Algebra Toolkit

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Preface

This ”Toolkit” accompanies ”Intermediate Algebra.” College students who take a developmental algebra course arrive with different levels of preparation, especially now that many institutions allow only a single math course below college level. The Toolkit provides students with a resource to support their knowledge of prerequisite skills. It can be used as a reference by individual students, or the instructor can use it as a supplement to the main text. We find that a brief review of half-forgotten skills, targeted specifically at new material, is often more effective than rehashing a previous course.

Each section of the Toolkit is aligned to the corresponding section in the text, and addresses the algebra skills used in that particular section. The Toolkit section includes one or two examples of each skill and several exercises (with answers) for students to try. We think this approach has a number of advantages.

- Students are usually more motivated to master a skill when they see an immediate need for it.
- Only two or three related skills are targeted in each lesson.
- We can return to broadly defined ideas in particular settings. For example, solving equations appears in several lessons, each with a different emphasis, which allows for reinforcement of the ideas.
- Another benefit of treating the same skill more than once is that similar algebraic computations can look quite different to students in different applications. Examples can help with transfer of learning.

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Atascadero, CA 2020

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Chapter 1

Linear Models

Linear Models

1. Write a linear model

When we say "Express y in terms of x," we mean to write an equation that looks like

$$y = \text{algebraic expression in } x$$

We say that x is the input variable, and y is the output variable.

In particular, a **linear model** has the form

$$\boldsymbol{y} = \text{starting value} + \text{rate} \times \boldsymbol{x}$$

Examples

Example 1.1 Steve bought a Blu-Ray player for \$269 and a number of discs at \$14 each. Write an expression for Steve's total bill, B (before tax), in terms of the number of discs he bought, d.

Solution. We want an equation of the form

$$B = \text{starting value} + \text{rate} \times d$$

where Steve's bill started with the Blu-Ray player or \$269, and then increased by a number of discs at a rate of \$14 each. Substituting those values, we have

$$B = 269 + 14d$$

□

Example 1.2 At 6 am the temperature was 50°, and it has been falling by 4° every hour. Write an equation for the temperature, T, after h hours.

Solution. We want an equation of the form

$$T = \text{starting value} + \text{rate} \times h$$

The temperature started at 50°, and then decreased each hour at the rate of 4° per hour, so we subtract $4h$ from 50 to get

$$T = 50 - 4h$$

□

Example 1.3 Kyli's electricity company charges her \$6 per month plus \$0.10 per kilowatt hour (kWh) of energy she uses. Write an equation for Kyli's electric bill, E, if she uses w kWh of electricity.

Solution. Kyli's bill starts a6 \$6 and increases by \$0.10 for each kWh, w. Thus,

$$E = 6 + 0.10w$$

□

Exercises

Checkpoint 1.4 Salewa saved \$5000 to go to school full time. She spends \$200 per week on living expenses. Write an equation for Salewa's savings, S, after w weeks.

Answer. $S = 5000 - 200w$

Checkpoint 1.5 As a student at City College, Delbert pays a \$50 registration fee plus \$15 for each unit he takes. Write an equation that gives Delbert's tuition, T, if he takes u units.

Answer. $T = 50 + 15u$

Checkpoint 1.6 Greta's math notebook has 100 pages, and she uses on average 6 pages per day for notes and homework. How many pages, P, will she have left after d days?

Answer. $P = 100 - 6d$

Checkpoint 1.7 Asa has typed 220 words of his term paper, and is still typing at a rate of 20 words per minute. How many words, W, will Asa have typed after m more minutes?

Answer. $W = 220 + 20m$

Checkpoint 1.8 The temperature in Nome was $-12°$ F at noon. It has been rising at a rate of $2°$ F per hour all day. Write an equation for the temperature, T, after h hours.

Answer. $T = -12 + 2h$

Checkpoint 1.9 Francine borrowed money from her mother, and she owes her \$750 right now. She has been paying off the debt at a rate of \$50 per month. Write an equation for Francine's financial status, F, in terms of m, the number of months from now.

Answer. $F = -750 + 50m$

2. Plot points and graph a line

The simplest way to graph a line is to make a table and plot points. We will learn more efficient methods short.ly.

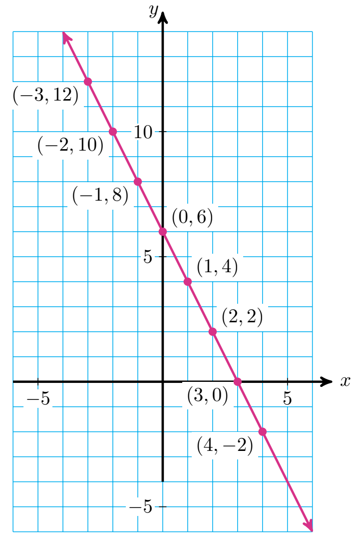
Examples

Example 1.10 Make a table of values, plot the points, and graph the equation $y = -2x + 6$.

Solution. Choose both positive and negative values for x. Calculate the y-value for each x-value by substituting the x-value into the equation.

x	y	
-3	12	$y = -2(\mathbf{-3}) + 6 = 12$
-2	10	$y = -2(\mathbf{-2}) + 6 = 10$
-1	8	$y = -2(\mathbf{-1}) + 6 = 8$
0	6	$y = -2(\mathbf{0}) + 6 = 6$
1	4	$y = -2(\mathbf{1}) + 6 = 4$
2	2	$y = -2(\mathbf{2}) + 6 = 2$
3	0	$y = -2(\mathbf{3}) + 6 = 0$
4	-2	$y = -2(\mathbf{4}) + 6 = -2$

Next, sketch a Cartesian coordinate system with appropriate scales on the x- and y-axes. Plot each of the points in the table of values and connect them with a straight line. The completed graph is shown at right.



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Example 1.11 Byron borrowed \$6000 from his uncle to help pay for his college education. Now that he has graduated and has a job, he is paying back the loan at \$100 per month.

a Write an equation showing the amount of money, y, that Byron still owes his uncle after x months.

b Graph your equation.

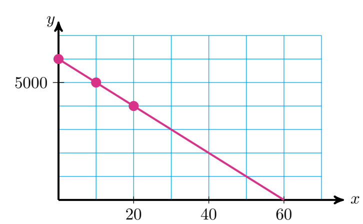
Solution.

a $y = 6000 - 100x$

b

x	y	
0	6000	$y = 6000 - 100(\mathbf{0}) = 6000$
10	5000	$y = 6000 - 100(\mathbf{10}) = 5000$
20	4000	$y = 6000 - 100(\mathbf{20}) = 4000$

Now choose appropriate scales for the axes. A good choice would be to scale the x-axis by 10's and the y-axis by 1000's.

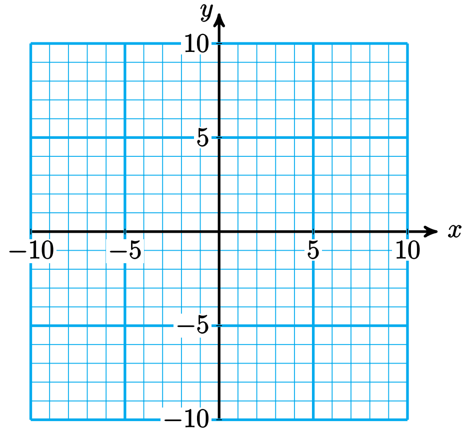


□

Exercises

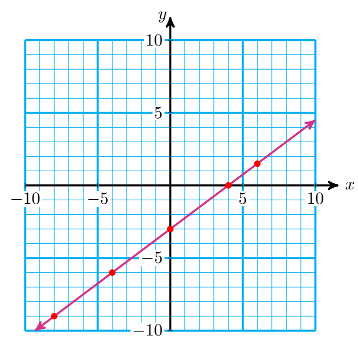
Checkpoint 1.12 Graph $y = \frac{3}{4}x - 3$.

x	-8	-4	0	4	6
y					



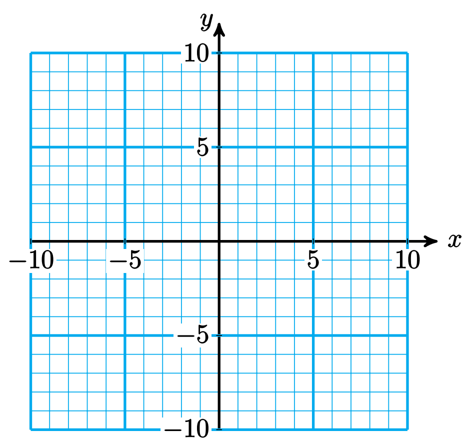
Answer.

x	-8	-4	0	4	6
y	-9	-6	-3	0	1.5



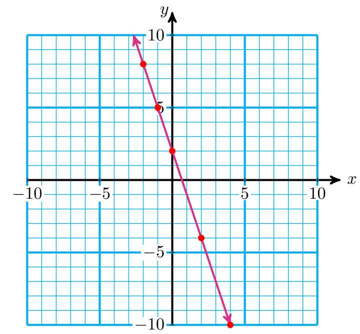
Checkpoint 1.13 Graph $y = -3x + 2$.

x	-2	-1	0	2	4
y					



Answer.

x	-2	-1	0	2	4
y	8	5	2	-4	-10

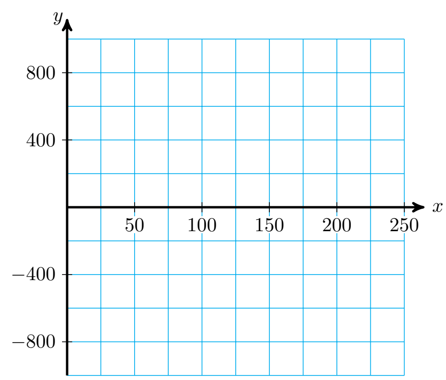


Checkpoint 1.14 Stuart invested \$800 in a computer and now makes \$5 a page typing research papers. Let x represent the number of pages Stuart has typed, and let y represent his profit.

a Write an equation for y in terms of x.

b Complete the table and graph your equation.

x	y
0	
50	
100	
200	

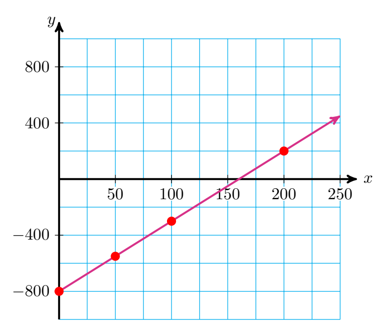


Answer.

a $y = 5x - 800$

b

x	y
0	-800
50	-550
100	-300
200	200

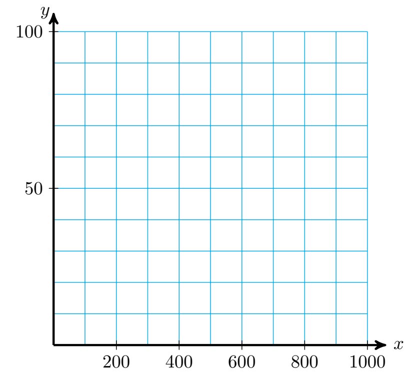


Checkpoint 1.15 Ludmilla earns a commission of 5% of her real estate sales. Let x represent her sales in thousands of dollars, and let y represent the commission she earns from her sales, in thousands of dollars.

a Write an equation for y in terms of x.

b Complete the table and graph the equation.

x	y
250	
600	
800	
1000	

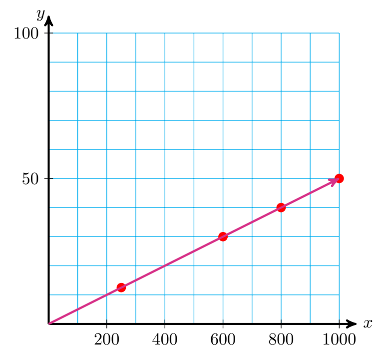


Answer.

a $y = 0.05x$

b

x	y
250	12.5
600	30
800	40
1000	50



3. Solve a linear equation

Recall that to solve an equation we want to "isolate" the variable on one side of the equals sign. We "undo" each operation performed on the variable by performing the opposite operation on both sides of the equation.

Examples

Example 1.16 Solve the equation $\frac{2}{3}x - 5 = 7$

Solution.

$$\frac{2}{3}x - 5 = 7 \qquad \text{Add 5 to both sides.}$$

$$\frac{2}{3}x = 12 \qquad \text{To divide both sides by } \mathbf{\frac{2}{3}}\text{, we:}$$

$$\frac{3}{2}\left(\frac{2}{3}x\right) = \frac{3}{2}(12) \qquad \text{Multiply by the reciprocal of } \mathbf{\frac{2}{3}}.$$

$$x = 18 \qquad \text{The solution is 18.}$$

□

Example 1.17 Solve the equation $2x + 7 = 4x - 3$

Solution. To begin, we must get both variable terms on the same side of the equation.

$$2x + 7 = 4x - 3 \qquad \text{Subtract } \boldsymbol{2x} \text{ from both sides.}$$

$$7 = 2x - 3 \qquad \text{Add 3 to both sides.}$$

$$10 = 2x \qquad \text{Divide both sides by 2.}$$
$$5 = x \qquad \text{The solution is 5.}$$

□

Exercises

Checkpoint 1.18 Solve the equation $10 = 1 - \dfrac{3x}{7}$

Answer. -21

Checkpoint 1.19 Solve the equation $6p - 8 = -3p - 26$

Answer. -2

Checkpoint 1.20 Solve the equation $12 = \dfrac{7u + 4}{5}$

Hint: Start by clearing the fraction: multiply both sides by 5.

Answer. 8

Checkpoint 1.21 Solve the equation $0 = 13q + 25 - 17q + 7$

Hint: Start by combining like terms.

Answer. 8

4. Solve a linear inequality

The rules for solving an inequality are the same as those for solving an equation, with one important difference:

Solving a Linear Inequality.

If we multiply or divide both sides of a linear inequality by a negative number, we must reverse the direction of the inequality symbol.

Examples

Example 1.22 Solve $-3x + 1 > 7$ and graph the solutions on a number line.

Solution.

$$-3x + 1 > 7 \qquad \text{Subtract 1 from both sides.}$$
$$-3x > 6 \qquad \text{Divide both sides by } \mathbf{-3}\text{, and reverse the direction of the inequality.}$$
$$x < -2$$

The solutions are all the numbers less than -2. The graph of the solutions is shown below.

□

Example 1.23 Solve $-3 < 2x - 5 \le 6$ and graph the solutions on a number line.

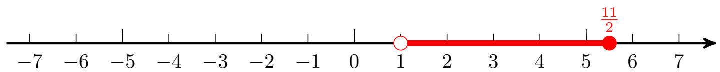
Solution.

$$-3 < 2x - 5 \le 6 \qquad \text{Add 5 on all three sides of the inequality.}$$

$$2 < 2x \leq 11 \qquad \text{Divide each side by 2.}$$

$$1 < x \leq \frac{11}{2} \qquad \text{Notice that we did not reverse the inequality.}$$

The solutions are all the numbers greater than 1 but less than 5.5. The graph of the solutions is shown below.



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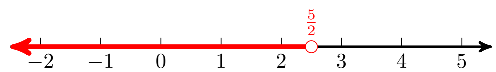
Recall that a solid dot on a number line indicates that the number is part of the solution; an open dot means that the number is not part of the solution.

Exercises

Checkpoint 1.24 Solve the inequality $8 - 4x > -2$ and graph the solutions on a number line.

Answer.

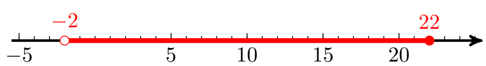
$x < \dfrac{5}{2}$



Checkpoint 1.25 Solve the inequality $-6 \leq \dfrac{4-x}{3} < 2$ and graph the solutions on a number line.

Answer.

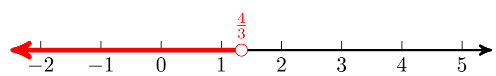
$22 \geq x > -2$



Checkpoint 1.26 Solve the inequality $3x - 5 < -6x + 7$ and graph the solutions on a number line.

Answer.

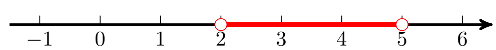
$x < \dfrac{4}{3}$



Checkpoint 1.27 Solve the inequality $-6 > 4 - 5b > -21$ and graph the solutions on a number line.

Answer.

$2 < b < 5$



Graphs and Equations

1. Verify a solution

We can always check a solution to an equation by verifying that it makes the equation true.

Examples

Example 1.28 Verify that $x = -5$ is a solution of the equation

$$x^2 + 2x - 15 = 0$$

Solution. We show that substituting -5 for x makes the equation true. When we substitute a negative number for a variable, we should enclose the number in parentheses.

$$\begin{aligned} x^2 + 2x - 15 &= (\mathbf{-5})^2 + 2(\mathbf{-5}) - 15 \\ &= 25 - 10 - 15 = 0 \end{aligned}$$

Because the expression does equal 0, we see that $x = -5$ is a solution. □

Example 1.29 Verify that $x = -3$ is not a solution of the equation

$$\sqrt{2x+10} - 3x = 8$$

Solution. We show that substituting -3 for x does not make the equation true.

$$\begin{aligned} \sqrt{2x+10} - 3x &= \sqrt{2(\mathbf{-3}) + 10} - 3(\mathbf{-3}) \\ &= \sqrt{4} + 9 = 2 + 9 = 11 \neq 8 \end{aligned}$$

The left side of the equation does not equal 8 when $x = -3$, so $x = -3$ is not a solution. □

Exercises

Checkpoint 1.30 Decide whether the given value is a solution of the equation.

$$x^3 - 3x^2 - 4x + 2 = 10; \qquad x = -2$$

Answer. Yes

Checkpoint 1.31 Decide whether the given value is a solution of the equation.

$$\sqrt{3x+5} = 10 + \sqrt{x+7}; \qquad x = 9$$

Answer. No

Checkpoint 1.32 Decide whether the given value is a solution of the equation.

$$\frac{2x-1}{x+1} + 2 = \frac{x+1}{x-1}; \qquad x = 2$$

Answer. Yes

Checkpoint 1.33 Decide whether the given value is a solution of the equation.

$$9 - 4x = 5\sqrt{x+3}; \qquad x = 6$$

Answer. No

2. Solve linear equations and inequalities with parentheses

Strategy for solving linear equations.

1 Simplify each side of the equation: apply the distributive law, combine like terms.

2 Use addition and subtraction to get all the variable terms on one side of the equation, and all constsnt terms on the other side.

3 Divide both sides by the coefficient of the variable.

Examples

Example 1.34 Solve $3(2a-4) \geq 4-(1-3a)$

Solution. First, we remove parentheses by applying the distributive law. Then we can combine like terms on each side of the equation.

Note that the minus sign in front of the parentheses on the right side of the equation applies to both terms inside the parentheses.

$$\begin{aligned} 3(2a-4) &\geq 4-(1-3a) && \text{Apply the distributive law.} \\ 6a-12 &\geq 4-1+3a && \text{Simplify the right side.} \\ 6a-12 &\geq 3+3a && \text{Subtract } \boldsymbol{3a} \text{ from both sides.} \\ 3a-12 &\geq 3 && \text{Add 12 to both sides.} \\ 3a &\geq 15 && \text{Divide both sides by 3.} \\ a &\geq 5 \end{aligned}$$

□

Example 1.35 Solve the inequality $25-6x > 3x-2(4-x)$

Solution. We begin by the same way we solve an equation. For this example, we start by removing the parentheses.

$$\begin{aligned} 25-6x &> 3x-2(4-x) && \text{Apply the distributive law.} \\ 25-6x &> 3x-8+2x && \text{Combine like terms.} \\ 25-6x &> 5x-8 && \text{Subtract } \boldsymbol{5x} \text{ from both sides.} \\ 25-11x &> -8 && \text{Subtract 25 from both sides.} \\ -11x &> -33 && \text{Divide both sides by -11.} \\ x &< 3 && \text{Don't forget to reverse the inequality symbol.} \end{aligned}$$

Recall that if we multiply or divide both sides of an inequality by a negative number, we must reverse the direction of the inequality symbol. □

Exercises

Checkpoint 1.36 Solve the inequality $-4(x+2)+3(x-2) \geq -2$

Answer. $x \leq -12$

Checkpoint 1.37 Solve the equation $4(2-3w) = 9-3(2w-1)$

Answer. $\dfrac{-1}{2}$

Checkpoint 1.38 Solve the inequality $2(3h-6) < 5-(h-4)$

Answer. $h < 3$

Checkpoint 1.39 Solve the equation $0.25(x+3) - 0.45(x-3) = 0.30$

Answer. 9

3. Solve an equation in two variables

A solution of an equation in two variables x and y is written as an **ordered pair**, (x, y). For example, the solution $(-2, 5)$ means that $x = -2$ and $y = 5$.

Examples

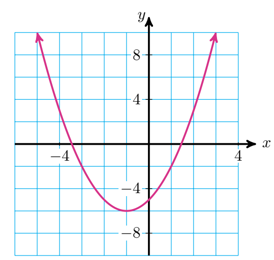
Example 1.40 Is $(-3, 2)$ a solution of the equation $x^2 + 4y^2 = 25$?

Solution. We substitute $x = \mathbf{-3}$ and $y = \mathbf{2}$ into the equation.

$$(\mathbf{-3})^2 + 4(\mathbf{2})^2 = 9 + 4(4) = 9 + 16 = 25$$

The ordered pair $(-3, 2)$ satisfies the equation, so it is a solution. □

Example 1.41
Which of the following ordered pairs are solutions of the equation whose graph is shown?



a $(-3, -2)$

b $(-5, 0)$

c $(1, -4)$

d $(-1, -6)$

Solution. The graph of an equation is just a picture of its solutions, so points that lie on the graph are solutions of the equation.

The points $(-3, -2)$ and $(-1, -6)$ lie on the graph, so they represent solutions of the equation. The points $(-5, 0)$ and $(1, -4)$ do not lie on the graph, so they are not solutions of the equation. □

Exercises

Checkpoint 1.42 Find a solution of the equation with the given coordinate.

$$6x - 5y = -3, \qquad (2, ?)$$

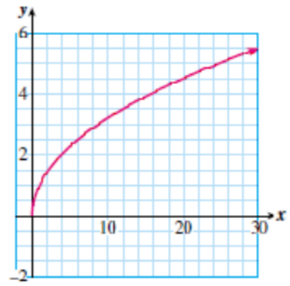
Answer. $(2, 3)$

Checkpoint 1.43 Find a solution of the equation with the given coordinate.

$$y = \frac{3}{4}x + 8, \qquad (?, -1)$$

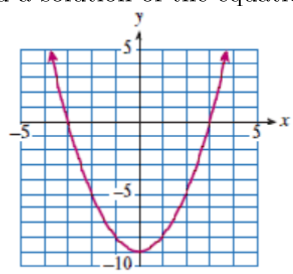
Answer. $(-12, -1)$

Checkpoint 1.44 Find a solution of the equation with the given coordinate.



Answer. $(16, 4)$

Checkpoint 1.45 Find a solution of the equation with the given coordinate.



Answer. $(2, -5)$

Intercepts

1. Graph a linear equation by the intercept method

To graph a line by the intercept method, we find the x- and y-intercepts of the line and plot those points.

Example 1.46 Graph the equation $3x + 2y = 7$ by the intercept method.

Solution. First, we find the x- and y-intercepts of the graph. To find the y-intercept, we substitute 0 for x and solve for y:

$$\begin{aligned} 3(\mathbf{0}) + 2y &= 7 && \text{Simpify the left side.} \\ 2y &= 7 && \text{Divide both sides by 2.} \\ y &= \frac{7}{2} = 3\frac{1}{2} \end{aligned}$$

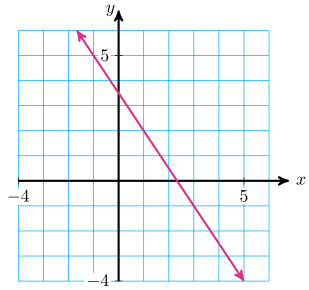
The y-intercept is the point $\left(0, 3\frac{1}{2}\right)$. To find the x-intercept, we substitute 0 for y and solve for x:

$$\begin{aligned} 3x + 2(\mathbf{0}) &= 7 && \text{Simpify the left side.} \\ 3x &= 7 && \text{Divide both sides by 3.} \\ x &= \frac{7}{3} = 2\frac{1}{3} \end{aligned}$$

The x-intercept is the point $\left(2\frac{1}{3}, 0\right)$.

A table with the two intercepts is shown below. We plot the intercepts and connect them with a straight line.

x	y
0	$3\frac{1}{2}$
$2\frac{1}{3}$	0

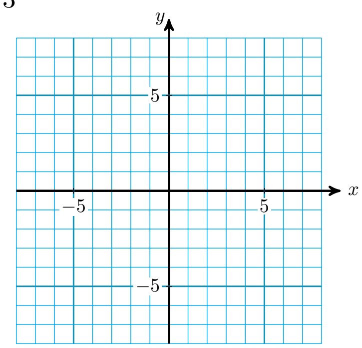


□

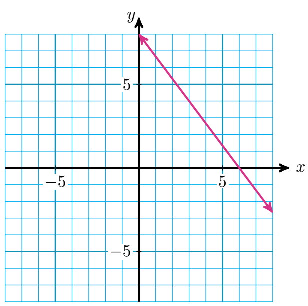
Exercises

Checkpoint 1.47 Graph the line $y = \dfrac{-4}{3}x + 8$ by the intercept method.

x	y

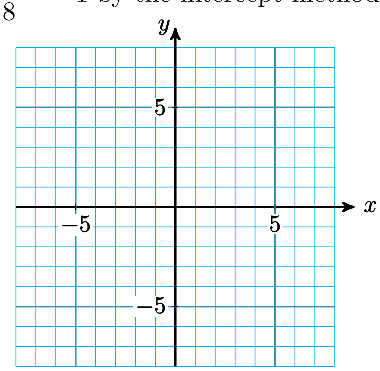


Answer.

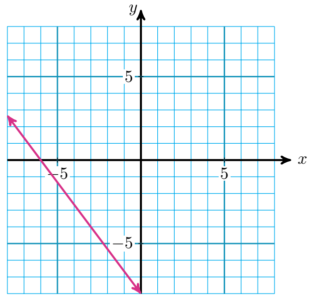


Checkpoint 1.48 Graph the line $\dfrac{x}{6} + \dfrac{y}{8} = -1$ by the intercept method.

x	y



Answer.



2. Interpret the intercepts

The values of the variables at the intercepts often tell us something important about a linear model

Example 1.49 The temperature, T, in Nome was $-12°$ at noon and has been rising at a rate of $2°$ per hour all day.

- Write and graph an equaton for T in terms of h, the number of hours after noon.
- Find the intercepts of the graph and interpret their meaning in the context of the problem situation.

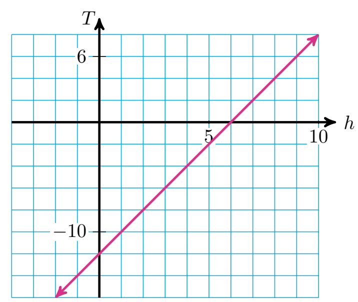
Solution.

An equation for T at time h is

$$T = -12 + 2h$$

To find the T-intercept, we set $h = 0$ and solve for T.

$$T = -12 + 2(\mathbf{0}) = -12$$



The T-intercept is $(0, -12)$. This point tells us that when $h = 0, T = -12$, or the temperature at noon was $-12°$. To find the h-intercept, we set $T = 0$ and solve for h.

$$\begin{aligned} \mathbf{0} &= -12 + 2h && \text{Add 12 to both sides.} \\ 12 &= 2h && \text{Divide both sides by 2.} \\ 6 &= h \end{aligned}$$

The h-intercept is the point $(6, 0)$. This point tells us that when $h = 6, T = 0$, or the temperature will reach zero degrees at six hours after noon, or 6 pm. □

Exercises

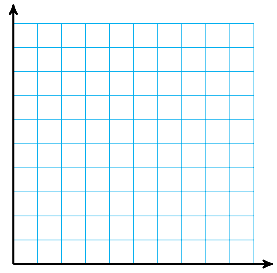
Checkpoint 1.50 Sheri bought a bottle of multivitamins for her family. The number of vitamins lt in the bottle after d days is given by

$$N = 300 - 5d$$

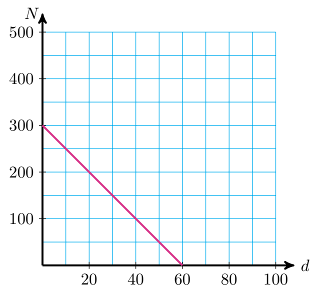
a Find the intercepts and use them to make a graph of the equation.

d	N

b Explain what each intercept tells us about the vitamins.



Answer.

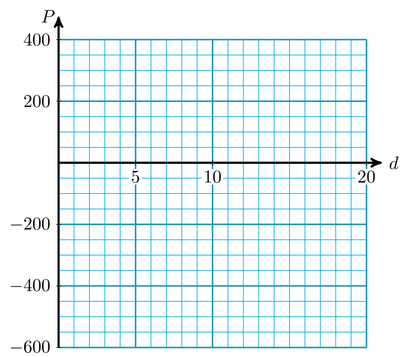


- $(0, 300)$ There were 300 vitamins to start.
- $(60, 0)$ The vitamin bottle is empty after 60 days.

Checkpoint 1.51 Delbert bought some equipment and went into the dog-grooming business. His profit is increasing according to the equation

$$P = -600 + 40d$$

where d is the number of dogs he has groomed.

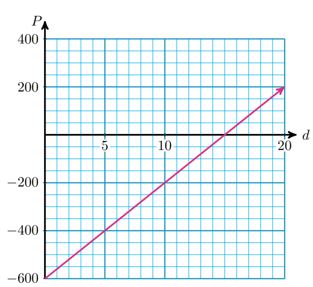


a Find the intercepts and use them to make a graph of the equation.

d	P

b Explain what each intercept tells us about Delbert's dog-grooming business.

Answer.



- $(0, -600)$ To start, Delbert's profit is $-\$600$. (He is \$600 in debt.)
- $(15, 0)$ Delbert breaks even after grooming 15 dogs.

3. Solve an equation for one of the variables

It is usually easier to study a model and draw its graph if it is in the form

$$y = \text{starting value} + \text{rate} \times x$$

To put an equation into this form, we want to "isolate" the output variable on one side of the equation.

Examples

Example 1.52 Solve the equation $2x - 3y = 8$ for y.

Solution.

$$\begin{aligned} 2x - 3y &= 8 && \text{Subtract } \mathbf{2x} \text{ from both sides.} \\ -3y &= 8 - 2x && \text{Divide both sides by } \mathbf{-3.} \\ y &= \frac{8-2x}{-3} && \text{Divide each term of the numerator by } \mathbf{-3.} \\ y &= \frac{8}{3} - \frac{2}{3}x \end{aligned}$$

□

Example 1.53 Solve the equation $A = \frac{h}{2}(b + c)$ for b.

Solution. It is nearly always best to clear fractions from an equation first, so we begin by multiplying both sides by 2.

$$\begin{aligned} 2A &= \cancel{2}\left(\frac{h}{\cancel{2}}(b+c)\right) && \text{Multiply both sides by } \mathbf{2.} \\ 2A &= h(b+c) && \text{Divide both sides by } \boldsymbol{h}\mathbf{.} \\ \frac{2A}{h} &= b + c && \text{Subtract } \boldsymbol{c} \text{ from both sides.} \\ \frac{2A}{h} - c &= b \end{aligned}$$

□

Exercises

Checkpoint 1.54 Solve $f = s + at$ for t

Answer. $t = \frac{f-a}{s}$

Checkpoint 1.55 Solve $2x - 4y = k$ for y

Answer. $y = \frac{-1}{4}k + \frac{1}{2}x$

Checkpoint 1.56 Solve $P = 2l + 2w$ for l

Answer. $l = \frac{P}{2} - w$

Checkpoint 1.57 Solve $\frac{x}{a} + \frac{y}{b} = 1$ for x

Answer. $x = a - \frac{ay}{b}$

Slope

1. Use ratios for comparison

Slope is a type of ratio that compares vertical distance per unit of horizontal distance. We use ratios for comparison in other situations, for example, when shopping we might compute price per unit.

Examples

Example 1.58 You are choosing between two brands of iced tea. Which is a better bargain: a 28-ounce bottle of Teatime for $1.82, or a 36-ounce bottle of Leafdream for $2.25?

Solution. Compute the ratio price per ounce for each brand.

$$\text{Teatime:} \quad \frac{182 \text{ cents}}{28 \text{ ounces}} = 6.5 \text{ cents per ounce}$$

$$\text{Leafdream:} \quad \frac{225 \text{ cents}}{36 \text{ ounces}} = 6.25 \text{ cents per ounce}$$

Leafdream is the better bargain. □

Example 1.59 The trail to Lookout Point gains 780 feet in elevation over a distance of 1.3 miles. The trail to Knife Edge gains 950 feet in elevation over a distance of 1.6 miles. Which trail is steeper?

Solution. Compute the ratio of elevation gain to horizontal distance traveled for each trail.

$$\text{Lookout Point:} \quad \frac{780 \text{ feet}}{1.3 \text{ miles}} = 600 \text{ feet per mile}$$

$$\text{Knife Edge:} \quad \frac{950 \text{ feet}}{1.6 \text{ miles}} = 593.75 \text{ feet per mile}$$

The Lookout Point trail is steeper. □

Exercises

Checkpoint 1.60 Rachel drove 292.4 miles on 8.6 gallons of gasoline. Reuben drove 390 miles on 12 gallons of gasoline. Who got the better gas mileage?

Hint: Compute the ratio miles per gallon.

Answer. Rachel: 34 miles per gallon; Reuben: 32.5 miles per gallon

Checkpoint 1.61 Leslie drove 168 miles in 2.8 hours, and Mark drove 224 miles in 3.5 hours. Who drove at the greater average speed?

Hint: Compute the ratio miles per hour.

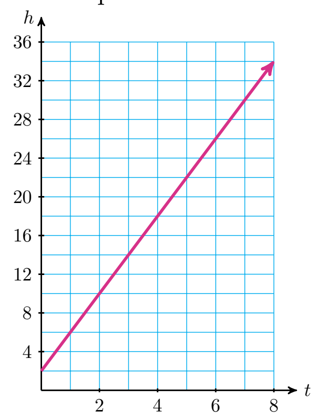
Answer. Mark: 64 miles per hour; Leslie: 60 miles per hour

2. Calculate slope from a graph

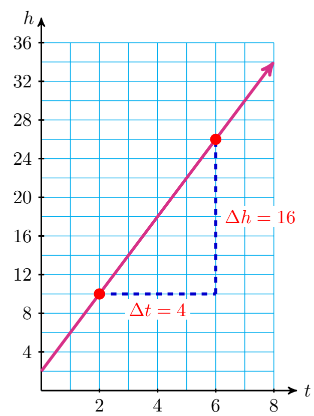
We often think of slope as measuring the ”steepness” of a graph, but the appearance of steepness is also affected by the scales on the axes.

Examples

Example 1.62 Calculate the slope of the line.

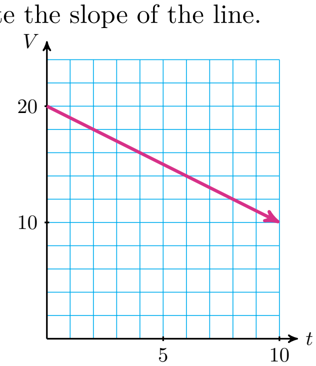


Solution. Choose two points on the line, and calculate the ratio of vertical change to horizontal change. Use the grid lines on the graph, but don't forget to note the scales on the axes.

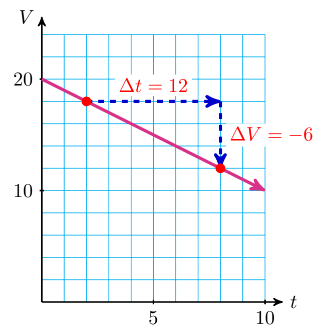


The slope is the ratio $\dfrac{\Delta h}{\Delta t}$. The variable on the horizontal axis increases by 4 units, from 2 to 6, so $\Delta t = 4$. The variable on the vertical axis increases by 8 grid lines, but each grid line represents 2 units, so $\Delta h = 16$. Thus, the slope is $\dfrac{\Delta h}{\Delta t} = \dfrac{16}{4} = 4$. $\square$

Example 1.63 Calculate the slope of the line.



Solution. Choose two points on the line, and calculate the ratio of vertical change to horizontal change. Use the grid lines on the graph, but don't forget to note the scales on the axes.

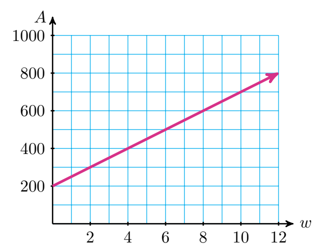


The slope is the ratio $\dfrac{\Delta V}{\Delta t}$. The horizontal variable, t, increases by 6 grid lines, but each grid line represents 2 units, so $\Delta t = 12$. The vertical variable, V, decreases by 3 grid lines, or 6 units, so $\Delta V = -6$. Thus, $\dfrac{\Delta V}{\Delta t} = \dfrac{-6}{12} = \dfrac{-1}{2}$. □

Exercises

Checkpoint 1.64 Calculate the slope of the line.

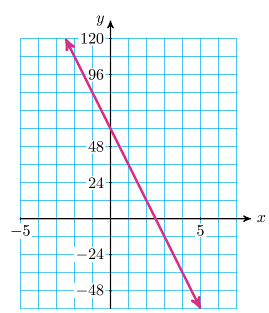
Hint: Find two points that lie on the intersection of grid lines, so that it's easy to read their coordinates. For example, you could use $(2, 300)$ and $(8, 600)$.



Answer. 50

Checkpoint 1.65 Calculate the slope of the line.

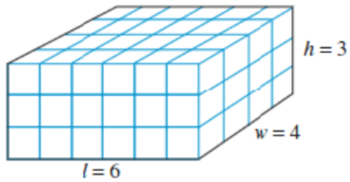
Hint: Find two points that lie on the intersection of grid lines. For example, you could use $(0, 60)$ and $(3, -12)$.



Answer. -24

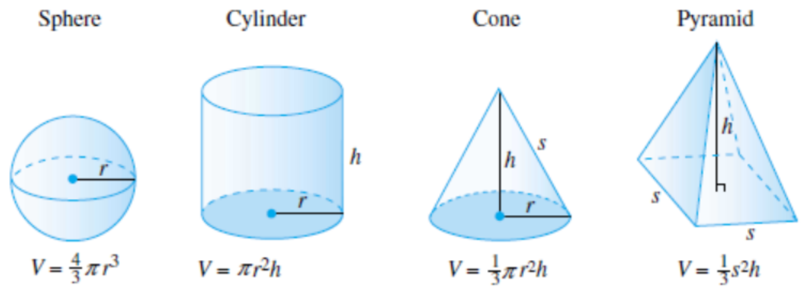
3. Calculate volume

The volume of a box is measured in cubic units and can be calculated using the formula $V = lwh$, where l, w, and h stand for the length, width, and height of the box. Volume measures the amount of space inside an object by telling us how many blocks 1 unit on a side will fit inside the space. For example, the volume of the box below, whose dimensions are given in centimeters, is



$$\begin{aligned} V &= lwh \\ &= (6)(4)(3) = 72 \text{ cubic centimeters} \end{aligned}$$

It may seem difficult to measure the inside of a round object like a sphere or a cone in cubic units, but you can imagine filling the object with liquid and then pouring the liquid into a box to measure its volume.



Examples

Example 1.66 An aquarium is 24 inches long and 10 inches wide. What is the area of its base? How much water is needed to fill it to a depth of 5 inches?

Solution. The area of the base is

$$A = lw = (24 \text{ in})(10 \text{ in}) = 240 \text{ in}^2$$

To calculate the volume of water, we can multiply the area of the base, lw, by the height of the water, h.

$$V = (lw)h = Ah = (240 \text{ in}^2)(5 \text{ in}) = 1200 \text{ in}^3$$

□

Example 1.67 The diameter of a spherical wax candle is 5 inches. What is the volume of wax in the candle?

Solution. The radius of the candle is half its diameter, or 2.5 inches. The volume of the candle is

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(2.5 \text{ in})^3 = 65.45 \text{ in}^3$$

□

Exercises

Checkpoint 1.68 Find the volume of a cylindrical water tank whose diameter is 20 feet and whose height is 20 feet.

Answer. 6283.2 cubic feet

Checkpoint 1.69 The diameter of the Earth is about 7920 miles. Find its volume.

Answer. About 260,120,000,000 cubic miles

Equations of Lines

1. Calculate slope using a formula

Recall that the subscripts on the coordinates in $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ just mean "first point" and "second point".

Two-Point Formula for Slope.

The slope of the line joining points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{if} \quad x_2 \neq x_1$$

Example

Example 1.70 Compute the slope of the line joining $(-6, 2)$ and $(3, -1)$.

Solution. It doesn't matter which point is P_1 and which is P_2, so we choose P_1 to be $(-6, 2)$. Then $(x_1, y_1) = (-6, 2)$ and $(x_2, y_2) = (3, -1)$. Thus,

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 - 2}{3 - (-6)} = \frac{-3}{9} = \frac{-1}{3} \end{aligned}$$

□

Caution 1.71 Make sure that you subtract both the x and y coordinates in the same order! That is, do NOT calculate

$$\frac{y_{\mathbf{2}} - y_{\mathbf{1}}}{x_{\mathbf{1}} - x_{\mathbf{2}}} \qquad \text{Incorrect!}$$

or your slope will have the wrong sign.

Exercises

Checkpoint 1.72 Compute the slope of the line joining the points $(5, 2)$ and $(8, 7)$.

Answer. $\frac{5}{3}$

Checkpoint 1.73 Compute the slope of the line joining the points $(-3, -4)$ and $(-7, 1)$.

Answer. $\frac{-5}{4}$

2. Slope-Intercept Form

Because the y-intercept $(0, b)$ is the "starting value" of a linear model, and its rate of change is measured by its slope,m, the equation for a linear model

$$y = \text{starting value} + \text{rate} \times x$$

can be expressed in symbols as

$$y = b + mx$$

Slope-Intercept Form.

If we write the equation of a linear function in the form,

$$f(x) = b + mx$$

then m is the **slope** of the line, and b is the **y-intercept**.

Examples

Example 1.74 The temperature inside a pottery drying oven starts at 70 degrees and is rising at a rate of 0.5 degrees per minute. Write a function for the temperature, H, inside the oven after t minutes.

Solution. At $t = 0$, the temperature is 70 degrees, so $b = 70$.

The slope is given by the rate of increase of H, so $m = 0.5$.

Thus, the function is

$$H = 70 + 0.5t$$

□

Example 1.75 A perfect score on a driving test is 120 points, and you lose 4 points for each wrong answer. Write a function for your score, S, if you give n wrong answers.

Solution. If $n = 0$, your score is 120, so $b = 120$.

Your score decreases by 4 points per wrong answer, so $m = \dfrac{\Delta S}{\Delta n} = 4$.

The function is

$$S = 120 - 4n$$

□

Exercises

Checkpoint 1.76 Monica has saved \$7800 to live on while she attends college. She spends \$600 a month. Write a function for the amount, S, in Monica's savings account after t months.

Answer. $b = 7800$ and $m = -400$, so $S = 7800 - 600t$

Checkpoint 1.77 Jesse opened a new doughnut shop in an old store-front. He invested \$2400 in remodeling and set-up, and he makes about \$400 per week from the business. Write a function giving the shop's financial standing, F, after w weeks.

Answer. $b = -2400$ and $m = 400$, so $F = -2400 + 400w$

3. Point-Slope Form

If we don't know the y-intercept of a line but we do know one other point and the slope, we can still find an equation for the line.

Point-Slope Formula.

To find an equation for the line of slope m passing through the point (x_1, y_1), use the point-slope formula

$$\frac{y - y_1}{x - x_1} = m$$

or

$$y - y_1 = m(x - x_1)$$

Example

Example 1.78 Find an equation for the line that passes through $(1, 3)$ and has slope -2.

Solution. We substitute $x_1 = \mathbf{1}$, $y_1 = \mathbf{3}$, and $m = \mathbf{-2}$ into the point-slope formula.

$$\begin{aligned} y - \mathbf{3} &= \mathbf{-2}(x - \mathbf{1}) && \text{Apply the distributive law.} \\ y - 3 &= -2x + 2 && \text{Add 3 to both sides.} \\ y &= -2x + 5 \end{aligned}$$

□

Example 1.79 Find an equation for the line of slope $\dfrac{-1}{2}$ that passes through $(-3, -2)$.

Solution. We substitute $x_1 = \mathbf{-3}$, $y_1 = \mathbf{-2}$, and $m = \dfrac{\mathbf{-1}}{\mathbf{2}}$ into the point-slope formula.

$$\begin{aligned} \frac{y - (\mathbf{-2})}{x - (\mathbf{-3})} &= \frac{\mathbf{-1}}{\mathbf{2}} && \text{Simplify the left side.} \\ \frac{y + 2}{x + 3} &= \frac{-1}{2} && \text{Cross-multiply.} \\ 2(y + 2) &= -1(x + 3) && \text{Apply the distributive law.} \\ 2y + 4 &= -x - 3 && \text{Subtract 4 from both sides.} \\ 2y &= -x - 7 && \text{Divide both sides by 2.} \\ y &= \frac{-1}{2}x - \frac{7}{2} \end{aligned}$$

□

Exercises

Checkpoint 1.80 Find an equation for the line of slope -4 that passes through $(2, -5)$.

Answer. $y = -4x + 3$

Checkpoint 1.81 Find an equation for the line of slope $\frac{2}{3}$ that passes through $(-6, 1)$.

Answer. $y = \frac{2}{3}x + 5$

4. Graphing a linear equation by the point-slope method

If we know one point on a line and its slope, we can sketch its graph without having to make a table of values.

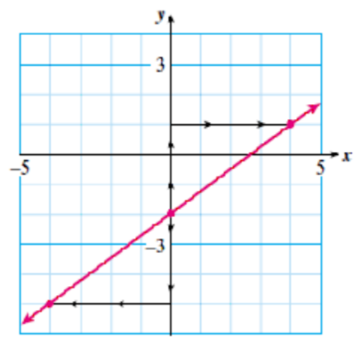
Examples

Example 1.82 Graph the line $y = \frac{3}{4}x - 2$

Solution. Step1: Begin by plotting the y-intercept, $(0, -2)$.

Step 2: We use the slope, $\frac{\Delta y}{\Delta x} = \frac{3}{4}$, to find another point on the line, as follows. Start at the point $(0, -2)$ and move 3 units up and 4 units to the right. Plot a second point here, at $(4, 1)$.

Step 3: Find a third point by writing the slope as $\frac{\Delta y}{\Delta x} = \frac{-3}{-4}$: from $(0, -2)$, move down 3 units and 4 units to the left. Plot a third point here, at $(-4, -5)$.



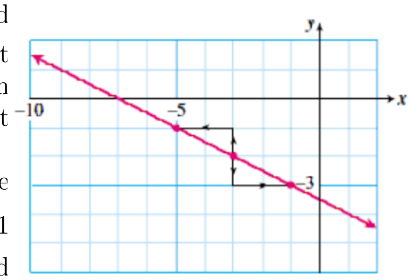
Finally, draw a line through the three points. □

Example 1.83 Graph the line of slope $\frac{-1}{2}$ that passes through $(-3, -2)$.

Solution. Step 1: Begin by plotting the point $(-3, -2)$.

Step 2: Use the slope, $\frac{\Delta y}{\Delta x} = \frac{-1}{2}$, to find another point on the line, as follows. Start at the point $(-3, -2)$ and move 1 unit down and 2 units to the right. Plot a second point here, at $(-1, -3)$.

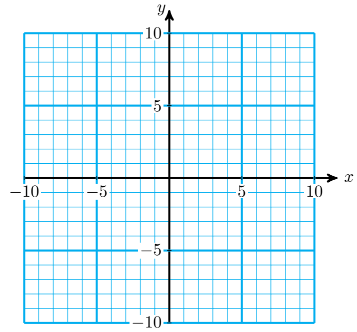
Step 3: Find a third point by writing the slope as $\frac{\Delta y}{\Delta x} = \frac{1}{-2}$: from $(-3, -2)$, move 1 unit up and 2 units to the left. Plot a third point here, at $(-5, -1)$.



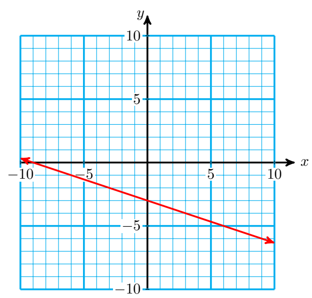
Finally, draw a line through the three points. □

Exercises

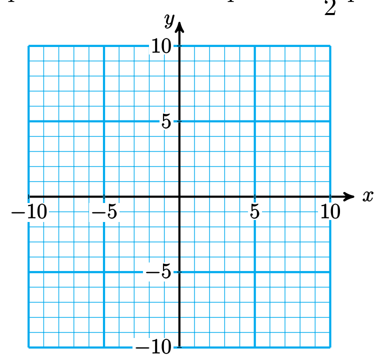
Checkpoint 1.84 Graph the line $y = \frac{-1}{3}x - 3$



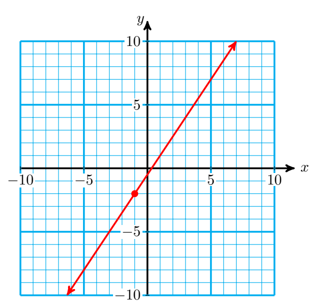
Answer.



Checkpoint 1.85 Graph the line with slope $m = \dfrac{3}{2}$ passing through $(-1, -2)$.



Answer.



Chapter 2

Applications of Linear Models

Linear Regression

1. Read a scatterplot

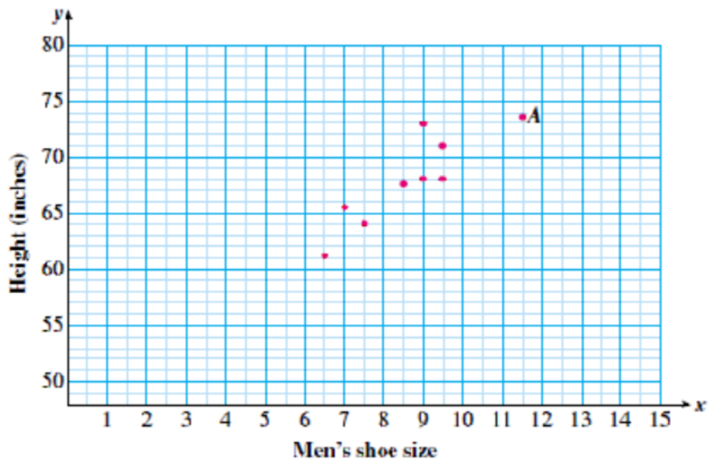
We read the coordinates of points on a scatterplot the same way we do for any other graph.

Example

Example 2.1 The scatterplot shows the height and shoe size of a group of men.

a State the height and shoe size of the man represented by point A.

b Find the heights of two men with the same shoe size.



Solution.

a The man represented by point A has shoe size $11\frac{1}{2}$ and is $73\frac{1}{2}$ inches tall.

b There are two men with shoe size 9, with heights 68 and 73 inches. There are also two men with shoe size $9\frac{1}{2}$, with heights 68 and 71 inches.

□

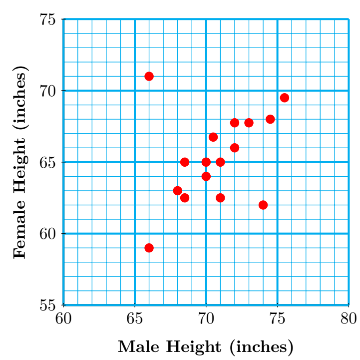
Exercise

Checkpoint 2.2

The scatterplot shows the heights of dance partners in a ballroom dance class.

a How tall is the shortest woman?

b What are the heights of the three partners of the 65-inch tall women?



Answer.

a 59 in

b $68\frac{1}{2}$, 70, and 71 in

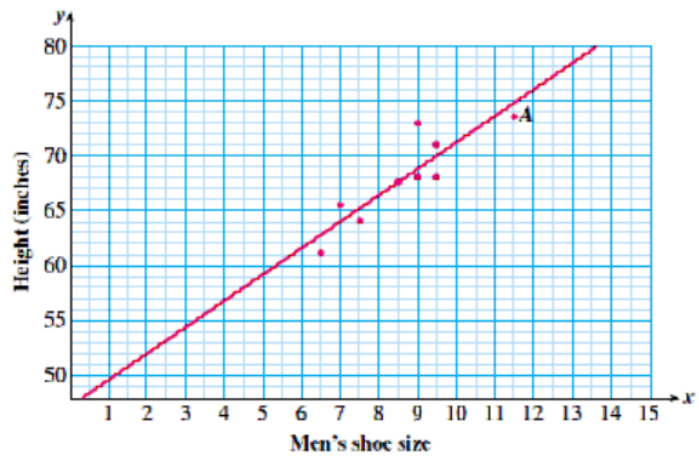
2. Sketch a line of best fit

Of course, the points on a scatterplot may not lie on a straight line. But if they seem to cluster near a line, we can try to find that line.

Example

Example 2.3 Sketch a line of best fit for the scatterplot in part 1.

Solution. We draw a line that lies as close as possible to all of the data points. As a rule of thumb, we try to keep equal numbers of points on each side of the line.

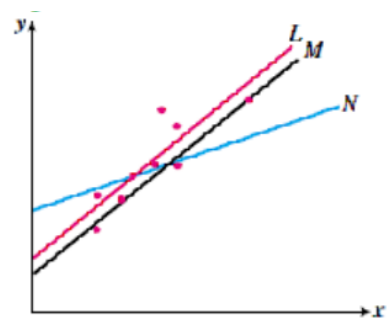


□

Exercise

Checkpoint 2.4

Which of the lines fits the scatterplot best?



Answer. Line L

3. Fit a line through two points

If we don't know the slope of a line, but we do know two points on the line, we can calculate the slope first and then use the point-slope formula.

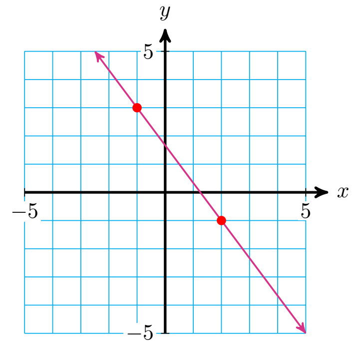
Example

Example 2.5 Find an equation for the line that passes through $(2, -1)$ and $(-1, 3)$.

Solution.

We solve this problem in two steps: First, find the slope of the line, and then use the point-slope formula.
Step 1: Let $(x_1, y_1) = (2, -1)$ and $(x_2, y_2) = (-1, 3)$. Use the slope formula to find

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3-(-1)}{-1-2} = \frac{4}{-3} = \frac{-4}{3} \end{aligned}$$



Step 2: Apply the point-slope formula with $m = \dfrac{-4}{3}$ and $(x_1, y_1) = (2, -1)$. (We can use either point in the formula.) Then

$$\frac{y - y_1}{x - x_1} = m \qquad \text{becomes} \qquad \frac{y-(-1)}{x-2} = \frac{-4}{3}$$

Cross-multiply to find

$$\begin{aligned} 3(y+1) &= -4(y-2) && \text{Apply the distributive law.} \\ 3y + 3 &= -4x + 8 && \text{Solve for } \boldsymbol{y}\textbf{.} \\ 3y &= -4x + 5 \\ y &= \frac{-4}{3}x + \frac{5}{3} \end{aligned}$$

□

Exercise

Checkpoint 2.6 Find an equation for the line that passes through $(-6, -1)$ and $(1, 3)$.

Answer. Step 1: Compute the slope.
Step 2: Use the point-slope formula.

$$y = \frac{4}{7}x + \frac{17}{7}$$

Linear Systems

1. Identify the solution of a system

Recall that a solution to a system makes each equation in the system true.

Examples

Example 2.7 Decide whether $(3, -2)$ is a solution of the system

$$\begin{aligned} x &= 5y + 13 \\ 2x + 3y &= 0 \end{aligned}$$

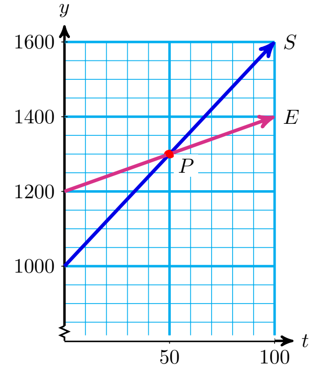
Solution. A solution must satisfy both equations. We substitute $x = \mathbf{3}$ and $y = \mathbf{-2}$ into the equations.

$$\mathbf{3} = 5(\mathbf{-2}) + 13? \qquad \text{Yes}$$

$$2(\mathbf{3}) + 3(\mathbf{-2}) = 0? \qquad \text{Yes}$$

Yes, $(3, -2)$ is a solution □

Example 2.8 Find the solution of the system graphed below.



Solution. The solution must lie on both graphs, so it is the intersection point, P. The coordinates of point P are $(50, 1300)$, so the solution of the system is $t = 50,\ y = 1300$. □

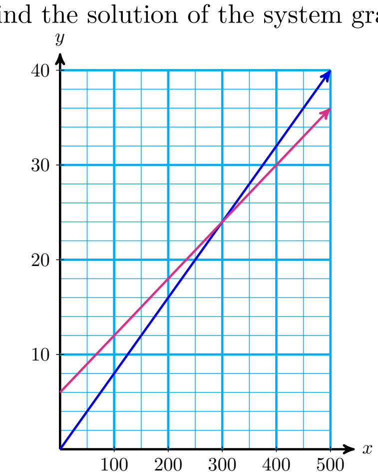
Exercises

Checkpoint 2.9 Decide whether $(-3, -2)$ is a solution of the system

$$\begin{aligned} x + 3y &= -9 \\ 3x + 2y &= -5 \end{aligned}$$

Answer. No

Checkpoint 2.10 Find the solution of the system graphed below.



Answer. $(300, 24)$

2. Use the formula for profit, $P = R - C$

Profit.

To find the profit earned by a company we subtract the costs from the revenue.

$$\text{Profit} = \text{Revenue} - \text{Cost} \qquad P = R - C$$

"Revenue" is the amount of money a company takes in from selling its product. A negative profit is the same as a loss.

Examples

Example 2.11 The owner of a sandwich shop spent \$800 last week for labor and supplies. She received \$1150 in revenue. What was her profit?

Solution. We evaluate the formula with $R = 1150$ and $C = 800$ to find

$$\begin{aligned} P &= R - C \\ &= 1150 - 800 = 350 \end{aligned}$$

The owner's profit was \$350. □

Example 2.12 EcoGreen made \$1848 profit on low-flow shower heads last year, and spent \$3426 in costs. What was their revenue from shower heads?

Solution. We usse the profit formula and solve the equation

$$1848 = R - 3246$$

to find that $R = 5094$. Their revenue was \$5094. □

Exercises

Checkpoint 2.13

a The Earth Alliance made \$6000 in revenue from selling tickets to Earth Day, an educational event for children. Write an expression for their profit in terms of their costs.

b What was their profit if their costs were \$2500?

Answer.

a $P = 6000 - C$

b \$3500

Checkpoint 2.14 Last week Moe's Auto Shop took in \$5400 in revenue, but experienced a loss of \$800. What were Moe's costs last week?

Answer. \$6200

3. Write equations in two variables

Applied problems that involve more than one unknown are often easier to model and solve with a system of equations.

Examples

Example 2.15 Write equations about the number of tables and the number of chairs:

a There are four chairs for each table.

b Chairs cost \$125 each; a table costs \$350. Darryl spent \$10,200 on tables and chairs.

Solution. Let x be the number of tables and y the number of chairs.

a The number of chairs is 4 times the number of tables: $y = 4x$.

b $125y + 350x = 10,200$

□

Example 2.16 Write equations about the dimensions of a rectangle:

a The perimeter of the rectangle is 42 meters.

b The length is 3 meters more than twice the width.

Solution. Let x be the width of the rectangle and y its length.

a $2x + 2y = 42$

b $y = 3 + 2x$

□

Exercises

Checkpoint 2.17 Write equations about the number of calories in a hamburger and in a chocolate shake.

a A hamburger and a chocolate shake together contain 1020 calories.

b Two shakes and three hamburgers contain 2710 calories.

Answer.

a $x + y = 1020$

b $3x + 2y = 2710$

Checkpoint 2.18 Write equations about the vertex angle and the base angles of an isosceles triangle.

a The vertex angle is 15° less than each base angle.

b The sum of the angles in a triangle is 180°.

Answer.

a $y = x - 5$

b $2x + y = 180$

Algebraic Solution of Systems

Some familiar formulas are useful in writing equations to solve a problem.

1. Use the interest formula, $I = Pr$

Example

Example 2.19 You have \$5000 to invest for one year. You want to put part of the money into bonds that pay 7% interest, and the rest of the money into stocks that involve some risk but will pay 12% if successful. Now suppose you decide to invest x dollars in stocks and y dollars in bonds.

a Use the interest formula, $I = Pr$, to write expressions for the interest earned on the bonds and on the stocks.

b Write an equation about the amount invested.

c Write an equation to say that the total interest earned was \$400.

Solution.

a Stocks: $I = 0.12x$; Bonds: $I = 0.07y$

b $x + y = 500$

c $0.12x + 0.07y = 400$

□

Exercise

Checkpoint 2.20 Jerry invested \$2000, part in a CD at 4% interest and the remainder in a business venture at 9%. After one year, his income from the business venture was \$37 more than his income from the CD. Now suppose Jerry invested x dollars in the CD and y dollars in the business venture.

a Use the interest formula to write expressions for the interest Jerry earned on the CD and the interest he earned on the business venture.

b Write an equation about the amount Jerry invested.

c Write an equation about the interest Jerry earned.

Answer.

1 $0.04x$; $0.09y$

2 $x + y = 2000$

3 $0.09y = 37 + 0.04x$

2. Use the percent formula, $P = rW$

Example

Example 2.21 A chemist wants to produce 45 quarts of a 40% solution of carbolic acid by mixing a 20% solution with a 50% solution. She uses x quarts of the 20% solution and y quarts of the 50% solution.

a Write an equation about the total amount of solution.

b Use the percent formula, $P = rW$, to write expressions about the amount of carbolic acid in each original solution.

c How many quarts of carbolic acid are in the mixture?

d Write an equation about the amount of carbolic acid.

Solution.

a $x + y = 45$

b 20% solution: $0.20x$; 50% solution: $0.50y$

c $P = rW = 0.40(45)$

d $0.20x + 0.50y = 0.40(45)$

□

Exercise

Checkpoint 2.22 A pet store owner wants to mix a 12% saltwater solution and a 30% saltwater solution to obtain 90 liters of a 24% solution. He uses x quarts of the 12% solution and y quarts of the 30% solution.

a Write an equation about the total amount of saltwater.

b Use the percent formula to write expressions about the amount of salt in each original solution.

c How many liters of salt are in the mixture?

d Write an equation about the amount of salt.

Answer.

1 $x + y = 90$

2 $0.20x; \quad 0.30y$

3 $0.24(90)$

4 $0.20x + 0.30y = 0.24(90)$

3. Use the distance formula, $d = rt$

Example

Example 2.23 A river steamer requires 3 hours to travel 24 miles upstream and 2 hours for the return trip downstream. Let x be the speed of the current and y the speed of the steamer in still water.

a Write an equation about the upstream trip.

b Write an equation about the downstream trip.

Solution.

a The speed of the steamer aginst the current is $r = y - x$, so $3(y - x) = 24$

b The speed of the steamer with the current is $r = y + x$, so $2(y + x) = 24$

□

Exercise

Checkpoint 2.24 A yacht leaves San Diego and heads south, traveling at 25 miles per hour. Six hours later a Coast Guard cutter leaves San Diego traveling at 40 miles per hour and pursues the yacht. Let x be the time it takes the cutter to catch the yacht, and y the distance it traveled.

a Write an equation about the yacht's journey.

b Write an equation about the cutter's journey.

Answer.

1 $25(x + 6) = y$

2 $40x = y$

Gaussian Reduction

1. Write an equation in standard form

Before we can use Gaussian reduction, we must write each equation in standard form.

Examples

Example 2.25 Write the equation in standard form.

a $3y - 7 = 4z + x$

b $6 = -5z + 2x$

Solution. The standard form is $ax + by + cz = d$. We add or subtract appropriate terms on both sides of the equation.

a $-x + 3y - 4z = 7, \quad \text{or} \quad x - 3y + 4z = -7$

b $2x + 0y - 5z = 6, \quad \text{or} \quad -2x + 0y + 5z = -6$

□

Exercise

Checkpoint 2.26 Write the equation in standard form.

a $5 - 3x + 4y = 2z$ b $y = 8 - 2z$

Answer.

a $-3x + 4y - 2z = -5$

b $0x + y + 2z = 8$

2. Clear fractions from an equation

It is easier to use Gaussian reduction if the equations have integer coefficients.

Examples

Example 2.27 Write the equation with integer coefficients.

a $\dfrac{1}{4}x + z = \dfrac{3}{4}$

b $\dfrac{2}{3}x - 2y + \dfrac{1}{2}z = 3$

Solution.

a We multiply both sides of the equation by **4**.

$$\mathbf{4} \cdot \left(\frac{1}{4}x + z\right) = \left(\frac{3}{4}\right) \cdot \mathbf{4}$$
$$x + 4z = 3$$

b We multiply both sides of the equation by the LCD of the fractions, **6**.

$$\mathbf{6}\cdot\left(\frac{2}{3}x-2y+\frac{1}{2}z\right)=(3)\cdot\mathbf{6}$$
$$4x-12y+3z=18$$

□

Exercise

Checkpoint 2.28 Write the equation with integer coefficients.

a $\frac{1}{5}x-\frac{2}{5}y+z=-1$

b $\frac{3}{4}x-y+\frac{5}{6}z=6$

Answer.

a $x-2y+5z=-5$

b $9x-12y+10z=72$

Linear Inequalities in Two Variables

1. Solve a linear inequality

Before we solve inequalities in two variables, let's review solving linear inequalities in one variable.

Example

Example 2.29 Solve $3k-13<5+6k$

Solution. We begin just as we do to solve an equation. The only difference is that we must reverse the direction of the inequality if we multiply or divide by a negative number.

$$\begin{aligned} 3k-13 &< 5+6k && \text{Subtract } \boldsymbol{6k} \text{ from both sides.}\\ -3k &< 18 && \text{Divide both sides by } \mathbf{-3.}\\ k &> -6 && \text{Don't forget to reverse the inequality.} \end{aligned}$$

In interval notation, the solution set is $(-6,\infty)$. □

Exercise

Checkpoint 2.30 Solve $4(3a-7)>-18+2a$. Write the solution with interval notation.

Answer. $(1,\infty)$

Checkpoint 2.31 Solve $4\leq\frac{-3x}{4}-2$. Write the solution with interval notation.

Answer. $(-\infty,8]$

Checkpoint 2.32 Solve $15 \geq -6 + 3m \geq -6$. Write the solution with interval notation.

Answer. $[0, 7]$

Checkpoint 2.33 Solve $\dfrac{-9}{2} < \dfrac{5-2n}{-4} \leq -1$. Write the solution with interval notation.

Answer. $\left(\dfrac{-13}{2}, \dfrac{1}{2}\right]$

2. Graph a line

The boundary of the solution set for a linear inequality in two variables is made up of portions of straight lines.

Examples

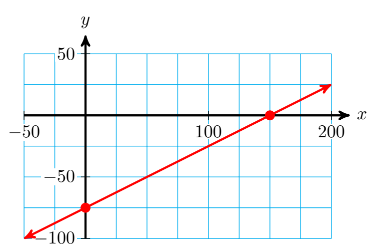
Example 2.34 Use the most convenient method to graph the equation.

a $5x - 10y = 750$

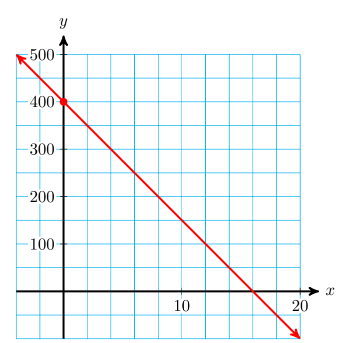
b $y = 400 - 25x$

Solution.

a This equation is in the form $Ax + By = C$, so the intercept method of graphing is convenient. The intercepts are $(150, 0)$ and $(0, -75)$. The graph is shown below.



b This equation is in the form $y = mx + b$, so the slope-intercept method of graphing is convenient. The y-intercept is $(0, 400)$, and the slope is -25. The graph is shown below.

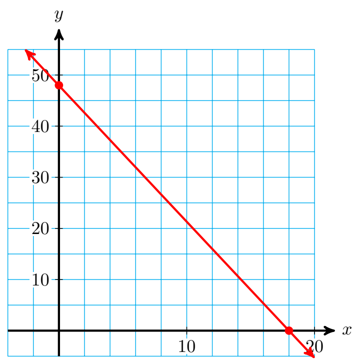


□

Exercise

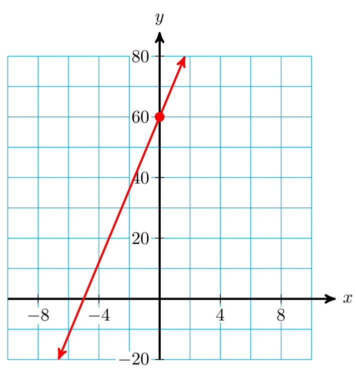
Checkpoint 2.35 Graph the equation $24x + 9y = 432$

Answer.



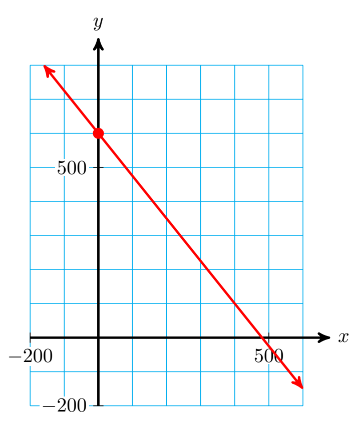
Checkpoint 2.36 Graph the equation $y = 12x + 60$

Answer.



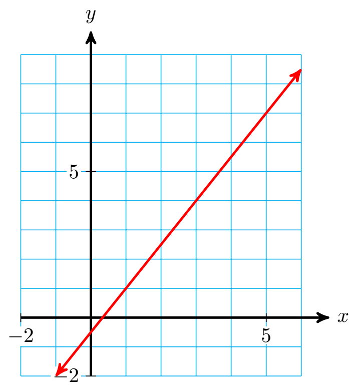
Checkpoint 2.37 Graph the equation $y = 600 - 1.25x$

Answer.



Checkpoint 2.38 Graph the equation $45x - 30y = 15$

Answer.



3. Solve a 2x2 system

To find the vertices of the boundary of the solution set, we solve a linear 2x2 system.

Examples

Example 2.39 Use substitution to solve the system:

$$\begin{aligned} 3y - 2x &= 3 \\ x - 2y &= -2 \end{aligned}$$

Solution. We start by solving the second equation for x to get $x = 2y - 2$. Then we substitute this expression for x into the first equation, which gives us

$$3y - 2(2y - 2) = 3$$

We solve this equation for y to find $y = 1$. Finally, we substitute $y = 1$ into our first step to find

$$x = 2(1) - 2 = 0$$

The solution is $x = 0,\ y = 1$, or $(0, 1)$. □

Example 2.40 Use elimination to solve the system:

$$\begin{aligned} 2x + 3y &= -1 \\ 3x + 5y &= -2 \end{aligned}$$

Solution. We multiply the first equation by 3 and the second equation by -2 in order to eliminate x.

$$\begin{aligned} 6x + 9y &= -3 \\ -6x - 10y &= 4 \end{aligned}$$

Adding these two equations gives us $-y = 1$, or $y = -1$. Finally, we substitute $y = -1$ into either equation (we choose the first equation), and solve for x.

$$\begin{aligned} 2x + 3(-1) &= -1 \\ 2x - 3 &= -1 \end{aligned}$$

We find$x = 1$, so the solution is $x = 1,\ y = -1$, or $(1, -1)$. □

Exercise

Checkpoint 2.41 Solve the system:

$$\begin{aligned} y &= 2x + 1 \\ 2x + 3y &= -21 \end{aligned}$$

Answer. $(-3, -5)$

Checkpoint 2.42 Solve the system:

$$\begin{aligned} x + 4y &= 1 \\ 2x + 3y &= -3 \end{aligned}$$

Answer. $(-3, 1)$

Checkpoint 2.43 Solve the system:

$$\begin{aligned} 2x + 7y &= -19 \\ 5x - 3y &= 14 \end{aligned}$$

Answer. $(1, -3)$

Checkpoint 2.44 Solve the system:

$$\begin{aligned} 4x + 3y &= -19 \\ 5x + 15 &= -2y \end{aligned}$$

Answer. $(-1, -5)$

Chapter 3

Quadratic Models

Extraction of Roots

1. Evaluate quadratic expressions

When squaring a negative number, don't forget to enclose it in parentheses. For example, if $x = -4$, then

$$x^2 = (-4)^2 = (-4)(-4) = 16$$

If we write -4^2, then only the 4 is squared, so we have

$$-4^2 = -(4)(4) = -16$$

Examples

Example 3.1 Evaluate for $x = -6$.

a $2x^2$

b $2 - x^2$

c $(2x)^2$

d $(2 - x)^2$

Solution. Enclose -6 in parentheses, and follow the order of operations.

a Square first, then multiply by 2: $2x^2 = 2(-6)^2 = 2(36) = 72$

b Square first, then subtract from 2: $2 - x^2 = 2 - (-6)^2 = 2 - 36 = -34$

c Multiply by 2 first, then square: $(2x)^2 = [2(-6)]^2 = [-12]^2 = 144$

d Subtract from 2 first, then square: $(2 - x)^2 = [2 - (-6)^2 = [8]^2 = 64$

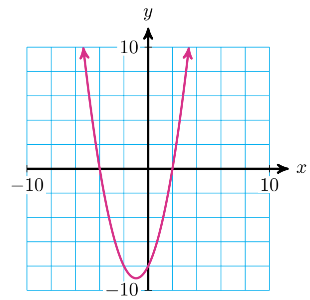
□

Example 3.2 Make a table of values for $y = x^2 + 2x - 8$, and graph the equation.

Solution. We plot the points from the table and connect them with a smooth curve.

x	y
-5	7
-4	0
-3	-5
-2	-8
-1	-9
0	-8
1	-5
2	0
3	7

$(-5)^2+2(-5)-8=25-10-8$
$(-4)^2+2(-4)-8=16-8-8$
$(-3)^2+2(-3)-8=9-6-8$
$(-2)^2+2(-2)-8=4-4-8$
$(-1)^2+2(-1)-8=1-2-8$
$(0)^2+2(0)-8=0+0-8$
$(1)^2+2(1)-8=1+2-8$
$(2)^2+2(2)-8=4+4-8$
$(3)^2+2(3)-8=9+6-8$



□

Exercises

Checkpoint 3.3 Evaluate for $w=-9$

a $(2w)^2$

b $36-(2w)^2$

c $-2(4-w)^2$

d $2-w^2$

Answer.

a 324

b -288

c -338

d -79

Checkpoint 3.4 Evaluate for $a=-3,\ b=-4$

a ab^2

b $a-b^2$

c $(a-b^2)^2$

d $ab(a^2-b^2)$

Answer.

a -48

b -19

c 361

d -84

Checkpoint 3.5 Evaluate for $h=-2,\ g=-5$

a $h^2-2hg+g^2$

b $(h-g)^2$

c h^2-g^2

d $(h-g)(h+g)$

Answer.

a 9

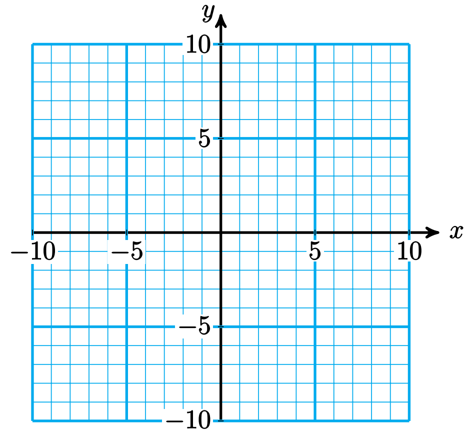
b 9

c -21

d -21

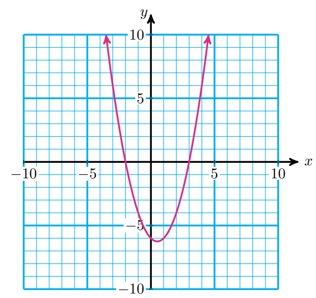
Checkpoint 3.6 Make a table of values for $y = x^2 - x - 6$, and graph the equation.

x	y
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	



Answer.

x	y
-3	6
-2	0
-1	-4
0	-6
1	-6
2	-4
3	0
4	6
5	14



2. Use square roots

The simplest quadratic equations have the form

$$ax^2 + c = 0$$

They are missing a linear term, bx, and are not difficult to solve.

Examples

Example 3.7 Solve $3y^2 - 40 = 35$ by extraction of roots.

Solution. We isolate the quadratic term, y^2, and then extract roots.

$$\begin{aligned} 3y^2 - 40 &= 35 && \text{Add 40 to both sides.} \\ 3y^2 &= 75 && \text{Divide both sides by 3.} \\ y^2 &= 25 && \text{Take the square root of both sides.} \\ y &= \pm\sqrt{25} && \text{Simplify.} \\ y &= \pm 5 && \text{Remeber that a positive number has two square roots.} \end{aligned}$$

The solutions are 5 and -5. □

Example 3.8 A cone is 16 cm tall and its volume is 500 cm^3. What is the radius of the base of the cone?

Solution. We substitute $h = 16$ and $V = 500$ into the formula for the volume of a cone, and solve for r.

$$\begin{aligned} \frac{1}{3}\pi r^2(\mathbf{16}) &= \mathbf{500} && \text{Multiply both sides by 3.} \\ 16\pi r^2 &= 1500 && \text{Divide both sides by } \boldsymbol{16\pi}\text{.} \\ r^2 &= 29.842 && \text{Take the square root of both sides.} \\ r &= 5.46 \end{aligned}$$

The radius of the cone is 5.46 cm. □

Exercises

Checkpoint 3.9 Solve by extraction of roots.

a $98 = 2a^2$

b $0 = 3n^2 - 15$

c $144 = \dfrac{h^2}{9} = 169$

d $12 - 5v^2 = 2$

Answer.

a ± 7

b $\pm\sqrt{5}$

c ± 15

d $\pm\sqrt{2}$

Checkpoint 3.10 A cylindrical syringe holds 100 cc (cubic centimeters) of fluid. If the syringe is 10 centimeters long, what is its radius?

Answer. 1.78 cm

3. Use the Pythagorean theorem

If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then

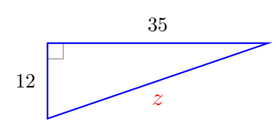
$$a^2 + b^2 = c^2$$

Note that the theorem is true only for right triangles -- ones that have a 90° angle.

Examples

Example 3.11

Find the unknown side in the right triangle.



Solution. The unknown side is the hypotenuse, so we apply the Pythagorean theorem with $c = z$, $a = 12$, and $b = 35$.

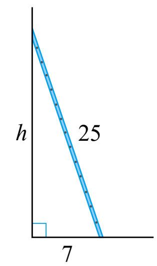
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 12^2 + 35^2 &= z^2 && \text{Simplify the left side.} \\ 1369 &= z^2 && \text{Take the square root of both sides.} \\ \pm 37 &= z \end{aligned}$$

The length of the hypotenuse is a positive number, so $z = 37$. □

Example 3.12 A 25-foot ladder is placed against a wall so that its foot is 7 feet from the base of the wall. How far up the wall does the ladder reach?

Solution.

We make a sketch and label the known dimensions, calling the unknown height h. The ladder forms the hypotenuse of a right triangle, so we apply the Pythagorean theorem, substituting 25 for c, 7 for b, and h for a.



$$a^2 + b^2 = c^2$$
$$h^2 + 7^2 = 25^2$$

We solve the equation by extraction of roots:

$$\begin{aligned} h^2 + 49 &= 625 && \text{Subtract 49 from both sides.} \\ h^2 &= 576 && \text{Extract roots.} \\ h &= \pm\sqrt{576} && \text{Simplify the radical.} \\ h &\pm 24 \end{aligned}$$

The height must be a positive number, so the ladder reaches 24 feet up the wall. □

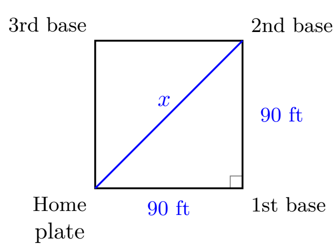
Exercises

Checkpoint 3.13 A baseball diamond is a square whose sides are 90 feet long. Find the straight-line distance from home plate to second base.

a Make a sketch of the situation and label a right triangle.

b Write an equation and solve.

Answer.

a 

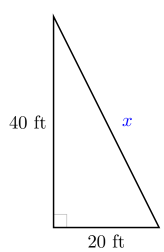
b 127.28 ft

Checkpoint 3.14 How long a wire is needed to stretch from the top of a 40-foot telephone pole to a point on the ground 20 feet from the base of the pole?

a Make a sketch of the situation and label a right triangle.

b Write an equation and solve.

Answer.

a 

b 44.72 ft

Intercepts and Factors

In this lesson we review the skills we need to solve quadratic equations by factoring.

1. Multiply binomials

Examples

Example 3.15 Expand the product $(2x + 3)(x - 6)$.

Solution. Multiply each term of the first binomial be each term of the second binomial. This gives four multiplications, often denoted by "FOIL," which stands for First terms, Outside terms, Inside terms, and Last terms.

$$\begin{aligned}(2x+3)(x-6) &= \underbrace{x \cdot x}_{\text{F}} + \underbrace{2x \cdot (-6)}_{\text{O}} + \underbrace{(-3) \cdot x}_{\text{I}} + \underbrace{(-3) \cdot (-6)}_{\text{L}} \\ &= 2x^2 - 12x - 3x + 18 \qquad \text{Combine like terms.} \\ &= 2x^2 - 15x + 18\end{aligned}$$

□

Example 3.16 Expand the product $-2(3x - 4)(3x - 5)$.

Solution. First, multiply the binomial factors together.

$$(3x - 4)(3x - 5) = 9x^2 - 27x + 20$$

Then use the distributive law to multiply the result by the monomial factor, -2.

$$\mathbf{-2}(9x^2 - 27x + 20) = -18x^2 + 54x - 40$$

□

Exercises

Checkpoint 3.17 Expand the product $(2x + 1)(3x - 2)$.

Answer. $6x^2 - x - 2$

Checkpoint 3.18 Expand the product $(2t + 5)(2t + 5)$.

Answer. $4t^2 + 20t + 25$

Checkpoint 3.19 Expand the product $4(a - 3)(3a - 5)$.

Answer. $12a^2 - 56a + 60$

Checkpoint 3.20 Expand the product $-3(2b - 3)(5b + 1)$.

Answer. $-30b^2 + 39b + 9$

2. Factor quadratic trinomials

To factor the trinomial $x^2 + bx + c$, we look for two numbers p and q whose product pq is the constant term and whose sum $p + q$ is the coefficient of the middle term.

$$\begin{aligned}(x + p)(x + q) &= x^2 + qx + px + pq \\ &= x^2 + \boldsymbol{(p + q)}x + \boldsymbol{pq} = x^2 + \boldsymbol{b}x + \boldsymbol{c}\end{aligned}$$

Sign Patterns for Quadratic Trinomials.

Assume that $b,\ c,\ p$ and q are positive integers. Then

1 $x^2 + bx + c = (x + p)(x + q)$

If all the coefficients of the trinomial are positive, then both p and q are positive.

2 $x^2 - bx + c = (x - p)(x - q)$

If the middle term of the trinomial is negative and the other two terms are positive, then p and q are both negative.

3 $x^2 \pm bx - c = (x + p)(x - q)$

If the constant term of the trinomial is negative, then p and q have opposite signs.

Examples

Example 3.21 Factor $t^2 + 7t + 12$ as a product of two binomials,

$$t^2 + 7t + 12 = (t + p)(t + q)$$

Solution. The constant term is 12, so we look for two numbers p and q whose product is 12. There are three possibilities:

$$1 \text{ and } 12,\ 2 \text{ and } 6, \text{ or } 3 \text{ and } 4$$

Because the middle term is $7t$, we must have $p + q = 7$. We check each possibility and find that $p = 3$ and $q = 4$. Thus,

$$t^2 + 7t + 12 = (t + 3)(t + 4)$$

□

Example 3.22 Factor $x^2 - 12x + 20$.

Solution. For this example we must find two numbers p and q for which $pq = 20$ and $p + q = -12$. These two conditions tell us that p and q must both be negative. We start by listing all the ways to factor 20 with negative factors:

$$-1 \text{ and } -20, \qquad -2 \text{ and } -10, \qquad -4 \text{ and } -5$$

We check $p + q$ for each possibility to see which one gives the correct middle term. Because $-2 + (-10) = -12$, the factorization is

$$x^2 - 12x + 20 = (x - 2)(x - 10)$$

□

Example 3.23 Factor $x^2 + 2x - 15$.

Solution. This time the product pq must be negative, so p and q must have opposite signs, one positive and one negative. There are only two ways to factor 15, either 1 times 15 or 3 times 5. We just "guess" that the second factor is negative, and check $p + q$ for each possibility:

$$1 - 15 = -14 \qquad \text{or} \qquad 3 - 5 = -2$$

The middle term we want is $2x$, not $-2x$, so we change the signs of p and q:

we use -3 and $+5$. The correct factorization is

$$x^2 + 2x - 15 = (x - 3)(x + 5)$$

□

Exercises

Checkpoint 3.24 Factor $x^2 + 8x + 15$
Answer. $(x + 3)(x + 5)$

Checkpoint 3.25 Factor $y^2 + 14y + 49$
Answer. $(y + 7)(y + 7)$

Checkpoint 3.26 Factor $m^2 - 10m + 24$
Answer. $(m - 4)(m - 6)$

Checkpoint 3.27 Factor $m^2 - 11m + 24$
Answer. $(m - 3)(m - 8)$

Checkpoint 3.28 Factor $t^2 + 8t - 48$
Answer. $(t + 12)(t - 4)$

Checkpoint 3.29 Factor $t^2 - 8t - 48$
Answer. $(t - 12)(t + 4)$

3. Write algebraic expressions

Examples

Example 3.30 Ralph and Wanda together weigh 320 pounds. If Ralph weighs x pounds, how much does Wanda weigh?

Solution. We subtract Ralph's weight from the total; the remainder is Wanda's weight: $320 - x$ pounds □

Example 3.31 Delbert and Francine live 24 miles apart on Route 30. They meet at a cafe between their houses. If Delbert drove d miles, how far did Francine drive?

Solution. We subtract Delbert's distance from the total; the remainder is Francine's distance: $24 - d$ miles □

Example 3.32 Three eggs and two slices of buttered toast contain 526 calories. If one egg contains c calories, how many calories are in a slice of buttered toast?

Solution. We subtract the calories in three eggs from the total; the remainder is the number of calories in two slices of toast, so one slice has half that many calories: $\frac{1}{2}(526 - 3c)$ □

Example 3.33 The perimeter of a large rectangular playground is 124 yards. If its width is w yards, what is its length?

Solution. We subtract twice the length from the perimeter; the remainder is twice the width, so the width is half that: $\frac{1}{2}(124 - 2w) = 62 - w$ yards □

Exercises

Checkpoint 3.34 Garth and Taylor together made $86,000 last year. If Garth made d dollars, how much did Taylor make?

Answer. $86,000 - d$ dollars

Checkpoint 3.35 Six coffees and four pastries cost the office manager $21. If a pastry costs x dollars, how much does a coffee cost?

Answer. $\dfrac{1}{6}(21 - 4x)$ dollars

Checkpoint 3.36 The vertex angle of an isosceles triangle is v degrees. What is the measure of each of the two base angles?

Answer. $\dfrac{1}{2}(180 - v)$ degrees

Checkpoint 3.37 The perimeter of a rectangular swimming pool is 260 feet. If the length of the pool is L feet, what is its width?

Answer. $\dfrac{1}{2}(260 - 2L)$ feet

Graphing Parabolas

1. Find the average of two numbers

The average of two numbers lies half-way between them on a number line. To find their average, we take one-half of their sum. That is, the average of p and q is

$$\frac{1}{2}(p+q) \quad \text{or} \quad \frac{p+q}{2}$$

Example

Example 3.38 The average of 4 and 9 is

$$\frac{1}{2}(4+9) = \frac{1}{2}(13) = \frac{13}{2}, \quad \text{or} \quad 6\frac{1}{2}$$

□

Example 3.39 The average of -8 and 4 is

$$\frac{1}{2}(-8+4) = \frac{1}{2}(-4) = -2$$

□

Example 3.40 The average of $\dfrac{5}{2}$ and $\dfrac{-3}{4}$ is

$$\frac{1}{2}\left(\frac{5}{2} - \frac{3}{4}\right) = \frac{1}{2}\left(\frac{10}{4} - \frac{3}{4}\right) = \frac{1}{2}\left(\frac{10}{4}\right) = \frac{7}{8}$$

□

Exercises

Checkpoint 3.41 Find the average of -12 and -7.

Answer. $\dfrac{-19}{2}$

Checkpoint 3.42 Find the average of -4 and $\dfrac{1}{2}$.

Answer. $\dfrac{-7}{4}$

Checkpoint 3.43 Find the average of $\dfrac{3}{2}$ and $\dfrac{9}{2}$.

Answer. 3

Checkpoint 3.44 Find the average of $\dfrac{9}{4}$ and $\dfrac{-3}{4}$.

Answer. $\dfrac{3}{4}$

2. Solve quadratic equations

Examples

Example 3.45 Solve $3x^2 = 48$

Solution. We use extraction of roots. We first divide by 3 to isolate the squared expression.

$$\begin{aligned} x^2 &= 16 && \text{Take square roots.} \\ x &= \pm 4 \end{aligned}$$

The solutions are $x = 4$ and $x = -4$. □

Example 3.46 Solve $3x^2 = 12x$

Solution. We solve by factoring. First, we get zero on one side of the equation.

$$\begin{aligned} 3x^2 - 12x &= 0 && \text{Factor the left side.} \\ 3x(x-4) &= 0 && \text{Set each factor equal to zero.} \\ 3x = 0, \quad x - 4 &= 0 \end{aligned}$$

The solutions are $x = 0$ and $x = 4$. □

Example 3.47 Solve $3x^2 - 10x - 8 = 0$

Solution. We solve by factoring. We factor the left side.

$$\begin{aligned} (x-4)(3x+2) &= 0 && \text{Set each factor equal to zero.} \\ x - 4 = 0, \quad 3x + 2 &= 0 && \text{Solve each equation.} \\ x = 4, \quad x &= \frac{-2}{3} \end{aligned}$$

The solutions are $x = 0$ and $x = \dfrac{-2}{3}$. □

Exercises

Checkpoint 3.48 Solve $5x^2 - 30 = 0$

Answer. $x = \pm\sqrt{6}$

Checkpoint 3.49 Solve $\dfrac{1}{3}(x-2)^2 = 8$

Answer. $x = 2 \pm \sqrt{24}$

Checkpoint 3.50 Solve $x^2 - 5x = 300$

Answer. $x = 20,\ x = -15$

Checkpoint 3.51 Solve $4x^2 + 13x - 12 = 0$

Answer. $x = \dfrac{3}{4},\ x = -4$

3. Find the coordinates of points on a parabola

To find the x-coordinate of a point on a parabola, we usually need to solve a quadratic equation.

Examples

Example 3.52 Find the y-coordinate of the point on the graph of $y = 2x^2 - 3x + 5$ with x-coordinate -3.

Solution. Substitute $x = \mathbf{-3}$ into the equation, and evaluate.

$$y = 2(\mathbf{-3})^2 - 3(\mathbf{-3}) + 5 = 18 + 9 + 5 = 32$$

The y-coordinate is 32, and the point is $(-3, 32)$. □

Example 3.53 Find the x-coordinates of all points on the graph of $y = 20 - 3x^2$ with y-coordinate -28.

Solution. Substitute $y = \mathbf{-28}$ into the equation, and solve.

$$\begin{aligned} \mathbf{-28} &= 20 - 3x^2 && \text{Subtract 20 from both sides.}\\ -48 &= -3x^2 && \text{Divide both sides by } \mathbf{-3.}\\ 16 &= x^2 && \text{Extract roots.}\\ \pm 4 &= x \end{aligned}$$

The points are $(4, -28)$ and $(-4, -28)$. □

Exercises

Checkpoint 3.54 Find the y-coordinate of the point on the graph of $y = -x^2 + 6x + 2$ with x-coordinate -2.

Answer. -14

Checkpoint 3.55 The x-coordinate of the vertex of $y = 2x^2 - 6x + 1$ is $\dfrac{3}{2}$. Find the y-coordinate of the vertex.

Answer. $\dfrac{-7}{2}$

Checkpoint 3.56 Find the x-coordinates of all points on the graph of $y = x^2 - 2x + 5$ with y-coordinate 8.

Answer. $-1,\ 3$

Checkpoint 3.57 Find the x-intercepts of the graph of $y = \dfrac{1}{4}x^2 - 5x + 24$.

Answer. $(8, 0),\ (12, 0)$

Completing the Square

To solve a quadratic equation by completing the square, we often have to work with fractions.

1. Multiply fractions

To multiply two fractions together, we multiply their numerators together, and multiply their denominators together. We can divide out any common factors in numerator and denominator before we multiply.

Examples

Example 3.58 Multiply $\dfrac{3}{8} \cdot \dfrac{6}{5}$

Solution. We can divide out a factor of 2.

$$\frac{3}{8} \cdot \frac{6}{5} = \frac{3}{\cancel{2} \cdot 4} \cdot \frac{\cancel{2} \cdot 3}{5} = \frac{9}{20}$$

□

Example 3.59 Multiply $\dfrac{ab}{6} \cdot \dfrac{3a}{2b}$

Solution. We divide out common factors before multiplying.

$$\frac{ab}{6} \cdot \frac{3a}{2b} = \frac{a\cancel{b}}{2 \cdot \cancel{3}} \cdot \frac{\cancel{3}a}{2\cancel{b}} = \frac{a^2}{4}$$

□

Exercises

Checkpoint 3.60 Multiply.

a $\dfrac{2}{3} \cdot \dfrac{5}{7}$

b $\dfrac{6}{7} \cdot \dfrac{14}{15}$

Answer.

a $\dfrac{10}{21}$

b $\dfrac{4}{5}$

Checkpoint 3.61 Multiply.

a $\dfrac{12x}{16y} \cdot \dfrac{18}{27xy}$

b $\dfrac{9c^2}{10c} \cdot \dfrac{25cd}{12d^2}$

Answer.

a $\dfrac{1}{2y^2}$

b $\dfrac{15c^2}{8d}$

1. Add fractions

To add or subtract unlike fractions.

1 Find the LCD for the fractions.

2 Build each fraction to an equivalent one with the LCD as its denominator.

3 Add or subtract the numerators. Keep the same denominator.

Examples

Example 3.62 Add.

a $\dfrac{7}{10}+\dfrac{5}{6}$

b $6+\dfrac{4}{9}$

Solution.

a Step1: Find the LCD. Factor each denominator.

$$\begin{aligned}10 &= 2\cdot 5\\ 6 &= 2\cdot 3\end{aligned}$$

The LCD is $2\cdot 3\cdot 5 = 30$.

Step2: Build each fraction to a denominator of 30. The building factor for the first fraction is **3**, and **5** for the second fraction.

$$\frac{7}{10}\cdot\frac{\mathbf{3}}{\mathbf{3}}=\frac{21}{30}\quad\text{and}\quad\frac{5}{6}\cdot\frac{\mathbf{5}}{\mathbf{5}}=\frac{25}{30}$$

Step 3: Add the two like fractions, and reduce.

$$\begin{aligned}\frac{7}{10}+\frac{5}{6} &= \frac{21}{30}+\frac{25}{30}=\frac{46}{30}\\ &= \frac{\cancel{2}\cdot 23}{\cancel{2}\cdot 15}=\frac{23}{15}\end{aligned}$$

b Step1: The LCD is 9.

Step 2: Build the whole number to a denominator of 9.

$$\frac{6}{1}\cdot\frac{\mathbf{9}}{\mathbf{9}}=\frac{54}{9}$$

Step 3: Add the two like fractions.

$$6+\frac{4}{9}=\frac{54}{9}+\frac{4}{9}=\frac{58}{9}$$

□

Exercises

Checkpoint 3.63 Add or subtract.

a $\dfrac{5}{8}+\dfrac{1}{12}$

b $3-\dfrac{3}{4}$

Answer.

a $\dfrac{17}{24}$

b $\dfrac{9}{4}$

Checkpoint 3.64 Add or subtract.

a $\dfrac{5}{2}-\dfrac{5}{3}$

b $4+\dfrac{3}{8}$

Answer.

a $\dfrac{5}{6}$

b $\dfrac{35}{8}$

3. Recognize squares of binomials

To solve a quadratic equation by completint the square, we create the square of a binomial:

$$\boldsymbol{(x+p)^2 = x^2+2px+p^2}$$

Example

Example 3.65 Write each trinomial as the square of a binomial.

a x^2+6x+9

b $x^2-5x+\dfrac{25}{4}$

Solution.

a In the formula above, note that the coefficient of x is $2p$ and the constant term is p^2. In this example, $2p=6$ and $p^2=9$, so $p=\mathbf{3}$, and

$$x^2+6x+9=(x+\mathbf{3})^2$$

b The coefficient of x is $2p=-5$ and the constant terms is $p^2=\dfrac{25}{4}$, so $p=\dfrac{\mathbf{-5}}{\mathbf{2}}$, and

$$x^2-5x+\frac{25}{4}=\left(x-\frac{\mathbf{5}}{\mathbf{2}}\right)^2$$

□

Exercises

Checkpoint 3.66 Write $x^2+12x+36$ as the square of a binomial.

Answer. $(x+6)^2$

Checkpoint 3.67 Write $x^2 - 26x + 169$ as the square of a binomial.
Answer. $(x - 13)^2$

Checkpoint 3.68 Write $a^2 - 9a + \frac{81}{4}$ as the square of a binomial.
Answer. $(a - \frac{9}{2})^2$

Checkpoint 3.69 Write $t^2 - \frac{4}{3}t + \frac{4}{9}$ as the square of a binomial.
Answer. $(t - \frac{2}{3})^2$

Chapter 4

Applications of Quadratic Models

Quadratic Formula

1. Multiply algebraic fractions

To multiply two fractions together, we multiply their numerators together, and multiply their denominators together. We can divide out any common factors in numerator and denominator before we multiply.

Examples

Example 4.1 Multiply $\frac{1}{2}\left(\frac{P}{Q}\right)$

Solution. $\frac{1}{2}\left(\frac{P}{Q}\right) = \frac{1 \cdot P}{2 \cdot Q} = \frac{P}{2Q}$ □

Example 4.2 Multiply $4\left(\frac{b}{c}\right)$

Solution. $4\left(\frac{b}{c}\right) = \frac{4}{1}\left(\frac{b}{c}\right) = \frac{4 \cdot b}{1 \cdot c} = \frac{4b}{c}$ □

Example 4.3 Multiply $\frac{t}{w} \cdot \frac{w}{3t^2}$

Solution. $\frac{t}{w} \cdot \frac{w}{3t^2} = \frac{\cancel{t}}{\cancel{w}} \cdot \frac{\cancel{w}}{3t \cdot \cancel{t}} = \frac{1}{3t}$ □

Example 4.4 Multiply $\frac{a}{2} \cdot \frac{1-b}{b}$

Solution. $\frac{a}{2} \cdot \frac{1-b}{b} = \frac{a(1-b)}{2 \cdot b} = \frac{a-ab}{2b}$ □

Exercises

Checkpoint 4.5 Multiply $\frac{3}{2a}\left(\frac{a^2}{6}\right)$

Answer. $\frac{a}{9}$

Checkpoint 4.6 Multiply $8\left(\dfrac{m}{4x^2}\right)$

Answer. $\dfrac{2m}{x^2}$

Checkpoint 4.7 Multiply $\dfrac{3}{4} \cdot \dfrac{t^2-2}{cw^2}$

Answer. $\dfrac{3t^2-6}{4cw^2}$

Checkpoint 4.8 Multiply $\dfrac{n}{n-1} \cdot \dfrac{p+1}{n^2}$

Answer. $\dfrac{p+1}{n^2-n}$

2. Add or subtract algebraic fractions

To add or subtract unlike fractions.

1 Find the LCD for the fractions.

2 Build each fraction to an equivalent one with the LCD as its denominator.

3 Add or subtract the numerators. Keep the same denominator.

Examples

Example 4.9 Subtract $\dfrac{2x}{w} - \dfrac{3x}{w}$

Solution. These are like fractions, so we need only combine their numerators.

$$\frac{2x}{w} - \frac{3x}{w} = \frac{2x-3x}{w} = \frac{-x}{w}$$

□

Example 4.10 Subtract $\dfrac{3}{a} - \dfrac{a+2}{a}$

Solution. These are like fractions, so we need only combine their numerators. Be careful to subtract both terms of the second numerator.

$$\frac{3}{a} - \frac{a+2}{a} = \frac{3-(a+2)}{a} = \frac{1-a}{a}$$

□

Example 4.11 Add $2 + \dfrac{1}{x}$

Solution. We write 2 as a fraction, $\dfrac{2}{1}$, and build it to the LCD, x.

$$\frac{2}{1} \cdot \frac{\boldsymbol{x}}{\boldsymbol{x}} + \frac{1}{x} = \frac{2x}{x} + \frac{1}{x} = \frac{2x+1}{x}$$

□

Example 4.12 Subtract $\dfrac{a}{2b} - \dfrac{3}{b^2}$

Solution. We build each fraction to the LCD, $2b^2$.

$$\begin{aligned}\frac{a}{2b} - \frac{3}{b^2} &= \frac{a}{2b} \cdot \frac{\mathbf{b}}{\mathbf{b}} - \frac{3}{b^2} \cdot \frac{\mathbf{2}}{\mathbf{2}}\\ &= \frac{ab}{2b^2} - \frac{6}{2b^2} = \frac{ab-6}{2b^2}\end{aligned}$$

□

Exercises

Checkpoint 4.13 Subtract $\dfrac{4a}{b^2} - \dfrac{c}{b^2}$

Answer. $\dfrac{4a-c}{b^2}$

Checkpoint 4.14 Subtract $\dfrac{p+2}{2q} - \dfrac{p-1}{2q}$

Answer. $\dfrac{3}{2q}$

Checkpoint 4.15 Add $N + \dfrac{2}{N}$

Answer. $\dfrac{N^2+2}{N}$

Checkpoint 4.16 Add $\dfrac{2}{xy} - \dfrac{y}{3x}$

Answer. $\dfrac{6+y^2}{3xy}$

3. Simplify square roots

Be careful when simplifying radicals after extracting roots.

Example

Example 4.17 Can you simplify the first expression into the second expression? (Decide whether the expressions are equivalent.)

a Is $\sqrt{4+x^2}$ equivalent to $2+x$?

b Is $\sqrt{\dfrac{x^2}{9}}$ equivalent to $\dfrac{x}{3}$ for $x \geq 0$?

c Is $\sqrt{w-3}$ equivalent to $\sqrt{w} - \sqrt{3}$?

Solution.

a If the expressions are equivalent, they must be equal for every value of the variable. Let's test with $x = 3$. Then

$$\begin{aligned} &\sqrt{4+x^2} = \sqrt{4+9} = \sqrt{13} \approx 3.6\\ \text{but} \qquad &2 + x = 2 + 3 = 5\end{aligned}$$

No, the expressions are not equivalent.

b Because $\left(\dfrac{x}{3}\right)^2 = \dfrac{x}{3} \cdot \dfrac{x}{3} = \dfrac{x^2}{3^2} = \dfrac{x^2}{9}$, it is also true that $\sqrt{\dfrac{x^2}{9}} = \dfrac{x}{3}$. Yes, the expressions are equivalent.

c Let $w = 16$. Then

$$\begin{aligned} \sqrt{w-3} &= \sqrt{16-3} = \sqrt{13} \approx 3.6 \\ \text{but} \quad \sqrt{w} - \sqrt{3} &= \sqrt{16} - \sqrt{3} \approx 4 - 1.7 = 2.3 \end{aligned}$$

No, the expressions are not equivalent.

□

Exercises

Decide whether the expressions are equivalent. Assume all variables are positive.

Checkpoint 4.18 $\sqrt{b^2 - 81}$ and $b - 9$

Answer. No

Checkpoint 4.19 $\sqrt{64x^2y^2}$ and $8xy$

Answer. Yes

Checkpoint 4.20 $\sqrt{64 + x^2y^2}$ and $8 + xy$

Answer. No

Checkpoint 4.21 $\sqrt{\dfrac{c^2+d^2}{4b^2}}$ and $\dfrac{\sqrt{c^2+d^2}}{2b}$

Answer. Yes

The Vertex

1. Graph parabolas in vertex form

We can sketch the graph of a parabola with the vertex, the y-intercept, and its symmetric point.

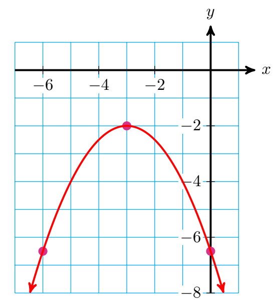
Examples

Example 4.22 Graph the equation $y = \dfrac{-1}{2}(x+3)^2 - 2$

Solution. The vertex is the point $(-3, 2)$. We can find the y-intercept by setting $x = 0$.

$$y = \frac{-1}{2}(\mathbf{0} + 3)^2 - 2 = \frac{-9}{2} - 2 = -6\frac{1}{2}$$

The y-intercept is the point $\left(0, -6\dfrac{1}{2}\right)$. The axis of symmetry is the vertical line $x = -3$, and there is a symmetric point equidistant from the axis, namely $\left(-6, -6\dfrac{1}{2}\right)$. We plot these three points and sketch the parabola through them.



□

Example 4.23 Graph the equation $y = 2(x - 15)^2 - 72$

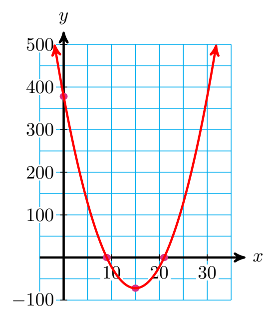
Solution. The vertex is $(15, -72)$. We find the y-intercept by setting $x = 0$:

$$y = 2(\mathbf{0} - 15)^2 - 72 = 378$$

The y-intercept is $(0, 378)$. We find the x-intercepts by setting $y = 0$:

$$\begin{aligned} 2(x - 15)^2 - 72 &= \mathbf{0} \\ (x - 15)^2 &= 36 \\ x &= \pm 6 + 15 \end{aligned}$$

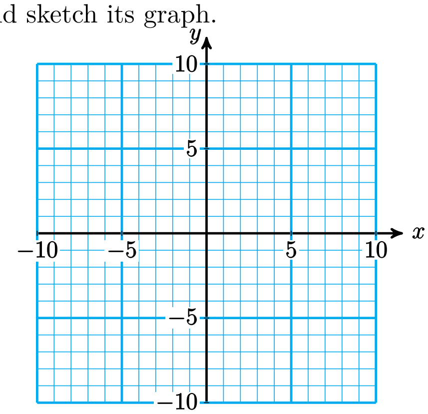
The x-intercepts are $(9, 0)$ and $(21, 0)$. We plot these three points and sketch the parabola through them.



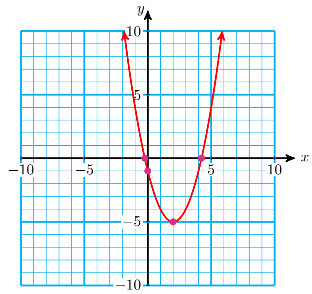
□

Exercises

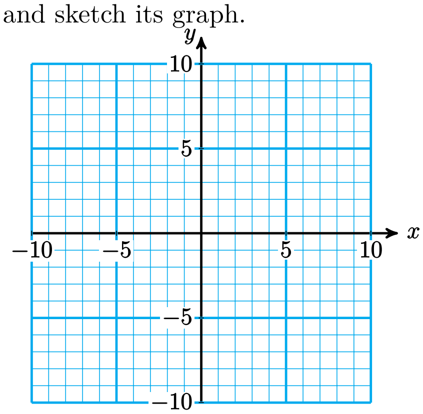
Checkpoint 4.24 Find the vertex, the y-intercept, and the x-intercepts of $y = (x - 2)^2 - 5$, and sketch its graph.



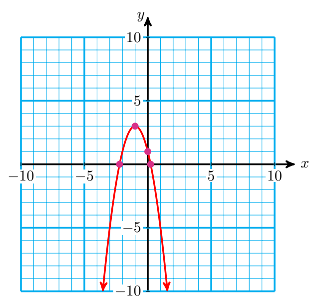
Answer.



Checkpoint 4.25 Find the vertex, the y-intercept, and the x-intercepts of $y = -2(x+1)^2 + 3$, and sketch its graph.



Answer.



2. Expand a product of binomials

With practice, you will be able to multiply binomials mentally.

Examples

Example 4.26 Multiply $(x-6)(x+3)$.

Solution. We apply the distributive law to multiply each term of the first factor by each term of the second factor. (The "FOIL" method.)

$$\begin{aligned}(x-6)(x+3) &= x \cdot x + 3 \cdot x - 6 \cdot x - 6 \cdot 3\\ &= x^2 + 3x - 6x - 18 && \text{Combine like terms.}\\ &= x^2 - 3x - 18\end{aligned}$$

□

Example 4.27 We multiply $(2t^3 - 5)(3t^3 - 2)$.

Solution. Multiply each term of the first factor by each term of the second factor.

$$\begin{aligned}(2t^3-5)(3t^3-2) &= 6t^6-4t^3-15t^3+10 && \text{Combine like terms.}\\ &= 6t^6-19t^3+10\end{aligned}$$

□

Exercises

Checkpoint 4.28 Multiply $(4a-7)(3a+8)$.

Answer. $12a^2+11a-56$

Checkpoint 4.29 Multiply $(1-3b^2)(3-4b^2)$.

Answer. $12b^4-13b^2+3$

3. Solve an equation for a parameter

We can find the equation for a parabola if we know, for example, the vertex and one other point.

Examples

Example 4.30 The point $(6,2)$ lies on the graph of $y=a(x-4)^2+1$. Solve for a.

Solution. Substitute 6 for x and 2 for y, then solve for a.

$$\begin{aligned}\mathbf{2} &= a(\mathbf{6}-4)^2+1\\ 2 &= a(4)+1\\ 1 &= 4a\end{aligned}$$

The solution is $a=\dfrac{1}{4}$. □

Example 4.31 The point $(-2,11)$ lies on the graph of $y=x^2+bx-3$. Solve for b.

Solution. Substitute -2 for x and 11 for y, then solve for b.

$$\begin{aligned}(\mathbf{-2})^2+b(\mathbf{-2})-3 &= 11\\ 4-2b-3 &= 11\\ -2b &= 10\end{aligned}$$

The solution is $b=-5$. □

Exercises

Checkpoint 4.32 The point $(-6,10)$ lies on the graph of $y=a(x+3)^2-2$. Solve for a.

Answer. $a=\dfrac{4}{3}$

Checkpoint 4.33 The point $(-3,8)$ lies on the graph of $y=-x^2+bx+5$. Solve for b.

Answer. $b=-4$

Checkpoint 4.34 The point $(8, -12)$ lies on the graph of $y = ax^2 - 4x + 36$. Solve for a.

Answer. $a = \dfrac{-1}{3}$

Checkpoint 4.35 The point $(60, -480)$ lies on the graph of $y = \dfrac{-2}{3}(x-h)^2 + 120$. Solve for h.

Answer. $h = 30$

Curve Fitting

1. Write an equation for a point on a graph

If a curve passes through a given point, the coordinates of the point satisfy the equation of the curve.

Example

Example 4.36 Write an equation to say that $(-3, 8)$ lies on the graph of $y = ax^2 + bx + c$.

Solution. Substitute -3 for x and 8 for y.

$$8 = a(-3)^2 + b(-3) + c \qquad \text{Simplify.}$$
$$8 = 9a - 3b + c$$

□

Exercises

Checkpoint 4.37 Write an equation to say that $(-4, -18)$ lies on the graph of $y = ax^2 + bx + c$.

Answer. $-16a - 4b + c = -18$

Checkpoint 4.38 Write an equation to say that $(8, 0)$ lies on the graph of $y = ax^2 + bx + c$.

Answer. $64a + 8b + c = 0$

Checkpoint 4.39 Write an equation to say that $(0, -5)$ lies on the graph of $y = ax^2 + bx + c$.

Answer. $c = -5$

Checkpoint 4.40 Write an equation to say that $(-60, 400)$ lies on the graph of $y = ax^2 + bx + c$.

Answer. $3600a - 60b + c = 400$

2. Solve a 2x2 linear system

For fitting a parabola through given points, we'll solve systems using the method of elimination.

Example

Example 4.41 Solve the system by elimination.

$$5x - 2y = 22$$
$$2x - 5y = 13$$

Solution. To eliminate the x-terms,look for the smallest integer that both 2 and 5 divide into evenly, namely, 10. Multiply the first equation by 2 and the second equation by -5.

$$\begin{aligned} \mathbf{2}(5x - 2y = 22) &\quad \rightarrow \quad & 10x - 4y = 44 \\ \mathbf{-5}(2x - 5y = 13) &\quad \rightarrow \quad & -10x + 25y = -65 \end{aligned}$$

Add these new equations to obtain an equation in y.

$$\begin{array}{r} 10x - 4y = 44 \\ \underline{-10x + 25y = -65} \\ 21y = -21 \end{array}$$

Solve for y to find $y = -1$. Finally, substitute $y = \mathbf{-1}$ into the first equation and solve for x.

$$\begin{aligned} 5x - 2(\mathbf{-1}) &= 22 \\ 5x + 2 &= 22 \\ x &= 4 \end{aligned}$$

The solution to the system is $(4, -1)$. □

Exercises

Checkpoint 4.42 Solve the system by elimination.

$$2x - 9y = 3$$
$$4x - 5y = -7$$

Answer. $(-3, -1)$

Checkpoint 4.43 Solve the system by elimination.

$$5x + 2y = 5$$
$$4x + 3y = -3$$

Answer. $(3, -5)$

3. Solve a (special) 3x3 linear system

In this special case of solving a 3x3 system, we can eliminate c to create a 2x2 system.

Example

Example 4.44 Solve the system by elimination.

$$\begin{aligned} a+b+c &= 3 && (1)\\ 4a-2b+c &= 18 && (2)\\ 9a+3b+c &= 13 && (3) \end{aligned}$$

Solution. Eliminate c by subtracting (1) from (2), then eliminate c again by subtracting (1) from (3), to get a 2x2 system:

$$\begin{aligned} 3a-3b &= 15\\ 8a+2b &= 10 \end{aligned}$$

Divide the first equation by 3 and the second equation by 2, then add.

$$\begin{aligned} a-b &= 5\\ \underline{4a+b} &\underline{= 5}\\ 5a &= 10 \end{aligned}$$

We see that $a = 2$. Substituting $a = 2$ into the equation $a - b = 5$, we find that $b = -3$. Finally, we substitue $a = 2$ and $b = -3$ into equation (1) to find

$$\begin{aligned} 2-3+c &= 3\\ c &= 4 \end{aligned}$$

The solution is $a = 2,\ b = -3$, and $c = 4$. □

Exercise

Checkpoint 4.45 Solve the system by elimination.

$$\begin{aligned} a+b+c &= 5\\ 4a-2b+c &= -7\\ 16a+4b+c &= -37 \end{aligned}$$

Answer. $a = -3,\ b = 1,\ c = 7$

Quadratic Inequalities

1. Solve a linear inequality

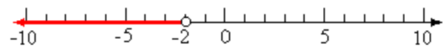
First, let's review solving linear inqualities.

Examples

Example 4.46 Solve $-3x+1 > 7$ and graph the solutions on a number line.
Solution.

$$\begin{aligned} -3x+1 &> 7 && \text{Subtract 1 from both sides.}\\ -3x &> 6 && \text{Divide both sides by -3.}\\ x &< -2 && \text{Reverse the direction of the inequality.} \end{aligned}$$

The graph of the solutions is shown below.



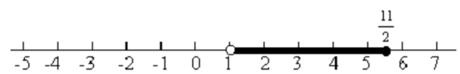
□

Example 4.47 Solve $-3 < 2x - 5 \leq 6$ and graph the solutions on a number line.

Solution.

$$\begin{aligned} -3 < 2x - 5 &\leq 6 && \text{Add 5 on all three sides.} \\ 2 < 2x &\leq 11 && \text{Divide each side by 2.} \\ 1 < x &\leq \frac{11}{2} && \text{Do not reverse the inequality.} \end{aligned}$$

The graph of the solutions is shown below.



□

Exercises

Checkpoint 4.48 Solve the inequality $8 - 4x > -2$

Answer. $x < \dfrac{5}{2}$

Checkpoint 4.49 Solve the inequality $-6 \leq \dfrac{4 - x}{3} < 2$

Answer. $22 \geq x > -2$

Checkpoint 4.50 Solve the inequality $3x - 5 < -6x + 7$

Answer. $x < \dfrac{4}{3}$

Checkpoint 4.51 Solve the inequality $-6 > 4 - 5b > -21$

Answer. $2 < b < 5$

2. Find the x-intercepts of a parabola

To solve a quadratic inequality, we first find the x-intercepts of the graph. Remember that there are four different methods for solving a quadratic equation.

Examples

Example 4.52 Find the x-intercepts of the parabola $y = 4x^2 - 12$

Solution. Set $y = 0$ and solve for x. Use extraction of roots.

$$\begin{aligned} 4x^2 - 12 &= 0 \\ 4x^2 &= 12 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \end{aligned}$$

The x-intercepts are $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$, or about $(1.7, 0)$ and $(-1.7, 0)$. □

Example 4.53 Find the x-intercepts of the parabola $y = -4x^2 - 12x$

Solution. Set $y = 0$ and solve for x. Factor the right side.

$$\begin{aligned} 0 &= -4x^2 - 12x \\ 0 &= -4x(x+3) && \text{Set each factor equal to 0.} \\ 4x = 0 &\qquad x + 3 = 0 \\ x = 0 &\qquad x = -3 \end{aligned}$$

The x-intercepts are $(0, 0)$ and $(-3, 0)$. □

Example 4.54 Find the x-intercepts of the parabola $y = 4x^2 - 12x + 8$

Solution. Set $y = 0$ and solve for x. Factor the right side.

$$\begin{aligned} 0 &= 4x^2 - 12x + 8 \\ 0 &= 4(x^2 - 3x + 2) \\ 0 &= 4(x-2)(x-1) && \text{Set each factor equal to 0.} \\ x - 2 = 0 &\qquad x - 1 = 0 \\ x = 2 &\qquad x = 1 \end{aligned}$$

The x-intercepts are $(2, 0)$ and $(1, 0)$. □

Example 4.55 Find the x-intercepts of the parabola $y = 12 - 12x - 4x^2$

Solution. Set $y = 0$ and solve for x. Use the quadratic formula.

$$\begin{aligned} 0 &= -4x^2 - 12x + 12 && \boldsymbol{a = -4,\ b = -12,\ c = 12} \\ x &= \frac{12 \pm \sqrt{(-12)^2 - 4(-4)(12)}}{2(-4)} \\ &= \frac{12 \pm \sqrt{144 + 96}}{-8} \\ &= \frac{12 \pm \sqrt{240}}{-8} = \frac{12 \pm 4\sqrt{15}}{-8} && \boldsymbol{\sqrt{240} = \sqrt{16 \cdot 15} = 4\sqrt{15}} \\ &= \frac{-3 \pm \sqrt{15}}{2} \end{aligned}$$

The x-intercepts are $\left(\frac{-3 + \sqrt{15}}{2}, 0\right)$ and $\left(\frac{-3 - \sqrt{15}}{2}, 0\right)$, or about $(0.44, 0)$ and $(-3.44, 0)$. □

Exercises

Find the x-intercepts of the parabola.

Checkpoint 4.56 $y = 2x^2 - 7x + 3$

Answer. $\left(\frac{1}{2}, 0\right)$, $(3, 0)$

Checkpoint 4.57 $y = 7x - 2x^2$

Answer. $(0, 0)$, $\left(\frac{7}{2}, 0\right)$

Checkpoint 4.58 $y = 10 - 2x^2$

Answer. $(2.24, 0)$, $(-2.24, 0)$

Checkpoint 4.59 $y = 2x^2 + 10x + 3$

Answer. $(-0.32, 0),\ (-4.68, 0)$

Chapter 5

Functions

Functions

1. New vocabulary

Definitions

Write a definition or description for each term. You can find answers in Section 1.2 of your textbook.

1 Function

2 Input variable

3 Output variable

4 Function value

5 Function notation

Exercise

Checkpoint 5.1 Identify each term above, or give an example, for this situation: At time t seconds, the height of a basketball above the ground, h, in feet, is given by

$$h = -16t^2 + 20t + 5.$$

Answer.

1 h is a function of t.

2 The input variable is t.

3 The output variable is h.

4 The function value for $t = 1$ is $h = 9$.

5 $h = f(t)$

2. Solve non-linear equations

To solve simple non-linear equations, we ”undo” the operation performed on the variable.

Examples

Example 5.2 Solve the equation $5\sqrt{t} = 83$

Solution. To "undo" a square root, we square both sides of the equation. First, we isolate the square root.

$$\begin{aligned} \frac{5\sqrt{t}}{\mathbf{5}} &= \frac{83}{\mathbf{5}} && \text{Divide both sides by 5.} \\ (\sqrt{t})^2 &= (16.6)^2 && \text{Square both sides.} \\ t &= 275.56 \end{aligned}$$

□

Example 5.3 Solve the equation $\dfrac{15}{y} = 45$

Solution. If the variable is in the denominator of a fraction, we must first clear the fraction.

$$\begin{aligned} \boldsymbol{y}\left(\frac{15}{y}\right) &= 45 \cdot \boldsymbol{y} && \text{Multiply both sides by } \boldsymbol{y}\text{.} \\ 15 &= 45y && \text{Divide both sides by 45.} \\ y &= \frac{15}{45} = \frac{1}{3} \end{aligned}$$

□

Exercises

Checkpoint 5.4 Solve the equation $\dfrac{4.8}{w} = 3$

Answer. 1.6

Checkpoint 5.5 Solve the equation $18 = 36\sqrt{q}$

Answer. $\dfrac{1}{4}$

Graphs of Functions

1. New vocabulary

We can use function notation to describe a graph.

Fill in the blanks:

1 The point (a, b) lies on the graph of f if and only if ______.

2 Each point on the graph of $y = f(x)$ has coordinates ______.

3 A graph of a function is increasing if the ______ values get larger as we read from left to right.

4 The maximum value of a function is the ____ of the highest point on the graph.

Exercises

Checkpoint 5.6 How do you know that the point $(1, 9)$ lies on the graph of $f(t) = -16t^2 + 20t + 5$?

Answer. Because $f(1) = 9$.

Checkpoint 5.7 What are the coordinates of any point on the graph of $h = f(t)$?

Answer. $(t, f(t))$

2. Solve equations and inequalities graphically

Examples

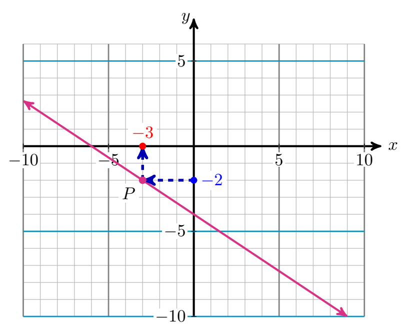
Every point on the graph of an equation $y = f(x)$ tells us the solution of the equation for a particular value of y.

Example 5.8 Use the graph of $y = \frac{-2}{3}x - 4$ to solve the equation

$$\frac{-2}{3}x - 4 = -2.$$

Solution.

We see that y has been replaced by -2 in the equation for the graph. So we look for the point on the graph that has y-coordinate -2.
This point, labeled P on the graph at right, has x-coordinate -3. Because it lies on the graph, the point $P(-3, -2)$ is a solution of the equation $y = \frac{-2}{3}x - 4$.



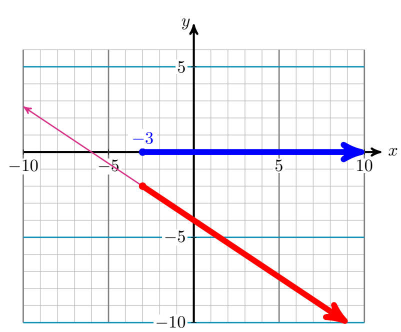
But this statement also tells us that -3 is a solution of the equation $\frac{-2}{3}x - 4 = 2$. You can check that substituting $x = -3$ into this equation produces a true statement. □

Example 5.9 Use the graph of $y = \frac{-2}{3}x - 4$ to solve the inequality

$$\frac{-2}{3}x - 4 \leq -2.$$

Solution.

We would like to find the x-coordinates of all points on the graph that have y-coordinate less than or equal to -2. These points on the graph are indicated by the heavy portion of the line.
The x-coordinates of these points are shown by the heavy portion of the x-axis. The solution is $x \geq -3$, or in interval notation, $[-3, \infty)$.



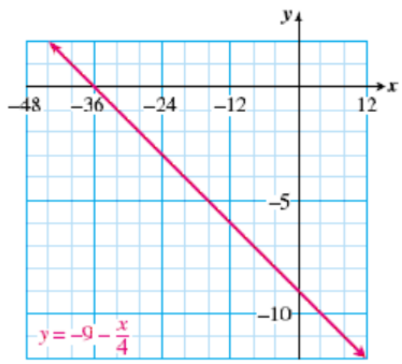
□

Exercises

Checkpoint 5.10 Use the graph to solve the equation or inequality. (Note the scales on the axes.) Show your solutions on the graph. Then verify your solutions by solving algebraically.

a $-9 - \dfrac{x}{4} = -2$

b $-9 - \dfrac{x}{4} \geq -5$



Answer.

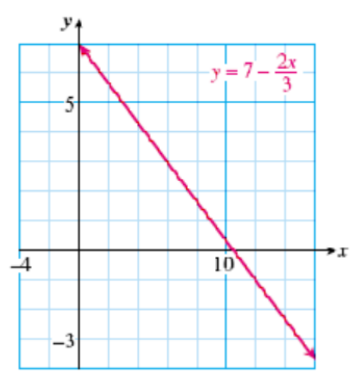
a $x = -28$

b $x \leq -16$

Checkpoint 5.11 Use the graph to solve the equation or inequality. (Note the scales on the axes.) Show your solutions on the graph. Then verify your solutions by solving algebraically.

a $7 - \dfrac{2x}{3} = 3$

b $7 - \dfrac{2x}{3} < -1$



Answer.

a $x = 6$

b $x > 12$

Some Basic Functions

1. Evaluate cube roots

Examples

It is a good idea to become familiar with the first few perfect cubes:

$$2^3 = 8 \qquad 3^3 = 27 \qquad 4^3 = 64 \qquad 5^3 = 125 \qquad 6^3 = 216$$

and so on.

Example 5.12 Evaluate each cube root.

a $\sqrt[3]{64}$

b $\sqrt[3]{-125}$

c $\sqrt[3]{1}$

d $\sqrt[3]{\dfrac{-1}{8}}$

Solution.

a $4^3 = 64, \quad \text{so} \quad \sqrt[3]{64} = 4$

b The cube root of a negative number is negative.

$$(-5)^3 = -125, \quad \text{so} \quad \sqrt[3]{-125} = -5$$

c $1^3 = 1, \quad \text{so} \quad \sqrt[3]{1} = 1$

d We can take the cube root of a fraction by taking the cube root of its numerator and denominator.

$$\sqrt[3]{\frac{-1}{8}} = \frac{\sqrt[3]{-1}}{\sqrt[3]{8}} = \frac{-1}{2}$$

□

Example 5.13 Use a calculator to evaluate the cube root. Round to thousandths.

a $\sqrt[3]{347}$

b $\sqrt[3]{0.85}$

c $\sqrt[3]{-9}$

Solution. On a scientific calculator, look for the key labeled $\boxed{\sqrt[3]{}}$. On a graphing calculator, press MATH 4

a $\sqrt[3]{347} \approx 7.027$

b $\sqrt[3]{0.85} \approx 0.947$

c $\sqrt[3]{-9} \approx -2.080$

□

Exercises

Checkpoint 5.14 Evaluate $\sqrt[3]{-0.5}$. Round to thousandths.

Answer. -0.794

Checkpoint 5.15 Evaluate $\sqrt[3]{81}$. Round to thousandths.

Answer. 4.327

2. Evaluate absolute values

Examples

The definition of how to take an absolute value may look complicated, but it just says two things:

1 If the number is positive, leave it alone.

2 If the number is negative, put another negative in front, which will make the number positive.

Example 5.16 Simplify each expression.

a $|-3|$

b $-|3|$

c $-(-3)$

d $-|-3|$

Solution. The absolute value of any number is positive (or zero). We can think of the absolute value of a number as its distnce from on a number line.

a -3 is 3 units from 0, so $|-3| = 3$.

b $-|3|$ is the opposite of $|3|$, so $-|3| = -3$.

c The opposite of -3 is 3, so $-(-3) = 3$.

d $-|-3|$ is the opposite of $|-3|$, so $-|-3| = -3$.

□

Example 5.17 Suppose x represents -8. Evaluate each expression.

a $-x$

b $|x|$

c $|-x|$

Solution.

a $-x = -(-8) = 8$

b $|x| = |-8| = 8$

c $|-x| = |-(-8)| = 8$

□

Exercises

Checkpoint 5.18 Simplify $-|-12|$.

Answer. -12

Checkpoint 5.19 Simplify $|-25|$.

Answer. 25

Checkpoint 5.20 Simplify $-(-90)|$.

Answer. 90

3. Use the order of operations in evaluation

Recall the order of operations:

1 Simplify what's inside parentheses (or absolute value bars) first.

2 Next evaluate all powers and roots.

3 Then perform all multiplications and divisions in order from left to right.

4 Finally, perform all additionas and subtractions in order from left to right.

Examples

Example 5.21 Simplify $|2|-4|3-8|$

Solution. Absolute value bars are a grouping device. We simplify expressions within absolute value bars first.

$$\begin{aligned} |2|-4|\mathbf{3-8}| &= |2|-4|\mathbf{-5}| && \text{Evaluate absolute values.} \\ &= 2-4(5) && \text{Multiply.} \\ &= 2-20=18 \end{aligned}$$

□

Example 5.22 Simplify $\dfrac{8-2\sqrt[3]{11.375+2.5^3}}{8-4}$

Solution. Simplify the expression under the radical first.

$$\frac{8-2\sqrt[3]{\mathbf{11.375+2.5^3}}}{8-4} = \frac{8-2\sqrt[3]{\mathbf{27}}}{8-4} \qquad \text{Evaluate the radical.}$$

$$amp = \frac{8-2(3)}{8-4} \qquad \text{Simplify numerator and denominator.}$$

$$= \frac{8-6}{4}$$

$$= \frac{2}{4} = \frac{1}{2}$$

□

Exercises

Checkpoint 5.23 Simplify $3\sqrt[3]{\dfrac{125}{216}}+\dfrac{4}{5}\sqrt[3]{-512}$. Follow the order of operations.

Answer. $\dfrac{-39}{10}$

Checkpoint 5.24 Simplify $-3|3-6|-4|-4-3|$. Follow the order of operations.

Answer. -37

Direct Variation

1. Solve a variation equation

In these examples, we assume all variables are positive. We round answers to tenths.

Examples

Example 5.25 Solve $231.90 = 18.85r^2$

Solution. The equation is quadratic. We solve by extraction of roots.

$$\begin{aligned} 231.90 &= 18.85r^2 && \text{Isolate the squared expression.} \\ 12.302 &= r^2 && \text{Take square roots.} \\ r &= 35 \end{aligned}$$

□

Example 5.26 Solve $62x^3 = 4860.8$

Solution. This equation is cubic. We isolate the variable, then take cube roots.

$$\begin{aligned} 62x^3 &= 4860.8 && \text{Divide both sides by 62.} \\ x^3 &= 78.4 && \text{Take cube roots.} \\ x &= 16.9 \end{aligned}$$

□

Exercises

Checkpoint 5.27 Solve $1371.8 = 25R^3$

Answer. 3.8

Checkpoint 5.28 Solve $6.3t^2 = 18.4$

Answer. 1.7

2. Sketch a variation graph

The graphs of direct variations are transformations of the basic graphs $y = x^n$.

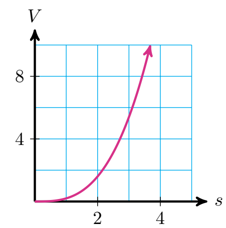
Example

Example 5.29 Sketch a graph of $V = 0.2s^3$.

Solution. We know that the graph has the shape of the basic function $y = x^3$, so all we need are a few points to "anchor" the graph.

$$\begin{aligned} &\text{If } s = 1, V = 0.2(1)^3 = 0.2 \\ &\text{If } s = 2, V = 0.2(2)^3 = 1.6 \\ &\text{If } s = 3, V = 0.2(3)^3 = 5.4 \end{aligned}$$

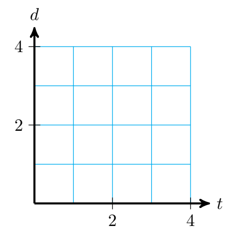
The graph is shown below.



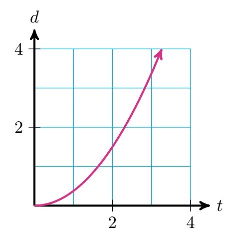
□

Exercise

Checkpoint 5.30 Plot three points and sketch a graph of $d = \dfrac{3}{8}t^2$.



Answer. $\left(1, \dfrac{3}{8}\right)$, $\left(2, \dfrac{3}{2}\right)$, $(4, 6)$



3. Find the constant of variation

If we know the type of variation and the coordinates of one point ont the graph, we can find the variation equation.

Example

Example 5.31 Find the constant of variation and the variation equation:

y varies directly with the square of x, and $y = 100$ when $x = 2.5$.

Solution. Because y varies directly with the square of x, we know that $y = kx^2$. We substitute the given values to find

$$100 = k(2.5)^2 \qquad \text{Solve for } \boldsymbol{k}.$$

$$k = \frac{100}{2.5^2} = 16$$

The constant of variation is 16, and the variation equation is $y = 16x^2$. □

Exercise

Checkpoint 5.32 Find the constant of variation and the variation equation:
y varies directly with the cube of x, and $y = 119,164$ when $x = 6.2$.

Answer. $k = 500$ and $y = 500x^3$

Inverse Variation

1. Solve a variation equation

In these examples, we assume all variables are positive. We round answers to tenths.

Examples

Example 5.33 Solve $2.8125 = \dfrac{36}{n}$

Solution. We must first get the variable out of the denominator.

$$\begin{aligned} n(2.8125) &= \frac{36}{n}n && \text{Multiply boyh sides by } \boldsymbol{n}. \\ 2.8125n &= 36 && \text{Divide both sides by 2.8125.} \\ n &= 12.8 \end{aligned}$$

□

Example 5.34 Solve $0.5547 = \dfrac{1500}{d^2}$

Solution. We must first get the variable out of the denominator.

$$\begin{aligned} d^2(0.5547) &= \frac{1500}{d^2}d^2 && \text{Multiply boyh sides by } \boldsymbol{d^2}. \\ 0.5547d^2 &= 1500 && \text{Divide both sides by 0.5547.} \\ d^2 &= 2704.16 && \text{Take square roots.} \\ d &= 52 \end{aligned}$$

□

Exercises

Checkpoint 5.35 Solve $13.03 = \dfrac{380}{h^2}$

Answer. 5.4

Checkpoint 5.36 Solve $0.065 = \dfrac{12}{p}$

Answer. 184.6

2. Sketch a variation graph

The graphs of inverse variations are transformations of the basic graphs $y = \dfrac{1}{x^n}$.

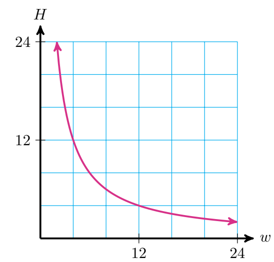
Example

Example 5.37 Sketch a graph of $H = \dfrac{48}{w}$.

Solution. We know that the graph has the shape of the basic function $y = \dfrac{1}{x}$, so all we need are a few points to "anchor" the graph.

$$\text{If } w = 2, H = \frac{48}{2} = 24$$
$$\text{If } w = 6, H = \frac{48}{6} = 8$$
$$\text{If } w = 12, H = \frac{48}{12} = 4$$

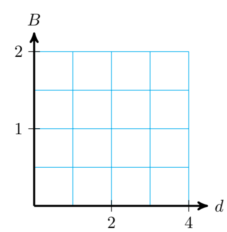
The graph is shown below.



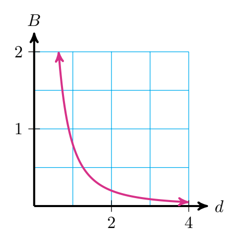
□

Exercise

Checkpoint 5.38 Plot three points and sketch a graph of $B = \dfrac{0.8}{d^2}$.



Answer. $(1, 0.8)$, $(2, 0.2)$, $(4, 0.05)$



3. Find the constant of variation

If we know the type of variation and the coordinates of one point on the graph, we can find the variation equation.

Example

Example 5.39 Find the constant of variation and the variation equation:
y varies inversely with the square of x, and $y = 4687.5$ when $x = 0.16$.

Solution. Because y varies inversely with the square of x, we know that $y = \dfrac{k}{x^2}$. We substitute the given values to find

$$\begin{aligned} 4687.5 &= \frac{k}{0.16^2} && \text{Solve for } \boldsymbol{k}. \\ k &= 4687.5(0.16)^2 = 120 \end{aligned}$$

The constant of variation is 120, and the variation equation is $y = \dfrac{120}{x^2}$. □

Exercise

Checkpoint 5.40 Find the constant of variation and the variation equation:
y varies inversely with x, and $y = 31.25$ when $x = 640$.

Answer. $k = 20,000$ and $y = \dfrac{20,000}{x}$

Functions as Mathematical Models

1. Properties of the basic functions

The eight basic functions are often used as models.

For these Examples, refer to the eight basic functions:

$$\begin{array}{llll} f(x) = x & f(x) = |x| & f(x) = x^2 & f(x) = x^3 \\ f(x) = \sqrt{x} & f(x) = \sqrt[3]{x} & f(x) = \dfrac{1}{x} & f(x) = \dfrac{1}{x^2} \end{array}$$

Examples

Example 5.41 Which of the eight basic functions are always increasing?

Solution. $f(x) = x$, $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$ □

Example 5.42 Which of the eight basic functions are concave up for positive x?

Solution. $f(x) = x^2$, $f(x) = x^3$, $f(x) = \dfrac{1}{x}$, $f(x) = \dfrac{1}{x^2}$ □

Exercises

Checkpoint 5.43 Which of the eight basic functions are undefined at $x = 0$?

Answer. $f(x) = \dfrac{1}{x}$, $f(x) = \dfrac{1}{x^2}$

Checkpoint 5.44 Which of the eight basic functions are always non-negative?

Answer. $f(x) = |x|$, $f(x) = x^2$, $f(x) = \sqrt{x}$, $f(x) = \dfrac{1}{x^2}$

2. Use familiar formulas

Here are some useful formulas often used in models.

Examples

Example 5.45 Write a formula for the volume of a rectangular box, and identify the variables.

Solution. $V = lwh$

V stands for volume, and l, w, and h stand for, respectively, the length, width, and height of the box. □

Example 5.46 Write a formula for the average of a number of scores, and identify the variables.

Solution. $A = \dfrac{S}{n}$

A stands for the average, S stands for the sum of the scores, and n stands for the number of scores. □

Exercises

Checkpoint 5.47 Choose the correct formula from the list below, and identify the variables.

- $I = Prt$
- $d = rt$
- $P = R - C$
- $P = rW$

a The distance traveled at a constant speed.

b The simple interest on an investment.

c The part specified by a percentage.

d The profit on sales of an item.

Answer.

a $d = rt$

d stands for the distance traveled at speed r for time t.

b $I = Prt$

I stands for the interest earned on an investment P at interest rate r after a time period t.

c $P = rW$

P stands for the quantity r percent of a whole amount W.

d $P = R - C$

P stands for the profit left after the costs C are subtracted from the revenue R.

Checkpoint 5.48 Choose the correct geometric formula from the list below, and identify the variables.

- $A = lw$
- $P = 2l + 2w$
- $A = \pi r^2$
- $C = \pi d$
- $V = \pi r^2 h$
- $V = \frac{4}{3}\pi r^3$

a The volume of a cylinder.

b The area of a circle.

c The area of a rectangle.

d The perimeter of a rectangle.

e The volume of a sphere.

f The circumference of a circle.

Answer.

a $V = \pi r^2 h$

V stands for the volume, r for the radius, and h for the height of the cylinder.

b $A = \pi r^2$

A stands for the area and r for the radius of the circle.

c $A = lw$

A stands for the area of the rectangle, l and w stand for its length and width.

d $P = 2l + 2w$

P stands for the perimeter of the rectangle, l and w stand for its length and width.

e $V = \frac{4}{3}\pi r^3$

V stands for the volume and r for the radius of the sphere.

f $C = \pi d$

C stands for the circumference and d for the diameter of the circle.

Chapter 6

Powers and Roots

Integer Exponents

1. Use the laws of exponents

Recall the five Laws of Exponents.

Laws of Exponents.

I $a^m \cdot a^n = a^{m+n}$

II $\dfrac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{if } m > n \\ \dfrac{1}{a^{n-m}} & \text{if } n > m \end{cases}$

III $(a^m)^n = a^{mn}$

IV $(ab)^n = a^n b^n$

V $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Examples

Example 6.1 Multiply $(2x^2y)(5x^4y^3)$.

Solution. Rearrange the factors to group together the numerical coefficients and the powers of each base.

$$(2x^2y)(5x^4y^3) = (2)(5)x^2x^4yy^3$$

Multiply the coefficients together, and use the first law of exponents to find the products of the variable factors.

$$(2)(5)x^2x^4yy^3 = 10x^6y^4 \qquad \text{Add exponents on each base.}$$

□

Example 6.2 Divide $\dfrac{3x^2y^4}{6x^3y}$.

Solution. Consider the numerical coefficients and the powers of each base separately. Use the second law of exponents to simplify each quotient of powers.

$$\begin{aligned}\frac{3x^2y^4}{6x^3y} &= \frac{3}{6}\cdot\frac{x^2}{x^3}\cdot\frac{y^4}{y} && \text{Subtract exponents on each base.}\\ &= \frac{1}{2}\cdot\frac{1}{x^{3-2}}\cdot y^{4-1} && \text{Multiply factors.}\\ &= \frac{1}{2}\cdot\frac{1}{x}\cdot\frac{y^3}{1} = \frac{y^3}{2x}\end{aligned}$$

□

Example 6.3 Simplify $(5a^3b)^2$.
Solution. Apply the fourth law of exponents and square each factor.

$$(5a^3b)^2 = 5^2(a^3)^2b^2 = 25a^6b^2 \qquad \text{Apply the third law: multiply exponents.}$$

□

Example 6.4 Simplify $\left(\dfrac{2}{y^3}\right)^3$.
Solution. Apply the fifth law of exponents.

$$\begin{aligned}\left(\frac{2}{y^3}\right)^3 &= \frac{2^3}{(y^3)^3} && \text{Cube numerator and denominator.}\\ &= \frac{2^3}{y^{3(3)}} = \frac{8}{y^9} && \text{Apply the third law.}\end{aligned}$$

□

Exercises

Checkpoint 6.5 Multiply $-3a^4b(-4a^3b)$.
Answer. $12a^7b^2$

Checkpoint 6.6 Divide $\dfrac{8x^2y}{12x^5y^3}$.

Answer. $\dfrac{2}{3x^3y^2}$

Checkpoint 6.7 Simplify $(6pq^4)^3$.
Answer. $216p^3q^{12}$

Checkpoint 6.8 Simplify $\left(\dfrac{n^3}{k^4}\right)^8$.

Answer. $\dfrac{n^{24}}{k^{32}}$

2. Evaluate powers with negative exponents

Remember that a negative exponent indicates a reciprocal, so for example

$$x^{-2} = \frac{1}{x^2}$$

A negative exponent does *not* mean that the power is negative. So for example

$$4^{-2} = \frac{1}{16};$$

4^{-2} does *not* mean -16.

Examples

Example 6.9 Write each expression without using negative exponents.

a 10^{-4}

b $\left(\dfrac{x}{4}\right)^{-3}$

Solution.

a $10^{-4} = \dfrac{1}{10^4} = \dfrac{1}{10,000}$, or 0.0001.

b To compute a negative power of a fraction, we compute the corresponding positive power of its reciprocal. Thus,

$$\left(\frac{x}{4}\right)^{-3} = \left(\frac{4}{x}\right)^{3} = \frac{64}{x^3}$$

□

Example 6.10 Write each expression using negative exponents.

a $\dfrac{1}{3a^4a^2}$

b $\dfrac{8}{x^4}$

Solution.

a $\dfrac{1}{3a^4a^2} = 3^{-4}a^{-2}$

b $\dfrac{8}{x^4} = 8x^{-4}$

□

Exercises

Checkpoint 6.11 Write each expression using negative exponents and evaluate.

a $(-6)^{-2}$

b $\left(\dfrac{3}{5}\right)^{-2}$

Answer.

a $\dfrac{1}{6^2} = \dfrac{1}{36}$

b $\dfrac{5^2}{3^2} = \dfrac{25}{9}$

Checkpoint 6.12 Write each expression using negative exponents.

a $4t^{-2}$

b $(4t)^{-2}$

Answer.

a $\dfrac{4}{t^2}$

b $\dfrac{1}{16t^2}$

3. Use scientific notation

If we move the decimal point to the left, we are making a number smaller, so we must multiply by a positive power of 10 to compensate. If we move the decimal point to the right, we must multiply by a negative power of 10.

Example

Example 6.13 Write each number in scientific notation.

a $62,000,000$

b 0.000431

Solution.

a First, we position the decimal point so that there is just one nonzero digit to the left of the decimal.

$$62,000,000 = 6.2 \times \underline{\qquad}$$

To recover $62,000,000$ from 6.2, we must move the decimal point seven places to the right. Therefore, we multiply 6.2 by 10^7.

$$62,000,000 = 6.2 \times 10^7$$

b First, we position the decimal point so that there is just one nonzero digit to the left of the decimal.

$$0.000431 = 4.31 \times \underline{\qquad}$$

To recover 0.000431 from 4.31, we must move the decimal point seven places to the right. Therefore, we multiply 4.31 by 10^{-4}.

$$0.000431 = 4.31 \times 10^{-4}$$

□

Exercise

Checkpoint 6.14 Write each number in scientific notation.

a The largest living animal is the blue whale, with an average weight of $120,000,000$ grams.

b The smallest animal is the fairy fly beetle, which weighs about 0.000005 grams.

Answer.

a 1.2×10^8

b 5×10^{-6}

Roots and Radicals

1. Use the definition of root

Because $(\sqrt{a})(\sqrt{a}) = a$, it is also true that $\dfrac{a}{\sqrt{a}} = \sqrt{a}$.

Examples

Example 6.15 Simplify. Do not use a calculator!

a $\left(\sqrt{7}\right)\left(\sqrt{7}\right)$

b $\sqrt{n}(\sqrt{n})$

Solution. By the definition of square root, $\sqrt{a}$ is a number whose square is a.

a $\left(\sqrt{7}\right)\left(\sqrt{7}\right) = 7$

b $\sqrt{n}(\sqrt{n}) = n$

□

Example 6.16 Simplify. Do not use a calculator!

a $\left(\sqrt[3]{5}\right)^3$

b $\left(\sqrt[3]{4}\right)\left(\sqrt[3]{4}\right)\left(\sqrt[3]{4}\right)$

Solution. By the definition of cube root, $\sqrt[3]{a}$ is a number whose cube is a.

a $\left(\sqrt[3]{5}\right)^3 = 5$

b $\left(\sqrt[3]{4}\right)\left(\sqrt[3]{4}\right)\left(\sqrt[3]{4}\right) = 4$

□

Example 6.17 Simplify. Do not use a calculator!

a $\dfrac{3}{\sqrt{3}}$

b $\dfrac{p}{\sqrt{p}}$

Solution.

a $\dfrac{3}{\sqrt{3}} = \dfrac{\sqrt{3}\sqrt{3}}{\sqrt{3}} = \sqrt{3}$

b $\dfrac{p}{\sqrt{p}} = \dfrac{\sqrt{p}\sqrt{p}}{\sqrt{p}} = \sqrt{p}$

□

Exercises

Checkpoint 6.18 Simplify. Do not use a calculator!

a $\sqrt{5}(\sqrt{5})$

b $\sqrt{x}(\sqrt{x})$

Answer.

a 5

b x

Checkpoint 6.19 Simplify. Do not use a calculator!

a $\left(\sqrt[3]{9}\right)\left(\sqrt[3]{9}\right)\left(\sqrt[3]{9}\right)$

b $\left(\sqrt[3]{20}\right)^3$

Answer.

a 9

b 20

Checkpoint 6.20 Simplify. Do not use a calculator!

a $\dfrac{10}{\sqrt{10}}$

b $\dfrac{H}{\sqrt{H}}$

Answer.

a $\sqrt{10}$

b $\sqrt{H}$

2. Approximate rational numbers

Rational numbers are the integers and common fractions; we can represent them precisely in decimal form. But the best we can do for an irrational number is to write an approximate decimal form by rounding.

Examples

Example 6.21 Identify each number as rational or irrational.

a $\sqrt{6}$

b $\dfrac{-5}{3}$

c $\sqrt{16}$

d $\sqrt{\dfrac{5}{9}}$

Solution.

a Irrational: $\sqrt{6}$ is not the quotient of two integers.

b Rational: $\dfrac{-5}{3}$ is the quotient of two integers.

c Rational: $\sqrt{16} = 4$ is an integer.

d Irrational: $\sqrt{\dfrac{5}{9}} = \dfrac{\sqrt{5}}{3}$, but $\sqrt{5}$ is irrational.

□

Example 6.22 Give a decimal approximation rounded to thousandths.

a $5\sqrt{3}$

b $\dfrac{-2}{3}\sqrt{21}$

c $2 + \sqrt[3]{5}$

Solution. Use a calculator to evaluate.

a Enter 5 $\boxed{\sqrt{}}$ 3 $\boxed{\text{ENTER}}$ and round to three decimal places: 8.660

b Enter $\boxed{\text{(-)}}$ 2 $\boxed{\sqrt{}}$ 21 $\boxed{)}$ $\boxed{\div}$ 3 $\boxed{\text{ENTER}}$ and round to three decimal places: -3.055

c Enter 2 $\boxed{+}$ $\boxed{\text{MATH}}$ $\boxed{4}$ 5 $\boxed{\text{ENTER}}$ and round to three decimal places: 3.710

□

Exercises

Checkpoint 6.23 Identify each number as rational or irrational.

a $\sqrt{250}$

b $\dfrac{\sqrt{3}}{2}$

c $\dfrac{\sqrt{81}}{4}$

d $\sqrt[3]{16}$

Answer.

a Irrational

b Irrational

c Rational

d Irrational

Checkpoint 6.24 Give a decimal approximation rounded to thousandths.

a $-6\sqrt[3]{5}$

b $\frac{3}{5}\sqrt{76}$

c $7-\sqrt{19}$

Answer.

a -10.260

b 5.231

c 2.641

3. Use the order of operations

In the order of operations, simplifying radicals comes after what's inside parentheses (or fraction bars) and before products and quotients.

Examples

Example 6.25 Simplify each expression. Do not use a calculator!

a $\dfrac{4-\sqrt{64}}{2}$

b $-2\left(3\sqrt{16}-\sqrt{3(27)}\right)$

c $6-3\sqrt[3]{27-7(5)}$

Solution.

a We start by simplifying the numerator.

$$\begin{aligned} \frac{4-\sqrt{64}}{2} &= \frac{4-8}{2} && \text{Evaluate the radical, then subtract.} \\ &= \frac{-4}{2} = -2 && \text{Reduce the fraction.} \end{aligned}$$

b We start by simplifyng what's inside parentheses.

$$\begin{aligned} &-2\left(3\sqrt{16}-\sqrt{3(27)}\right) && \text{Evaluate the radicals.} \\ &= -2(3\cdot 4-\sqrt{81}) && \text{Simplify inside the parentheses.} \\ &= -2(12-9) = -6 \end{aligned}$$

c We start by simplifying the radicand.

$$\begin{aligned} 6-3\sqrt[3]{27-7(5)} &= 6-3\sqrt[3]{27-35} && \text{Subtract under the radical.} \\ &= 6-3\sqrt[3]{-8} && \text{Evaluate the radical.} \\ &= 6-3(-2) = 12 \end{aligned}$$

□

Example 6.26 Simplify each expression. Round your answer to hundredths.

a $\dfrac{8-2\sqrt{2}}{4}$

b $2+6\sqrt[3]{-25}$

Solution.

a Do not start with "8 − 2"! Evaluate $\sqrt{2}$ first, then multiply by 2, and subtract the result from 8. Once the numerator is simplified, divide by 4.

On a calculator, enter

$$\boxed{(}\ 8\ \boxed{-}\ 2\ \boxed{\sqrt{}}\ \boxed{)}\ \boxed{)}\ \boxed{\div}\ 4\ \boxed{\text{ENTER}}$$

and round to two decimal places: 1.29

b Evaluate the cube root, multiply by 6, then add the result to 2.

On a calculator, enter

$$2\ \boxed{+}\ 6\ \boxed{\text{MATH}}\ \boxed{4}\ \boxed{-}\ 25\ \boxed{\text{ENTER}}$$

and round to two decimal places: −15.54

□

Exercise

Checkpoint 6.27 Simplify each expression. Do not use a calculator!

a $\dfrac{36}{6+\sqrt{36}}$

b $10+2(3-\sqrt{169})$

c $\dfrac{3+\sqrt[3]{-729}}{6-\sqrt[3]{-27}}$

Answer.

a 3

b −10

c $\dfrac{-2}{3}$

Checkpoint 6.28 Simplify each expression. Round your answer to hundredths.

a $\dfrac{6+9\sqrt{3}}{3}$

b $-1-3\sqrt[3]{120}$

Answer.

a 7.20

b −15.80

Rational Exponents

1. Perform operations on fractions

When working with rational exponents, we will need to perform operations on fractions.

Examples

Example 6.29 Add $\dfrac{-3}{4}+\left(\dfrac{-5}{8}\right)$

Solution. The LCD for the fractions is 8, so we build the first fraction:

$$\frac{-3}{4}\cdot\frac{\mathbf{2}}{\mathbf{2}}=\frac{-6}{8}$$

Then we combine like fractions:

$$\frac{-6}{8}+\left(\frac{-5}{8}\right)=\frac{-6+(-5)}{8}=\frac{-11}{8}$$

□

Example 6.30 Subtract $\dfrac{-5}{6}-\left(\dfrac{-3}{4}\right)$

Solution. The LCD for the fractions is 12, so we build each fraction:

$$\frac{-5}{6}\cdot\frac{\mathbf{2}}{\mathbf{2}}=\frac{-10}{12};\qquad\frac{-3}{4}\cdot\frac{\mathbf{3}}{\mathbf{3}}=\frac{-9}{12}$$

Then we combine like fractions:

$$\frac{-10}{12}-\left(\frac{-9}{12}\right)=\frac{-10+9}{12}=\frac{-1}{12}$$

□

Example 6.31 Multiply $\dfrac{-2}{3}\left(\dfrac{5}{4}\right)$

Solution. We multiply numerators together, and multiply denominators together:

$$\frac{-2}{3}\left(\frac{5}{4}\right)=\frac{-2\cdot 5}{3\cdot 4}=\frac{-10}{12}$$

Then we reduce:

$$\frac{-10}{12}=\frac{-5\cdot\cancel{2}}{6\cdot\cancel{2}}=\frac{-5}{6}$$

□

Exercises

Checkpoint 6.32 Add $\dfrac{-3}{4}+\dfrac{1}{3}$

Answer. $\dfrac{-5}{12}$

Checkpoint 6.33 Subtract $\dfrac{3}{8}-\left(\dfrac{-1}{6}\right)$

Answer. $\dfrac{13}{24}$

Checkpoint 6.34 Multiply $\dfrac{3}{8}\cdot\left(\dfrac{-1}{6}\right)$

Answer. $\dfrac{-1}{16}$

2. Convert between fractions and decimals

Rational exponents may also be written in decimal form.

Examples

Example 6.35 Convert 0.016 to a common fraction.

Solution. The numerator of the fraction is 016, or 16. The last digit, 6, is in the thousandths place, so the denominator of the fraction is 1000. Thus, $0.016 = \dfrac{16}{1000}$. We can reduce this fraction by dividing top and bottom by 8:

$$\frac{16}{1000} = \frac{\cancel{8} \cdot 2}{\cancel{8} \cdot 125} = \frac{2}{125}$$

□

Example 6.36 Convert $\dfrac{5}{16}$ to a decimal fraction.

Solution. Using a calculator, divide 5 by 16:
$5 \boxed{\div} 16 = 0.3125$ □

Example 6.37 Convert $\dfrac{5}{11}$ to a decimal fraction.

Solution. Using a calculator, divide 5 by 11:
$5 \boxed{\div} 11 = 0.45454545...$

This is a nonterminating decimal, which we indicate by a repeater bar:

$$\frac{5}{11} = 0.45454545... = 0.\overline{45}$$

□

Exercises

Checkpoint 6.38 Convert 0.1062 to a common fraction.

Answer. $\dfrac{531}{5000}$

Checkpoint 6.39 Convert 2.08 to a common fraction.

Answer. $\dfrac{52}{25}$

Checkpoint 6.40 Convert $\dfrac{4}{15}$ to a decimal fraction.

Answer. $0.2\overline{6}$

3. Solve equations

To solve an equation of the form $x^n = k$, we can raise both sides to the reciprocal of the exponent:

$$\begin{aligned} (x^n)^{1/n} &= k^{1/n} \\ x &= k^{1/n} \end{aligned}$$

because $(x^n)^{1/n} = x^{n(1/n)} = x^1$.

Examples

Example 6.41 Solve $0.6x^4 = 578$. Round your answer to hundredths.

Solution. First, we isolate the power.

$$\begin{aligned} 0.6x^4 &= 578 && \text{Divide both sides by 0.6.} \\ x^4 &= 963.\overline{3} \end{aligned}$$

We raise both sides to the reciprocal of the power.

$$\begin{aligned} (x^4)^{1/4} &= (963.\overline{3})^{1/4} && \text{By the third law of exponents, } \boldsymbol{(x^4)^{1/4} = x}. \\ x &= 5.57 \end{aligned}$$

To evaluate $(963.\overline{3})^{1/4}$, enter ANS $\boxed{\wedge}$ .25 $\boxed{\text{ENTER}}$ □

Example 6.42 Solve $x^{2/3} - 4 = 60$.

Solution. First, we isolate the power.

$$\begin{aligned} x^{2/3} - 4 &= 60 && \text{Add 4 to both sides.} \\ x^{2/3} &= 64 \end{aligned}$$

We raise both sides to the reciprocal of the power.

$$\begin{aligned} \left(x^{2/3}\right)^{3/2} &= 64^{3/2} && \boldsymbol{64^{3/2} = \left(64^{1/2}\right)^3 = 8^3} \\ x &= 512 \end{aligned}$$

Or we can evaluate $64^{3/2}$ by entering 64 $\boxed{\wedge}$ 1.5 $\boxed{\text{ENTER}}$ □

Example 6.43 Solve $18x^{0.24} = 6.5$. Round your answer to thousandths.

Solution. First, we isolate the power.

$$\begin{aligned} 18x^{0.24} &= 6.5 && \text{Divide both sides by 18.} \\ x^{0.24} &= 0.36\overline{1} \end{aligned}$$

We raise both sides to the reciprocal of the power.

$$\begin{aligned} \left(x^{0.24}\right)^{1/0.24} &= (0.36\overline{1})^{1/0.24} \\ x &= 0.014 \end{aligned}$$

We evaluate $(0.36\overline{1})^{1/0.24}$ by entering ANS $\boxed{\wedge}$ $\boxed{(}$ 1 $\boxed{\div}$ .24 $\boxed{)}$ $\boxed{\text{ENTER}}$ □

Exercises

Checkpoint 6.44 Solve $4x^5 = 1825$. Round your answer to thousandths.

Answer. 3.403

Checkpoint 6.45 Solve $\dfrac{3}{4}x^{3/4} = 36$. Round your answer to thousandths.

Answer. 174.444

Checkpoint 6.46 Solve $0.2x^{1.4} + 1.8 = 12.3$. Round your answer to thousandths.

Answer. 16.931

Working with Radicals

1. Factor

To simplify a radical, we factor out the largest perfect square.

Examples

Example 6.47 Find the missiing factor.

a $60x^9 = 3x^3 \cdot \ \ ?$

b $9x^3 - 3x^9 \cdot \ \ ?$

Solution.

a We mentally divide $60x^9$ by $3x^3$ to find $\dfrac{60x^9}{3x^3} = 20x^6$. The missing factor is $20x^6$.

b We mentally divide each term by $3x^3$ to find $\dfrac{9x^3}{3x^3} = 3$ and $\dfrac{3x^9}{3x^3} = x^6$. The missing factor is $3 - x^6$.

□

Example 6.48 Factor out the largest perfect square.

a $108a^5b^2$

b $\dfrac{a^2 + 4a^4}{8}$

Solution.

a By trial and error, we find that 36 is the largest square that divides 108. From each power, we can factor out the power with the largest possible even exponent, namely a^4 and b^2. Thus, we factor out $36a^4b^2$ to find $108a^5b^2 = 36a^4b^2 \cdot 3a$.

b The largest even power that divides into both a^2 and a^4 is a^2, so we factor a^2 from the numerator:

$$a^2 + a^4 = a^2(1 + 4a^2)$$

The largest perfect square that divides into the denominator is 4. Thus, we factor out $\dfrac{a^2}{4}$ from the fraction to find

$$\frac{a^2 + 4a^4}{8} = \frac{a^2}{4} \cdot \frac{1 + 4a^2}{2}$$

□

Exercises

Checkpoint 6.49 Find the missiing factor.

a $16z^{16} + 4z^6 = 4z^4 \cdot \ \ ?$

b $\dfrac{20}{7}m^7 = 4m^6 \cdot \ \ ?$

Answer.

a $4z^{12} + z^2$

b $\dfrac{5}{7}m$

Checkpoint 6.50 Factor out the largest perfect square.

a $\dfrac{5k^5}{9n}$

b $32a^{10} - 48a^9$

Answer.

a $\dfrac{k^4}{9} \cdot \dfrac{5k}{n}$

b $16a^8(2a^2 - 3a)$

2. Apply properties of radicals

We have a Product Rule and a Quotient Rule for radicals.

Rules for Radicals.

- $\sqrt{ab} = \sqrt{a}\,\sqrt{b} \qquad \text{if } a,\ b \geq 0$
- $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}} \qquad \text{if } a \geq 0,\ b > 0$

Examples

Example 6.51 Decide whether each statement is true or false. Then use a calculator to verify your answer.

a $\sqrt{6} = \sqrt{2}\,\sqrt{3}$

b $\sqrt{6} = \sqrt{2} + \sqrt{4}$

Solution.

a Yes: we can multiply (or divide) radicals together, if they have the same index. You can check that $\sqrt{6} \approx 2.4495$, and

$$\sqrt{2}\,\sqrt{3} \approx (1.4142)(1.7321) = 2.4495$$

rounded to four decimal places.

b No: we cannot combine radicals with addition or subtraction. You can check that $\sqrt{6} \approx 2.4495$, but $\sqrt{2} + \sqrt{4} \approx 1.4142 + 2 = 3.4142$.

□

Example 6.52 Find and correct the error in each calculation.

a $\sqrt{36 + 64} \to 6 + 8$

b $\sqrt{3} + \sqrt{3} \to \sqrt{6}$

Solution.

a We cannot split radicals with addition or subtraction; we must follow the order of operations:

$$= \sqrt{36 + 64} = \sqrt{100} = 10$$

b We cannot combine radicals with addition or subtraction. However, we can add like terms:

$$\sqrt{3} + \sqrt{3} = 2\sqrt{3}$$

□

Exercises

Checkpoint 6.53 Decide whether each statement is true or false. Then use a calculator to verify your answer.

a $\sqrt{16} = \sqrt{18} - \sqrt{2}$

b $\sqrt{8} = \dfrac{\sqrt{72}}{\sqrt{9}}$

c $\sqrt{5} + \sqrt{5} = \sqrt{10}$

d $\sqrt{2}\,\sqrt{9} = \sqrt{18}$

Answer.

a False

b True

c False

d True

Checkpoint 6.54 Find and correct the error in each calculation.

a $\sqrt{25+5} \to 5 + \sqrt{5}$

b $\sqrt{10} + \sqrt{15} \to 5$

c $\sqrt{9+x^2} \to 3 + x$

d $\sqrt{a^2 - b^2} \to a - b$

Answer.

a $\sqrt{30}$

b cannot be simplified

c cannot be simplified

d cannot be simplified

3. Simplify radicals

We simplify square roots by factoring out any perfect squares.

Examples

Example 6.55 Simplify $\sqrt{45}$

Solution. The largest perfect square that divides evenly into 45 is 9, so we factor 45 as $9 \cdot 5$. We use the product rule to write

$$\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9}\,\sqrt{5}$$

Finally, we simplify to get

$$\sqrt{45} = \sqrt{9}\,\sqrt{5} = 3\sqrt{5}$$

□

Example 6.56 Simplify $\sqrt{20x^2y^3}$

Solution. The largest perfect square that divides 20 is 4. We write the radicand as the product of two factors, one containing the perfect square and the largest possible even powers of the variables. That is,

$$20x^2y^3 = 4x^2y^2 \cdot 5y$$

Then we write the radical as a product.

$$\sqrt{20x^2y^3} = \sqrt{4x^2y^2 \cdot 5y} = \sqrt{4x^2y^2}\,\sqrt{5y}$$

Finally, we simplify the first of the two factors to find

$$\sqrt{20x^2y^3} = \boldsymbol{\sqrt{4x^2y^2}}\,\sqrt{5y} = \boldsymbol{2xy}\,\sqrt{5y}$$

□

Exercises

Checkpoint 6.57 Simplify $\sqrt{75}$

Answer. $3\sqrt{5}$

Checkpoint 6.58 Simplify $\sqrt{72u^6v^9}$

Answer. $6u^3v^4\sqrt{2v}$

Radical Equations

1. Solve radical equations

To solve a simple radical equation, we raise both sides to the index of the radical.

Examples

Example 6.59 Solve $\sqrt{x-3}=4$

Solution. We square both sides of the equation to produce an equation without radicals.

$$\begin{aligned}\left(\sqrt{x-3}\right)^2 &= 4^2\\ x-3 &= 16\\ x &= 19\end{aligned}$$

You can check that $x=19$ satisfies the original equation. ☐

Example 6.60 Solve $-2\sqrt[3]{x-4}=-6$

Solution. We first isolate the cube root.

$$\begin{aligned}-2\sqrt[3]{x-4} &= -6 && \text{Divide both sides by } \mathbf{-2}.\\ \sqrt[3]{x-4} &= 3\end{aligned}$$

Next, we undo the cube root by cubing both sides of the equation.

$$\begin{aligned}\left(\sqrt[3]{x-4}\right)^3 &= 3^3\\ x-4 &= 27\end{aligned}$$

Finally, we add 4 to both sides to find the solution, $x=31$. We do not have to check for extraneous solutions when we cube both sides of an equation, but it is a good idea to check the solution for accuracy anyway.

Check: We substitute **31** for x into the left side of the equation.

$$2\sqrt[3]{\mathbf{31}-4}=-2\sqrt[3]{27}=-2(3)=-6$$

The solutions checks. ☐

Exercises

Checkpoint 6.61 Solve $\sqrt{x-6}=2$

Answer. $x=10$

Checkpoint 6.62 Solve $3\sqrt[3]{4x-1} = -15$

Answer. $x = -31$

2. Square binomials containing radicals

We may encounter binomials when squaring both sides of an equation.

Examples

Example 6.63 Expand $(\sqrt{x}-3)^2$

Solution. $(\sqrt{x}-3)^2 = (\sqrt{x}-3)(\sqrt{x}-3)$, so we apply "FOIL" to get

$$\begin{aligned}(\sqrt{x}-3)(\sqrt{x}-3) &= \sqrt{x}\sqrt{x} - 3\sqrt{x} - 3\sqrt{x} - 3(-3) && \text{Simplify.}\\ &= x - 6\sqrt{x} + 9\end{aligned}$$

□

Example 6.64 Expand $(8+\sqrt{t-2})^2$

Solution. We multiply $(8+\sqrt{t-2})(8+\sqrt{t-2})$ to get

$$\begin{aligned}&8\cdot 8 + 8\sqrt{t-2} + 8\sqrt{t-2} + \left(\sqrt{t-2}\right)\left(\sqrt{t-2}\right)\\ &= 64 + \sqrt{t-2} + (t+2)\\ &= 66 + 16\sqrt{t-2} + t\end{aligned}$$

□

Exercises

Checkpoint 6.65 Expand $\left(6-\sqrt{3a+1}\right)^2$

Answer. $37 - 12\sqrt{3a+1} + 3a$

Checkpoint 6.66 Expand $\left(2\sqrt{z+4}-5\right)^2$

Answer. $4z + 41 - 20\sqrt{z+4}$

3. Use absolute value

Examples

Example 6.67 Explain why $\sqrt{x^2} = x$ is not true for all values of x.

Solution. Recall that the symbol $\sqrt{a}$ means the *non-negative* square root of a. If x is a negative number, for example $x = -6$, then $x^2 = (-6)^2 = 36$, and not $\sqrt{x^2} = \sqrt{36} = 6$, not -6. So if x is a negative number, $\sqrt{x^2} \neq x$ In fact, $\sqrt{x^2} = |x|$. □

Example 6.68 For what values of x is $\sqrt{(x-5)^2} = x-5$?

Solution. $\sqrt{(x-5)^2} = x-5$ when $x-5$ is positive or zero, that is for $x \geq 5$. If $x < 5$, then $x-5$ is negative. But the $\sqrt{}$ symbol returns only the positive root, so we use absolute value bars to indicate that the root is positive:

$$\sqrt{(x-5)^2} = |x-5|$$

□

Exercises

Checkpoint 6.69 For what values of x is $\sqrt{(2x+8)^2} = 2x+8$?
Answer. $x \geq -4$

Checkpoint 6.70 For what values of x is $\sqrt{(x-9)^2} = 9-x$?
Answer. $x \leq 9$

Chapter 7

Exponential Functions

Exponential Growth and Decay

1. Compute percent increase and decrease

To calculate an increase of $r\%$, we write the percent as a decimal and multiply the old amount by $1 + r$. To calculate a decrease we multiply the old amount by $1 - r$.

Examples

Example 7.1 A loaf of bread cost \$3.00 last month, but this year the price rose by 6%. What should you multiply by to find the new price? What is the new price?

Solution. To get the new price, we multiply by 1.06 to get

$$1.06(3.00) = 3.18$$

The new price is \$3.18. □

Example 7.2 Priceco is offering a 15% discount off the regular price of \$180 for a ceiling fan. What should you multiply by to find the new price? What is the new price?

Solution. To get the new price, we multiply by $1 - 0.15$, or 0.85, to get

$$0.85(180) = 153$$

so the new price is \$153. □

Exercises

Checkpoint 7.3 Muriel's rent was increased by 8% from \$650 per month. What should you multiply by to find her new rent? What is her new rent?

Answer. 1.08, \$702

Checkpoint 7.4 A brand new SUV loses 18% of its value as soon as you drive it off the lot. If your SUV cost \$35,000, what should you multiply to find its new value? What is its new value?

Answer. 0.82, \$28,700

2. Use the order of operations

Recall that evaluating powers comes before multiplication in the order of operations.

Examples

Example 7.5 Simplify.
a $-4-2^3$ b $-4(-2)^3$ c $(-4-2)^3$

Solution.

a Compute 2^3 first, then subtract the result from -4:

$$-4-2^3 = -4-8 = -12$$

b Compute $(-2)^3$ first, then multiply the result by -4:

$$-4(-2)^3 = -4(-8) = 32$$

c Compute $(-4-2)$ first, then cube the result:

$$(-4-2)^3 = (-6)^3 = -216$$

□

Example 7.6 Evaluate for$x = 6$. Round your answers to hundredths.
a $12(1.05)^x$ b $12(1 + x/100)^5$

Solution.

a Follow the order of operations. Compute the power first:

$$12(1.05)^{6} = 12(1.3400956...) = 16.08$$

b Follow the order of operations. Compute the power first:

$$12(1 + 6/100)^5 = 12(1.06)^5 = 12(1.3382255...) = 16.06$$

□

Exercises

Checkpoint 7.7 Simplify. Round your answers to the nearest whole number.
a $450(1-0.12)^4$ b $180-80(1+0.25)^3$

Answer.

a 270

b 24

Checkpoint 7.8 Evaluate for $x = -3,\ y = -2$.
a $-2x^2 + y^3$ b $4(x-y)(x+2y)$

Answer.

a -26

b 28

3. Raise fractions to powers

Examples

Example 7.9 Complete the table of powers. As the exponent increases, do the powers increase or decrease?

a

x	1	2	3	4
$\left(\frac{2}{3}\right)^x$				

b

x	1	2	3	4
$\left(\frac{5}{4}\right)^x$				

Solution.

a

x	1	2	3	4
$\left(\frac{2}{3}\right)^x$	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$	$\frac{16}{81}$

When we multiply a number by $\frac{2}{3}$, the product is smaller than the original number, so the powers of $\frac{2}{3}$ decrease as the exponent increases. We can compare the powers more easily by converting the fractions to decimals, rounded to three places:

$$\frac{2}{3} = 0.667,\ \frac{4}{9} = 0.444,\ \frac{8}{27} = 0.296,\ \frac{16}{81} = 0.198$$

b

x	1	2	3	4
$\left(\frac{5}{4}\right)^x$	$\frac{5}{4}$	$\frac{25}{16}$	$\frac{125}{64}$	$\frac{625}{256}$

When we multiply a number by $\frac{5}{4}$, the product is larger than the original number, so the powers of $\frac{5}{4}$ increase as the exponent increases. We can compare the powers more easily by converting the fractions to decimals, rounded to three places:

$$\frac{5}{4} = 1.25,\ \frac{25}{16} = 1.563,\ \frac{125}{64} = 1.953,\ \frac{625}{256} = 2.441$$

□

Example 7.10 Complete the table of powers. As the exponent increases, do the powers increase or decrease?

a

x	1	2	3	4
0.2^x				

b

x	1	2	3	4
1.2^x				

Solution.

a

x	1	2	3	4
0.2^x	0.2	0.04	0.008	0.0016

The powers decrease.

b

x	1	2	3	4
1.2^x	1.2	1.44	1.728	2.0736

The powers increase.

□

Exercise

Checkpoint 7.11 Complete the table of powers. As the exponent increases, do the powers increase or decrease?

a

x	1	2	3	4
$\left(\frac{3}{4}\right)^x$				

b

x	1	2	3	4
$\left(\frac{4}{3}\right)^x$				

Answer.

a

x	1	2	3	4
$\left(\frac{3}{4}\right)^x$	$\frac{3}{4}$	$\frac{9}{16}$	$\frac{27}{64}$	$\frac{81}{256}$

Decrease

b

x	1	2	3	4
$\left(\frac{4}{3}\right)^x$	$\frac{4}{3}$	$\frac{16}{9}$	$\frac{64}{27}$	$\frac{257}{81}$

Increase

Checkpoint 7.12 Complete the table of powers. As the exponent increases, do the powers increase or decrease?

a

x	1	2	3	4
0.8^x				

b

x	1	2	3	4
1.5^x				

Answer.

a

x	1	2	3	4
0.8^x	0.8	0.64	0.512	0.4096

Decrease

b

x	1	2	3	4
1.5^x	1.5	2.25	3.375	5.0625

Increase

Exponential Functions

1. Evaluate exponential functions

Examples

Powers come before products in the order of operations, so to evaluate an exponential function $f(x) = ab^x$ we evaluate b^x before multiplying by a.

Example 7.13 Evaluate $f(x) = 8 \cdot 4^x$.

a $f(2)$

b $f(-2)$

c $f\left(\frac{1}{2}\right)$

d $f\left(-\frac{1}{2}\right)$

Solution. Follow the order of operations: compute powers before products.

a $f(2) = 8 \cdot 4^2 = 8 \cdot 16 = 128$

b $f(-2) = 8 \cdot 4^{-2} = 8 \cdot \dfrac{1}{16} = \dfrac{1}{2}$

c $f\left(\dfrac{1}{2}\right) = 8 \cdot 4^{1/2} = 8 \cdot 2 = 16$

d $f\left(-\dfrac{1}{2}\right) = 8 \cdot 4^{-1/2} = 8 \cdot \dfrac{1}{2} = 4$

□

Example 7.14 Evaluate $g(x) = 120(0.65)^x$. Round your answers to thousandths.

a $g(2.3)$

b $g(-1.8)$

c $g(0.4)$

d $g(-0.25)$

Solution. Follow the order of operations: use your calculator to compute powers before products. Do not round off at intermediate steps!

a $g(2.3) = 120(0.65)^{2.3} = 120(0.37127...) = 44.554$

b $g(-1.8) = 120(0.65)^{-1.8} = 120(2.17148...) = 260.578$

c $g(0.4) = 120(0.65)^{0.4} = 120(0.84171...) = 101.006$

d $g(-0.25) = 120(0.65)^{-0.25} = 120(1.11370...) = 133.645$

□

Exercises

Checkpoint 7.15 Evaluate each function. Give your answers as common fractions.

a $G(t) = 15(5)^t$. Find $G(-3)$

b $H(n) = 4\left(\dfrac{1}{27}\right)^n$. Find $H\left(\dfrac{2}{3}\right)$

c $F(x) = \dfrac{1}{2} \cdot 8^x$. Find $F\left(-\dfrac{1}{3}\right)$

Answer.

a $\dfrac{3}{25}$

b $\dfrac{4}{9}$

c $\dfrac{1}{4}$

Checkpoint 7.16 Evaluate each function. Round your answers to hundredths.

a $G(t) = 15(1.5)^t$. Find $G(-3)$

b $h(z) = 1.8(0.8)^z$. Find $h(4)$

c $F(w) = 2500(1.03)^w$. Find $F(25)$

Answer.

a 4.44

b 0.745

c 5234.44

2. Interpret function notation

The definitions of the variables help us interpret function notation.

Examples

Example 7.17 The number of students at Salt Creek Elementary School is growing according to the formula $f(t) = 500(1.08)^t$, where t is the number of years since the school opened in 2005.

a What does the equation $f(6) = 500(1.08)^6$ tell us about the school?

b Use function notation to say that the student population was 583 in 2007.

Solution.

a In this equation, $t = 6$ and $f(6) = 793$. In 2011 (six years after the school opened), the student population was 793.

b In 2007, $t = 2$, so $f(2) = 500(1.08)^2 = 583$.

□

Example 7.18 The value of Digicorp stock has been falling according to the formula $V(w) = 48(0.96)^w$, where w is the number of weeks since its peak value of \$48 per share.

a Use function notation to say that 8 weeks later the value of a share of Digicorp stock was \$34.63.

b What does the equation $V(12) = 48(0.96)^{12} = 29.41$ tell us about the stock?

Solution.

a We evaluate the function at $w = 8$ to get $V(8) = 48(0.96)^8 = 34.63$.

b In this equation, $w = 12$ and $V(12) = 29.41$, so 12 weeks after the peak value a share of Digicorp stock was worth \$29.41.

□

Exercises

Checkpoint 7.19 The number of internet users in the United States is given by $I(t) = 95,331,000(1.09)^t$, where $t = 0$ in 2000. Use function notation to say that the number of internet users in 2005 was 146,679,000.

Answer. $I(t) = 146,679,000$

Checkpoint 7.20 The percent of U.S. households that maintain a landline telephone is decreasing according to the formula $L(t) = 95(0.96)^t$, where $t = 0$ in 2004. What does the equation $L(t) = 95(0.96)^{10} = 63$ tell us about landlines?

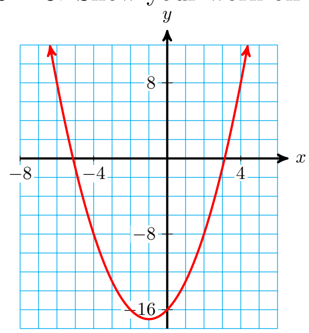
Answer. In 2014, 63% of households maintained a landline.

3. Solve equations graphically

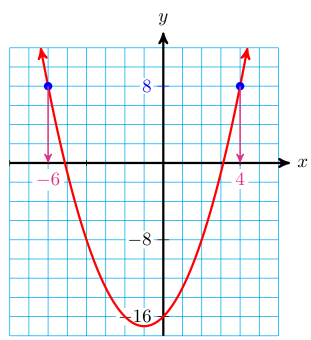
We first solve equations graphically in Section 1.3, so you might want to review that section.

Examples

Example 7.21 Here is a graph of $f(x) = x^2 + 2x - 16$. Use the graph to solve the equation $x^2 + 2x - 16 = 8$. Show your work on the graph.



Solution. To solve the equation, we want to find x-values that produce a function value of 8. The vertical coordinate of each point on the graph is given by the function value, $f(x)$. So we look for points on the graph with vertical coordinate $f(x) = 8$.

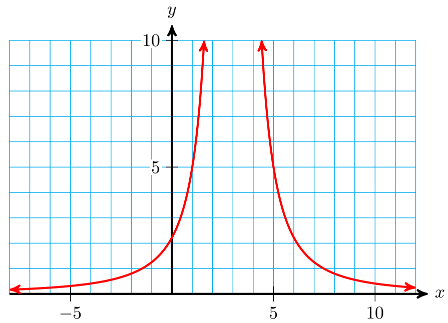


There are two such points, $(-6, 8)$ and $(4, 8)$. Those points tell us that $f(-6) = 8$ and $f(4) = 8$. Thus, the x-coordinates of the points, namely -6 and 4, are the solutions. To check algebraically, we can verify that $f(-6) = 8$ and $f(4) = 8$:

$$f(\mathbf{-6}) = (\mathbf{-6})^2 + 2(\mathbf{-6}) - 16 = 36 - 12 - 16 = 8$$
$$f(\mathbf{4}) = \mathbf{4}^2 + 2(\mathbf{4}) - 16 = 16 + 8 - 16 = 8$$

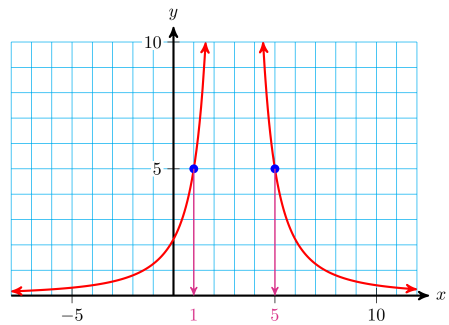
□

Example 7.22 Here is a graph of $G(x) = \dfrac{20}{(x-3)^2}$. Use the graph to solve the equation $\dfrac{20}{(x-3)^2} = 5$. Show your work on the graph.



Solution.

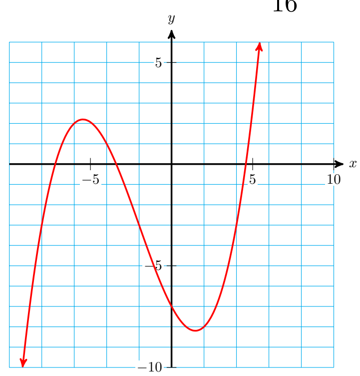
We find any points on the graph with vertical coordinate $G(x) = 5$. There are two points, $(1, 5)$ and $(5, 5.)$ The x-coordinates of those points, namely 1 and 5, are the solutions.



□

Exercise

Checkpoint 7.23 Here is a graph of $F(x) = \frac{1}{16}x^3 + \frac{3}{8}x^2 - \frac{3}{2} - 7$



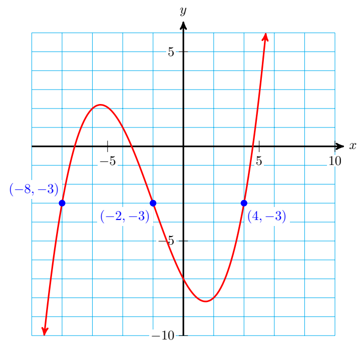
Use the graph to solve the equation

$$\frac{1}{16}x^3 + \frac{3}{8}x^2 - \frac{3}{2} - 7 = -3$$

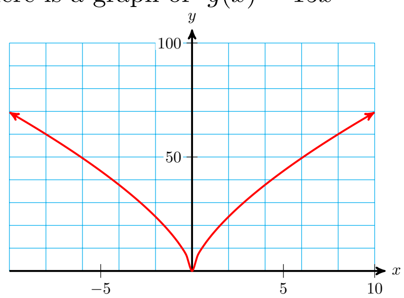
Show your work on the graph.

Answer.

$-8, -2, 4$



Checkpoint 7.24 Here is a graph of $g(x) = 15x^{2/3}$



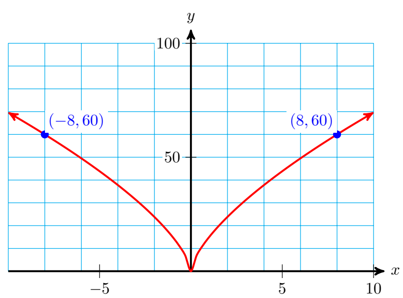
Use the graph to solve the equation

$$15x^{2/3} = 60$$

Show your work on the graph.

Answer.

$-8,\ 8$



Logarithms

1. Convert between radicals and powers

Examples

Because a logarithm is an exponent, it is helpful to convert easily between radical notation and exponent noatation.

Example 7.25 Write each power as a radical.

a $x^{2/3}$

b $t^{-3/2}$

c $w^{1.25}$

d $z^{-0.4}$

Solution. Recall that the numerator of the exponent is the power and the denominator is the root. A negative exponent indicates a reciprocal.

a $x^{2/3} = \sqrt[3]{x^2}$

b $t^{-3/2} = \dfrac{1}{t^{3/2}} = \dfrac{1}{\sqrt{t^3}}$

c $w^{1.25} = w^{5/4} = \sqrt[4]{w^5}$

d $z^{-0.4} = \dfrac{1}{z^{4/10}} = \dfrac{1}{z^{2/5}} = \dfrac{1}{\sqrt[5]{z^2}}$

□

Example 7.26 Write each radical expression in exponential form and simplify.

a $\sqrt[4]{b^3}$

b $\dfrac{1}{\sqrt[6]{a^3}}$

c $x^2\sqrt[4]{x}$

d $\dfrac{\sqrt[3]{v}}{\sqrt{v}}$

Solution.

a $\sqrt[4]{b^3} = b^{3/4}$

b $\dfrac{1}{\sqrt[6]{a^3}} = \dfrac{1}{a^{3/6}} = a^{-3/6} = a^{-1/2}$

c $x^2\sqrt[4]{x} = x^2x^{1/4} = a^{2+1/4} = a^{9/4}$

d $\dfrac{\sqrt[3]{v}}{\sqrt{v}} = \dfrac{v^{1/3}}{v^{1/2}} = v^{1/3-1/2} = v^{-1/6}$

□

Exercises

Checkpoint 7.27 Write each power as a radical.

a $m^{-3/5}$

b $p^{2.75}$

c $x^{0.18}$

Answer.

a $\dfrac{1}{\sqrt[5]{m^3}}$

b $\sqrt[4]{p^{11}}$

c $\sqrt[50]{x^9}$

Checkpoint 7.28 Write each radical expression in exponential form and simplify.

a $\sqrt[10]{n^9}$

b $\sqrt{h}\sqrt[4]{h}$

c $\left(\sqrt[3]{t^2}\right)^4$

Answer.

a $n^{9/10}$

b $h^{3/4}$

c $t^{8/3}$

2. Find an unknown exponent

If we can write both sides of an equation as powers with the same base, we can equate the exponents.

Examples

Example 7.29 Find the value of the exponent.

a $3^x = 81$

b $5^x = \dfrac{1}{125}$

c $\left(\dfrac{3}{4}\right)^x = \dfrac{16}{9}$

d $64^x = 16$

Solution.

a We can write both sides with base 3.

$$81 = 3^4, \quad \text{so} \quad x = 4.$$

b We can write both sides with base 5.

$$125 = 5^3, \quad \text{so} \quad 5^{-3} = \frac{1}{125}, \quad \text{and} \quad x = -3.$$

c $\left(\dfrac{3}{4}\right)^2 = \dfrac{9}{16}$, so $\left(\dfrac{3}{4}\right)^{-2} = \dfrac{16}{9}$, and $x = -2$.

d We can write both sides with base 4.

$$64 = 4^3 \quad \text{and} \quad 16 = 4^2,$$

so

$$\begin{aligned} (4^3)^x &= 4^2 && \text{Multiply exponents.} \\ 4^{3x} &= 4^2 && \text{Equate exponents.} \\ 3x &= 2 \\ x &= \frac{2}{3} \end{aligned}$$

□

Example 7.30 By using trial and error, estimate the value of the exponent to the nearest tenth.

a $2^x = 15$

b $3^x = 65$

c $10^x = 0.03$

d $0.5^x = 0.20$

Solution.

a $2^4 = 16$, so we try a slightly smaller exponent and find that $2^{3.9} = 14.9285$, so $x \approx 3.9$.

b 65 is between $3^3 = 27$ and $3^4 = 81$, so x must be between 3 and 4. By trying exponents 3.1, 3.2, 3.3, and so on, we find that $3^{3.8} = 65.022$, so $x \approx 3.8$.

c $10^{-1} = 0.1$ and $10^{-2} = 0.01$, so $-2 < x < -1$. By trying exponents between -2 and -1, we find that $10^{-1.5} = 0.0316$, so $x \approx -1.5$.

d $0.5^2 = 0.25$, and as we increase the exponent on 0.5, the result will be smaller. By trial and error we find that $0.5^{2.3} = 0.2031$, so $x \approx 2.3$.

□

Exercises

Checkpoint 7.31 Find the value of the exponent.

a $2^x = \dfrac{1}{1024}$

b $125^x = 25$

Answer.

a 10

b $\dfrac{2}{3}$

Checkpoint 7.32 By using trial and error, estimate the value of the exponent to the nearest tenth.

a $10^x = 50$

b $1.08^x = 1.5$

Answer.

a 1.7

b 5.3

3. Apply the laws of exponents

The laws of exponents still apply to variable exponents. (If you would like to review the laws of exponents, they are listed in Section 3.2.)

Examples

Example 7.33 Use the laws of exponents to simplify.

a $1.35^6(1.35^4)$

b $0.64^5(0.64^n)$

Solution. When multiplying two powers with the same base, we add the exponents. Notice that the base does not change.

a $1.35^6(1.35^4) = 1.35^{6+4} = 1.35^{10}$

b $0.64^5(0.64^n) = 0.64^{5+n}$

□

Example 7.34 Use the laws of exponents to simplify.

a $\dfrac{0.32^8}{0.32^2}$

b $\dfrac{0.32^t}{0.32^x}$

Solution. When dividing two powers with the same base, we subtract the exponents.

a $\dfrac{0.32^8}{0.32^2} = 0.32^{8-2} = 0.32^6$

b $\dfrac{0.32^t}{0.32^x} = 0.32^{t-x}$

□

Example 7.35 Use the laws of exponents to simplify.

a $\left(1.07^5\right)^3$ b $\left(1.07^4\right)^p$

Solution. When raising a power to a power, we multiply the exponents.

a $\left(1.07^5\right)^3 = 1.07^{15}$

b $\left(1.07^4\right)^p = 1.07^{4p}$

□

Exercise

Checkpoint 7.36 Use the laws of exponents to simplify $2.5^{2t}(2.5^3)$.

Answer. 2.5^{2t+3}

Checkpoint 7.37 Use the laws of exponents to simplify $\left(0.94^4\right)^{m-2}$.

Answer. 0.94^{4m-8}

Checkpoint 7.38 Use the laws of exponents to simplify $\dfrac{1.13^{8x}}{1.13^{5x}}$.

Answer. 1.13^{3x}

Properties of Logarithms

1. Apply the distributive law

We have met several types of algebraic properties before treating logarithms. Here is a review of the most common ones.

Example

Example 7.39 Which equation is a correct application of the distributive law?

a $2(5 \cdot 3^x) = 10 \cdot 6^x$ or $2(5 + 3^x) = 10 + 2 \cdot 3^x$

b $\log(x + 10) = \log x + \log 10$ or $\dfrac{1}{x}(x + 10) = 1 + \dfrac{10}{x}$

Solution.

a The distributive law applies to multiplying a sum or difference, not a product. In the first equation, $5 \cdot 3^x$ is a product, so the distributive law does not apply. (We can, however, simplfy that expression with the associative law:

$$2(5 \cdot 3^x) = (2 \cdot 5) \cdot 3^x = 10 \cdot 3^x$$

The second equation is a correct application of the distributive law. You can check that the first equation is false and the second equation is true by substituting $x = 1$.

b The distributive law applies only to multiplying a sum or product, not to other operations, such as taking logs. You can check that the first equation is false by substituting $x = 10$.

The second equation is a correct application of the distributive law.

□

Exercises

Decide whether each equation is a correct application of the distributive law. Write a correct statement if possible.

Checkpoint 7.40 $\dfrac{x+6}{3} \to \dfrac{x}{3} + \dfrac{6}{3}$

Answer. Correct

Checkpoint 7.41 $\dfrac{6}{x+3} \to \dfrac{6}{x} + \dfrac{6}{3}$

Answer. Not correct

Checkpoint 7.42 $2(P_0a^t) \to 2P_0 + 2a^t$

Answer. Not correct. $2(P_0a^t) = 2P_0a^t$

Checkpoint 7.43 $25(1 + r)^8 \to (25 + 25r)^8$

Answer. Not correct

2. Apply the laws of exponents

Be careful to avoid tempting but false operations with exponents.

Example

Example 7.44 Which equation is a correct application of the laws of exponents?

a $20(1 + r)^4 = 20 + 20r^4$ or $(ab^t)^3 = a^3b^{3t}$

b $2^{t/5} = (2^{1/5})^t$ or $6.8(10)^t = 68^t$

Solution.

a The first statement is not correct. There is no law that says $(a + b)^n$ is equivalent to $a^n + b^n$, so $(1 + r)^4$ is not equivalent to $1^4 + r^4$ or $1 + r^4$.

However, it is true that $(ab)^n = a^n b^n$, so in particular the second statement is true:

$$(ab^t)^3 = a^3(b^t)^3 = a^3b^{3t}$$

b The first statement is correct. If we start with $(2^{1/5})^t$, we can apply the third law, $(a^m)^n = a^{mn}$, to find

$$(2^{1/5})^t = 2^{(1/5)t} = 2^{t/5}.$$

In the second statement, 6.8 is not raised to power t, so we cannot multiply 6.8 times 10.

□

Exercises

Decide whether each equation is a correct application of the laws of exponents. Write a correct statement if possible.

Checkpoint 7.45 $P(1-r)^6 \to P - Pr^6$

Answer. Not correct

Checkpoint 7.46 $25(2^t)\cdot 4(2^t) \to 100\cdot 2^{t^2}$

Answer. Not correct. $25(2^t)\cdot 4(2^t) = 100(2^{2t})$

Checkpoint 7.47 $a\left(b^{1/8}\right)^{2t} \to ab^{t/4}$

Answer. Correct

Checkpoint 7.48 $N(0.94)^{1/8.3} \to \dfrac{N}{(0.94)^{8.3}}$

Answer. Not correct, but $N(0.94)^{-8.3} = \dfrac{N}{(0.94)^{8.3}}$

3. Apply the properties of radicals

Rules for Radicals.

Product Rule

$$\sqrt[n]{ab} = \sqrt[n]{a}\ \sqrt[n]{b} \qquad \text{for } a, b \geq 0$$

Quotient Rule

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad \text{for } a \geq 0,\quad b > 0$$

In general, it is *not* true that $\sqrt[n]{a+b}$ is equivalent to $\sqrt[n]{a} + \sqrt[n]{b}$, or that $\sqrt[n]{a-b}$ is equivalent to $\sqrt[n]{a} - \sqrt[n]{b}$.

Examples

Example 7.49 Which equation is a correct application of the properties of radicals?

a $\sqrt{x^4+81} = x^2+9$ or $\sqrt[3]{P^2}\ \sqrt[3]{1+r} = \sqrt[3]{P^2(1+r)}$

b $\dfrac{\sqrt{x+y}}{\sqrt{x}} = \sqrt{y}$ or $\dfrac{x+y}{\sqrt{x+y}} = \sqrt{x+y}$

Solution.

a The first statement is incorrect. There is no property that says $\sqrt[n]{a+b} = \sqrt[n]{a} + \sqrt[n]{b}$.

However, it is true that $\sqrt[n]{a}\ \sqrt[n]{b} = \sqrt[n]{ab}$, so the second statement is correct.

b The first statement is incorrect, because $\dfrac{x+y}{x}$ is not equivalent to y.

The second statement is correct, because $\sqrt{x+y}\ \sqrt{x+y} = x+y$.

□

Exercises

Decide whether each equation is a correct application of the properties of radicals. Write a correct statement if possible.

Checkpoint 7.50 $\sqrt[4]{a^2-a^4} \to \sqrt[4]{a^2} - a$

Answer. Not correct

Checkpoint 7.51 $\sqrt{b^4-16} \to \sqrt{b^2-4}\sqrt{b^2+4}$

Answer. Correct

Checkpoint 7.52 $\sqrt[3]{t^4} + \sqrt[3]{t^4} \to \sqrt[3]{2t^4}$

Answer. Not correct. $\sqrt[3]{t^4} + \sqrt[3]{t^4} = 2\sqrt[3]{t^4}$

Checkpoint 7.53 $\dfrac{\sqrt{2p}}{\sqrt{4p+8p^2}} \to \dfrac{1}{\sqrt{2+4p}}$

Answer. Correct

Exponential Models

1. Solve power and exponential equations

Compare the procedures for solving power equations and exponential equations.

Examples

Example 7.54 Solve $3x^{1.05} = 18$. Round your answer to hundredths.

Solution. This is a power equation. We divide both sides by 3 to isolate the variable, then raise both sides to the reciprocal of the exponent.

$$(x^{1.05})^{1/1.05} = 6^{1/1.05}$$
$$x = 5.51$$

□

Example 7.55 Solve $3(1.05)^x = 18$. Round your answer to hundredths.

Solution. This is an exponential equation. We divide both sides by 3, then take logarithms.

$$\begin{aligned} \log(1.05^x) &= \log 6 && \text{Apply the third log property.} \\ x \log 1.05 &= \log 6 \\ x &= \frac{\log 6}{\log 1.05} = 36.72 \end{aligned}$$

□

Example 7.56 Solve $9x^{3/5} = 36$. Round your answer to hundredths.

Solution. This is a power equation. We divide both sides by 9 to isolate the variable, then raise both sides to the reciprocal of the exponent.

$$\begin{aligned} (x^{3/5})^{5/3} &= 4^{5/3} \\ x &= 10.08 \end{aligned}$$

□

Example 7.57 Solve $1.5(3^{x/5}) = 12$. Round your answer to hundredths.

Solution. This is an exponential equation. We divide both sides by 1.5, then take logarithms.

$$\begin{aligned} \log 3^{x/5} &= \log 8 && \text{Apply the third log property.} \\ \frac{x}{5} \log 3 &= \log 8 \\ x &= \frac{5 \log 8}{\log 3} = 9.46 \end{aligned}$$

□

Exercises

Checkpoint 7.58 Solve $6x^{3/4} - 8 = 76$. Round your answer to hundredths.

Answer. 33.74

Checkpoint 7.59 Solve $6\left(\frac{3}{4}\right)^x - 8 = 76$. Round your answer to hundredths.

Answer. -9.17

Checkpoint 7.60 Solve $13.2(1.36)^x = 284.8$. Round your answer to hundredths.

Answer. 9.99

Checkpoint 7.61 Solve $13.2x^{1.26} = 284.8$. Round your answer to hundredths.

Answer. 11.45

2. Calculate gowth and decay rates

Doubling Time and Half-Life.

If D is the doubling time for an exponential function $P(t)$, then

$$P(t) = P_0 2^{t/D}$$

If H is the half-life for an exponential function $Q(t)$, then

$$Q(t) = Q_0(0.5)^{t/H}$$

Examples

Example 7.62 The half-life of a cold medication in the body is 6 hours. Find its decay rate.

Solution. The decay law for the medication is

$$N = N_0(0.5)^{t/8}$$

We can rewrite this expression as

$$N = N_0(0.5^{1/8})^t$$

so $b = 0.5^{1/8} = 0.9170$, and $r = 1 - b = 0.083$. The decay rate is 8.3%. □

Example 7.63 The growth rate of a population of badgers is 3.8% per year. Find its doubling time.

Solution. The growth law for the population is $P = P_0(1.038)^t$. We set $P = 2P_0$ and solve for t.

$$\begin{aligned} 2P_0 &= P_0(1.038)^t && \text{Divide both sides by } \boldsymbol{P_0}\text{.} \\ 2 &= (1.038)^t && \text{Take the log of both sides.} \\ \log 2 &= t \log 1.038 && \text{Apply the third log property.} \\ t &= \frac{\log 2}{\log 1.038} = 18.59 \end{aligned}$$

The doubling time is 18.59 years. □

Exercises

Checkpoint 7.64 The doubling time for a population is 18 years. Find its annual growth rate.

Answer. 3.9%

Checkpoint 7.65 A radioactive isotope decays by 0.04% per second. What is its half-life?

Answer. 4.81 hrs

3. Analyze graphs of exponential functions

From a graph, we can read the initial value of an exponential function and then its doubling time or hlaf-life. From there we can calculate the growth or decay law.

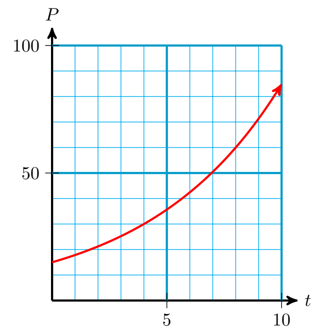
Examples

Example 7.66 The graph shows the population, P, of a herd of llamas t years after 2000.

a How many llamas were there in 2000?

b What is the doubling time for the population?

c What is the annual growth rate for the population?



Solution.

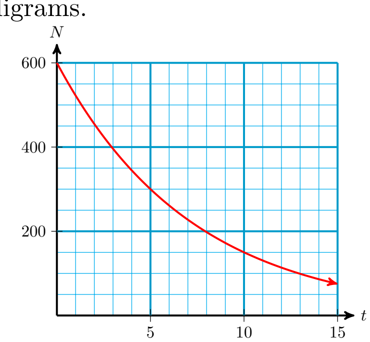
a The initial value of the population is given by the P-intercept of the graph, $(0, 15)$. There were 15 llamas in 2000.

b Look for the time when the initial llama population doubles. When $t = 4,\ P = 30$, and when $t = 8,\ P = 60$, so the llama population doubles every 4 years.

c The growth factor for the population is $2^{1/4} = 1.189$, so the annual growth rate is 18.9%.

□

Example 7.67 Write a decay law for the graph shown below, where t is in hours and N is in milligrams.



Solution. The initial value is given by the vertical intercept of the graph, $(0, 600)$, so $N_0 = 600$.

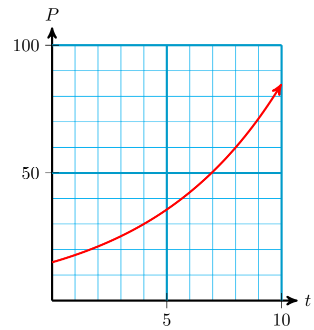
When $t = 5,\ N = 300$, so the half-life of the substance is 5 hours. Thus the decay law is $N(t) = 600(0.5)^{t/5}$, or $N(t) = 600(0.87)^t$. □

Exercises

Checkpoint 7.68

a Write a growth law for the population whose graph is shown, where t is in years.

b What is the annual growth rate for the population?



Answer.

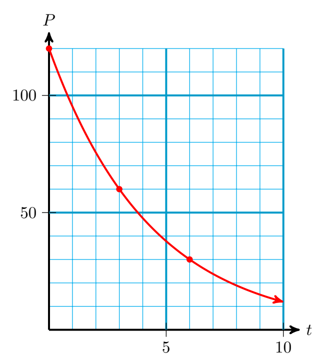
a $P(t) = 10(2^{t/2.5})$

b 32.0%

Checkpoint 7.69

a Write a decay law for the population whose graph is shown, where t is in days.

b What is the daily decay rate for the population?



Answer.

a $P(t) = 120(0.5^{t/3})$

b 20.6%

Chapter 8

Polynomial and Rational Functions

Polynomial Functions

1. Compute sums and products

Compare the rules for simplilfying products to the rules for simplifying sums.

Examples

Example 8.1 Simplify each expression if possible.
a $3x^2 - 5x^3$ b $3x^2(-5x^3)$

Solution.

a This expression is a difference of terms, but they are not like terms (because the variable has different exponents), so we cannot combine them.

b This expression is a product, and the powers have the same base, so we can apply the first law of exponents to get $3x^2(-5x^3) = -15x^5$.

□

Example 8.2 Simplify each expression if possible.
a $-6t^4 - 8t^4$ b $-6t^4(-8t^4)$

Solution.

a This expression is a difference of like terms, so we can combine their coefficients to get $-6t^4 - 8t^4 = -14t^4$

b This expression is a product, and the powers have the same base, so we can apply the first law of exponents to get $-6t^4(-8t^4) = 48t^8$

□

Exercises

Checkpoint 8.3 Simplify each expression if possible.
a $2a^2 - 9a^3 + a^2$
b $2a^2(-9a^3 + a^2)$

Answer.

a $3a^2 - 9a^3$

b $-18a^5 + 2a^4$

Checkpoint 8.4 Simplify each expression if possible.
a $7 - 4a^3 + 2a^3$
b $7 - 4a^2(2a^3)$

Answer.

a $7 - 2a^3$

b $7 - 8a^5$

2. Use formulas

There are several useful formulas for simplifying polynomials.

Examples

Example 8.5 If $a = 5t^4$, find a^3 and $3a^2$.
Solution. We substitute $5t^4$ for a to find

$$a^3 = (\boldsymbol{5t^4})^3 = 5^3(t^4)^3 = 125t^{12} \qquad \text{Apply the third law of exponents.}$$
$$3a^2 = 3(\boldsymbol{5t^4})^2 = 3 \cdot 5^2(t^4)^2 = 75t^8$$

□

Example 8.6 If $a = 2y$ and $b = -3z^2$, find b^3 and $3a^2b$.
Solution. We substitute $2y$ for a and $-3z^2$ for b to find

$$b^3 = (\boldsymbol{-3z^2})^3 = (-3)^3(z^2)^3 = -27z^6$$
$$3a^2b = 3(\boldsymbol{2y})^2(\boldsymbol{-3z^2}) = 3(4y^2)(-3z^2) = -36y^2z^2$$

□

Exercises

Checkpoint 8.7 If $a = -4x^3$ and $b = 3h$, find a^3 and ab^2.
Answer. $-64x^9$; $-36x^3h^2$

Checkpoint 8.8 If $x = 6p^2$ and $y = mq^2$, find y^3 and x^2y.
Answer. m^3q^6; $-36mp^4q^2$

3. Square binomials

Sometimes it is easier to use formulas to square binomials.

Special Products of Binomials.

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$
$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

Examples

Example 8.9 Use the identity $(a+b)^2 = a^2+2ab+b^2$ to expand $(3h^2+4k^3)^2$.

Solution. We substitute $3h^2$ for a and $4k^3$ for b into the identity.

$$\begin{aligned}(3h^2+4k^3)^2 &= (3h^2)^2 + 2(3h^2)(4k^3) + (4k^3)^2 \\ &= 9h^4 + 24h^2k^3 + 16k^6\end{aligned}$$

□

Example 8.10 Use the identity $(a-b)^2 = a^2-2ab+b^2$ to expand $(2xy^2-5)^2$.

Solution. We substitute $2xy^2$ for a and 5 for b into the identity.

$$\begin{aligned}(2xy^2-5)^2 &= (2xy^2)^2 - 2(2xy^2)(5) + 5^2 \\ &= 4x^2y^4 - 20xy^2 + 25\end{aligned}$$

□

Exercises

Checkpoint 8.11 Expand $(8w^4 - 3w^3)^2$

Answer. $64w^8 - 48w^7 + 9w^6$

Checkpoint 8.12 Expand $(a^3b + 9ab^3)^2$

Answer. $a^6b^2 + 18a^4b^4 + 81a^2b^6$

Algebraic Fractions

1. Factor a polynomial

To reduce an algebraic fraction, we must factor its numerator and denominator.

Examples

Example 8.13 Factor.

a $4x^2 - 4x$ b $4x^2 - 1$ c $4x^2 - 4x + 2$

Solution.

a We factor out a common factor of $4x$ to get $4x(x-1)$.

b This is a difference of two squares that factors as $(2x-1)(2x+1)$.

c This is the square of a binomial, $(2x-1)^2$.

□

Example 8.14 Factor.
a $27a^2 - 3$ b $27a^3 - 1$ c $81a^3 - a$

Solution.

a We first factor out 3 to find $3(9a^2 - 1)$, then factor the difference of two squares to get $3(3a - 1)(3a + 1)$.

b This is a differece of two cubes, which factors as $(3a - 1)(9a^2 + 3a + 1)$.

c We first factor out a to get $a(81a^2 - 1)$, then factor the difference of two squares to get $a(9a - 1)(9a + 1)$.

□

Exercises

Checkpoint 8.15 Factor completely $2x^3 + 16y^3$
Answer. $2(x + 2y)(x^2 - 2xy + 4y^2)$

Checkpoint 8.16 Factor completely $4x^2y - 36y^3$
Answer. $4y(x - 3y)(x + 3y)$

Checkpoint 8.17 Factor completely $2b^3 - 6b^2 - 36b$
Answer. $2b(b - 6)(b + 3)$

Checkpoint 8.18 Factor completely $9b^4 + 9b^2$
Answer. $9b^2(b^2 + 1)$

2. Find the opposite of a binomial

To find the opposite or negative of a binomial we multiply by -1.

Examples

Example 8.19 Which of these is the opposite of $m^2 - p$?
a $m^2 + p$ b $m - p^2$ c $p - m^2$

Solution. The opposite of $m^2 - p$ is $-(m^2 - p) = -m^2 + p$, or $p - m^2$. □

Example 8.20 Which of these pairs of binomials are opposites?

a $3c - 5$ and $5 + 3c$

b $5 - 3c$ and $3 - 5c$

c $5c - 3$ and $3 - 5c$

Solution. The opposite of $5c - 3$ is $-(5c - 3) = -5c + 3$, or $3 - 5c$, so (c) is correct. □

Exercises

Checkpoint 8.21 Find the opposite of the binomial $2x + 1$
Answer. $-2x - 1$

Checkpoint 8.22 Find the opposite of the binomial $b^2 - b$
Answer. $b - b^2$

Checkpoint 8.23 Find the opposite of the binomial $-4n + 8$

Answer. $4n - 8$

Checkpoint 8.24 Find the opposite of the binomial $-3z^2 - 2$

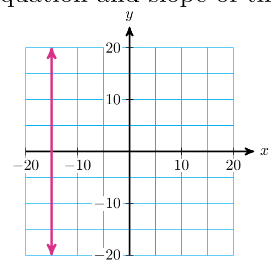
Answer. $3z^2 + 2$

3. Use horizontal and vertical lines

The asymptotes of rational functions are horizontal and vertical lines.

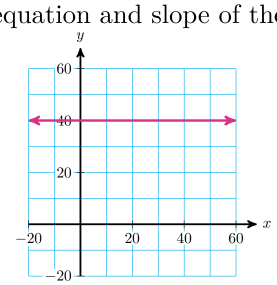
Examples

Example 8.25 Give the equation and slope of the line.



Solution. This line is a vertical line. All the points on the line have x-coordinate -15, so the equation of the line is $x = -15$. Because $\Delta x = 0$ between any two points on the line, its slope is undefined. □

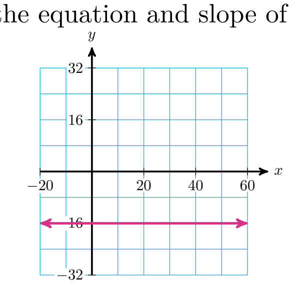
Example 8.26 Give the equation and slope of the line.



Solution. This line is a horizontal line. All the points on the line have y-coordinate 04, so the equation of the line is $y = 40$. Because $\Delta y = 0$ between any two points on the line, its slope is 0. □

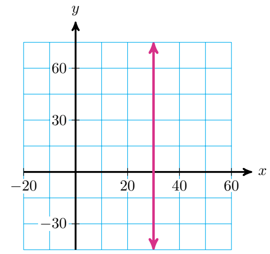
Exercises

Checkpoint 8.27 Give the equation and slope of the line.



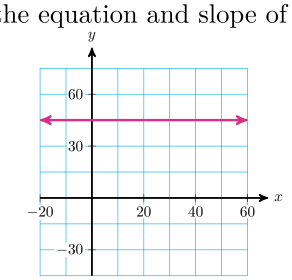
Answer. $y = -16;\ m = 0$

Checkpoint 8.28 Give the equation and slope of the line.



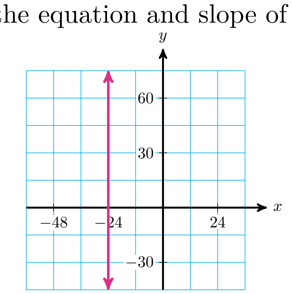
Answer. $x = 30$; m is undefined

Checkpoint 8.29 Give the equation and slope of the line.



Answer. $y = 45; m = 0$

Checkpoint 8.30 Give the equation and slope of the line.



Answer. $x = -24$; m is undefined

Operations on Algebraic Fractions

1. Use improper fractions

An improper fraction is one in which the numerator is larger than the denominator.

Examples

Example 8.31 Write an improper fraction for the sum $2 + \dfrac{3}{8}$

Solution. We write the whole number with the same denominator as the fraction. Thus,

$$\mathbf{2} + \frac{3}{8} = \frac{\mathbf{16}}{\mathbf{8}} + \frac{3}{8} = \frac{16+3}{8} = \frac{19}{8}$$

□

Example 8.32 Write an improper fraction for the difference $3 - \dfrac{1}{5}$

Solution. We write the whole number with the same denominator as the fraction. Thus,

$$\mathbf{3} - \frac{1}{5} = \frac{\mathbf{15}}{\mathbf{5}} - \frac{1}{5} = \frac{15-1}{5} = \frac{14}{5}$$

□

Exercises

Checkpoint 8.33 Write an improper fraction for each sum or difference.

a $1 + \dfrac{3}{4}$

b $2 - \dfrac{3}{10}$

Answer.

a $\dfrac{7}{4}$

b $\dfrac{17}{10}$

Checkpoint 8.34 Write an improper fraction for each sum or difference.

a $1 - \dfrac{23}{100}$

b $3 + \dfrac{3}{25}$

Answer.

a $\dfrac{77}{100}$

b $\dfrac{78}{25}$

2. Find an LCD

The first step in adding unlike fractions is to find the lowest common denominator, or LCD.

Examples

Example 8.35 Find the LCD for the fractions $\dfrac{2}{5} + \dfrac{3}{10}$

Solution. We factor each denominator and line up any common factors vertically. We use one factor from each column in the LCD.

$$\begin{aligned} 15 &= \mathbf{3} \cdot 5 \\ 10 &= \phantom{3 \cdot{}} \mathbf{5} \cdot \mathbf{2} \end{aligned}$$

The LCD is $3 \cdot 5 \cdot 2$, or 30. □

Example 8.36 Find the LCD for the fractions $\dfrac{5}{6a} - \dfrac{2}{9a^2}$

Solution. We factor each denominator and line up any common factors vertically. We use one factor from each column in the LCD.

$$\begin{aligned} 6a &= \mathbf{2} \cdot 3 \cdot a \\ 9a^2 &= \phantom{2 \cdot{}} \mathbf{3} \cdot \mathbf{3} \cdot \boldsymbol{a} \cdot \boldsymbol{a} \end{aligned}$$

The LCd is $2 \cdot 3 \cdot 3 \cdot a \cdot a$, or $18a^2$. □

Exercises

Checkpoint 8.37 Find the LCD for the fractions

a $\dfrac{7}{8} - \dfrac{1}{6}$

b $\dfrac{52}{75} + \dfrac{13}{24}$

Answer.

a 24 b 600

Checkpoint 8.38 Find the LCD for the fractions

a $\dfrac{3}{4a^2b^2} + \dfrac{7}{10ab^3}$ b $\dfrac{3}{8t^4} - \dfrac{3}{5t^2}$

Answer.

a $20a^2b^3$ b $40t^4$

3. Build fractions

Before we can add unlike fractions, we must build each fraction to an equivalent one with the LCD as denominator.

Examples

Example 8.39 Write an equivalent fraction with the new denominator:

$$\frac{3}{s} = \frac{?}{2s^2}$$

Solution. We first find the building factor for the fraction: what must we mulitply the old denominator by to get the new denominator? We factor the new denominator to see what factors are missing.

The new denominator is $2 \cdot s \cdot 3$, so we need to multiply the old denominator by $2s$. This is the building factor. We multiply top and bottom of the old fraction by the building factor:

$$\frac{3}{s} \cdot \frac{\mathbf{2s}}{\mathbf{2s}} = \frac{6}{2s^2}$$

□

Example 8.40 Write an equivalent fraction with the new denominator:

$$\frac{2}{t+1} = \frac{?}{t^2+t}$$

Solution. The new denominator factors as $t(t+1)$, so the building factor is t. We multiply top and bottom of the old fraction by t to obtain:

$$\frac{2}{t+1} \cdot \frac{\mathbf{t}}{\mathbf{t}} = \frac{2t}{t^2+t}$$

□

Exercises

Checkpoint 8.41 Write an equivalent fraction with the new denominator.

a $\dfrac{1}{n} = \dfrac{?}{n^2-n}$ b $\dfrac{4}{a-1} = \dfrac{?}{a^2-1}$

Answer.

a $\dfrac{n-1}{n^2-n}$ b $\dfrac{4a+4}{a^2-1}$

Checkpoint 8.42 Write an equivalent fraction with the new denominator.

a $\dfrac{b}{b-2} = \dfrac{?}{(b+3)(b-2)}$

b $\dfrac{x-1}{x^2+2x} = \dfrac{?}{(x^2-x)(x+2)}$

Answer.

a $\dfrac{b^2+3b}{(b+3)(b-2)}$

b $\dfrac{x^2-2x+1}{(x^2-x)(x+2)}$

More Operations on Algebraic Fractions

1. Work with radicals

Rationalizing the denominator of a fraction helps maintain accuracy.

Examples

Example 8.43 Simplify $\dfrac{3\sqrt{2}}{\sqrt{3}}$

Solution. We can rationalize the denominator by multiplying mumerator and denominator by $\boldsymbol{\sqrt{3}}$:

$$\frac{3\sqrt{2}}{\sqrt{3}} \cdot \frac{\boldsymbol{\sqrt{3}}}{\boldsymbol{\sqrt{3}}} = \frac{3\sqrt{6}}{3} = \sqrt{6}$$

or we can divide 3 by $\sqrt{3}$ to get $\sqrt{3}$. (Remember that $\sqrt{3}\,\sqrt{3} = 3$.)

$$\frac{\mathbf{3}\sqrt{2}}{\boldsymbol{\sqrt{3}}} = \boldsymbol{\sqrt{3}}\,\sqrt{2} = \sqrt{6}$$

□

Example 8.44 Combine $\dfrac{3}{\sqrt{2}} + \dfrac{5}{2}$

Solution. The LCD for the two fractions is 2, and the building factor for the first fraction is $\boldsymbol{\sqrt{2}}$.

$$\begin{aligned} \frac{3}{\sqrt{2}} + \frac{5}{2} &= \frac{3}{\sqrt{2}} \cdot \frac{\boldsymbol{\sqrt{2}}}{\boldsymbol{\sqrt{2}}} + \frac{5}{2} \\ &= \frac{3\sqrt{2}}{2} + \frac{5}{2} = \frac{3\sqrt{2}+5}{2} \end{aligned}$$

□

Exercises

Checkpoint 8.45 Simplify $\dfrac{8}{2\sqrt{2}}$

Answer. $2\sqrt{2}$

Checkpoint 8.46 Simplify $\sqrt{6} \cdot \dfrac{\sqrt{3}}{2\sqrt{2}}$

Answer. $\dfrac{3}{2}$

Checkpoint 8.47 Simplify $\dfrac{\sqrt{2}}{6} - \dfrac{2}{\sqrt{3}}$

Answer. $\dfrac{\sqrt{2} - 4\sqrt{3}}{6}$

Checkpoint 8.48 Simplify $\dfrac{1}{\sqrt{6}} + \dfrac{3}{\sqrt{2}}$

Answer. $\dfrac{\sqrt{6} + 9\sqrt{2}}{6}$

2. Use negative exponents

Recall that a negative exponent indicates a reciprocal.

Examples

Example 8.49 Write each expression without negative exponents.

a $\dfrac{1}{4}xy^{-2}$

b $a^{-3}b^{-2}$

c $\dfrac{2a^{-1}}{bc^{-2}}$

d $\dfrac{x}{y^{-2}} + \dfrac{x^{-2}}{y}$

Solution. We use the fact that $a^{-n} = \dfrac{1}{a^n}$, and consequently that $\dfrac{1}{a^{-n}} = a^n$.

a $\dfrac{x}{4y^2}$

b $\dfrac{1}{a^3b^2}$

c $\dfrac{2c^2}{ab}$

d $xy^2 + \dfrac{1}{x^2y}$

□

Example 8.50 Simplify where possible using the laws of exponents.

a $3x^{-3}x^5$

b $\dfrac{4a^{-4}}{8a^{-8}}$

c $(2bc^{-3})^{-2}$

d $3x^{-4} - 2x^{-3}$

Solution.

a Add the exponents: $3x^{-3}x^5 = 3x^2$

b Subtract the exponents: $\dfrac{4a^{-4}}{8a^{-8}} = \dfrac{1}{2}a^{-4-(-8)} = \dfrac{a^4}{2}$

c Raise each factor to the power -2. Multiply exponents:

$$(2bc^{-3})^{-2} = 2^{-2}b^{-2}(c^{-3})^{-2} = \frac{c^6}{4b^2}$$

d We cannot add or subtract powers with different exponents.

$$3x^{-4} - 2x^{-3} = \frac{3}{x^4} - \frac{2}{x^3}$$

□

Exercises

Checkpoint 8.51 Simplify where possible. Write your answer without negative exponents.

$$(2x^3y^{-4})(\frac{3}{4}x^{-2}y^2)$$

Answer. $\dfrac{3x}{2y^2}$

Checkpoint 8.52 Simplify where possible. Write your answer without negative exponents.

$$\frac{ab^{-3}}{(3ab)^{-2}}$$

Answer. $\dfrac{9a^3}{b}$

Checkpoint 8.53 Simplify where possible. Write your answer without negative exponents.

$$2x^{-2} - (2x)^{-2}$$

Answer. $\dfrac{8x^4-1}{4x^2}$

Checkpoint 8.54 Simplify where possible. Write your answer without negative exponents.

$$2x^{-2}(-2x)^{-2}$$

Answer. $\dfrac{1}{2}$

3. Check a division

Remember that division is the inverse operation for multiplication.

Examples

Example 8.55 Check that the division is correct: $536 \div 15 = 35\frac{11}{15}$

Solution. The quotient tells us that 15 divides into 536 thirty-five times, with a remainder of 11. This in turn means that if we multiply 15 by 35, and then add 11, we should get 536 back again.

$$15 \times 35 + 11 = 525 + 11 = 536$$

Note the pattern: divisor × quotient + remainder = starting number □

Example 8.56 Check that the division is correct:

$$(3n^2 + n - 6) \div (n + 2) = 3n - 5 + \frac{4}{n+2}$$

Solution. The answer tells us that $n + 2$ divides into $3n^2 + n - 6$ to give a quotient of $3n - 5$, with a remainder of 4. If we multiply $n+2$ by $3n-5$, and then add 4, we should get $3n^2 + n - 6$ back again.

$$(n + 2)(3n - 5) + 4 = (3n^2 + n - 10) + 4 = 3n^2 - n - 6$$

□

Exercises

Checkpoint 8.57 Check the division.

$$25 \div 4 = 6\frac{1}{4}$$

Answer. $4(6) + 1 = 25$

Checkpoint 8.58 Check the division.

$$1331 \div 28 = 47\frac{15}{28}$$

Answer. $28(47) + 15 = 1331$

Checkpoint 8.59 Check the division.

$$(n^2 + 3n + 6) \div (n + 1) = n + 2 + \frac{4}{n + 1}$$

Answer. $(n + 1)(n + 2) + 4 = n^2 + 3n + 6$

Checkpoint 8.60 Check the division.

$$(2x^3 + 7x^2 + 9x + 40) \div (2x - 3) = x^2 + 5x + 12 + \frac{40}{2x - 3}$$

Answer. $(2x - 3)(x^2 + 5x + 12) + 40 = 2x^3 + 7x^2 + 9x + 40$

Equations with Algebraic Fractions

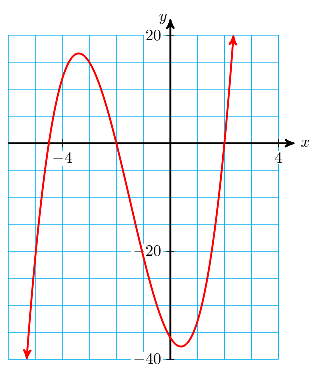
1. Solve equations graphically

If we can't solve an equation algebraically, we may be able use a graph to find at least an approximation for the solution.

Examples

Example 8.61 Use a graph to solve the equation $2x^3 + 9x^2 - 8x + 36 = 0$

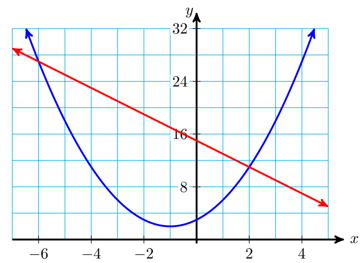
Solution. We graph the equation $y = 2x^3 + 9x^2 - 8x - 36$ and look for the points where $y = 0$ (the x-intercepts).



From the graph, we estimate the solutions at $x = -4.5$, $x = -2$, and $x = 2$. By substituting each of these values into the original equation, you can verify that they are indeed solutions. □

Example 8.62 Use a graph to solve the equation $x^2 + 2x + 3 = 15 - 2x$

Solution. We graph the equations $y_1 = x^2 + 2x + 3$ and $y_2 = 15 - 2x$ and look for points on the two graphs where the coordinates are equal (intersection points).



From the graph, we see that the points with $x = -6$ and $x = 2$ have the same y-coordinate on both graphs. In other words, $y_1 = y_2$ when $x = -6$ or $x = 2$, so $x = -6$ and $x = 2$ are the solutions. □

Exercises

Checkpoint 8.63 Use a graph to solve the equation $2x^3 + 7x^2 - 7x - 12 = 0$

Answer. $x = -4, \ -1, \ \dfrac{3}{2}$

Checkpoint 8.64 Use a graph to solve the equation $\dfrac{24}{x+4} = 11 + 2x - x^2$

Answer. $x = -1, \ 4$

2. Choose the correct technique

We have learned a number of algebraic skills to deal with fractions. Each type of problem has a particular method.

Example

Example 8.65 Choose the appropriate technique for each problem.

I Cross-multiply

II Multiply each term by the LCD

III Multiply top and bottom by the LCD

IV Find building factors

a Combine $\dfrac{8}{x+2} + \dfrac{x}{x-3}$

b Solve $\dfrac{8}{x+2} = \dfrac{x}{x-3}$

c Solve $\dfrac{8}{x+2} + 1 = \dfrac{x}{x-3}$

d Simplify $\dfrac{\dfrac{8}{x} + 1}{\dfrac{x}{x-3} + \dfrac{2}{x}}$

Solution.

a To add fractions, we find an LCD and build each fraction, so choice IV is correct.

b To solve a proportion, we can cross-multiply, so choice I is correct.

c To clear fractions from an equation, we multiply by the LCD, so choice II is correct.

d To simplify a complex fraction, we apply the fundamental pricnicple of fractions, so choice III is correct.

□

Exercises

Checkpoint 8.66 Write the first step for the problem.

Solve $\dfrac{3}{x}+3=\dfrac{1}{x+3}$

Answer. $3(x+3)+3x(x+3)=x$

Checkpoint 8.67 Write the first step for the problem.

Combine $\dfrac{3}{x}+3-\dfrac{1}{x+3}$

Answer. $\dfrac{3(x+3)}{x(x+3)}+\dfrac{3x(x+3)}{x(x+3)}-\dfrac{x}{x(x+3)}$

Checkpoint 8.68 Write the first step for the problem.

Simplify $\dfrac{\dfrac{3}{x}+1}{3-\dfrac{1}{x+3}}$

Answer. $\dfrac{3(x+3)-x(x+3)}{3x(x+3)-x}$

Checkpoint 8.69 Write the first step for the problem.

Solve $\dfrac{3}{x}=\dfrac{1}{x+3}$

Answer. $3(x+3)=x$

3. Solve quadratic equations

Once we have cleared the fractions from an equation, we may have a quadratic equation to solve. We can choose the easiest method to solve: factoring, extracting roots, or the quadratic formula.

Example

Example 8.70 Solve each quadratic equation by the easiest method.

a $2x^2-2x=3$

b $(2x-1)^2=3$

c $2x^2-x=3$

Solution.

a Because $2x^2-2x-3$ does not factor, we use the quadratic formula.

$$x=\frac{2\pm\sqrt{(-2)^2-4(2)(-3)}}{2(2)}=\frac{2\pm\sqrt{28}}{4}=\frac{1\pm\sqrt{7}}{2}$$

b We use extraction of roots.

$$2x - 1 = \pm\sqrt{3}$$
$$x = \frac{1 \pm \sqrt{3}}{2}$$

c We write the equation in standard form and factor the left side.

$$\begin{aligned} 2x^2 - x - 3 &= 0 \\ (2x-3)(x+1) &= 0 \\ 2x - 3 = 0 \quad x + 1 &= 0 \\ x = \frac{3}{2} \quad x &= -1 \end{aligned}$$

□

Exercise

Checkpoint 8.71 Solve each equation by the easiest method.

a $3x^2 + 10x = 8$

b $x^2 + 6x + 9 = 8$

c $81x^2 - 18x + 1 = 0$

d $9x^2 + 18x = 27$

Answer.

a $x = -4, \ \frac{2}{3}$

b $x = -2 \pm 2\sqrt{2}$

c $x = \frac{1}{9}, \ \frac{1}{9}$

d $x = -3, \ 1$

Chapter 9

Equations and Graphs

Properties of Lines

1. Find the slope of a line

We can tell whether two lines are parallel, perpendicular, or neither by comparing their slopes.

Examples

Example 9.1 Find the slope of the line $6x - 8y = 9$

Solution. The easiest way to find the slope of this line is to put its equation into slope-intercept form by solving for y.

$$\begin{aligned} -8y &= -6x + 9 \\ y &= \frac{-6x+9}{-8} = \frac{3}{4}x - \frac{9}{8} \end{aligned}$$

The slope of the line is $\dfrac{3}{4}$. $\square$

Example 9.2 Find the slope of the line whose intercepts are $(112, 0)$ and $(0, 140)$.

Solution. We use the slope formula.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{140 - 0}{0 - 112} = \frac{140}{-112} = \frac{-5}{4}$$

$\square$

Exercises

Checkpoint 9.3 Find the slope of the line $2.8x + 3.6y = 1.2$

Answer. $\dfrac{7}{9}$

Checkpoint 9.4 Find the slope of the line $6x = 72$

Answer. undefined

Checkpoint 9.5 Find the slope of the line that passes through $(15, -6)$ and $(10, -3)$.

Answer. $\dfrac{-3}{5}$

Checkpoint 9.6 Find the slope of a line that describes a 35% grade (or incline).

Answer. $\dfrac{7}{20}$

2. Use the point-slope formula

Recall the point-slope formula for finding the equation of a line:

$$y - y_1 = m(x - x_1)$$

Examples

Example 9.7 Find the y-intercept of the line of slope $\dfrac{-2}{3}$ that passes through $(-8, 9)$.

Solution. We first use the point-slope formula to find the equation of the line.

$$\begin{aligned} y - 9 &= \frac{-2}{3}(x+8) && \text{Multiply by 3 to clear the fraction.} \\ 3y - 27 &= -2x - 16 && \text{Add 27 to both sides.} \\ 3y &= -2x + 11 && \text{Divide both sides by 3.} \\ y &= \frac{-2}{3}x + \frac{11}{3} \end{aligned}$$

The y-intercept of the line is $\left(0, \dfrac{11}{3}\right)$. □

Example 9.8 Find the equation of the line with x-intercept $(2, 0)$ that passes through $(-3, -3)$.

Solution. We first compute the slope, using the two points given.

$$m = \frac{0+3}{2+3} = \frac{3}{5}$$

Now we can use the point-slope formula.

$$\begin{aligned} y - 0 &= \frac{3}{5}(x-2) && \text{Multiply by 5 to clear the fraction.} \\ 5y &= 3x - 6 && \text{Divide both sides by 5.} \\ y &= \frac{3}{5}x - \frac{6}{5} \end{aligned}$$

□

Exercises

Checkpoint 9.9 Find the equation of the line that has slope $\dfrac{-1}{3}$ and passes through $(-4, -6)$.

Answer. $y = \dfrac{1}{3}x = \dfrac{22}{3}$

Checkpoint 9.10 Find the equation of the line that has slope 0 and passes through $(25, 64)$.

Answer. $y = 64$

Checkpoint 9.11 Find the y-intercept of the line that passes through $(-5, 49)$ and has slope $\dfrac{-4}{5}$.

Answer. $(0, 0)$

Checkpoint 9.12 Find the y-intercept of the line that has slope $\dfrac{-7}{3}$ and passes through $(-6, 0)$.

Answer. $(0, -14)$

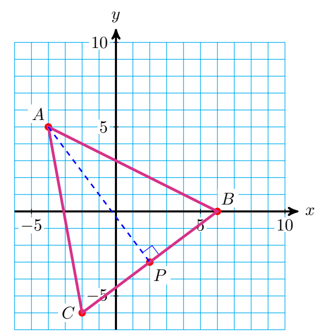
3. Use properties of geometric figures.

Analytic geometry uses algebra to help solve geometric problems.

Example

Example 9.13

The figure shows isosceles triangle ABC and its altitude $\overline{AP}$. (Recall that the altitude of a triangle is the segment perpendicular to the base that passes through the opposite vertex.) For this problem, we'll use the following property:
In an isosceles triangle, the altitude bisects the base.



a Find the equation of the line that includes $\overline{AP}$.

b Find the coordinates of point P.

c Find the length of the segment $\overline{AP}$.

Solution.

a Because $\overline{AP}$ is perpendicular to $\overline{BC}$, we can find its slope. The coordinates of B and C are $(6, 0)$ and $(2, -6)$, so the slope of $\overline{BC}$ is

$$\frac{-6-0}{-2-6} = \frac{3}{4}$$

The slope of $\overline{AP}$ is the negative reciprocal of $\dfrac{3}{4}$, or $\dfrac{-4}{3}$.

Now we can use the point-slope formula with $m = \dfrac{-4}{3}$ and the coordinates of $A(-4, 5)$ to calculate the equation of the line.

$$\begin{aligned} \frac{-4}{3} &= \frac{y-5}{x+4} && \text{Cross-multiply.} \\ 3(y-5) &= -4(x+4) && \text{Apply the distributive law.} \\ 3y - 15 &= -4x - 16 && \text{Add 15 to both sides.} \\ 3y &= -4x - 1 && \text{Divide both sides by 3.} \\ y &= \frac{-4}{3}x - \frac{1}{3} \end{aligned}$$

b Because the altitude bisects the base, point P is the midpoint of $\overline{BC}$. The coordinates of B and C are $(6, 0)$ and $(2, -6)$, so we use the midpoint formula to find the coordinates of P.

$$x = \frac{-2+6}{2} = 2$$
$$y = \frac{-6+0}{2} = -3$$

The coordinates of P are $(2, -3)$.

c The coordinates of P are $(2, -3)$, and the coordinates of A are $(-4, 5)$. We use the distance formula to find the length of $\overline{AP}$.

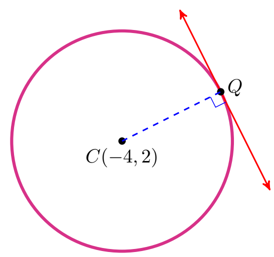
$$\overline{AP} = \sqrt{(-4-2)^2 + (5+3)^2} = \sqrt{36+64} = 10$$

□

Exercises

Checkpoint 9.14

The line $y = -2x + 9$ is tangent at point Q to a circle with center $C(-4, 2)$. For this problem, use the following property:
The tangent to a circle is perpendicular to the radius through the point of tangency.



a Find the equation of the line through C and Q.

b Find the coordinates of Q.

c Find the radius of the circle, $r = CQ$.

d Find the equation of the circle.

Answer.

a $y = \frac{1}{2}x = 4$

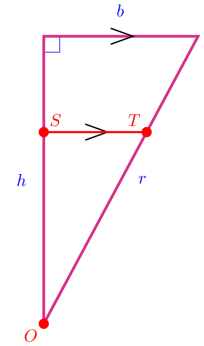
b $(2, 5)$

c $3\sqrt{5}$

d $(x+4)^2 + (y-2)^2 = 45$

Checkpoint 9.15

In the right triangle shown, $\overline{ST}$ is parallel to the shorter leg, and $h = \frac{9}{10}r$. For this problem, use the following property:
A line parallel to the base of a triangle cuts off a similar triangle. (Recall that the sides of similar triangles are proportional.)



a Write an expression for r in terms of h.

b Find an expression for b in terms of h.

c If $OS = \frac{2}{3}h$ find an expression for ST in terms of h.

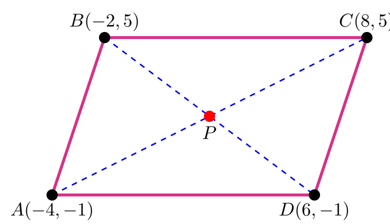
Answer.

a $r = \frac{10}{9}h$

b $b = \frac{\sqrt{19}}{9}h$

c $ST = \frac{2\sqrt{19}}{27}h$

Checkpoint 9.16 The quadrilateral $ABCD$ has vertices $A(-4,-1)$, $B(-2,5)$, $C(8,5)$ and $D(6,-1)$.



a Show that $ABCD$ is a parallelogram (its opposite sides are parallel).

b Find equations for the lines through the diagonals, $\overline{AC}$ and $\overline{BD}$.

c Find the intersection of the diagonals, P.

d Find the lengths of $\overline{AP}$ and $\overline{PC}$, and the lengths of $\overline{BP}$ and $\overline{PD}$.

e This example illustrates the following property of parallelograms:
The diagonals of a parallelogram ______ each other.

Answer.

a $m_{\overline{AB}} = m_{\overline{CD}} = \frac{3}{2}$; $m_{\overline{AD}} = m_{\overline{BC}} = 0$

b $y = \frac{1}{2}x + 1$; $y = \frac{-3}{4}x + \frac{7}{2}$

c $(2,2)$

d $AP = PC = 3\sqrt{5}$; $BP = PD = 5$

e bisect

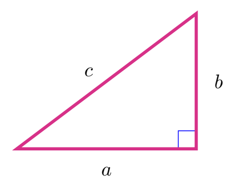
Distance and Midpoint Formulas

1. Use radicals

Because the distance formula is derived from the Pythagorean theorem, using it involves working with radicals.

Examples

Example 9.17
In the right triangle shown, $a = 8$ and $b = 6$.



a Is $c = a + b$? Why or why not?

b Find the length of c.

Solution.

a No, the length of c is shorter than the lengths of a and b combined, so $c < a + b$.

b We use the Pythagorean theorem:

$$c^2 = a^2 + b^2 \qquad \text{Substitute } \boldsymbol{a = 8} \text{ and } \boldsymbol{b = 6} \text{ and evaluate.}$$
$$c^2 = 8^2 + 6^2 = 64 + 36 = 100 \qquad \text{Take square roots.}$$
$$c = \sqrt{100} = 10$$

The hypotenuse is 10 cm long.

□

Example 9.18 Is $\sqrt{a^2 + b^2} = a + b$?

Solution. No. In the previous example, $c = \sqrt{a^2 + b^2}$, and we saw that $c < a + b$. We cannot simplify $\sqrt{a^2 + b^2}$ by taking the square root of each term.

□

Example 9.19 If $r = \sqrt{14}$, write and simplify expressions for:

a $r \cdot r$

b $r + r$

Solution.

a $r \cdot r = \left(\sqrt{14}\right)\left(\sqrt{14}\right) = 14$. Or, $r \cdot r = r^2 = \left(\sqrt{14}\right)^2 = 14$.

b $r + r = \sqrt{14} + \sqrt{14} = 2\sqrt{14}$. Or, $r + r = 2r = 2\sqrt{14}$.

□

Exercises

Checkpoint 9.20 Simplify if possible: $\sqrt{(x-4)^2 + (y-2)^2}$

Answer. cannot be simplified

Checkpoint 9.21 Simplify if possible: $\sqrt{7^2 + w^2}$

Answer. $\sqrt{49 + w^2}$

Checkpoint 9.22 Simplify if possible: $\sqrt{7+w}\sqrt{7+w}$

Answer. $7+w$

2. Complete the square

To find the center and radius of a circle, we may need to complete the square.

Examples

Example 9.23 Write an equivalent equation in which the left side is a perfect square: $x^2-6x=2$

Solution. We want to find a constant p^2 so that $x^2\boldsymbol{-6x}+p^2$ is a perfect square, namely $(x+p)^2$. Now, $(x+p)^2=x^2\boldsymbol{+2px}+p^2$, so $2p=-6$, and $p=-3$. Thus, we add $p^2=\mathbf{9}$ to both sides of the equation.

$$x^2-6x+\mathbf{9}=2+\mathbf{9}$$

Now we can write the left side as a perfect square:

$$(x-3)^2=11$$

□

Example 9.24 Write an equivalent equation in which the left side is a perfect square: $y^2+7y=5$

Solution. For this equation, $2p=7$ so $p=\dfrac{7}{2}$. We add $p^2=\left(\dfrac{7}{2}\right)^2=\dfrac{\mathbf{49}}{\mathbf{4}}$ to both sides of the equation.

$$y^2+7y+\frac{\mathbf{49}}{\mathbf{4}}=5+\frac{\mathbf{49}}{\mathbf{4}}$$

We write the left sides as a perfect square, $(x+p)^2$, and simplify the right side.

$$\left(x+\frac{7}{2}\right)^2=\frac{69}{4} \qquad \mathbf{5+\frac{49}{4}=\frac{20}{4}+\frac{49}{4}=\frac{69}{4}}$$

□

Exercises

Checkpoint 9.25 Write an equivalent equation in which the left side is a perfect square: $x^2+12x=-6$

Answer. $(x+6)^2=30$

Checkpoint 9.26 Write an equivalent equation in which the left side is a perfect square: $y^2-5y=3$

Answer. $\left(y-\dfrac{5}{2}\right)^2=\dfrac{37}{4}$

3. Use the equation for a circle

The equation for a circle of radius r centered at (h,k) is

$$\boldsymbol{(x-h)^2+(y-k)^2=r^2}$$

Examples

Example 9.27 Does $(-1, 5)$ lie on the circle $(x-1)^2 + (y-3)^2 = 8$?

Solution. We substitute $x = \mathbf{-1}$, $y = \mathbf{5}$ into the equation for the circle.

$$(\mathbf{-1} - 1)^2 + (\mathbf{5} - 3)^2 = (-2)^2 + 2^2 = 4 + 4 = 8$$

The point $(-1, 5)$ satisfies the equation, so it does lie on the circle. □

Example 9.28 Find a point with x-coordinate -3 that lies on the circle $(x+2)^2 + (y-4)^2 = 5$.

Solution. We substitute $x = \mathbf{-3}$ into the equation, and solve for y.

$$\begin{aligned} (\mathbf{-3} + 2)^2 + (y-4)^2 &= 5 && \text{Simplify.} \\ (-1)^2 + (y-4)^2 &= 5 && \text{Solve by extracting roots.} \\ (y-4)^2 &= 4 && \text{Take square roots.} \\ y - 4 &= \pm 2 \end{aligned}$$

We see that the solutions are $y = 6$ and $y = 2$. Thus, the points $(-3, 6)$ and $(-3, 2)$ lie on the circle. □

Exercises

Checkpoint 9.29 Does $(2, 6)$ lie on the circle $(x+3)^2 + (y-5)^2 = 26$?

Answer. Yes

Checkpoint 9.30 Find a point with y-coordinate -2 that lies on the circle $(x-2)^2 + (y+1)^2 = 9$.

Answer. $(2 + \sqrt{8}, -2)$ and $(2 - \sqrt{8}, -2)$

Conic Sections: Ellipses

1. Complete the square

To graph a conic section, we complete the square in each variable to put the equation in standard form.

Example

Example 9.31 Solve $2x^2 + 3 = 8x$ by completing the square.

Solution. We begin by isolating the constant term on the right side of the equation.

$$\begin{aligned} 2x^2 - 8x &= -3 && \text{Factor out the coefficient of } \boldsymbol{x^2}\text{.} \\ 2(x + 2 - 4x \quad &= -3) && \text{Complete the square.} \\ 2(x^2 - 4x\mathbf{+4}) &= -3\mathbf{+4} && \text{Simplify each side.} \\ 2(x-2)^2 &= 5 && \text{Isolate the perfect square.} \\ (x-2)^2 &= \frac{5}{2} && \text{Extract roots.} \\ x - 2 &= \pm\sqrt{\frac{5}{2}} && \text{Sovle for } \boldsymbol{x}\text{.} \end{aligned}$$

$$x = 2 \pm \sqrt{\frac{5}{2}}$$

□

Exercises

Checkpoint 9.32 Solve $x^2 - 7x = 4$ by completing the square.

Answer. $\dfrac{7 \pm \sqrt{65}}{2}$

Checkpoint 9.33 Solve $3x^2 + 6x - 2 = 0$ by completing the square.

Answer. $-1 \pm \sqrt{\dfrac{5}{3}}$

Checkpoint 9.34 Solve $2x^2 - 8 = 3x$ by completing the square.

Answer. $\dfrac{3 \pm \sqrt{73}}{4}$

2. Find points on a graph

Points on the graph of a conic section satisfy a quadratic equation in two variables.

Example

Example 9.35 Find all points on the graph of $x^2 + y^2 = 12$ with x-coordinate 2.

Solution. We substitute $x = 2$ into the equation to obtain

$$2^2 + y^2 = 12$$

and simplify to $y^2 = 8$. Solving for y, we find $y = \pm\sqrt{8} = \pm 2\sqrt{2}$. Thus, the points on the graph of $x^2 + y^2 = 12$ with x-coordinate 2 are $(2, 2\sqrt{2})$ and $(2, -2\sqrt{2})$. □

Exercises

Checkpoint 9.36 Find all points on the graph of $y = 2x^2 - 4x + 3$ with y-coordinate 3.

Answer. $(0, 3)$, $(2, 3)$

Checkpoint 9.37 Find all points on the graph of $(x - 4)^2 + (y + 1)^2 = 25$ with y-coordinate 3.

Answer. $(1, 3)$ $(7, 3)$

Checkpoint 9.38 Find all points on the graph of $\dfrac{(x-2)^2}{16} + \dfrac{(y+1)^2}{9} = 1$ with x-coordinate -1.

Answer. $\left(-1, -1 + \dfrac{3\sqrt{7}}{4}\right)$, $\left(-1, -1 - \dfrac{3\sqrt{7}}{4}\right)$

3. Divide by a fraction

We may encounter coefficients that are fractions when putting anequation in standard form.

Example

Example 9.39 Solve for y: $\frac{6}{5}y = \frac{2}{3}x^2 - 8x + 12$

Solution. We divide both sides of the equation by $\frac{6}{5}$, or equivalently, we multiply both sides by $\frac{5}{6}$.

$$\begin{aligned} \frac{5}{6}\left(\frac{6}{5}y\right) &= \frac{5}{6}\left(\frac{2}{3}x^2 - 8x + 12\right) && \text{Apply the distributive law.} \\ y &= \frac{5}{6}\left(\frac{2}{3}x^2\right) - \frac{5}{6}(8x) + \frac{5}{6}(12) && \text{Simplify each term.} \\ y &= \frac{5}{9}x^2 - \frac{20}{3}x + 10 \end{aligned}$$

□

Exercises

Checkpoint 9.40 Solve for y: $\frac{2}{3}y - (x+5)^2 = 1$

Answer. $y = \frac{3}{2}(x+5)^2 + \frac{3}{2}$

Checkpoint 9.41 Solve for y: $2(x-6)^2 = \frac{4}{5}y$

Answer. $y = \frac{5}{2}(x-6)^2$

Checkpoint 9.42 Solve for z: $2(x+1)^2 + 6(y-w)^2 = \frac{24}{9}z$

Answer. $z = \frac{3}{4}(x+1)^2 + \frac{9}{4}(y-3)^2$

Conic Sections: Hyperbolas

1. Write a quadratic equation in standard form

The parameters in the standard form determine the shape of the graph.

Example

Example 9.43 Write the equation $4x^2 - 3y^2 = 1$ in the form $\frac{x^2}{A} - \frac{y^2}{B} = 1$

Solution. Recall that dividing by a fraction is equivalent to multiplying by its reciprocal. For instance, $\frac{x^2}{\frac{1}{2}} = 2x$, and $\frac{a}{\frac{3}{4}} = \frac{4}{3}a$

Thus, for our example,

$$4x^2 = \frac{x^2}{\frac{1}{4}}, \quad \text{and} \quad 3y^2 = \frac{y^2}{\frac{1}{3}}$$

So we can write the equation as $\dfrac{x^2}{\frac{1}{4}} - \dfrac{y^2}{\frac{1}{3}} = 1$. □

Exercises

Write each equation in the form $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

Checkpoint 9.44 $\frac{4}{9}(x-2)^2 - \frac{1}{5}(y+2)^2 = 1$

Answer. $\dfrac{(x-2)^2}{\frac{9}{4}} - \dfrac{(y+2)^2}{5} = 1$

Checkpoint 9.45 $(x+3)^2 - \frac{1}{5}(y+2)^2 = 1$

Answer. $\dfrac{(x+3)^2}{3} - \dfrac{(y-2)^2}{\frac{3}{4}} = 1$

Checkpoint 9.46 $2x^2 - (y-1)^2 = \frac{8}{9}$

Answer. $\dfrac{(x^2}{\frac{4}{9}} - \dfrac{(y-1)^2}{\frac{5}{9}} = 1$

Checkpoint 9.47 $\frac{2}{3}(x-5)^2 - \frac{5}{3}(y-6)^2 = \frac{6}{5}$

Answer. $\dfrac{((x-5)^2}{\frac{9}{5}} - \dfrac{(y-6)^2}{\frac{18}{25}} = 1$

2. Solve a quadratic equation for y

Example

Example 9.48 Solve for y: $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$

Solution. We begin by isolating y^2.

$$y^2 = 4\left(1 + \frac{x^2}{9}\right)$$

Before extracting roots, we simplify the right side of the equation.

$$y^2 = 4\left(\frac{9+x^2}{9}\right)$$

$$y^2 = \frac{4}{9}(9+x^2)$$

Finally, we take square roots of both sides, and simplify.

$$y = \pm\sqrt{\frac{4}{9}(9+x^2)}$$
$$y = \pm\frac{2}{3}\sqrt{9+x^2}$$

□

Exercises

Solve each equation for y.

Checkpoint 9.49 $y^2 - 4x^2 = 4$
Answer. $y = \pm 2\sqrt{1+x^2}$

Checkpoint 9.50 $(x-2)^2 - y^2 = 4$
Answer. $y = \pm\sqrt{x^2-4x}$

Checkpoint 9.51 $\dfrac{x^2}{3} + \dfrac{y^2}{12} = 1$
Answer. $y = \pm 2\sqrt{3-x^2}$

Checkpoint 9.52 $y^2 - 6y = x + 3$ (Hint: complete the square in y.)
Answer. $y = 3 \pm \sqrt{x+12}$

3. Find an asymptote

Example

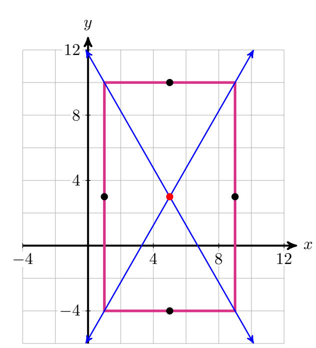
Example 9.53 The points $(1,3)$, $(9,3)$, $(5,10)$, and $(5,-4)$ are the midpoints of the four sides of a rectangle.

a Sketch the rectangle.

b Find equations for the diagonals of the rectangle.

Solution.

a The easiest way find the diagonals of the rectangle is to sketch it first. From the sketch at right, we can see that the center of the rectangle is $(5,3)$, and the two upper vertices are $(1,10)$ and $(9,10)$.



b Both diagonals pass through the center, $(5,3)$. The diagonal that passes through $(9,10)$ has slope

$$m = \frac{10-3}{9-5} = \frac{7}{4}$$

We use the point-slope formula to find its equation.

$$\begin{aligned}
\frac{y-3}{x-5} &= \frac{7}{4} && \text{Cross-multiply.}\\
4(y-3) &= 7(x-5) && \text{Apply the distributive law.}\\
4y - 12 &= 7x - 35 && \text{Add 12 to both sides.}\\
4y &= 7x - 23 && \text{Divide both sides by 4.}
\end{aligned}$$

$$y = \frac{7}{4}x - \frac{23}{4}$$

Similarly, you can check that the diagonal that passes through $(1, 10)$ has slope $\frac{-7}{4}$, and its equation is $y = \frac{-7}{4}x + \frac{47}{4}$ □

Exercises

The four points given are the midpoints of the four sides of a rectangle.

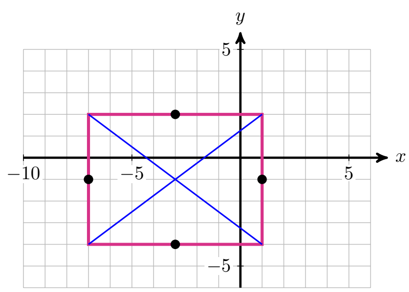
a Sketch the rectangle.

b Find equations for the diagonals of the rectangle.

Checkpoint 9.54 $(-7, -1),\ (1, -1),\ (-3, -4),\ (-3, 2)$

Answer.

a

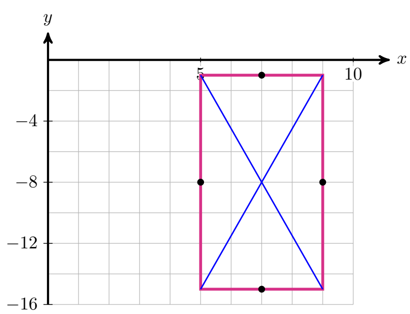


b $y = \frac{3}{4}x + \frac{5}{4};\ y = \frac{-3}{4}x - \frac{13}{4}$

Checkpoint 9.55 $(5, -8),\ (9, -8),\ (7, -15),\ (7, -1)$

Answer.

a



b $y = \frac{7}{2}x - \frac{65}{2};\ y = \frac{-7}{2}x + \frac{61}{2}$

Nonlinear Systems

1. Write a system of equations

Problems that involve two variables can sometimes be described by a system of equations.

Examples

Example 9.56 The area of a rectangle is 874 square millimeters, and its perimeter is 122 millimeters. Write a system of equations for the dimensions of the rectangle.

Solution. We use the formulas for area and perimeter, where w stands for the width of the rectangle, and l stands for its length.

$$\begin{aligned} lw &= 874 \\ 2l + 2w &= 122 \end{aligned}$$

□

Example 9.57 A small plane flies at a constant speed. In 5 hours, it travels 750 miles with a tailwind, but it travels only 600 miles in 5 hours against the wind. Write a system of equations for the speed of the plane and the speed of the wind.

Solution. We let v stand for the speed of the plane, and w stand for the speed of the wind. We use the formula $RT = D$.

$$\begin{aligned} (v + w)5 &= 750 \\ (v - w)5 &= 600 \end{aligned}$$

□

Exercises

Write a system of equations for the problem.

Checkpoint 9.58 Darryl plans to mix some 50% solution with some 75% solution to make 20 liters of 60% solution. How much of each should he use?

Answer.

$$\begin{aligned} x + y &= 20 \\ 0.5x + 0.75y &= 12 \end{aligned}$$

Checkpoint 9.59 The perimeter of a rectangle is 38 inches. If we triple the width and decrease the length by 8 inches, we increase the area of the rectangle by 40%. What are the dimensions of the original rectangle?

Answer.

$$\begin{aligned} 2l + 2w &= 38 \\ 3w(l - 8) &= 1.4lw \end{aligned}$$

Checkpoint 9.60 The express train travels 15 miles per hour faster than the local. The local takes 10 minutes longer to travel 30 miles than the express takes. Find the speed of each.

Answer.

$$\begin{aligned} v &= 15 + l \\ \frac{30}{l} &= \frac{30}{v} + \frac{1}{6} \end{aligned}$$

Checkpoint 9.61 Delbert made \$65 interest on his money market account this year. Francine invested \$100 less than Delbert at 0.5% higher interest rate and earned \$66. How much did Delbert invest, and at what interest rate?

Answer.

$$Pr = 65$$
$$(P - 100)(r + 0.005) = 66$$

2. Find the vertex and intercepts of a parabola

We review some earlier skills for graphing parabolas.

Example

Example 9.62 Find the vertex and intercepts of the parabola $y = -x^2 - 2x + 8$.

Solution. To find the x-intercepts of the graph, we set $y = 0$ and solve for x. We begin by factoring -1, the coefficient of x^2, from the equation.

$$-(x^2 + 2x - 8) = 0$$
$$-(x + 4)(x - 2) = 0$$
$$x = -4, \; x = 2$$

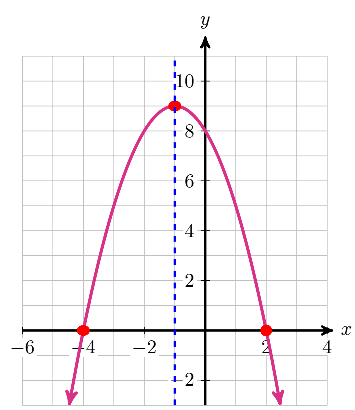
The x-intercepts are $(-4, 0)$ and $(2, 0)$. The y-intercept, $(0, 8)$, is given by the constant term. We find the $x$coordinate of the vertex by using the formula

$$x_v = \frac{-b}{2a} = \frac{1(-2)}{2(-1)} = -1$$

To find the y-coordinate, we substitute $\mathbf{-1}$ into the equation.

$$y_v = -(\mathbf{-1})^2 - 2(\mathbf{-1}) + 8 = 9$$

The vertex is the point $(-1, 9)$. The graph of the parabola is shown below.



□

Exercises

Find the intercepts and the vertex of the parabola.

Checkpoint 9.63 $y = x^2 - 8x + 15$

Answer. $(5, 0), \; (3, 0), \; (0, 15), \; (4, -1)$

Checkpoint 9.64 $y = 8 - 2x^2$

Answer. $(2, 0), \; (-2, 0), \; (0, 8), \; (0, 8)$

Checkpoint 9.65 $y = 3x^2 + 6x + 3$

Answer. $(-1, 0),\ (-1, 0),\ (0, 3),\ (-1, 0)$

Checkpoint 9.66 $y = x^2 - 4x - 1$

Answer. $(2 + \sqrt{5}, 0),\ (2 - \sqrt{5}, 0),\ (0, -1),\ (2, -5)$

3. Use substitution

We can use substitution to solve some systems of equations.

Example

Example 9.67 Use substitution to write an equation in one variable.

$$\begin{aligned} x^2 - 8x + y^2 + 2y &= 23 \\ x + 2y &= 12 \end{aligned}$$

Solution. We solve the second equation for x to get $x = \mathbf{12 - 2y}$, and then substitute this expression into the first equation.

$$\begin{aligned} (\mathbf{12 - 2y})^2 - 8(\mathbf{12 - 2y}) + y^2 + 2y &= 23 \\ 144 - 48y + 4y^2 - 96 + 16y + y^2 + 2y &= 23 \\ 5y^2 - 30y + 25 &= 0 \end{aligned}$$

□

Exercises

Use substitution to write an equation in one variable.

Checkpoint 9.68

$$\begin{aligned} x^2 - 2x + y^2 + 4y &= 20 \\ y - x &= 4 \end{aligned}$$

Answer. $2x^2 + 10x + 12 = 0$

Checkpoint 9.69

$$\begin{aligned} x^2 - y^2 &= 16 \\ xy &= 6 \end{aligned}$$

Answer. $x^4 - 16x^2 - 36 = 0$

Checkpoint 9.70

$$\begin{aligned} x^2 - 12x + y^2 + 2 &= 0 \\ y &= 4x - 7 \end{aligned}$$

Answer. $17x^2 - 68x + 51 = 0$

Checkpoint 9.71

$$\begin{aligned} x^2 - 4x + y^2 + 2y &= 5 \\ x + 2y &= -5 \end{aligned}$$

Answer. $5y^2 + 30y + 40 = 0$

Chapter 10

Logarithmic Functions

Logarithmic Functions

1. Estimate logs

Examples

It is useful to be able to estimate mentally the value of a log.

Example 10.1 Write each log equation in exponential form. Then use trial and error to estimate the log between two integers.

a $\log_2 6 = x$

b $\log_2 24 = x$

c $\log_2 100 = x$

d $\log_2 0.3 = x$

Solution.

a $2^x = 6$. Because $2^2 = 4$ and $2^3 = 8$, $2 < x < 3$.

b $2^x = 24$. Because $2^4 = 16$ and $2^5 = 32$, $4 < x < 5$.

c $2^x = 100$. Because $2^6 = 64$ and $2^7 = 128$, $6 < x < 7$.

d $2^x = 0.3$. Because $2^{-2} = \dfrac{1}{4} = 0.25$ and $2^{-1} = \dfrac{1}{2} = 0.5$, $-2 < x < -1$.

□

Example 10.2

a Use computing technology to complete the table for $f(x) = 5^x$. Round the function values to tenths.

x	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
$f(x)$											

b Use your table from part (a) to make a table of values for the function $g(x) = \log_5 x$.

x											
$g(x)$											

Solution.

a

x	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
$f(x)$	25	29.4	34.5	40.5	47.6	55.9	65.7	77.1	90.6	106.4	125

b

x	25	29.4	34.5	40.5	47.6	55.9	65.7	77.1	90.6	106.4	125
$g(x)$	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0

□

Exercises

Checkpoint 10.3 Write each log equation in exponential form. Then use trial and error to estimate the log, first between two integers, and then to the nearest tenth.

a $\log_3 10 = x$

b $\log_3 20 = x$

c $\log_3 150 = x$

d $\log_3 0.5 = x$

Answer.

a $3^x = 10$, between 2 and 3, 2.1

b $3^x = 20$, between 2 and 3, 2.7

c $3^x = 150$, between 4 and 5, 4.6

d $3^x = 0.5$, between -1 and 0, -0.6

Checkpoint 10.4

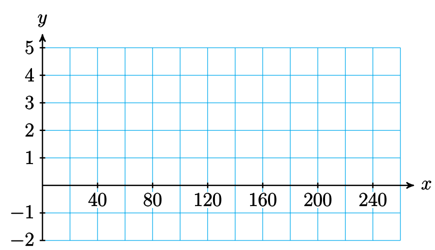
a Use computing technology to complete the table for $f(x) = 4^x$.

x	-1	-0.5	0	0.5	1	1.5	2	2.5	3	3.5	4.0
$f(x)$											

b Use your table from part (a) to make a table of values for the function $g(x) = \log_4 x$.

x											
$g(x)$											

c Use your table from part (b) to make a graph of $g(x) = \log_4 x$.



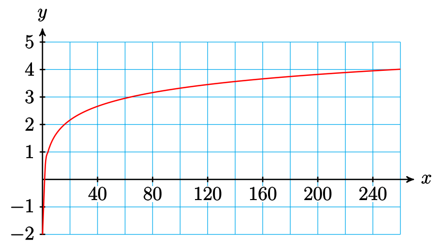
Answer.

a

x	-1	-0.5	0	0.5	1	1.5	2	2.5	3	3.5	4.0
$f(x)$	0.25	0.5	1	2	4	8	16	32	64	128	256

b

x	0.25	0.5	1	2	4	8	16	32	64	128	256
$g(x)$	-1	-0.5	0	0.5	1	1.5	2	2.5	3	3.5	4.0

c 

2. Use function notation

A log function is the inverse of the exponential function with the same base, and vice versa.

Examples

Example 10.5 $f(x) = \log_6 x$

a What is $f^{-1}(x)$?

b Evaluate and simplify $f\left(f^{-1}(4)\right)$

c Evaluate and simplify $f^{-1}\left(f(5)\right)$

Solution.

a $f^{-1}(x) = 6^x$

b $f(f^{-1}(4)) = f(6^4) = \log_6(6^4) = 4$

c $f^{-1}(\log_6 5) = 6^{\log_6 5} = 5$

□

Example 10.6 For each function $f(x)$, decide whether $f(a+b) = f(a)+f(b)$.

a $f(x) = 3^x$

b $f(x) = \log_3 x$

Solution.

a $f(a+b) = 3^{a+b}$, and $f(a)+f(b) = 3^a + 3^b$.

But 3^{a+b} is not equivalent to $3^a + 3^b$; in fact $3^{a+b} = 3^a \cdot 3^b$.

So for this function, $f(a+b) \neq f(a)+f(b)$.

b $f(a+b) = \log_3(a+b)$, and $f(a)+f(b) = \log_3 a + \log_3 b$.

But $\log_3(a+b)$ is not equivalent to $\log_3 a + \log_3 b$; in fact $\log_3(ab) = \log_3 a + \log_3 b$.

So for this function, $f(a+b) \neq f(a)+f(b)$.

□

Exercises

Checkpoint 10.7 $h(x) = \log_4 x$. Evaluate if possible.

a $h(4)$

b $h^{-1}(4)$

c $h(0)$

d $h^{-1}(0)$

Answer.

a 1

b 256

c undefined

d 1

Checkpoint 10.8 $q(x) = 9^x$. Evaluate if possible.

a $q\left(\frac{1}{2}\right)$

c $q(0)$

b $q^{-1}(3)$

d $q^{-1}(0)$

Answer.

a 3

b $\frac{1}{2}$

c 1

d undefined

Checkpoint 10.9 $g(x) = 5^x$. Evaluate and simplify if possible.

a $g(3 + t)$

b $g(3t)$

Answer.

a $125 \cdot 5^t$

b 125^t

Checkpoint 10.10 $f(x) = \log_8 x$. Evaluate and simplify if possible.

a $f(64p)$

b $f(64 + p)$

Answer.

a $2 + \log_8 p$

b cannot be simplified

3. Graph log functions

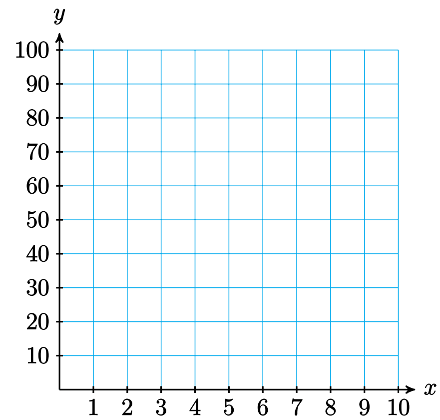
One way to graph a log function is to first make a table of values for its inverse function, the exponential function with the same base, then interchange the variables.

Exercise

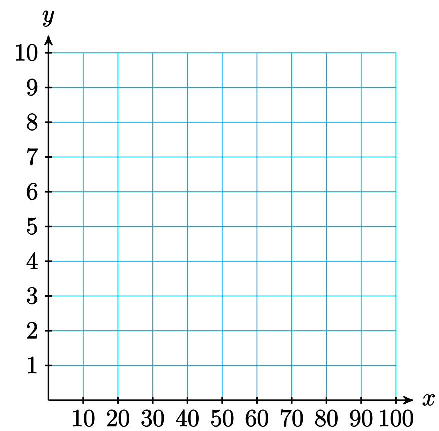
Checkpoint 10.11

a Complete the table of values and graph on the same grid: $f(x) = x^2$ and $g(x) = 2^x$

x	0	1	2	3	4	5	6	8	10
$f(x)$									
$g(x)$									

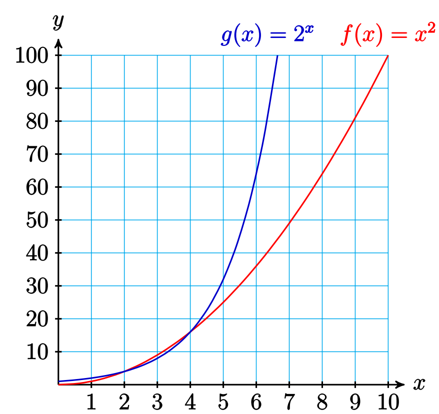


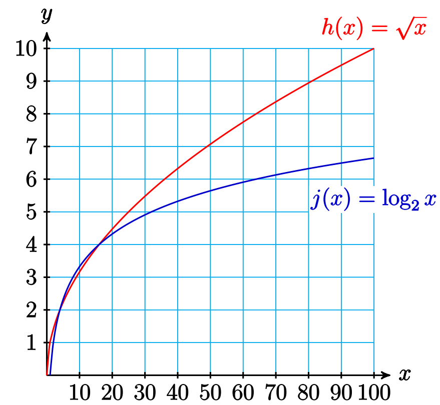
b Use your tables from part (a) to graph $h(x) = \sqrt{x}$ and $j(x) = \log_2 x$ on the same grid.



Answer.

a



b 

Log Scales

1. Compare quantities with logarithms

Because $\log x$ grows very slowly, we can use logs to compare quantities that vary greatly in magnitude.

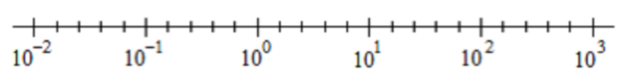
Example

Example 10.12

a Complete the table. Round the values to one decimal place.

x	1	5	25	125	625
$\log x$					

b Plot the values of x on a log scale.

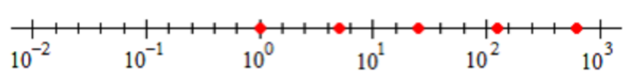


c Each time we multiply x by 5, how much does the logarithm increase? What is $\log 5$, to one decimal place?

Solution.

a

x	1	5	25	125	625
$\log x$	0	0.7	1.4	2.1	2.8

b 

c Each time we multiply x by 5, the log of x increases by 0.7, because $\log 5 = 0.7$. This is an application of the log properties:

$$\log 5x = \log x + \log 5 = \log x + 0.7$$

□

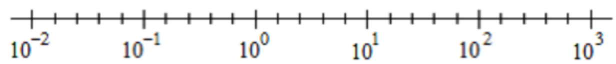
Exercises

Checkpoint 10.13

a Complete the table. Round the values to one decimal place.

x	5	10	20	40	80
$\log x$					

b Plot the values of x on a log scale.



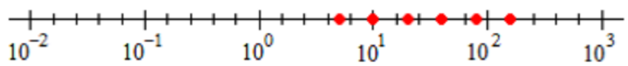
c Each time we multiply x by 2, how much does the logarithm increase? What is $\log 2$, to one decimal place?

Answer.

a

x	5	10	20	40	80
$\log x$	0.7	1	1.3	1.6	1.9

b



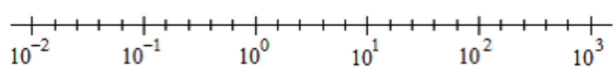
c 0.3; 0.3

Checkpoint 10.14

a Complete the table. Round the values to one decimal place.

x	0.25	1	4	16	64	256
$\log x$						

b Plot the values of x on a log scale.



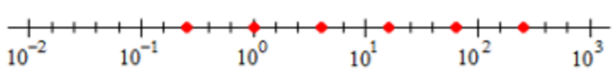
c Each time we multiply x by 4, how much does the logarithm increase? What is $\log 4$, to one decimal place?

Answer.

a

x	0.25	1	4	16	64	256
$\log x$	−0.6	0	0.6	1.2	1.8	2.4

b



c 0.6; 0.6

2. Use the properties of logarithms

The three properties of lagarithms are helpful in making computations involving logs.

Properties of Logarithms.

If x, y, $b > 0$, and $b \neq 1$, then

1 $\log_b (xy) = \log_b x + \log_b y$

2 $\log_b \dfrac{x}{y} = \log_b x - \log_b y$

3 $\log_b x^k = k \log_b x$

Examples

Example 10.15 If $\log_b 10 = 2.303$ and $\log_b 2 = 0.693$, what is $\log_b 5$?

Solution. Because $5 = \dfrac{10}{2}$,

$$\log_b 5 = \log_b \left(\frac{10}{2}\right) = \log_b 10 - \log_b 2 = 2.303 - 0.693 = 1.61$$

□

Example 10.16 If $\log_b 10 = 2.303$ and $\log_b 2 = 0.693$, what is $\log_b 20$?

Solution. Because $20 = 10 \cdot 2$,

$$\log_b 20 = \log_b (10 \cdot 2) = \log_b 10 + \log_b 2 = 2.303 + 0.693 = 2.996$$

□

Exercises

Checkpoint 10.17 Take the log of each number. What do you notice?

a $8 \cdot 100 = 800$

b $12 \cdot 1000 = 12,000$

c $20 \cdot 25 = 500$

d $200 \cdot 250 = 50,000$

Answer.

a $\log 8 + \log 10 = \log 800$

b $\log 12 + \log 100 = \log 12,000$

c $\log 20 + \log 25 = \log 500$

d $\log 200 + \log 250 = \log 50,000$

Checkpoint 10.18 Compare the two operations. What do you notice?

a (i) Compute $10^{2.68}$ (ii) Solve for x : $\log x = 2.68$

b (i) Compute $10^{-0.75}$ (ii) Solve for x : $\log x = -0.75$

Answer.

a (i) and (ii) have the same answer: 478.63

b (i) and (ii) have the same answer: 0.1778

Checkpoint 10.19

a The ratio of N to P is 32.6. Compute $\log N - \log P$.

b $\log z - \log t = 2.5$. Compute $\frac{z}{t}$.

Answer.

a 1.5132

b 316.2278

3. Write expressions to compare quantities

There is often more than one way to express a comparison with mathematical notation.

Example

Example 10.20 When we say that "A is 3 times larger than B," we mean that $A = 3B$. □

Example 10.21 When we say that "A is 3 more than B," we mean that $A = B + 3$. □

Exercises

a $x = 5H$

b $x = \frac{5}{H}$

c $x = 5 + H$

d $H = x + 5$

e $H = 5x$

f $H = \frac{5}{x}$

g $x - H = 5$

h $H - x = 5$

i $\frac{x}{H} = 5$

j $\frac{H}{x} = 5$

k $\frac{\log x}{\log H} = 5$

l $\log x - \log H = \log 5$

m $\log x + \log 5 = \log H$

Checkpoint 10.22 From the list above, match all the correct algebraic expressions to the phrase "x is 5 times as large as H."

Answer. (a), (i), (l)

Checkpoint 10.23 From the list above, match all the correct algebraic expressions to the phrase "x is 5 more than H."

Answer. (c), (g)

The Natural Base

1. Using growth and decay laws with base e

Examples

We can write exponential growth and decay laws using bas e.

Exponential Growth and Decay.

The function

$$P(t) = P_0 e^{kt}$$

describes exponential growth if $k > 0$, and exponential decay if $k < 0$.

Example 10.24 A colony of bees grows at a rate of 8% annually. Write its growth law using base e.

Solution. The growth factor is $b = 1 + r = 1.08$, so the growth law can be written as

$$P(t) = P_0(1.08)^t$$

Using base e, we write $P(t) = P_0e^{kt}$, where $e^k = 1.08$. (You can see this by evaluating each growth law at $t = 1$.) So we solve for k.

$$\begin{aligned} e^k &= 1.08 && \text{Take the natural log of both sides.} \\ \ln(e^k) &= \ln(1.08) && \text{Simplify both sides.} \\ k &= 0.0770 \end{aligned}$$

The growth law is $P(t) = P_0e^{0.077t}$. □

Example 10.25 A radioactive isotope decays according to the formula $N(t) = N_0e^{-0.016t}$, where t is in hours. Find its percent rate of decay.

Solution. First we write the decay law in the form $N(t) = N_0b^t$, where $b = e^k$.

In this case, $k = -0.016$, so $b = e^{-0.016} = 0.9841$. Now, $b = 1 - r$, and solving for r we find $r = -0.0159$. The rate of decay is approximately 16% per hour. □

Exercises

Checkpoint 10.26 A virus spreads in the population at a rate of 19.5% daily. Write its growth law using base e.

Answer. $P(t) = P_0e^{0.178t}$

Checkpoint 10.27 Sea ice is decreasing at a rate of 12.85% per decade. Write its decay law using base e.

Answer. $Q(t) = Q_0e^{-0.1375t}$

Checkpoint 10.28 In 2020, the world population was growing according to the formula $P(t) = P_0e^{0.0488t}$, where t is in years. Find its percent rate of growth.

Answer. 5%

Checkpoint 10.29 Since 1984, the population of cod has decreased annually according to the formula $N(t) = N_0e^{-0.1863t}$. Find its percent rate of decay.

Answer. 17%

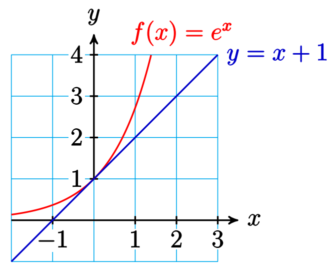
2. Graphing $y = e^x$ and $y = \ln x$

The graphs of the natural exponential function and the natural log function have some special properties.

Exercises

Checkpoint 10.30 Use technology to graph $f(x) = e^x$ and $y = x + 1$ in a window with $-2 \le x \le 3$ and $-1 \le y \le 4$. What do you notice about the two graphs?

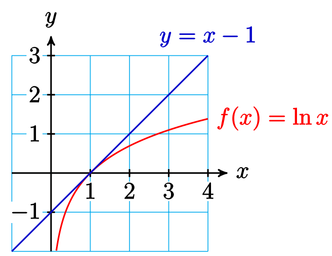
Answer.



The line is tangent to the graph at $(0, 1)$.

Checkpoint 10.31 Use technology to graph $f(x) = \ln x$ and $y = x - 1$ in a window with $-1 \le x \le 4$ and $-2 \le y \le 3$. What do you notice about the two graphs?

Answer.



The line is tangent to the graph at $(1, 0)$.