is, the coefficient of $x^n/n!$ is the number of distinct F-structures on n labels. For example, for $n \geq 1$ there are $(n-1)!$ distinct labelled cycles (necklaces), so the egf for the species of cycles is

$$C(x) = \sum_{n \geq 1} (n-1)! \frac{x^n}{n!} = \sum_{n \geq 1} \frac{x^n}{n}.$$

The *ordinary generating function* (ogf) associated to a species F is defined by

$$\tilde{F}(x) = \sum_{n \geq 0} |F[n]/\!\sim\!| x^n$$

where $\sim$ is the equivalence relation on F-structures induced by permuting the labels. That is, $\tilde{F}$ counts *equivalence classes* of F-structures up to relabelling. For example, there is only one equivalence class of cycles for each n (any two cycles "have the same shape" and are thus related by some relabelling), so the ogf for the species of cycles is

$$\tilde{C}(x) = \sum_{n \geq 1} x^n = \frac{x}{1-x}.$$

Egfs are quite natural, and the mapping from species to their associated egf is a homomorphism that preserves many operations such as sum, product, composition, and derivative. ogfs, it turns out, are not quite as nice; the mapping from species to ogfs preserves sum and product but does not, in general, preserve composition or derivative. In some sense ogfs throw away too much information. Thus it is not possible to compositionally determine the ogf for a species defined in terms of these operations—at least not directly. The solution is to turn to a richer class of generating functions that do preserve the necessary information.

2 Cycle index series

For a species F and a permutation $\sigma \in \mathcal{S}_n$, let $\operatorname{Fix} F[\sigma]$ denote the number of F-structures that are fixed by the action of σ, that is,

$$\operatorname{Fix} F[\sigma] = \#\{f \in F[n] \mid F[\sigma]f = f\}.$$

The *cycle index series* of a combinatorial species F is a power series in an infinite set of variables $x_1, x_2, x_3, \dots$ defined by

$$Z_F(x_1, x_2, x_3, \dots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \operatorname{Fix} F[\sigma] x_1^{\sigma_1} x_2^{\sigma_2} \dots$$

We also sometimes write x^σ as an abbreviation for $x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \dots$. As a simple example, consider the species of lists, *i.e.* linear orderings. For each n, the identity permutation (with cycle type $n, 0, 0, \dots$) fixes all $n!$ lists of length n, whereas all other permutations do not fix any lists. Therefore

$$Z_L(x_1, x_2, x_3, \dots) = \sum_{n \geq 0} \frac{1}{n!} n! x_1^n = \sum_{n \geq 0} x_1^n = \frac{1}{1 - x_1}.$$

Cycle index series are linked to both egfs and ogfs by the identities

$$F(x) = Z_F(x, 0, 0, \dots) \tag{1}$$

$$\tilde{F}(x) = Z_F(x, x^2, x^3, \dots) \tag{2}$$

To show (1), note that setting all x_i to 0 other than x_1 means that the only terms that survive are terms with only x_1 raised to some power. These correspond to permutations with only 1-cycles, that is, identity permutations. Identity permutations fix *all* F-structures of a given size, so we have

$$\begin{aligned} Z_F(x, 0, 0, \dots) &= \sum_{n \geq 0} \frac{1}{n!} \operatorname{Fix} F[id] x^n \\ &= \sum_{n \geq 0} |F[n]| \frac{x^n}{n!}. \end{aligned}$$

The proof of (2) depends on Burnside's Lemma. Note that for any permutation $\sigma \in \mathcal{S}_n$ with cycle type $\sigma_1, \sigma_2, \sigma_3, \dots$ we have $\sigma_1 + 2\sigma_2 + 3\sigma_3 + \dots = n$. Thus:

$$\begin{aligned} Z_F(x, x^2, x^3, \dots) &= \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \operatorname{Fix} F[\sigma] x^{\sigma_1} x^{2\sigma_2} x^{3\sigma_3} \dots \\ &= \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \operatorname{Fix} F[\sigma] x^n \\ &= \sum_{n \geq 0} |F[n]/\sim| x^n \end{aligned}$$

where the final step is an application of Burnside's Lemma.

Crucially, the mapping from species to cycle index series is again a homomorphism for many of the operations we care about, including composition. So one can compute with cycle index series and project down to ogfs at the end.

3 Cycle index series for bracelets

We now work out the cycle index series for the species B of bracelets, where a bracelet structure is construed as a sequence of (uniquely labelled) beads, considered equivalent up to rotation and reflection.

Note first that bracelets of size n have the dihedral group D_{2n} as their symmetry group. That is, every n-bracelet is fixed by the action of any element of D_{2n}, and no bracelets are fixed by the action of any other permutation. There is a single n-bracelet for each $n \in \{1, 2\}$ (by convention there are no 0-bracelets, just as there are no 0-cycles/necklaces), and for $n \geq 3$ there are $(n-1)!/2$ labelled bracelets.

Thus, the cycle index series is given by

$$\begin{aligned} Z_B(x_1, x_2, x_3, \dots) &= x_1 + \frac{1}{2}(x_1^2 + x_2) + \sum_{n \geq 3} \frac{1}{n!} \sum_{\sigma \in D_{2n}} \frac{(n-1)!}{2} x^\sigma \\ &= x_1 + \frac{1}{2}(x_1^2 + x_2) + \sum_{n \geq 3} \frac{1}{2n} \sum_{\sigma \in D_{2n}} x^\sigma. \end{aligned}$$

Our remaining task is thus to compute $\sum_{\sigma \in D_{2n}} x^\sigma$, that is, to compute the cycle types of elements of D_{2n}.