Folding Conic Sections

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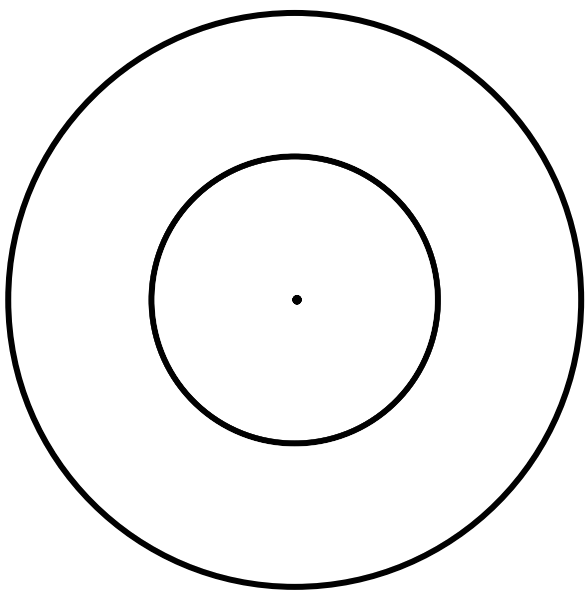
Bruce Yoshiwara

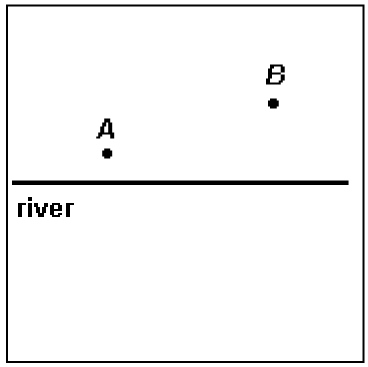
yoshiwbw@piercecollege.edu

http://www.piercecollege.edu/faculty/yoshibw/



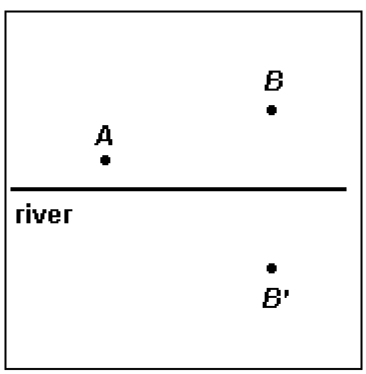
Folding Conic Sections

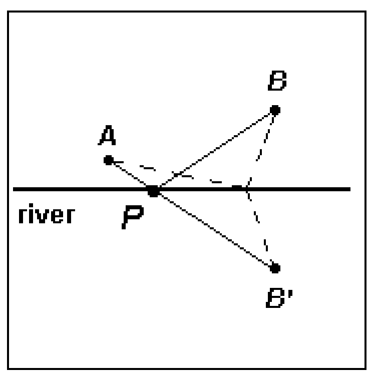
Here's a classic problem. (Heron of Alexandria gave a solution ca. 100 C.E.) A campsite is located in a large, flat clearing near a straight river. The camper is at point *A* and the tent is at point *B*. The tent catches fire, and fortunately the camper already has a bucket in hand. At what point *P* on the river should the camper fill the bucket with water to make the shortest possible path to douse the fire?



Activity 1

Copy the points *A* and *B* and the line *l* representing the river onto a clean sheet of (patty) paper. Fold the paper along the river, and mark the point *B'* which lies directly on *B* when the paper is folded. Clearly all the points on *l* are equidistant to *B* and *B'* (even after we unfold the paper).





In other words, the total distance of traveling from *A* to a point *P* on the river and then to *B* is the same distance as traveling from *A* to the point *P* and then to *B'*. The camper should head to the point on the river in line with the point *B'* because the shortest distance from *A* to *B'* is the straight path.

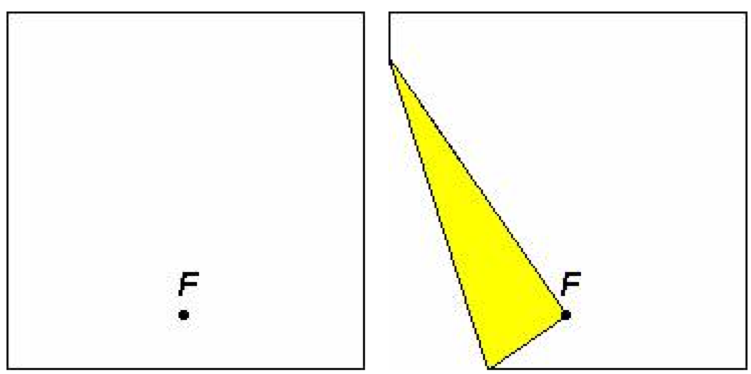
Nature uses optimal paths. The point *P* is where a billiards player should aim a cue ball positioned at *A* if the cue ball is to bounce off the cushion (in place of the river) to strike a ball at *B* (assuming the player is not putting "English" on the ball). Light travels in paths to minimize time, so the path the camper should take corresponds to the path light will take if river is replaced by a mirror and the camper simply wants to see the reflection of the burning tent. The point *B'* is called the *reflection of B across l.*

The vertical angles formed by the river and the line from *A* to *B'* are congruent. Also, the angle made with the river and the ray from *P* to *B* is congruent to the angle made with the river and the ray from *P* to *B'* , so the minimum path through the point *P* satisfies the property that "the angle of incidence equals the angle of reflection."

Of course *l* is the perpendicular bisector of the line segment joining *B* to *B'*, but the important property for us is that all points on *l* are equidistant to *B* and *B'*.

Activity 2

On a clean sheet of (patty) paper, mark a black dot about 2 centimeters above the bottom edge of the paper, and label the dot with an "*F*." Mark any point on the bottom edge of the paper and fold the paper until that new point lies directly above the first dot. Make a neat crease in the paper.

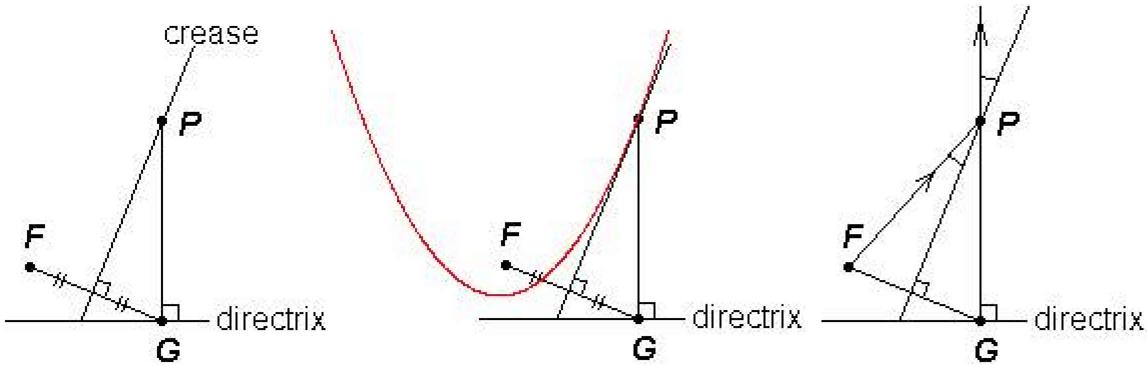


Unfold the paper, choose a second point on the bottom edge, and fold to put this new point directly over the point *F*. Crease carefully. Repeat this creasing process for a few dozen relatively evenly spaced points across the bottom edge. The creases should create a curve winding partially around *F*. This curve is a *parabola*.

What's going on?

When we fold the sheet so that one point lies directly over another, the crease is along the line equidistant from the two points (the perpendicular bisector of the segment joining the two points).

If we examine a single crease from Activity 2, we have a line that is equidistant to the point *F* and a point on the bottom edge of the paper. We call *F* our **focus** and the bottom edge is part of the line we call our **directrix**. A parabola consists of all the points that are equidistant to a focus and a directrix. (The distance from a point to a line is the distance to the nearest point on the line.)



Let us call our point on the bottom edge *G*. Of all the points along the crease, the point *P* that is directly above *G* has a special property, namely, that *P* is the same distance from *F* as it is from the bottom edge. Or in other words, *P* lies on our parabola.

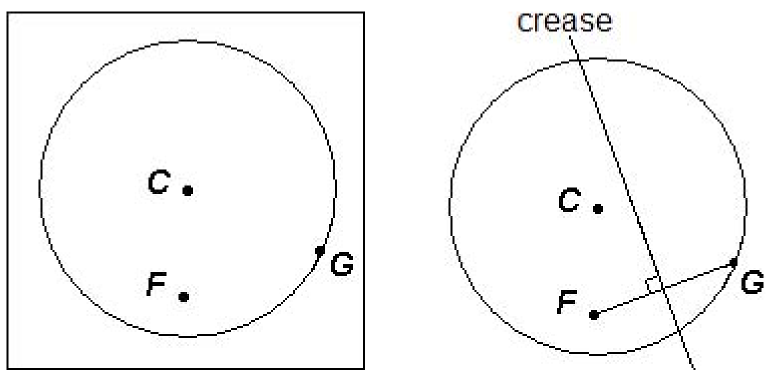
If we take any other point on the crease, its closest point on the bottom edge is no longer *G*, so the new point is closer to the bottom edge than it is to *G*. On the other hand, the new point is still equidistant to *F* and *G*, so it is closer to the bottom edge than it is to *F*. This means that the new point on the crease is too low to be on the parabola. So aside from the point *P*, all the points on the parabola are above the crease. The parabola touches but does not cross the crease at *P*, or in other words, <u>the crease is a tangent line</u> to the parabola.

A light ray or a billiard ball will reflect off a curve as if reflecting off the tangent line. So a ray of light emanating from the focus will reflect off the parabola as if it is reflecting off the crease. The fold and the vertical line through P form congruent vertical angles. Also, the fold bisects the angle FPG. These two facts give us that the acute angle that FP makes with the fold is congruent to the acute angle that the fold makes with the vertical ray going up from P. In other words, a ray of light emanating from the focus reflects off the parabola to travel straight up (perpendicular to the directrix, parallel to the *axis* of the parabola).

A light bulb placed at the focal point of a parabolic mirror will have its light reflected in parallel lines, a property used in flashlights and automobile headlights. Light or other electromagnetic waves traveling in lines parallel to the axis will reflect off a parabolic dish towards a collector at the focus, which is how satellite dishes and reflecting telescopes work.

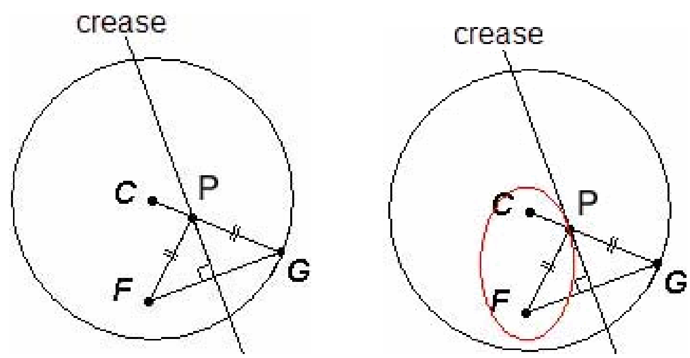
Activity 3

On a clean sheet of (patty) paper, draw a large circle and mark a black dot inside the circle, but off center. Label the dot with an "*F*." Pick a point on the circle and fold the paper until *F* lies directly above that new point, making a neat crease in the paper.

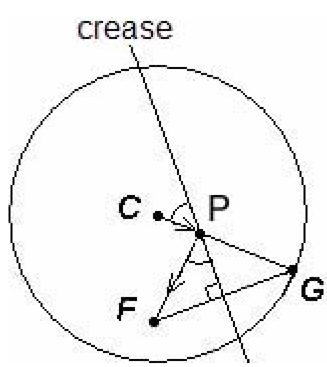


Unfold the paper, choose a second point on the circle, and fold to put *F* directly above this new point. Crease and repeat for a few dozen relatively evenly spaced points on the circle. The creases create a curve inside the circle. This oval is an *ellipse*.

An ellipse is the set of all points whose sum of distances to two foci is a constant. The constant is the length of the major axis of the ellipse. In Activity 3, one focus is the point *F* and the other focus is *C*, the center of the circle. A crease is a line that is equidistant to the focus *F* and a point *G* on the circle.



Consider the point *P* that is on both the crease and also on the radius from *C* to *G*. The distance from *C* to *G* is the radius *r*. On the other hand, because *P* is equidistant to *F* and *G*, the sum of the distances from *P* to *C* and from *P* to *F* is *r*. Or in other words, *P* lies on an ellipse whose major axis shares the same length as the radius of the given circle. (We could also describe an ellipse as the set of points that are equidistant to a circle and a point inside the circle.)



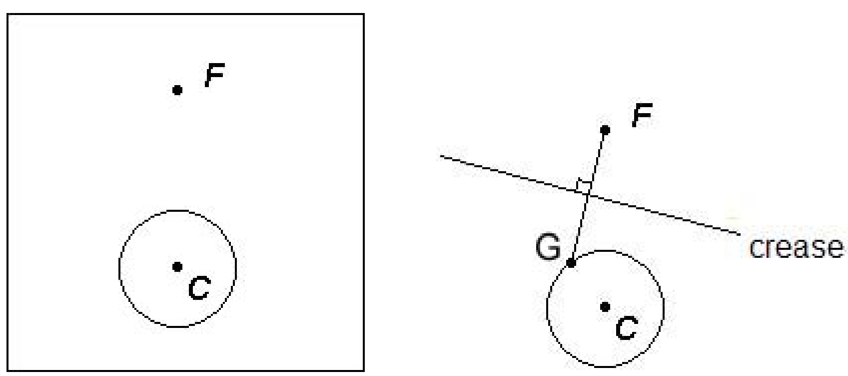
If we take any other point on the crease, it is no longer on the radius to *G*, so the sum of its distances to the two foci exceeds the radius of the circle (because of the triangle inequality). Thus any other point on the crease is outside the ellipse, and so the crease is a tangent line to the ellipse.

As with the parabola, the crease bisects the angle *FPG*, and the crease also creates vertical angles with the radius to *G*. The angle made by the segment *FP* with the crease is congruent to the angle made by *CP* with the crease. So a ray emanating from one focus reflects off the point *P* on the ellipse towards the other focus.

Elliptical domes, such as the Mormon Tabernacle in Salt Lake City or the Capitol in Washington DC, create "whispering galleries" where even a pin dropped at one focus can be heard more than a hundred feet away at the other focus. The sound waves from the whisper all travel the same distance to bounce off the ceiling to meet simultaneously at other focus. A non-surgical treatment of kidney stones also uses the reflection property of the ellipse. In Extracorporeal Shock Wave Lithotripsy, the patient is placed so that the kidney stone positioned at one focus of an ellipse. A high energy sound wave is created at the other focus, and it reflects off all parts of an elliptical tank wall to break up the kidney stone.

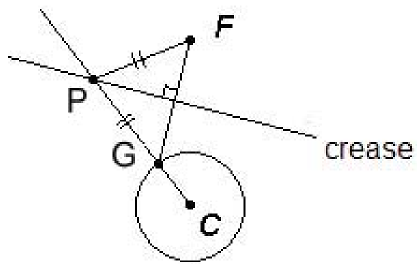
Activity 4

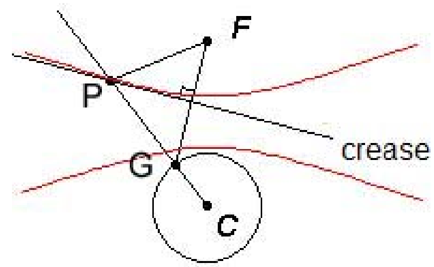
On a clean sheet of (patty) paper, draw a small circle and mark a black dot outside the circle, labeling the dot "*F*." Mark a point on the circle, and fold the paper until *F* lies directly above that new point, making a neat crease in the paper.



Repeat for a few dozen relatively evenly spaced points on the circle. The creases create two curves. The two pieces together form a *hyperbola*.

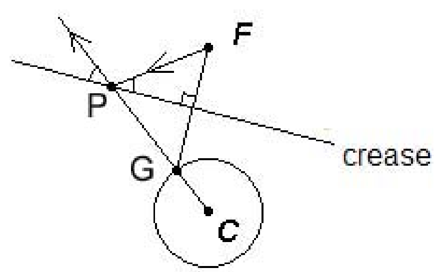
A hyperbola is the set of all points whose difference of distances to two foci is a constant. In Activity 4, the foci are again the point *F* and the center of the circle, *C*. As with the ellipse, a crease is a line that is equidistant to the focus *F* and a point *G* on the circle.





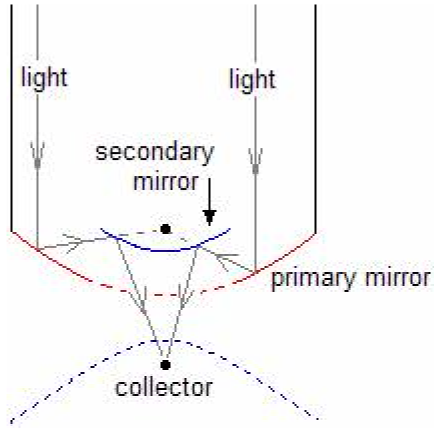
Consider the point *P* that is on both the crease and also on the line through C and G. The radius *r* is the difference in distances between *C* and *P* and between *P* and *G*. On the other hand, because *P* is equidistant to *F* and *G*, *r* is the difference in distances between *C* and *P* and between *P* and *F*. Or in other words, *P* lies on a hyperbola, and the radius of the given circle is the difference between the distances to the two foci.

If we take any other point on the crease, it is no longer on the ray through *G*, so the difference of its distances to the two foci is less than the radius of the circle. Thus any other point on the crease lies too far from the near focus to be on the hyperbola, and so the crease is a tangent line to the hyperbola.



As with the other two conic sections, the crease bisects the angle *FPG*, and the crease also creates vertical angles with the ray through *G*. The angle made by the segment *FP* with the crease is congruent to the angle made by *CP* with the crease. So a ray emanating from one focus reflects off the point *P* on the hyperbola as if it had emanated from the other focus. This has the effect of spreading out any waves emanating from a focus. Hyperbolic shapes are used for horns, street lamps, and space heaters.

In the Cassegrain telescope design, the primary *parabolic* mirror reflects light towards a focal point. A secondary, *hyperbolic* mirror is positioned so that one of its foci coincides with the focus of the parabola and the other is the collector behind a hole in the primary mirror. The advantage of the Cassegrain design over the traditional reflection telescope is that the Cassegrain telescope can be much more compact.



Disclaimer: The drawings are not accurate!

References

Chakerian, Crabill and Stein, *Geometry: A Guided Inquiry*, Davis Mathematics Press, (1984).

Kriegler et al, *More (Advanced) Topics in Geometry Participant Reader* (Draft), UCLA, (2004).

COMAP, *For All Practical Purposes*, 3rd, W.H. Freeman and Company, (1994).

(For Origami: http://kahuna.merrimack.edu/~thull/)