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Katherine Yoshiwara did her undergraduate work at Michigan State and her graduate work at UCLA. She has received teaching awards from the Mathematical Association of America, the American Mathematical Association of Two-Year Colleges, and the National Institute of Staff and Organizational Development.

She retired from Los Angeles Pierce College, is learning to play the cello, and likes gardening.

Acknowledgements

I would like to thank my cats.

I would also like to acknowledge Bruce Yoshiwara for helpful comments and suggestions.

Preface

Mathematics, as we all know, is the language of science, and fluency in algebraic skills has always been necessary for anyone aspiring to disciplines based on calculus. But in the information age, increasingly sophisticated mathematical methods are used in all fields of knowledge, from archaeology to zoology. Consequently, there is a new focus on the courses before calculus. The availability of calculators and computers allows students to tackle complex problems involving real data, but requires more attention to analysis and interpretation of results. All students, not just those headed for science and engineering, should develop a mathematical viewpoint, including critical thinking, problem-solving strategies, and estimation, in addition to computational skills. *Modeling, Functions and Graphs* employs a variety of applications to motivate mathematical thinking.

0.1 MODELING

The ability to model problems or phenomena by algebraic expressions and equations is the ultimate goal of any algebra course. Through a variety of applications, we motivate students to develop the skills and techniques of algebra. Each chapter includes an interactive Investigation that gives students an opportunity to explore an openended modeling problem. These Investigations can be used in class as guided explorations or as projects for small groups. They are designed to show students how the mathematical techniques they are learning can be applied to study and understand new situations.

0.2 Functions

The fundamental concept underlying calculus and related disciplines is the notion of function, and students should acquire a good understanding of functions before they embark on their study of college-level mathematics. While the formal study of functions is usually the content of precalculus, it is not too early to begin building an intuitive understanding of functional relationships in the preceding algebra courses. These ideas are useful not only in calculus but in practically any field students may pursue. We begin working with functions in Chapter 1 and explore the different families of functions in subsequent chapters.

In all our work with functions and modeling we employ the "Rule of Four,"" that all problems should be considered using algebraic, numerical, graphical, and verbal methods. It is the connections between these approaches that we have endeavored to establish in this course. At this level it is crucial that students learn to write an algebraic expression from a verbal description, recognize trends in a table of data, and extract and interpret information from the graph of a function.

0.3 Graphs

No tool for conveying information about a system is more powerful than a graph. Yet many students have trouble progressing from a point-wise understanding of graphs to a more global view. By taking advantage of graphing calculators, we examine a large number of examples and study them in more detail than is possible when every graph is plotted by hand. We can consider more realistic models in which calculations by more traditional methods are difficult or impossible.

We have incorporated graphing calculators into the text wherever they can be used to enhance understanding. Calculator use is not simply an add-on, but in many ways shapes the organization of the material. The text includes instructions for the TI-84 graphing calculator, but these can easily be adapted to any other graphing utility. We have not attempted to use all the features of the calculator or to teach calculator use for its own sake, but in all cases have let the mathematics suggest how technology should be used.

Katherine Yoshiwara
Atascadero, CA 2016

Contributors to the 5$^{\text{th}}$ Edition

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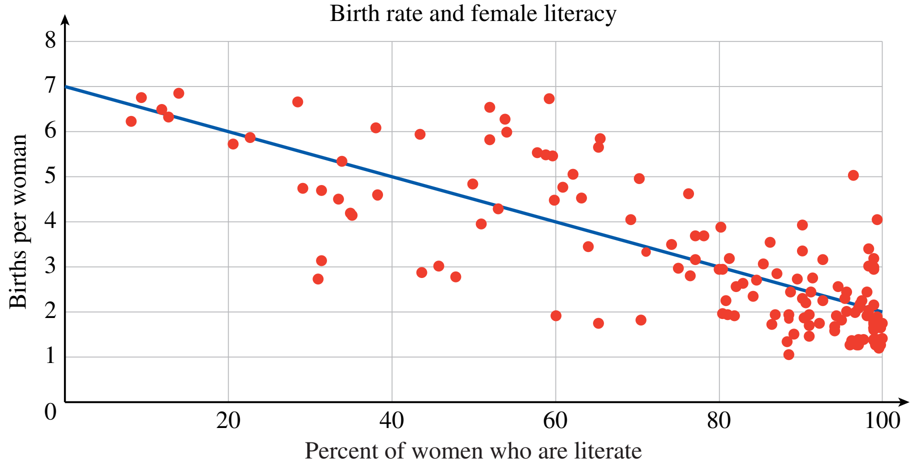
Chapter 1

Functions and Their Graphs

You may have heard that mathematics is the language of science. In fact, professionals in nearly every discipline take advantage of mathematical methods to analyze data, identify trends, and predict the effects of change. This process is called **mathematical modeling**. A model is a simplified representation of reality that helps us understand a process or phenomenon. Because it is a simplification, a model can never be completely accurate. Instead, it should focus on those aspects of the real situation that will help us answer specific questions. Here is an example.

The world's population is growing at different rates in different nations. Many factors, including economic and social forces, influence the birth rate. Is there a connection between birth rates and education levels? The figure shows the birth rate plotted against the female literacy rate in 148 countries. Although the data points do not all lie precisely on a line, we see a generally decreasing trend: the higher the literacy rate, the lower the birth rate. The **regression line** provides a model for this trend, and a tool for analyzing the data. In this chapter we study the properties of linear models and some techniques for fitting a linear model to data.

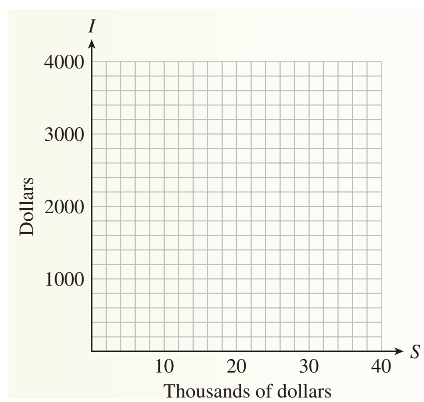


Investigation 1.1 (Sales on Commission)**.** Delbert is offered a part-time job selling restaurant equipment. He will be paid \$1000 per month plus a 6% commission on his sales. The sales manager tells Delbert he can expect to sell about \$8000 worth of equipment per month. To help him decide whether to accept the job, Delbert does a few calculations.

1. Based on the sales manager's estimate, what monthly income can Delbert expect from this job? What annual salary would that provide?

2. What would Delbert's monthly salary be if he sold only \$5000 of equipment per month? What would his salary be if he sold \$10,000 worth per month? Compute monthly incomes for each sales total shown in the table.

Sales	Income
5000	
8000	
10,000	
12,000	
15,000	
18,000	
20,000	
25,000	
30,000	
35,000	



3. Plot your data points on a graph, using sales, S, on the horizontal axis and income, I, on the vertical axis, as shown in the figure. Connect the data points to show Delbert's monthly income for all possible monthly sales totals.

4. Add two new data points to the table by reading values from your graph.

5. Write an algebraic expression for Delbert's monthly income, I, in terms of his monthly sales, S. Use the description in the problem to help you:

He will be paid: \$1000 . . . plus a 6% commission on his sales.

Income = ______________

6. Test your formula from part (5) to see if it gives the same results as those you recorded in the table.

7. Use your formula to find out what monthly sales total Delbert would need in order to have a monthly income of \$2500.

8. Each increase of \$1000 in monthly sales increases Delbert's monthly income by ______________.

9. Summarize the results of your work: In your own words, describe the relationship between Delbert's monthly sales and his monthly income. Include in your discussion a description of your graph.

1.1 Linear Models

1.1.1 Tables, Graphs and Equations

The first step in creating a model is to describe relationships between variables. In Investigation 1.1, we analyzed the relationship between Delbert's sales and his income. Starting from the verbal description of his income, we represented the relationship by a table of values, a graph, and an algebraic equation. Each of these mathematical tools is useful in a different way.

1. A **table of values** displays specific data points with precise numerical values.

2. A **graph** is a visual display of the data. It is easier to spot trends and describe the overall behavior of the variables from a graph.

3. An **algebraic equation** is a compact summary of the model. It can be used to analyze the model and to make predictions

We begin our study of modeling with some examples of **linear models**. In the examples that follow, observe the interplay among the three modeling tools, and how each contributes to the model.

Example 1.1. Annelise is on vacation at a seaside resort. She can rent a bicycle from her hotel for \$3 an hour, plus a \$5 insurance fee. (A fraction of an hour is charged as the same fraction of \$3.)

a Make a table of values showing the cost, C, of renting a bike for various lengths of time, t.

b Plot the points on a graph. Draw a curve through the data points.

c Write an equation for C in terms of t.

Solution.

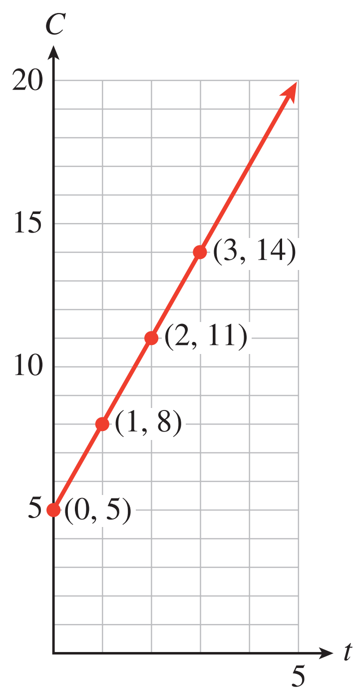
a To find the cost, we multiply the time by \$3, and add the result to the \$5 insurance fee. For example, the cost of a 1-hour bike ride is

$$\begin{aligned} \text{Cost} &= (\$5 \text{ insurance fee}) + (\$3 \text{ per hour}) \times (\mathbf{1} \text{ hour}) \\ C &= 5 + 3(\mathbf{1}) = 8 \end{aligned}$$

A 1-hour bike ride costs \$8. Record the results in a table, as shown here:

Length of rental (hours)	Cost of rental (dollars)		(t, C)
1	8	$C = 5 + 3(1)$	$(1, 8)$
2	11	$C = 5 + 3(2)$	$(2, 11)$
3	14	$C = 5 + 3(3)$	$(3, 14)$

b Each pair of values represents a point on the graph. The first value gives the horizontal coordinate of the point, and the second value gives the vertical coordinate. The points lie on a straight line, as shown in the figure. The line extends infinitely in only one direction, because negative values of t do not make sense here.



c To write an equation, let C represent the cost of the rental, and use t for the number of hours:

$$\begin{aligned} \text{Cost} &= (\$5 \text{ insurance fee}) + (\$3 \text{ per hour}) \times (\text{number of hours}) \\ C &= 5 + 3 \cdot t = 8 \end{aligned}$$

Example 1.2. Use the equation $C = 5 + 3 \cdot t$ you found in Example 1.1 to answer the following questions. Then show how to find the answers by using the graph.

a How much will it cost Annelise to rent a bicycle for 6 hours?

b How long can Annelise bicycle for \$18.50?

Solution.

a Substitute $t = \mathbf{6}$ into the expression for C to find

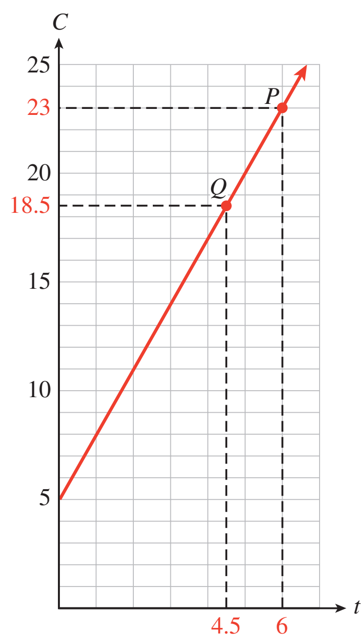
$$C = 5 + 3(\mathbf{6}) = 23$$

A 6-hour bike ride will cost \$23. The point P on the graph in the figure represents the cost of a 6-hour bike ride. The value on the C-axis at the same height as point P is 23, so a 6-hour bike ride costs \$23.

b Substitute $C = \mathbf{18.50}$ into the equation and solve for t.

$$\begin{aligned} \mathbf{18.50} &= 5 + 3t \\ 13.50 &= 3t \\ t &= 4.5 \end{aligned}$$

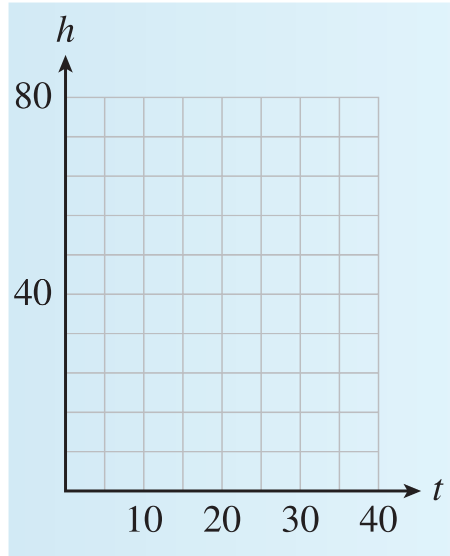
For \$18.50 Annelise can bicycle for $4\frac{1}{2}$ hours. The point Q on the graph represents an \$18.50 bike ride. The value on the t-axis below point Q is 4.5, so \$18.50 will buy a 4.5 hour bike ride.



In Example 1.2, notice the different algebraic techniques we used in parts (a) and (b). In part (a), we were given a value of t and we **evaluated the expression** $5 + 3t$ to find C. In part (b) we were given a value of C and we **solved the equation** $C = 5 + 3t$ to find t.

Exercise 1.3. Frank plants a dozen corn seedlings, each 6 inches tall. With plenty of water and sunlight they will grow approximately 2 inches per day. Complete the table of values for the height, h, of the seedlings after t days.

t	0	5	10	15	20
h					



a Write an equation for the height of the seedlings in terms of the number of days since they were planted.

b Graph the equation.

Exercise 1.4. Use your equation from Exercise 1.3 to answer the questions. Illustrate each answer on the graph.

a How tall is the corn after 3 weeks?

b How long will it be before the corn is 6 feet tall?

For part (b), convert feet to inches.

1.1.2 Choosing Scales for the Axes

To create a useful graph, we must choose appropriate scales for the axes. The axes must extend far enough to show the values of the variables, and the tick marks should be equally spaced. Usually we should use no more than 10 or 15 tick marks.

Example 1.5. In 1990, the median price of a home in the US was \$92,000. The median price increased by about \$4700 per year over the next decade.

a Make a table of values showing the median price of a house in 1990, 1994, 1998, and 2000.

b Choose suitable scales for the axes and plot the values you found in part (a) on a graph. Use t, the number of years since 1990, on the horizontal axis and the price of the house, P, on the vertical axis. Draw a curve through the points.

c Write an equation that expresses P in terms of t.

d How much did the price of the house increase from 1990 to 1996? Illustrate the increase on your graph.

Solution.

a In 1990 the median price was \$92,000. Four years later, in 1994, the price had increased by $4(4700) = 18,800$ dollars, so

$$P = 92,000 + 4(4700) = 110,800$$

In 1998 the price had increased by $8(4700) = 37,600$ dollars, so

$$P = 92,000 + 8(4700) = 129,600$$

You can verify the price of the house in 2000 by a similar calculation.

Year	Price of House)	(t, P)
1990	$92,000$	$(0,\ 92,000)$
1994	$110,800$	$(4,\ 110,800)$
1998	$129,600$	$(8,\ 129,600)$
2000	$139,000$	$(10,\ 139,000)$

b Let t stand for the number of years since 1990, so that $t = 0$ in 1990, $t = 4$ in 1994, and so on. To choose scales for the axes, look at the values in the table. For this graph we scale the horizontal axis, or t-axis, in 1-year intervals and the vertical axis, or P-axis, for \$90,000 to \$140,000 in intervals of \$5,000. The points in Figure 1.6. lie on a straight line.

c Look back at the calculations in part (a). The price of the house started at \$92,000 in 1990 and increased by $t \times 4700$ dollars after t years. Thus,

$$P = 92,000 + 4700t$$

d Find the points on the graph corresponding to 1990 and 1996.

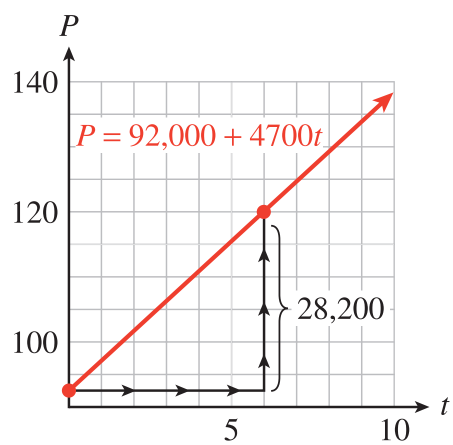


Figure 1.6

These points lie above $t = 0$ and $t = 6$ on the t-axis. Now find the values on the P-axis corresponding to the two points. The values are $P = 92,000$ in 1990 and $P = 120,200$ in 1996. The increase in price is the difference of the two P-values.

$$\begin{aligned} \text{increase in price} &= 120,200 - 92,000 \\ &= 28,200 \end{aligned}$$

The price of the home increased \$28,200 between 1990 and 1996. This increase is indicated by the arrows in Figure 1.6.

The graphs in the preceding examples are **increasing graphs**. As we move along the graph from left to right (in the direction of increasing t), the second coordinate increases as well. Try Exercise 1.7, which illustrates a **decreasing graph**.

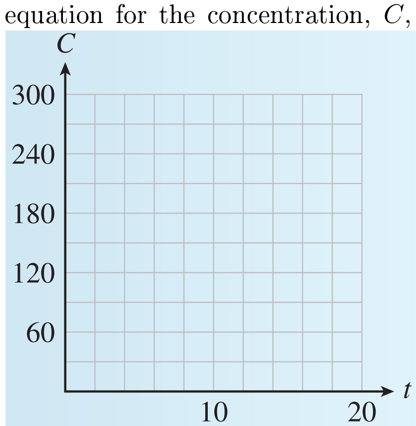
Exercise 1.7. Silver Lake has been polluted by industrial waste products. The concentration of toxic chemicals in the water is currently 285 parts per million (ppm). Local environmental officials would like to reduce the concentration by 15 ppm each year

a Complete the table of values showing the desired concentration, C, of toxic chemicals t years from now. For each t-value, calculate the corresponding value for C. Write your answers as ordered pairs.

t	C		(t, C)
0		$C = 285 - 150(\mathbf{0})$	(0,)
5		$C = 285 - 150(\mathbf{5})$	(5,)
10		$C = 285 - 150(\mathbf{10})$	(10,)
15		$C = 285 - 150(\mathbf{15})$	(15,)

b To choose scales for the axes, notice that the value of C starts at 285 and decreases from there. We'll scale the vertical axis up to 300, and use 10 tick marks at intervals of 30. Graph the ordered pairs on the grid, and connect them with a straight line. Extend the graph until it reaches the horizontal axis, but no farther. Points with negative C-coordinates have no meaning for the problem.

c Write an equation for the concentration, C, of toxic chemicals t years from now.



Hint. For part (c): The concentration is initially 8 ppm, and we subtract 15 ppm for each year that passes, or $15 \times t$.

Remark 1.8 (Graphing an Equation)**.** We can use a graphing calculator to graph an equation. On most calculators, we follow three steps.

To Graph an Equation:

1. Press `Y=` and enter the equation you wish to graph.
2. Press `WINDOW` and select a suitable graphing window.
3. Press `GRAPH`

Example 1.9 (Using a Graphing Calculator)**.** In Example 1.5, we found the equation $P = 92,000 + 4700t$ for the median price of a house t years after 1990. Graph this equation on a calculator.

Solution. To begin, we press `Y=` and enter

$$Y1 = 92,000 + 4700X$$

For this graph, we'll use the grid in Example 1.5 for our window settings, so we press `WINDOW` and enter

Xmin= 0	Xmax= 10
Ymin= 90,000	Ymax= 140,000

Finally, we press `GRAPH`. The calculator's graph is shown in Figure 1.10.

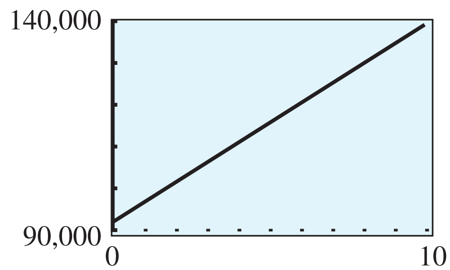


Figure 1.10

Exercise 1.11.

a Solve the equation $2y - 1575 = 45x$ for y in terms of x.

b Graph the equation on a graphing calculator. Use the window

Xmin= −50	Xmax= 50	Xscl= 5
Ymin= −500	Ymax= 1000	Yscl= 100

c Sketch the graph on paper. Use the window settings to choose appropriate scales for the axes.

1.1.3 Linear Equations

All the models in the preceding examples have equations with a similar form:

$$y = (\text{starting value}) + (\text{rate of change}) \cdot x$$

(We'll talk more about rate of change in Section 1.4.) Their graphs were all portions of straight lines. For this reason such equations are called **linear equations**. The order of the terms in the equation does not matter. For example, the equation in Example 1.1,

$$C = 5 + 3t$$

can be written equivalently as

$$-3t + C = 5$$

and the equation in Example 1.5,

$$P = 92,000 + 4700t$$

can be written as

$$-4700t + P = 92,000$$

This form of a linear equation,

$$Ax + By = C$$

is called the **general form**.

General Form for a Linear Equation

The graph of any equation

$$Ax + By = C$$

where A and B are not both equal to zero, is a straight line.

Example 1.12. The manager at Albert's Appliances has \$3000 to spend on advertising for the next fiscal quarter. A 30-second spot on television costs \$150 per broadcast, and a 30-second radio ad costs \$50.

a The manager decides to buy x television ads and y radio ads. Write an equation relating x and y.

b Make a table of values showing several choices for x and y.

c Plot the points from your table, and graph the equation.

Solution.

a Each television ad costs \$150, so x ads will cost \$150x. Similarly, y radio ads will cost \$50y. The manager has \$3000 to spend, so the sum of the costs must be \$3000. Thus,

$$150x + 50y = 3000$$

b Choose some values of x, and solve the equation for the corresponding value of y. For example, if $x = \mathbf{10}$ then

$$\begin{aligned} 150(\mathbf{10}) + 50y &= 300 \\ 1500 + 50y &= 3000 \\ 50y &= 1500 \\ y &= 30 \end{aligned}$$

If the manager buys 10 television ads, she can also buy 30 radio ads. You can verify the other entries in the table.

x	8	10	12	14
y	36	30	24	18

c Plot the points from the table. All the solutions lie on a straight line, as shown in Figure 1.13 .

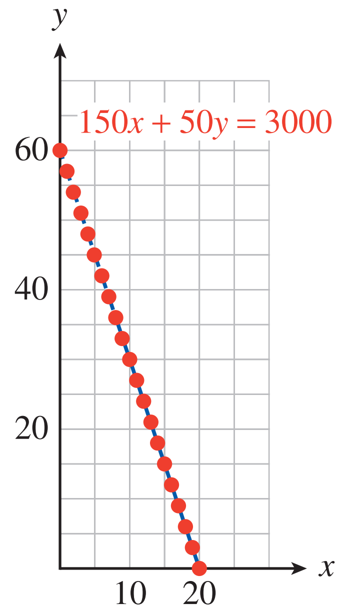


Figure 1.13

Exercise 1.14. In central Nebraska, each acre of corn requires 25 acre-inches of water per year, and each acre of winter wheat requires 18 acre-inches of water. (An acre-inch is the amount of water needed to cover one acre of land to a depth of one inch.) A farmer can count on 9000 acre-inches of water for the coming year. (Source: Institute of Agriculture and Natural Resources, University of Nebraska)

a Write an equation relating the number of acres of corn, x, and the number of acres of wheat, y, that the farmer can plant.

b Complete the table.

x	50	100	150	200
y				

1.1.4 Intercepts

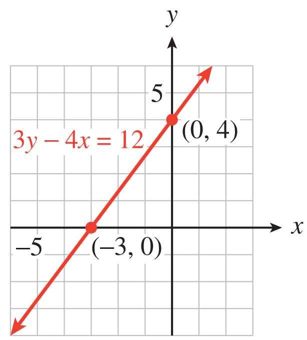


Figure 1.15

Consider the graph of the equation

$$3x - 4y = 12$$

shown in Figure 1.15. The points where the graph crosses the axes are called the **intercepts** of the graph. The coordinates of these points are easy to find. The y-coordinate of the x-intercept is zero, so we set $y = \mathbf{0}$ in the equation to get

$$\begin{aligned} 3(\mathbf{0}) - 4x &= 12 \\ x &= -3 \end{aligned}$$

The x-intercept is the point $(-3, 0)$. Also, the x-coordinate of the y-intercept is zero, so we set $x = \mathbf{0}$ in the equation to get

$$\begin{aligned} 3y - 4(\mathbf{0}) &= 12 \\ y &= 4 \end{aligned}$$

The y-intercept is $(0, 4)$.

Intercepts of a Graph

The points where a graph crosses the axes are called the **intercepts of the graph**.

1. To find the y-intercept, set $x = 0$ and solve for y.
2. To find the x-intercept, set $y = 0$ and solve for x

The intercepts of a graph tell us something about the situation it models.

Example 1.16.

a Find the intercepts of the graph in Exercise 1.7, about the pollution in Silver Lake.

b What do the intercepts tell us about the problem?

Solution.

a An equation for the concentration of toxic chemicals is

$$C = 285 - 15t$$

To find the C-intercept, set t equal to zero.

$$C = 285 - 15(0) = 285$$

The C-intercept is the point $(0, 285)$, or simply 285. To find the t-intercept, set C equal to zero and solve for t.

$\mathbf{0}$& $= 285 - 15t$&&Add $15t$ to both sides.\\$15t$& $= 285$&&Divide both sides by 15.\\$t$& $= 19$&&

The t-intercept is the point $(19, 0)$, or simply 19.

b The C-intercept represents the concentration of toxic chemicals in Silver Lake now: When $t = 0$, $C = 285$, so the concentration is currently 285 ppm. The t-intercept represents the number of years it will take for the concentration of toxic chemicals to drop to zero: When $C = 0$, $t = 19$, so it will take 19 years for the pollution to be eliminated entirely.

Exercise 1.17.

a Find the intercepts of the graph in Example 1.12, about the advertising budget for Albert's Appliances: $150x + 50y = 3000$.

b What do the intercepts tell us about the problem?

1.1.5 Intercept Method for Graphing Lines

Because we really only need two points to graph a linear equation, we might as well find the intercepts first and use them to draw the graph. The values of the intercepts will also help us choose suitable scales for the axes. It is always a good idea to find a third point as a check.

Example 1.18.

a Find the x- and y-intercepts of the graph of $150x - 180y = 9000$.

b Use the intercepts to graph the equation. Find a third point as a check.

Solution.

a To find the x-intercept, set $y = \mathbf{0}$.

$150x - 18(\mathbf{0})$& $= 9000$&&Simpify.\\$150x$& $= 9000$&&Divide both sides by 150.\\$x$& $= 60$&&

The x-intercept is the point $(60, 0)$. To find the y-intercept, set $x = \mathbf{0}$.

$150(\mathbf{0}) - 18y$& $= 9000$&&Simpify.\\ $-180y$& $= 9000$&&Divide both sides by -180.\\y& $= -50$&&

The y-intercept is the point $(0, -50)$.

b Scale both axes in intervals of 10 and then plot the two intercepts, $(60, 0)$ and $(0, -50)$. Draw the line through them, as shown in Figure 1.19. Now find another point and check that it lies on this line. We choose $x = \mathbf{20}$ and solve for y.

$$\begin{aligned}150(\mathbf{20}) - 180y &= 9000\\ 3000 - 180y &= 9000\\ -180y &= 6000\\ y &= -33.\overline{3}\end{aligned}$$

Plot the point $(20, -33\frac{1}{3})$. Because this point lies on the line, we can be reasonably confident that our graph is correct.

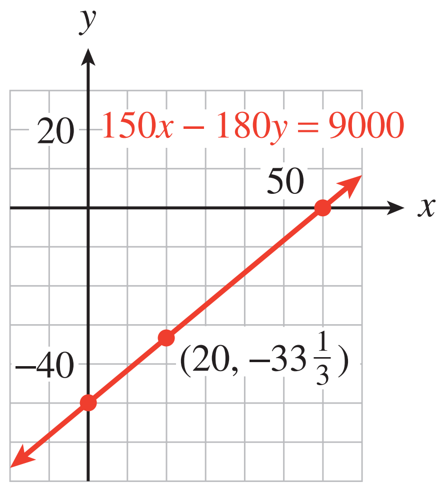


Figure 1.19

Remark 1.20 (Choosing a Graphing Window). Knowing the intercepts can also help us choose a suitable window on a graphing calculator. We would like the window to be large enough to show the intercepts. For the graph in Figure 1.19, we can enter the equation

$$Y = (9000 - 150X)/-180$$

in the window

Xmin$= -20$	Xmax$= 70$
Ymin$= -70$	Ymax$= 30$

To Graph a Line Using the Intercept Method:

1 Find the intercepts of the line.

++a To find the x-intercept, set $y = 0$ and solve for x.

++b To find the y-intercept, set $x = 0$ and solve for y.

2 Plot the intercepts.

3 Choose a value for x and find a third point on the line.

4 Draw a line through the points.

Exercise 1.21.

a In Exercise 1.14, you wrote an equation about crops in Nebraska. Find the intercepts of the graph.

b Use the intercepts to help you choose appropriate scales for the axes, and then graph the equation.

c What do the intercepts tell us about the problem?

The examples in this section model simple linear relationships between two variables. Such relationships, in which the value of one variable is determined by the value of the other, are called **functions**. We will study various kinds of functions throughout the course.

1.1.6 Section Summary

1.1.6.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Variable
- Mathematical model
- Increasing graph
- Linear equation
- Solve an equation
- Evaluate an expression
- Intercept
- Decreasing graph

1.1.6.2 CONCEPTS

1 We can describe a relationship between variables with a table of values, a graph, or an equation.

2 Linear models have equations of the following form:

$$y = (\text{starting value}) + (\text{rate of change}) \cdot x$$

3 To make a useful graph, we must choose appropriate scales for the axes.

General Form for a Linear Equation

4 The graph of any equation

$$Ax + By = C$$

where A and B are not both equal to zero, is a straight line.

5 The intercepts of a graph are the points where the graph crosses the axes.

6 We can use the intercepts to graph a line.

To Graph a Line Using the Intercept Method:

1 Find the intercepts of the line.

++a To find the x-intercept, set $y = 0$ and solve for x.

++b To find the y-intercept, set $x = 0$ and solve for y.

2 Plot the intercepts.

3 Choose a value for x and find a third point on the line.

4 Draw a line through the points.

7 The intercepts are also useful for interpreting a model.

1.1.6.3 STUDY QUESTIONS

1 Name three ways to represent a relationship between two variables.

2 If C is expressed in terms of H, which variable goes on the horizontal axis?

3 Explain the difference between evaluating an expression and solving an equation.

4 How many points do you need to graph a linear equation?

5 Explain how the words **intercept** and **intersect** are related; explain how they are different.

6 Delbert says that the intercepts of the line $3x+5y = 30$ are $(10, 6)$. What is wrong with his answer?

1.1.6.4 SKILLS

Practice each skill in the Homework 1.1.7 problems listed.

1 Make a table of values: #1–4, 7 and 8

2 Plot points and draw a graph: #1– 4, 7 and 8

3 Choose appropriate scales for the axes: #5–12

4 Write a linear model of the form $y =$ (starting value)+(rate of change)$\cdot x$: #1–8

5 Write a linear model in general form: #25–28, 33–36

6 Evaluate a linear expression, algebraically and graphically: #1–4

7 Solve a linear equation, algebraically and graphically: #1–4

8 Find the intercepts of a graph: #5 and 6, 13–24, 45–52

9 Graph a line by the intercept method: #5 and 6, 13–24

10 Interpret the meaning of the intercepts: #5 and 6, 25–28

11 Use a graphing calculator to graph a line: #37–52

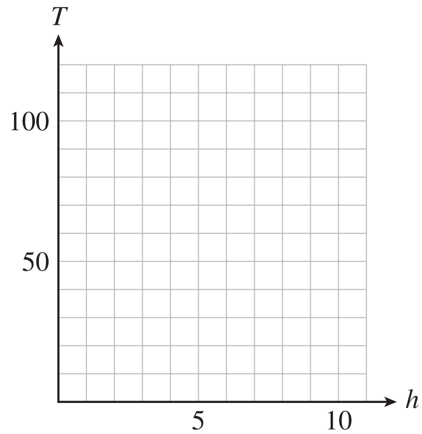
12 Sketch on paper a graph obtained on a calculator: #37–44

1.1.7 Homework

1. The temperature in the desert at 6 a.m., just before sunrise, was 65°F. The temperature rose 5 degrees every hour until it reached its maximum value at about 5 p.m. Complete the table of values for the temperature, T, at h hours after 6 a.m.

h	0	3	6	9	10
T					

a Write an equation for the temperature, T, in terms of h.



b Graph the equation.

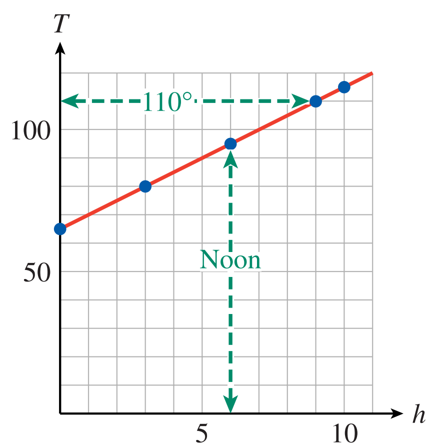
c How hot is it at noon? Illustrate the answer on your graph.

d When will the temperature be 110°F? Illustrate the answer on your graph.

Answer.

h	0	3	6	9	10
T	65	80	95	110	115

a $T = 65 + 5h$

b 

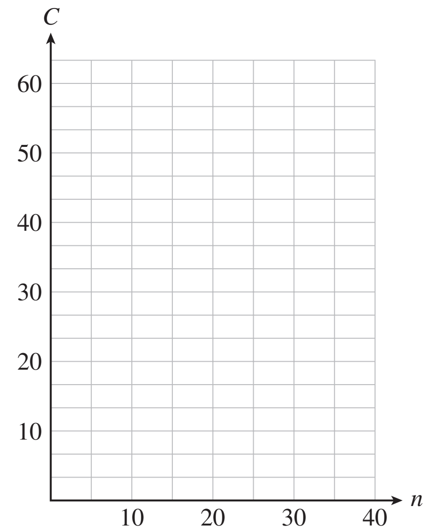
c 95°

d 3 p.m.

2. The taxi out of Dulles Airport charges a traveler with one suitcase an initial fee of \$2.00, plus \$1.50 for each mile traveled. Complete the table of values showing the charge, C, for a trip of n miles.

n	0	5	10	15	20	25
C						

(a) Write an equation for the charge, C, in terms of the number of miles traveled, n.



(b) Graph the equation.

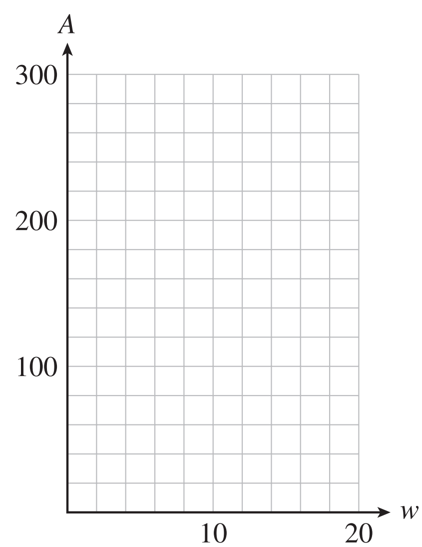
(c) What is the charge for a trip to Mount Vernon, 40 miles from the airport? Illustrate the answer on your graph.

(d) If a ride to the National Institutes of Health (NIH) costs \$39.50, how far is it from the airport to the NIH? Illustrate the answer on your graph.

3. On October 31, Betty and Paul fill their 250-gallon oil tank for their heater. Beginning in November, they use an average of 15 gallons of oil per week. Complete the table of values for the amount of oil, A, left in the tank after w weeks.

w	0	4	8	12	16
A					

(a) Write an equation that expresses the amount of oil, A, in the tank in terms of the number of weeks, w, since October 31.



(b) Graph the equation.

(c) How much did the amount of fuel oil in the tank decrease between the third week and the eighth week? Illustrate this amount on the graph.

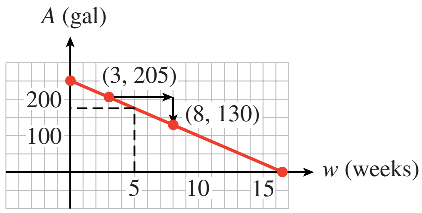
(d) When will the tank contain more than 175 gallons of fuel oil? Illustrate on the graph.

Answer.

w	0	4	8	12	16
A	250	190	130	70	10

(a) $A = 250 - 15w$

(b)



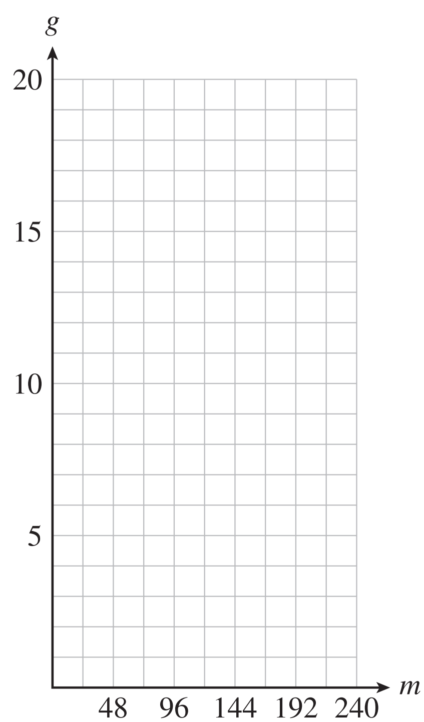
(c) 75 gallons

(d) Until the fifth week

4. Leon's camper has a 20-gallon gas tank, and he gets 12 miles to the gallon. (That is, he uses 1/12 gallon per mile.) Complete the table of values for the amount of gas, g, left in Leon's tank after driving m miles.

m	0	48	96	144	192
g					

(a) Write an equation that expresses the amount of gas, g, in Leon's fuel tank in terms of the number of miles, m, he has driven.



(b) Graph the equation.

(c) How much gas will Leon use between 8 a.m., when his odometer reads 96 miles, and 9 a.m., when the odometer reads 144 miles? Illustrate on the graph.

(d) If Leon has less than 5 gallons of gas left, how many miles has he driven? Illustrate on the graph.

5. Phil and Ernie buy a used photocopier for \$800 and set up a copy service on their campus. For each hour that the copier runs, Phil and Ernie make \$40.

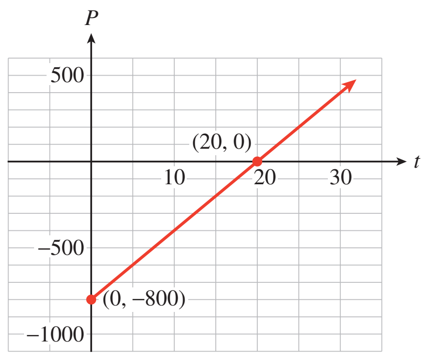
(a) Write an equation that expresses Phil and Ernie's profit (or loss), P, in terms of the number of hours, t, they run the copier.

(b) Find the intercepts and sketch the graph. (Suggestion: Scale the horizontal axis from 0 to 40 in increments of 5, and scale the vertical axis from -1000 to 400 in increments of 100.)

(c) What do the intercepts tell us about the profit?

Answer.

(a) $P = -800 + 40t$



(b) $(0, -800)$, $(20, 0)$

(c) The P-intercept, 800, is the initial ($t = 0$) value of the profit. Phil and Ernie start out \$800 in debt. The t-intercept, 20, is the number of hours required for Phil and Ernie to break even.

6. A deep-sea diver is taking some readings at a depth of 400 feet. He begins rising at 20 feet per minute.

(a) Write an equation that expresses the diver's altitude, h, in terms of the number of minutes, m, elapsed. (Consider a depth of 400 feet as an altitude of -400 feet.)

(b) Find the intercepts and sketch the graph. (Suggestion: Scale the horizontal axis from 0 to 24 in increments of 2, and scale the vertical axis from -500 to 100 in increments of 50.)

(c) What do the intercepts tell us about the diver's depth?

7. There are many formulas for estimating the annual cost of driving. The Automobile Club estimates that fixed costs for a small car—including insurance, registration, depreciation, and financing—total about \$5000 per year. The operating costs for gasoline, oil, maintenance, tires, and so forth are about 12.5 cents per mile. (Source: Automobile Association of America)

(a) Write an equation for the annual driving cost, C, in terms of d, the number of miles driven.

(b) Complete the table of values.

Miles Driven	4000	8000	12,000	16,000	20,000
Cost (\$)					

(c) Choose scales for the axes and graph the equation.

(d) How much does the annual cost of driving increase when the mileage increases from 8000 to 12,000 miles? Illustrate this amount on the graph.

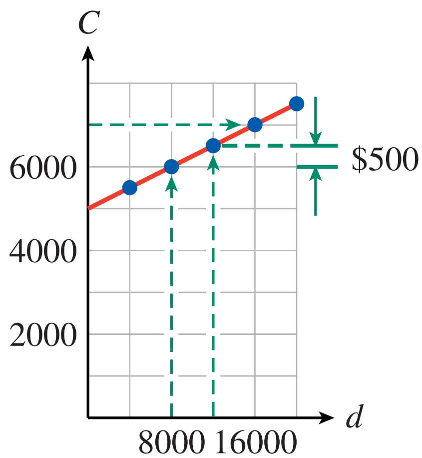
(e) How much mileage will cause the annual cost to exceed \$7000? Illustrate on the graph.

Answer.

(a) $C = 5000 + 0.125d$

(b) Complete the table of values.

Miles Driven	4000	8000	12,000	16,000	20,000
Cost (\$)	5500	6000	6500	7000	7500

(c) 

(d) \$500

(e) More than 16,000 miles

8. The boiling point of water changes with altitude. At sea level, water boils at 212°F, and the boiling point diminishes by approximately 0.002°F for each 1-foot increase in altitude.

(a) Write an equation for the boiling point, B, in terms of a, the altitude in feet.

(b) Complete the table of values.

Altitude (ft)	−500	0	1000	2000	3000	4000	5000
Boiling point (°F)							

(c) Choose scales for the axes and graph the equation.

(d) How much does the boiling point decrease when the altitude increases from 1000 to 3000 feet? Illustrate this amount on the graph.

(e) At what altitudes is the boiling point less than 204°F? Illustrate on the graph.

For each table, choose appropriate scales for the axes and plot the given points.

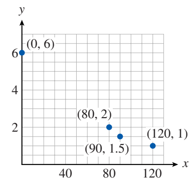
9.

x	0	80	90	120
y	6	2	1.5	1

10.

x	300	500	800	1100
y	1.2	1.3	1.5	1.9

Answer.



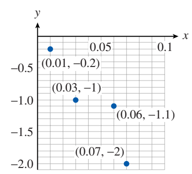
11.

x	0.01	0.03	0.06	0.07
y	-0.2	-1	-1.1	-2

12.

x	0.003	0.005	0.008	0.011
y	6	2	1.5	1

Answer.



For Problems 13-18,

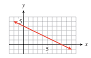
(a) Find the intercepts of the graph.

(b) Graph the equation by the intercept method.

13. $x + 2y = 8$

Answer.

a $(8, 0), (0, 4)$

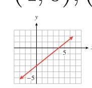
b 

14. $2x - y = 6$

15. $3x - 4y = 12$

Answer.

a $(4, 0), (0, -3)$

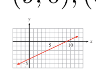
b 

16. $2x + 6y = 6$

17. $\dfrac{x}{9} - \dfrac{y}{4} = 1$

Answer.

a $(9, 0), (0, -4)$

b 

18. $\dfrac{x}{5} + \dfrac{y}{8} = 1$

For Problems 19-24,

(a) Find the intercepts of the graph.

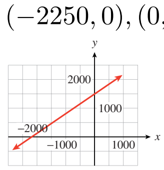
(b) Use the intercepts to choose scales for the axes, and then graph the equation by the intercept method.

19. $20x = 30y - 45,000$

20. $30x = 45y + 60,000$

Answer.

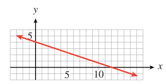
a $(-2250, 0), (0, 1500)$

b 

21. $0.4x + 1.2y = 4.8$

Answer.

a $(12, 0), (0, 4)$

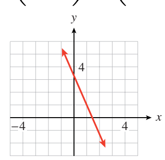
b 

22. $3.2x - 0.8y = 12.8$

23. $\dfrac{2x}{3} + \dfrac{3y}{11} = 1$

Answer.

a $\left(\dfrac{3}{2}, 0\right), \left(0, \dfrac{11}{3}\right)$

b 

24. $\dfrac{8x}{7} - \dfrac{2y}{7} = 1$

25. The owner of a gas station has $\$19,200$ to spend on unleaded gas this month. Regular unleaded costs him $\$2.40$ per gallon, and premium unleaded costs $\$3.20$ per gallon.

a How much do x gallons of regular cost? How much do y gallons of premium cost?

b Write an equation in general form that relates the amount of regular unleaded gasoline, , the owner can buy and the amount of premium unleaded, y.

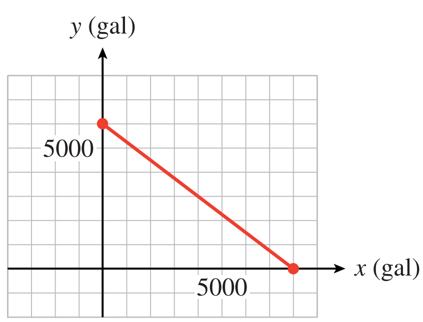
c Find the intercepts and sketch the graph.

d What do the intercepts tell us about the amount of gasoline the owner can purchase?

Answer.

a $\$2.40x, \$3.20y$

b $2.40x + 3.20y = 19,200$

c 

d The y-intercept, 6000 gallons, is the amount of premium that the gas station owner can buy if he buys no regular. The x-intercept, 8000 gallons, is the amount of regular he can buy if he buys no premium.

26. Five pounds of body fat is equivalent to $16,000$ calories. Carol can burn 600 calories per hour bicycling and 400 calories per hour swimming.

a How many calories will Carol burn in hours of cycling? How many calories will she burn in y hours of swimming?

b Write an equation in general form that relates the number of hours, x, of cycling and the number of hours, y, of swimming Carol needs to perform in order to lose 5 pounds.

c Find the intercepts and sketch the graph.

d What do the intercepts tell us about Carol's exercise program?

27. Delbert must increase his daily potassium intake by 1800 mg. He decides to eat a combination of figs and bananas, which are both low in sodium. There are 9 mg potassium per gram of fig, and 4 mg potassium per gram of banana.

a How much potassium is in x grams of fig? How much potassium is in y grams of banana?

b Write an equation in general form that relates the number of grams, x, of fig and the number of grams, y, of banana Delbert needs to get 1800 mg of potassium.

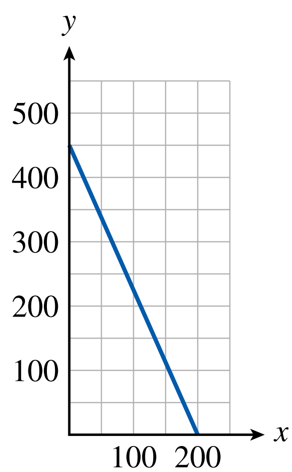
c Find the intercepts and sketch the graph.

d What do the intercepts tell us about Delbert's diet?

Answer.

a $9x$ mg, $4y$ mg

b $9x + 4y = 1800$

c 

d The x-intercept, 200 grams, tells how much fig Delbert should eat if he has no bananas, and the y-intercept, 450 grams, tells how much banana he should eat if he has no figs.

28. Leslie plans to invest some money in two CD accounts. The first account pays 3.6% interest per year, and the second account pays 2.8% interest per year. Leslie would like to earn \$500 per year on her investment.

a If Leslie invests x dollars in the first account, how much interest will she earn? How much interest will she earn if she invests y dollars in the second account?

b Write an equation in general form that relates x and y if Leslie earns \$500 interest.

c Find the intercepts and sketch the graph.

d What do the intercepts tell us about Leslie's investments?

29. Find the intercepts of the graph for each equation.

a $\dfrac{x}{3}+\dfrac{y}{5}=1$

b $2x-4y=1$

c $\dfrac{2x}{5}-\dfrac{2y}{3}=1$

d $\dfrac{x}{p}+\dfrac{y}{q}=1$

e Why is the equation $\dfrac{x}{a}+\dfrac{y}{b}=1$ called the **intercept form** for a line?

Answer.

a $(3,0),(0,5)$

b $\left(\dfrac{1}{2},0\right),\left(0,\dfrac{-1}{4}\right)$

c $\left(\dfrac{5}{2},0\right),\left(0,\dfrac{-3}{2}\right)$

d $(p,0),(0,q)$

e The value of a is the x-intercept, and the value of b is the y-intercept.

30. Write an equation in intercept form (see Problem 29) for the line with the given intercepts. Then write the equation in general form.

a $(6,0),(0,2)$

b $(-3,0),(0,8)$

c $\left(\dfrac{3}{4},0\right),\left(0,\dfrac{-1}{4}\right)$

d $(v,0),(0,-w)$

e $\left(\dfrac{1}{H},0\right),\left(0,\dfrac{1}{T}\right)$

31.

a Find the y-intercept of the line $y=mx+b$.

b Find the x-intercept of the line $y=mx+b$.

Answer.

a $(0,b)$

b $\left(\dfrac{-b}{m},0\right)$, if $m\neq 0$

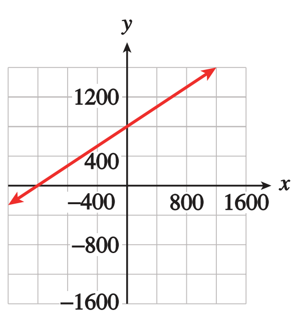
32.

a Find the y-intercept of the line $Ax+By=C$.

b Find the x-intercept of the line $Ax+By=C$.

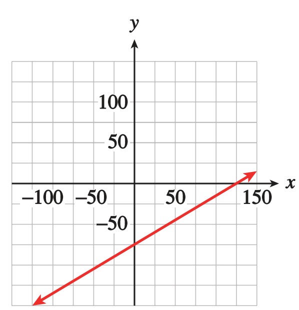
Write an equation in general form for each line.

33.

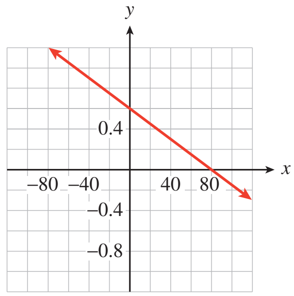


Answer. $-2x+3y=2400$

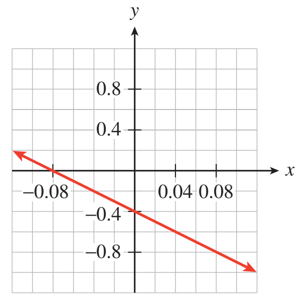
34.



35.



36.



Answer. $3x + 400y = 240$

For Problems 37–44,

a Solve each equation for y in terms of x. (See the Algebra Skills Refresher Section **??** to review this skill.)

b Graph the equation on your calculator in the specified window. (See Appendix **??** for help with the graphing calculator.)

c Make a pencil and paper sketch of the graph. Label the scales on your axes, and the coordinates of the intercepts.

37. $2 + y = 6$

Xmin $= -10$	Ymin $= -10$
Xmax $= 10$	Ymax $= 10$
Xscl $= 1$	Yscl $= 1$

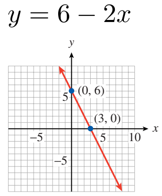
38. $8 - y + 3x = 0$

Xmin $= -10$	Ymin $= -10$
Xmax $= 10$	Ymax $= 10$
Xscl $= 1$	Yscl $= 1$

Answer.

a. $y = 6 - 2x$

c.



39. $3x - 4y = 1200$

Xmin $= -1000$	Ymin $= -1000$
Xmax $= 1000$	Ymax $= 1000$
Xscl $= 100$	Yscl $= 100$

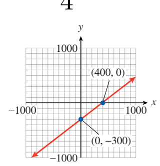
40. $x + 2y = 500$

Xmin $= -1000$	Ymin $= -1000$
Xmax $= 1000$	Ymax $= 1000$
Xscl $= 100$	Yscl $= 100$

Answer.

a. $y = \frac{3}{4}x - 300$

c.



41. $0.2x + 5y = 0.1$

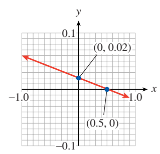
$$\begin{array}{rl} \text{Xmin} = -1 & \text{Ymin} = -0.1 \\ \text{Xmax} = 1 & \text{Ymax} = 0.1 \\ \text{Xscl} = 0.1 & \text{Yscl} = 0.01 \end{array}$$

42. $1.2x - 4.2y = 3.6$

$$\begin{array}{rl} \text{Xmin} = -1 & \text{Ymin} = -1 \\ \text{Xmax} = 4 & \text{Ymax} = 1 \\ \text{Xscl} = 0.2 & \text{Yscl} = 0.1 \end{array}$$

Answer.

a. $y = 0.02 - 0.04x$

c. 

43. $70x + 3y = y + 420$

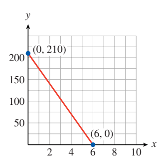
$$\begin{array}{rl} \text{Xmin} = 0 & \text{Ymin} = 0 \\ \text{Xmax} = 10 & \text{Ymax} = 250 \\ \text{Xscl} = 1 & \text{Yscl} = 25 \end{array}$$

44. $40y - 5x = 780 - 20y$

$$\begin{array}{rl} \text{Xmin} = -200 & \text{Ymin} = 0 \\ \text{Xmax} = 0 & \text{Ymax} = 20 \\ \text{Xscl} = 20 & \text{Yscl} = 2 \end{array}$$

Answer.

a. $y = 210 - 35x$

c. 

For Problems 45–52,

a Find the - and y-intercepts.

b Solve the equation for y.

c Choose a graphing window in which both intercepts are visible, and graph the equation on your calculator.

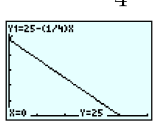
45. $x + 4y = 100$

46. $2x - 3y = -72$

Answer.

a $(100, 0), (0, 25)$

b $y = 25 - \dfrac{1}{4}x$

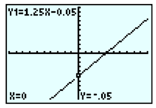
c 

47. $25x - 20y = 1$

Answer.

a $(0.04, 0), (0, -0.05)$

b $y = 1.25x - 0.05$

c 

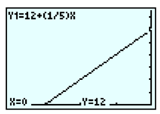
48. $4x + 75y = 60,000$

49. $\dfrac{y}{12} - \dfrac{x}{60} = 1$

Answer.

a $(-60, 0), (0, 12)$

b $y = 12 + \dfrac{1}{5}x$

c 

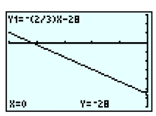
50. $\dfrac{x}{80} + \dfrac{y}{400} = 1$

51. $-2x = 3y + 84$

Answer.

a $(-42, 0), (0, -28)$

b $y = \dfrac{-2}{3}x - 28$

c 

52. $7x = 91 - 13y$

1.2 Functions

1.2.1 Definition of Function

We often want to predict values of one variable from the values of a related variable. For example, when a physician prescribes a drug in a certain dosage, she needs to know how long the dose will remain in the bloodstream. A sales manager needs to know how the price of his product will affect its sales. A **function** is a special type of relationship between variables that allows us to make such predictions.

Suppose it costs \$800 for flying lessons, plus \$30 per hour to rent a plane. If we let C represent the total cost for t hours of flying lessons, then

$$C = 800 + 305t \quad (t \geq 0)$$

Thus, for example

$$\begin{array}{lll} \text{when} & t = \mathbf{0}, & C = 800 + 30(\mathbf{0}) = 800 \\ \text{when} & t = \mathbf{4}, & C = 800 + 30(\mathbf{4}) = 920 \\ \text{when} & t = \mathbf{10}, & C = 800 + 30(\mathbf{10}) = 1100 \end{array}$$

The variable t is called the **input** or **independent** variable, and C is the **output** or **dependent** variable, because its values are determined by the value of t. We can display the relationship between two variables by a table or by ordered pairs. The input variable is the first component of the ordered pair, and the output variable is the second component.

t	C	(t, C)
0	800	$(0, 800)$
4	920	$(4, 920)$
10	1100	$(10, 1100)$

For this relationship, we can find the value C of associated with any given value of t. All we have to do is substitute the value of t into the equation and solve for C. The result has no ambiguity: Only one value for C corresponds to each value of t. This type of relationship between variables is called a **function**. In general, we make the following definition.

Definition of Function

A **function** is a relationship between two variables for which a unique value of the **output** variable can be determined from a value of the **input** variable.

What distinguishes functions from other variable relationships? The definition of a function calls for a *unique value*—that is, *exactly one value* of the output variable corresponding to each value of the input variable. This property makes functions useful in applications because they can often be used to make predictions.

Example 1.22.

a The distance, d, traveled by a car in 2 hours is a function of its speed, r. If we know the speed of the car, we can determine the distance it travels by the formula $d = r \cdot 2$.

b The cost of a fill-up with unleaded gasoline is a function of the number of gallons purchased. The gas pump represents the function by displaying the corresponding values of the input variable (number of gallons) and the output variable (cost).

c Score on the Scholastic Aptitude Test (SAT) is not a function of score on an IQ test, because two people with the same score on an IQ test may score differently on the SAT; that is, a person's score on the SAT is not uniquely determined by his or her score on an IQ test.

Exercise 1.23.

a As part of a project to improve the success rate of freshmen, the counseling department studied the grades earned by a group of students in English and algebra. Do you think that a student's grade in algebra is a function of his or her grade in English? Explain why or why not.

b Phatburger features a soda bar, where you can serve your own soft drinks in any size. Do you think that the number of calories in a serving of Zap Kola is a function of the number of fluid ounces? Explain why or why not.

1.2.2 Functions Defined by Tables

When we use a table to describe a function, the first variable in the table (the left column of a vertical table or the top row of a horizontal table) is the input variable, and the second variable is the output. We say that the output variable is a function of the input.

Example 1.24.

a Table 1.25 shows data on sales compiled over several years by the accounting office for Eau Claire Auto Parts, a division of Major Motors. In this example, the year is the input variable, and total sales is the output. We say that total sales, S, is a function of t.

Year (t)	Total sales (S)
2000	\$612,000
2001	\$663,000
2002	\$692,000
2003	\$749,000
2004	\$904,000

Table 1.25

b Table 1.26 gives the cost of sending printed material by first-class mail in 2016.

Weight in ounces (w)	Postage (P)
$0 < w \leq 1$	\$0.47
$1 < w \leq 2$	\$0.68
$2 < w \leq 3$	\$0.89
$3 < w \leq 4$	\$1.10
$4 < w \leq 5$	\$1.31
$5 < w \leq 6$	\$1.52
$6 < w \leq 7$	\$1.73

Table 1.26

If we know the weight of the article being shipped, we can determine the required postage from Table 1.26. For instance, a catalog weighing 4.5 ounces would require \$1.31 in postage. In this example, w is the input variable and p is the output variable. We say that *p is a function of w.*

c Table 1.27 records the age and cholesterol count for 20 patients tested in a hospital survey.

Age	Cholesterol count		Age	Cholesterol count
53	217		**51**	**209**
48	232		53	241
55	198		49	186
56	238		**51**	**216**
51	**227**		57	208
52	264		52	248
53	195		50	214
47	203		56	271
48	212		53	193
50	234		48	172

Table 1.27

According to these data, cholesterol count is *not* a function of age, because several patients who are the same age have different cholesterol levels. For example, three different patients are 51 years old but have cholesterol counts of 227, 209, and 216, respectively. Thus, we cannot determine a *unique* value of the output variable (cholesterol count) from the value of the input variable (age). Other factors besides age must influence a person's cholesterol count.

Exercise 1.28. Decide whether each table describes y as a function of x. Explain your choice.

a

x	3.5	2.0	2.5	3.5	2.5	4.0	2.5	3.0
y	2.5	3.0	2.5	4.0	3.5	4.0	2.0	2.5

b

x	-3	-2	-1	0	1	2	3
y	17	3	0	-1	0	3	17

1.2.3 Functions Defined by Graphs

A graph may also be used to define one variable as a function of another. The input variable is displayed on the horizontal axis, and the output variable on the vertical axis.

Example 1.29. Figure 1.30 shows the number of hours, H, that the sun is above the horizon in Peoria, Illinois, on day t, where January 1 corresponds to $t = 0$.

a Which variable is the input, and which is the output?

b Approximately how many hours of sunlight are there in Peoria on day 150?

c On which days are there 12 hours of sunlight?

d What are the maximum and minimum values of H, and when do these valuesoccur?

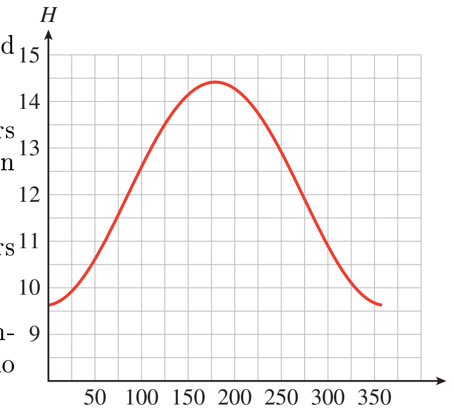


Figure 1.30

Solution.

a The input variable, t, appears on the horizontal axis. The number of daylight hours, H, is a function of the date. The output variable appears on the vertical axis.

b The point on the curve where $t = 150$ has $H \approx 14.1$, so Peoria gets about 14.1 hours of daylight when $t = 150$, which is at the end of May.

c $H = 12$ at the two points where $t \approx 85$ (in late March) and $t \approx 270$ (late September).

d The maximum value of 14.4 hours occurs on the longest day of the year, when $t \approx 170$, about three weeks into June. The minimum of 9.6 hours occurs on the shortest day, when $t \approx 355$, about three weeks into December.

Exercise 1.31. Figure 1.32 shows the elevation in feet, a, of the Los Angeles Marathon course at a distance d miles into the race. (Source: *Los Angeles Times*, March 3, 2005)

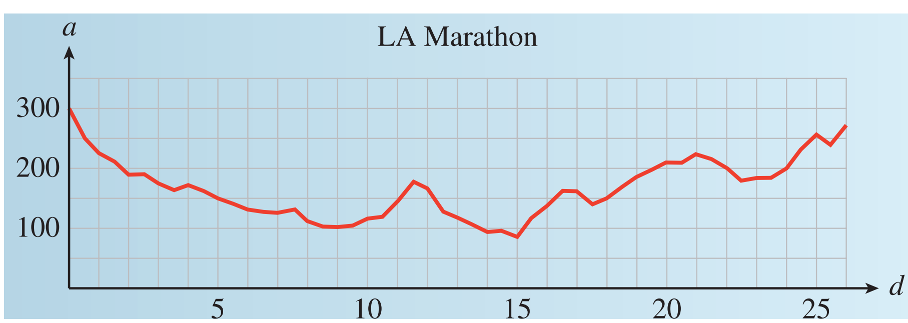


Figure 1.32

a Which variable is the input, and which is the output?

b What is the elevation at mile 20?

c At what distances is the elevation 150 feet?

d What are the maximum and minimum values of a, and when do these values occur?

e The runners pass by the Los Angeles Coliseum at about 4.2 miles into the race. What is the elevation there?

1.2.4 Functions Defined by Equations

Example 1.33 illustrates a function defined by an equation.

Example 1.33. As of 2016, One World Trade Center in New York City is the nation's tallest building, at 1776 feet. If an algebra book is dropped from the top of the Sears Tower, its height above the ground after t seconds is given by the equation

$$h = 1776 - 16t^2$$

Thus, after **1** second the book's height is

$$h = 1776 - 16(\mathbf{1})^2 = 1760 \text{ feet}$$

After **2** seconds its height is

$$h = 1776 - 16(\mathbf{2})^2 = 1712 \text{ feet}$$

For this function, t is the input variable and h is the output variable. For any value of t, a unique value of h can be determined from the equation for h. We say that *h is a function of t*.

Exercise 1.34. Write an equation that gives the volume, V, of a sphere as a function of its radius, r.

Remark 1.35 (Making a Table of Values with a Calculator)**.** We can use a graphing calculator to make a table of values for a function defined by an equation. For the function in Example 1.33,

$$h = 1776 - 16t^2$$

we begin by entering the equation: Press the `Y=` key, clear out any other equations, and define $Y_1 = 1776 - 16X^2$.

Next, we choose the x-values for the table. Press `2ndWINDOW` to access the TblSet (Table Setup) menu and set it to look like Figure 1.36. This setting will give us an initial x-value of 0 ($TblStart = 0$) and an increment of one unit in the x-values, ($\Delta Tbl = 1$). It also fills in values of both variables automatically. Now press `2nd GRAPH` to see the table of values, as shown in Figure 1.37. From this table, we can check the heights we found in Example 1.33.

Now try making a table of values with $TblStart = 0$ and $\Delta Tbl = 0.5$. Use the â and â arrow keys to scroll up and down the table.

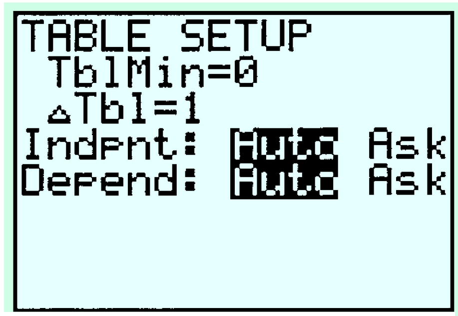


Figure 1.36

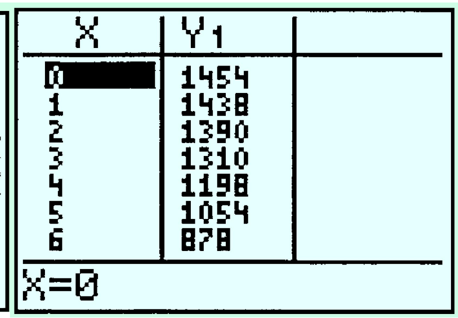


Figure 1.37

1.2.5 Function Notation

There is a convenient notation for discussing functions. First, we choose a letter, such as f, g, or h (or F, G, or H), to name a particular function. (We can use any letter, but these are the most common choices.) For instance, in Example 1.33, the height, h, of a falling algebra book is a function of the elapsed time, t. We might call this function f. In other words, f is the name of the relationship between the variables h and t. We write

$$h = f(t)$$

which means "h is a function of t, and f is the name of the function."

The new symbol $f(t)$, read "f of t," is another name for the height, h. The parentheses in the symbol $f(t)$ do not indicate multiplication. (It would not make sense to multiply the name of a function by a variable.) Think of the symbol $f(t)$ as a single variable that represents the output value of the function.

With this new notation we may write

$$h = f(t) = 1776 - 16t^2$$

or just

$$f(t) = 1776 - 16t^2$$

instead of

$$h = 1776 - 16t^2$$

to describe the function.

Perhaps it seems complicated to introduce a new symbol for h, but the notation $f(t)$ is very useful for showing the correspondence between specific values of the variables h and t.

Example 1.38. In Example 1.33, the height of an algebra book dropped from the top of the Sears Tower is given by the equation

$$h = 1776 - 16t^2$$

We see that

$$\begin{array}{ll} \text{when } t = 1 & h = 1760 \\ \text{when } t = 2 & h = 1712 \end{array}$$

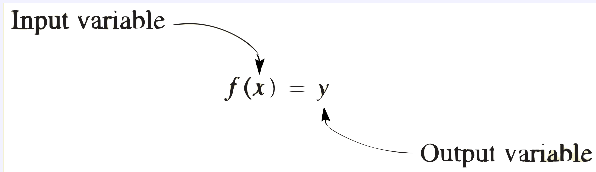
Using function notation, these relationships can be expressed more concisely as

$$f(1) = 1760 \quad \text{and} \quad f(2) = 1712$$

which we read as "f of 1 equals 1760" and "f of 2 equals 1712." The values for the input variable, t, appear *inside* the parentheses, and the values for the output variable, h, appear on the other side of the equation.

Remember that when we write $y = f(x)$, the symbol $f(x)$ is just another name for the output variable.

Function Notation



Exercise 1.39. Let F be the name of the function defined by the graph in Example 1.29, the number of hours of daylight in Peoria.

a Use function notation to state that H is a function of t.

b What does the statement $F(15) = 9.7$ mean in the context of the problem?

1.2.6 Evaluating a Function

Finding the value of the output variable that corresponds to a particular value of the input variable is called **evaluating the function**.

Example 1.40. Let g be the name of the postage function defined by Table 1.26 in Example 1.22. Find $g(1)$, $g(3)$, and $g(6.75)$.

Solution. According to the table,

$$\begin{array}{llll} \text{when } w = 1, & p = 0.47 & \text{so} & g(1) = 0.47 \\ \text{when } w = 3, & p = 0.89 & \text{so} & g(3) = 0.89 \\ \text{when } w = 6.75, & p = 1.73 & \text{so} & g(6.75) = 1.73 \end{array}$$

Thus, a letter weighing 1 ounce costs \$0.47 to mail, a letter weighing 3 ounces costs \$0.89, and a letter weighing 6.75 ounces costs \$1.73.

Exercise 1.41. When you exercise, your heart rate should increase until it reaches your target heart rate. The table shows target heart rate, $r = f(a)$, as a function of age.

a	20	25	30	35	40	45	50	55	60	65	70
r	150	146	142	139	135	131	127	124	120	116	112

a Find $f(25)$ and $f(50)$.

b Find a value of a for which $f(a) = 135$.

If a function is described by an equation, we simply substitute the given input value into the equation to find the corresponding output, or function value.

Example 1.42. The function H is defined by $H = f(s) = \dfrac{\sqrt{s+3}}{s}$. Evaluate the function at the following values.

a $s = 6$

b $s = -1$

Solution.

a $f(\mathbf{6}) = \frac{\sqrt{\mathbf{6}+3}}{\mathbf{6}} = \frac{\sqrt{9}}{6} = \frac{3}{6} = \frac{1}{2}$. Thus, $f(6) = \frac{1}{2}$.

b $f(\mathbf{-1}) = \frac{\sqrt{\mathbf{-1}+3}}{\mathbf{-1}} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$. Thus, $f(-1) = -\sqrt{2}$.

Exercise 1.43. Complete the table displaying ordered pairs for the function $f(x) = 5 - x^3$. Evaluate the function to find the corresponding $f(x)$-value for each value of x.

x	$f(x)$	
-2		$f(\mathbf{-2}) = 5 - (\mathbf{-2})^3 =$
0		$f(\mathbf{0}) = 5 - \mathbf{0}^3 =$
1		$f(\mathbf{1}) = 5 - \mathbf{1}^3 =$
3		$f(\mathbf{3}) = 5 - \mathbf{3}^3 =$

Remark 1.44 (Evaluating a Function)**.** We can use the table feature on a graphing calculator to evaluate functions. Consider the function of Exercise 1.43, $f(x) = 5 - x^3$.

Press `Y=`, clear any old functions, and enter

$$Y_1 = 5 - X\ {\scriptstyle 3}$$

Then press TblSet (`2nd WINDOW`) and choose Ask after Indpnt, as shown in Figure 1.45, and press `ENTER`. This setting allows you to enter any x-values you like. Next, press TABLE (using `2nd GRAPH`).

To follow Exercise 1.43, key in `(-)` 2 `ENTER` for the x-value, and the calculator will fill in the y-value. Continue by entering 0, 1, 3, or any other x-values you choose. One such table is shown in Figure 1.46.

If you would like to evaluate a new function, you do not have to return to the `Y=` screen. Use the â and â arrow keys to highlight Y_1 at the top of the second column. The definition of Y_1 will appear at the bottom of the display, as shown in Figure 1.46. You can key in a new definition here, and the second column will be updated automatically to show the y-values of the new function.

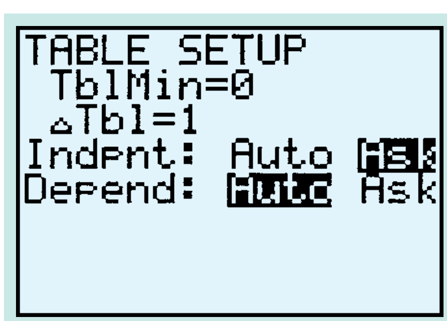


Figure 1.45

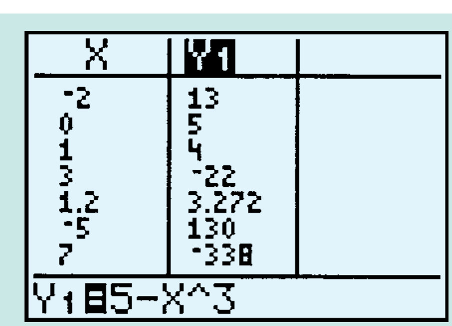


Figure 1.46

To simplify the notation, we sometimes use the same letter for the output variable and for the name of the function. In the next example, C is used in this way.

Example 1.47. TrailGear decides to market a line of backpacks. The cost, C, of manufacturing backpacks is a function of the number, x, of backpacks produced, given by the equation

$$C(x) = 3000 + 20x$$

where $C(x)$ is measured in dollars. Find the cost of producing 500 backpacks.

Solution. To find the value of C that corresponds to $x = \mathbf{500}$, evaluate $C(500)$.

$$C(\mathbf{500}) = 3000 + 20(\mathbf{500}) = 13,000$$

The cost of producing 500 backpacks is \$13,000.

Exercise 1.48. The volume of a sphere of radius r centimeters is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

Evaluate $V(10)$ and explain what it means.

1.2.7 Operations with Function Notation

Sometimes we need to evaluate a function at an algebraic expression rather than at a specific number.

Example 1.49. TrailGear manufactures backpacks at a cost of

$$C(x) = 3000 + 20x$$

for x backpacks. The company finds that the monthly demand for backpacks increases by 50

a If each co-op usually produces b backpacks per month, how many should it produce during the summer months?

b What costs for producing backpacks should the company expect during the summer?

Solution.

a An increase of 50

b The cost of producing $1.5b$ backpacks will be

$$C(\mathbf{1.5b}) = 3000 + 20(\mathbf{1.5b}) = 3000 + 30b$$

Exercise 1.50. A spherical balloon has a radius of 10 centimeters.

a If we increase the radius by h centimeters, what will the new volume be?

b If $h = 2$, how much did the volume increase?

Example 1.51. Evaluate the function $f(x) = 4x^2 - x + 5$ for the following expressions.

a $x = 2h$

b $x = a + 3$

Solution.

a
$$\begin{aligned} f(\mathbf{2h}) &= 4(\mathbf{2h})^2 - (\mathbf{2h}) + 5 \\ &= 4(4h^2) - 2h + 5 \\ &= 16h^2 - 2h + 5 \end{aligned}$$

b
$$\begin{aligned} f(\boldsymbol{a+3}) &= 4(\boldsymbol{a+3})^2 - (\boldsymbol{a+3}) + 5 \\ &= 4(a^2 + 6a + 9) - a - 3 + 5 \\ &= 4a^2 + 24a + 36 - a + 2 \\ &= 4a^2 + 23a + 38 \end{aligned}$$

CAUTION In Example 1.51, notice that

$$f(2h) \neq 2f(h)$$

and

$$f(a+3) \neq f(a) + f(3)$$

To compute $f(a) + f(3)$, we must first compute $f(a)$ and $f(3)$, then add them:

$$f(a) + f(3)\& = (4a^2 - a + 5) + (4 \cdot 3^2 - 3 + 5)\backslash\backslash\& = 4a^2 - a + 43\backslash\backslash$$

In general, it is not true that $f(a+b) = f(a) + f(b)$. Remember that the parentheses in the expression $f(x)$ do not indicate multiplication, so the distributive law does not apply to the expression $f(a+b)$.

Exercise 1.52. Let $f(x) = x^3 - 1$ and evaluate each expression.

a $f(2) + f(3)$

b $f(2+3)$

c $2f(x) + 3$

1.2.8 Section Summary

1.2.8.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Function
- Input variable
- Independent variable
- Dependent variable
- Function value
- Output variable

1.2.8.2 CONCEPTS

1 A function is a rule that assigns to each value of the input variable a unique value of the output variable.

2 Functions may be defined by words, tables, graphs, or equations.

3 Function notation: $y = f(x)$, where x is the input and y is the output.

1.2.8.3 STUDY QUESTIONS

1 What property makes a relation between two variables a function?

2 Name three ways to define a function.

3 Give an example of a function in which two distinct values of the input variable correspond to the same value of the output variable.

4 Use function notation to write the statement "G defines w as a function of p."

5 Give an example of a function for which $f(2+3) \neq f(2)+f(3)$.

1.2.8.4 SKILLS

Practice each skill in the Homework 1.2.9 problems listed.

1 Decide whether a relationship between two variables is a function: #1–26

2 Evaluate a function defined by a table, a graph, or an equation: #27–54

3 Choose appropriate scales for the axes: #5–12

4 Interpret function notation: #31–34, 49–54

5 Simplify expressions involving function notation: #59–76

1.2.9 Homework

For which of the following pairs is the second quantity a function of the first? Explain your answers.

1. Price of an item; sales tax on the item at 4%

Answer. Function; the tax is determined by the price of the item.

2. Time traveled at constant speed; distance traveled

3. Number of years of education; annual income

Answer. Not a function; incomes may differ for same number of years of education.

4. Distance flown in an airplane; price of the ticket

5. Volume of a container of water; the weight of the water

Answer. Function; weight is determined by volume.

6. Amount of a paycheck; amount of Social Security tax withheld

Each of the following objects establishes a correspondence between two variables. Suggest appropriate input and output variables and decide whether the relationship is a function.

7. An itemized grocery receipt

Answer. Input: items purchased; output: price of item. Yes, a function because each item has only one price.

8. An inventory list

9. An index

Answer. Input: topics; output: page or pages on which topic occurs. No, not a function because the same topic may appear in more than one page.

10. A will

11. An instructor's grade book

Answer. Input: students' names; output: students' scores on quizzes, tests, etc. No, not a function because the same student can have different grades on different tests.

12. An address book

13. A bathroom scale

Answer. Input: person stepping on scales; output: person's weight. Yes, a function because a person cannot have two different weights at the same time.

14. A radio dial

Which of the following tables define the second variable as a function of the first variable? Explain why or why not.

15.

x	t
-1	2
0	9
1	-2
0	-3
-1	5

Answer. No

16.

y	w
0	8
1	12
3	7
5	-3
7	4

17.

x	y
-3	8
-2	3
-1	0
0	-1
1	0
2	3
3	8

Answer. Yes

18.

s	t
2	5
4	10
6	15
8	20
6	25
4	30
2	35

19.

r	-4	-2	0	2	4
v	6	6	3	6	8

Answer. Yes

20.

p	-5	-4	-3	-2	-1
d	-5	-4	-3	-2	-1

21.

Pressure (p)	Volume (v)
15	100.0
20	75.0
25	60.0
30	50.0
35	42.8
40	37.5
45	33.3
50	30.0

22.

Frequency (f)	Wavelength (w)
5	60.0
10	30.0
20	15.0
30	10.0
40	7.5
50	6.0
60	5.0
70	4.3

Answer. Yes

23.

Temperature (T)	Humidity (h)
Jan. 1 34°F	42%
Jan. 2 36°F	44%
Jan. 3 35°F	47%
Jan. 4 29°F	50%
Jan. 5 31°F	52%
Jan. 635°F 49%	51$Jan.$734°F

24.

Inflation rate (I)		Unemployment rate (U)
1972	5.6%	5.1%
1973	6.2%	4.5%
1974	10.1%	4.9%
1975	9.2%	7.4%
1976	5.8%	6.7%
1977	5.6%	6.8%
1978	6.7%	7.4%

Answer. No

25.

Adjusted gross income (I)	Tax bracket (T)
\$0 – 2479	0%
\$2480 – 3669	4.5%
\$3670 – 4749	12%
\$4750 – 7009	14%
\$7010 – 9169	15%
9170-11,649	16%
11,650-13,919	18%

26.

Cost of merchandise (M)	Shipping charge (C)
\$0.01 – 10.00	\$2.50
10.01 – 20.00	3.75
20.01 – 35.00	4.85
30.01 – 50.00	5.95
50.01 – 75.00	6.95
75.01 – 100.00	7.95
Over 100.00	8.95

Answer. Yes

27. The function described in Problem 21 is called g, so that $v = g(p)$. Find the following:

a $g(25)$

b $g(40)$

c x so that $g(x) = 50$

Answer.

a 60

b 37.5

c 30

28. The function described in Problem 22 is called h, so that $w = h(f)$. Find the following:

a $h(20)$

b $h(60)$

c x so that $h(x) = 10$

29. The function described in Problem 25 is called T, so that $T = T(I)$. Find the following:

a $T(8750)$

b $T(6249)$

c x so that $T(x) = 15\%$

Answer.

a 15%

b 14%

c $7010˘$9169

30. The function described in Problem 26 is called C, so that $C = C(M)$. Find the following:

a $C(11.50)$

b $C(47.24)$

c x so that $C(x) = 7.95$

31. Data indicate that U.S. women are delaying having children longer than their counterparts 50 years ago. The table shows $f(t)$ the percent of 20–24-year-old women in year t who had not yet had children. (Source: U.S. Dept of Health and Human Services)

Year (t)	1960	1965	1970	1975	1980	1985	1990	1995	2000
Percent of women	47.5	51.4	47.0	62.5	66.2	67.7	68.3	65.5	66.0

a Evaluate $f(1985)$ and explain what it means.

b Estimate a solution to the equation $f(t) = 68$ and explain what it means.

c In 1997, 64.9% of 20–24-year-old women had not yet had children. Write an equation with function notation that states this fact.

Answer.

a 67.7: In 1985, 67.7% of 20–24 year old women had not yet had children.

b 1987: 1987 is approximately when 68% of 20–24 year old women had not yet had children.

c $f(1997) = 64.9$

32. The table shows $f(t)$, the death rate (per 100,000 people) from HIV among 15–24-year-olds, and $g(t)$, the death rate from HIV among 25–34-year-olds, for selected years from 1997 to 2002. (Source: U.S. Dept of Health and Human Services)

Year	1987	1988	1989	1990	1992	1994	1996	1998	2000	2002
15–24-year-olds	1.3	1.4	1.6	1.5	1.6	1.8	1.1	0.6	0.5	0.4
25–34-year-olds	11.7	14.0	17.9	19.7	24.2	28.6	19.2	8.1	6.1	4.6

a Evaluate $f(1995)$ and explain what it means.

b Find a solution to the equation $g(t) = 28.6$ and explain what it means.

c In 1988, the death rate from HIV for 25–34-year-olds was 10 times the corresponding rate for 15–24-year-olds. Write an equation with function notation that states this fact.

33. When you exercise, your heart rate should increase until it reaches your target heart rate. The table shows target heart rate, $r = f(a)$, as a function of age.

a	20	25	30	35	40	50	55	60	65	70
r	150	146	142	139	131	127	124	120	116	112

a Does $f(50) = 2f(25)$?

b Find a value of a for which $f(a) = 2a$. Is $f(a) = 2a$ for all values of a?

c Is $r = f(a)$ an increasing function or a decreasing function?

Answer.

a No

b 60; no

c Decreasing

34. The table shows $M = f(d)$, the men's Olympic record time, and $W = g(d)$, the women's Olympic record time, as a function of the length, d, of the race. For example, the women's record in the 100 meters is 10.62 seconds, and the men's record in the 800 meters is 1 minute, 42.58 seconds. (Source: www.hickoksports.com)

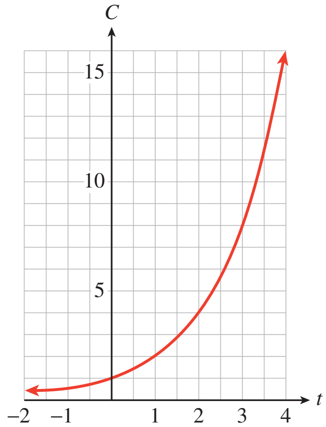
Distance (meters)	100	200	400	800	1500	5000	10,000
Men	9.63	19.30	43.03	1 : 40.91	3 : 32.07	12 : 57.82	27 : 01.17
Women	10.62	21.34	48.25	1 : 53.43	3 : 53.96	14 : 26.17	29 : 17.45

a Does $f(800) = 2f(400)$? Does $g(400) = 2g(200)$?

b Find a value of d for which $f(2d) < 2f(d)$. Is there a value of d for which $g(2d) < 2g(d)$?

In Problems 35—40, use the graph of the function to answer the questions.

35. The graph shows C as a function of t. C stands for the number of students (in thousands) at State University who consider themselves computer literate, and t represents time, measured in years since 1990.



a When did 2000 students consider themselves computer literate?

b How long did it take that number to double?

c How long did it take for the number to double again?

d How many students became computer literate between January 1992 and June 1993?

Answer.

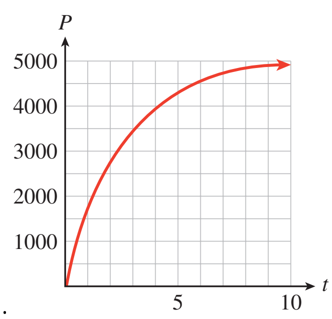
a 1991

b 1 yr

c 1 yr

d About 7300

36. The graph shows P as a function of t. P is the number of people in Cedar Grove who owned a portable DVD player t years after 2000.



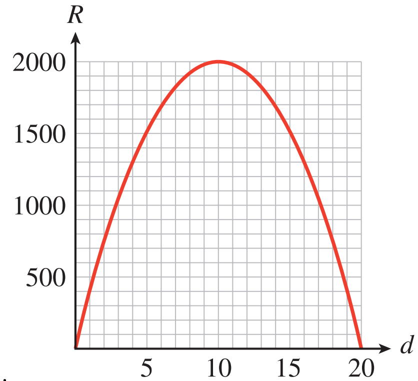
a When did 3500 people own portable DVD players?

b How many people owned portable DVD players in 2005?

c The number of owners of portable DVD players in Cedar Grove seems to be leveling off at what number?

d How many people acquired portable DVD players between 2001 and 2004?

37. The graph shows the revenue, R, a movie theater collects as a function



of the price, d, it charges for a ticket.

a What is the revenue if the theater charges \$12.00 for a ticket?

b What should the theater charge for a ticket in order to collect \$1500 in revenue?

c For what values of d is $R > 1875$?

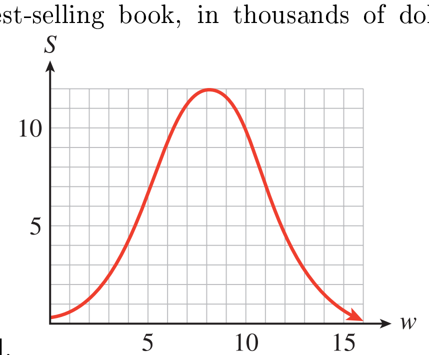
Answer.

a Approximately \$1920

b \$5 or \$15

c $7.50 < d < 12.50$

38. The graph shows S as a function of w. S represents the weekly sales of a best-selling book, in thousands of dollars, w weeks after it is



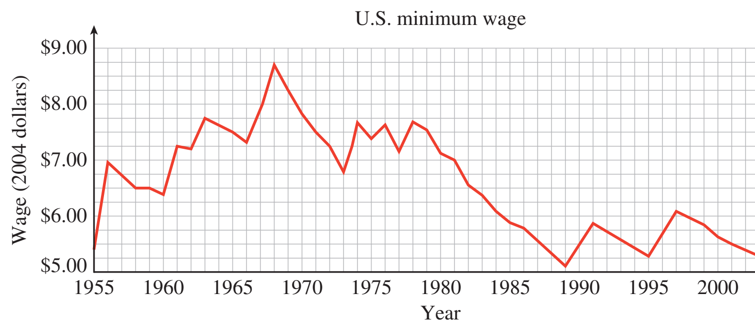
released.

a In which weeks were sales over \$7000?

b In which week did sales fall below \$5000 on their way down?

c For what values of w is $S > 3.4$?

39. The graph shows the federal minimum wage, M, as a function of time, t, adjusted for inflation to reflect its buying power in 2004 dollars.



(Source: www.infoplease.com)

a When did the minimum wage reach its highest buying power, and what was it worth in 2004 dollars?

b When did the minimum wage fall to its lowest buying power after its peak, and what was its worth at that time?

c Give two years in which the minimum wage was worth \$8 in 2004 dollars.

Answer.

a 1968, about \$8.70

b 1989, about \$5.10

c 1967, approximately 1970

40. The graph shows the U.S. unemployment rate, U, as a function of time, t, for the years 1985–2004. (Source: U.S. Bureau of Labor Statistics)

a When did the unemployment rate reach its highest value, and what was its highest value?

b When did the unemployment rate fall to its lowest value, and what was its lowest value?

c Give two years in which the unemployment rate was 4.5%.

In Problems 41–48, evaluate each function for the given values.

41. $f(x) = 6 - 2x$

a $f(3)$

b $f(-2)$

c $f(12.7)$

d $f\left(\dfrac{2}{3}\right)$

Answer.

a 0

b 10

c -19.4

d $\dfrac{14}{3}$

42. $g(t) = 5t - 3$

a $g(1)$

b $g(-4)$

c $g(14.1)$

d $g\left(\dfrac{3}{4}\right)$

43. $h(v) = 2v^2 - 3v + 1$

a $h(0)$

b $h(-1)$

c $h\left(\dfrac{1}{4}\right)$

d $h(-6.2)$

Answer.

a 1

b 6

c $\dfrac{3}{8}$

d 96.48

44. $r(s) = 2s - s^2$

a $r(2)$

b $r(-4)$

c $r\left(\dfrac{1}{3}\right)$

d $r(-1.3)$

45. $H(z) = \dfrac{2z-3}{z+2}$

a $H(4)$

b $H(-3)$

c $H\left(\dfrac{4}{3}\right)$

d $H(4.5)$

Answer.

a $\dfrac{5}{6}$

b 9

c $\dfrac{-1}{10}$

d $\dfrac{12}{13} \approx 0.923$

46. $F(x) = \dfrac{1-x}{2x-3}$

a $F(0)$

b $F(-3)$

c $F\left(\dfrac{5}{2}\right)$

d $F(9.8)$

47. $E(t) = \sqrt{t-4}$

a $E(16)$

b $E(4)$

c $E(7)$

d $E(4.2)$

Answer.

a $\sqrt{12}$

b 0

c $\sqrt{3}$

d $\sqrt{0.2} \approx 0.447$

48. $D(r) = \sqrt{5-r}$

a $D(4)$

b $D(-3)$

c $D(-9)$

d $D(4.6)$

49. A sport utility vehicle costs $\$28,000$ and depreciates according to the formula $V(t) = 28,000(1 - 0.08t)$,where V is the value of the vehicle after t years.

a Evaluate $V(12)$ and explain what it means.

b Solve the equation $V(t) = 0$ and explain what it means.

c If this year is $t = n$, what does $V(n + 2)$ mean?

Answer.

a $V(12) = 1120$: After 12 years, the SUV is worth \$1120.

b $t = 12.5$: The SUV has zero value after $12\frac{1}{2}$ years.

c The value 2 years later

50. In a profit-sharing plan, an employee receives a salary of $S(x) = 20,000 + 0.01x$, where x represents the company's profit for the year.

a Evaluate $S(850,000)$ and explain what it means.

b Solve the equation $S(x) = 30,000$ and explain what it means.

c If the company made a profit of p dollars this year, what does $S(2p)$ mean?

51. The number of compact cars that a large dealership can sell at price p is given by $N(p) = \dfrac{12,000,000}{p}$.

a Evaluate $N(6000)$ and explain what it means.

b As p increases, does $N(p)$ increase or decrease? Why is this reasonable?

c If the current price for a compact car is D, what does $2N(D)$ mean?

Answer.

a $N(6000) = 2000$: 2000 cars will be sold at a price of \$6000.

b $N(p)$ decreases with increasing p because fewer cars will be sold when the price increases.

c $2N(D)$ represents twice the number of cars that can be sold at the current price.

52. A department store finds that the market value of its Christmas-related merchandise is given by $M(t) = \dfrac{600,000}{t}$, $t \leq 30$, where t is the number of weeks after Christmas.

a Evaluate $M(2)$ and explain what it means.

b As t increases, does $M(t)$ increase or decrease? Why is this reasonable?

c If this week $t = n$, what does $M(n + 1)$ mean?

53. The velocity of a car that brakes suddenly can be determined from the length of its skid marks, d, by $v(d) = \sqrt{12d}$, where d is in feet and v is in miles per hour.

a Evaluate $v(250)$ and explain what it means.

b Estimate the length of the skid marks left by a car traveling at 100 miles per hour.

c Write your answer to part (b) with function notation.

Answer.

a $v(250) = 54.8$ is the speed of a car that left 250-foot skid marks.

b $833\frac{1}{3}$ feet

c $v\left(833\frac{1}{3}\right) = 100$

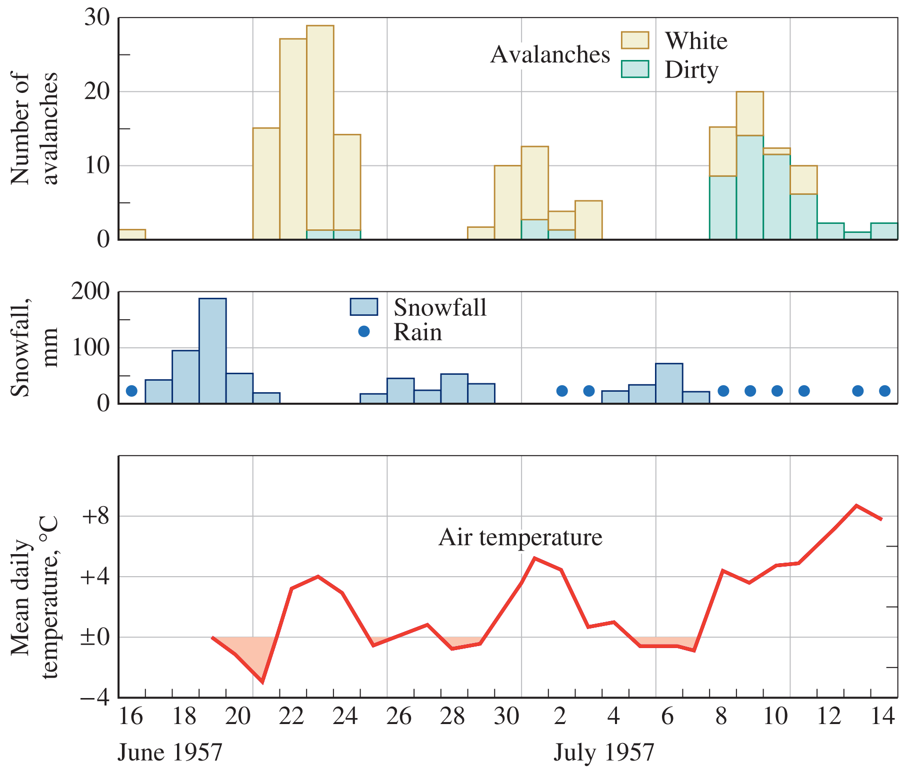
54. The distance, d, in miles that a person can see on a clear day from a height, h, in feet is given by $d(h) = 1.22\sqrt{h}$.

a Evaluate $d(20,320)$ and explain what it means.

b Estimate the height you need in order to see 100 miles.

c Write your answer to part (b) with function notation.

55. The figure gives data about snowfall, air temperature, and number of avalanches on the Mikka glacier in Sarek, Lapland, in 1957. (Source: Leopold,



Wolman, Miller, 1992)

a During June and July, avalanches occurred over three separate time intervals. What were they?

b Over what three time intervals did snow fall?

c When was the temperature above freezing (0°C)?

d Using your answers to parts (a)–(c), make a conjecture about the conditions that encourage avalanches.

Answer.

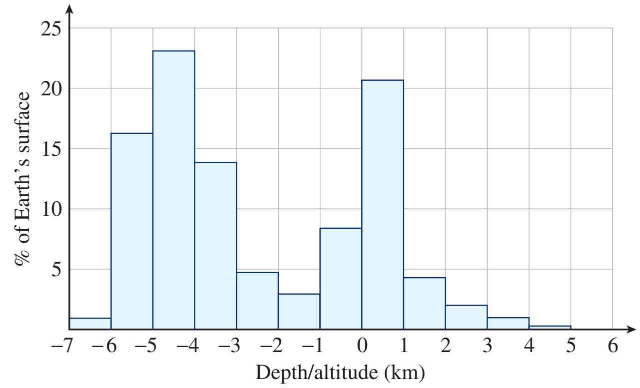
a June 21–24, June 29–July 3, July 8–14

b June 17–21, June 25–29, July 4–7

c June 22–24, June 27, June 29–July 4, July 8–14

d Avalanches occur when temperatures rise above freezing immediately after snowfall.

56. The bar graph shows the percent of Earth's surface that lies at various altitudes or depths below the surface of the oceans. (Depths are given as negative altitudes.) (Source: Open University)



a Read the graph and complete the table.

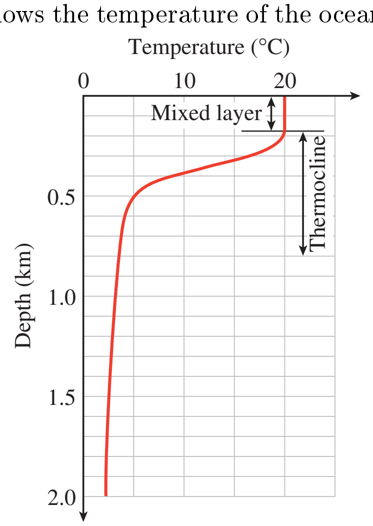
Altitude (km)	Percent of Earth's surface
-7 to -6	
-6 to -5	
-5 to -4	
-4 to -3	
-3 to -2	
-2 to -1	
-1 to 0	
0 to 1	
1 to 2	
2 to 3	
3 to 4	
4 to 5	

b What is the most common altitude? What is the second most common altitude??

c Approximately what percent of the Earth's surface is below sea level?

d The height of Mt. Everest is 8.85 kilometers. Can you think of a reason why it is not included in the graph?

57. The graph shows the temperature of the ocean at various depths. (Source: Open University)



a Is depth a function of temperature?

b Is temperature a function of depth?

c The axes are scaled in an unusual way. Why is it useful to present the graph in this way?

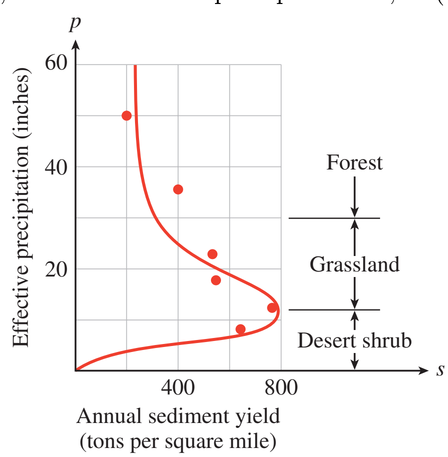
Answer.

a No

b Yes

c Moving downwards on the graph corresponds to moving downwards in the ocean.

58. The graph shows the relationship between annual precipitation, p, in a region and the amount of erosion, measured in tons per square mile, s. (Source:



Leopold, Wolman, Miller, 1992)

a Is the amount of erosion a function of the amount of precipitation?

b At what annual precipitation is erosion at a maximum, and what is that maximum?

c Over what interval of annual precipitation does erosion decrease?

d An increase in vegetation inhibits erosion, and precipitation encourages vegetation. What happens to the amount of erosion as precipitation increases in each of these three environments?

desert shrub	$0 < 0 < 12$
grassland	$12 < 0 < 30$
forest	$30 < 0 < 60$

Evaluate the function and simplify.

59. $G(s) = 3s^2 - 6s$

a $G(3a)$ c $G(a) + 2$

b $G(a + 2)$ d $G(-a)$

60. $h(x) = 2x^2 + 6x - 3$

a $h(2a)$ c $h(a) + 3$

b $h(a + 3)$ d $h(-a)$

Answer.

a $27a^2 - 18a$ 2

b $3a^2 + 6a$

c $3a^2 - 6a +$ d $3a^2 + 6a$

61. $g(x) = 8$

a $g(2)$
b $g(8)$
c $g(a+1)$
d $g(-x)$

Answer.

a 8
b 8
c 8
d 8

62. $f(t) = -3$

a $f(4)$
b $f(-3)$
c $f(b-2)$
d $f(-t)$

63. $P(x) = x^3 - 1$

a $P(2x)$
b $2P(x)$
c $P(x^2)$
d $[P(x)]^2$

Answer.

a $8x^3 - 1$
b $2x^3 - 2$
c $x^6 - 1$
d $x^6 - 2x^3 + 1$

64. $Q(t) = 5t^3$

a $Q(2t)$
b $2Q(t)$
c $Q(t^2)$
d $[Q(t)]^2$

Evaluate each function for the given expressions and simplify.

65. $f(x) = x^3$

a $f(a^2)$
b $a^3 \cdot f(a^3)$
c $f(ab)$
d $f(a+b)$

Answer.

a a^6
b a^{12}
c a^3b^3
d $a^3+3a^2b+3ab^2+b^3$

66. $g(x) = x^4$

a $g(a^3)$
b $a^4 \cdot g(a^4)$
c $g(ab)$
d $g(a+b)$

67. $F(x) = 3x^5$

a $F(2a)$
b $2F(a)$
c $F(a^2)$
d $[F(a)]^2$

Answer.

a $96a^5$
b $6a^5$
c $3a^{10}$
d $9a^{10}$

68. $G(x) = 4x^3$

a $G(3a)$
b $3G(a)$
c $G(a^4)$
d $[G(a)]^4$

For the functions in Problems 69–76, compute the following:

a $f(2)+f(3)$ b $f(2+3)$ c $f(a)+f(b)$ d $f(a+b)$

For which functions does $f(a+b)=f(a)+f(b)$ for all values of a and b?

69. $f(x)=3x-2$

Answer.

a 11
b 13
c $3a+3b-4$
d $3a+3b-2$

This function does NOT satisfy $f(a+b)=f(a)+f(b)$.

70. $f(x)=1-4x$

71. $f(x)=x^2+3$

Answer.

a 19
b 28
c a^2+b^2+6
d $a^2+2ab+b^2+3$

This function does NOT satisfy $f(a+b)=f(a)+f(b)$.

72. $f(x)=x^2-1$

73. $f(x)=\sqrt{x+1}$

Answer.

a $\sqrt{3}+2$
b $\sqrt{6}$
c $\sqrt{a+1}+\sqrt{b+1}$
d $\sqrt{a+b+1}$

This function does NOT satisfy $f(a+b)=f(a)+f(b)$.

74. $f(x)=\sqrt{6-x}$

75. $f(x)=\dfrac{-2}{x}$

Answer.

a $\dfrac{-5}{3}$
b $\dfrac{-2}{5}$
c $\dfrac{-2}{a}-\dfrac{-2}{b}$
d $\dfrac{-2}{a+b}$

This function does NOT satisfy $f(a+b)=f(a)+f(b)$.

76. $f(x)=\dfrac{3}{x}$

77. Use a table of values to estimate a solution to $f(x)=800+6x-0.2x^2=500$ as follows:

a Make a table starting at $x=0$ and increasing by $\Delta x=10$, as shown in the accompanying tables. Find two x-values a and b so that $f(a)>500>f(b)$.

x	0	10	20	30	40	50	60	70	80	90	100
$f(x)$											

b Make a new table starting at $x=a$ and increasing by $\Delta x=1$. Find two x-values, c and d, so that $f(c)>500>f(d)$.

c Make a new table starting at $x = c$ and increasing by $\Delta x = 0.1$. Find two x-values, p and q, so that $f(p) > 500 > f(q)$.

d Take the average of p and q, that is, set $s = \dfrac{p+q}{2}$. Then s is an approximate solution that is off by at most 0.05.

e Evaluate $f(s)$ to check that the output is approximately 500.

Answer.

a $x = 50$ and $x = 60$

x	0	10	20	30	40	50	60	70	80	90	100
$f(x)$	800	840	840	800	720	600	440	240	0	−280	−600

b $x = 56$ and $x = 57$

x	50	51	52	53	54	55	56	57	58
$f(x)$	600	585.8	571.2	556.2	540.8	525	508.8	492.2	475.2

c $x = 56.5$ and $x = 56.6$

x	56	56.1	56.2	56.3	56.4	56.5	56.6
$f(x)$	508.8	507.158	505.512	503.862	502.208	500.55	498.888

d $s = 56.55$

e $f(56.55) = 499.7195$

78. Use a table of values to estimate a solution to $f(x) = x^3 - 4x^2 + 5x = 18,000$ as follows:

a Make a table starting at $x = 0$ and increasing by $\Delta x = 10$, as shown in the accompanying tables. Find two x-values a and b so that $f(a) < 18,000 < f(b)$.

x	0	10	20	30	40	50	60	70	80	90	100
$f(x)$											

b Make a new table starting at $x = a$ and increasing by $\Delta x = 1$. Find two x-values, c and d, so that $f(c) < 18,000 < f(d)$.

c Make a new table starting at $x = c$ and increasing by $\Delta x = 0.1$. Find two x-values, p and q, so that $f(p) < 18,000 < f(q)$.

d Take the average of p and q, that is, set $s = \dfrac{p+q}{2}$. Then s is an approximate solution that is off by at most 0.05.

e Evaluate $f(s)$ to check that the output is approximately 18,000.

79. Use tables of values to estimate the positive solution to $f(x) = x^2 - \dfrac{1}{x} = 9000$, accurate to within 0.05.

Answer. 94.85

80. Use tables of values to estimate the positive solution to $f(x) = \dfrac{8}{x} + 500 - \dfrac{x^2}{9} = 300$, accurate to within 0.05.

Answer. 94.85

1.3 Graphs of Functions

1.3.1 Reading Function Values from a Graph

The graph in Figure 1.53 shows the Dow-Jones Industrial Average (the average value of the stock prices of 500 major companies) during the stock market correction of October 1987. The Dow-Jones Industrial Average (DJIA) is given as a function of time during the 8 days from October 15 to October 22; that is, $f(t)$ is the DJIA recorded at noon on day t.

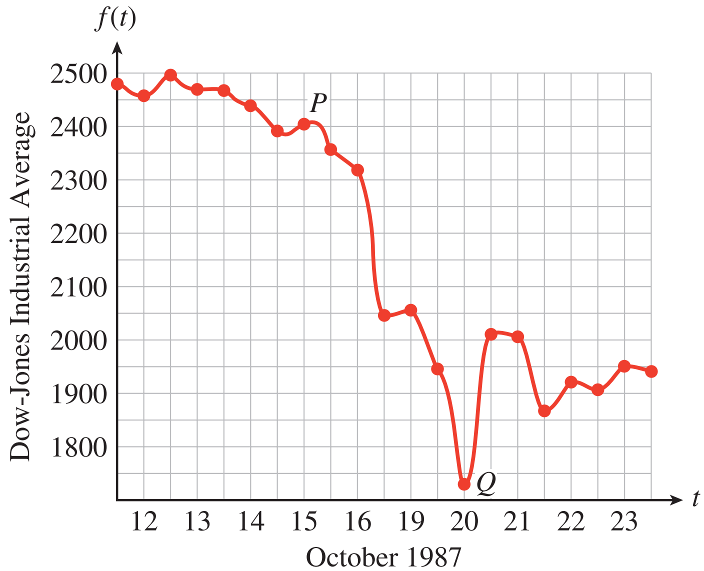


Figure 1.53

The values of the input variable, time, are displayed on the horizontal axis, and the values of the output variable, DJIA, are displayed on the vertical axis. There is no formula that gives the DJIA for a particular day; but it is still a function, defined by its graph. The value of $f(t)$ is specified by the vertical coordinate of the point with the given t-coordinate.

Example 1.54.

a The coordinates of point P in Figure 1.53 are $(15, 2412)$. What do the coordinates tell you about the function f?

b If the DJIA was 1726 at noon on October 20, what can you say about the graph of f?

Solution.

a The coordinates of point P tell us that $f(15) = 2412$, so the DJIA was 2412 at noon on October 15.

b We can say that $f(20) = 1726$, so the point $(20, 1726)$ lies on the graph of f. This point is labeled Q in Figure 1.53.

Thus, the coordinates of each point on the graph of the function represent a pair of corresponding values of the two variables. In general, we can make the following statement.

Graph of a Function

The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

Exercise 1.55. The water level in Lake Huron alters unpredictably over time. The graph in Figure 1.56 gives the average water level, $L(t)$, in meters in the year t over a 20-year period. (Source: The Canadian Hydrographic Service)

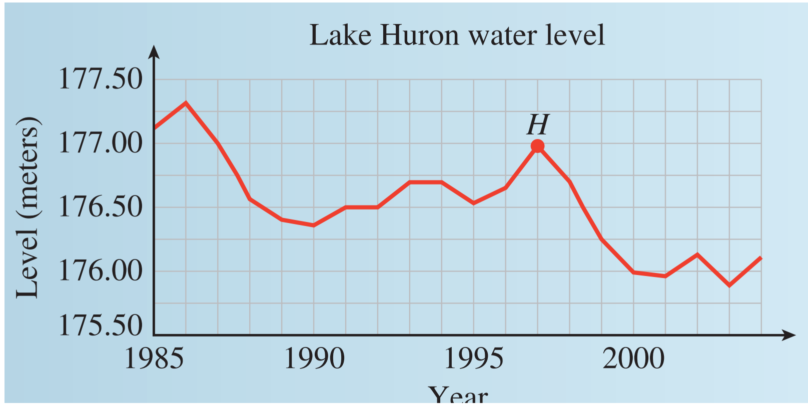


Figure 1.56

a The coordinates of point H in Figure 1.56 are $(1997, 176.98)$. What do the coordinates tell you about the function L?

b The average water level in 2004 was 176.11 meters. Write this fact in function notation. What can you say about the graph of L?

Another way of describing how a graph depicts a function is as follows:

Functions and Coordinates

Each point on the graph of the function f has coordinates $(x, f(x))$ for some value of x.

Example 1.57. Figure 1.58 shows the graph of a function g.

a Find $g(-2)$ and $g(5)$.

b For what value(s) of t is $g(t) = -2$?

c What is the largest, or maximum, value of $g(t)$? For what value of t does the function take on its maximum value?

d On what intervals is g increasing?

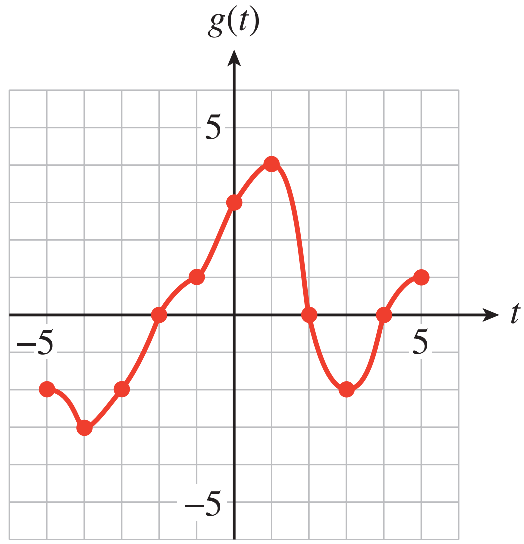


Figure 1.58

Solution.

a To find $g(-2)$, we look for the point with t-coordinate -2. The point $(-2,0)$ lies on the graph of g, so $g(-2)=0$. Similarly, the point $(5,1)$ lies on the graph, so $g(5)=1$.

b We look for points on the graph with y-coordinate -2. Because the points $(-5,-2)$, $(-3,-2)$, and $(3,-2)$ lie on the graph, we know that $g(-5)=-2$, $g(-3)=-2$, and $g(3)=-2$. Thus, the t-values we want are -5, -3, and 3.

c The highest point on the graph is $(1,4)$, so the largest y-value is 4. Thus, the maximum value of $g(t)$ is 4, and it occurs when $t=1$.

d A graph is increasing if the y-values get larger as we read from left to right. The graph of g is increasing for t-values between -4 and 1, and between 3 and 5. Thus, g is increasing on the intervals $(-4,1)$ and $(3,5)$.

Exercise 1.59. Refer to the graph of the function g shown in Figure 1.58 in Example 1.57.

a Find $g(0)$.

b For what value(s) of t is $g(t)=0$?

c What is the smallest, or minimum, value of $g(t)$? For what value of t does the function take on its minimum value?

d On what intervals is g decreasing?

Remark 1.60 (Finding Coordinates with a Graphing Calculator)**.** We can use the TRACE feature of the calculator to find the coordinates of points on a graph. For example, graph the equation $y=-2.6x-5.4$ in the window

Xmin= -5	Xmax= 4.4
Ymin= -20	Ymax= 15

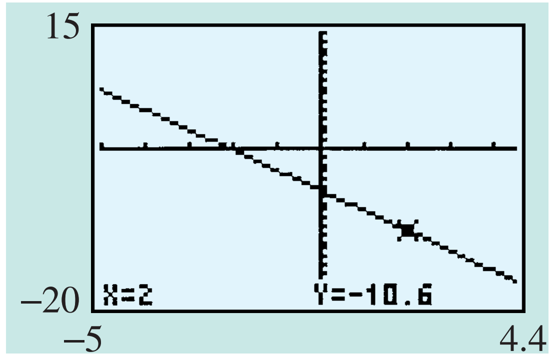


Figure 1.61

Press TRACE, and a "bug" begins flashing on the display. The coordinates of the bug appear at the bottom of the display, as shown in Figure 1.61. Use the left and right arrows to move the bug along the graph. You can check that the coordinates of the point $(2, -10.6)$ do satisfy the equation $y = -2.6x - 5.4$.

The points identified by the Trace bug depend on the window settings and on the type of calculator. If we want to find the y-coordinate for a particular x-value, we enter the x-coordinate of the desired point and press ENTER.

1.3.2 Constructing the Graph of a Function

Although some functions are defined by their graphs, we can also construct graphs for functions described by tables or equations. We make these graphs the same way we graph equations in two variables: by plotting points whose coordinates satisfy the equation.

Example 1.62. Graph the function $f(x) = \sqrt{x+4}$.

Solution. Choose several convenient values for x and evaluate the function to find the corresponding $f(x)$-values. For this function we cannot choose x-values less than -4, because the square root of a negative number is not a real number.

$$f(\mathbf{-4}) = \sqrt{\mathbf{-4}+4} = \sqrt{0} = 0$$
$$f(\mathbf{-3}) = \sqrt{\mathbf{-3}+4} = \sqrt{1} = 1$$
$$f(\mathbf{0}) = \sqrt{\mathbf{0}+4} = \sqrt{4} = 2$$
$$f(\mathbf{2}) = \sqrt{\mathbf{2}+4} = \sqrt{6} \approx 2.45$$
$$f(\mathbf{5}) = \sqrt{\mathbf{5}+4} = \sqrt{9} = 3$$

The results are shown in the table.

x	$f(x)$
-4	0
-3	1
0	2
2	$\sqrt{6}$
5	3

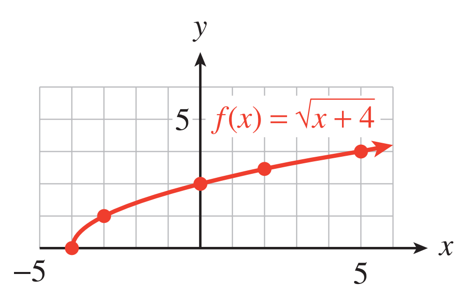


Figure 1.63

Remark 1.64 (Using a Calculator to Graph a Function)**.** We can also use a graphing calculator to obtain a table and graph for the function in Example 1.62. We graph a function just as we graphed an equation. For this function, we enter

$$Y_1 = \sqrt{\quad (X+4)}$$

and press `ZOOM` 6 for the standard window. (See ⟨⟨appendix-b⟩⟩ for details.) The calculator's graph is shown in Figure 1.65.

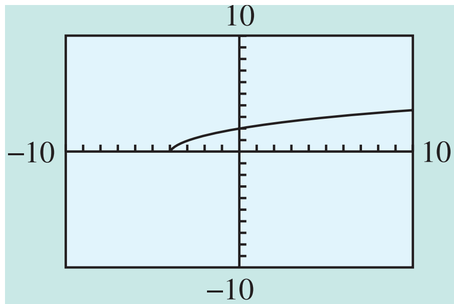


Figure 1.65

Exercise 1.66. $f(x) = x^3 - 2$

a Complete the table of values and sketch a graph of the function.

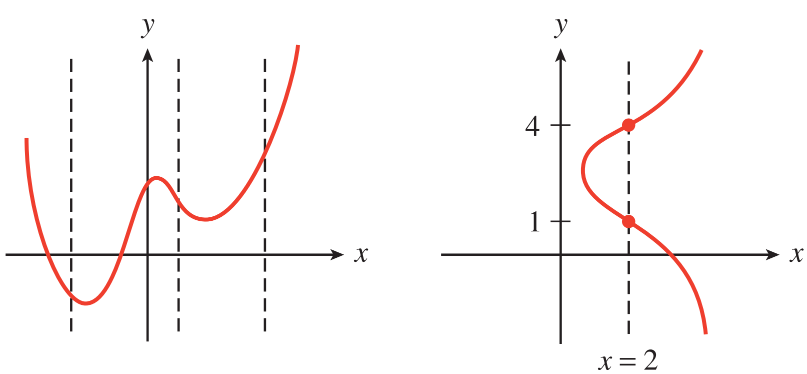
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x)$						

b Use your calculator to make a table of values and graph the function.

1.3.3 The Vertical Line Test

In a function, two different outputs cannot be related to the same input. This restriction means that two different ordered pairs cannot have the same first coordinate. What does it mean for the graph of the function?

Consider the graph shown in Figure 1.67a. Every vertical line intersects the graph in at most one point, so there is only one point on the graph for each x-value. This graph represents a function. In Figure 1.67b, however, the line $x = 2$ intersects the graph at two points, $(2, 1)$ and $(2, 4)$. Two different y-values, 1 and 4, are related to the same x-value, 2. This graph cannot be the graph of a function.



(a) **(b)**

Figure 1.67

We summarize these observations as follows.

The Vertical Line Test

A graph represents a function if and only if every vertical line intersects the graph in at most one point.

Example 1.68. Use the vertical line test to decide which of the graphs in Figure 1.69 represent functions.

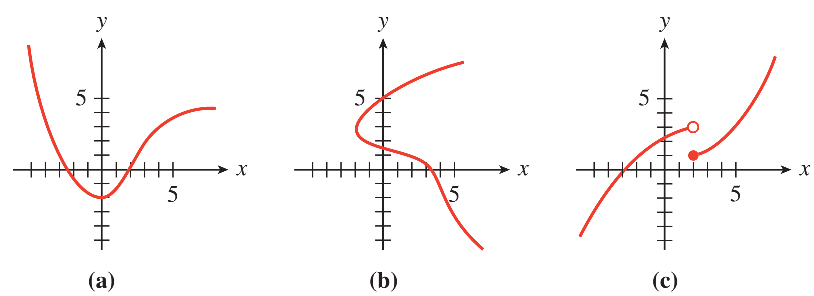


Figure 1.69

Solution. Graph (a) represents a function, because it passes the vertical line test. Graph (b) is not the graph of a function, because the vertical line at (for example) $x = 1$ intersects the graph at two points. For graph (c), notice the break in the curve at $x = 2$: The solid dot at $(2, 1)$ is the only point on the graph with $x = 2$; the open circle at $(2, 3)$ indicates that $(2, 3)$ is not a point on the graph. Thus, graph (c) is a function, with $f(2) = 1$.

Exercise 1.70. Use the vertical line test to determine which of the graphs in Figure 1.71 represent functions.

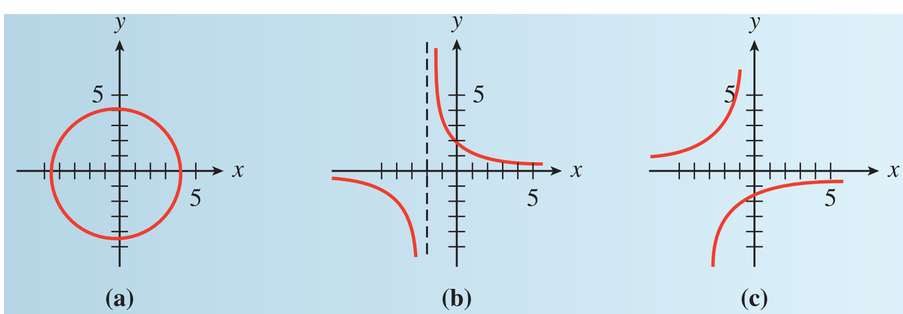


Figure 1.71

1.3.4 Graphical Solution of Equations and Inequalities

The graph of an equation in two variables is just a picture of its solutions. When we read the coordinates of a point on the graph, we are reading a pair of x- and y-values that make the equation true.

For example, the point $(2, 7)$ lies on the graph of $y = 2x+3$ shown in Figure 1.72 , so we know that the ordered pair $(2, 7)$ is a solution of the equation $y = 2x+3$. You can verify algebraically that $x = \mathbf{2}$ and $y = \mathbf{7}$ satisfy the equation:

$$\text{Does } \mathbf{7} = 2(\mathbf{2}) + 3? \text{ Yes}$$

We can also say that $x = 2$ is a solution of the one-variable equation $2x+3 = 7$. In fact, we can use the graph of $y = 2x + 3$ to solve the equation $2x + 3 = k$ for any value of k. Thus, we can use graphs to find solutions to equations in one variable.

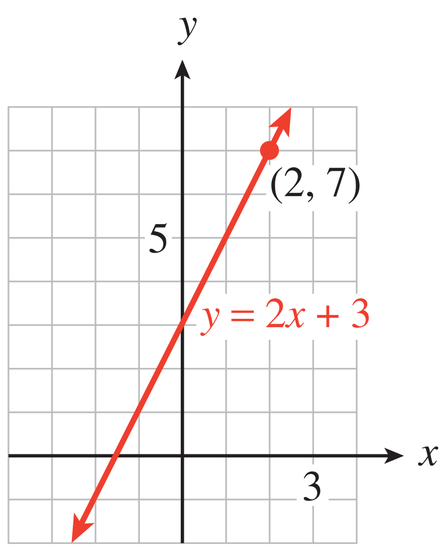


Figure 1.72

Example 1.73. Use the graph of $y = 285 - 15x$ to solve the equation $150 = 285 - 15x$.

Solution.

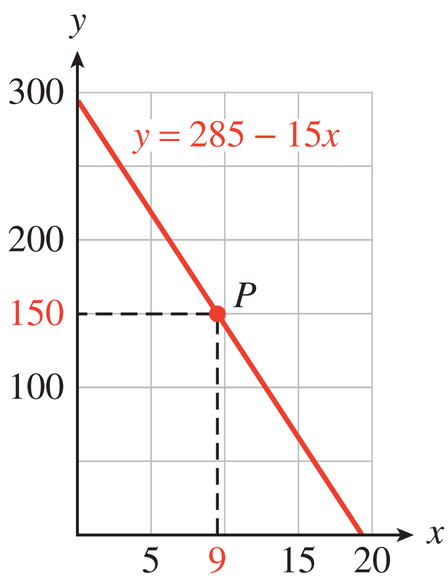


Figure 1.74

Begin by locating the point P on the graph for which $y = 150$, as shown in Figure 1.74. Now find the x-coordinate of point P by drawing an imaginary line from P straight down to the x-axis. The x-coordinate of P is $x = 9$. Thus, P is the point $(9, 150)$, and $x = 9$ when $y = 150$. The solution of the equation $150 = 285 - 15x$ is $x = 9$. You can verify the solution algebraically by substituting $x = \mathbf{9}$ into the equation:

Does $150 = 285 - 15(\mathbf{9})$?

$$285 - 15(\mathbf{9}) = 285 - 135 = 150. \quad \textbf{\textit{Yes}}$$

The relationship between an equation and its graph is an important one. For the previous example, make sure you understand that the following three statements are equivalent:

1. The point $(9, 150)$ lies on the graph of $y = 285 - 15x$.

2. The ordered pair $(9, 150)$ is a solution of the equation $y = 285 - 15x$.

3. $x = 9$ is a solution of the equation $150 = 285 - 15x$.

Exercise 1.75.

a Use the graph of $y = 30 - 8x$ shown in Figure 1.76 to solve the equation

$$30 - 8x = 50$$

b Verify your solution algebraically.

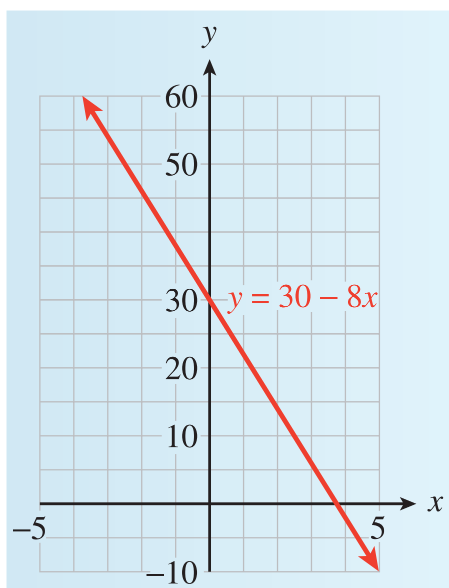


Figure 1.76

In a similar fashion, we can solve inequalities with a graph. Consider again the graph of $y = 2x + 3$, shown in Figure 1.77. We saw that $x = 2$ is the solution of the equation $2x + 3 = 7$. When we use $x = 2$ as the input for the function $f(x) = 2x + 3$, the output is $y = 7$. Which input values for x produce output values greater than 7? You can see in Figure 1.77 that x-values greater than 2 produce y-values greater than 7, because points on the graph with x-values greater than 2 have y-values greater than 7. Thus, the solutions of the inequality $2x + 3 > 7$ are $x > 2$. You can verify this result by solving the inequality algebraically.

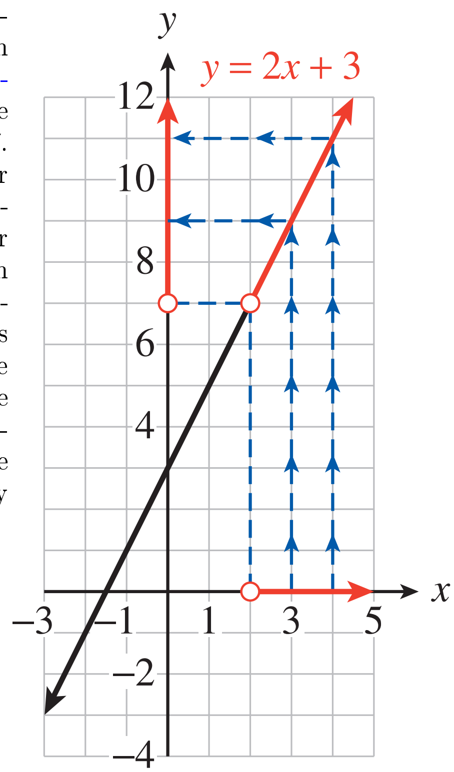


Figure 1.77

Example 1.78. Use the graph of $y = 285 - 15x$ to solve the inequality

$$285 - 15x > 150$$

Solution. We begin by locating the point P on the graph for which $y = 150$ and $x = 9$ (its x-coordinate). Now, because $y = 285 - 15x$ for points on the graph, the inequality $285 - 15x > 150$ is equivalent to $y > 150$. So we are looking for points on the graph with y-coordinate greater than 150. These points are shown in Figure 1.79. The x-coordinates of these points are the x-values that satisfy the inequality. From the graph, we see that the solutions are $x < 9$.

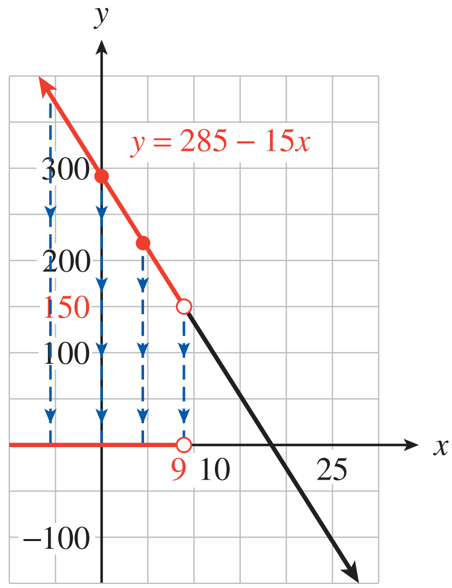


Figure 1.79

Exercise 1.80.

a Use the graph of $y = 30 - 8x$ in Figure 1.76 to solve the inequality

$$30 - 8x \leq 50$$

b Solve the inequality algebraically.

We can also use this graphical technique to solve nonlinear equations and inequalities.

Example 1.81. Use a graph of $f(x) = -2x^3 + x^2 + 16x$ to solve the equation

$$-2x^3 + x^2 + 16x = 15$$

Solution. If we sketch in the horizontal line $y = 15$, we can see that there are three points on the graph of f that have y-coordinate 15, as shown in Figure 1.82. The x-coordinates of these points are the solutions of the equation

$$-2x^3 + x^2 + 16x = 15$$

From the graph, we see that the solutions are $x = -3$, $x = 1$, and approximately $x = 2.5$. We can verify the solutions algebraically. For example, if $x = \mathbf{-3}$, we have

$$f(\mathbf{-3})\& - 2(\mathbf{-3})3 + (\mathbf{-3})^2 + 16(\mathbf{-3})\backslash\backslash\& = -2(-27) + 9 - 48\backslash\backslash\& = 54 + 9 - 48 = 15$$

so -3 is a solution.

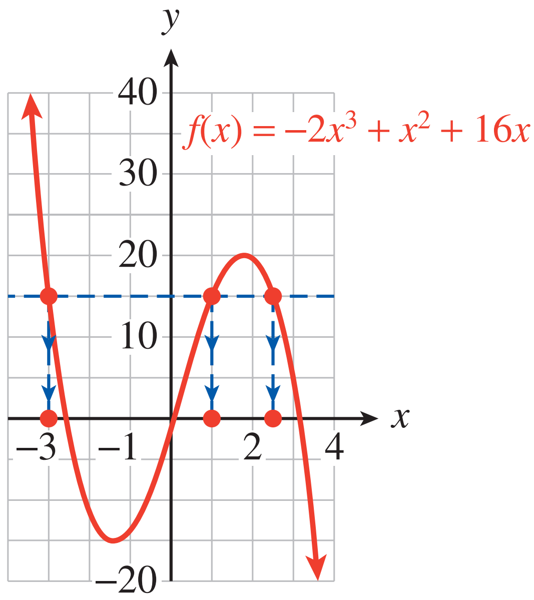


Figure 1.82

Exercise 1.83. Use the graph of $y = \frac{1}{2}n^2 + 2n - 10$ shown in Figure 1.84 to solve

$$\frac{1}{2}n^2 + 2n - 10 = 6$$

and verify your solutions algebraically.

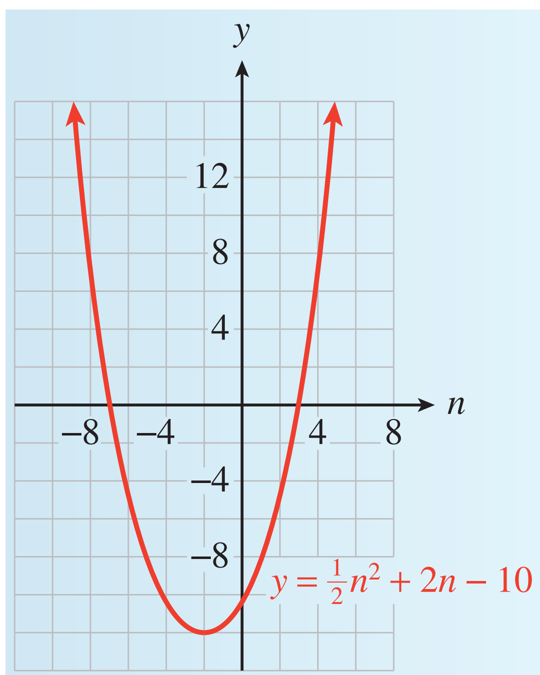


Figure 1.84

Remark 1.85 (Using the Trace Feature)**.** You can use the Trace feature on a graphing calculator to approximate solutions to equations. Graph the function $f(x)$ in Example 1.81 in the window

Xmin$= -4$	Xmax$= 4$
Ymin$= -20$	Ymax$= 40$

and trace along the curve to the point $(2.4680851, 15.512401)$. We are close to a solution, because the y-value is close to 15. Try entering x-values close to 2.4680851, for instance, $x = 2.4$ and $x = 2.5$, to find a better approximation for the solution.

We can use the intersect feature on a graphing calculator to obtain more accurate estimates for the solutions of equations. See ⟨⟨appendix-b⟩⟩ for details.

Example 1.86. Use the graph in Example 1.81 to solve the inequality

$$-2x^3 + x^2 + 16x \geq 15$$

Solution. We first locate all points on the graph that have y-coordinates greater than or equal to 15. The x-coordinates of these points are the solutions of the inequality. Figure 1.87 shows the points, and their x-coordinates as intervals on the x-axis. The solutions are $x \leq -3$ and $1 \leq x \leq 2.5$, or in interval notation, $(-, -3][1, 2.5]$.

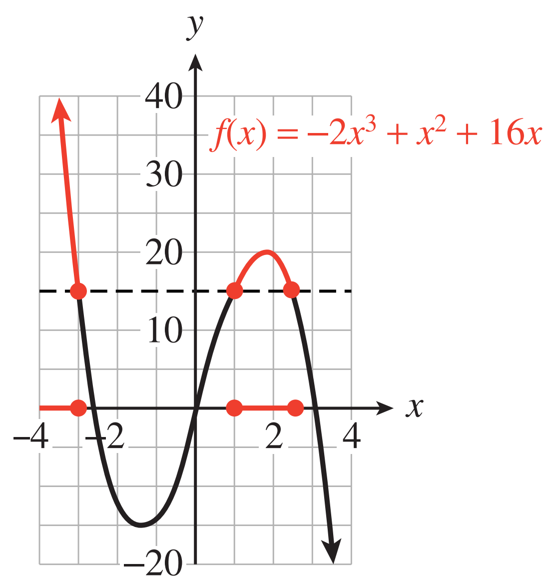


Figure 1.87

Exercise 1.88. Use Figure 1.84 in Exercise 1.83 to solve the inequality

$$\frac{1}{2}n^2 + 2n - 10 < 6$$

1.3.5 Section Summary

1.3.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Coordinates
- Algebraic solution
- Vertical line test
- Minimum
- Interval
- Maximum
- Graphical solution
- Inequality

1.3.5.2 CONCEPTS

1 The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

2 Each point on the graph of the function f has coordinates $(x, f(x))$ for some value of x.

3 The vertical line test tells us whether a graph represents a function.

4 We can use a graph to solve equations and inequalities in one variable.

1.3.5.3 STUDY QUESTIONS

1 How can you find the value of $f(3)$ from a graph of f?

2 If $f(8) = 2$, what point lies on the graph of f?

3 Explain how to construct the graph of a function from its equation.

4 Explain how to use the vertical line test.

5 How can you solve the equation $x + \sqrt{x} = 56$ using the graph of $y = x + \sqrt{x}$?

1.3.5.4 SKILLS

Practice each skill in the Homework 1.3.6 problems listed.

1 Read function values from a graph: #1–8, 17–20, 33–36

2 Recognize the graph of a function: #9–10, 31 and 32

3 Construct a table of values and a graph of a function: #11–16

4 Solve equations and inequalities graphically: #21–30, 41–50

1.3.6 Homework

In Problems 1–8, use the graphs to answer the questions about the functions.

1.

a Find $h(-3)$, $h(1)$, and $h(3)$.

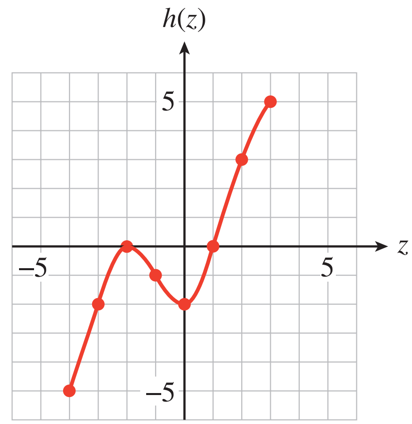
b For what value(s) of z is $h(z) = 3$?

c Find the intercepts of the graph. List the function values given by the intercepts.

d What is the maximum value of $h(z)$?

e For what value(s) of z does h take on its maximum value?

f On what intervals is the function increasing? Decreasing?



Answer.

a June 21–24, June 29–July 3, July 8–14

b June 17–21, June 25–29, July 4–7

c June 22–24, June 27, June 29–July 4, July 8–14

d Avalanches occur when temperatures rise above freezing immediately after snowfall.

2.

a Find $G(-3)$, $G(-1)$, and $G(2)$.

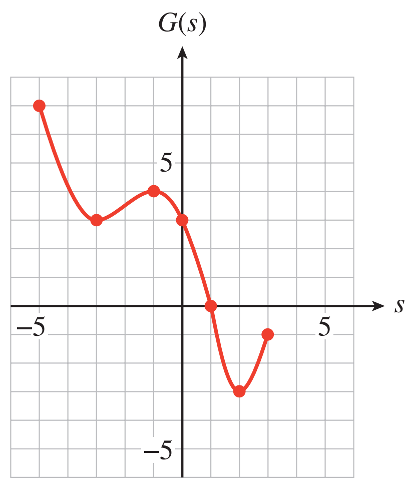
b For what value(s) of s is $G(s) = 3$??

c Find the intercepts of the graph. List the function values given by the intercepts.

d What is the minimum value of $G(s)$?

e For what value(s) of s does G take on its minimum value?

f On what intervals is the function increasing? Decreasing?



3.

a Find $R(1)$ and $R(3)$.

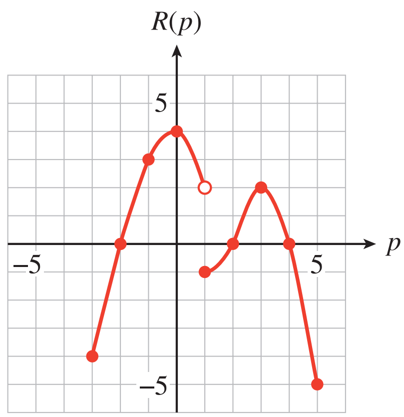
b For what value(s) of p is $R(p) = 2$?

c Find the intercepts of the graph. List the function values given by the intercepts.

d Find the maximum and minimum values of $R(p)$.

e For what value(s) of p does R take on its maximum and minimum values?

f On what intervals is the function increasing? Decreasing?



Answer.

a $-1, 2$

b $3, -1.3$

c $R(-2) = 0,\ R(2) = 0,\ R(4) = 0,\ R(0) = 4$

d Max: 4; Min: -5

e Max at $p = 0$; Min at $p = 5$

f Increasing: $(-3, 0)$ and $(1, 3)$; Decreasing: $(0, 1)$ and $(3, 5)$

4.

a Find $f(-1)$ and $f(3)$.

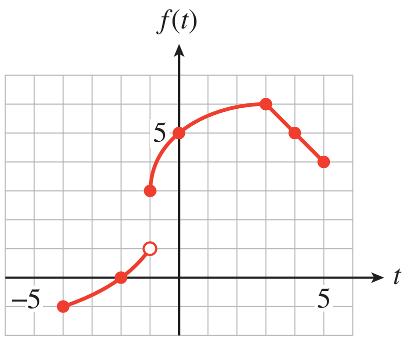
b For what value(s) of t is $f(t) = 5$?

c Find the intercepts of the graph. List the function values given by the intercepts.

d Find the maximum and minimum values of $f(t)$.

e For what value(s) of t does f take on its maximum and minimum values?

f On what intervals is the function increasing? Decreasing?



5.

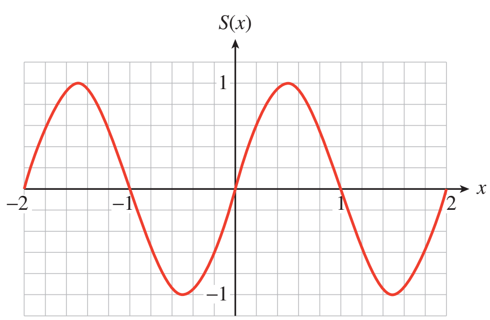
a Find $S(0)$, $S\left(\frac{1}{6}\right)$, and $S(-1)$.

b Estimate the value of $S\left(\frac{1}{3}\right)$ from the graph.

c For what value(s) of x is $S(x) = -\frac{1}{2}$?

d Find the maximum and minimum values of $S(x)$.

e For what value(s) of x does S take on its maximum and minimum values?



Answer.

a $0, \dfrac{1}{2}, 0$

b 0.9

c $\dfrac{-5}{6}, \dfrac{-1}{6}, \dfrac{7}{6}, \dfrac{11}{6}$

d Max: 1; Min: -1

e Max at $x = -1.5, 0.5$; Min at $x = -0.5, 1.5$

6.

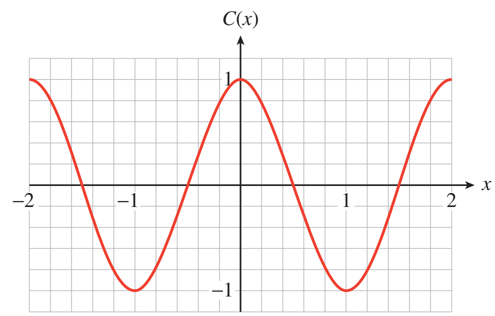
a Find $C(0)$, $C\left(-\frac{1}{3}\right)$, and $C(1)$.

b Estimate the value of $C\left(\frac{1}{6}\right)$ from the graph.

c For what value(s) of x is $C(x) = \frac{1}{2}$?

d Find the maximum and minimum values of $C(x)$.

e For what value(s) of x does C take on its maximum and minimum values?



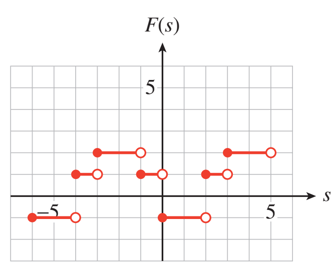
7.

a Find $F(-3)$, $F(-2)$, and $F(2)$.

b For what value(s) of s is $F(s) = -1$?

c Find the maximum and minimum values of $F(s)$.

d For what value(s) of s does F take on its maximum and minimum values?



Answer.

a $2, 2, 1$

b $-6 \le st - 4$ or $0 \le s < 2$

c Max: 2; Min: -1

d Max for $-3 \le s < -1$ or $3 \le s < 5$; Min for $-6 \le s < -4$ or $0 \le s < 2$

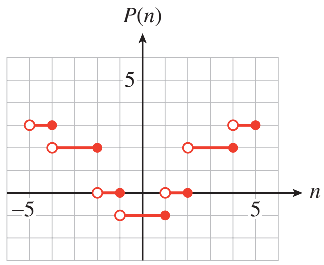
8.

a Find $P(-3)$, $P(-2)$, and $P(1)$.

b For what value(s) of n is $P(n) = 0$?

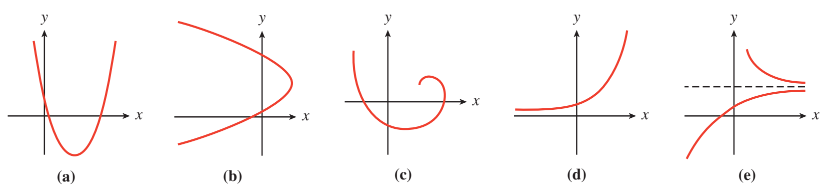
c Find the maximum and minimum values of $P(n)$.

d For what value(s) of n does P take on its maximum and minimum values?



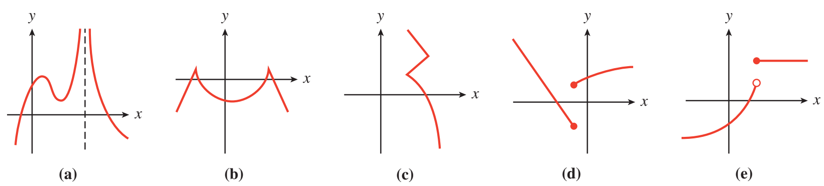
Which of the graphs in Problems 9 and 10 represent functions?

9.



Answer. (a) and (d)

10.



In Problems 11–16,

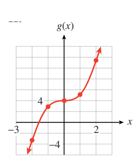
a Make a table of values and sketch a graph of the function by plotting points. (Use the suggested x-values.)

b Use your calculator to graph the function.

Compare the calculator's graph with your sketch.

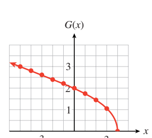
11. $g(x) = x^3+4$; $x = -2, -1, \ldots$ **12.** $h(x) = 2+\sqrt{x}$; $x = 0, 1, \ldots, 9$

Answer.



13. $G(x) = \sqrt{4-x}$; $x = -5, -4, \ldots$ **14.** $F(x) = \sqrt{x-1}$; $x = 1, 2, \ldots, 10$

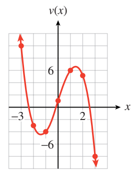
Answer.



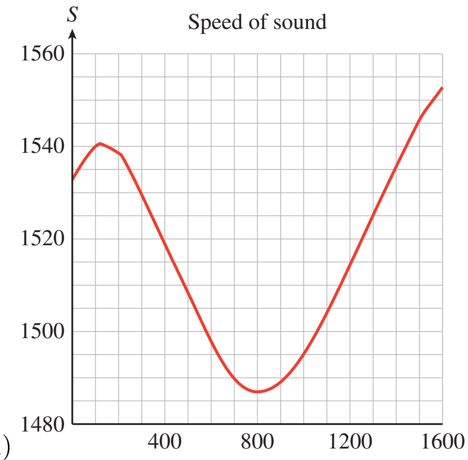
15. $v(x) = 1 + 6x - x^3$; $x = -3, -2, \ldots, 3$

16. $w(x) = x^3 - 8x$; $x = -4, -3, \ldots, 4$

Answer.



17. The graph shows the speed of sound in the ocean as a function of depth, S = f (d). The speed of sound is affected both by increasing water pressure and by dropping temperature. (Source: Scientific American)



a Evaluate $f(1000)$ and explain its meaning.

b Solve $f(d) = 1500$ and explain its meaning.

c At what depth is the speed of sound the slowest, and what is the speed? Write your answer with function notation.

d Describe the behavior of $f(d)$ as d increases.

Answer.

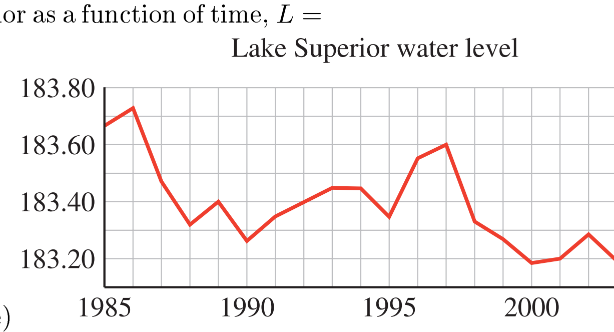
a $f(1000) = 1495$: The speed of sound at a depth of 1000 meters is approximately 1495 meters per second.

b $d = 570$ or $d = 1070$: The speed of sound is 1500 meters per second at both a depth of 570 meters and a depth of 1070 meters.

c The slowest speed occurs at a depth of about 810 meters and the speed is about 1487 meters per second, so $f(810) = 1487$.

d f increases from about 1533 to 1541 in the first 110 meters of depth, then drops to about 1487 at 810 meters, then rises again, passing 1553 at a depth of about 1600 meters.

18. The graph shows the water level in Lake Superior as a function of time, $L = f(t)$. (Source: The Canadian Hydrographic Service)



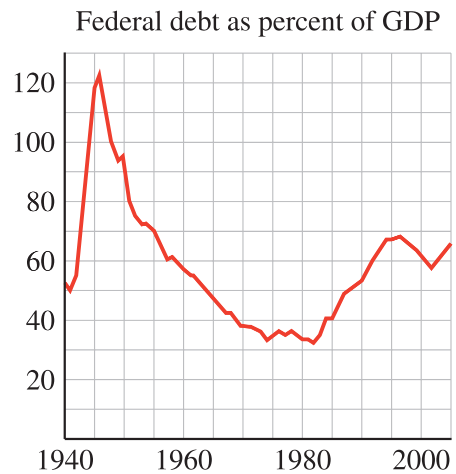
a Evaluate $f(1997)$ and explain its meaning.

b Solve $f(t) = 183.5$ and explain its meaning.

c In which two years did Lake Superior reach its highest levels, and what were those levels? Write your answers with function notation.

d Over which two-year period did the water level drop the most?

19. The graph shows the federal debt as a percentage of the gross domestic product (GDP), as a function of time, $D = f(t)$. (Source: Office of Management and Budget)



a Evaluate $f(1985)$ and explain its meaning.

b Solve $f(t) = 70$ and explain its meaning.

c When did the federal debt reach its highest level since 1960, and what was that level? Write your answer with function notation.

d What is the longest time interval over which the federal debt was decreasing?

Answer.

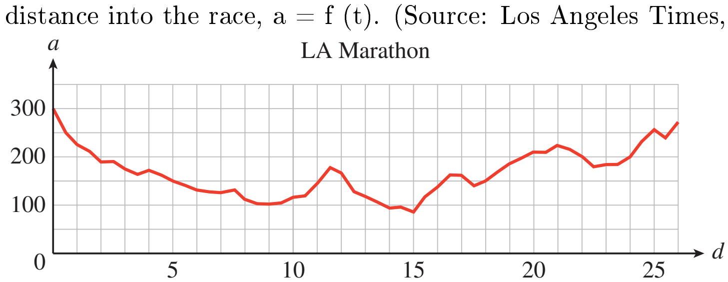
a $f(1985) = 41$: The federal debt in 1985 was about 41% of the gross domestic product.

b $t = 1942$ or $t = 1955$: The federal debt was 70% of the gross domestic product in 1942 and 1955.

c In about 1997, the debt was about 67% of the gross domestic product, so $f(1997) \approx 67.3$.

d The percentage basically dropped from 1946 to 1973, but there were small rises around 1950, 1954, 1958, and 1968, so the longest time interval was from 1958 to 1967.

20. The graph shows the elevation of the Los Angeles Marathon course as a function of the distance into the race, a = f (t). (Source: Los Angeles Times, March 3, 2005)

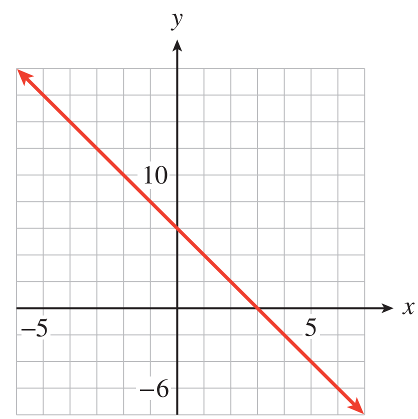


a Evaluate $f(5)$ and explain its meaning.

b Solve $f(d) = 200$ and explain its meaning.

c When does the marathon course reach its lowest elevation, and what is that elevation? Write your answer with function notation.

d Give three intervals over which the elevation is increasing.



21. The figure shows a graph of $y = -2x+6$.

a Use the graph to find all values of x for which

i $y = 12$

ii $y > 12$

iii $y < 12$

b Use the graph to solve

i $-2x + 6 = 12$

ii $-2x + 6 > 12$

iii $-2x + 6 < 12$

c Explain why your answers to parts (a) and (b) are the same.

Answer.

a *i* $x = -3$

ii $x < -3$

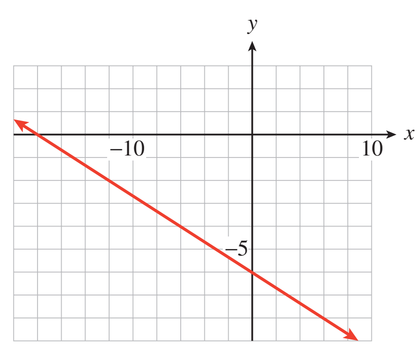
iii $x > -3$

b *i* $x = -3$

ii $x < -3$

iii $x > -3$

c On the graph of $y = -2x + 6$, a value of y is the same as a value of $-2x + 6$, so parts (a) and (b) are asking for the same x's.



22. The figure shows a graph of $y = \dfrac{-x}{3} - 6$.

a Use the graph to find all values of x for which

i $y = -4$

ii $y > -4$

iii $y < -4$

b Use the graph to solve

i $\dfrac{-x}{3} - 6 = -4$

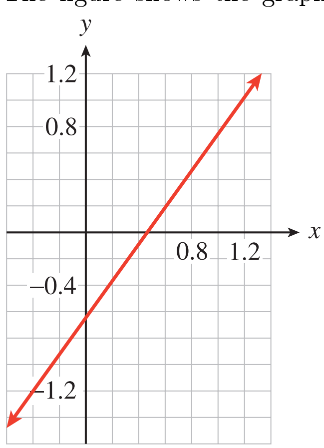
ii $\dfrac{-x}{3} - 6 > -4$

iii $\dfrac{-x}{3} - 6 < -4$

c Explain why your answers to parts (a) and (b) are the same.

In Problems 23 and 24, use the graph to solve the equation or inequality, and then solve algebraically. (To review solving linear inequalities algebraically, see Algebra Skills Refresher **??**.)

23. The figure shows the graph of $y = 1.4x - 0.64$. Solve the following:

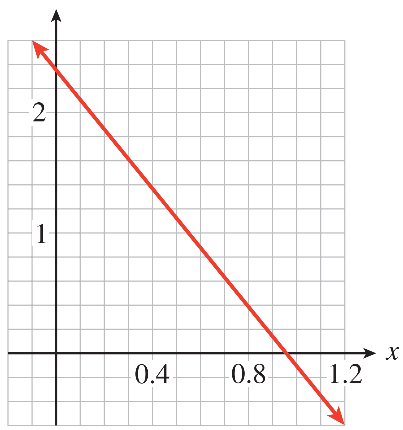


a $1.4x - 0.64 = 0.2$
b $-1.2 = 1.4x - 0.64$
c $1.4x - 0.64 > 0.2$
d $-1.2 > 1.4x - 0.64$

Answer.

a $x = 0.6$
b $x = -0.4$
c $x > 0.6$
d $x < -0.4$

24. The figure shows the graph of $y = -2.4x + 2.32$. Solve the following:



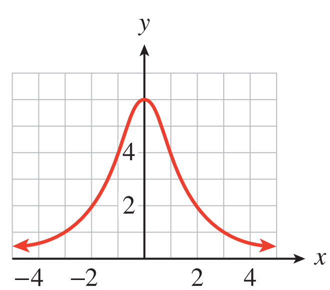
a $1.6 = -2.4x + 2.32$

b $-2.4x + 2.32 = 0.4$

c $-2.4x + 2.32 \geq 1.6$

d $0.4 \geq -2.4x + 2.32$

For Problems 25–30, use the graphs to estimate solutions to the equations and inequalities.



25. The figure shows the graph of $g(x) = \dfrac{12}{2 + x^2}$.

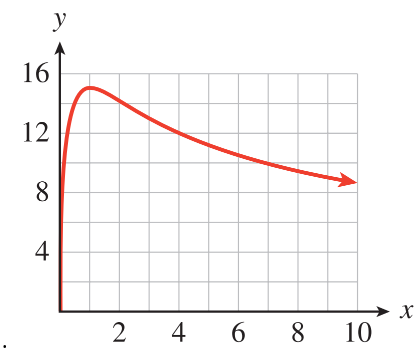
a Solve $\dfrac{12}{2 + x^2} = 4$

b Solve $1 \leq \dfrac{12}{2 + x^2} \leq 2$

Answer.

a $x = -1$ or $x = 1$

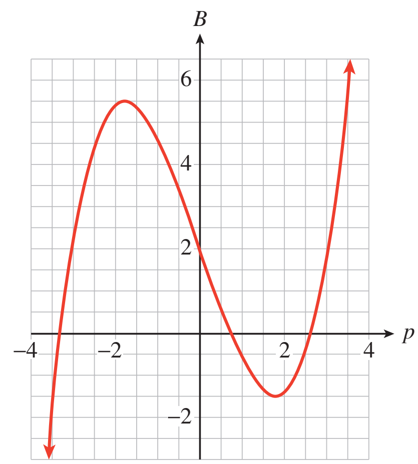
b Approximately $-3 \leq x \leq -2$ or $2 \leq x \leq 3$



26. The figure shows the graph of $f(x) = \dfrac{30\sqrt{x}}{1+x}$.

a Solve $\dfrac{30\sqrt{x}}{1+x} = 15$

b Solve $\dfrac{30\sqrt{x}}{1+x} < 12$



27. The figure shows a graph of $B = \dfrac{1}{3}p^3 - 3p + 2$.

a Solve $\dfrac{1}{3}p^3 - 3p + 2 = 6$

b Solve $\dfrac{1}{3}p^3 - 3p + 2 = 5$

c Solve $\dfrac{1}{3}p^3 - 3p + 2 < 1$

d What range of values does B have for p between -2.5 and 0.5?

e For what values of p is B increasing?

Answer.

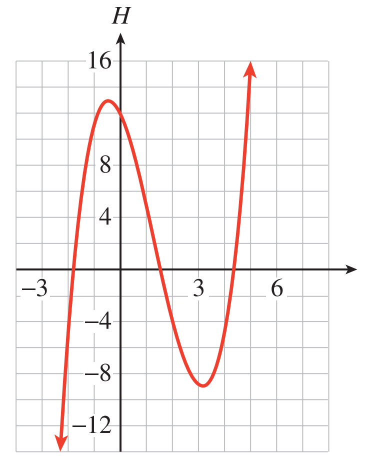
a 3.5

b $-2.2, -1.2, 3.4$

c $p < -3.1$ or $0.3 < p < 2.8$

d $0.5 < B < 5.5$

e $p < -1.7$ or $p > 1.7$



28. The figure shows a graph of $H = t^3 - 4t^2 - 4t + 12$.

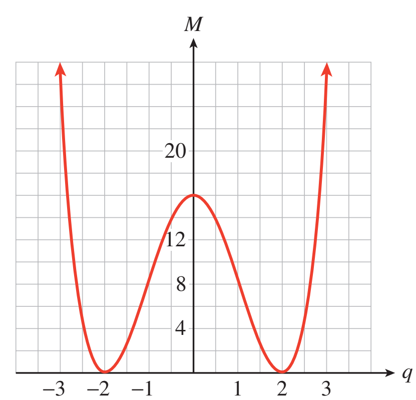
a Solve $t^3 - 4t^2 - 4t + 12 = -4$

b Solve $t^3 - 4t^2 - 4t + 12 = 16$

c Solve $t^3 - 4t^2 - 4t + 12 > 6$

d Estimate the horizontal and vertical intercepts of the graph.

e For what values of t is H increasing?



29. The figure shows a graph of $M = g(q)$.

a Find all values of q for which

i $g(q) = 0$

ii $g(q) = 16$

iii $g(q) < 6$

b For what values of q is $g(q)$ increasing?

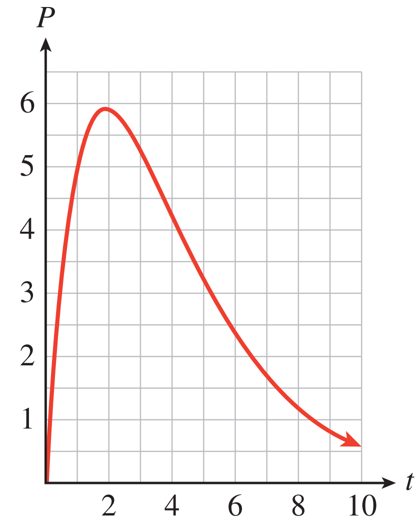
Answer.

a *i* $-2, 2$

ii $-2.8, 0, 2.8$

iii $-2.5 < q < -1.25$ or $1.25 < q < 2.5$

b $-2 < q < 0$ or $q > 2$



30. The figure shows a graph of $P = f(t)$.

a Find all values of q for which

i $f(t) = 3$

ii $f(t) > 4.5$

iii $2 \leq f(t) \leq 4$

b For what values of t is $f(t)$ decreasing?

31.

a Delbert reads the following values from the graph of a function:

$$f(-3) = 5,\ f(-1) = 2,\ f(1) = 0,$$

$$f(-1) = -4,\ f(-3) = -2$$

Can his readings be correct? Explain why or why not.

b Francine reads the following values from the graph of a function:

$$g(-2) = 6,\ g(0) = 0,\ g(2) = 6,$$

$$g(4) = 0,\ g(6) = 6$$

Can her readings be correct? Explain why or why not.

Answer.

a He has an error: $f(-3)$ cannot have both the value 5 and also the value -2 and $f(-1)$ cannot have both the value of 2 and -4.

b Her readings are possible for a function: each input has only one output.

32.

a Sketch the graph of a function that has the following values:

$$F(-2) = 3,\ F(-1) = 3,\ F(0) = 3,$$

$$F(1) = 3,\ F(2) = 3$$

b Sketch the graph of a function that has the following values:

$$G(-2) = 1,\ G(-1) = 0,\ G(0) = -1,$$

$$G(1) = 0,\ G(2) = 1$$

For Problems 33–36, graph each function in the friendly window

Xmin $= -9.4$	Xmax $= 9.4$
Ymin $= -10$	Ymax $= 10$

Then answer the questions about the graph. (See Appendix **??** for an explanation of friendly windows.)

33. $g(x) = \sqrt{36 - x^2}$

a Complete the table. (Round values to tenths.)

x	-4	-2	3	5
$g(x)$				

b Find all points on the graph for which $g(x) = 3.6$.

Answer.

a

x	-4	-2	3	5
$g(x)$	4.5	5.7	5.2	3.3

b $-4.8, 4.8$

34. $g(x) = \sqrt{x^2} - 6$

a Complete the table. (Round values to tenths.)

x	-8	-2	3	6
$f(x)$				

b Find all points on the graph for which $f(x) = -2$.

35. $F(x) = 0.5x^3 - 4x$

a Estimate the coordinates of the turning points of the graph, that is, where the graph changes from increasing to decreasing or vice versa.

b Write an equation of the form $F(a) = b$ for each turning point.

Answer.

a $(-1.6, 4.352), (1.6, -4.352)$

b $F(-1.6) = 4.352$; $F(1.6) = -4.352$

36. $G(x) = 2 + 4x - x^3$

a Estimate the coordinates of the turning points of the graph, that is, where the graph changes from increasing to decreasing or vice versa.

b Write an equation of the form $G(a) = b$ for each turning point.

For Problems 37–40, graph each function

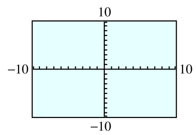
a First using the standard window.

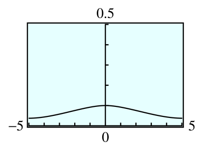
b Then using the suggested window. Explain how the window alters the appearance of the graph in each case.

37. $h(x) = \dfrac{1}{x^2+10}$

Xmin $= -5$ Xmax $= 5$
Ymin $= 0$ Ymax $= 0.5$

Answer.

a 

b 

The curve cannot be distinguished from the x-axis in the standard window because the values of y are closer to zero than the resolution of the calculator can display. The second window provides sufficient resolution to see the curve.

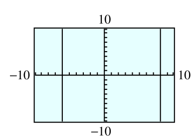
38. $H(x) = \sqrt{1-x^2}$

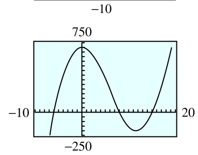
Xmin $= -2$ Xmax $= 2$
Ymin $= -2$ Ymax $= 2$

39. $P(x) = (x-8)(x+6)(x-15)$

Xmin $= -10$ Xmax $= 20$
Ymin $= -250$ Ymax $= 750$

Answer.

a 

b 

The curve looks like two vertical lines in the standard window because that window covers too small a region of the plane. The second window allows us to see the turning points of the curve.

40. $p(x) = 200x^3$

Xmin $= -5$ Xmax $= 5$
Ymin $= -10000$ Ymax $= 10000$

Graph each equation with the ZInteger setting. (Press `ZOOM` 6,then `ZOOM` 8 `ENTER`.) Use the graph to answer each question. Use the equation to verify your answers.

41. Graph $y = 2x - 3$.

a For what value of x is $y = 5$?
b For what value of x is $y = -13$?
c For what value of x is $y > -1$?
d For what value of x is $y < 25$?

Answer.

a $x = 4$ b $x = -5$ c $x > 1$ d $x < 14$

42. Graph $y = 4 - 2x$.

a For what value of x is $y = 6$?
b For what value of x is $y = -4$?
c For what value of x is $y > -12$?
d For what value of x is $y < 18$?

43. Graph $y = 6.5 - 1.8x$.

a For what value of x is $y = -13.3$?
b For what value of x is $y = 24.5$?
c For what value of x is $y \leq 15.5$?
d For what value of x is $y \geq -7.9$?

Answer.

a $x = 11$ b $x = -10$ c $x \geq -5$ d $x \leq 8$

44. Graph $y = 0.2x + 1.4$..

a For what value of x is $y = -5.2$?
b For what value of x is $y = 2.8$?
c For what value of x is $y \leq -3.2$?
d For what value of x is $y \geq 4.4$?

In Problems 45–48, graph each equation with the ZInteger setting. Use the graph to solve each equation or inequality. Check your solutions algebraically.

45. Graph $y = -0.4x + 3.7$.

a Solve $-0.4x + 3.7 = 2.1$.
b Solve $-0.4x + 3.7 > -5.1$.

Answer.

a $x = 4$ b $x < 22$

46. Graph $y = 0.4(x - 1.5)7$.

a Solve $0.4(x - 1.5) = -8.6$.

b Solve $0.4(x - 1.5) < 8.6$.

47. Graph $y = \frac{2}{3}x - 24$.

a Solve $\frac{2}{3}x - 24 = -10\frac{2}{3}$.

b Solve $\frac{2}{3}x - 24 \leq -19\frac{1}{3}$.

Answer.

a $x = 20$

b $x \leq 7$

48. Graph $y = \dfrac{80 - 3x}{5}$.

a Solve $\dfrac{80 - 3x}{5} = 22\frac{3}{5}$.

b Solve $\dfrac{80 - 3x}{5} \leq -9\frac{2}{5}$.

49. Graph $y = 0.01x^3 - 0.1x^2 - 2.75x + 15$.

a Use your graph to solve $0.01x^3 - 0.1x^2 - 2.75x + 15 = 0$.

b Press `Y=` and enter $Y_2 = 10$. Press `GRAPH`, and you should see the horizontal line $y = 10$ superimposed on your previous graph. How many solutions does the equation $0.01x^3 - 0.1x^2 - 2.75x + 15 = 10$ have? Estimate each solution to the nearest whole number.

Answer.

a $-15, 5, 20$

b $-13, 2, 22$

50. Graph $y = 2.5x - 0.025x^2 - 0.005x^3$.

a Use your graph to solve $2.5x - 0.025x^2 - 0.005x^3 = 0$.

b Press `Y=` and enter $Y_2 = -5$. Press `GRAPH`, and you should see the horizontal line $y = -5$ superimposed on your previous graph. How many solutions does the equation $2.5x - 0.025x^2 - 0.005x^3 = -5$ have? Estimate each solution to the nearest whole number.

1.4 Slope and Rate of Change

1.4.1 Using Ratios for Comparison

Which is more expensive, a 64-ounce bottle of Velvolux dish soap that costs \$3.52, or a 60-ounce bottle of Rainfresh dish soap that costs \$3.36?

You are probably familiar with the notion of comparison shopping. To decide which dish soap is the better buy, we compute the unit price, or price per ounce, for each bottle. The unit price for Velvolux is

$$\frac{352 \text{ cents}}{64 \text{ ounces}} = 5.5 \text{ cents per ounce}$$

and the unit price for Rainfresh is

$$\frac{336 \text{ cents}}{60 \text{ ounces}} = 5.6 \text{ cents per ounce}$$

The Velvolux costs less per ounce, so it is the better buy. By computing the price of each brand for *the same amount of soap*, it is easy to compare them.

In many situations, a ratio, similar to a unit price, can provide a basis for comparison. Example 1.89 uses a ratio to measure a rate of growth.

Example 1.89. Which grow faster, Hybrid A wheat seedlings, which grow 11.2 centimeters in 14 days, or Hybrid B seedlings, which grow 13.5 centimeters in 18 days?

Solution. We compute the growth rate for each strain of wheat. Growth rate is expressed as a ratio, $\frac{\text{centimeters}}{\text{days}}$, or centimeters per day. The growth rate for Hybrid A is

$$\frac{11.2 \text{ centimeters}}{14 \text{ days}} = 0.8 \text{ centimeters per day}$$

and the growth rate for Hybrid B is

$$\frac{13.5 \text{ centimeters}}{18 \text{ days}} = 0.75 \text{ centimeters per day}$$

Because their rate of growth is larger, the Hybrid A seedlings grow faster.

By computing the growth of each strain of wheat seedling over the same unit of time, a single day, we have a basis for comparison. In this case, the ratio $\frac{\text{centimeters}}{\text{day}}$ measures the rate of growth of the wheat seedlings.

Exercise 1.90. Delbert traveled 258 miles on 12 gallons of gas, and Francine traveled 182 miles on 8 gallons of gas. Compute the ratio $\frac{\text{miles}}{\text{gallon}}$ for each car. Whose car gets the better gas mileage?

In Exercise 1.90, the ratio $\frac{\text{miles}}{\text{gallon}}$ measures the rate at which each car uses gasoline. By computing the mileage for each car for the same amount of gas, we have a basis for comparison. We can use this same idea, finding a common basis for comparison, to measure the steepness of an incline.

1.4.2 Measuring Steepness

Imagine you are an ant carrying a heavy burden along one of the two paths shown in Figure 1.91. Which path is more difficult? Most ants would agree that the steeper path is more difficult.

But what exactly is steepness? It is not merely the gain in altitude, because even a gentle incline will reach a great height eventually. Steepness measures how sharply the altitude increases. An ant finds the second path more difficult, or steeper, because it rises 5 feet while the first path rises only 2 feet over the same horizontal distance.

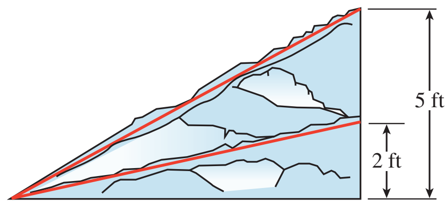


Figure 1.91

To compare the steepness of two inclined paths, we compute the ratio of change in altitude to change in horizontal distance for each path.

Example 1.92. Which is steeper, Stony Point trail, which climbs 400 feet over a horizontal distance of 2500 feet, or Lone Pine trail, which climbs 360 feet over a horizontal distance of 1800 feet?

Solution. For each trail, we compute the ratio of vertical gain to horizontal distance. For Stony Point trail, the ratio is

$$\frac{400 \text{ feet}}{2500 \text{ feet}} = 0.16$$

and for Lone Pine trail, the ratio is

$$\frac{360 \text{ feet}}{1800 \text{ feet}} = 0.20$$

Lone Pine trail is steeper, because it has a vertical gain of 0.20 foot for every foot traveled horizontally. Or, in more practical units, Lone Pine trail rises 20 feet for every 100 feet of horizontal distance, whereas Stony Point trail rises only 16 feet over a horizontal distance of 100 feet.

Exercise 1.93. Which is steeper, a staircase that rises 10 feet over a horizontal distance of 4 feet, or the steps in the football stadium, which rise 20 yards over a horizontal distance of 12 yards?

1.4.3 Definition of Slope

To compare the steepness of the two trails in Example 1.92, it is not enough to know which trail has the greater gain in elevation overall. Instead, we compare their elevation gains over the same horizontal distance. Using the same horizontal distance provides a basis for comparison. The two trails are illustrated in Figure 1.94 as lines on a coordinate grid.

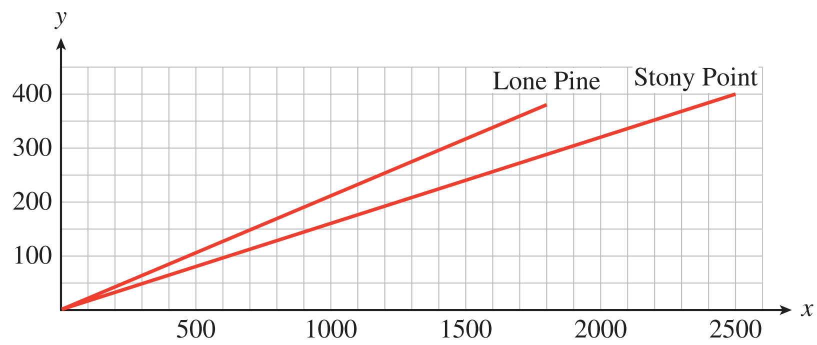


Figure 1.94

The ratio we computed in Example 1.92,

$$\frac{\text{change in elevation}}{\text{change in horizontal position}}$$

appears on the graphs in Figure 1.94 as

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

For example, as we travel along the line representing Stony Point trail, we move from the point$0, 0)$ to the point $(2500, 400)$. The y-coordinate changes by 400 and the x-coordinate changes by 2500, giving the ratio 0.16 that we found in Example 1.922. We call this ratio the **slope** of the line.

Definition of Slope

The **slope** of a line is the ratio

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

as we move from one point to another on the line.

Example 1.95. Compute the slope of the line that passes through points A and B in Figure 1.96.

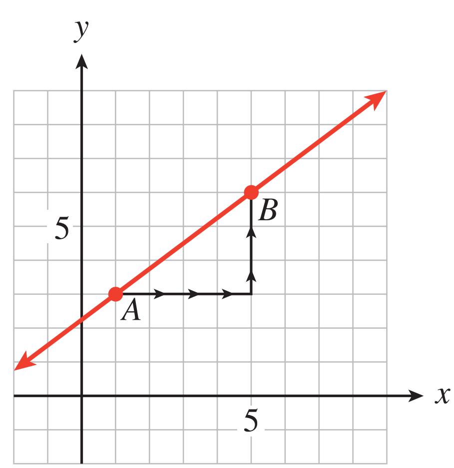


Figure 1.96

Solution. As we move along the line from A to B, the y-coordinate changes by 3 units, and the x-coordinate changes by 4 units. The slope of the line is thus

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} = \frac{3}{4}$$

Exercise 1.97.

Compute the slope of the line through the indicated points in Figure 1.98. On both axes, one square represents one unit.

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} =$$

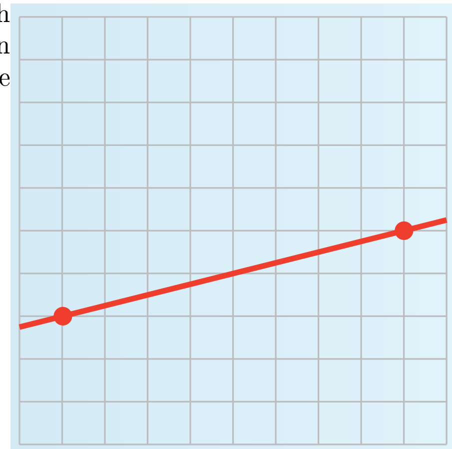


Figure 1.98

The slope of a line is a number. It tells us how much the y-coordinates of points on the line increase when we increase their x-coordinates by 1 unit. For instance, the slope $\frac{3}{4}$ in Example 1.95 means that the y-coordinate increases by $\frac{3}{4}$ unit when the x-coordinate increases by 1 unit. For increasing graphs, a larger slope indicates a greater increase in altitude, and hence a steeper line.

1.4.4 Notation for Slope

We use a shorthand notation for the ratio that defines slope,

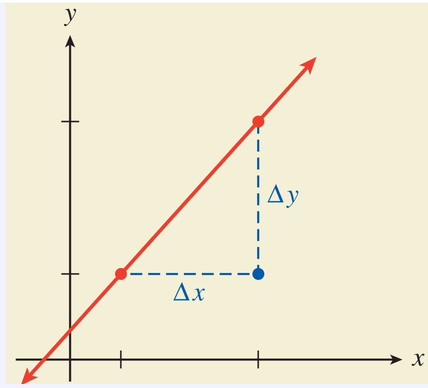
$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

The symbol Δ (the Greek letter delta) is used in mathematics to denote *change in*. In particular, Δy means *change in y-coordinate*, and Δx means *change in x-coordinate*. We also use the letter m to stand for slope. With these symbols, we can write the definition of slope as follows.

Notation for Slope

The **slope** of a line is given by

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}, \quad x \neq 0$$



Example 1.99. The Great Pyramid of Khufu in Egypt was built around 2550 B.C. It is 147 meters tall and has a square base 229 meters on each side. Calculate the slope of the sides of the pyramid, rounded to two decimal places.

Solution.

From Figure 1.100, we see that Δx is only half the base of the Great Pyramid, so

$$\Delta x = 0.5(229) = 114.5$$

and the slope of the side is

$$m = \frac{\Delta y}{\Delta x} = \frac{147}{114.5} = 1.28$$

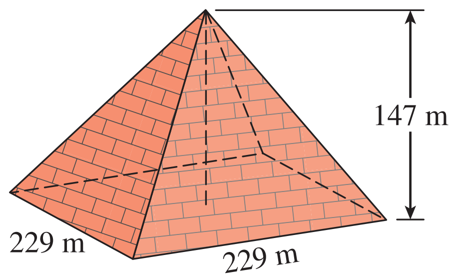


Figure 1.100

Exercise 1.101.

The Kukulcan Pyramid at Chichen Itza in Mexico was built around 800 A.D. It is 24 meters high, with a temple built on its top platform, as shown in Figure 1.102. The square base is 55 meters on each side, and the top platform is 19.5 meters on each side. Calculate the slope of the sides of the pyramid. Which pyramid is steeper, Kukulcan or the Great Pyramid?

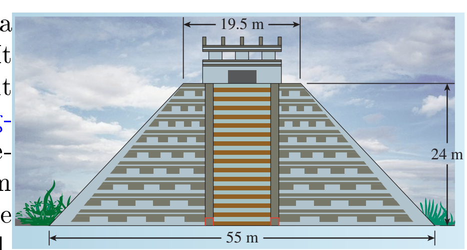


Figure 1.102

So far, we have only considered examples in which Δx and Δy are positive numbers, but they can also be negative.

$$\Delta x = \begin{cases} \text{positive if } x \text{ increases (move to the right)} \\ \text{negative if } x \text{ decreases (move to the left)} \end{cases}$$

$$\Delta y = \begin{cases} \text{positive if } y \text{ increases (move up)} \\ \text{negative if } y \text{ decreases (move down)} \end{cases}$$

Example 1.103.

Compute the slope of the line that passes through the points $P(-4, 2)$ and $Q(5, -1)$ shown in Figure 1.104 Illustrate Δy and Δx on the graph.

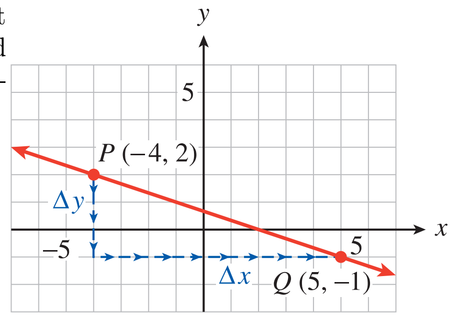


Figure 1.104

Solution. As we move from the point $P(-4, 2)$ to the point $Q(5, -1)$, we move 3 units *down*, so $\Delta y = -3$. We then move 9 units to the right, so $\Delta x = 9$. Thus, the slope is

$$m = \frac{\Delta y}{\Delta x} = \frac{-3}{9} = \frac{-1}{3}$$

Δy and Δx are labeled on the graph.

We can move from point to point in either direction to compute the slope. The line graphed in Example 1.103 decreases as we move from left to right and hence has a negative slope. The slope is the same if we move from point Q to point P instead of from P to Q. (See Figure 1.105.) In that case, our computation looks like this:

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{-9} = \frac{-1}{3}$$

Δy and Δx are labeled on the graph.

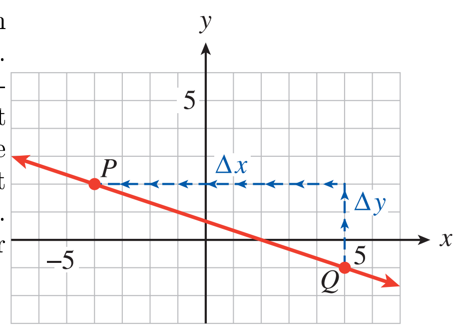


Figure 1.105

1.4.5 Lines Have Constant Slope

How do we know which two points to choose when we want to compute the slope of a line? It turns out that any two points on the line will do.

Exercise 1.106.

a Graph the line $4x - 2y = 8$ by finding the x- and y-intercepts

b Compute the slope of the line using the x-intercept and y-intercept.

c Compute the slope of the line using the points $(4, 4)$ and $(1, -2)$.

Exercise 1.106 illustrates an important property of lines: They have constant slope. No matter which two points we use to calculate the slope, we will always get the same result. We will see later that lines are the only graphs that have this property. We can think of the slope as a scale factor that tells us how many units y increases (or decreases) for each unit of increase in x. Compare the lines in Figure 1.107

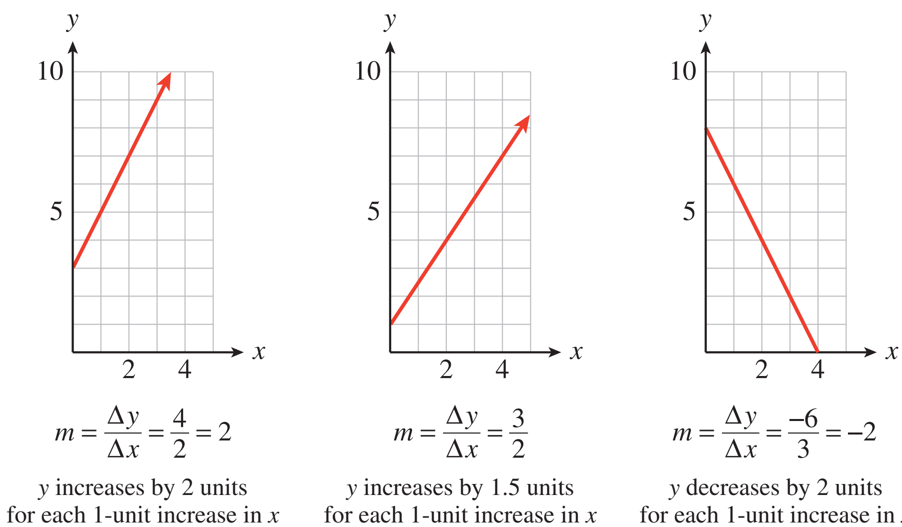


Figure 1.107

Observe that a line with positive slope increases from left to right, and one with negative slope decreases. What sort of line has slope $m = 0$?

1.4.6 Meaning of Slope

In Example 1 of Section 1.1, we graphed the equation $C = 5 + 3t$ showing the cost of a bicycle rental in terms of the length of the rental. The graph is reproduced in Figure 1.108. We can choose any two points on the line to compute its slope. Using points P and Q as shown, we find that

$$m = \frac{\Delta C}{\Delta t} = \frac{9}{3} = 3$$

The slope of the line is 3.

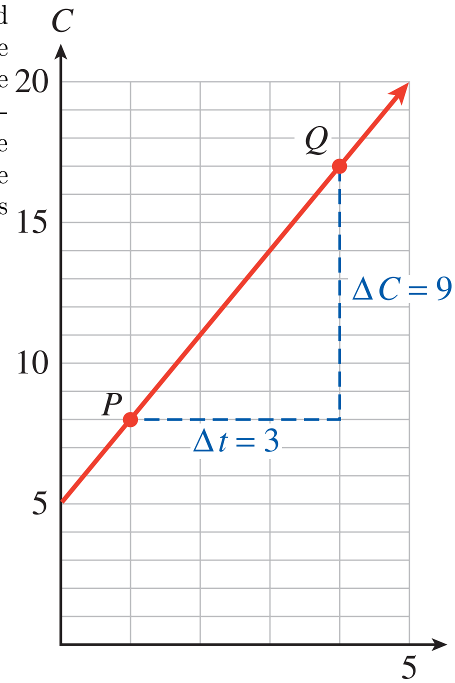


Figure 1.108

What does this value mean for the cost of renting a bicycle? The expression

$$\frac{\Delta C}{\Delta t} = \frac{9}{3}$$

stands for

$$\frac{\text{change in cost}}{\text{change in time}} = \frac{9 \text{ dollars}}{3 \text{ hours}}$$

If we increase the length of the rental by 3 hours, the cost of the rental increases by 9 dollars. The slope gives the rate of increase in the rental fee, 3 dollars per hour. In general, we can make the following statement.

Rate of Change

The slope of a line measures the *rate of change* of the output variable with respect to the input variable.

Depending on the variables involved, this rate might be interpreted as a rate of growth or a rate of speed. A negative slope might represent a rate of decrease or a rate of consumption. The slope of a graph can give us valuable information about the variables.

Example 1.109. The graph in Figure 1.110 shows the distance in miles traveled by a big-rig truck driver after t hours on the road.

a Compute the slope of the graph.

b What does the slope tell us about the problem?

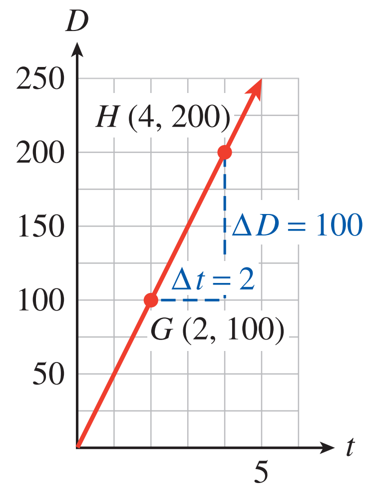


Figure 1.110

Solution.

a Choose any two points on the line, say $G(2, 100)$ and $H(4, 200)$, in Figure 1.110. As we move from G to H, we find

$$m = \frac{\Delta D}{\Delta t} = \frac{100}{2} = 50$$

The slope of the line is 50.

b The best way to understand the slope is to include units in the calculation. For our example,

$$\frac{\Delta D}{\Delta t} \text{ means } \frac{\text{change in distance}}{\text{change in time}}$$

or

$$\frac{\Delta D}{\Delta t} = \frac{100 \text{ miles}}{2 \text{ hours}} = 50 \text{miles per hour}$$

The slope represents the trucker's average speed or velocity.

Exercise 1.111. The graph in Figure 1.112 shows the altitude, a (in feet), of a skier t minutes after getting on a ski lift.

a Choose two points and compute the slope (including units).

b What does the slope tell us about the problem?

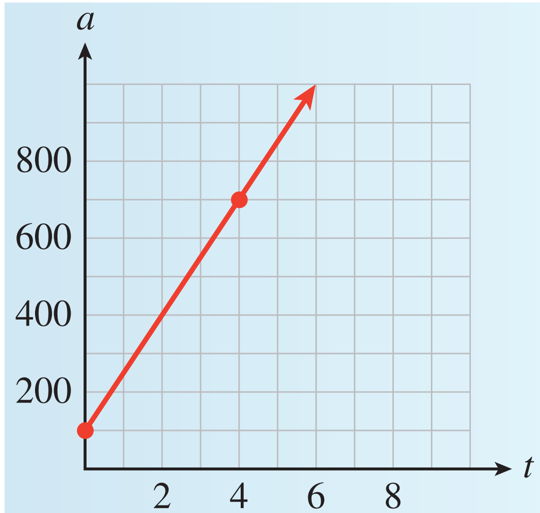


Figure 1.112

1.4.7 A Formula for Slope

We have defined the slope of a line to be the ratio $m = \frac{\Delta y}{\Delta x}$ as we move from one point to another on the line. So far, we have computed Δy and Δx by counting squares on the graph, but this method is not always practical. All we really need are the coordinates of two points on the graph.

We will use **subscripts** to distinguish the two points:

P_1 means *first point* and P_2 means *second point*.

We denote the coordinates of P_1 by (x_1, y_1) and the coordinates of P_2 by (x_2, y_2).

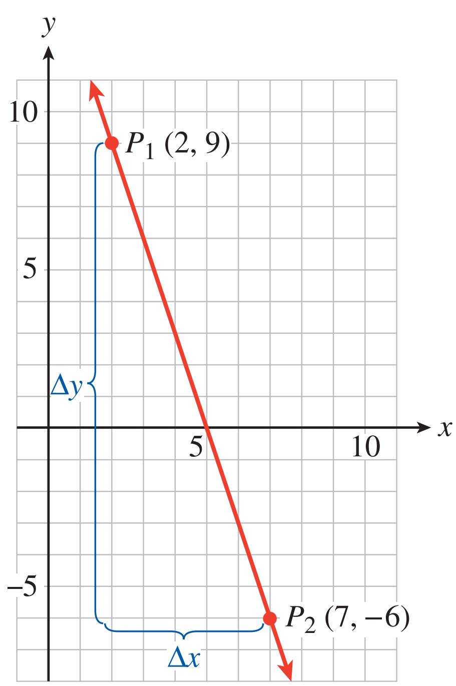


Figure 1.113

Now consider a specific example. The line through the two points $P_1(2, 9)$ and $P_2(7, -6)$ is shown in Figure 1.113. We can find Δx by subtracting the x-coordinates of the points:

$$\Delta x = 7 - 2 = 5$$

In general, we have

$$\Delta x = x_2 - x_1$$

and similarly

$$\Delta y = y_2 - y_1$$

These formulas work even if some of the coordinates are negative; in our example

$$\Delta y = y_2 - y_1 = -6 - 9 = -15$$

By counting squares *down* from P_1 to P_2, we see that Δy is indeed -15. The slope of the line in Figure 1.113 is

$$m\& = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \backslash\backslash\& = \frac{-15}{5} = -3$$

We now have a formula for the slope of a line that works even if we do not have a graph.

Two-Point Slope Formula

The slope of the line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_2 \neq x_1$$

Example 1.114. Compute the slope of the line in Figure 1.113 using the points $Q_1(6, -3)$ and $Q_2(4, 3)$.

Solution. Substitute the coordinates of Q_1 and Q_2 into the slope formula to find

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-3)}{4 - 6} = \frac{6}{-2} = -3$$

This value for the slope, -3, is the same value found above.

Exercise 1.115.

a Find the slope of the line passing through the points $(2, -3)$ and $(-2, -1)$.

b Sketch a graph of the line by hand.

It will also be useful to write the slope formula with function notation. Recall that $f(x)$ is another symbol for y, and, in particular, that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Thus, if $x_2 \neq x_1$, we have

Slope Formula in Function Notation

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad x_2 \neq x_1$$

Figure 1.117 shows a graph of

$$f(x) = x_2 - 6x$$

Example 1.116. a Compute the slope of the line segment joining the points at $x = 1$ and $x = 4$.

b Compute the slope of the line segment joining the points at $x = 2$ and $x = 5$.

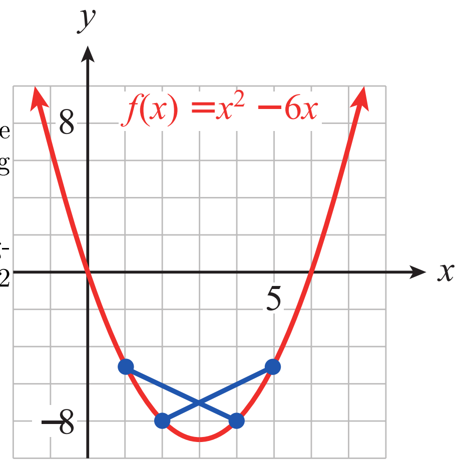


Figure 1.117

Solution.

a We set $x_1 = \mathbf{1}$ and $x_2 = \mathbf{4}$ and find the function values at each point.

$$f(x_1) = f(\mathbf{1}) = \mathbf{1}^2 - 6(\mathbf{1}) = -5$$

$$f(x_2) = f(\mathbf{4}) = \mathbf{4}^2 - 6(\mathbf{4}) = -8$$

Then

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{-8 - (-5)}{4 - 1} = \frac{-3}{3} = -1$$

b We set $x_1 = \mathbf{2}$ and $x_2 = \mathbf{5}$ and find the function values at each point.

$$f(x_1) = f(\mathbf{2}) = \mathbf{2}^2 - 6(\mathbf{2}) = -8$$

$$f(x_2) = f(\mathbf{5}) = \mathbf{5}^2 - 6(\mathbf{5}) = -5$$

Then

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{-5 - (-8)}{5 - 2} = \frac{3}{3} = 1$$

Note that the graph of f is not a straight line and that the slope is not constant.

Exercise 1.118.

Figure 1.119 shows the graph of a function f.

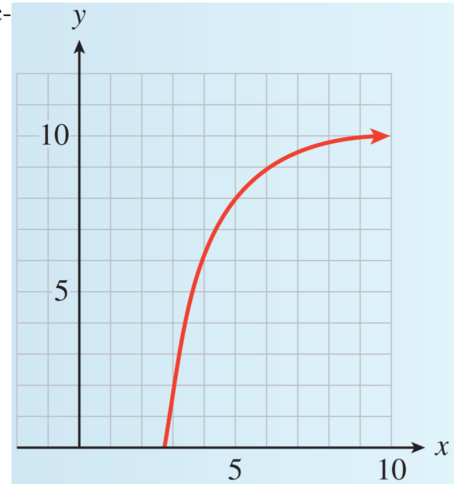


Figure 1.119

a Find $f(3)$ and $f(5)$.

b Compute the slope of the line segment joining the points at $x = 3$ and $x = 5$.

c Write an expression for the slope of the line segment joining the points at $x = a$ and $x = b$.

1.4.8 Section Summary

1.4.8.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Ratio
- Scale factor
- Rate of change
- Slope

1.4.8.2 CONCEPTS

1 We can use ratios to compare quantities.

2 The slope ratio, $\dfrac{\text{change in y-coordinate}}{\text{change in } x\text{-coordinate}}$, measures the steepness of a line.

3 Notation for slope: $m = \dfrac{\Delta y}{\Delta x}$, $\Delta x \neq 0$.

4 Formula for slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}, \quad x_2 \neq x_1$

5 Formula for slope: $m = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad x_2 \neq x_1$

6 Lines have constant slope.

7 Slope is a scale factor that tells us how many units Δy increases for each unit increase in Δx as we move along the line.

8 The slope gives us the rate of change.

1.4.8.3 STUDY QUESTIONS

1 Explain how to compare prices with unit pricing.

2 Why is Δy the numerator of the slope ratio and Δx the denominator?

3 Which line is steeper, one with $m = -2$ or one with $m = -5$?

4 A classmate says that you must always use the intercepts to calculate the slope of a line. Do you agree? Explain.

5 In an application, what does the slope of the graph tell you about the situation?

1.4.8.4 SKILLS

Practice each skill in the Homework 1.4.9 problems listed.

1 Use ratios for comparison: #1–4

2 Compute slope from a graph: #5–16, 23–26

3 Use slope to find Δy or Δx: #17–20, 27–30

4 Use slope to compare steepness: #21 and 22

5 Decide whether data points lie on a straight line: #41–46

6 Interpret slope as a rate of change: #31–40

7 Use function notation to discuss graphs and slope: #53–62

1.4.9 Homework

Compute ratios to answer the questions in Problems 1–4.

1. Carl runs 100 meters in 10 seconds. Anthony runs 200 meters in 19.6 seconds. Who has the faster average speed?

Answer. Anthony

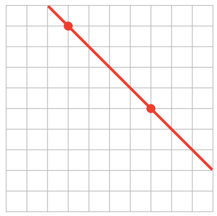
2. On his 512-mile round trip to Las Vegas and back, Corey needed 16 gallons of gasoline. He used 13 gallons of gasoline on a 429-mile trip to Los Angeles. On which trip did he get better fuel economy?

3. Grimy Gulch Pass rises 0.6 miles over a horizontal distance of 26 miles. Bob's driveway rises 12 feet over a horizontal distance of 150 feet. Which is steeper?

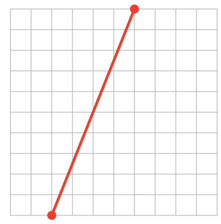
Answer. Bob's driveway

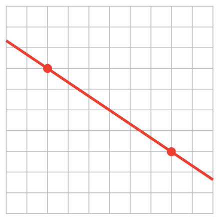
4. Which is steeper, the truck ramp for Acme Movers, which rises 4 feet over a horizontal distance of 9 feet, or a toy truck ramp, which rises 3 centimeters over a horizontal distance of 7 centimeters?

Compute the slope of the line through the indicated points. On both axes, one square represents one unit.

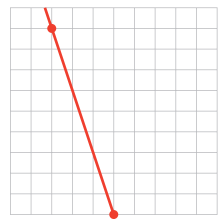
5. 

Answer. -1

6. 

7. 

Answer. $\dfrac{-2}{3}$

8. 

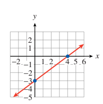
For Problems 9–14,

a Graph each line by the intercept method.

b Use the intercepts to compute the slope.

9. $3x - 4y = 12$

Answer.

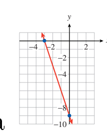
a 

b $\dfrac{3}{4}$

10. $2y - 5x = 10$

11. $2y + 6x = -18$

Answer.

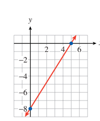
a 

b -3

12. $9x + 12y = 36$

13. $\dfrac{x}{5} - \dfrac{y}{8} = 1$

Answer.

a 

b $\dfrac{3}{4}$

14. $\dfrac{x}{7} - \dfrac{y}{4} = 1$

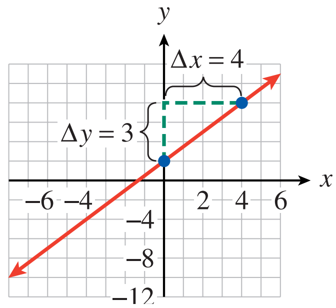
15.

a Use the points $(0, 2)$ and $(4, 8)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.

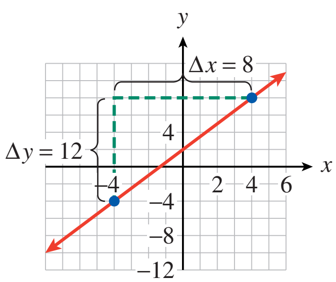
b Use the points $(-4, -4)$ and $(4, 8)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.

c Use the points $(0, 2)$ and $(-6, -7)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.

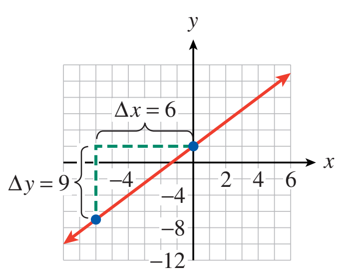
Answer.



a



b



c

16.

a Use the points $(0, -6)$ and $(8, -12)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.

b Use the points $(-8, 0)$ and $(4, -9)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.

c Use the points $(4, -9)$ and $(0, -6)$ to compute the slope of the line. Illustrate Δy and Δx on the graph.

For Problems 17–20, use the formula $m = \dfrac{\Delta y}{\Delta x}$.

17. A line has slope $\frac{-3}{4}$.

a Find the vertical change associated with each horizontal change along the line.

i $\Delta x = 4$	*iii* $\Delta x = 2$
ii $\Delta x = -8$	*iv* $\Delta x = -6$

b Find the horizontal change associated with each vertical change along the line.

i $\Delta y = 3$	*iii* $\Delta y = -2$
ii $\Delta y = -6$	*iv* $\Delta y = 1$

Answer.

a *i* -3	*ii* 6	*iii* $\dfrac{-3}{2}$	*iv* $\dfrac{9}{2}$
b *i* -4	*ii* 8	*iii* $\dfrac{8}{3}$	*iv* $\dfrac{4}{3}$

18. A line has slope $\frac{5}{3}$.

a Find the vertical change associated with each horizontal change along the line.

i $\Delta x = 3$	*iii* $\Delta x = 1$
ii $\Delta x = -6$	*iv* $\Delta x = -24$

b Find the horizontal change associated with each vertical change along the line.

i $\Delta y = -5$	*iii* $\Delta y = -1$
ii $\Delta y = -2.5$	*iv* $\Delta y = 3$

19. Residential staircases are usually built with a slope of 70%, or $\frac{7}{10}$. If the vertical distance between stories is 10 feet, how much horizontal space does the staircase require?

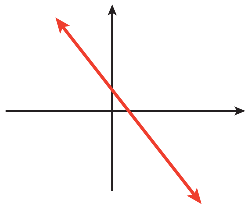
Answer. $\dfrac{100}{7}$ ft ≈ 14.286 ft ≈ 14 ft 3.4 in

20. A straight section of highway in the Midwest maintains a grade (slope) of 4%, or $\frac{1}{25}$, for 12 miles. How much does your elevation change as you travel the road?

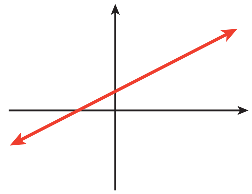
21. Choose the line with the correct slope. The scales are the same on both axes.

a $m = 2$	b $m = -\frac{1}{2}$	c $m = \frac{2}{3}$	d $m = -\frac{5}{3}$

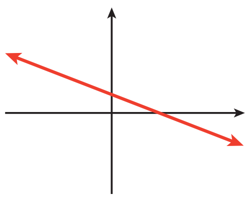
I



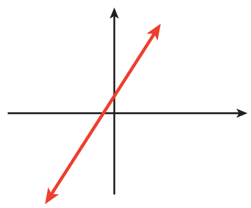
II



III



IV



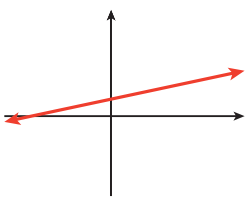
Answer.

a IV b III c II d I

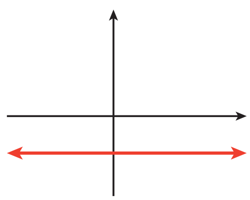
22. Choose the line with the correct slope. The scales are the same on both axes.

a $m = 2$ b $m = -\frac{1}{2}$ c $m = \frac{2}{3}$ d $m = -\frac{5}{3}$

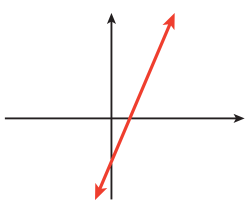
I



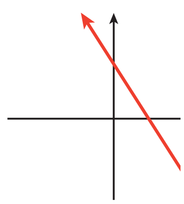
II



III

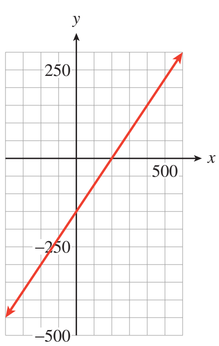


IV



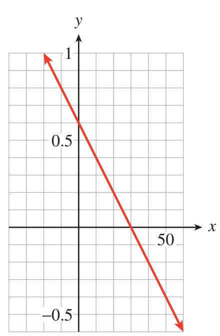
Compute the slope of each line. Note the scales on the axes.

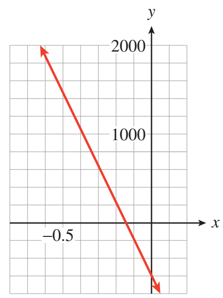
23.



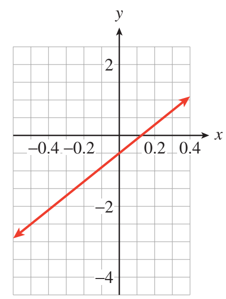
Answer. $\dfrac{3}{4}$

24.



25. 

Answer. -4000

26. 

Each table in Problems 27–30 gives the coordinates of points on a line.

a Find the slope of the line.

b Fill in the missing table entries.

27.

x	y
-4	-14
-2	-9
2	1
3	
	11

28.

x	y
-5	-3.8
-1	-0.6
2	1.8
	4.2
7	

29.

x	y
-3	36
-1	
	12
6	9
10	-3

30.

x	y
-10	800
-2	
5	440
	368
16	176

Answer. (27)

a $\frac{5}{2}$

b

x	y
3	$\mathbf{\frac{7}{2}}$
6	11

Answer. (29)

a -3

b

x	y
-1	**30**
5	12

31. A temporary typist's paycheck (before deductions) is given, in dollars, by $S = 8t$, where t is the number of hours she worked.

(a) Make a table of values for the function.

t	4	8	20	40
S				

(b) Graph the function.

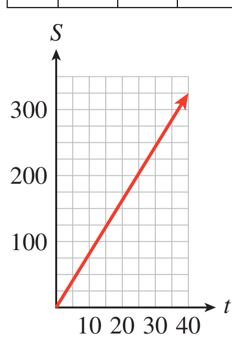
(c) Using two points on the graph, compute the slope $\frac{\Delta S}{\Delta t}$, including units.

(d) What does the slope tell us about the typist's paycheck?

Answer.

a

t	4	8	20	40
S	32	64	160	320

b 

c 8 dollars/hour

d The typist is paid \$8 per hour.

32. The distance (in miles) covered by a cross-country competitor is given by $d = 6t$, where t is the number of hours she runs.

(a) Make a table of values for the function.

t	2	4	6	8
d				

(b) Graph the function.

(c) Using two points on the graph, compute the slope $\frac{\Delta d}{\Delta t}$, including units.

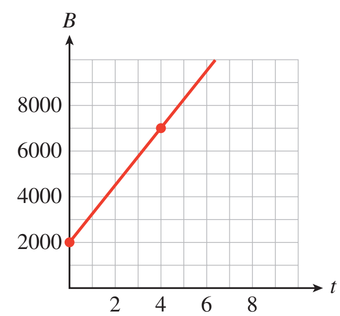
(d) What does the slope tell us about the cross-country runner?

In Problems 33–40,

a Choose two points and compute the slope of the graph (including units).

b Explain what the slope measures in the context of the problem.

33. The graph shows the number of barrels of oil, B, that has been pumped at a drill site t days after a new drill is installed.

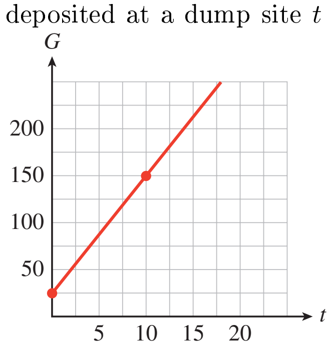


Answer.

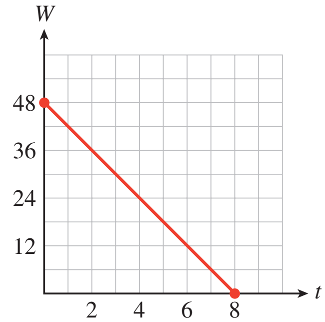
a 1250 barrels/day

b The slope indicates that oil is pumped at a rate of 1250 barrels per day.

34. The graph shows the amount of garbage, G (in tons), that has been deposited at a dump site t years after new regulations go into effect.



35. The graph shows the amount of emergency water, W (in liters), remaining in a southern California household t days after an earthquake.

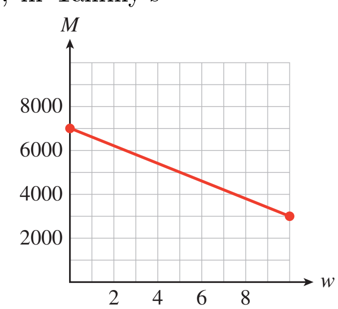


Answer.

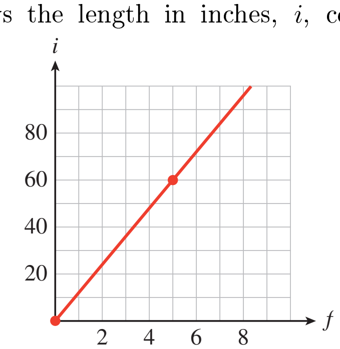
a -6 liters/day

b The slope indicates that the water is diminishing at a rate of 6 liters per day.

36. The graph shows the amount of money, M (in dollars), in Tammy's bank account w weeks after she loses all sources of income.



37. The graph shows the length in inches, i, corresponding to various lengths in feet f.

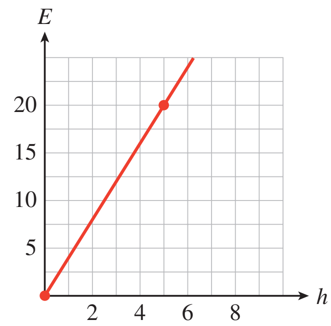


Answer.

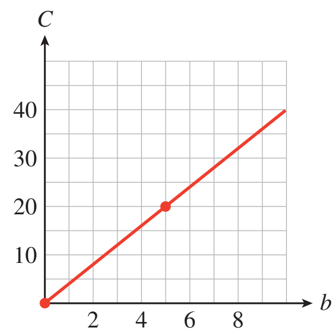
a 12 inches/foot

b The slope gives the conversion rate of 12 inches per foot.

38. The graph shows the number of ounces, z, that correspond to various weights measured in pounds, p.



39. The graph shows the cost, C (in dollars), of coffee beans in terms of the amount of coffee, b (in kilograms).

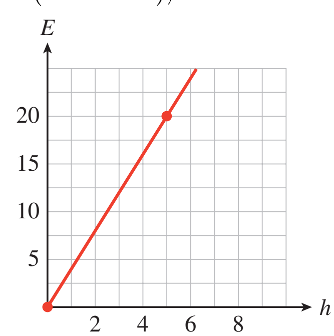


Answer.

a 4 dollars/kilogram

b The slope gives the unit price of \$4 per kilogram

40. The graph shows Tracey's earnings, E (in dollars), in terms of the number of hours, h, that she babysits.



Which of the tables in Problems 41 and 42 represent variables that are related by a linear function? (Hint: Which relationships have constant slope?)

41.

a

x	y
2	12
3	17
4	22
5	27
6	32

b

t	P
2	4
3	9
4	16
5	25
6	36

Answer. (a)

42.

a

h	w
-6	20
-3	18
0	16
3	14
6	12

b

t	d
5	0
10	3
15	6
20	12
25	24

43. The table shows the amount of ammonium chloride salt, in grams, that can be dissolved in 100 grams of water at different temperatures.

Temperature,°C
10
12
15
21
25
40
52
Grams of salt
33
34
35.5
38.5
40.5
48
54

a If you plot the data, will the points lie on a straight line? Why or why not?

b Calculate the rate of change of salt dissolved with respect to temperature.

Answer.

a Yes, the slope between any two points is $\frac{1}{2}$.

b 0.5 grams of salt per degree Celsius

44. A spring is suspended from the ceiling. The table shows the length of the spring, in centimeters, as it is stretched by hanging various weights from it.

Weight, kg
3
4
8
10
12
15
22
Length, cm
25.87
25.88
26.36
26.6
26.84
27.2
28.04

a If you plot the data, will the points lie on a straight line? Why or why not?

b Calculate the rate of change of length with respect to weight.

45. The table gives the radius and circumference of various circles, rounded to three decimal places.

r	C
4	25.133
6	37.699
10	62.832
15	94.248

a If we plot the data, will the points lie on a straight line?

b What familiar number does the slope turn out to be? (Hint: Recall a formula from geometry.)

Answer.

a Yes

b 2π

46. The table gives the side and the diagonal of various squares, rounded to three decimal places.

s	d
3	4.243
6	8.485
8	11.314
10	14.142

a If we plot the data, will the points lie on a straight line?

b What familiar number does the slope turn out to be? (Hint: Draw a picture of one of the squares and use the Pythagorean theorem to compute its diagonal.)

47. Geologists can measure the depth of the ocean at different points using a technique called echo-sounding. Scientists on board a ship send a pulse of sound toward the ocean floor and measure the time interval until the echo returns to the ship. The speed of sound in seawater is about 1500 meters per second.

a Write the speed of sound as a ratio.

b If the echo returns in 4.5 seconds, what is the depth of the ocean at that point?

Answer.

a $\dfrac{1500 \text{ meters}}{1 \text{ second}}$

b 3375 meters

48. Niagara Falls was discovered by Father Louis Hennepin in 1682. In 1952, much of the water of the Niagara River was diverted for hydroelectric power, but until that time erosion caused the Falls to recede upstream by 3 feet per year.

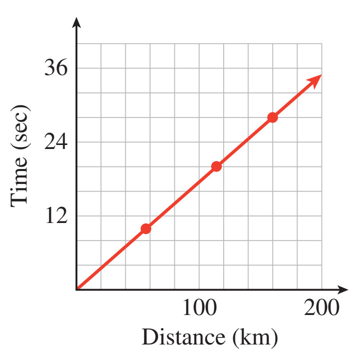
a How far did the Falls recede from 1682 to 1952?

b The Falls were formed about 12,000 years ago during the end of the last ice age. How far downstream from their current position were they then? (Give your answer in miles.)

49. Geologists calculate the speed of seismic waves by plotting the travel times for waves to reach seismometers at known distances from the epicenter. The speed of the wave can help them determine the nature of the material it passes through. The graph shows a travel-time graph for P-waves from a shallow earthquake.

a Why do you think the graph is plotted with distance as the input variable?

b Use the graph to calculate the speed of the wave.



Answer.

a The distances are known.

b 5.7 km per second

50. Energy (supplied by heat) is required to raise the temperature of a substance, and it is also needed to melt a solid substance to a liquid. The table shows data from heating a solid sample of stearic acid. Heat was applied at a constant rate throughout the experiment. (Source: J. A. Hunt and A. Sykes, 1984)

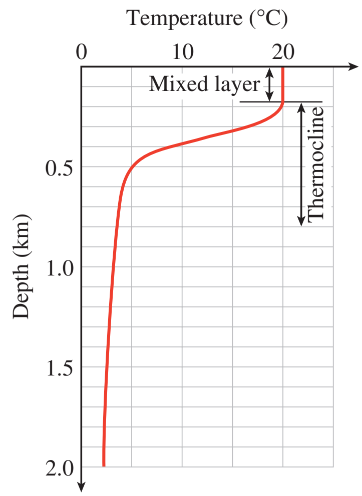
Time (minutes
0
0.5
1.5
2
2.5
3
4
5
6
7
8
8.5
9
9.5
10
Temperature, °C
19
29
40
48
53
55
55
55
55
55
55
64
70
73
74

a Did the temperature rise at a constant rate? Describe the temperature as a function of time.

b Graph temperature as a function of time.

c What is the melting point of stearic acid? How long did it take the sample to melt?

51. The graph shows the temperature of the ocean as a function of depth.



a What is the difference in temperature between the surface of the ocean and the deepest level shown?

b Over what depths does the temperature change most rapidly?

c What is the average rate of change of temperature with respect to depth in the region called the thermocline?

Answer.

a About 18°C

b 0.3 km to 0.4 km

c About −28°C per kilometer

52. The graph shows the average air temperature as a function of altitude. (Figure (b) is an enlargement of the indicated region of Figure (a).) (Source: Ahrens, 1998)

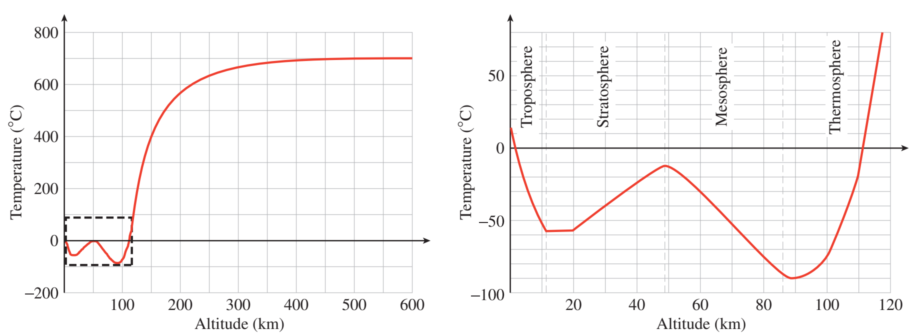
a Is temperature a decreasing function of altitude?

b The **lapse rate** is the rate at which the temperature changes with altitude. In which regions of the atmosphere is the lapse rate positive?

c The region where the lapse rate is zero is called the isothermal zone. Give an interval of altitudes that describes the isothermal zone.

d What is the lapse rate in the mesosphere?

e Describe the temperature for altitudes greater than 90 kilometers.



(a) (b)

In Problems 53–56, evaluate the function at $x = a$ and $x = b$, and then find the slope of the line segment joining the two corresponding points on the graph. Illustrate the line segment on a graph of the function.

53. $f(x) = x^2 - 2x - 8$

a $a = -2, b = 1$

b $a = -1, b = 5$

Answer.

a -3

b 2

54. $g(x) = \sqrt{x+4}$

a $a = -2, b = 0$

b $a = 0, b = 5$

55. $h(x) = \dfrac{4}{x+2}$

a $a = 0, b = 6$

b $a = -1, b = 2$

Answer.

a $\dfrac{-1}{4}$

b -1

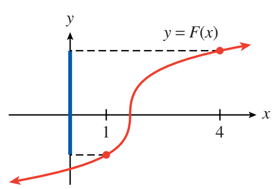
56. $q(x) = x^3 - 4x$

a $a = -1, b = 2$

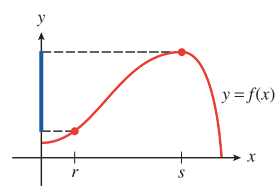
b $a = -1, b = 3$

In Problems 57–62, find the coordinates of the indicated points, and then write an algebraic expression using function notation for the indicated quantity.

57. The length of the vertical line segment on the y-axis



a



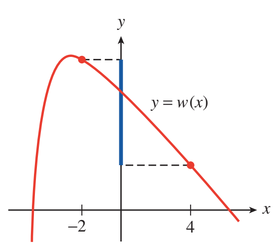
b

Answer.

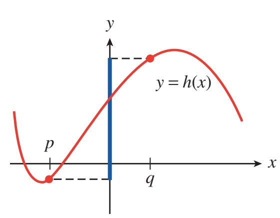
a $(1, F(1)), (4, F(4))$; $F(4) - F(1)$

b $(r, f(r)), (s, f(s))$; $f(s) - f(r)$

58. The length of the vertical line segment on the y-axis



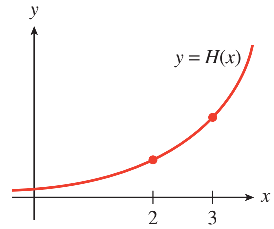
a



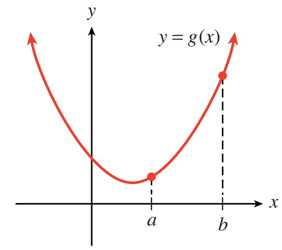
b

59.

a The increase in y as x increases from 2 to 3



b The increase in y as x increases from a to b



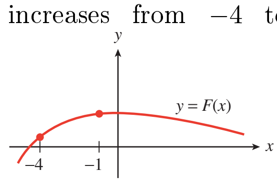
Answer.

a $(2, H(2)), (3, H(3))$; $H(3) - H(2)$

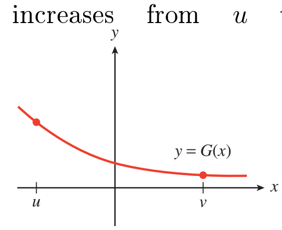
b $(a, g(a)), (b, g(b))$; $g(b) - g(a)$

60.

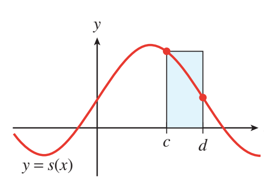
a The increase in y as x increases from -4 to -1

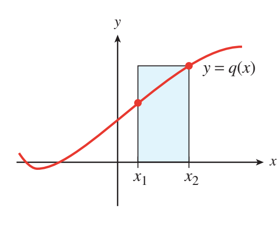


b The increase in y as x increases from u to v



61. The shaded area

a 

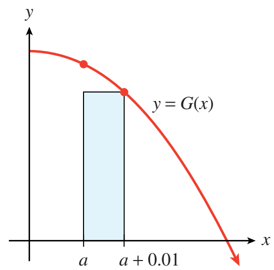
b 

Answer.

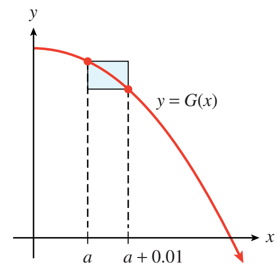
a $(c, s(c)), (d, s(d))$; $s(c)(d - c)$

b $(x_1, q(x_1)), (x_2, q(x_2))$; $q(x_2)(x_2 - x_1)$

62. The shaded area



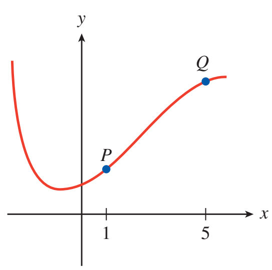
a



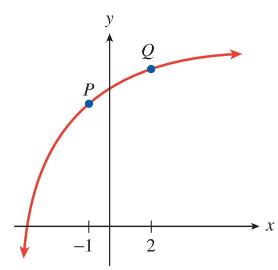
b

In Problems 63–66, find the coordinates of the indicated points on the graph of $y = f(x)$ and write an algebraic expression using function notation for the slope of the line segment joining points P and Q.

63.



a

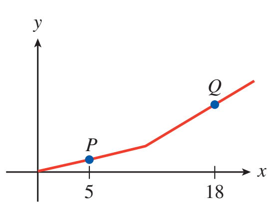


b

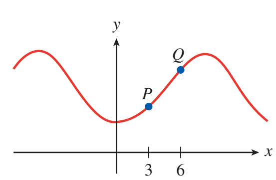
Answer.

a $(1, f(1)), (5, f(5))$; $\dfrac{f(5) - f(1)}{4}$ b $(-1, f(-1)), (2, f(2))$; $\dfrac{f(2) - f(-1)}{3}$

64.

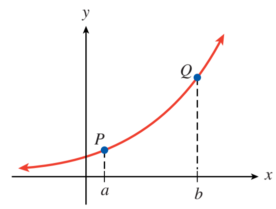


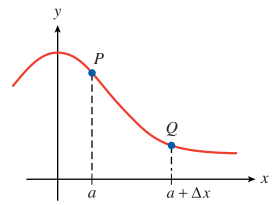
a



b

65.

a 

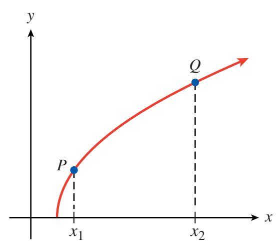
b 

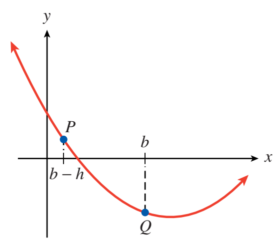
Answer.

a $(a, f(a)), (b, f(b))$; $\dfrac{f(b) - f(a)}{b - a}$

b $(a, f(a)), (a+\Delta x, f(a+\Delta x))$; $\dfrac{f(a + \Delta x) - f(a)}{\Delta x}$

66.

a 

b 

1.5 Linear Functions

1.5.1 Slope-Intercept Form

As we saw in Section 1.1, many linear models $y = f(x)$ have equations of the form

$$f(x) = (\text{starting value}) + (\text{rate of change}) \cdot x$$

The starting value, or the value of y at $x = 0$, is the y-intercept of the graph, and the rate of change is the slope of the graph. Thus, we can write the equation of a line as

$$f(x) = b + mx$$

where the constant term, b, is the y-intercept of the line, and m, the coefficient of x, is the slope of the line. This form for the equation of a line is called the **slope-intercept form**.

Slope-Intercept Form

If we write the equation of a linear function in the form,

$$f(x) = b + mx$$

then m is the **slope** of the line, and b is the **y-intercept**.

(You may have encountered the slope-intercept equation in the equivalent form $y = mx + b$.) For example, consider the two linear functions and their graphs shown in Figure 1.121 and Figure 1.123.

x	$f(x)$
0	10
1	7
2	4
3	1
4	-2

Table 1.120: $f(x) = 10 - 3x$

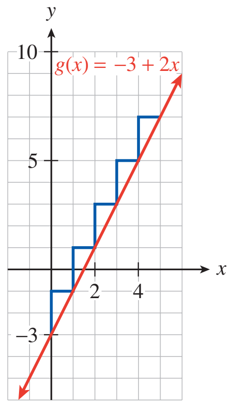


Figure 1.121

x	$f(x)$
0	-3
1	-1
2	1
3	3
4	5

Table 1.122: $g(x) = -3 + 2x$

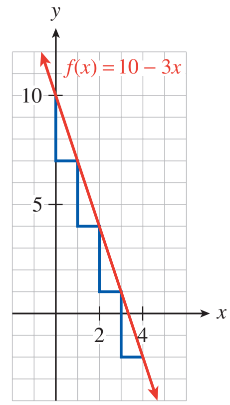


Figure 1.123

We can see that the y-intercept of each line is given by the constant term, b. By examining the table of values, we can also see why the coefficient of $x$gives the slope of the line: For $f(x)$, each time x increases by 1 unit, y decreases by 3 units. For $g(x)$, each time x increases by 1 unit, y increases by 2 units. For each graph, the coefficient of x is a scale factor that tells us how many units y changes for 1 unit increase in x. But that is exactly what the slope tells us about a line.

Example 1.124. Francine is choosing an Internet service provider. She paid \$30 for a modem, and she is considering three companies for dialup service: Juno charges \$14.95 per month, ISP.com charges \$12.95 per month, and peoplepc charges \$15.95 per month. Match the graphs in Figure 1.125 to Francine's Internet cost with each company.

Solution. Francine pays the same initial amount, \$30 for the modem, under each plan. The monthly fee is the rate of change of her total cost, in dollars per month.

We can write a formula for her cost under each plan.

Juno: $f(x) = 30 + 14.95x$

ISP.com: $g(x) = 30 + 12.95x$

peoplepc: $h(x) = 30 + 15.95x$

The graphs of these three functions all have the same y-intercept, but their slopes are determined by the monthly fees. The steepest graph, III, is the one with the largest monthly fee, peoplepc, and ISP.com, which has the lowest monthly fee, has the least steep graph, I.

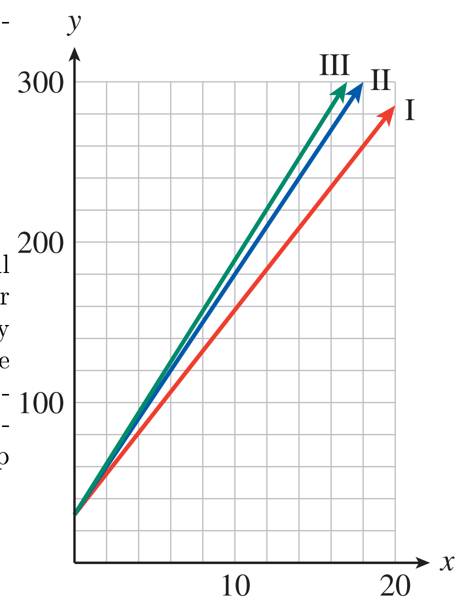


Figure 1.125

Exercise 1.126. Delbert decides to use DSL for his Internet service. Earthlink charges a \$99 activation fee and \$39.95 per month, DigitalRain charges \$50 for activation and \$34.95 per month, and FreeAmerica charges \$149 for activation and \$34.95 per month.

a Write a formula for Delbert's Internet costs under each plan.

b Match Delbert's Internet cost under each company with its graph in Figure 1.127.

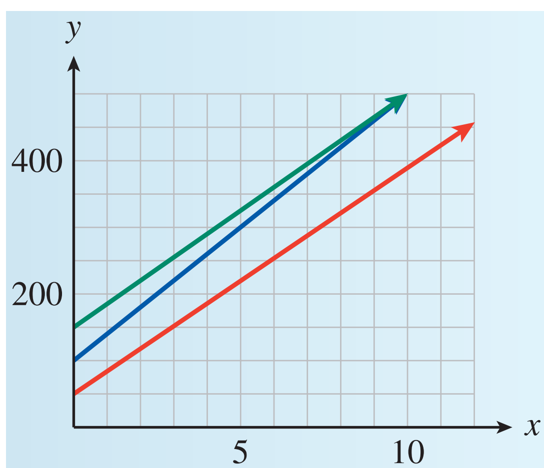


Figure 1.127

In the equation $f(x) = b + mx$, we call m and b **parameters**. Their values are fixed for any particular linear equation; for example, in the equation $y = 2x + 3$, $m = 2$ and $b = 3$, and the variables are x and y. By changing the values of m and b, we can write the equation for any line except a vertical line (see Figure 1.128). The collection of all linear functions $f(x) = b + mx$ is called a **two-parameter** family of functions.

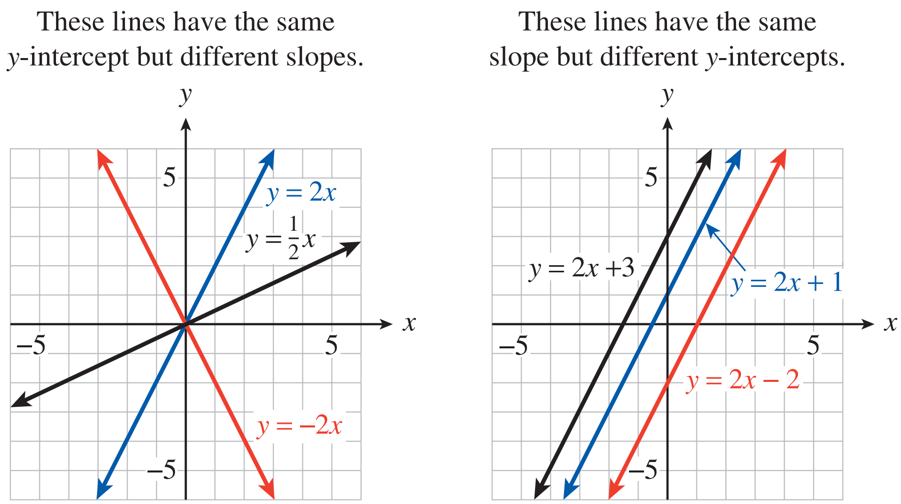


Figure 1.128

1.5.2 Slope-Intercept Method of Graphing

Look again at the lines in Figure 1.128: There is only one line that has a given slope and passes through a particular point. That is, the values of m and b determine the particular line. The value of b gives us a starting point, and the value of m tells us which direction to go to plot a second point. Thus, we can graph a line given in slope-intercept form without having to make a table of values.

Example 1.129.

a Write the equation $4x - 3y = 6$ in slope-intercept form.

b Graph the line by hand.

Solution.

a We solve the equation for y in terms of x.

$$-3y\& = 6 - 4x\backslash\backslash y\& = \frac{6-4x}{-3} = \frac{6}{-3} + \frac{-4x}{-3}\backslash\backslash y\& = -2 + \frac{4}{3}x$$

b We see that the slope of the line is $m = \frac{4}{3}$ and its y-intercept is $b = -2$. We begin by plotting the y-intercept, $(0, -2)$. We then use the slope to find another point on the line. We have

$$m = \frac{\Delta y}{\Delta x} = \frac{4}{3}$$

so starting at $(0, -2)$, we move 4 units in the y-direction and 3 units in the x-direction, to arrive at the point $(3, 2)$. Finally, we draw the line through these two points. (See 1.130.)

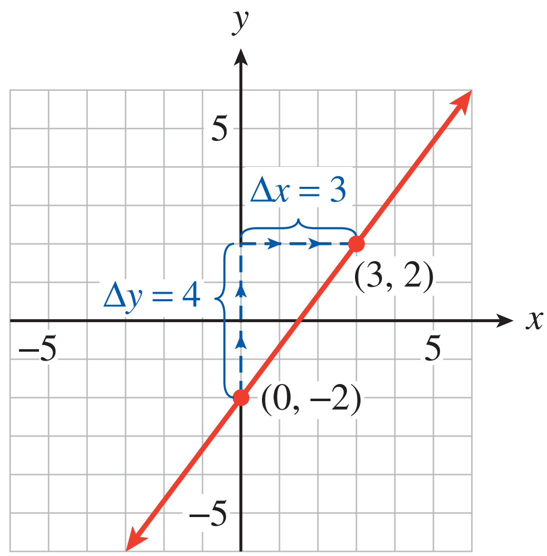


Figure 1.130

The slope of a line is a ratio and can be written in many equivalent ways. In Example 1.129, the slope is equal to $\frac{8}{6}$, $\frac{12}{9}$, and $\frac{-4}{-3}$. We can use any of these fractions to locate a third point on the line as a check. If we use $m = \frac{\Delta y}{\Delta x} = \frac{-4}{-3}$, we move down 4 units and left 3 units from the y-intercept to find the point $(-3, -6)$ on the line.

Slope-Intercept Method for Graphing a Line

a Plot the y-intercept $(0, b)$.

b Use the definition of slope to find a second point on the line: Starting at the y-intercept, move Δy units in the y-direction and Δx units in the x-direction. Plot a second point at this location.

c Use an equivalent form of the slope to find a third point, and draw a line through the points.

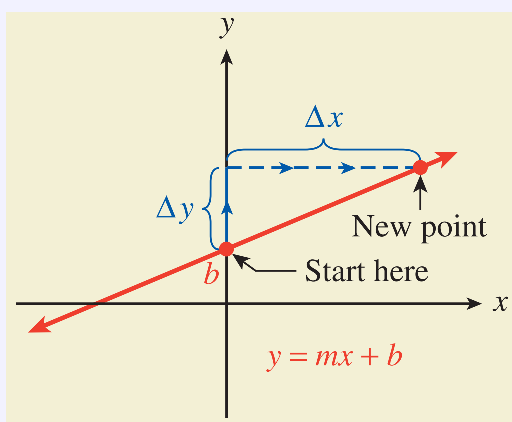


Figure 1.131

Exercise 1.132.

a Write the equation $2y + 3x + 4 = 0$ in slope-intercept form.

b Use the slope-intercept method to graph the line.

1.5.3 Finding a Linear Equation from a Graph

We can also use the slope-intercept form to find the equation of a line from its graph. First, note the value of the y-intercept from the graph, and then calculate the slope using two convenient points.

Example 1.133. Find an equation for the line shown in Figure 1.134.

Solution.

The line crosses the y-axis at the point $(0, 3200)$, so the y-intercept is 3200. To calculate the slope of the line, locate another point, say $(20, 6000)$, and compute:

$$m\& = \frac{\Delta y}{\Delta x} = \frac{6000 - 3200}{20 - 0}\backslash\backslash\& = \frac{2800}{20}$$

The slope-intercept form of the equation, with $m = 140$ and $b = 3200$, is $y = 3200 + 140x$.

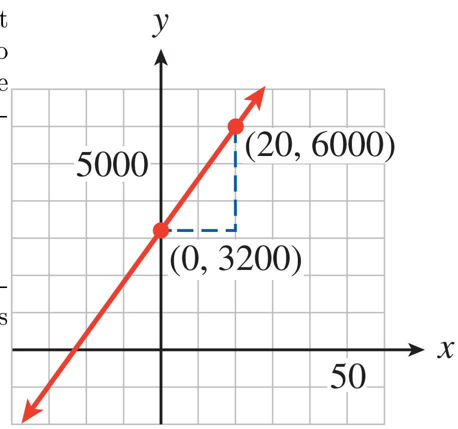


Figure 1.134

Exercise 1.135.

Find an equation for the line shown in Figure 1.136

$$b\& = \qquad \backslash\backslash m\& = \backslash\backslash y\& = \backslash\backslash$$

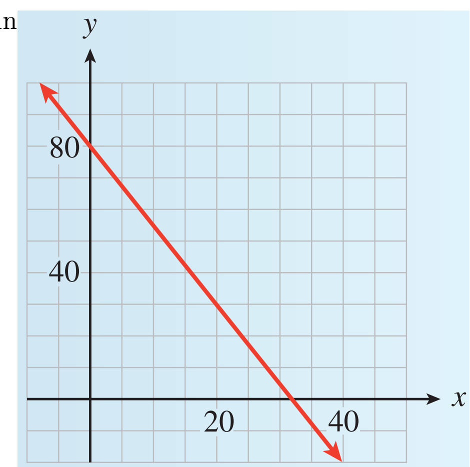


Figure 1.136

$$b = 80,\ m = \frac{-5}{2},\ y = 80 - \frac{5}{2}x$$

1.5.4 Point-Slope Form

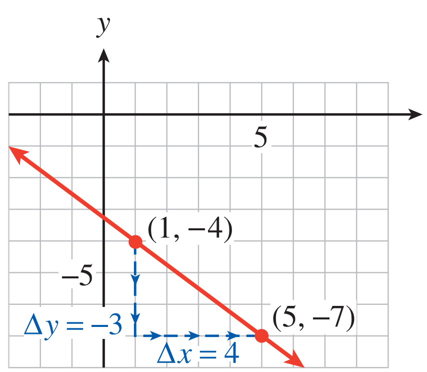


Figure 1.137

We can find the equation for a line if we know its slope and y-intercept. What if we do not know the y-intercept, but instead know some other point on the line? There is only one line that passes through a given point and has a given slope. For example, we can graph the line of slope $\frac{-3}{4}$ that passes through the point $(1, -4)$.

We first plot the given point, $(1, -4)$, as shown in Figure 1.137. Then we use the slope to find another point on the line. The slope is

$$m = \frac{-3}{4} = \frac{\Delta y}{\Delta x}$$

so we move down 3 units and then 4 units to the right, starting from $(1, -4)$. This brings us to the point $(5, -7)$. We can then draw the line through these two points.

We can also find an equation for the line, as shown in Example 4.

Example 1.138. Find an equation for the line that passes through $(1, -4)$ and has slope $\frac{-3}{4}$.

Solution. We will use the formula for slope,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

We substitute $\frac{-3}{4}$ for the slope, m, and $(1, -4)$ for (x_1, y_1). For the second point, (x_2, y_2), we will use the variable point (x, y). Substituting these values into the slope formula gives us

$$\frac{-3}{4} = \frac{y - (-4)}{x - 1} = \frac{y + 4}{x - 1}$$

To solve for y we first multiply both sides by $x - 1$.

$$(x - 1)\frac{-3}{4} \& = \frac{y + 4}{x - 1}(x - 1)\&\&\backslash\backslash\frac{-3}{4}(x - 1)\& = y + 4\&\&\text{Apply the distributive law.}\backslash\backslash\frac{-3}{4}x + \frac{3}{4}\& =$$

When we use the slope formula in this way to find the equation of a line, we substitute a variable point (x, y) for the second point. This version of the formula,

$$m = \frac{y - y_1}{x - x_1}$$

is called the **point-slope form** for a linear equation. It is sometimes stated in another form obtained by clearing the fraction to get

$$(x - x_1)m\& = \frac{y - y_1}{x - x_1}(x - x_1)\&\&\text{Multiply both sides by } (x - x_1)\backslash\backslash(x - x_1)m\& = y - y_1\&\&\text{Clear fr}$$

Point-Slope Form

The equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y = y_1 + m(x - x_1)$$

Exercise 1.139. Use the point-slope form to find the equation of the line that passes through the point $(-3, 5)$ and has slope -1.4.

$y = y_1 + m(x - x_1)$	Substitute -1.4 for m and $(-3, 5)$ for (x_1, y_1).
	Simplify: Apply the distributive law.

The point-slope form is useful for modeling linear functions when we do not know the initial value but do know some other point on the line.

Example 1.140. Under a proposed graduated income tax system, single taxpayers would owe \$1500 plus 20% of the amount of their income over \$13,000. (For example, if your income is \$18,000, you would pay \$1500 plus 20% of \$5000.)

a Complete the table of values for the tax, T, on various incomes, I.

I	$15,000$	$20,000$	$22,000$
T			

b Write a linear equation in point-slope form for the tax, T, on an income I.

c Write the equation in slope-intercept form.

Solution.

a On an income of \$15,000, the amount of income over \$13,000 is \$15,000 - \$13,000 = \$2000, so you would pay \$1500 plus 20% of \$2000, or

$$T = 1500 + 0.20(2000) = 1900$$

You can compute the other function values in the same way.

I	$15,000$	$20,000$	$22,000$
T	1900	2900	3300

b On an income of I, the amount of income over \$13,000 is I−13,000, so you would pay \$1500 plus 20% of $I - 13,000$, or

$$T = 1500 + 0.20(I - 13,000)$$

c Simplify the right side of the equation to get

$$T\& = 1500 + 0.20I - 2600\backslash\backslash T\& = -1100 + 0.20I$$

Exercise 1.141. A healthy weight for a young woman of average height, 64 inches, is 120 pounds. To calculate a healthy weight for a woman taller than 64 inches, add 5 pounds for each inch of height over 64.

a Write a linear equation in point-slope form for the healthy weight, W, for a woman of height, H, in inches.

b Write the equation in slope-intercept form.

1.5.5 Section Summary

1.5.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Slope-intercept form
- Point-slope form
- Parameter

1.5.5.2 CONCEPTS

1 Linear functions form a two-parameter family, $f(x) = b + mx$.

2 The initial value of the function and the y-intercept of its graph are given by b. The rate of change of the function and the slope of its graph are given by m.

3 The slope-intercept form, $y = b + mx$, is useful when we know the initial value and the rate of change.

4 The point-slope form, $y = y_1 + m(x - x_1)$, is useful when we know the rate of change and one point on the line.

1.5.5.3 STUDY QUESTIONS

1 How can you put a linear equation into slope-intercept form?

2 What do the coefficients in the slope-intercept form tell you about the line?

3 Explain how to graph a line using the slope-intercept method.

4 Explain how to find an equation for a line from its graph.

5 Explain how to use the point-slope form for a linear equation.

6 Francine says that the slope of the line $y = 4x - 6$ is $4x$. Is she correct? Explain your answer

7 Delbert says that the slope of the line $3x - 4y = 8$ is 3. Is he correct? Explain your answer.

1.5.5.4 SKILLS

Practice each skill in the Homework 1.5.6 problems listed.

1 Write a linear equation in slope-intercept form: #1–14

2 Identify the slope and y-intercept: #1–10

3 Graph a line by the slope-intercept method: #11–14

4 Find a linear equation from its graph: #21–26, 29–32, 53–56

5 Interpret the slope and y-intercept: #21–28, 63 and 64

6 Find a linear equation from one point and the slope: #33–50

1.5.6 Homework

In Problems 1–10,

a Write each equation in slope-intercept form.

b State the slope and y-intercept of the line.

1. $3x + 2y = 1$

Answer.

a $y = \frac{1}{2} - \frac{3}{2}x$

b Slope $\frac{-3}{2}$, y-intercept $\frac{1}{2}$

2. $5x - 4y = 0$

3. $\frac{1}{4}x + \frac{3}{2}y = \frac{1}{6}$

Answer.

a $y = \frac{1}{9} - \frac{1}{6}x$

b Slope $\frac{-1}{6}$, y-intercept $\frac{1}{9}$

4. $\frac{7}{6}x - \frac{2}{9}y = 3$

5. $4.2x - 0.3y = 6.6$

Answer.

a $y = -22+14x$

b Slope 14, y-intercept -22

6. $0.8x + 0.004y = 0.24$

7. $y + 29 = 0$

Answer.

a $y = -29$

b Slope 0, y-intercept -29

8. $y - 37 = 0$

9. $250x + 150y = 2450$

Answer.

a $y = \frac{49}{3} - \frac{5}{3}x$

b Slope $\frac{-5}{3}$, y-intercept $\frac{49}{3}$

10. $280x - 360y = 6120$

In Problems 11–14,

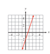
a Sketch by hand the graph of the line with the given slope and y-intercept.

b Write an equation for the line.

c Find the x-intercept of the line.

11. $m = 3$ and $b = -2$

Answer.

a 

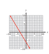
b $y = -2 + 3x$

c $\frac{2}{3}$

12. $m = -4$ and $b = 1$

13. $m = -\frac{5}{3}$ and $b = -6$

Answer.

a 

b $y = -6 + \frac{5}{3}x$

c $\frac{-18}{5}$

14. $m = \frac{3}{4}$ and $b = -2$

15. The point $(2, -1)$ lies on the graph of $f(x) = -3x + b$. Find b.

Answer. 5

16.

17. The point $(8, -5)$ lies on the graph of $f(x) = mx - 3$. Find m.

Answer. $\frac{-1}{4}$

18. The point $(-5, -6)$ lies on the graph of $f(x) = mx + 2$. Find m.

19. Find the slope and intercepts of the line $Ax + By = C$.

Answer. $m = \frac{-A}{B}$, x-intercept $\left(\frac{C}{A}, 0\right)$, y-intercept $\left(0, \frac{C}{B}\right)$

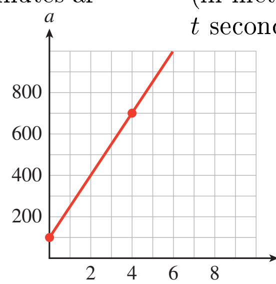
20. Find the slope and intercepts of the line $\frac{x}{a} + \frac{y}{b} = 1$

In Problems 21–26,

a Find a formula for the function whose graph is shown.

b Say what the slope and the vertical intercept tell us about the problem.

21. The graph shows the altitude, a (in feet), of a skier t minutes after getting on a ski lift.

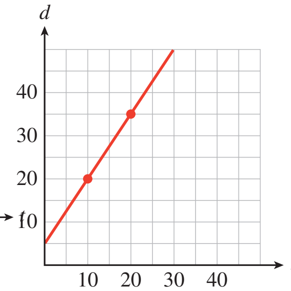


Answer.

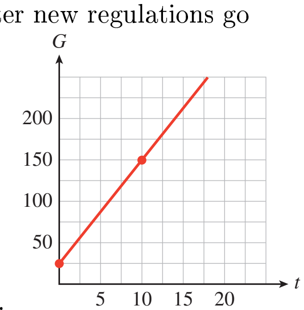
a $a = 100 + 150t$

b The slope tells us that the skier's altitude is increasing at a rate of 150 feet per minute, the vertical intercept that the skier began at an altitude of 200 feet.

22. The graph shows the distance, d (in meters), traveled by a train t seconds after it passes an observer.



23. The graph shows the amount of garbage, G (in tons), that has been deposited at a dump site t years after new regulations go into effect.

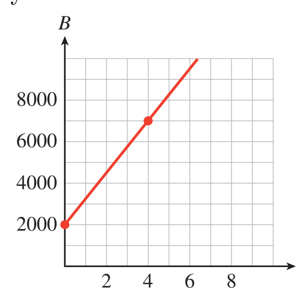


Answer.

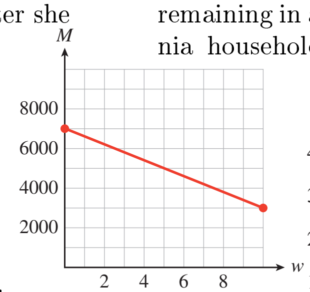
a $G = 25 + 12.5t$

b The slope tells us that the garbage is increasing at a rate of 12.5 tons per year, the vertical intercept that the dump already had 25 tons (when the new regulations went into effect).

24. The graph shows the number of barrels of oil, B, that has been pumped at a drill site t days after a new drill is installed.



25. The graph shows the amount of money, M (in dollars), in Tammy's bank account w weeks after she loses all sources of income.

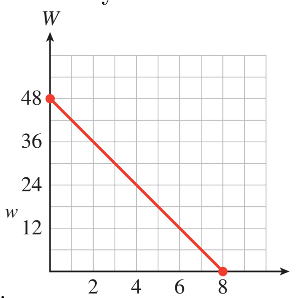


Answer.

a $M = 7000 - 400w$

b The slope tells us that Tammy's bank account is diminishing at a rate of \$400 per week, the vertical intercept that she had \$7000 (when she lost all sources of income).

26. The graph shows the amount of emergency water, W (in liters), remaining in a southern California household t days after an earthquake.



27. The formula $F = \frac{9}{5}C + 32$ converts the temperature in degrees Celsius to degrees Fahrenheit.

a What is the Fahrenheit temperature when it is $10°$ Celsius?

b What is the Celsius temperature when it is $-4°$ Fahrenheit?

c Choose appropriate `WINDOW` settings and graph the equation $y = \frac{9}{5}x + 32$.

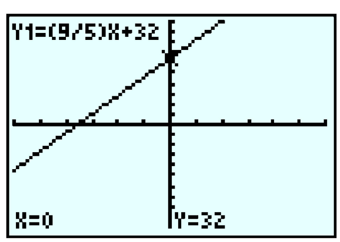
d Find the slope and explain its meaning for this problem.

e Find the intercepts and explain their meanings for this problem.

Answer.

a $50°$F

b $-20°$C

c 

d The slope, $\frac{9}{5} = 1.8$, tells us that Fahrenheit temperatures increase by $1.8°$ for each increase of $1°$ Celsius.

e C-intercept $\left(-17\frac{7}{9}, 0\right)$: $-17\frac{7}{9}^{\circ}$ C is the same as $0°$F; F-intercept $(0, 32)$: $0°$C is the same as $32°$F.

28. If the temperature on the ground is $70°$ Fahrenheit, the formula $T = 70 - \frac{3}{820}h$ gives the temperature at an altitude of h feet.

a What is the temperature at an altitude of 4100 feet?

b At what altitude is the temperature $34°$ Fahrenheit?

c Choose appropriate `WINDOW` settings and graph the equation $y = 70 - \frac{3}{820}x$.

d Find the slope and explain its meaning for this problem.

e Find the intercepts and explain their meanings for this problem.

29. In England, oven cooking temperatures are often given as Gas Marks rather than degrees Fahrenheit. The table shows the equivalent oven temperatures for various Gas Marks.

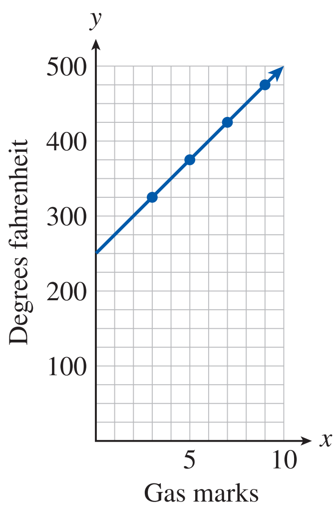
Gas Mark	3	5	7	9
Degrees (F)	325	375	425	475

a Plot the data and draw a line through the data points.

b Calculate the slope of your line. Estimate the y-intercept from the graph.

c Find an equation that gives the temperature in degrees Fahrenheit in terms of the Gas Mark.

Answer.

a 

b $m = 25, \; b = 250$

c $y = 250 + 25x$

30. European shoe sizes are scaled differently than American shoe sizes. The table shows the European equivalents for various American shoe sizes.

American shoe size	5.5	6.5	7.5	8.5
European shoe size	37	38	39	40

a Plot the data and draw a line through the data points.

b Calculate the slope of your line. Estimate the y-intercept from the graph.

c Find an equation that gives the European shoe size in terms of American shoe size.

31. A spring is suspended from the ceiling. The table shows the length of the spring in centimeters as it is stretched by hanging various weights from it.

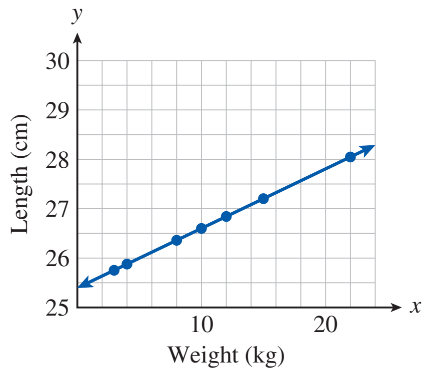
Weight, kg	3	4	8	10	12	15	27
Length, cm	25.76	25.88	26.36	26.6	26.84	27.2	28.04

a Plot the data on graph paper and draw a straight line through the points. Estimate the y-intercept of your graph.

b Find an equation for the line.

c If the spring is stretched to 27.56 cm, how heavy is the attached weight?

Answer.

a 

b $y = 0.12x + 25.4$

c 18 kg

32. The table shows the amount of ammonium chloride salt, in grams, that can be dissolved in 100 grams of water at different temperatures.

Temperature, °C	10	12	15	21	25	40	52
Grams of salt	33	34	35.5	38.5	40.5	48	54

a Plot the data on graph paper and draw a straight line through the points. Estimate the y-intercept of your graph.

b Find an equation for the line.

c At what temperature will 46 grams of salt dissolve?

In Problems 33–36,

a Sketch by hand the graph of the line that passes through the given point and has the given slope.

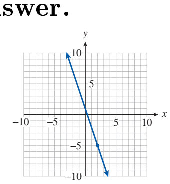
b Write an equation for the line in point-slope form.

c Put your equation from part (b) into slope-intercept form.

33. $(2, -5)$; $m = -3$

34. $(-6, -1)$; $m = 4$

Answer.

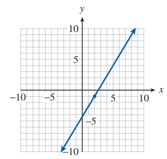
a 

b $y + 5 = -3(x - 2)$

c $y = 1 - 3x$

35. $(2, -1)$; $m = \frac{5}{3}$

Answer.

a 

b $y + 1 = \frac{5}{3}(x - 2)$

c $y = \frac{-13}{3} + \frac{5}{3}x$

36. $(-1, 2)$; $m = -\frac{3}{2}$

For Problems 37–40,

a Write an equation in point-slope form for the line that passes through the given point and has the given slope.

b Put your equation from part (a) into slope-intercept form.

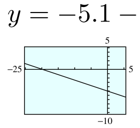
c Use your graphing calculator to graph the line.

37. $(-6.4, -3.5)$, $m = -0.25$

Answer.

a $y + 3.5 = -0.25(x + 6.4)$

b $y = -5.1 - 0.25x$

c 

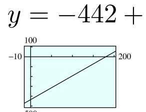
38. $(7.2, -5.6)$, $m = 1.6$

39. $(80, -250)$, $m = 2.4$

Answer.

a $y + 250 = 2.4(x - 80)$

b $y = -442 + 2.4x$

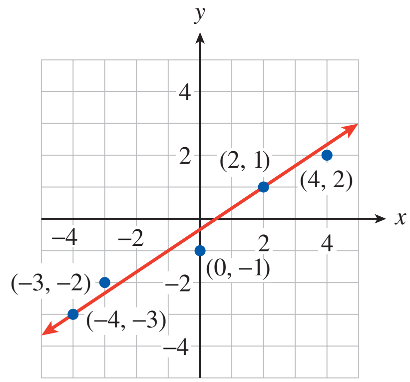
c 

40. $(-150, 1800)$, $m = -24$

For Problems 41 and 42,

a Find the slope of the line. (Note that not all the labeled points lie on the line.)

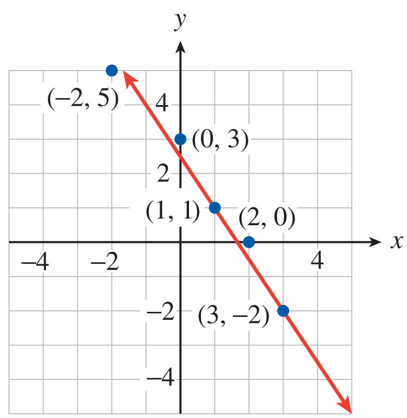
b Find an equation for the line.

41. 

Answer.

a $m = \dfrac{2}{3}$

b $y = \dfrac{-1}{3} + \dfrac{2}{3}x$

42. 

The equation of line l_1 is $y = q + px$, and the equation of line l_2 is $y = v + tx$.

a Decide whether the coordinates of each labeled point are

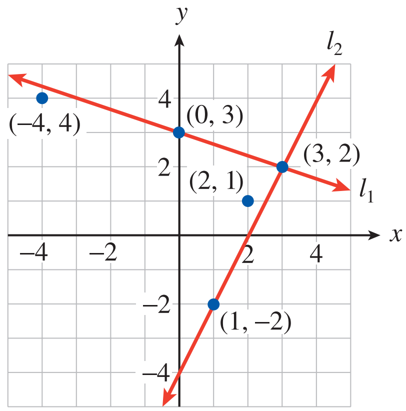
i a solution of $y = q + px$,

ii a solution of $y = v + tx$,

iii a solution of both equations, or

iv a solution of neither equation.

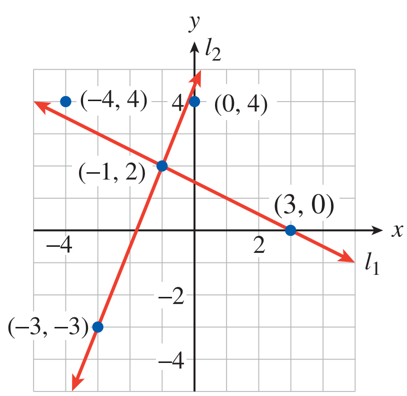
b Find p, q, t, and v.

43. 

Answer.

a $(-4,4)$: neither; $(0,3)$: $y = px+q$; $(3,2)$: both; $(2,1)$: neither; $(1,-2)$: $y = tx + v$

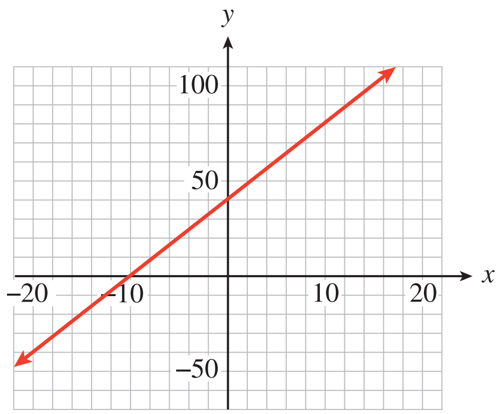
b $p = \dfrac{-1}{3}$, $q = 3$, $t = 2$, $v = -4$

44. 

For Problems 45–50,

a Estimate the slope and vertical intercept of each line. (Hint: To calculate the slope, find two points on the graph that lie on the intersection of grid lines.)

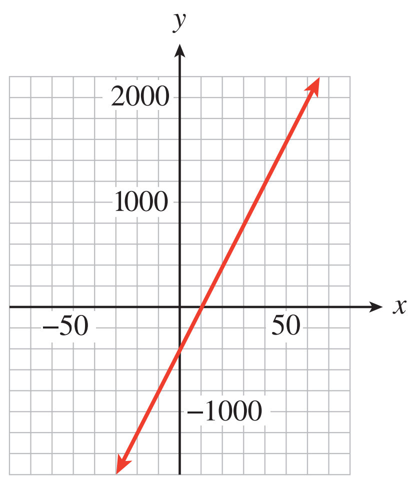
b Using your estimates from (a), write an equation for the line.

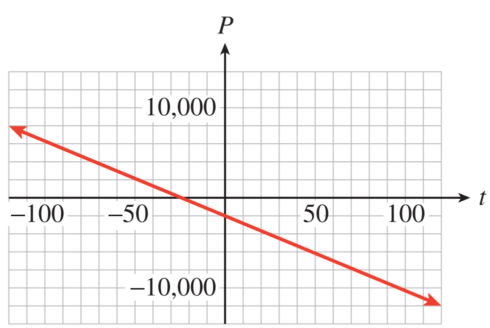
45. 

Answer.

a $m = 4, \ b = 40$

b $y = 40 + 4x$

46. 

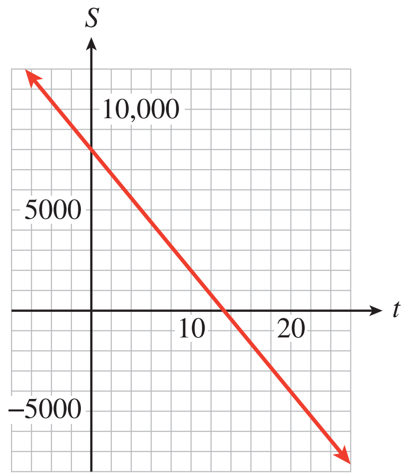
47. 

Answer.

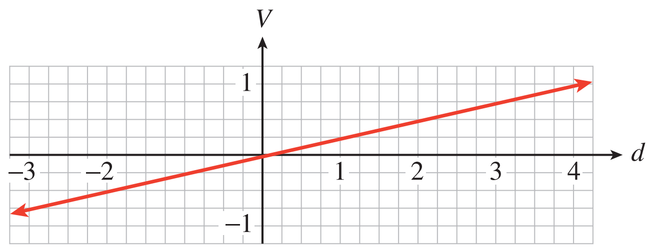
a $m = -80, \ b = -2000$

b $P = -2000 - 80t$

48.



49.

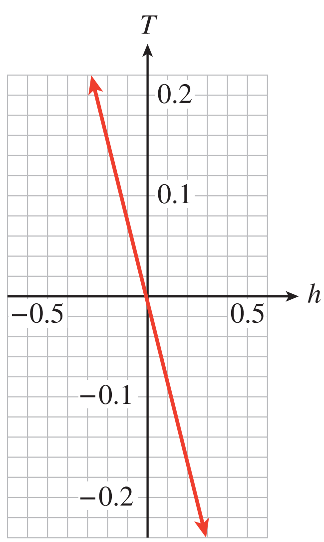


Answer.

a $m = \dfrac{1}{4},\ b = 0$

b $V = \dfrac{1}{4}d$

50.



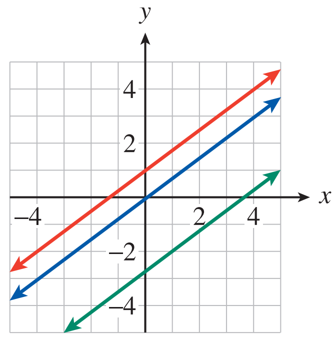
51.

a Write equations for three lines with slope $m = \frac{3}{4}$. (Many answers are possible.)

b Graph all three lines in the same window. What do you notice about the lines?

Answer.

a $y = \frac{3}{4}x$, $y = 1 + \frac{3}{4}x$, $y = -2.7 + \frac{3}{4}x$

b 

The lines are parallel.

52.

a Write equations for three lines with slope $m = 0$. (Many answers are possible.)

b Graph all three lines in the same window. What do you notice about the lines?

In Problems 53–56, choose the correct graph for each equation. The scales on both axes are the same.

53.

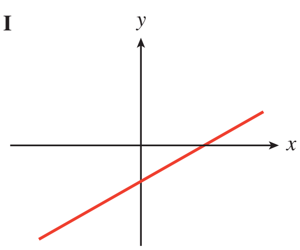
a $y = \dfrac{3}{4}x + 2$

b $y = \dfrac{-3}{4}x + 2$

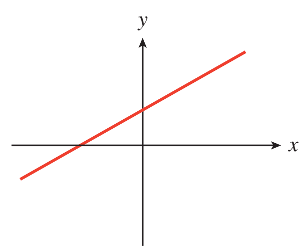
c $y = \dfrac{3}{4}x - 2$

d $y = \dfrac{-3}{4}x - 2$

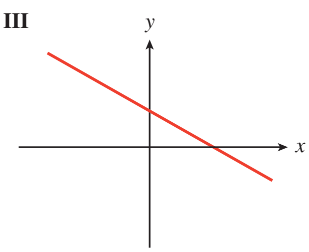
I



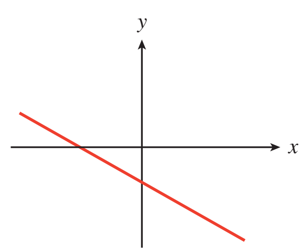
II



III



IV



Answer.

a II

b III

c I

d IV

54.

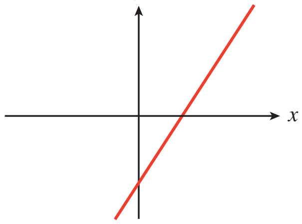
a $m < 0$, $b > 0$

b $m > 1$, $b < 0$

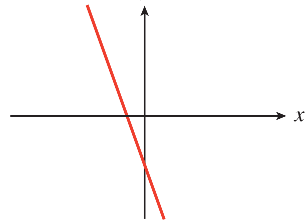
c $0 < m < 1$, $b < 0$

d $m < -1$, $b < 0$

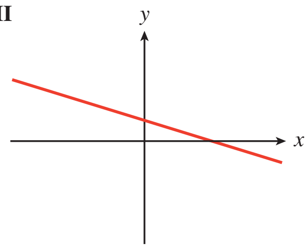
I



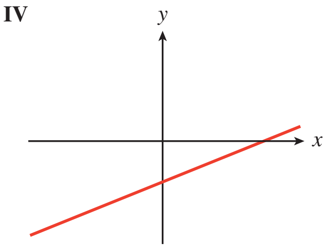
II



III



IV



55.

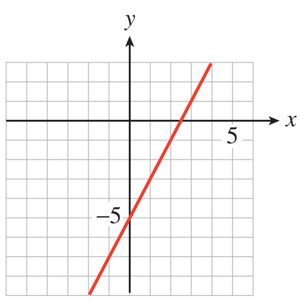
a $y = 1 + 2(x + 3)$

b $y = -1 + 2(x - 3)$

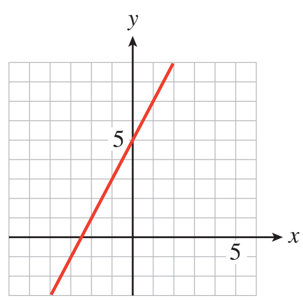
c $y = -1 + 2(x + 3)$

d $y = 1 + 2(x - 3)$

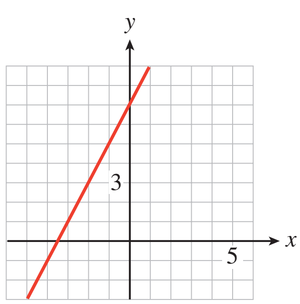
I



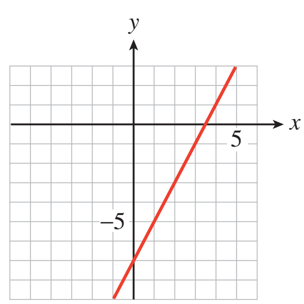
II



III



IV



Answer.

a III

b IV

c II

d I

56.

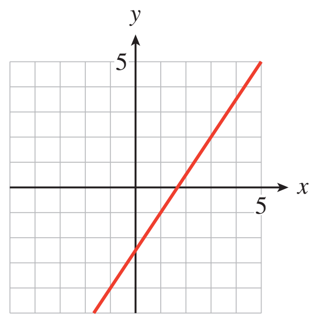
a $y = 2 - \frac{2}{3}(x-3)$

b $y = 2 - \frac{3}{2}(x+3)$

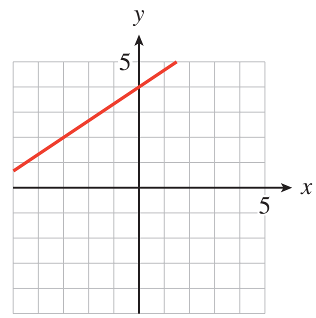
c $y = 2 + \frac{3}{2}(x-3)$

d $y = 2 + \frac{2}{3}(x+3)$

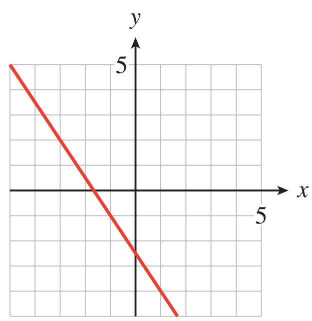
I



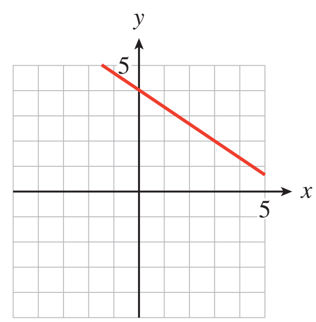
II



III



IV



In Problems 57–60, find the slope of each line and the coordinates of one point on the line. (No calculation is necessary!)

57. $y + 1 = 2(x - 6)$

Answer. $m = 2$; $(6, -1)$

58. $2(y - 8) = 5(x + 2)$

59. $y = 3 - \frac{4}{3}(x + 5)$

Answer. $m = \frac{-4}{3}$; $(-5, 3)$

60. $7x = -3y$

61.

a Draw a set of coordinate axes with a square grid (i.e., with units the same size in both directions). Sketch four lines through the point (0, 4) with the following slopes:

$$m = 3, \quad m = -3, \quad m = \frac{1}{3}, \quad m = \frac{-1}{3}$$

b What do you notice about these lines?

Look for perpendicular lines.

Answer.

a

b The lines with slope 3 and $\frac{-1}{3}$ are perpendicular to each other, and the lines with slope -3 and $\frac{1}{3}$ are perpendicular to each other.

62.

a Draw a set of coordinate axes with a square grid (see Problem 61). Sketch four lines through the point (0, -3) with the following slopes:

$$m = \frac{2}{5}, \quad m = \frac{-2}{5}, \quad m = \frac{5}{2}, \quad m = \frac{-5}{2}$$

b What do you notice about these lines?

63. The boiling point of water changes with altitude and is approximated by the formula

$$B = f(h) = 212 - 0.0018h$$

where B is in degrees and h is in feet. State the slope and vertical intercept of the graph, including units, and explain their meaning in this context.

Answer. $m = -0.0018$ degree/foot, so the boiling point drops with altitude at a rate of 0.0018 degree per foot. $b = 212$, so the boiling point is 212° at sea level (where the elevation $h = 0$).

64. The height of a woman in centimeters is related to the length of her femur (in centimeters) by the formula $H = f(x) = 2.47x + 54.10$. State the slope and the vertical intercept of the graph, including units, and explain their meaning in this context.

1.6 Linear Regression

We have spent most of this chapter analyzing models described by graphs or equations. To create a model, however, we often start with a quantity of data. Choosing an appropriate function for a model is a complicated process. In this section, we consider only linear models and explore methods for fitting a linear function to a collection of data points. First, we fit a line through two data points.

1.6.1 Fitting a Line through Two Points

If we already know that two variables are related by a linear function, we can find a formula from just two data points. For example, variables that increase or decrease at a constant rate can be described by linear functions.

Example 1.142. In 1993, Americans drank 188.6 million cases of wine. Wine consumption increased at a constant rate over the next decade, and we drank 258.3 million cases of wine in 2003. (Source: Los Angeles Times, Adams Beverage Group)

a Find a formula for wine consumption, W, in millions of cases, as a linear function of time, t, in years since 1990.

b State the slope as a rate of change. What does the slope tell us about this problem?

Solution.

a We have two data points of the form (t, W), namely $(3, 188.6)$ and $(13, 258.3)$. We use the point-slope formula to fit a line through these two points. First, we compute the slope.

$$\frac{\Delta W}{\Delta t} = \frac{258.3 - 188.6}{13 - 3} = 6.97$$

Next, we use the slope m = 6.97 and either of the two data points in the point-slope formula.

$$W = W_1 + m(t - t_1)$$
$$W = 188.6 + 6.97(t - 3)$$
$$W = 167.69 + 6.97t$$

Thus, $W = f(t) = 167.69 + 6.97t$.

b The slope gives us the rate of change of the function, and the units of the variables can help us interpret the slope in context.

$$\frac{\Delta W}{\Delta t} = \frac{258.3 - 188.6\text{millions of cases}}{13 - 3 \text{ years}} = 6.97 millionsofcases/year$$

Over the 10 years between 1993 and 2003, wine consumption in the United States increased at a rate of 6.97 million cases per year.

To Fit a Line through Two Points:

1 Compute the slope between the two points.

2 Substitute the slope and either point into the point-slope formula

$$y = y_1 + m(x - x_1)$$

Exercise 1.143. In 1991, there were 64.6 burglaries per 1000 households in the United States. The number of burglaries reported annually declined at a roughly constant rate over the next decade, and in 2001 there were 28.7 burglaries per 1000 households. (Source: U.S. Department of Justice)

a Find a function for the number of burglaries, B, as a function of time, t, in years, since 1990.

b State the slope as a rate of change. What does the slope tell us about this problem?.

1.6.2 Scatterplots

Empirical data points in a linear relation may not lie exactly on a line. There are many factors that can affect experimental data, including measurement error, the influence of environmental conditions, and the presence of related variable quantities.

Example 1.144. A consumer group wants to test the gas mileage of a new model SUV. They test-drive six vehicles under similar conditions and record the distance each drove on various amounts of gasoline.

Gasoline used (gal)	9.6	11.3	8.8	5.2	10.3	6.7
Miles driven	155.8	183.6	139.6	80.4	167.1	99.7

a Are the data linear?

b Draw a line that fits the data.

c What does the slope of the line tell us about the data?

Solution.

a No, the data are not strictly linear. If we compute the slopes between successive data points, the values are not constant. We can see from an accurate plot of the data, shown in Figure 1.145, that the points lie close to, but not precisely on, a straight line.

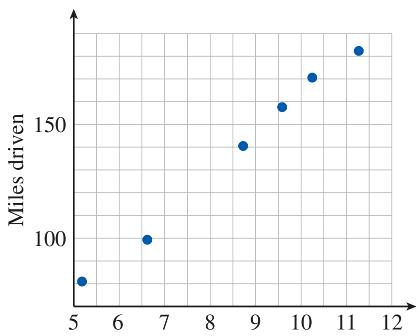


Figure 1.145

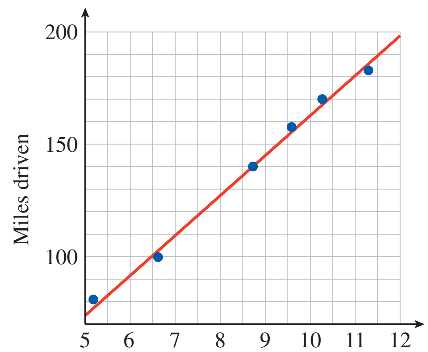


Figure 1.146

b We would like to draw a line that comes as close as possible to all the data points, even though it may not pass precisely through any of them. In particular, we try to adjust the line so that we have the same number of data points above the line and below the line. One possible solution is shown in Figure 1.146.

c To compute the slope of the line of best fit, we first choose two points on the line. Our line appears to pass through one of the data points,$(8.8, 139.6)$. Look for a second point on the line whose coordinates are easy to read, perhaps $(6.5, 100)$. The slope is

$$m = \frac{139.6 - 100}{8.8 - 6.5} = 17.2 \text{ miles per gallon}$$

According to our data, the SUV gets about 17.2 miles to the gallon.

Exercise 1.147.

a Plot the data points. Do the points lie on a line?

b Draw a line that fits the data.

x	1.49	3.68	4.95	5.49	7.88	8.41
y	2.69	3.7	4.6	5.2	7.2	7.3

The graph in Example 1.144 is called a **scatterplot**. The points on a scatterplot may or may not show some sort of pattern. Consider the three plots in Figure 1.148. In Figure 1.148a, the data points resemble a cloud of gnats; there is no apparent pattern to their locations. In Figure 1.148b, the data follow a generally decreasing trend, but certainly do not all lie on the same line. The points in Figure 1.148c are even more organized; they seem to lie very close to an imaginary line.

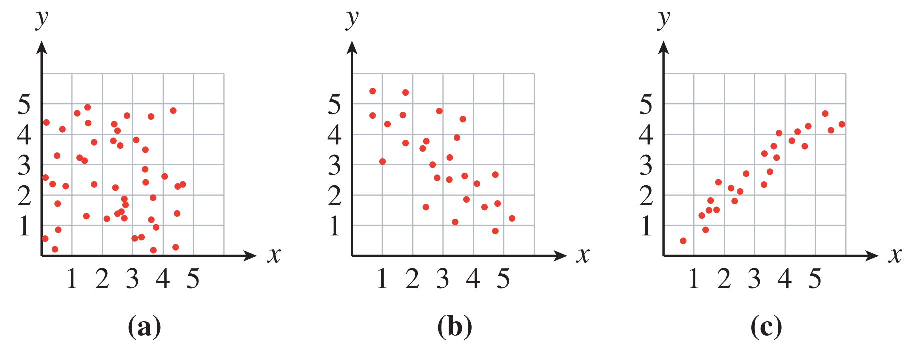


Figure 1.148

If the data in a scatterplot are roughly linear, we can estimate the location of an imaginary **line of best** fit that passes as close as possible to the data points. We can then use this line to make predictions about the data.

1.6.3 Linear Regression

One measure of a person's physical fitness is the **body mass index**, or BMI. Your BMI is the ratio of your weight in kilograms to the square of your height in centimeters. Thus, thinner people have lower BMI scores, and fatter people have higher scores. The Centers for Disease Control considers a BMI between 18.5 and 24.9 to be healthy. The points on the scatterplot in Figure 1.149 show the BMI of Miss America from 1918 to 1998. From the data in the scatterplot, can we see a trend in Americans' ideal of female beauty?

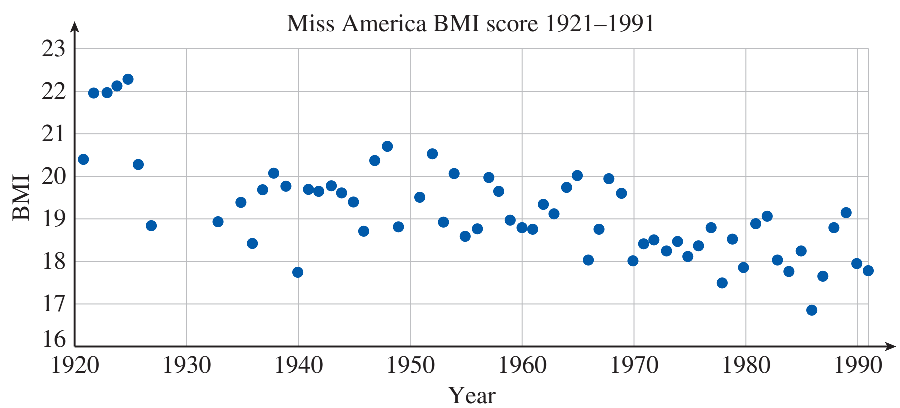


Figure 1.149

Example 1.150.

a Estimate a line of best fit for the scatterplot in Figure 1.149. (Source: http://www.pbs.org)

b Use your line to estimate the BMI of Miss America 1980.

Solution.

a We draw a line that fits the data points as best we can, as shown in Figure 1.151. (Note that we have set $t = 0$ in 1920 on this graph.) We try to end up with roughly equal numbers of data points above and below our line.

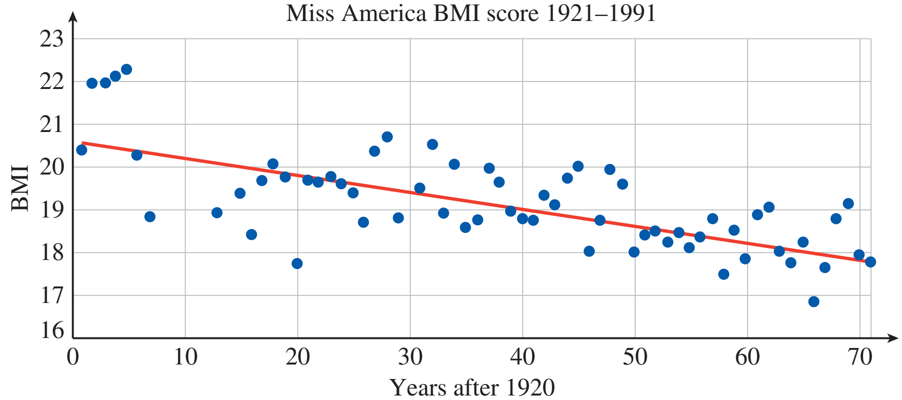


Figure 1.151

b We see that when $t = 60$ on this line, the y-value is approximately 18.3. We therefore estimate that Miss America 1980 had a BMI of 18.3. (Her actual BMI was 17.85.)

Exercise 1.152. Human brains consume a large amount of energy, about 16 times as much as muscle tissue per unit weight. In fact, brain metabolism accounts for about 25% of an adult human's energy needs, as compared to about 5% for other mammals. As hominid species evolved, their brains required larger and larger amounts of energy, as shown in Figure 1.153. (Source: Scientific American, December 2002)

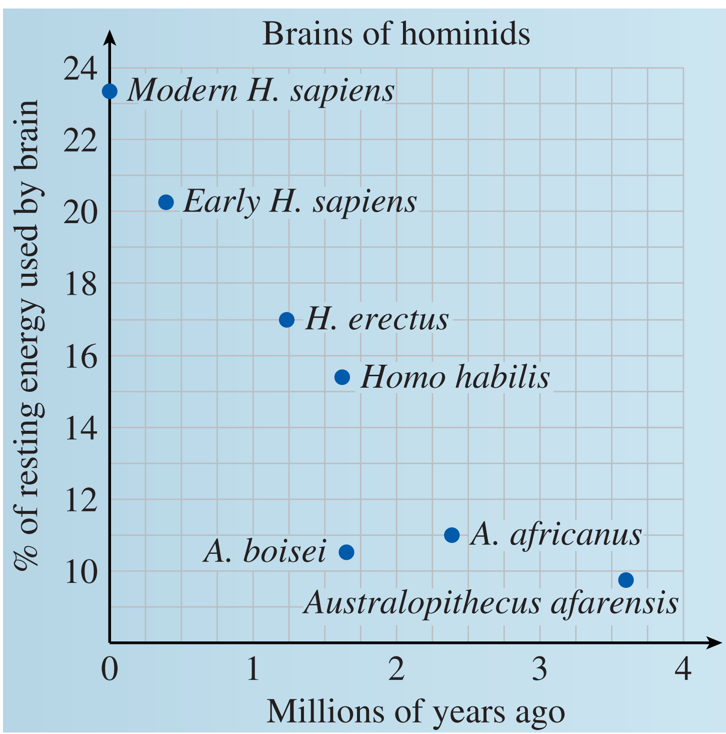


Figure 1.153

a Draw a line of best fit through the data points.

b Estimate the amount of energy used by the brain of a hominid species that lived three million years ago.

The process of predicting an output value based on a straight line that fits the data is called **linear regression**, and the line itself is called the **regression line**. The equation of the regression line is usually used (instead of a graph) to predict values.

Example 1.154.

a Find the equation of the regression line in Example 1.150, Figure 1.149.

b Use the regression equation to predict the BMI of Miss America 1980.

Solution.

a We first calculate the slope by choosing two points on the regression line we drew in Figure 1.151. The points we choose are not necessarily any of the original data points; instead they should be points on the regression line itself. The line appears to pass through the points $(17, 20)$ and $(67, 18)$. The slope of the line is then

$$m = \frac{18 - 20}{67 - 17} \approx -0.04$$

Now we use the point-slope formula to find the equation of the line. (If you need to review the point-slope formula, see Section 1.5.) We substitute $m = -0.04$ and use either of the two points for (x_1, y_1); we will choose $(17, 20)$. The equation of the regression line is

$$\begin{aligned} y &= y_1 + m(x - x_1) \\ y &= 20 - 0.04(x - 17) && \text{Simplify.} \\ y &= 20.68 - 0.04t \end{aligned}$$

b We will use the regression equation to make our prediction. For Miss America 1980, $t = 60$ and

$$y = 20.68 - 0.04(60) = 18.28$$

This value agrees well with the estimate we made in Example 1.150.

Exercise 1.155. The number of manatees killed by watercraft in Florida waters has been increasing since 1975. Data are given at 5-year intervals in the table. (Source: Florida Fish and Wildlife Conservation Commission)

Year	Manatee deaths
1975	6
1980	16
1985	33
1990	47
1995	42
2000	78

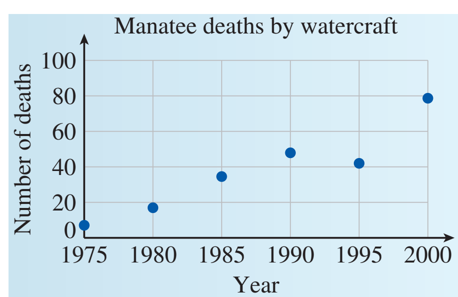


Figure 1.156

a Draw a regression line through the data points shown in Figure 1.156.

b Use the regression equation to estimate the number of manatees killed by watercraft in 1998.

1.6.4 Linear Interpolation and Extrapolation

Using a regression line to estimate values between known data points is called **interpolation**. Making predictions beyond the range of known data is called **extrapolation**.

Example 1.157.

a Use linear interpolation to estimate the BMI of Miss America 1960.

b Use linear extrapolation to predict the BMI of Miss America 2001.

Solution.

a For 1960, we substitute $t = 40$ into the regression equation we found in Example 1.154.

$$y = 20.68 - 0.04(40) = 19.08$$

We estimate that Miss America 1960 had a BMI of 19.08. (Her BMI was actually 18.79.)

b For 2001, we substitute $t = 81$ into the regression equation.

$$y = 20.68 - 0.04(81) = 17.44$$

Our model predicts that Miss America 2001 had a BMI of 17.44. In fact, her BMI was 20.25. By the late 1990s, public concern over the self-image of young women had led to a reversal of the trend toward ever-thinner role models.

Example 1.157b illustrates an important fact about extrapolation: If we try to extrapolate too far, we may get unreasonable results. For example, if we use our model to predict the BMI of Miss America 2520 (when $t = 600$), we get

$$y = 20.68 - 0.04(600) = -3.32$$

Even if the Miss America pageant is still operating in 600 years, the winner cannot have a negative BMI. Our linear model provides a fair approximation for 1920–1990, but if we try to extrapolate too far beyond the known data, the model may no longer apply.

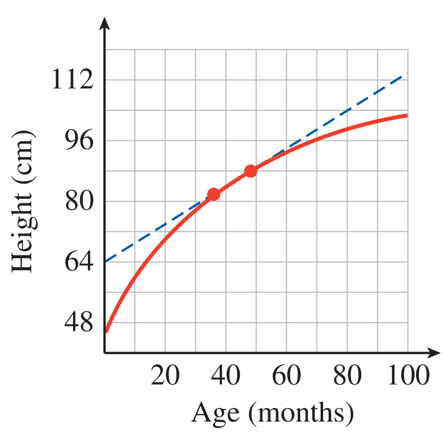


Figure 1.158

We can also use interpolation and extrapolation to make estimates for nonlinear functions. Sometimes a variable relationship is not linear, but a portion of its graph can be approximated by a line. The graph in Figure 1.158 shows a child's height each month. The graph is not linear because her rate of growth is not constant; her growth slows down as she approaches her adult height. However, over a short time interval the graph is close to a line, and that line can be used to approximate the coordinates of points on the curve.

Exercise 1.159. Emily was 82 centimeters tall at age 36 months and 88 centimeters tall at age 48 months.

a Find a linear equation that approximates Emily's height in terms of her age over the given time interval.

b Use linear interpolation to estimate Emily's height when she was 38 months old, and extrapolate to predict her height at age 50 months.

c Predict Emily's height at age 25 (300 months). Is your answer reasonable?

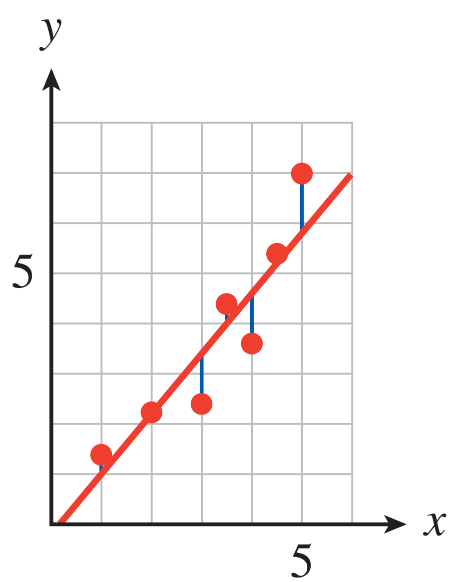


Figure 1.160

Estimating a line of best fit is a subjective process. Rather than base their estimates on such a line, statisticians often use the **least squares regression line**. This regression line minimizes the sum of the squares of all the vertical distances between the data points and the corresponding points on the line (see Figure 1.160). Many calculators are programmed to find the least squares regression line, using an algorithm that depends only on the data, not on the appearance of the graph.

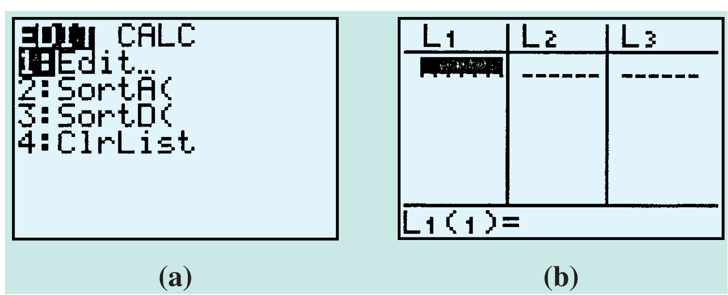


Figure 1.162

Remark 1.161 (Using a Calculator for Linear Regression)**.** You can use a graphing calculator to make a scatterplot, find a regression line, and graph the regression line with the data points. On the TI-83 calculator, we use the statistics mode, which you can access by pressing STAT. You will see a display that looks like Figure 1.162a. Choose 1 to Edit (enter or alter) data. Now follow the instructions in Example 1.163 for using your calculator's statistics features.

Example 1.163.

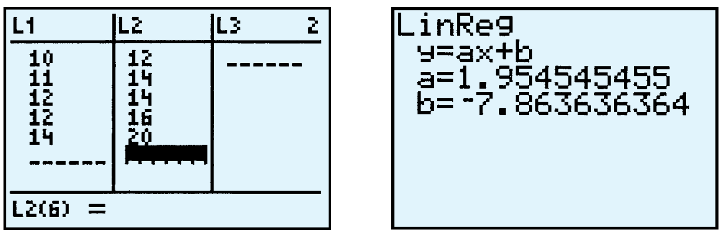
a Find the equation of the least squares regression line for the following data:

$$(10, 12), (11, 14), (12, 14), (12, 16), (14, 20)$$

b Plot the data points and the least squares regression line on the same axes.

Solution.

a We must first enter the data. Press STATENTER to select Edit. If there are data in column L_1 or L_2, clear them out: Use the â key to select L_1, press CLEAR, then do the same for L_2. Enter the x-coordinates of the data points in the L_1 column and enter the y-coordinates in the L_2 column, as shown in Figure 1.164a.

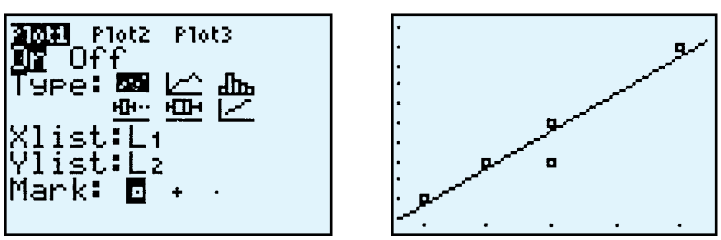


(a) (b)

Figure 1.164

Now we are ready to find the regression equation for our data. Press STAT â 4 to select linear regression, or LinReg (ax + b), then press ENTER. The calculator will display the equation $y = ax + b$ and the values for a and b, as shown in Figure 1.164b. You should find that your regression line is approximately $y = 1.95x - 7.86$.

b First, we first clear out any old definitions in the list. Position the cursor after Y_1 = and copy in the regression equation as follows: Press VARS5 â â ENTER. To draw a scatterplot, press 2ndY=1 and set the Plot1 menu as shown in Figure 1.165a. Finally, press ZOOM 9 to see the scatterplot of the data and the regression line. The graph is shown in Figure 1.165b.



(a) (b)

Figure 1.165

Caution 1.166. When you are through with the scatterplot, press Y= â ENTER to turn off the Stat Plot. If you neglect to do this, the calculator will continue to show the scatterplot even after you ask it to plot a new equation.

Exercise 1.167.

a Use your calculator's statistics features to find the least squares regression equation for the data in Exercise 1.147.

b Plot the data and the graph of the regression equation.

1.6.5 Section Summary

1.6.5.1 Vocabulary

Look up the definitions of new terms in the Glossary.

- Scatterplot
- Extrapolate
- Interpolate
- Least squares regression line
- Regression line
- Linear regression

1.6.5.2 CONCEPTS

1 Data points may not lie exactly on the graph of an equation.

2 Points in a scatterplot may or may not exhibit a pattern.

3 We can approximate a linear pattern by a regression line.

4 We can use interpolation or extrapolation to make estimates and predictions.

5 If we extrapolate too far beyond the known data, we may get unreasonable results.

1.6.5.3 STUDY QUESTIONS

1 What is a regression line?

2 State two formulas you will need to calculate the equation of a line through two points.

3 Explain the difference between interpolation and extrapolation.

4 In general, should you have more confidence in figures obtained by interpolation or by extrapolation? Why?

1.6.5.4 SKILLS

Practice each skill in the Homework 1.6.6 problems listed.

1 Find the equation of a line through two points: #1–6, 29–36

2 Draw a line of best fit: #7–18

3 Find the equation of a regression line: #11–28, 37–40

4 Use interpolation and extrapolation to make predictions: #11–40

1.6.6 Homework

In Problems 1–6, we find a linear model from two data points.

a Make a table showing the coordinates of two data points for the model. (Which variable should be plotted on the horizontal axis?)

b Find a linear equation relating the variables.

c State the slope of the line, including units, and explain its meaning in the context of the problem.

1. It cost a bicycle company \$9000 to make 40 touring bikes in its first month of operation and \$15,000 to make 125 bikes during its second month. Express the company's monthly production cost, C, in terms of the number, x, of bikes it makes.

Answer.

a

x	50	125
y	9000	$15,000$

b $C = 5000 + 80x$

c $m = 80$ dollars/bike, so it costs the company \$80 per bike it manufactures.

2. Flying lessons cost \$645 for an 8-hour course and \$1425 for a 20-hour course. Both prices include a fixed insurance fee. Express the cost, C, of flying lessons in terms of the length, h, of the course in hours.

3. Under ideal conditions, Andrea's Porsche can travel 312 miles on a full tank (12 gallons of gasoline) and 130 miles on 5 gallons. Express the distance, d, Andrea can drive in terms of the amount of gasoline, g, she buys.

Answer.

a

g	12	5
d	312	130

b $d = 26g$

c $m = 26$ miles/gallon, so the Porche's fuel efficiency is 26 miles per gallon.

4. On an international flight, a passenger may check two bags each weighing 70 kilograms, or 154 pounds, and one carry-on bag weighing 50 kilograms, or 110 pounds. Express the weight, p, of a bag in pounds in terms of its weight, k, in kilograms.

5. A radio station in Detroit, Michigan, reports the high and low temperatures in the Detroit/Windsor area as 59°F and 23°F, respectively. A station in Windsor, Ontario, reports the same temperatures as 15°C and −5°C. Express the Fahrenheit temperature, F, in terms of the Celsius temperature, C.

Answer.

a

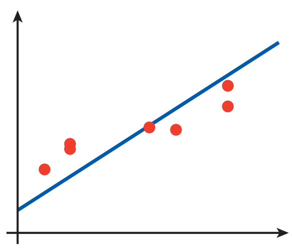
C	15	−5
F	59	23

b $F = 32 + \frac{9}{5}C$

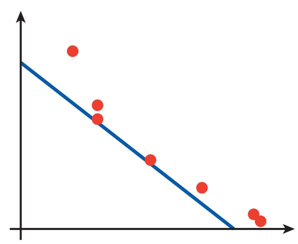
c $m = \frac{9}{5}$, so an increase of 1°C is equivalent to an increase of $\frac{9}{5}$°F.

6. Ms. Randolph bought a used car in 2000. In 2002, the car was worth \$9000, and in 2005 it was valued at \$4500. Express the value, V, of Ms. Randolph's car in terms of the number of years, t, she has owned it.

Each regression line can be improved by adjusting either m or b. Draw a line that fits the data points more closely.

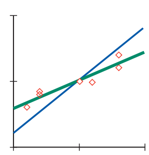


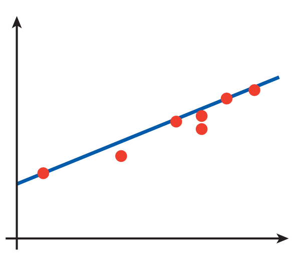
7.



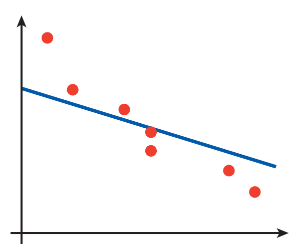
8.

Answer.



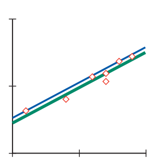


9.



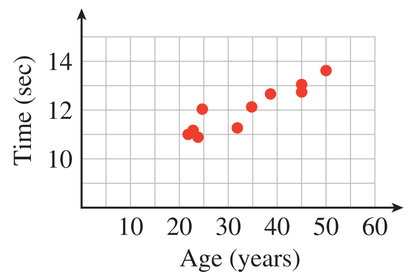
10.

Answer.



In Problems 11 and 12, use information from the graphs to answer the questions.

11. The scatterplot shows the ages of 10 army drill sergeants and the time it



took each to run 100 meters, in seconds.

a What was the hundred-meter time for the 25-year-old drill sergeant?

b How old was the drill sergeant whose hundred-meter time was 12.6 seconds?

c Use a straightedge to draw a line of best fit through the data points.

d Use your line of best fit to predict the hundred-meter time of a 28-year-old drill sergeant.

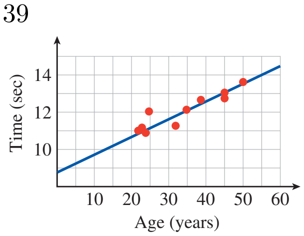
e Choose two points on your regression line and find its equation.

f Use the equation to predict the hundred-meter time of a 40-year-old drill sergeant and a 12 year-old drill sergeant. Are these predictions reasonable?

Answer.

a 12 seconds

b 39

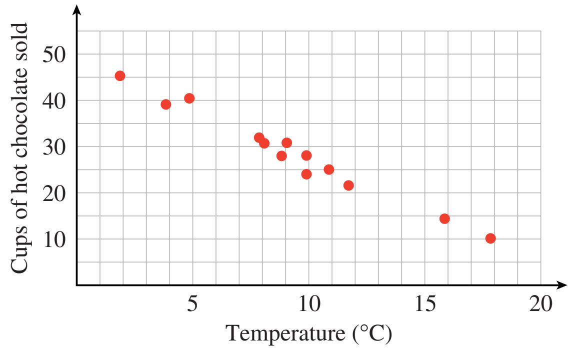
c 

d 11.6 seconds

e $y = 8.5 + 0.1x$

f 12.7 seconds; 10.18 seconds; The prediction for the 40-year-old is reasonable, but not the prediction for the 12-year-old. Yes, if you can find a gifted 12-year old sergeant!

12. The scatterplot shows the outside temperature and the number of cups of cocoa sold at an outdoor skating rink snack bar on 13 consecutive nights.



a How many cups of cocoa were sold when the temperature was 2°C?

b What was the temperature on the night when 25 cups of cocoa were sold?

c Use a straightedge to draw a line of best fit through the data points

d Use your line of best fit to predict the number of cups of cocoa that will be sold at the snack bar if the temperature is 7°C.

e Choose two points on your regression line and find its equation.

f Use the equation to predict the number of cups of cocoa that will be sold when the temperature is 10°C and when the temperature is 24°C. Are these predictions reasonable?

13. With Americans' increased use of faxes, pagers, and cell phones, new area codes are being created at a steady rate. The table shows the number of area codes in the United States each year. (Source: USA Today, NeuStar, Inc.)

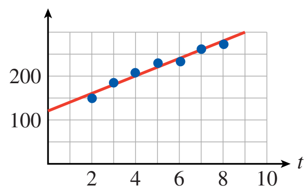
Year	Number of area codes
1997	151
1998	186
1999	204
2000	226
2001	239
2002	262
2003	274

a Let t represent the number of years after 1995 and plot the data. Draw a line of best fit for the data points.

b Find an equation for your regression line.

c How many area codes do you predict for 2010?

Answer.

a 

b $y = 121 + 19.86t$

c 419

14. The number of mobile homes in the United States has been increasing since 1960. The data in the table are given in millions of mobile homes. (Source: USA Today, U.S. Census Bureau)

Year 1960 1970 1980 1990 2000
Number of mobile homes 0.8 2.1 4.7 7.4 8.8

a Let t represent the number of years after 1960 and plot the data. Draw a line of best fit for the data points

b Find an equation for your regression line.

c How many mobile homes do you predict for 2010?

15. Teenage birth rates in the United States declined from 1991 to 2000. The table shows the number of births per 1000 women in selected years. (Source: U.S. National Health Statistics)

Year 1998	1991	1993	1995	1996	1997
Births 51.1	62.1	59.6	56.8	54.4	52.3

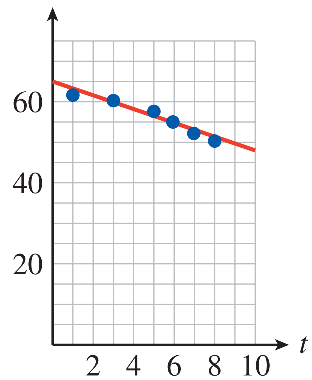
a Let t represent the number of years after 1990 and plot the data. Draw a line of best fit for the data points.

b Find an equation for your regression line.

c Estimate the teen birth rate in 1994.

d Predict the teen birth rate in 2010.

Answer.

a 

b $y = 64.2 - 1.63t$

c 58 births per 1000 women

d 32 births per 1000 women

16. The table shows the minimum wage in the United States at five-year intervals. (Source: Economic Policy Institute)

Year	1960	1965	1970	1975	1980	1985	1990	1995	2000
Minimum wage	1.00	1.25	1.60	2.10	3.10	3.35	3.80	4.25	5.15

a Let t represent the number of years after 1960 and plot the data. Draw a line of best fit for the data points.

b Find an equation for your regression line.

c Estimate the minimum wage in 1972.

d Predict the minimum wage in 2010.

17. Life expectancy in the United States has been rising since the nineteenth century. The table shows the U.S. life expectancy in selected years. (Source: http://www.infoplease.com)

Year	1950	1960	1970	1980	1990	2000
Life expectancy at birth	68.2	69.7	70.8	73.7	75.4	77

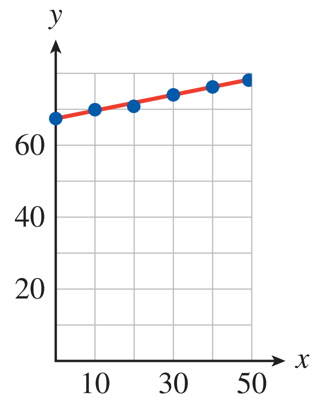
a Let t represent the number of years after 1950, and plot the data. Draw a line of best fit for the data points.

b Find an equation for your regression line.

c Estimate the life expectancy of someone born in 1987.

d Predict the life expectancy of someone born in 2010.

Answer.

a 

b $y = 0.18x + 68.2$

c 74.9 years

d 79 years

18. The table shows the per capita cigarette consumption in the United States at five-year intervals. (Source: http://www.infoplease.com)

Year	1980	1985	1990	1995	2000
Per capita cigarette consumption	3,851	3,461	2,827	2,515	2,092

a Let t represent the number of years after 1980, and plot the data. Draw a line of best fit for the data points.

b Find an equation for your regression line.

c Estimate the per capita cigarette consumption in 1998.

d Predict the per capita cigarette consumption in 2010.

19. "The earnings gap between high-school and college graduates continues to widen, the Census Bureau says. On average, college graduates now earn just over \$51,000 a year, almost twice as much as high-school graduates. And those with no high-school diploma have actually seen their earnings drop in recent years." The table shows the unemployment rate and the median weekly earnings for employees with different levels of education. (Source: Morning Edition, National Public Radio, March 28, 2005)

	Years of education	Unemployment rate	Weekly earnings (\$)
Some high school no diploma	10	8.8	396
High-school graduate	12	5.5	554
Some college no degree	13	5.2	622
Associate's degree	14	4.0	672
Bachelor's degree	16	3.3	900
Master's degree	18	2.9	1064
Professional degree	20	1.7	1307

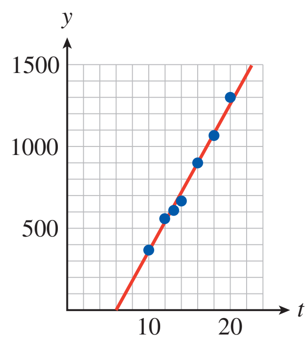
a Plot years of education on the horizontal axis and weekly earnings on the vertical axis.

b Find an equation for the regression line.

c State the slope of the regression line, including units, and explain what it means in the context of the data.

d Do you think this model is useful for extrapolation or interpolation? For example, what weekly earnings does the model predict for someone with 15 years of education? For 25 years? Do you think these predictions are valid? Why or why not?

Answer.

a 

b $y = 90.49t - 543.7$

c 90.49 dollars/year: Each additional year of education corresponds to an additional \$90.49 in weekly earnings.

d No: The degree or diploma attained is more significant than the number of years. So, for example, interpolation for the years of education between a bachelor's and master's degree may be inaccurate because earnings with just the bachelor's degree will not change until the master's degree is attained. And the years after the professional degree will not add significantly to earnings, so extrapolation is inappropriate.

20. The table shows the birth rate (in births per woman) and the female literacy rate (as a percent of the adult female population) in a number of nations. (Source: UNESCO, The World Fact Book, EarthTrends)

Country	Literacy rate	Birth rate
Brazil	88.6	1.93
Egypt	43.6	2.88
Germany	99	1.39
Iraq	53	4.28
Japan	99	1.39
Niger	9.4	6.75
Pakistan	35.2	4.14
Peru	82.1	2.56
Philippines	92.7	3.16
Portugal	91	1.47
Russian Federation	99.2	1.27
Saudi Arabia	69.3	4.05
United States	97	2.08

a Plot the data with literacy rate on the horizontal axis. Draw a line of best fit for the data points.

b Find an equation for the regression line.

c What values for the input variable make sense for the model? What are the largest and smallest values predicted by the model for the output variable?

d State the slope of the regression line, including units, and explain what it means in the context of the data.

21. The table shows the amount of carbon released into the atmosphere annually from burning fossil fuels, in billions of tons, at 5-year intervals from 1950 to 1995. (Source: www.worldwatch.org)

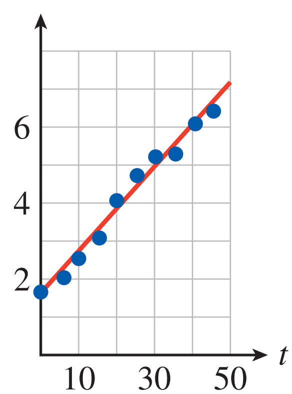
Year	1950	1955	1960	1965	1970	1975	1980	1985	1990	1995
Carbon emissions	1.6	2.0	2.5	3.1	4.0	4.5	5.2	5.3	5.9	6.2

a Let t represent the number of years after 1950 and plot the data. Draw a line of best fit for the data points.

b Find an equation for your regression line.

c Estimate the amount of carbon released in 1992.

Answer.

a 

b $y = 1.6 + 0.11t$

c 6.2 billion tons

22. High-frequency radiation is harmful to living things because it can cause changes in their genetic material. The data below, collected by C. P. Oliver in 1930, show the frequency of genetic transmutations induced in fruit flies by doses of X-rays, measured in roentgens. (Source: C. P. Oliver, 1930)

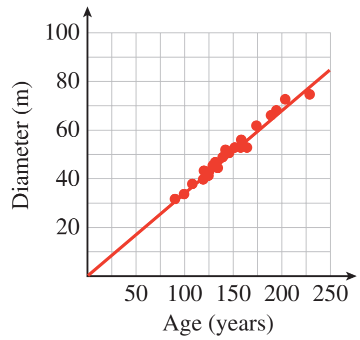
Dosage (roentgens)	285	570	1640	3280	6560
Percentage of mutated genes	1.18	2.99	4.56	9.63	15.85

a Plot the data and draw a line of best fit through the data points.

b Find an equation for your regression line.

c Use the regression equation to predict the percent of mutations that might result from exposure to 5000 roentgens of radiation.

23. Bracken, a type of fern, is one of the most successful plants in the world, growing on every continent except Antarctica. New plants, which are genetic clones of the original, spring from a network of underground stems, or rhizomes, to form a large circular colony. The graph shows the diameters of various colonies plotted against their age. (Source: Chapman et al.,1992)



a Calculate the rate of growth of the diameter of a bracken colony, in meters per year.

b Find an equation for the line of best fit. (What should the vertical intercept of the line be?)

c In Finland, bracken colonies over 450 meters in diameter have been found. How old are these colonies?

Answer.

a 0.34 meters per year

b $y = 0.34x$ ($b = 0$ because the plant has zero size until it begins.)

c Over 1300 years

24. The European sedge warbler can sing several different songs consisting of trills, whistles, and buzzes. Male warblers who sing the largest number of songs are the first to acquire mates in the spring. The data below show the number of different songs sung by several male warblers and the day on which they acquired mates, where day 1 is April 20. (Source: Krebs and Davies, 1993)

Number of songs	41	38	34	32	30	25	24	24	23	14
Pairing day	20	24	25	21	24	27	31	35	40	42

a Plot the data points, with number of songs on the horizontal axis. A regression line for the data is $y = -0.85x + 53$. Graph this line on the same axes with the data.

b What does the slope of the regression line represent?

c When can a sedge warbler that knows 10 songs expect to find a mate?

d What do the intercepts of the regression line represent? Do these values make sense in context?

25. The European sedge warbler can sing several different songs consisting of trills, whistles, and buzzes. Male warblers who sing the largest number of songs are the first to acquire mates in the spring. The data below show the number of different songs sung by several male warblers and the day on which they acquired mates, where day 1 is April 20. (Source: Krebs and Davies, 1993)

Women	Area (sq cm)	11.5	10.8	11.7	12.0	12.5	12.7	14.4	14.4	15.7
	Strength (kg)	11.3	13.2	13.2	14.5	15.6	14.8	15.6	16.1	18.4
Men	Area (sq cm)	13.5	13.8	15.4	15.4	17.7	18.6	20.8	–	–
	Strength (kg)	15.0	17.3	19.0	19.8	20.6	20.8	26.3	–	–

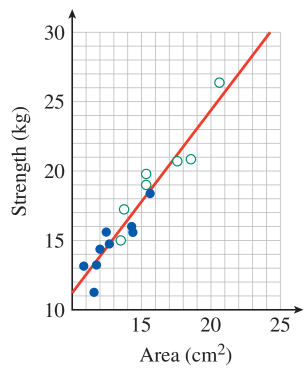
a Plot the data for both men and women on the same graph using different symbols for the data points for men and the data points for women.

b Are the data for both men and women described reasonably well by the same regression line? Draw a line of best fit through the data.

c Find the equation of your line of best fit, or use a calculator to find the regression line for the data.

d What does the slope mean in this context?

Answer.



a

b Yes

c $y = 1.29x - 1.62$

d The slope, 1.29 kg/sq cm, tells us that strength increases by 1.29 kg when the muscle cross-sectional area increases by 1 sq cm.

26. Astronomers use a numerical scale called magnitude to measure the brightness of a star, with brighter stars assigned smaller magnitudes. When we view a star from Earth, dust in the air absorbs some of the light, making the star appear fainter than it really is. Thus, the observed magnitude of a star, m, depends on the distance its light rays must travel through the Earth's atmosphere. The observed magnitude is given by

$$m = m_0 + kx$$

where m_0 is the actual magnitude of the star outside the atmosphere, x is the air mass (a measure of the distance through the atmosphere), and k is a constant called the **extinction coefficient**. To calculate m_0, astronomers observe the same object several times during the night at different positions in the sky, and hence for different values of x. Here are data from such observations. (Source: Karttunen et al., 1987)

Altitude	Air mass, x	Magnitude, m
50°	1.31	0.90
35°	1.74	0.98
25°	2.37	1.07
20°	2.92	1.17

a Plot observed magnitude against air mass, and draw a line of best fit through the data.

b Find the equation of your line of best fit, or use a calculator to find the regression line for the data.

c Find the equation of your line of best fit, or use a calculator to find the regression line for the data.

d What is the value of the extinction coefficient? What is the apparent magnitude of the star outside Earth's atmosphere?

27. Six students are trying to identify an unknown chemical compound by heating the substance and measuring the density of the gas that evaporates. (Density = mass/volume.) The students record the mass lost by the solid substance and the volume of the gas that evaporated from it. They know that the mass lost by the solid must be the same as the mass of the gas that evaporated. (Source: Hunt and Sykes, 1984)

Student	A	B	C	D	E	F
Volume of gas (cm^3)	48	60	24	81	76	54
Loss in mass (mg)	64	81	32	107	88	72

a Plot the data with volume on the horizontal axis. Which student made an error in the experiment?

b Ignoring the incorrect data point, draw a line of best fit through the other points.

c Find an equation of the form $y = kx$ for the data. Why should you expect the regression line to pass through the origin?

d Use your equation to calculate the mass of 1000 cm^3 (one liter) of the gas.

e Here are the densities of some gases at room temperature:

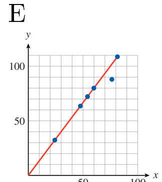
Hydrogen	8	mg/liter
Nitrogen	1160	mg/liter
Oxygen	1330	mg/liter
Carbon dioxide	1830	mg/liter

Which of these might have been the gas that evaporated from the unknown substance?

Use your answer to part (d) to calculate the density of the gas. 1 cm^3 = 1 milliliter.

Answer.

a E

b 

c $y = 1.33x$; There should be no loss in mass when no gas evaporates.

d 1333 mg

e 1333 mg

28. The formulas for many chemical compounds involve ratios of small integers. For example, the formula for water, H_20, means that two atoms of hydrogen combine with one atom of oxygen to make one water molecule. Similarly, magnesium and oxygen combine to produce magnesium oxide. In this problem, we will discover the chemical formula for magnesium oxide. (Source: Hunt and Sykes, 1984)

a Twenty-four grams of magnesium contain the same number of atoms as sixteen grams of oxygen. Complete the table showing the amount of oxygen needed if the formula for magnesium oxide is MgO, Mg_2O, or MgO_2.

Grams of Mg	Grams of O (if MgO)	Grams of O (if Mg_2O)	Grams of O (if MgO_2)
24	16		
48			
12			
6			

b Graph three lines on the same axes to represent the three possibilities, with grams of magnesium on the horizontal axis and grams of oxygen on the vertical axis.

c Here are the results of some experiments synthesizing magnesium oxide.

Experiment	Grams of Magnesium	Grams of oxygen
1	15	10
2	22	14
3	30	20
4	28	18
5	10	6

Plot the data on your graph from part (b). Which is the correct formula for magnesium oxide?

For Problems 29–32,

a Use linear interpolation to give approximate answers.

b What is the meaning of the slope in the context of the problem?

29. The temperature in Encino dropped from 81°F at 1 a.m. to 73°F at 5 a.m. Estimate the temperature at 4 a.m.

Answer.

a 75°F

b The slope of -2 degrees/hour says that temperatures are dropping at a rate of 2° per hour.

30. Newborn blue whales are about 24 feet long and weigh 3 tons. The young whale nurses for 7 months, at which time it is 53 feet long. Estimate the length of a 1-year-old blue whale.

31. A car starts from a standstill and accelerates to a speed of 60 miles per hour in 6 seconds. Estimate the car's speed 2 seconds after it began to accelerate.

Answer.

a 20 mph

b The slope of 10 mph/second says the car accelerates at a rate of 10 mph per second.

32. A truck on a slippery road is moving at 24 feet per second when the driver steps on the brakes. The truck needs 3 seconds to come to a stop. Estimate the truck's speed 2 seconds after the brakes were applied

In Problems 33–36, use linear interpolation or extrapolation to answer the questions.

33. The temperature of an automobile engine is 9° Celsius when the engine is started and 51°C 7 minutes later. Use a linear model to predict the engine temperature for both 2 minutes and 2 hours after it started. Are your predictions reasonable?

Answer. 2 min: 21°C; 2 hr: 729°C; The estimate at 2 minutes is reasonable; the estimate at 2 hours is not reasonable.

34. The temperature in Death Valley is 95° Fahrenheit at 5 a.m. and rises to 110° Fahrenheit by noon. Use a linear model to predict the temperature at 2 p.m. and at midnight. Are your predictions reasonable?

35. Ben weighed 8 pounds at birth and 20 pounds at age 1 year. How much will he weigh at age 10 if his weight increases at a constant rate?

Answer. 128 lb.

36. The elephant at the City Zoo becomes ill and loses weight. She weighed 10,012 pounds when healthy and only 9641 pounds a week later. Predict her weight after 10 days of illness.

37. Birds' nests are always in danger from predators. If there are other nests close by, the chances of predators finding the nest increase. The table shows the probability of a nest being found by predators and the distance to the nearest neighboring nest. (Source: Perrins, 1979)

Distance to nearest neighbor (meters)	20	40	60	80	100
Probability of predators (%)	47	34	32	17	1.5

a Plot the data and the least squares regression line.

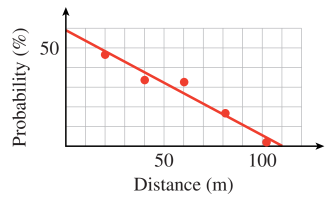
b Use the regression line to estimate the probability of predators finding a nest if its nearest neighbor is 50 meters away.

c If the probability of predators finding a nest is 10%, how far away is its nearest neighbor?

d What is the probability of predators finding a nest if its nearest neighbor is 120 meters away? Is your answer reasonable?

Answer.

a $y \approx -0.54x + 58.7$



b 31.7%

c 90 meters

d The regression line gives a negative probability, which is not reasonable.

38. A trained cyclist pedals faster as he increases his cycling speed, even with a multiple-gear bicycle. The table shows the pedal frequency, p (in revolutions per minute), and the cycling speed, c (in kilometers per hour), of one cyclist. (Source: Pugh, 1974)

Speed (km/hr)	8.8	12.5	16.2	24.4	31.9	35.0
Pedal frequency (rpm)	44.5	50.7	60.6	77.9	81.9	95.3

a Plot the data and the least squares regression line.

b Estimate the cyclist's pedal frequency at a speed of 20 kilometers per hour.

c Estimate the cyclist's speed when he is pedaling at 70 revolutions per minute.

d Does your regression line give a reasonable prediction for the pedaling frequency when the cyclist is not moving? Explain.

39. In this problem we will calculate the efficiency of swimming as a means of locomotion. A swimmer generates power to maintain a constant speed in the water. If she must swim against an opposing force, the power increases. The following table shows the power expended by a swimmer while working against different amounts of force. (A positive force opposes the swimmer, and a negative force helps her.) (Source: diPrampero et al., 1974, and Alexander, 1992)

Force (newtons)	−3.5	0	0	6	8	10	17	17
Metabolic power (watts)	100	190	230	320	380	450	560	600

a Plot the data on the grid, or use the **StatPlot** feature on your calculator. Use your calculator to find the least squares regression line. Graph the regression line on top of the data.

b Use your regression line to estimate the power needed for the swimmer to overcome an opposing force of 15 newtons.

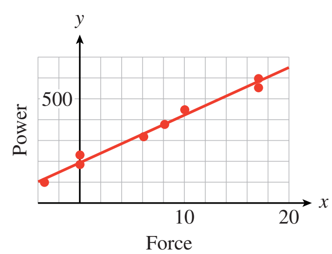
c Use your regression line to estimate the power generated by the swimmer when there is no force either hindering or helping her.

d Estimate the force needed to tow the swimmer at 0.4 meters per second while she rests. (If she is resting, she is not generating any power).

e The swimmer's **mechanical** power (or rate of work) is computed by multiplying her speed times the force needed to tow her at rest. Use your answer to part (d) to calculate the mechanical power she generates by swimming at 0.4 meters per second.

f The ratio of mechanical power to metabolic power is a measure of the swimmer's efficiency. Compute the efficiency of the swimmer when there is no external force opposing or helping her.

Answer.

a 

$y = 22.796x + 198.455$

b ≈ 540 watts

c 198.5 watts

d ≈ -8.7 newtons

e 3.48 watts

f about 0.018 or 1.8%

40. In this problem, we calculate the amount of energy generated by a cyclist. An athlete uses oxygen slowly when resting but more quickly during physical exertion. In an experiment, several trained cyclists took turns pedaling on a bicycle ergometer, which measures their work rate. The table shows the work rate of the cyclists, in watts, measured against their oxygen intake, in liters per minute. (Source: Pugh, 1974)

Oxygen consumption (liters/)	1	1.7	2	3.3	3.9	3.6	4.3	5
Work rate (watts)	40	100	180	220	280	300	320	410

a Plot the data on the grid, or use the **StatPlot** feature on your calculator. Use your calculator to find the least squares regression line. Graph the regression line on top of the data.

b Find the horizontal intercept of the regression line. What does the horizontal intercept tell you about this situation?

c Estimate the power produced by a cyclist consuming oxygen at 5.9 liters per minute.

d What is the slope of the regression line? The slope represents the amount of power, in watts, generated by a cyclist for each liter of oxygen consumed per minute. How many watts of power does a cyclist generate from each liter of oxygen?

e One watt of power represents an energy output of one joule per second. How many joules of energy does the cyclist generate in one minute?

f How many joules of energy can be extracted from each cubic centimeter of oxygen used? (One liter is equal to 1000 cubic centimeters.)

1.7 Chapter Summary and Review

1.7.1 Key Concepts

1 We can describe a relationship between variables with a table of values, a graph, or an equation.

2 Linear models have equations of the following form:

$$y = (\text{starting value}) + (\text{rate of change}) \cdot x$$

3 The general form for a linear equation is $Ax + By = C$.

4 We can use the **intercepts** to graph a line. The intercepts are also useful for interpreting a model.

5 A **function** is a rule that assigns to each value of the input variable a unique value of the output variable.

6 Function notation: $y = f(x)$, where x is the input and y is the output.

7 The point (a, b) lies on the graph of the function f if and only if $f(a) = b$

8 Each point on the graph of the function f has coordinates $(x, f(x))$ for some value of x.

9 The **vertical line test** tells us whether a graph represents a function.

10 Lines have constant slope.

11 The slope of a line gives us the **rate of change** of one variable with respect to another

12 **Formulas for Linear Functions**

Slope:&&m& $= \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$\\&&& $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$\\Slope-intercept form:&&y& $= b + mx$\\Point-slo

13 The **slope-intercept form** is useful when we know the initial value and the rate of change.

14 The **point-slope form** is useful when we know the rate of change and one point on the line.

15 Linear functions form a **two-parameter family**, $f(x) = b + mx$.

16 We can approximate a linear pattern by a **regression line**.

17 We can use **interpolation** or **extrapolation** to make estimates and predictions.

18 If we extrapolate too far beyond the known data, we may get unreasonable results.

.

1.7.2 Review Problems

Write and graph a linear equation for each situation. Then answer the questions.

1. Last year, Pinwheel Industries introduced a new model calculator. It cost \$2000 to develop the calculator and \$20 to manufacture each one.

a Complete the table of values showing the total cost, C, of producing n calculators.

n	100	500	800	1200	1500
C					

b Write an equation that expresses C in terms of n.

c Graph the equation by hand.

d What is the cost of producing 1000 calculators? Illustrate this as a point on your graph.

e How many calculators can be produced for \$10,000? Illustrate this as a point on your graph.

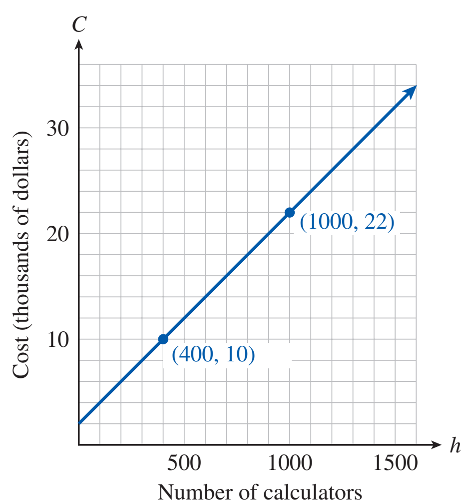
Answer.

a

n	100	500	800	1200	1500
C	4000	$12,000$	$18,000$	$26,000$	$32,000$

b $C = 20n + 2000$

c



d $\$22,000$

e 400

2. Megan weighed 5 pounds at birth and gained 18 ounces per month during her first year.

a Complete the table of values for Megan's weight, w, in terms of her age, m, in months.

m	2	4	6	9	12
w					

b Write an equation that expresses 2 in terms of m.

c Graph the equation by hand.

d How much did Megan weigh at 9 months? Illustrate this as a point on your graph.

e When did Megan weigh 9 pounds? Illustrate this as a point on your graph.

3. The total amount of oil remaining in 2005 is estimated at 2.1 trillion barrels, and total annual consumption is about 28 billion barrels.

a Assuming that oil consumption continues at the same level, write an equation for the remaining oil, R, as a function of time, t (in years since 2005).

b Find the intercepts and graph the equation by hand.

c What is the significance of the intercepts to the world's oil supply?

Answer.

a $R = 2100 - 28t$

b $(75, 0)$, $(0, 2100)$

c t-intercept: The oil reserves will be gone in 2080; R-intercept: There were 2100 billion barrels of oil reserves in 2005.

4. The world's copper reserves were 950 million tons in 2004; total annual consumption was 16.8 million tons.

a Assuming that copper consumption continues at the same level, write an equation for the remaining copper reserves, R, as a function of time, t (in years since 2004).

b Find the intercepts and graph the equation by hand.

c What is the significance of the intercepts to the world's copper supply?

5. The owner of a movie theater needs to bring in \$1000 at each screening in order to stay in business. He sells adult tickets at \$5 apiece and children's tickets at \$2 each.

a Write an equation that relates the number of adult tickets, A, he must sell and the number of children's tickets, C.

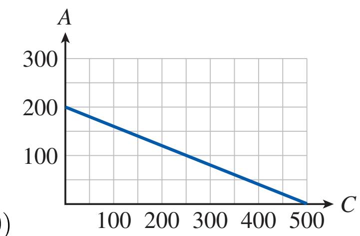
b Find the intercepts and graph the equation by hand.

c If the owner sells 120 adult tickets, how many children's tickets must he sell?

d What is the significance of the intercepts to the sale of tickets?

Answer.

a $2C + 5A = 1000$



b $(500, 0)$, $(0, 200)$

c C-intercept: If no adult tickets are sold, he must sell 500 children's tickets; A-intercept: If no children's tickets are sold, he must sell 200 adult tickets.

6. Alida plans to spend part of her vacation in Atlantic City and part in Saint-Tropez. She estimates that after airfare her vacation will cost \$60 per day in Atlantic City and 100 per day in Saint-Tropez. She has \$1200 to spend after airfare.

a Write an equation that relates the number of days, C, Alida can spend in Atlantic City and the number of days, T, in Saint-Tropez.

b Find the intercepts and graph the equation by hand.

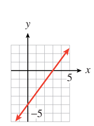
c If Alida spends 10 days in Atlantic City, how long can she spend in Saint-Tropez?

d What is the significance of the intercepts to Alida's vacation?

Graph each equation on graph paper. Use the most convenient method for each problem.

7. $4x - 3y = 12$

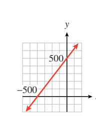
Answer.



8. $\dfrac{x}{6} - \dfrac{y}{12} = 1$

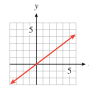
9. $50x = 40y - 20,000$

Answer.



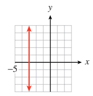
10. $1.4x + 2.1y = 8.4$

11. $3x - 4y = 0$

Answer. 

12. $x = -4y$

13. $4x = -12$

Answer. 

14. $2y - x = 0$

Which of the following tables describe functions? Explain.

15.

x	-2	-1	0	1	2	3
y	6	0	1	2	6	8

Answer. A function: Each x has exactly one associated y-value.

16.

p	3	-3	2	-2	-2	0
q	2	-1	4	-4	3	0

17.

Student	Score on IQ test	Score on SAT test
(A)	118	649
(B)	98	450
(C)	110	590
(D)	105	520
(E)	98	490
(F)	122	680

Answer. Not a function: The IQ of 98 has two possible SAT scores.

18.

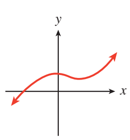
Student	Correct answers on math quiz	Quiz grade
(A)	13	85
(B)	15	89
(C)	10	79
(D)	12	82
(E)	16	91
(F)	18	95

19. The total number of barrels of oil pumped by the AQ oil company is given by the formula $N(t) = 2000 + 500t$, where N is the number of barrels of oil t days after a new well is opened. Evaluate $N(10)$ and explain what it means.

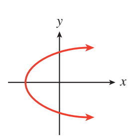
Answer. $N(10) = 7000$: Ten days after the new well is opened, the company has pumped a total of 7000 barrels of oil.

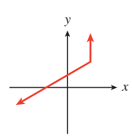
20. The number of hours required for a boat to travel upstream between two cities is given by the formula $H(v) = \dfrac{24}{v-8}$, where v represents the boat's top speed in miles per hour. Evaluate $H(16)$ and explain what it means.

Which of the following graphs represent functions?

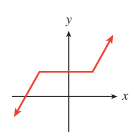
21. 

Answer. Function

22. 

23. 

Answer. Not a function

24. 

Evaluate each function for the given values.

25. $F(t) = \sqrt{1+4t^2}$, $F(0)$ and $F(-3)$

Answer. $F(0) = 1$, $F(-3) = \sqrt{37}$

26. $G(x) = \sqrt[3]{x-8}$, $G(0)$ and $G(20)$

27. $h(v) = 6 - |4 - 2v|$, $h(8)$ and $h(-8)$

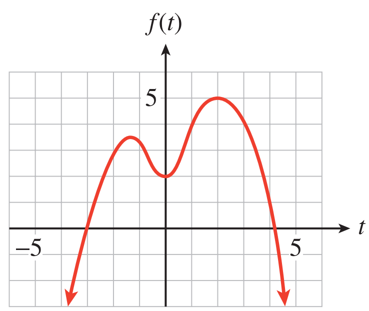
Answer. $h(8) = -6$, $h(-8) = -14$

28. $m(p) = \dfrac{120}{p+15}$, $m(5)$ and $m(-40)$

Refer to the graphs shown for Problems 29 and 30.

29.

a Find $f(-2)$ and $f(2)$.
b For what value(s) of t is $f(t) = 4$?
c Find the t- and $f(t)$-intercepts of the graph.
d What is the maximum value of f? For what value(s) of t does f take on its maximum value?

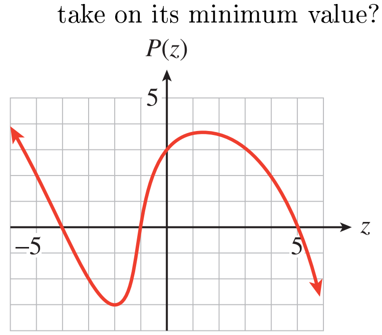


Answer.

a $f(-2) = 3$, $f(2) = 5$
b $t = 1$, $t = 3$
c t-intercepts $(-3, 0), (4, 0)$; $f(t)$-intercept: $(0, 2)$
d Maximum value of 5 occurs at $t = 2$

30.

a Find $P(-3)$ and $P(3)$.
b For what value(s) of z is $P(z) = 2$?
c Find the z- and $P(z)$-intercepts of the graph.
d What is the minimum value of P? For what value(s) of z does P take on its minimum value?



Graph the given function on a graphing calculator. Then use the graph to solve the equations and inequalities. Round your answers to one decimal place if necessary.

31. $y = \sqrt[3]{x}$

a Solve $\sqrt[3]{x} = 0.8$
b Solve $\sqrt[3]{x} = 1.5$
c Solve $\sqrt[3]{x} > 1.7$
d Solve $\sqrt[3]{x} \leq 1.26$

Answer.

a $x = \dfrac{1}{2} = 0.5$
b $x = \dfrac{27}{8} \approx 3.4$
c $x > 4.9$
d $x \leq 2.0$

32. $y = \dfrac{1}{x}$

a Solve $\dfrac{1}{x} = 2.5$
b Solve $\dfrac{1}{x} = 1.3125$
c Solve $\dfrac{1}{x} \geq 2.\overline{2}$
d Solve $\dfrac{1}{x} < 5$

33. $y = \dfrac{1}{x^2}$

a Solve $\dfrac{1}{x^2} = 0.03$
b Solve $\dfrac{1}{x^2} = 6.25$
c Solve $\dfrac{1}{x^2} > 0.16$
d Solve $\dfrac{1}{x^2} \leq 4$

Answer.

a $x \approx \pm 5.8$
b $x = \pm 0.4$
c $-2.5 < x < 0$ or $0 < x < 2.5$
d $x \leq -0.5$ or $x \geq 0.5$

34. $y = \sqrt{x}$

a Solve $\sqrt{x} = 0.707$
b Solve $\sqrt{x} = 1.7$
c Solve $\sqrt{x} < 1.5$
d Solve $\sqrt{x} \geq 1.3$

Evaluate each function.

35. $H(t) = t^2 + 2t, \quad H(2a) \quad \text{and} \quad H(a+1)$

Answer. $H(2a) = 4a^2 + 4a, \quad H(a+1) = a^2 + 4a + 3$

36. $F(x) = 2 - 3x, \quad F(2) + F(3) \quad \text{and} \quad F(2+3)$

37. $f(x) = 2x^2 - 4, \quad f(a) + f(b) \quad \text{and} \quad f(a+b)$

Answer. $f(a)+f(b) = 2a^2 + 2b^2 - 8, \quad f(a+b) = 2a^2 + 4ab + 2b^2 - 4$

38. $G(t) = 1 - t^2$, $\quad G(3w) \quad$ and $\quad G(s+1)$

39. A spiked volleyball travels 6 feet in 0.04 seconds. A pitched baseball travels 66 feet in 0.48 seconds. Which ball travels faster?

Answer. The volleyball

40. Kendra needs 412 gallons of Luke's Brand primer to cover 1710 square feet of wall. She uses 513 gallons of Slattery's Brand primer for 2040 square feet of wall. Which brand covered more wall per gallon?

41. Which is steeper, Stone Canyon Drive, which rises 840 feet over a horizontal distance of 1500 feet, or Highway 33, which rises 1150 feet over a horizontal distance of 2000 feet?

Answer. Highway 33

42. The top of Romeo's ladder is on Juliet's window sill that is 11 feet above the ground, and the bottom of the ladder is 5 feet from the base of the wall. Is the incline of this ladder as steep as a firefighter's ladder that rises a height of 35 feet over a horizontal distance of 16 feet?

43. The table shows the amount of oil, B (in thousands of barrels), left in a tanker t minutes after it hits an iceberg and springs a leak.

t	0	10	20	30
B	800	750	700	650

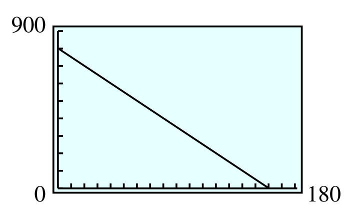
a Write a linear function for B in terms of t.

b Choose appropriate window settings on your calculator and graph your function.

c Give the slope of the graph, including units, and explain the meaning of the slope in terms of the oil leak.

Answer.

a $B = 800 - 5t$

b 

c $m = -5$ thousand barrels/minute: The amount of oil in the tanker is decreasing by 5000 barrels per minute.

44. A traditional first experiment for chemistry students is to make 98 observations about a burning candle. Delbert records the height, h, of the candle in inches at various times t minutes after he lit it.

t	0	10	30	45
h	12	11.5	10.5	9.75

a Write a linear function for h in terms of t.

b Choose appropriate window settings on your calculator and graph your function.

c Give the slope of the graph, including units, and explain the meaning of the slope in terms of the candle.

45. An interior decorator bases her fee on the cost of a remodeling job. The accompanying table shows her fee, F, for jobs of various costs, C, both given in dollars.

C	5000	10,000	20,000	50,000
F	1000	1500	2500	5500

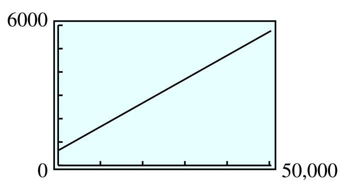
a Write a linear function for F in terms of C.

b Choose appropriate window settings on your calculator and graph your function.

c Give the slope of the graph, including units, and explain the meaning of the slope in terms of the the decorator's fee.

Answer.

a $F = 500 + 0.10C$

b 

c $m = 0.10$: The fee increases by \$0.10 for each dollar increase in the remodeling job.

46. Auto registration fees in Connie's home state depend on the value of the automobile. The table below shows the registration fee, R, for a car whose value is V, both given in dollars.

V	5000	10,000	15,000	20,000
R	135	235	335	435

a Write a linear function for R in terms of V.

b Choose appropriate window settings on your calculator and graph your function.

c Give the slope of the graph, including units, and explain the meaning of the slope in terms of the registration fee.

Find the slope of the line segment joining each pair of points.

47. $(-1, 4)$, $(3, -2)$

Answer. $\dfrac{-3}{2}$

48. $(5, 0)$, $(2, -6)$

49. $(6.2, 1.4)$, $(-2.1, 4.8)$

Answer. $\dfrac{-34}{83} \approx -0.4$

50. $(0, -6.4)$, $(-5.6, 3.2)$

51. The planners at AquaWorld want the small water slide to have a slope of 25%. If the slide is 20 feet tall, how far should the end of the slide be from the base of the ladder?

Answer. 80 ft

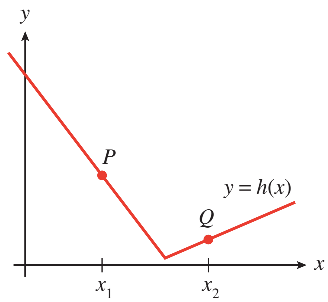
52. In areas with heavy snowfall, the pitch (or slope) of the roof of an A-frame house should be at least 1.2. If a small ski chalet is 40 feet wide at its base, how tall is the center of the roof?

Find the coordinates of the indicated points, and then write an algebraic expression using function notation for the indicated quantities.

53.

a Δy as x increases from x_1 to x_2

b The slope of the line segment joining P to Q



Answer.

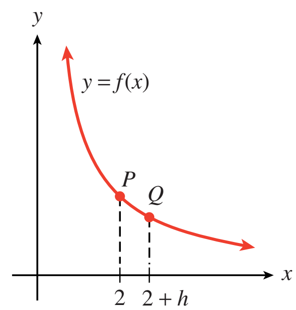
a $h(x_2) - h(x_1)$

b $\dfrac{h(x_2) - h(x_1)}{x_2 - x_1}$

54.

a Δy as x increases from 2 to $2 + h$

b The slope of the line segment joining P to Q



Answer.

a $h(x_2) - h(x_1)$

b $\dfrac{h(x_2) - h(x_1)}{x_2 - x_1}$

Which of the following tables represent linear functions?

55.

a

r	E
1	5
2	$\frac{5}{2}$
3	$\frac{5}{3}$
4	$\frac{5}{4}$
5	1

b

s	t
10	6.2
20	9.7
30	12.6
40	15.8
50	19.0

Answer. Neither

56.

a

w	A
2	-13
4	-23
6	-33
8	-43
10	-53

b

x	C
0	0
2	5
4	10
8	20
16	40

Each table gives values for a linear function. Fill in the missing values.

57.

d	V
-5	-4.8
-2	-3
	-1.2
6	1.8
10	

58.

q	S
-8	-8
-4	56
3	
	200
9	

Answer.

d	V
-5	-4.8
-2	-3
1	-1.2
6	1.8
10	**4.2**

Find the slope and y-intercept of each line.

59. $2x - 4y = 5$

Answer. $m = \dfrac{1}{2},\ b = \dfrac{-5}{4}$

60. $\dfrac{1}{2}x + \dfrac{2}{3}y = \dfrac{5}{6}$

61. $8.4x + 2.1y = 6.3$

Answer. $m = -4,\ b = 3$

62. $y - 3 = 0$

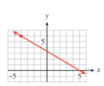
For Problems 63 and 64,

a Graph by hand the line that passes through the given point with the given slope.

b Find an equation for the line.

63. $(-4, 6)$; $m = \dfrac{-2}{3}$

Answer.

a 

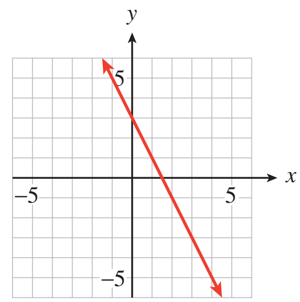
b $y = \dfrac{10}{3} - \dfrac{2}{3}x$

64. $(2, -5)$; $m = \dfrac{3}{2}$

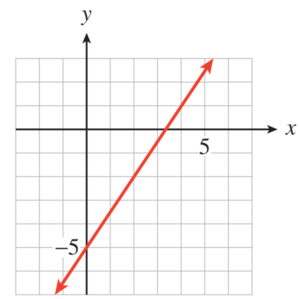
For Problems 65 and 66,

a Find the slope and y-intercept of each line.

b Write an equation for the line.



65.



66.

Answer.

a $m = -2,\ b = 3$

b $y = 3 - 2x$

67. What is the slope of the line whose intercepts are $(-5, 0)$ and $(0, 3)$?

Answer. $\dfrac{3}{5}$

68.

a Find the x- and y-intercepts of the line $\frac{x}{4} - \frac{y}{6} = 1$.

b What is the slope of the line in part (a)?

69.

a What is the slope of the line $y = 2 + \frac{3}{2}(x - 4)$?

b Find the point on the line whose x-coordinate is 4. Can there be more than one such point?

c Use your answers from parts (a) and (b) to find another point on the line.

Answer.

a $\dfrac{3}{2}$

b $(4, 2)$, no

c $(6, 5)$

70. A line passes through the point $(-2, -6)$ and has slope $\frac{2}{3}$. Find the coordinates of two more points on the line.

71. A line passes through the point $(-5, 3)$ and has slope $-\frac{8}{5}$. Find the coordinates of two more points on the line.

Answer. $(3, -14)$, $(-7, 2)$

72. Find an equation in point-slope form for the line of slope $\frac{6}{5}$ that passes through $(-3, -4)$.

73. The rate at which air temperature decreases with altitude is called the lapse rate. In the troposphere, the layer of atmosphere that extends from the Earth's surface to a height of about 7 miles, the lapse rate is about 3.6°F for every 1000 feet. (Source: Ahrens, 1998)

a If the temperature on the ground is 62°F, write an equation for the temperature, T, at an altitude of h feet.

b What is the temperature outside an aircraft flying at an altitude of $30,000$ feet? How much colder is that than the ground temperature?

c What is the temperature at the top of the troposphere?

Answer.

a $T = 62 - 0.0036h$ b $-46°$F; $108°$F c $-71°$F

74. In his television program *Notes from a Small Island*, aired in February 1999, Bill Bryson discussed the future of the British aristocracy. Because not all families produce an heir, 4 or 5 noble lines die out each year. At this rate, Mr. Bryson says, if no more peers are created, there will be no titled families left by the year 2175.

a Assuming that on average 4.5 titled families die out each year, write an equation for the number, N, of noble houses left in year t, where $t = 0$ in the year 1999.

b Graph your equation.

c According to your graph, how many noble families existed in 1999? Which point on the graph corresponds to this information?

Find an equation for the line passing through the two given points.

75. $(3, -5)$, $(-2, 4)$

Answer. $y = \dfrac{2}{5} - \dfrac{9}{5}x$

76. $(0, 8)$, $(4, -2)$

For Problems 77 and 78,

a Make a table of values showing two data points.

b Find a linear equation relating the variables.

c State the slope of the line, including units, and explain its meaning in the context of the problem.

77. The population of Maple Rapids was 4800 in 1990 and had grown to 6780 by 2005. Assume that the population increases at a constant rate. Express the population, P, of Maple Rapids in terms of the number of years, t, since 1990.

Answer.

a

t	0	15
P	4800	6780

b $P = 4800 + 132t$

c $m = 132$ people/year: the population grew at a rate of 132 people per year.

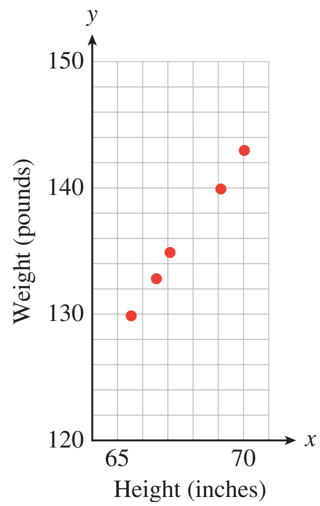
78. Cicely's odometer read 112 miles when she filled up her 14-gallon gas tank and 308 when the gas gauge read half full. Express her odometer reading, m, in terms of the amount of gas, g, she used

79. In 1986, the space shuttle Challenger exploded because of O-ring failure on a morning when the temperature was about 30°F. Previously, there had been one incident of O-ring failure when the temperature was 70°F and three incidents when the temperature was 54°F. Use linear extrapolation to estimate the number of incidents of O-ring failure you would expect when the temperature is 30°F.

Answer. 6

80. Thelma typed a 19-page technical report in 40 minutes. She required only 18 minutes for an 8-page technical report. Use linear interpolation to estimate how long Thelma would require to type a 12-page technical report.

81. The scatterplot shows weights (in pounds) and heights (in inches) for a team of distance runners.



a Use a straightedge to draw a line that fits the data.

b Use your line to predict the weight of a 65-inch-tall runner and the weight of a 71-inch-tall runner.

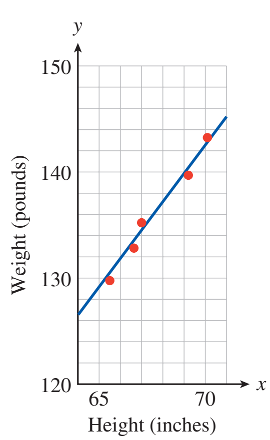
c Use your answers from part (b) to approximate the equation of a regression line.

d Use your answer to part (c) to predict the weight of a runner who is 68 inches tall.

e The points on the scatterplot are $(65.5, 130)$, $(66.5, 133)$, $(67, 135)$, $(69, 140)$, and $(70, 143)$. Use your calculator to find the least squares regression line.

f Use the regression line to predict the weight of a runner who is 68 inches tall.

Answer.

a 

b 129 lb, 145 lb

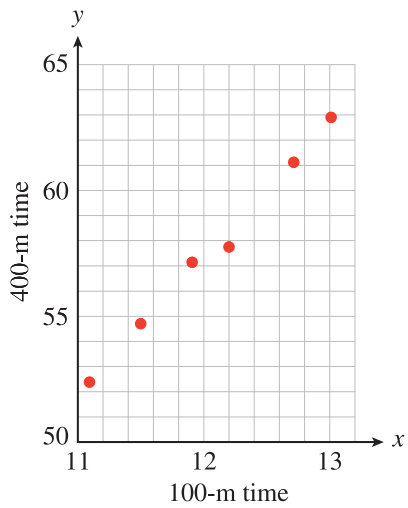
c $y = 2.\overline{6}x - 44.\overline{3}$

d 137 lb

e $y = 2.84x - 55.74$

f 137.33 lb

82. The scatterplot shows best times for various women running 400 meters



and 100 meters.

a Use a straightedge to draw a line that fits the data.

b Use your line to predict the 400-meter time of a woman who runs the 100-meter dash in 11.2 seconds and the 400-meter time of a woman who runs the 100-meter dash in 13.2 seconds.

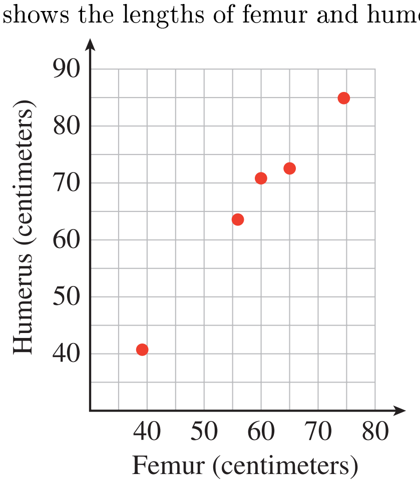
c Use your answers from part (b) to approximate the equation of a regression line.

d Use your answer to part (c) to predict the 400-meter time of a woman who runs the 100-meter dash in 12.1 seconds.

e The points on the scatterplot are $(11.1, 52.4)$, $(11.5, 54.7)$, $(11.9, 57.4)$, $(12.2, 57.9)$, $(12.7, 61.3)$, and $(13.0, 63.0)$. Use your calculator to find the least squares regression line.

f Use the regression line to predict the 400-meter time of a woman who runs the 100-meter dash in 12.1 seconds.

83. **Archaeopteryx** is an extinct creature with characteristics of both birds and reptiles. Only six fossil specimens are known, and only five of those include both a femur (leg bone) and a humerus (forearm bone) The scatterplot shows the lengths of femur and humerus for the five Archaeopteryx specimens.



a Use a straightedge to draw a line that fits the data.

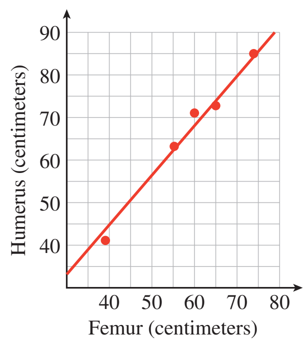
b Predict the humerus length of an Archaeopteryx whose femur is 40 centimeters c Predict the humerus length of an Archaeopteryx whose femur is 75 centimeters

d Use your answers from parts (b) and (c) to approximate the equation of a regression line.

e Use your answer to part (d) to predict the humerus length of an Archaeopteryx whose femur is 60 centimeters.

f Use your calculator and the given points on the scatterplot to find the least squares regression line. Compare the score this equation gives for part (d) with what you predicted earlier. The ordered pairs defining the data are $(38, 41)$, $(56, 63)$, $(59, 70)$, $(64, 72)$, $(74, 84)$.

Answer.



a

b 45 cm

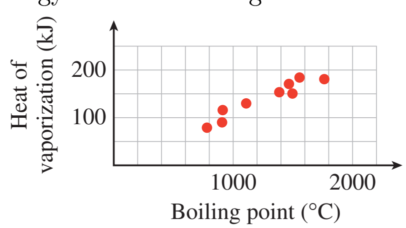
c 87 cm

d $y = 1.2x - 3$

e 69 cm

f $y = 1.197x - 3.660$; 68.16 cm

84. The scatterplot shows the boiling temperature of various substances on the horizontal axis and their heats of vaporization on the vertical axis. (The heat of vaporization is the energy needed to change the substance from liquid to gas at its boiling point.)



a Use a straightedge to estimate a line of best fit for the scatterplot.

b Use your line to predict the heat of vaporization of silver, whose boiling temperature is 2160°C.

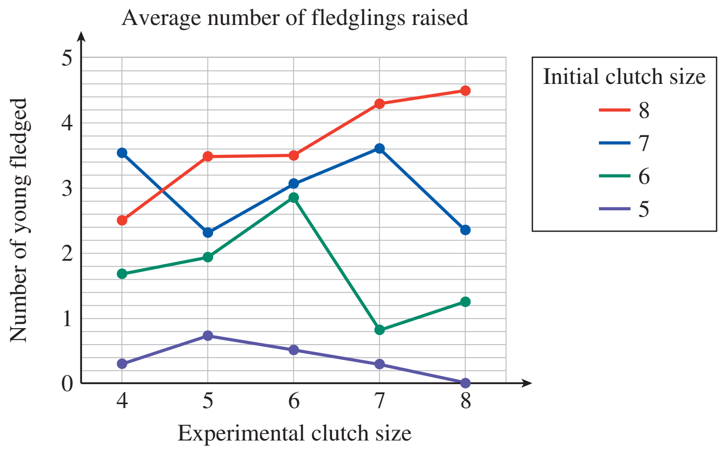
c Find the equation of the regression line.

d Use the regression line to predict the heat of vaporization of potassium bromide, whose boiling temperature is 1435°C.

1.8 Projects for Chapter 1

Project 1.2 (Optimal clutch size)**.** The number of eggs (clutch size) that a bird lays varies greatly. Is there an optimal clutch size for birds of a given species, or does it depend on the individual bird? In 1980, biologists in Sweden conducted an experiment on magpies as follows: They reduced or enlarged the natural clutch size by adding or removing eggs from the nests. They then computed the average number of fledglings successfully raised by the parent birds in each case. The graph shows the results for magpies that initially laid

5, 6, 7, or 8 eggs. (Source: Högstedt, 1980, via Krebs as developed in Davies,



1993)

a Use the graph to fill in the table of values for the number of fledglings raised in each situation.

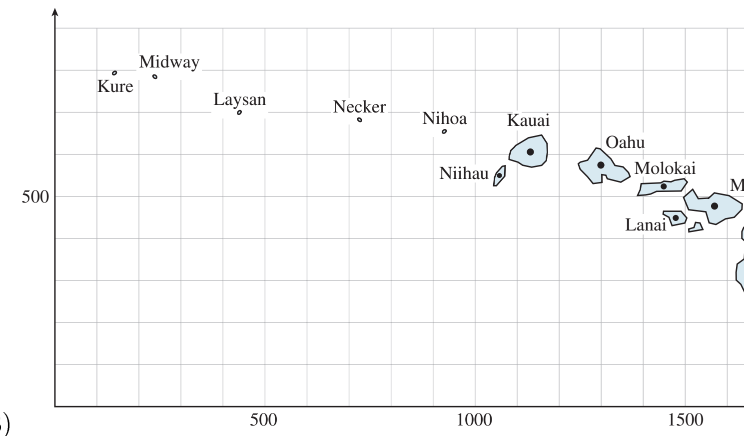
Initial clutch size laid	Experimental clutch size				
	4	5	6	7	8
5					
6					
7					
8					

b For each initial clutch size, which experimental clutch size produced the most fledglings? Record your answers in the table.

Initial clutch size	5	6	7	8
Optimum clutch size				

c What conclusions can you draw in response to the question in the problem?

Project 1.3 (Drift of Pacific tectonic plate)**.** The Big Island of Hawaii is the last island in a chain of islands and submarine mountain peaks that stretch almost 6000 kilometers across the Pacific Ocean. All are extinct volcanoes except for the Big Island itself, which is still active. The ages of the extinct peaks are roughly proportional to their distance from the Big Island. Geologists believe that the volcanic islands were formed as the tectonic plate drifted across a hot spot in the Earth's mantle. The figure shows a map of the islands, scaled in kilometers. (Source: Open University, 1998)



a The table gives the ages of the islands, in millions of years. Estimate the distance from each island to the Big Island, along a straight-line path through their centers. Fill in the third row of the table.

Island	Hawaii	Maui	Lanai	Molokai	Oahu	Kauai	Nihau	Nihoa	Necker	Laysan
Age	0.5	0.8	1.3	1.8	3.8	5.1	4.9	7.5	10	20
Distance										

b Make a scatterplot showing the age of each island along the horizontal axis and its distance from Hawaii on the vertical axis.

c Draw a line of best fit through the data.

d Calculate the slope of the line of best fit, including units.

e Explain why the slope provides an estimate for the speed of the Pacific plate.

Project 1.4 (Cross section of earth's surface)**.** The graph shows a cross section of Earth's surface along an east–west line from the coast of Africa through the Atlantic Ocean to South America. Both axes are scaled in kilometers. Use the figure to estimate the distances in this problem. (Source: Open University, 1998)

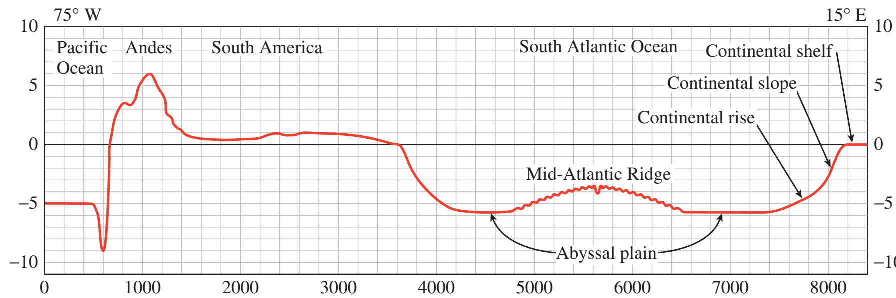
a What is the highest land elevation shown in the figure? What is the lowest ocean depth shown? Give the horizontal coordinates of these two points, in kilometers west of the 75°W longitude line.

b How deep is the Atlantic Ocean directly above the crest of the Mid-Atlantic Ridge? How deep is the ocean above the abyssal plain on either side of the ridge?

c What is the height of the Mid-Atlantic Ridge above the abyssal plain? What is the width of the Mid-Atlantic Ridge?

d Using your answers to part (c), calculate the slope from the abyssal plain to the crest of the Mid-Atlantic Ridge, rounded to five decimal places

e Estimate the slopes of the continental shelf, the continental slope, and the continental rise. Use the coordinates of the points indicated on the figure f Why do these slopes look much steeper in the accompanying figure than their numerical values suggest?



Project 1.5 (Mid-Atlantic Range)**.** The Mid-Atlantic Ridge is a mountain range on the sea floor beneath the Atlantic Ocean. It was discovered in the late nineteenth century during the laying of transatlantic telephone cables. The ridge is volcanic, and the ocean floor is moving away from the ridge on either side. Geologists have estimated the speed of this sea-floor spreading by recording the age of the rocks on the sea floor and their distance from the ridge. (The age of the rocks is calculated by measuring their magnetic polarity. At known intervals over the last four million years, the Earth reversed its polarity, and this information is encoded in the rocks.) (Source: Open University, 1998)

a According to the table, rocks that are 0.78 million years old have moved 17 kilometers from the ridge. What was the speed of spreading over the past 0.78 million years? (This is the rate of spreading closest to the ridge.)

b Plot the data in the table, with age on the horizontal axis and separation distance on the vertical axis. Draw a line of best fit through the data.

c Calculate the slope of the regression line. What are the units of the slope?

d The slope you calculated in part (c) represents the average spreading rate over the past 3.58 million years. Is the average rate greater or smaller than the rate of spreading closest to the ridge?

e Convert the average spreading rate to millimeters per year

Age (millions of years)	0.78	0.99	1.07	1.79	1.95	2.60	3.04	3.11	3.22	3.33	3.58
Distance (km)	17	18	21	32	39	48	58	59	62	65	66

Project 1.6 (Naismith's rule)**.** Naismith's rule is used by runners and walkers to estimate journey times in hilly terrain. In 1892, Naismith wrote in the *Scottish Mountaineering Club Journal* that a person "in fair condition should allow for easy expeditions an hour for every three miles on the map, with an additional hour for every 2000 feet of ascent." (Source: Scarf, 1998)

a According to Naismith, one unit of ascent requires the same time as how many units of horizontal travel? (Convert miles to feet.) This is called **Naismith's number**. Round your answer to one decimal place

b A walk in the Brecon Beacons in Wales covers 3.75 kilometers horizontally and climbs 582 meters. What is the equivalent flat distance?

c If you can walk at a pace of 15 minutes per kilometer over flat ground, how long will the walk in the Brecon Beacons take?

Project 1.7 (Improved Naismith's number)**.** Empirical investigations have improved Naismith's number (see Problem 5) to 8.0 for men and 9.5 for women. Part of the Karrimor International Mountain Marathon in the Arrochar Alps in Scotland has a choice of two routes. Route A is 1.75 kilometers long with a 240-meter climb, and route B is 3.25 kilometers long with a 90-meter climb. (Source: Scarf, 1998)

a Which route is faster for women?

b Which route is faster for men?

c At a pace of 6 minutes per flat kilometer, how much faster is the preferred route for women?

d At a pace of 6 minutes per flat kilometer, how much faster is the preferred route for men?

Appendix A

Algebra Skills Refresher

A.1 Numbers and Operations

A.1.1 Order of Operations

Numerical calculations often involve more than one operation. So that everyone agrees on how such expressions should be evaluated, we follow the order of operations.

Order of Operations

1 Simplify any expressions within grouping symbols (parentheses, brackets, square root bars, or fraction bars). Start with the innermost grouping symbols and work outward.

2 Evaluate all powers and roots.

3 Perform multiplications and divisions in order from left to right.

4 Perform additions and subtractions in order from left to right.

A.1.2 Parentheses and Fraction Bars

We can use parentheses to override the multiplication-first rule. Compare the two expressions below.

The sum of 4 times 6 and 10	$4 \cdot 6 + 10$
4 times the sum of 6 and 10	$4(6 + 10)$

In the first expression, we perform the multiplication 4×6 first, but in the second expression we perform the addition $6 + 10$ first, because it is enclosed in parentheses.

The location (or absence) of parentheses can drastically alter the meaning of an expression. In the following example, note how the location of the parentheses changes the value of the expression.

Example A.1.

a
$$\begin{aligned} 5 - 3 \cdot 4^2 &= 5 - 3 \cdot 16 \\ &= 5 - 48 = -43 \end{aligned}$$

b
$$\begin{aligned} 5 - (3 \cdot 4)^2 &= 5 - 12^2 \\ &= 5 - 144 = -139 \end{aligned}$$

c
$$\begin{aligned}(5-3\cdot 4)^2 &= (5-12)^2\\ &= (-7)^2 = 49\end{aligned}$$

d
$$\begin{aligned}(5-3)\cdot 4^2 &= 2\cdot 16\\ &= 32\end{aligned}$$

Caution A.2. In the expression $5-12^2$, which appears in Example **??**, the exponent 2 applies only to 12, not to -12. Thus, $5-12^2 \neq 5+144$.

The order of operations mentions other grouping devices besides parentheses: fraction bars and square root bars. Notice how the placement of the fraction bar affects the expressions in the next example.

Example A.3.

a
$$\begin{aligned}\frac{1+2}{3\cdot 4} &= \frac{3}{12}\\ &= \frac{1}{4}\end{aligned}$$

b
$$\begin{aligned}1+\frac{2}{3\cdot 4} &= 1+\frac{2}{12}\\ &= 1+\frac{1}{6} = \frac{7}{6}\end{aligned}$$

c
$$\begin{aligned}\frac{1+2}{3}\cdot 4 &= \frac{3}{3}\cdot 4\\ &= 1\cdot 4 = 4\end{aligned}$$

d
$$\begin{aligned}1+\frac{2}{3}\cdot 4 &= 1+\frac{8}{3}\\ &= \frac{3}{3}+\frac{8}{3} = \frac{11}{3}\end{aligned}$$

A.1.3 Radicals

You are already familiar with square roots. Every nonnegative number has two square roots, defined as follows:

$$s \quad \text{is a square root of} \quad n \quad \text{if} \quad s^2 = n$$

There are several other kinds of roots, one of which is called the **cube root**, denoted by $\sqrt[3]{n}$. We define the cube root as follows.

Cube Roots

$$b \quad \text{is a cube root of} \quad n \quad \text{if} \quad b \text{ cubed equals} \quad n.$$

In symbols, we write
$$b = \sqrt[3]{n} \quad \text{if} \quad b^3 = n$$

Although we cannot take the square root of a *negative number*, we can take the *cube root* of *any* real number. For example,

$$\sqrt[3]{64} = 4 \quad \text{because} \quad 4^3 = 64$$

and
$$\sqrt[3]{-27} = -3 \quad \text{because} \quad (-3)^3 = -27$$

In the order of operations, simplifying radicals and powers comes after parentheses but before products and quotients.

Example A.4. Simplify each expression.

a $3\sqrt[3]{-8}$

b $2 - \sqrt[3]{-125}$

c $\dfrac{6 - \sqrt[3]{-27}}{2}$

Solution.

a $3\sqrt[3]{-8} = 3(-2) = -6$

b $2 - \sqrt[3]{-125} = 2 - (-5) = 7$

c $\dfrac{6 - \sqrt[3]{-27}}{2} = \dfrac{6 - (-3)}{2} = \dfrac{9}{2}$

A.1.4 Scientific Notation

Scientists and engineers regularly encounter very large numbers such as

$$5,980,000,000,000,000,000,000,000$$

(the mass of the Earth in kilograms) and very small numbers such as

$$0.000\,000\,000\,000\,000\,000\,000\,001\,67$$

(the mass of a hydrogen atom in grams). These numbers can be written in a more compact and useful form by using powers of 10.

In our base 10 number system, multiplying a number by a positive power of 10 has the effect of moving the decimal place k places to the right, where k is the exponent in the power of 10. For example,

$$3.529 \times 10^2 = 352.9 \quad \text{and} \quad 25 \times 104 = 250,000$$

Multiplying by a power of 10 with a negative exponent moves the decimal place to the left. For example,

$$1728 \times 10^{-3} = 1.728 \quad \text{and} \quad 4.6 \times 10^{-5} = 0.000046$$

Using this property, we can write any number as the product of a number between 1 and 10 (including 1) and a power of 10. For example, the mass of the Earth and the mass of a hydrogen atom can be expressed as

$$5.98 \times 1024 \text{ kilograms} \qquad \text{and} \qquad 1.67 \times 10^{-24} \text{ gram}$$

respectively. A number written in this form is said to be expressed in **scientific notation**.

To Write a Number in Scientific Notation:

1 Locate the decimal point so that there is exactly one nonzero digit to its left.

2 Count the number of places you moved the decimal point: This determines the power of 10.

a If the original number is greater than 10, the exponent is positive.

b If the original number is less than 1, the exponent is negative.

Example A.5. Write each number in scientific notation.

a
$$\begin{aligned}478{,}000 &= 4.78000 \times 10^5\\ &= 4.78 \times 10^5\end{aligned}$$

b
$$\begin{aligned}0.00032 &= 00003.2 \times 10^{-4}\\ &= 3.2 \times 10^{-4}\end{aligned}$$

Example A.6. The average American eats 110 kilograms of meat per year. It takes about 16 kilograms of grain to produce 1 kilogram of meat, and advanced farming techniques can produce about 6000 kilograms of grain on each hectare of arable land. (The hectare is 10,000 square meters, or just under 21/2 acres.) Now, the total land area of the Earth is about 13 billion hectares, but only about 11% of that land is arable. Is it possible for each of the 5.5 billion people on Earth to eat as much meat as Americans do?

Solution. First we will compute the amount of meat necessary to feed every person on Earth 110 kilograms per year. There are 5.5×10^9 people on Earth.

$$(5.5 \times 10^9 \text{ people}) \times (110 \text{ kg/person}) = 6.05 \times 10^{11} \text{ kg of meat}$$

Next we will compute the amount of grain needed to produce that much meat.

$$(16 \text{ kg of grain/kg of meat}) \times (6.05 \times 10^{11} \text{ kg of meat}) = 9.68 \times 10^{12} \text{ kg of grain}$$

Next we will see how many hectares of land are needed to produce that much grain.

$$(9.68 \times 10^{12} \text{ kg of grain}) \div (6000 \text{ kg/hectare}) = 1.613 \times 10^9 \text{ hectares}$$

Finally, we will compute the amount of arable land available for grain production.

$$0.11 \times (13 \times 10^9 \text{ hectares}) = 1.43 \times 10^9 \text{ hectares}$$

Thus, even if we use every hectare of arable land to produce grain for livestock, we will not have enough to provide every person on Earth with 110 kilograms of meat per year.

A.2 Linear Equations and Inequalities

An **equation** is just a mathematical statement that two expressions are equal. Equations relating two variables are particularly useful. If we know the value of one of the variables, we can find the corresponding value of the other variable by solving the equation.

Example A.7. The equation $w = 6h$ gives Loren's wages, w, in terms of the number of hours she works, h. How many hours does Loren need to work next week if she wants to earn \$225?

Solution. We know that $w = 225$, and we would like to know the value of h. We substitute the value for w into our equation and then solve for h.

$$\begin{aligned} w &= 6h && \text{Substitute 225 for } w.\\ \mathbf{225} &= 6h && \text{Divide both sides by 6.}\\ \frac{225}{\mathbf{6}} &= \frac{6h}{\mathbf{6}} && \text{Simplify.}\end{aligned}$$

$$375.5 = h$$

Loren must work 37.5 hours in order to earn \$225. In reality, Loren will probably have to work for 38 hours, because most employers do not pay for portions of an hour's work. Thus, Loren needs to work for 38 hours.

To solve an equation we can generate simpler equations that have the same solutions. Equations that have identical solutions are called **equivalent equations**. For example,

$$3x - 5 = x + 3$$

and

$$2x = 8$$

are equivalent equations because the solution of each equation is 4. Often we can find simpler equivalent equations by undoing in reverse order the operations performed on the variable.

A.2.1 Solving Linear Equations

Linear, or first-degree, equations can be written so that every term is either a constant or a constant times the variable. The equations above are examples of linear equations. Recall the following rules for solving linear equations.

To Generate Equivalent Equations

1 We can add or subtract the *same* number on *both* sides of an equation.

2 We can multiply or divide *both* sides of an equation by the *same* number (except zero).

Applying either of these rules produces a new equation equivalent to the old one and thus preserves the solution.We use the rules to isolate the variable on one side of the equation.

Example A.8. Solve the equation $3x - 5 = x + 3$.

Solution. We first collect all the variable terms on one side of the equation, and the constant terms on the other side.

$3x - 5 \mathbf{- x} = x + 3 \mathbf{- x}$	Subtract x from both sides.
$2x - 5 = 3$	Simplify.
$2x - 5 \mathbf{+ 5} = 5 \mathbf{+ 5}$	Add 5 to both sides.
$2x = 8$	Simplify.
$\dfrac{2x}{\mathbf{2}} = \dfrac{8}{\mathbf{2}}$	Divide both sides by 2.
$x = 4$	Simplify.

The solution is 4. (You can check the solution by substituting 4 into the original equation to show that a true statement results.)

The following steps should enable you to solve any linear equation. Of course, you may not need all the steps for a particular equation.

To Solve a Linear Equation:

1 Simplify each side of the equation separately.

a Apply the distributive law to remove parentheses.

b Collect like terms.

2 By adding or subtracting appropriate terms to both sides of the equation, get all the variable terms on one side and all the constant terms on the other.

3 Divide both sides of the equation by the coefficient of the variable.

Example A.9. Solve $3(2x-5)-4x=2x-(6-3x)$.

Solution. We begin by simplifying each side of the equation.

$$\begin{aligned} 3(2x-5)-4x &= 2x-(6-3x) && \text{Apply the distributive law.} \\ 6x-15-4x &= 2x-6+3x && \text{Combine like terms on each side.} \\ 2x-15 &= 5x-6 \end{aligned}$$

Next, we collect all the variable terms on the left side of the equation, and all the constant terms on the right side.

$$\begin{aligned} 2x-15-5x+15 &= 5x-6-5x+15 && \text{Add } -5x+15 \text{ to both sides.} \\ -3x &= 9 \end{aligned}$$

Finally, we divide both sides of the equation by the coefficient of the variable.

$$\begin{aligned} -3x &= 9 && \text{Divide both sides by } -3. \\ x &= -3 \end{aligned}$$

The solution is -3.

A.2.2 Formulas

A **formula** is an equation that relates several variables. For example, the equation

$$P = 2l + 2w$$

gives the perimeter of a rectangle in terms of its length and width.

Suppose we have some wire fence to enclose an exercise area for rabbits, and we would like to see what dimensions are possible for different rectangles with that perimeter. In this case, it would be more useful to have a formula for the length of the rectangle in terms of its perimeter and its width. We can find such a formula by solving the perimeter formula for l in terms of P and w.

$$\begin{aligned} 2l+2w &= P && \text{Subtact } 2w \text{ from both sides.} \\ 2l &= P-2w && \text{Divide both sides by 2.} \\ l &= \frac{P-2w}{2} \end{aligned}$$

The result is a new formula that gives the length of a rectangle in terms of its perimeter and its width.

Example A.10. The formula $5F = 9C + 160$ relates the temperature in degrees Fahrenheit, F, to the temperature in degrees Celsius, C. Solve the formula for C in terms of F.

Solution. We begin by isolating the term that contains C.

$$\begin{aligned} 5F &= 9C + 160 && \text{Subtract 160 from both sides.} \\ 5F - 160 &= 9C && \text{Divide both sides by 9.} \\ \frac{5F - 160}{9} &= C \end{aligned}$$

We can also write the formula for C in terms of F as $C = \frac{5}{9}F - \frac{160}{9}$.

Example A.11. Solve $3x - 5y = 40$ for y in terms of x.

Solution. We isolate y on one side of the equation.

$$\begin{aligned} 3x - 5y &= 40 && \text{Subtract } 3x \text{ from both sides.} \\ -5y &= 40 - 3x && \text{Divide both sides by } -5. \\ \frac{-5y}{-5} &= \frac{40 - 3x}{-5} && \text{Simplify both sides.} \\ y &= -8 + \frac{3}{5}x \end{aligned}$$

A.2.3 Linear Inequalities

The symbol $>$ is called an **inequality symbol**, and the statement $a > b$ is called an **inequality**. There are four inequality symbols:

$>$	is greater than
$<$	is less than
$\geq$	is greater than or equal to
$\leq$	is less than or equal to

Inequalities that include the symbols $>$ or $\leq$ are called **strict inequalities**; those that include $\geq$ or $\leq$ are called **nonstrict**.

If we multiply or divide both sides of an inequality by a negative number, the direction of the inequality must be reversed. For example, if we multiply both sides of the inequality

$$2 < 5$$

by -3, we get

$$\begin{aligned} \mathbf{-3}(2) &> \mathbf{-3}(5) && \text{Change inequality symbol from } < \text{ to } >. \\ -6 &> -15. \end{aligned}$$

Because of this property, the rules for solving linear equations must be revised slightly for solving linear inequalities.

To Solve a Linear Inequality:

1 We may add or subtract the same number to both sides of an inequality without changing its solutions.

2 We may multiply or divide both sides of an inequality by a *positive* number without changing its solutions.

3 If we multiply or divide both sides of an inequality by a *negative* number, we must *reverse the direction of the inequality symbol.*

Example A.12. Solve the inequality $4 - 3x \geq -17$.

Solution. Use the rules above to isolate x on one side of the inequality.

$$\begin{aligned} 4 - 3x &\geq -17 && \text{Subtract 4 from both sides.} \\ -3x &\geq -21 && \text{Divide both sides by } -3. \\ x &\leq 7 \end{aligned}$$

Notice that we reversed the direction of the inequality when we divided by -3. Any number less than or equal to 7 is a solution of the inequality.

A **compound inequality** involves two inequality symbols.

Example A.13. Solve $4 \leq 3x + 10 \leq 16$.

Solution. We isolate x by performing the same operations on all three sides of the inequality.

$$\begin{aligned} 4 \leq 3x + 10 &\leq 16 && \text{Subtract 10.} \\ -6 \leq \quad 3x \quad &\leq 6 && \text{Divide by 3.} \\ -2 \leq \quad x \quad &\leq 2 \end{aligned}$$

The solutions are all numbers between -2 and 2, inclusive.

A.2.4 Interval Notation

The solutions of the inequality in Example **??** form an interval. An **interval** is a set that consists of all the real numbers between two numbers a and b.

The set $-2 \leq x \leq 2$ includes its endpoints -2 and 2, so we call it a **closed interval**, and we denote it by $[-2, 2]$ (see Figure **??**a). The square brackets tell us that the endpoints are included in the interval. An interval that does not include its endpoints, such as $-2 < x < 2$, is called an **open interval**, and we denote it with round brackets, $(-2, 2)$ (see Figure **??**b).

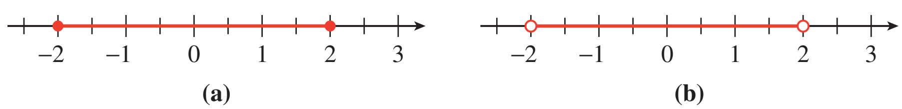


Figure A.14

Caution A.15. Do not confuse the open interval $(-2, 2)$ with the point $(-2, 2)$! The notation is the same, so you must decide from the context whether an interval or a point is being discussed.

We can also discuss **infinite intervals**, such as $x < 3$ and $x \geq -1$, shown in Figure **??**. We denote the interval $x < 3$ by $(-\infty, 3)$, and the interval $x \geq -1$ by $[-1, \infty)$. The symbol ∞, for infinity, does not represent a specific real number; it indicates that the interval continues forever along the real line.

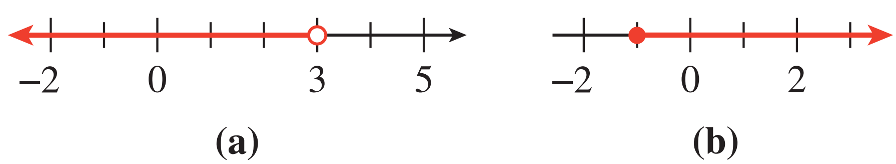


Figure A.16

Finally, we can combine two or more intervals into a larger set. For example, the set consisting of $x < -1$ or $x > 2$, shown in Figure **??**, is the **union** of two intervals and is denoted by $(-\infty, -2) \cup (2, \infty)$.

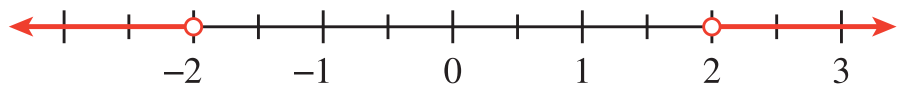


Figure A.17

Many solutions of inequalities are intervals or unions of intervals.

Example A.18. Write each of the solution sets with interval notation and graph the solution set on a number line.

a $3 \leq x < 6$

b $x \geq -9$

c $x \leq 1 \quad \text{or} \quad x > 4$

d $-8 < x \leq -5 \quad \text{or} \quad -1 \leq x < 3$

Solution.

a $[3, 6)$. This is called a **half-open** or **half-closed** interval. (See Figure **??**.)

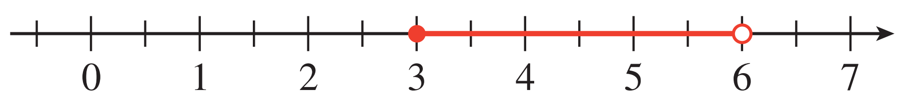


Figure A.19

b $[-9, \infty)$. We always use round brackets next to the symbol ∞ because ∞ is not a specific number and is not included in the set. (See Figure **??**.)

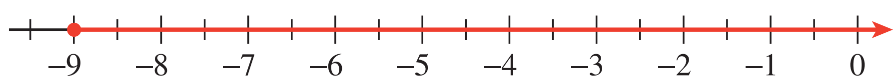


Figure A.20

c $(-\infty, 1] \cup (4, \infty)$. The word *or* describes the union of two sets. (See Figure **??**.)

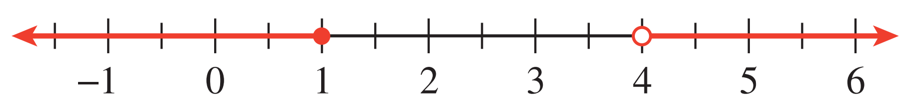


Figure A.21

d $(-8, -5] \cup [-1, 3)$. (See Figure **??**.)

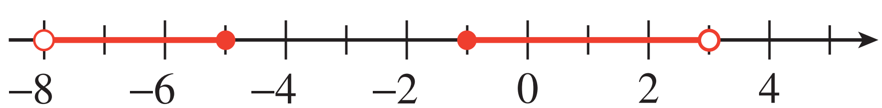


Figure A.22

A.3 Algebraic Expressions and Problem Solving

You are familiar with the use of letters, or **variables**, to stand for unknown numbers in equations or formulas. Variables are also used to represent numerical quantities that change over time or in different situations. For example, p might stand for the atmospheric pressure at different heights above the Earth's surface. Or N might represent the number of people infected with cholera t days after the start of an epidemic.

An **algebraic expression** is any meaningful combination of numbers, variables, and symbols of operation. Algebraic expressions are used to express relationships between variable quantities.

Example A.23. Loren makes \$6 an hour working at the campus bookstore.

a Choose a variable for the number of hours Loren works per week.

b Write an algebraic expression for the amount of Loren's weekly earnings.

Solution.

a Let h stand for the number of hours Loren works per week.

b The amount Loren earns is given by

$$6 \times (\text{number of hours Loren worked})$$

or $6 \cdot h$. Loren's weekly earnings can be expressed as $6h$.

The algebraic expression $6h$ represents the amount of money Loren earns *in terms of* the number of hours she works. If we substitute a specific value for the variable in an expression, we find a numerical value for the expression. This is called **evaluating** the expression.

Example A.24. If Loren from Example **??** works for 16 hours in the bookstore this week, how much will she earn?

Solution. Evaluate the expression $6h$ for $h = \mathbf{16}$.

$$6h = 6(\mathbf{16}) = 96$$

Loren will make \$96.

Example A.25. April sells environmentally friendly cleaning products. Her income consists of \$200 per week plus a commission of 9% of her sales.

a Choose variables to represent the unknown quantities and write an algebraic expression for April's weekly income in terms of her sales.

b Find April's income for a week in which she sells \$350 worth of cleaning products.

Solution.

a Let I represent April's total income for the week, and let S represent the total amount of her sales. We translate the information from the problem into mathematical language as follows:

$$\begin{array}{ccccccc} \text{Her income} & \text{consists of} & \$200... & \text{plus} & ...9\% & \text{of} & \text{her sales} \\ I & = & 200 & + & 0.09 & & S \end{array}$$

Thus, $I = 200 + 0.09S$.

b We want to evaluate our expression from part (a) with $S = 350$. We substitute **350** for S to find

$$I = 200 + 0.09(\mathbf{350})$$

Following the order of operations, we perform the multiplication before the addition. Thus, we begin by computing 0.09(350).

$$\begin{aligned} I &= 200 + 0.09(350) && \text{Multiply } 0.09(350) \text{ first.} \\ &= 200 + 31.5 \\ &= 231.50 \end{aligned}$$

April's income for the week is \$231.50.

Remark A.26 (Calculator Tip)**.** On a scientific or a graphing calculator, we can enter the expression from Example **??** just as it is written:

$$200 + 0.09 \boxed{\times} 350 \boxed{\text{ENTER}}$$

The calculator will perform the operations in the correct order—multiplication first.

Example A.27. Economy Parcel Service charges \$2.80 per pound to deliver a package from Pasadena to Cedar Rapids. Andrew wants to mail a painting that weighs 8.3 pounds, plus whatever packing material he uses.

a Choose variables to represent the unknown quantities and write an expression for the cost of shipping Andrew's painting.

b Find the shipping cost if Andrew uses 2.9 pounds of packing material.

Solution.

a Let C stand for the shipping cost and let w stand for the weight of the packing material. Andrew must find the total weight of his package first, then multiply by the shipping charge. The total weight of the package is $8.3 + w$ pounds. We use parentheses around this expression to show that it should be computed first, and the sum should be multiplied by the shipping charge of \$2.80 per pound. Thus,

$$C = 2.80(8.3 + w)$$

b Evaluate the formula from part (a) with $w = \mathbf{2.9}$.

$$\begin{aligned} C &= 2.80(8.3 + \mathbf{2.9}) && \text{Add inside parentheses.} \\ &= 2.80(11.2) && \text{Multiply.} \\ &= 31.36 \end{aligned}$$

The cost of shipping the painting is \$31.36.

Remark A.28 (Calculator Tip)**.** On a calculator, we enter the expression for C in the order it appears, including the parentheses. (Experiment to see whether your calculator requires you to enter the × symbol after 2.80.) The keying sequence

$$2.80 \times (8.3 + 2.9) \boxed{\text{ENTER}}$$

gives the correct result, 31.36.

Caution A.29. If we omit the parentheses, the calculator will perform the multiplication before the addition. Thus, the keying sequence

$$2.80 \times 8.3 + 2.9$$

gives an incorrect result for Example **??**. (The sequence

$$8.3 + 2.9 \times 2.80$$

does not work either!)

A.3.1 Problem Solving

Problem solving often involves translating a real-life problem into a computer programming language, or, in our case, into algebraic expressions. We can then use algebra to solve the mathematical problem and interpret the solution in the context of the original problem. Here are some guidelines for problem solving with algebraic equations.

Guidelines for Problem Solving

Step 1: Identify the unknown quantity and assign a variable to represent it.

Step 2: Find some quantity that can be expressed in two different ways and write an equation.

Step 3: Solve the equation.

Step 4: Interpret your solution to answer the question in the problem.

In step 1, begin by writing an English phrase to describe the quantity you are looking for. Be as specific as possible—if you are going to write an equation about this quantity, you must understand its properties! Remember that your variable must represent a numerical quantity. For example, x can represent the *speed* of a train, but not just "the train."

Writing an equation is the hardest part of the problem. Note that the quantity mentioned in step 2 will probably *not* be the same unknown quantity you are looking for, but the algebraic expressions you write *will* involve your variable. For example, if your variable represents the *speed* of a train, your equation might be about the *distance* the train traveled.

A.3.2 Supply and Demand

The law of supply and demand is fundamental in economics. If you increase the price of a product, the supply increases because its manufacturers are willing to provide more of the product, but the demand decreases because consumers are not willing to buy as much at a higher price. The price at which the demand for a product equals the supply is called the **equilibrium price**.

Example A.30. The Coffee Connection finds that when it charges p dollars for a pound of coffee, it can sell $800 - 60p$ pounds per month. On the other hand, at a price of p dollars a pound, International Food and Beverage will supply the Connection with $175 + 40p$ pounds of coffee per month. What price should the Coffee Connection charge for a pound of coffee so that its monthly inventory will sell out?

Solution.

Step 1: We are looking for the equilibrium price, p.

Step 2: The Coffee Connection would like the demand for its coffee to equal its supply. We equate the expressions for supply and for demand to obtain the equation

$$800 - 60p = 175 + 40p$$

Step 3: Solve the equation. To get all terms containing the variable, p, on one side of the equation, we add $60p$ to both sides and subtract 175 from both sides to obtain

$$\begin{aligned} 800 - 60p + \mathbf{60p - 175} &= 175 + 40p + \mathbf{60p - 175} \\ 625 &= 100p \qquad \text{Divide both sides by 100.} \\ 6.25 &= p \end{aligned}$$

Step 4: The Coffee Connection should charge \$6.25 per pound for its coffee.

A.3.3 Percent Problems

Recall the basic formula for computing percents.

Percent Formula

$$P = rW$$

the **P**art (or percent) = the percentage **r**ate × the **W**hole Amount

A **percent increase** or **percent decrease** is calculated as a fraction of the *original* amount. For example, suppose you make \$16.00 an hour now, but next month you are expecting a 5% raise. Your new salary should be

$$\overset{\text{Original salary}}{\$16.00} + \overset{\text{Increase}}{0.05(\$16.00)} = \overset{\text{New Salary}}{\$16.80}$$

Example A.31. The price of housing in urban areas increased 4% over the past year. If a certain house costs \$100, 000 today, what was its price last year?

Solution.

Step 1: Let c represent the cost of the house last year.

Step 2: Express the current price of the house in two different ways. During the past year, the price of the house increased by 4%, or $0.04c$. Its current price is thus

$$\overset{\text{Original cost}}{(1)c} + \overset{\text{Price increase}}{0.04c} = c(1 + 0.04) = 1.04c$$

This expression is equal to the value given for current price of the house:

$$1.04c = 100,000$$

Step 3: To solve this equation, we divide both sides by 1.04 to find

$$c = \frac{100,000}{1.04} = 96,153.846$$

Step 4: To the nearest cent, the cost of the house last year was \$96, 153.85.

Caution A.32. In Example **??**, it would be incorrect to calculate last year's price by subtracting 4% of \$100, 000 from \$100, 000 to get \$96, 000. (Do you see why?)

A.3.4 Weighted Averages

We find the **average**, or **mean**, of a set of values by adding up the values and dividing the sum by the number of values. Thus, the average, $\overline{x}$, of the numbers $x_1, x_2, \ldots, x_n$ is given by

$$\overline{x} = \frac{x_1 + x_2 + \quad + x_n}{n}$$

In a **weighted average**, the numbers being averaged occur with different frequencies or are weighted differently in their contribution to the average value. For instance, suppose a biology class of 12 students takes a 10-point quiz. Of the 12 students, 2 receive 10s, three receive 9s, 5 receive 8s, and 2 receive scores of 6. The average score earned on the quiz is then

$$\overline{x} = \frac{\mathbf{2}(10) + \mathbf{3}(9) + \mathbf{5}(8) + \mathbf{2}(6)}{12} = 8.25$$

The numbers in color are called the weights—in this example they represent the number of times each score was counted. Note that n, the total number of scores, is equal to the sum of the weights:

$$12 = 2 + 3 + 5 + 2$$

Example A.33. Kwan's grade in his accounting class will be computed as follows: Tests count for 50% of the grade, homework counts for 20%, and the final exam counts for 30%. If Kwan has an average of 84 on tests and 92 on homework, what score does he need on the final exam to earn a grade of 90?

Solution.

Step 1: Let x represent the final exam score Kwan needs.

Step 2: Kwan's grade is the weighted average of his test, homework, and final exam scores.

$$\frac{\mathbf{0.50}(84) + \mathbf{0.20}(92) + \mathbf{0.30}x}{1.00} = 90$$

(The sum of the weights is 1.00, or 100

$$0.50(84) + 0.20(92) + 0.30x = 1.00(90)$$

Step 3: Solve the equation. Simplify the left side first.

$$\begin{aligned} 60.4 + 0.30x &= 90 && \text{Subtract 60.4 from both sides.} \\ 0.30x &= 29.6 && \text{Divide both sides by 0.30.} \\ x &= 98.7 \end{aligned}$$

Step 4: Kwan needs a score of 98.7 on the final exam to earn a grade of 90.

In step 2 of Example **??**, we rewrote the formula for a weighted average in a simpler form.

Weighted Average

The sum of the weighted values equals the sum of the weights times the average value. In symbols,

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n = Wx$$

where W is the sum of the weights.

This form is particularly useful for solving problems involving mixtures.

Example A.34. The vet advised Delbert to feed his dog Rollo with kibble that is no more than 8% fat. Rollo likes JuicyBits, which are 15% fat. LeanMeal is more expensive, but it is only 5% fat. How much LeanMeal should Delbert mix with 50 pounds of JuicyBits to make amixture that is 8% fat?

Solution.

Step 1: Let p represent the number of pounds of LeanMeal needed.

Step 2: In this problem, we want the weighted average of the fat contents in the two kibbles to be 8%. The weights are the number of pounds of each kibble we use. It is often useful to summarize the given information in a table.

	% fat	Total pounds	Pounds of fat
Juicy Bits	15%	50	0.15(50)
LeanMeal	5%	p	0.05p
Mixture	8%	$50 + p$	0.08(50+p)

The amount of fat in the mixture must come from adding the amounts of fat in the two ingredients. This gives us an equation,

$$0.15(50) + 0.05p = 0.08(50 + p)$$

This equation is an example of the formula for weighted averages.

Step 3: Simplify each side of the equation, then solve.

$$\begin{aligned} 7.5 + 0.05p &= 4 + 0.08p \\ 3.5 &= 0.03p \\ p &= 116.\overline{6} \end{aligned}$$

Step 4: Delbert should mix $116\frac{2}{3}$ pounds of LeanMeal with 50 pounds of JuicyBits to make a mixture that is 8% fat.

A.4 Graphs and Equations

Graphs are useful tools for studying mathematical relationships. A graph provides an overview of a quantity of data, and it helps us identify trends or unexpected occurrences. Interpreting the graph can help us answer questions about the data.

For example, here are some data showing the atmospheric pressure at different altitudes. Altitude is given in feet, and atmospheric pressure is given in inches of mercury.

Altitude (ft)	0	5000	10,000	20,000	30,000	40,000	50,000
Pressure (in. Hg)	29.7	24.8	20.5	14.6	10.6	8.5	7.3

We observe a generally decreasing trend in pressure as the altitude increases, but it is difficult to say anything more precise about this relationship. A clearer picture emerges if we plot the data. To do this, we use two perpendicular number lines called axes. We use the horizontal axis for the values of the first variable, altitude, and the vertical axis for the values of the second variable, pressure.

The entries in the table are called **ordered pairs**, in which the **first component** is the altitude and the **second component** is the atmospheric pressure measured at that altitude. For example, the first two entries can be represented by $(0, 29.7)$ and $(5000, 24.8)$. We plot the points whose **coordinates** are given by the ordered pairs, as shown in Figure **??**a.

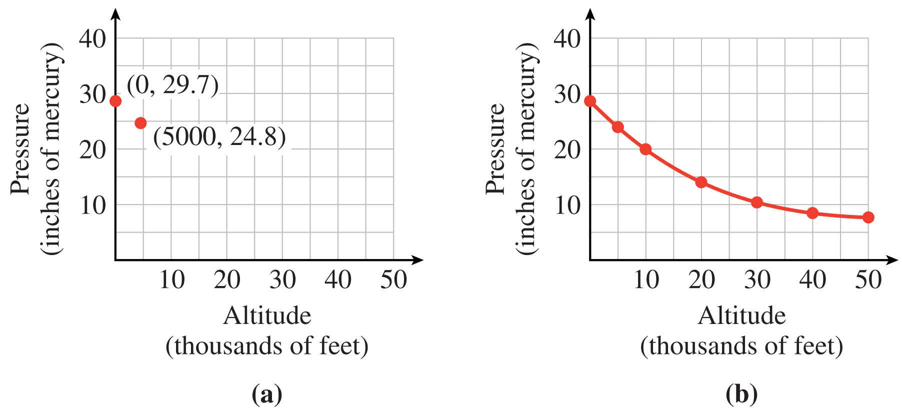


Figure A.35

We can connect the data points with a smooth curve as shown in Figure **??**b. In doing this, we are assuming that one variable changes smoothly with respect to the other, and in fact this is true for many physical situations. Thus, a smooth curve will thus serve as a good model.

A.4.1 Reading a Graph

Once we have constructed a graph, we can use it to estimate values of the variables between the known data points.

Example A.36. From the graph in Figure **??**b, estimate the following:

a The atmospheric pressure measured at an altitude of $15,000$ feet

b The altitude at which the pressure is 12 inches of mercury

Solution.

a The point with first coordinate $15,000$ on the graph in Figure **??** has second coordinate approximately 17.4. We estimate the pressure at $15,000$ feet to be 17.4 inches of mercury.

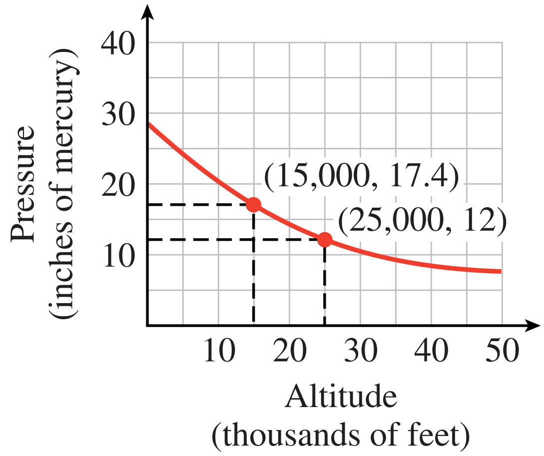


Figure A.37

b The point on the graph with second coordinate 12 has first coordinate approximately $25,000$, so an atmospheric pressure of 12 inches of mercury occurs at about $25,000$ feet.

We can also use the graph to obtain information about the relationship between altitude and pressure that would be difficult to see from the data alone.

Example A.38.

a For what altitudes is the pressure less than 18 inches of mercury?

b How much does the pressure decrease as the altitude increases from $15,000$ feet to $25,000$ feet?

c For which $10,000$-foot increase in altitude does the pressure change most rapidly?

Solution.

a From the graph in Figure **??**b, we see that the pressure has dropped to 18 inches of mercury at about $14,000$ feet, and that it continues to decrease as the altitude increases. Therefore, the pressure is less than 18 inches of mercury for altitudes greater than $14,000$ feet.

b The pressure at $15,000$ feet is approximately 17.4 inches of mercury, and at $25,000$ feet it is 12 inches. This represents a decrease in pressure of $17.4 - 12$, or 5.4, inches of mercury.

c By studying the graph we see that the pressure decreases most rapidly at low altitudes, so we conclude that the greatest drop in pressure occurs between 0 and $10,000$ feet.

A.4.2 Graphs of Equations

In Example **??**, we used a graph to illustrate data given in a table. Graphs can also help us analyze models given by equations. Let us first review some facts about solutions of equations in two variables.

An equation in two variables, such as $y = 2x + 3$, is said to be satisfied if the variables are replaced by a pair of numbers that make the statement true.

The pair of numbers is called a **solution** of the equation and is usually written as an ordered pair (x, y). (The first number in the pair is the value of x and the second number is the value of y.)

To find a solution of a given equation, we can assign a number to one of the variables and then solve for the second variable.

Example A.39. Find solutions to the equation $y = 2x + 3$.

Solution. We choose some values for x, say, -2, 0, and 1. Substitute these x-values into the equation to find a corresponding y-value for each.

$$\begin{aligned} &\text{When } x = -2, && y = 2(\mathbf{-2}) + 3 = -1 \\ &\text{When } x = 0, && y = 2(\mathbf{0}) + 3 = 3 \\ &\text{When } x = 1, && y = 2(\mathbf{1}) + 3 = 5 \end{aligned}$$

Thus, the ordered pairs $(-2, -1)$, $(0, 3)$, and $(1, 5)$ are three solutions of $y = 2x + 3$. We can also substitute values for y. For example, if we let $y = \mathbf{10}$, we have

$$\mathbf{10} = 2x + 3$$

Solving this equation for x, we find $7 = 2x$, or $x = 3.5$. This means that the ordered pair $(3.5, 10)$ is another solution of the equation $y = 2x + 3$.

An equation in two variables may have infinitely many solutions, so we cannot list them all. However, we can display the solutions on a graph. For this we use a **Cartesian** (or **rectangular**) **coordinate system**, as shown in Figure **??**.

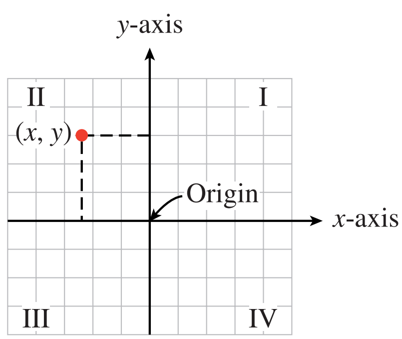


Figure A.40

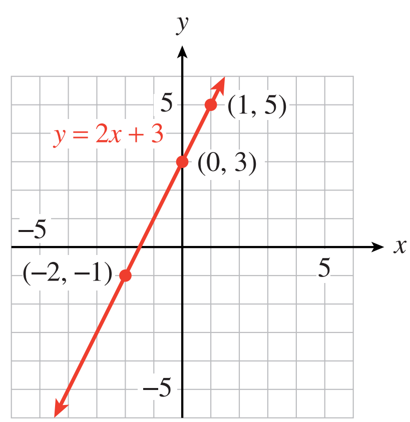


Figure A.41

The **graph of an equation** is a picture of its solutions. A point is included in the graph if its coordinates satisfy the equation, and if the coordinates do not satisfy the equation, the point is not part of the graph. A graph of $y = 2x + 3$ is shown in Figure **??**.

This graph does not display *all* the solutions of the equation, but it shows important features such as the intercepts on the x- and y-axes. Because there is a solution corresponding to every real number x, the graph extends infinitely in either direction, as indicated by the arrows.

Example A.42. Use the graph of $y = 0.5x^2 - 2$ in Figure **??** to decide whether the given ordered pairs are solutions of the equation. Verify your answers algebraically.

a $(-4, 6)$

b $(3, 0)$

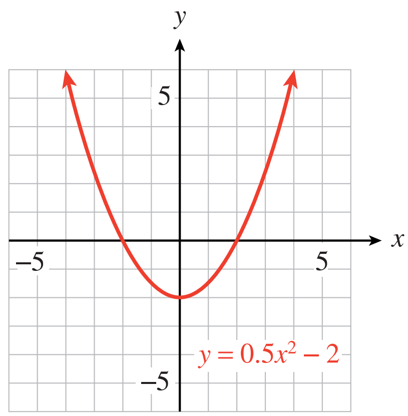


Figure A.43

Solution.

a Because the point $(-4, 6)$ does lie on the graph, the ordered pair $x = -4, y = 6$ is a solution of $y = 0.5x^2 - 2$. We can verify this by substituting $\mathbf{-4}$ for x and $\mathbf{6}$ for y:

$$\begin{aligned} 0.5(\mathbf{-4})^2 - 2 &= 0.5(16) - 2 \\ &= 8 - 2 = \mathbf{6} \end{aligned}$$

b Because the point $(3, 0)$ does not lie on the graph, the ordered pair $x = 3, y = 0$ is not a solution of $y = 0.5x^2 - 2$. We substitute $\mathbf{3}$ for x and $\mathbf{0}$ for y to verify this.

$$\begin{aligned} 0.5(\mathbf{3})^2 - 2 &= 0.5(9) - 2 \\ &= 4.5 - 2 = 2.5 \neq \mathbf{0} \end{aligned}$$

A.5 Linear Systems in Two Variables

A 2×2 **system** of equations is a set of 2 equations in the same 2 variables. A **solution** of a 2×2 system is an ordered pair that makes each equation in the system true. In this section, we review two algebraic methods for solving 2×2 linear systems: substitution and elimination.

A.5.1 Solving Systems by Substitution

The basic strategy for the **substitution** method can be described as follows.

Steps for Solving a System by Substitution

1 Solve one of the equations for one of the variables in terms of the other.

2 Substitute this expression into the second equation; doing so yields an equation in one variable.

3 Solve the new equation.

4 Use the result of step 1 to find the other variable.

Example A.44. Staci stocks two kinds of sleeping bags in her sporting goods store, a standard model and a down-filled model for colder temperatures. From past experience, she estimates that she will sell twice as many of the standard variety as of the down filled. She has room to stock 60 sleeping bags at a time. How many of each variety should Staci order?

Solution.

Step 1:

Number of standard sleeping bags: x

Number of down-filled sleeping bags: y

Step 2: Write two equations about the variables. Staci needs twice as many standard model as down filled, so

$$x = 2y \tag{1}$$

Also, the total number of sleeping bags is 60, so

$$x + y = 60 \tag{2}$$

Step 3: We will solve this system using substitution. Notice that Equation (1) is already solved for x in terms of y: $x = 2y$. Substitute $\mathbf{2y}$ for x in Equation (2) to obtain

$$\begin{aligned} \mathbf{2y} + y &= 60 \\ 3y &= 60 \end{aligned}$$

Solving for y, we find $y = \mathbf{20}$. Finally, substitute this value into Equation (1) to find

$$x = 2(\mathbf{20}) = 40$$

The solution to the system is $x = 40, y = 20$.

Step 4: Staci should order 40 standard sleeping bags and 20 down-filled bags.

A.5.2 Solving Systems by Elimination

The method of substitution is convenient if one of the variables in the system has a coefficient of 1 or -1, because it is easy to solve for that variable. If none of the coefficients is 1 or $-1s$, then a second method, called **elimination**, is usually more efficient.

The method of elimination is based on the following properties of linear equations.

Properties of Linear Systems

1 Multiplying a linear equation by a (nonzero) constant does not change its solutions. That is, any solution of the equation

$$ax + by = c$$

is also a solution of the equation

$$kax + kby = kc$$

2 Adding (or subtracting) two linear equations does not change their common solutions. That is, any solution of the system

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

is also a solution of the equation

$$(a_1 + a_2)x + (b_1 + b_2)y = c_1 + c_2$$

Example A.45. Solve the system by the method of elimination.

$$\begin{aligned} 2x + 3y &= 8 \qquad (1) \\ 3x - 4y &= -5 \qquad (2) \end{aligned}$$

Solution. We first decide which variable to eliminate, x or y. We can choose whichever looks easiest. In this problem, we choose to eliminate x. We next look for the smallest number that both coefficients, 2 and 3, divide into evenly. This number is 6. We want the coefficients of x to become 6 and -6, so we will multiply Equation (1) by 3 and Equation (2) by -2 to obtain

$$\begin{aligned} 6x + 9y &= 24 \qquad (1a) \\ -6x + 8y &= 10 \qquad (2a) \end{aligned}$$

Now add the corresponding terms of (1a) and (2a). The x-terms are eliminated, yielding an equation in one variable.

$$\begin{aligned} 6x + 9y &= 24 \qquad (1a) \\ -6x + 8y &= 10 \qquad (2a) \\ 17y &= 34 \qquad (3) \end{aligned}$$

Solve this equation for y to find $y = \mathbf{2}$. We can substitute this value of y into any of our equations involving both x and y. If we choose Equation (1), then

$$2x + 3(\mathbf{2}) = 8$$

and solving this equation yields $x = 1$. The ordered pair $(1, 2)$ is a solution to the system. You should verify that these values satisfy both original equations.

We summarize the strategy for solving a linear system by elimination.

Steps for Solving a 2×2 Linear System by Elimination

1 Choose one of the variables to eliminate. Multiply each equation by a suitable factor so that the coefficients of that variable are opposites.

2 Add the two new equations termwise.

3 Solve the resulting equation for the remaining variable.

4 Substitute the value found in step 3 into either of the original equations and solve for the other variable.

In Example **??**, we added 3 times the first equation to -2 times the second equation. The result from adding a constant multiple of one equation to a constant multiple of another equation is called a **linear combination** of the two equations. The method of elimination is also called the method of linear combinations.

If either equation in a system has fractional coefficients, it is helpful to clear the fractions before applying the method of linear combinations.

Example A.46. Solve the system by linear combinations.

$$\begin{aligned} \frac{2}{3}x - \quad y &= 2 && (1) \\ x + \frac{1}{2}y &= 7 && (2) \end{aligned}$$

Solution. Multiply each side of Equation (1) by 3 and each side of Equation (2) by 2 to clear the fractions:

$$\begin{aligned} 2x - 3y &= \ \ 6 && (1a) \\ 2x + \quad y &= 14 && (2a) \end{aligned}$$

To eliminate the variable x, multiply Equation (2a) by -1 and add the result to Equation (1a) to get

$$\begin{aligned} -4y &= -8 && \text{Divide both sides by -4.} \\ y &= 2 \end{aligned}$$

Substitute **2** for y in one of the original equations and solve for x. We use Equation (2).

$$\begin{aligned} x + \frac{1}{2}(\mathbf{2}) &= 7 && \text{Subtract 1 from both sides.} \\ x &= 6 \end{aligned}$$

Verify that $x = 6$ and $y = 2$ satisfy both Equations (1) and (2). The solution to the system is the ordered pair $(6, 2)$.

A.6 Laws of Exponents

In this section, we review the rules for performing operations on powers.

A.6.1 Product of Powers

Consider a product of two powers with the same base.

$$(a^3)(a^2) = aaa \cdot aa = a^5$$

because a occurs as a factor five times. The number of a's in the product is the *sum* of the number of a's in each factor.

First Law of Exponents: Product of Powers

To multiply two powers with the same base, add the exponents and leave the base unchanged.

$$a^m \cdot a^n = a^{m+n}$$

Example A.47.

a.

Add exponents.

$$5^3 \cdot 5^4 = 5^{3+4} = 5^7$$

Same base

b.

Add exponents.

$$x^4 \cdot x^2 = x^{4+2} = x^6$$

Same base

Here are some mistakes to avoid.

Caution A.48.

1 Note that we do not *multiply* the exponents when simplifying a product. For example,

$$b^4 \cdot b^2 \neq b^8$$

You can check this with your calculator by choosing a value for b, for instance, $b = 3$:

$$3^4 \cdot 3^2 \neq 3^8$$

2 In order to apply the first law of exponents, the bases must be the same. For example,

$$2^3 \cdot 3^5 \neq 6^8$$

(Check this on your calculator.)

3 We do not multiply the bases when simplifying a product. In Example **??**a, note that

$$5^3 \cdot 5^4 \neq 25^7$$

4 Although we can simplify the product x^2x^3 as x^5, we cannot simplify the sum $x^2 + x^3$, because x^2 and x^3 are not like terms.

Example A.49. Multiply $(-3x^4z^2)(5x^3z)$.

Solution. Rearrange the factors to group the numerical coefficients and the powers of each base. Apply the first law of exponents.

$$\begin{aligned}(-3x^4z^2)(5x^3z) &= (-3)(5)x^4x^3z^2z \\ &= -15x^7z^3\end{aligned}$$

A.6.2 Quotients of Powers

To reduce a fraction, we divide both numerator and denominator by any common factors.

$$cancelcolor\frac{x^7}{x^4} = \frac{xxx\cancel{x}\cancel{x}\cancel{x}\cancel{x}}{\cancel{x}\cancel{x}\cancel{x}\cancel{x}} = \frac{x^3}{1} = x^3$$

We can obtain the same result more quickly by *subtracting* the exponent of the denominator from the exponent of the numerator.

$$\frac{x^7}{x^4} = x^{7-4} = x^3$$

What if the larger power occurs in the denominator of the fraction?

$$\frac{x^4}{x^7} = \frac{blackxe/}{xx\,blackxe/} = \frac{1}{x^3}$$

In this case, we subtract the exponent of the numerator from the exponent of the denominator.

$$\frac{x^4}{x^7} = \frac{1}{x^{7-4}} = \frac{1}{x^3}$$

These examples suggest the following law.

Second Law of Exponents: Quotient of Powers

To divide two powers with the same base, subtract the smaller exponent from the larger one, keeping the same base.

1. If the larger exponent occurs in the numerator, put the power in the

numerator.

$$\text{If } m > n, \quad \text{then} \quad \frac{a^m}{a^n} = a^{m-n} \qquad (a \neq 0)$$

2. If the larger exponent occurs in the denominator, put the power in the denominator.

$$\text{If } m < n, \quad \text{then} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \qquad (a \neq 0)$$

Example A.50.

1. $\dfrac{3^8}{3^2} = 3^{8-2} = 3^6$ Subtract exponents: $8 > 2$.
2. $\dfrac{w^3}{w^6} = \dfrac{1}{w^{6-3}} = \dfrac{1}{w^3}$ Subtract exponents: $3 < 6$.

Example A.51. Divide $\dfrac{3x^2y^4}{6x^3y}$.

Solution. Consider the numerical coefficients and the powers of each variable separately. Use the second law of exponents to simplify each quotient of powers.

$$\begin{aligned} \frac{3x^2y^4}{6x^3y} &= \frac{3}{6} \cdot \frac{x^2}{x^3} \cdot \frac{y^4}{y} && \text{Subtract exponents.} \\ &= \frac{1}{2} \cdot \frac{1}{x^{3-2}} \cdot y^{4-1} \\ &= \frac{1}{2} \cdot \frac{1}{x} \cdot y^3 = \frac{y^3}{2x} \end{aligned}$$

A.6.3 Power of a Power

Consider the expression $\left(a^4\right)^3$, the third power of a^4.

$$\left(a^4\right)^3 = \left(a^4\right)\left(a^4\right)\left(a^4\right) = a^{4+4+4} = a^{12} \qquad \text{Add exponents.}$$

We can obtain the same result by multiplying the exponents together.

$$\left(a^4\right)^3 = a^{4 \cdot 3} = a^{12}$$

Third Law of Exponents: Power of a Power

To raise a power to a power, keep the same base and multiply the exponents.

$$(a^m)^n = a^{mn}$$

Example A.52.

Multiply exponents.

a. $(4^3)^5 = 4^{3 \cdot 5} = 4^{15}$

Multiply exponents.

b. $(y^5)^2 = y^{5 \cdot 2} = y^{10}$

Caution A.53. Notice the difference between the expressions

$$(x^3)(x^4) = x^{3+4} = x^7$$

and

$$\left(x^3\right)^4 = x^{3\cdot 4} = x^{12}$$

The first expression is a product, so we add the exponents. The second expression raises a power to a power, so we multiply the exponents.

A.6.4 Power of a Product

To simplify the expression $(5a)^3$, we use the associative and commutative laws to regroup the factors as follows.

$$\begin{aligned}(5a)^3 &= (5a)(5a)(5a)\\ &= 5\cdot 5\cdot 5\cdot a\cdot a\cdot a\\ &= 5^3a^3\end{aligned}$$

Thus, to raise a product to a power, we can simply raise each factor to the power.

Fourth Law of Exponents: Power of a Product

A power of a product is equal to the product of the powers of each of its factors.

$$(ab)^n = a^nb^n$$

Example A.54.

a $(5a)^3 = 5^3a^3 = 125a^3$ Cube each factor.

b $\left(-xy^2\right)^4 = (-x)^4\left(y^2\right)^4$ Raise each factor to the fourth power.

$= x^4y^8$ Apply the third law of exponents.

Caution A.55.

1 Compare the two expressions $3a^2$ and $(3a)^2$; they are not the same. In the expression $3a^2$, only the factor a is squared. But in $(3a)^2$, both 3 and a are squared. Thus,

$$3a^2 \quad \text{cannot be simplified}$$

but

$$(3a)^2 = 3^2a^2 = 9a^2$$

2 Compare the two expressions $(3a)^2$ and $(3+a)^2$. The fourth law of exponents applies to the *product* $3a$, but not to the *sum* $3+a$. Thus,

$$(3+a)^2 \neq 3^2 + a^2$$

In order to simplify $(3+a)^2$, we must expand the binomial product:

$$(3+a)^2 = (3+a)(3+a) = 9 + 6a + a^2$$

A.6.5 Power of a Quotient

To simplify the expression $\left(\frac{x}{3}\right)^4$, we multiply together 4 copies of the fraction $\frac{x}{3}$.

$$\begin{aligned}\left(\frac{x}{3}\right)^4 &= \frac{x}{3}\cdot\frac{x}{3}\cdot\frac{x}{3}\cdot\frac{x}{3} = \frac{x\cdot x\cdot x\cdot x}{3\cdot 3\cdot 3\cdot 3}\\ &= \frac{x^4}{3^4} = \frac{x^4}{81}\end{aligned}$$

In general, we have the following rule.

Fifth Law of Exponents: Power of a Quotient

To raise a quotient to a power, raise both the numerator and denominator to the power.

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

For reference, we state all of the laws of exponents together. All the laws are valid when a and b are not equal to zero and when the exponents m and n are whole numbers.

Laws of Exponents

I $a^m \cdot a^n = a^{m+n}$

II a $\frac{a^m}{a^n} = a^{m-n}$ $\quad m > n$

b $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ $\quad m < n$

III $(a^m)^n = a^{m+n}$

IV $(ab)^n = a^n b^n$

V $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example A.56. Simplify $5x^2y^3\left(2xy^2\right)^4$.

Solution. According to the order of operations, we should perform any powers before multiplications. Thus, we begin by simplifying $(2xy^2)^4$. We apply the fourth law.

$$\begin{aligned}5x^2y^3\left(2xy^2\right)^4 &= 5x^2y^3 2^4x^4\left(y^2\right)^4 && \text{Apply the fourth law.}\\ &= 5x^2y^3 2^4x^4y^8\end{aligned}$$

Finally, multiply powers with the same base. Apply the first law.

$$5x^2y^3\cdot 2^4x^4y^8 = 5\cdot 2^4x^2x^4y^3y^8 = 80x^6y^{11}$$

Example A.57. Simplify $\left(\frac{2x}{z^2}\right)^3$.

Solution. Begin by applying the fifth law.

$\left(\frac{2x}{z^2}\right)^3 = \frac{(2x)^3}{(z^2)^2}$Apply the fourth law to the numerator and the third law to the denominator.
$= \frac{2^3x^3}{z^6} = \frac{8x^3}{z^6}$

A.7 Polynomials and Factoring

In Section **??**, we used the first law of exponents to multiply two or more monomials. In this section, we review techniques for multiplying and factoring polynomials of several terms.

A.7.1 Polynomials

A **polynomial** is a sum of terms in which all the exponents on the variables are whole numbers and no variables appear in the denominator or under a radical. The expressions

$$0.1R^4, \quad d^2 + 32d - 21, \quad \text{and} \quad 128x^3 - 960x^2 + 8000$$

are all examples of polynomials in one variable.

An algebraic expression consisting of one term of the form cx^n, where c is a constant and n is a whole number, is called a **monomial**. For example,

$$y3, \quad -3x^8, \quad \text{and} \quad 0.1R^4$$

are monomials. A polynomial is just a sum of one or more monomials.

A polynomial with exactly two terms, such as $\frac{1}{2}n^2 + \frac{1}{2}n$, is called a **binomial**. A polynomial with exactly three terms, such as $d^2 + 32d - 21$ or $128x^3 - 960x^2 + 8000$, is called a **trinomial**. We have no special names for polynomials with more than three terms.

Example A.58. Which of the following expressions are polynomials?

a πr^2

b $23.4s^6 - 47.9s^4$

c $\frac{2}{3}w^3 - \frac{7}{3}w^2 + \frac{1}{3}w$

d $7 + m^{-2}$

e $\frac{x-2}{x+2}$

f $\sqrt[3]{4y}$

Solution. The first three are all polynomials. In fact, (a) is a monomial, (b) is a binomial, and (c) is a trinomial. The last three are not polynomials. The variable in (d) has a negative exponent, the variable in (e) occurs in the denominator, and the variable in (f) occurs under a radical.

In a polynomial containing only one variable, the greatest exponent that appears on the variable is called the **degree** of the polynomial. If there is no variable at all, then the polynomial is called a constant, and the degree of a constant is zero.

Example A.59. Give the degree of each polynomial.

a $b^3 - 3b^2 + 3b - 1$

b 10^{10}

c $-4w^3$

d $s^2 - s^6$

Solution.

a This is a polynomial in the variable b, and because the greatest exponent on b is 3, the degree of this polynomial is 3.

b This is a constant polynomial, so its degree is 0. (The exponent on a constant does not affect the degree.)

c This monomial has degree 3.

d This is a binomial of degree 6.

We can evaluate a polynomial just as we evaluate any other algebraic expression: We replace the variable with a number and simplify the result.

Example A.60. Let $p(x) = -2x^2 + 3x - 1$. Evaluate each of the following.

a $p(2)$

b $p(-1)$

c $p(t)$

d $p(t+3)$

Solution. In each case, we replace x by the given value.

a $p(\mathbf{2}) = -2(\mathbf{2})^2 + 3(\mathbf{2}) - 1 = -8 + 6 - 1 = -3$

b $p(\mathbf{-1}) = -2(\mathbf{-1})^2 + 3(\mathbf{-1}) - 1 = -2 + (-3) - 1 = -6$

c $p(\boldsymbol{t}) = -2(\boldsymbol{t})^2 + 3(\boldsymbol{t}) - 1 = - - 2t^2 + 3t - 1$

d
$$\begin{aligned} p(\boldsymbol{t+3}) &= -2(\boldsymbol{t+3})^2 + 3(\boldsymbol{t+3}) - 1 \\ &= -2(t^2 + 6t + 9) + 3(t+3) - 1 \\ &= -2t^2 - 9t - 10 \end{aligned}$$

A.7.2 Products of Polynomials

To multiply polynomials, we use a generalized form of the distributive property:

$$a(b + c + d + \cdots) = ab + ac + ad + \cdots$$

To multiply a polynomial by a monomial, we multiply each term of the polynomial by the monomial.

Example A.61.

a
$$\begin{aligned} 3x(x + y + z) &= 3x(x) + 3x(y) + 3x(z) \\ &= 3x^2 + 3xy + 3xz \end{aligned}$$

b
$$\begin{aligned} -2ab^2(3a^2 - ab + 2b^2) &= -2ab^2(3a^2) - 2ab^2(-ab) - 2ab^2(2b^2) \\ &= -6a^3b^2 + 2a^2b^3 - 4ab^4 \end{aligned}$$

A.7.3 Products of Binomials

Products of binomials occur so frequently that it is worthwhile to learn a shortcut for this type of multiplication. We can use the following scheme to perform the multiplication mentally. (See Figure **??**.)

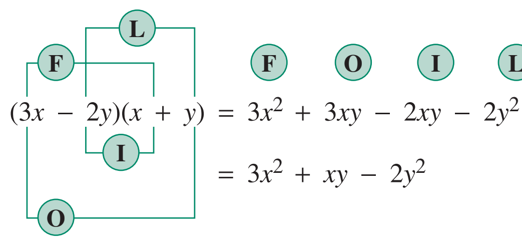


Figure A.62

Example A.63.

$$\begin{aligned}(2x-1)(x+3) &= 2x^2+6x-x-3\\ &= 2x^2+5x-3\end{aligned}$$

A.7.4 Factoring

We sometimes find it useful to write a polynomial as a single *term* composed of two or more *factors*. This process is the reverse of multiplication and is called **factoring**. For example, observe that

$$3x^2+6x = 3x(x+2)$$

We will only consider factorization in which the factors have integer coefficients.

A.7.5 Common Factors

We can factor a common factor from a polynomial by using the distributive property in the form

$$ab+ac = a(b+c)$$

We first identify the common factor. For example, each term of the polynomial

$$6x^3+9x^2-3x$$

contains the monomial $3x$ as a factor; therefore,

$$6x^3+9x^2-3x = 3x(\underline{\hspace{3cm}})$$

Next, we insert the proper polynomial factor within the parentheses. This factor can be determined by inspection. We ask ourselves for monomials that, when multiplied by $3x$, yield $6x^3$, $9x^2$, and $-3x$, respectively, and obtain

$$6x^3+9x^2-3x = 3x(2x^2+3x-1)$$

We can check the result of factoring an expression by multiplying the factors. In the example above,

$$3x(2x^2+3x-1) = 6x^3+9x^2-3x$$

Example A.64.

a
$$\begin{aligned} 18x^2y - 24xy2 &= 6xy(? - ?) \\ &= 6xy(3x - 4y) \end{aligned}$$

because
$$6xy(3x - 4y) = 18x^2y - 24xy^2$$

b
$$\begin{aligned} y(x-2) + z(x-2) &= (x-2)(? - ?) \\ &= (x-2)(y+z) \end{aligned}$$

because
$$(x-2)(y+z) = y(x-2) + z(x-2)$$

A.7.6 Opposite of a Binomial

It is often useful to factor -1 from the terms of a binomial.

$$\begin{aligned} a - b &= (-1)(-a + b) \\ &= (-1)(b - a) = -(b - a) \end{aligned}$$

Hence, we have the following important relationship.

Opposite of a Binomial

$$a - b = -(b - a)$$

That is, $a - b$ and $b - a$ are opposites or negatives of each other.

Example A.65.

a $3x - y = -(y - 3x)$

b $a - 2b = -(2b - a)$

A.7.7 Polynomial Division

We can divide one polynomial by a polynomial of lesser degree. The quotient will be the sum of a polynomial and a simpler algebraic fraction.

If the divisor is a monomial, we can simply divide the monomial into each term of the numerator.

Example A.66. Divide $\dfrac{9x^3 - 6x^2 + 4}{3x}$

Solution. Divide $3x$ into each term of the numerator.

$$\begin{aligned} \frac{9x^3 - 6x^2 + 4}{3x} &= \frac{9x^3}{3x} - \frac{6x^2}{3x} + \frac{4}{3x} \\ &= 3x^2 - 2x + \frac{4}{3x} \end{aligned}$$

The quotient is the sum of a polynomial, $3x^2 - 2x$, and an algebraic fraction, $\frac{4}{3x}$.

If the denominator is not a monomial, we can use a method similar to the long division algorithm used in arithmetic.

Example A.67. Divide $\dfrac{2x^2+x-7}{x+3}$

Solution. First write

$$x+3\overline{)2x^2+x-7}$$

and divide $2x^2$ (the first term of the numerator) by x (the first term of the denominator) to obtain $2x$. (It may be helpful to write down the division: $\frac{2x^2}{2x}=x$.) Write $2x$ above the quotient bar as the first term of the quotient, as shown below.

Next, multiply $x+3$ by $2x$ to obtain $2x^2+6x$, and subtract this product from $2x^2+x-7$:

$$\begin{array}{rl}
 & 2x \\
x+3 & \overline{)2x^2+\;\;x-7} \\
 & \underline{-(2x^2+6x)} \\
 & \qquad\;\; -5x-7
\end{array}$$

Repeating the process, divide $-5x$ by x to obtain -5. Write -5 as the second term of the quotient. Then multiply $x+3$ by -5 to obtain $-5x-15$, and subtract:

$$\begin{array}{rl}
 & 2x-5 \\
x+3 & \overline{)2x^2+x-7} \\
 & \underline{-(2x^2+6x)} \\
 & \qquad\;\; -5x-7 \\
 & \qquad -\underline{(-5x-15)} \\
 & \qquad\qquad\qquad 8
\end{array}$$

Because the degree of 8 is less than the degree of $x+3$, the division is finished. The quotient is $2x-5$, with a remainder of 8. We write the remainder as a fraction to obtain

$$\frac{2x^2+x-7}{x+3}=2x-5+\frac{8}{x+3}$$

When using polynomial division, it helps to write the polynomials in descending powers of the variable. If the numerator is missing any terms, we can insert terms with zero coefficients so that like powers will be aligned. For example, to perform the division

$$\frac{3x-1+4x^3}{2x-1}$$

we first write the numerator in descending powers as $4x^3+3x-1$. We then insert $0x^2$ between $4x^3$ and $3x$ and set up the quotient as

$$2x-1\overline{)4x^3+0x^2+3x-1}$$

We then proceed as in Example ??. You can check that the quotient is

$$2x^2 + x + 2 + \frac{1}{2x - 1}$$

A.8 Factoring Quadratic Trinomials

Consider the trinomial

$$x^2 + 10x + 16$$

Can we find two binomial factors,

$$(x + a)(x + b)$$

whose product is the given trinomial? The product of the binomials is

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

Thus, we are looking for two numbers, a and b, that satisfy

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab = x^2 + 10x + 16$$

By comparing the coefficients of the terms in the two trinomials, we see that $a + b = 10$ and $ab = 16$. That is, the sum of the two numbers is the coefficient of the linear term, 10, and their product is the constant term, 16.

To find the numbers, we list all the possible integer factorizations of 16:

$$1 \cdot 16, \quad 2 \cdot 8, \quad \text{and} \quad 4 \cdot 4$$

We see that only one combination gives the correct linear term: 8 and 2. These are the numbers a and b, so

$$x^2 + 10x + 16 = (x + 8)(x + 2)$$

In Example ?? we factor quadratic trinomials in which one or more of the coefficients is negative.

Example A.68. Factor.

a $x^2 - 7x + 12$

b $x^2 - x - 12$

Solution.

a Find two numbers whose product is 12 and whose sum is -7. Because the product is positive and the sum is negative, the two numbers must both be negative. The possible factors of 12 are -1 and -12, -2 and -6, or -3 and -4. Only -4 and -3 have the correct sum, -7. Hence,

$$x^2 - 7x + 12 = (x - 4)(x - 3)$$

b Find two numbers whose product is -12 and whose sum is -1. Because the product is negative, the two numbers must be of opposite sign and their sum must be -1. By listing the possible factors of -12, we find that the two numbers are -4 and 3. Hence,

$$x^2 - x - 12 = (x - 4)(x + 3)$$

If the coefficient of the quadratic term is not 1, we must also consider its factors.

Example A.69. Factor $8x^2 - 9 - 21x$.

Solution.

Step 1 Write the trinomial in decreasing powers of x.

$$8x^2 - 21x - 9$$

Step 2 List the possible factors for the quadratic term.

$$(8x \qquad)(x \qquad)$$
$$(4x \qquad)(2x \qquad)$$

Step 3 Consider possible factors for the constant term: 9 may be factored as $9 \cdot 1$ or as $3 \cdot 3$. Form all possible pairs of binomial factor using these factorizations. (See Figure **??**.)

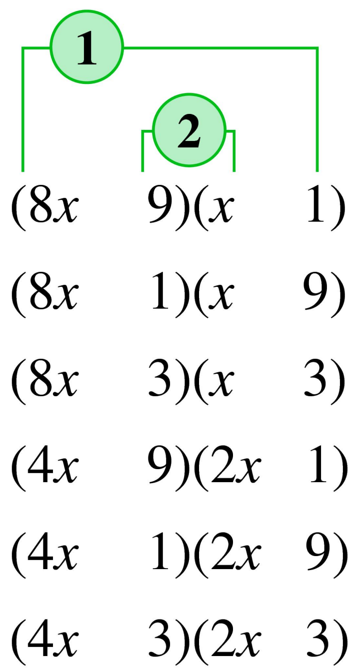


Figure A.70

Step 4 Select the combinations of the products and whose sum or difference could be the linear term, $-21x$.

$$(8x \quad 3)(x \quad 3)$$

Step 5 Insert the proper signs:

$$(8x + 3)(x - 3)$$

With practice, you can usually factor trinomials of the form $Ax^2 + Bx + C$ mentally. The following observations may help.

1 If A, B and C are all positive, both signs in the factored form are positive. For example, as a first step in factoring $6x^2 + 11x + 4$, we could write

$$(\quad + \quad)(\quad + \quad)$$

2 If A and C are positive and B is negative, both signs in the factored form are negative. Thus as a first step in factoring $6x^2 - 11x + 4$, we could write

$$(\quad - \quad)(\quad - \quad)$$

3 If C is negative, the signs in the factored form are opposite. Thus as a first step in factoring $6x^2 - 5x - 4$, we could write

$$(\quad + \quad)(\quad - \quad) \quad \text{or} \quad (\quad - \quad)(\quad + \quad)$$

Example A.71.

a
$$\begin{aligned} 6x^2 + 5x + 1 &= \quad + \quad)(\quad + \quad) \\ &= (3x+1)(2x+1) \end{aligned}$$

b
$$\begin{aligned} 6x^2 - 5x + 1 &= (\quad - \quad)(\quad - \quad) \\ &= (3x-1)(2x-1) \end{aligned}$$

c
$$\begin{aligned} 6x^2 - x - 1 &= (\quad + \quad)(\quad - \quad) \\ &= (3x+1)(2x-1) \end{aligned}$$

d
$$\begin{aligned} 6x^2 - xy - y^2 &= (\quad + \quad)(\quad - \quad) \\ &= (3x+y)(2x-y) \end{aligned}$$

A.8.1 Special Products and Factors

The products below are special cases of the multiplication of binomials. They occur so often that you should learn to recognize them on sight.

Special Products

I $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$

II $(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$

III $(a+b)(a-b) = a^2 - b^2$

Caution A.72. Notice that in (I) $(a+b)^2 \neq a^2+b^2$, and that in (II) $(a-b)^2 \neq a^2 - b^2$. For example,

$(x+4)^2 \neq x^2 + 16,$	instead	$(x+4)^2 = x^2 + 8x + 16$
$(t-5)^2 \neq t^2 - 16,$	instead	$(t-5)^2 = t^2 - 10t + 25$

Example A.73.

a
$$\begin{aligned} 3(x+4)^2 &= 3(x^2 + 2\cdot 4x + 4^2) \\ &= 3x^2 + 24x + 48 \end{aligned}$$

b
$$\begin{aligned} (y+5)(y-5) &= y^2 - 5^2 \\ &= y^2 - 25 \end{aligned}$$

c
$$\begin{aligned}(3x-2y)^2 &= (3x)^2 - 2(3x)(2y) + (2y)^2\\ &= 9x^2 - 12xy + 4y^2\end{aligned}$$

Each of the formulas for special products, when viewed from right to left, also represents a special case of factoring quadratic polynomials.

Special Factorizations

I $a^2 + 2ab + b^2 = (a+b)^2$

II $a^2 - 2ab + b^2 = (a-b)^2$

III $a^2 - b^2 = (a+b)(a-b)$

IV $a^2 + b^2$ cannot be factored

The trinomials in (I) and (II) are sometimes called **perfect-square trinomials** because they are squares of binomials. Note that the sum of two squares, $a^2 + b^2$, cannot be factored.

Example A.74. Factor.

a $x^2 + 8x + 16$

b $y^2 - 10y + 25$

c $4a^2 - 12ab + 9b^2$

d $25m^2n^2 + 20mn + 4$

Solution.

a Because 16 is equal to 4^2 and 8 is equal to $2 \cdot 4$,

$$\begin{aligned}x^2 + 8x + 16 &= x^2 - 2 \cdot 4x + 4^2\\ &= (x+4)^2\end{aligned}$$

b Because $25 = 5^2$ and $10 = 2 \cdot 5$,

$$\begin{aligned}y^2 - 10y + 25 &= y^2 - 2 \cdot 5y + 5^2\\ &= (y-5)^2\end{aligned}$$

c Because $4a^2 = (2a)^2$, $9b^2 = (3b)^2$, and $2ab = 2(2a)(3b)$,

$$\begin{aligned}4a^2 - 12ab + 9b^2 &= (2a)^2 - 2(2a)(3b) + (3b)^2\\ &= (2a-3b)^2\end{aligned}$$

d Because $25m^2n^2 = (5mn)^2$, $4 = 2^2$, and $20mn = 2(5mn)(2)$,

$$\begin{aligned}25m^2n^2 + 20mn + 4 &= (5mn)^2 + 2(5mn)(2) + 2^2\\ &= (5mn+2)^2\end{aligned}$$

Binomials of the form $a^2 - b^2$ are often called the **difference of two squares**.

Example A.75. Factor if possible.

a $x^2 - 81$

b $4x^2 - 9y^2$

c $x^2 + 81$

Solution.

a The expression $x^2 - 81$ is the difference of two squares, $x^2 - 9^2$, and thus can be factored according to Special Factorization (III) above.

$$\begin{aligned} x^2 - 81 &= x^2 - 9^2 \\ &= (x+9)(x-9) \end{aligned}$$

b Because $4x^2 - 9y^2$ can be written as $(2x)^2 - (3y)^2$,

$$\begin{aligned} 4x^2 - 9y^2 &= (2x^2) - (3y)^2 \\ &= (2x+3y)(2x-3y) \end{aligned}$$

c The expression $x^2 + 81$, or $x^2 + 0x + 81$, is *not* factorable, because no two real numbers have a product of 81 and a sum of 0.

Caution A.76. $x^2 + 81 \neq (x+9)(x+9)$, which you can verify by multiplying

$$(x+9)(x+9) = x^2 + 18x + 8$$

The factors $x + 9$ and $x - 9$ in Example **??**a are called **conjugates** of each other. In general, any binomials of the form $a + b$ and $a - b$ are called a **conjugate pair**.

A.9 Working with Algebraic Fractions

A quotient of two polynomials is called a **rational expression** or an **algebraic fraction**. Operations on algebraic fractions follow the same rules as operations on common fractions.

A.9.1 Reducing Fractions

When we reduce an ordinary fraction such as $\frac{24}{36}$, we are using the fundamental principle of fractions.

Fundamental Principle of Fractions

If we multiply or divide the numerator and denominator of a fraction by the same (nonzero) number, the new fraction is equivalent to the old one. In symbols,

$$\frac{ac}{bc} = \frac{a}{b}, \qquad (b,\ c \neq 0)$$

Thus, for example,

$$\frac{24}{36} = \frac{2 \cdot 12}{3 \cdot 12} = \frac{2}{3}$$

We use the same procedure to reduce algebraic fractions: We look for common factors in the numerator and denominator and then apply the fundamental principle.

Example A.77. Reduce each algebraic fraction.

a $\dfrac{8x^3y}{6x^2y^3}$

b $\dfrac{6x-3}{3}$

Solution. Factor out any common factors from the numerator and denominator. Then divide numerator and denominator by the common factors.

a
$$\begin{aligned}\frac{8x^3y}{6x^2y^3} &= \frac{4x\cdot 2x^2y}{3y^2\cdot 2x^2y}\\ &= \frac{4x}{3y^2}\end{aligned}$$

b
$$\begin{aligned}\frac{6x-3}{3} &= \frac{\cancel{3}(2x+1)}{\cancel{3}}\\ &= 2x+1\end{aligned}$$

If the numerator or denominator of the fraction contains more than one term, it is especially important to *factor* before attempting to apply the fundamental principle. We can divide out common *factors* from the numerator and denominator of a fraction, but the fundamental principle does *not* apply to common *terms*.

Caution A.78. We can reduce

$$\frac{2xy}{3y} = \frac{2x}{3}$$

because y is a common factor in the numerator and denominator. However,

$$\frac{2x+y}{3+y} \neq \frac{2x}{3}$$

because y is a common term but is *not a common factor* of the numerator and denominator. Furthermore,

$$\frac{5x+3}{5y} \neq \frac{x+3}{y}$$

because 5 is not a factor of the *entire* numerator.

Example A.79. Reduce each fraction.

a $\dfrac{4x+2}{4}$

b $\dfrac{9x^2+3}{6x+3}$

Solution. Factor the numerator and denominator. Then divide numerator and denominator by the common factors.

a
$$\begin{aligned}\frac{4x+2}{4} &= \frac{\cancel{2}(2x+1)}{\cancel{2}(2)}\\ &= \frac{2x+1}{2}\end{aligned}$$

b
$$\begin{aligned}\frac{9x^2+3}{6x+3} &= \frac{\cancel{3}(3x^2+1)}{\cancel{3}(2x+1)}\\ &= \frac{3x^2+1}{2x+1}\end{aligned}$$

Caution A.80. Note that in Example **??**a above,

$$\frac{4x+2}{4} \neq x+2$$

and in Example **??**b,

$$\frac{9x^2+3}{6x+3} \neq \frac{9x^2}{6x}$$

We summarize the procedure for reducing algebraic fractions as follows.

To Reduce an Algebraic Fraction:

1 Factor the numerator and denominator.

2 Divide the numerator and denominator by any common factors.

Example A.81. Reduce each fraction.

a $\dfrac{x^2-7x+6}{36-x^2}$

b $\dfrac{27x^3-1}{9x^2-1}$

Solution.

a Factor numerator and denominator to obtain

$$\frac{(x-6)(x-1)}{(6-x)(6+x)}$$

The factor $x-6$ in the numerator is the opposite of the factor $6-x$ in the denominator. That is, $x-6=-1(6-x)$. Thus,

$$\frac{-1\,blue\cancel{(6-x)}(x-1)}{blue\cancel{(6-x)}(6+x)} = \frac{-1(x-1)}{6+x} = \frac{1-x}{6+x}$$

b The numerator of the fraction is a difference of two cubes, and the denominator is a difference of two squares. Factor each to obtain

$$\frac{blue\cancel{(3x-1)}(9x^2+3x+1)}{blue\cancel{(3x-1)}(3x+1)} = \frac{9x^2+3x+1}{3x+1}$$

A.9.2 Products of Fractions

To multiply two or more common fractions together, we multiply their numerators together and multiply their denominators together. The same is true for a product of algebraic fractions. For example, xy

$$\frac{6x^2}{y}\cdot\frac{xy}{2} = \frac{6x^2\cdot}{y\cdot 2} = \frac{6x^3y}{2y} \qquad \text{Reduce.}$$

$$= \frac{3x^3(2y)}{2y} = 3x^3$$

We can simplify the process by first factoring each numerator and denominator and dividing out any common factors.

$$\frac{6x^2}{y}\cdot\frac{blue\cancel{2}\cdot 3x^2}{blue\cancel{y}}\cdot\frac{blue\,\cancel{xy}}{blue\,\cancel{2}} = 3x^3$$

In general, we have the following procedure for finding the product of algebraic fractions.

To Multiply Algebraic Fractions:

1 Factor each numerator and denominator.

2 Divide out any factors that appear in both a numerator and a denominator.

3 Multiply together the numerators; multiply together the denominators.

Example A.82. Find each product.

a $\dfrac{5}{x^2-1} \cdot \dfrac{x+2}{x}$

b $\dfrac{4y^2-1}{4-y^2} \cdot \dfrac{y^2-2y}{4y+2}$

Solution.

a The denominator of the first fraction factors into $(x+1)(x-1)$. There are no common factors to divide out, so we multiply the numerators together and multiply the denominators together.

$$\frac{5}{x^2-1} \cdot \frac{x+2}{x} = \frac{5(x+2)}{x(x^2-1)} = \frac{5x+10}{x^3-x}$$

b Factor each numerator and each denominator. Look for common factors.

$$\frac{4y^2-1}{4-y^2} \cdot \frac{y^2-2y}{4y+2} = \frac{(2y-1)\cancel{(2y+1)}}{\cancel{(2-y)}(2+y)} \cdot \frac{y\overset{-1}{\cancel{(y-2)}}}{2\cancel{(2y+1)}} \quad \text{Divide out common factors.}$$

$$= \frac{-y(2y-1)}{y(y+2)} \quad \text{Note: } y-2 = -(2-y)$$

A.9.3 Quotients of Fractions

To divide two algebraic fractions we multiply the first fraction by the reciprocal of the second fraction. For example,

$$\frac{2x^3}{3y} \div \frac{4x}{5y^2} = \frac{2x^3}{3y} \cdot \frac{5y^2}{4x}$$

$$= \frac{\cancel{2x}\,x^2}{3\cancel{y}} \cdot \frac{5y\,\cancel{y}}{2\cdot\cancel{2x}} = \frac{5x^2y}{6}$$

If the fractions involve polynomials of more than one term, we may need to factor each numerator and denominator in order to recognize any common factors. This suggests the following procedure for dividing algebraic fractions.

To Divide Algebraic Fractions:

1 Multiply the first fraction by the reciprocal of the second fraction.

2 Factor each numerator and denominator.

3 Divide out any factors that appear in both a numerator and a denominator.

4 Multiply together the numerators; multiply together the denominators.

Example A.83. Find each quotient.

a $\dfrac{x^2-1}{x+3} \div \dfrac{x^2-x-2}{x^2+5x+6}$

b $\dfrac{6ab}{2a+b} \div (4a^2b)$

Solution.

a Multiply the first fraction by the reciprocal of the second fraction.

$$\begin{aligned} \frac{x^2-1}{x+3} \div \frac{x^2-x-2}{x^2+5x+6} &= \frac{x^2-1}{x+3} \cdot \frac{x^2+5x+6}{x^2-x-2} && \text{Factor.} \\ &= \frac{(x-1)\cancel{(x+1)}}{\cancel{x+3}} \cdot \frac{\cancel{(x+3)}(x+2)}{\cancel{(x+1)}(x-2)} \\ &= \frac{(x-1)(x+2)}{x-2} \end{aligned}$$

b Multiply the first fraction by the reciprocal of the second fraction.

$$\begin{aligned} \frac{6ab}{2a+b} \div (4a^2b) &= \frac{\overset{3}{\cancel{6ab}}}{2a+b} \cdot \frac{1}{\underset{2}{\cancel{4a^2b}}} && \text{Divide out common factors.} \\ &= \frac{3}{2a(2a+b)} = \frac{3}{4a^2+2ab} \end{aligned}$$

A.9.4 Sums and Differences of Like Fractions

Algebraic fractions with the same denominator are called **like fractions**. To add or subtract like fractions, we combine their numerators and keep the same denominator for the sum or difference. This method is an application of the distributive law.

Example A.84. Find each sum or difference.

a $\dfrac{2x}{9z^2} + \dfrac{5x}{9z^2}$

b $\dfrac{2x-1}{x+3} - \dfrac{5x-3}{x+3}$

Solution.

a Because these are like fractions, we add their numerators and keep the same denominator.

$$\frac{2x}{9z^2} + \frac{5x}{9z^2} = \frac{2x+5x}{9z^2} = \frac{7x}{9z^2}$$

b Be careful to subtract the *entire* numerator of the second fraction: Use parentheses to show that the subtraction applies to both terms of $5x-3$.

$$\begin{aligned} \frac{2x-1}{x+3} - \frac{5x-3}{x+3} &= \frac{2x-1-(5x-3)}{x+3} \\ &= \frac{2x-1-5x+3}{x+3} = \frac{-3x+2}{x+3} \end{aligned}$$

A.9.5 Lowest Common Denominator

To add or subtract fractions with different denominators, we must first find a **common denominator**. For arithmetic fractions, we use the smallest natural number that is exactly divisible by each of the given denominators. For example, to add the fractions $\frac{1}{6}$ and $\frac{3}{8}$, we use 24 as the common denominator because 24 is the smallest natural number that both 6 and 8 divide into evenly.

We define the **lowest common denominator** (**LCD**) of two or more algebraic fractions as the polynomial of least degree that is exactly divisible by each of the given denominators.

Example A.85. Find the LCD for the fractions $\dfrac{3x}{x+2}$ and $\dfrac{2x}{x-3}$

Solution. The LCD is a polynomial that has as factors both $x+2$ and $x-3$. The simplest such polynomial is $(x+2)(x-3)$, or x^2-x-6. For our purposes, it will be more convenient to leave the LCD in factored form, so the LCD is $(x+2)(x-3)$.

The LCD in Example **??** was easy to find because each original denominator consisted of a single factor; that is, neither denominator could be factored. In that case, the LCD is just the product of the original denominators. We can always find a common denominator by multiplying together all the denominators in the given fractions, but this may not give us the *simplest* or *lowest* common denominator. Using anything other than the simplest possible common denominator will complicate our work needlessly.

If any of the denominators in the given fractions can be factored, we factor them before looking for the LCD.

To Find the LCD of Algebraic Fractions:

1 Factor each denominator completely.

2 Include each different factor in the LCD as many times as it occurs in any *one* of the given denominators.

Example A.86. Find the LCD for the fractions $\dfrac{2x}{x^2-1}$ and $\dfrac{x+3}{x^2+x}$.

Solution. Factor the denominators of each of the given fractions.

$$x^2-1=(x-1)(x+1) \quad \text{and} \quad x^2+x=x(x+1)$$

The factor $(x-1)$ occurs once in the first denominator, the factor x occurs once in the second denominator, and the factor $(x+1)$ occurs once in each denominator. Therefore, we include in our LCD one copy of each of these factors. The LCD is $x(x+1)(x-1)$.

Caution A.87. In Example **??**, we do not include two factors of $(x+1)$ in the LCD. We need only one factor of $(x+1)$ because $(x+1)$ occurs only once in either denominator. You should check that each original denominator divides evenly into our LCD, $x(x+1)(x-1)$.

A.9.6 Building Fractions

After finding the LCD, we **build** each fraction to an equivalent one with the LCD as its denominator. The new fractions will be like fractions, and we can combine them as explained above.

Building a fraction is the opposite of reducing a fraction, in the sense that we multiply, rather than divide, the numerator and denominator by an appropriate factor. To find the **building factor**, we compare the factors of the original denominator with those of the desired common denominator.

Example A.88. Build each of the fractions $\dfrac{3x}{x+2}$ and $\dfrac{2x}{x-3}$ to equivalent fractions with the LCD $(x+2)(x-3)$ as denominator.

Solution. Compare the denominator of the given fraction to the LCD. We see that the fraction $\dfrac{3x}{x+2}$ needs a factor of $(x-3)$ in its denominator, so $(x-3)$ is the building factor for the first fraction. We multiply the numerator and denominator of the first fraction by $(x-3)$ to obtain an equivalent fraction:

$$\frac{3x}{x+2} = \frac{3x\boldsymbol{(x-3)}}{(x+2)\boldsymbol{(x-3)}} = \frac{3x^2-9x}{x^2-x-6}$$

The fraction $\dfrac{2x}{x-3}$ needs a factor of $(x+2)$ in the denominator, so we multiply numerator and denominator by $(x+2)$:

$$\frac{2x}{x-3} = \frac{2x\boldsymbol{(x+2)}}{(x-3)\boldsymbol{(x+2)}} = \frac{2x^2+4x}{x^2-x-6}$$

The two new fractions we obtained in Example **??** are like fractions; they have the same denominator.

A.9.7 Sums and Differences of Unlike Fractions

We are now ready to add or subtract algebraic fractions with unlike denominators. We will do this in four steps.

To Add or Subtract Fractions with Unlike Denominators:

1 Find the LCD for the given fractions.

2 Build each fraction to an equivalent fraction with the LCD as its denominator.

3 Add or subtract the numerators of the resulting like fractions. Use the LCD as the denominator of the sum or difference.

4 Reduce the sum or difference, if possible.

Example A.89. Subtract $\dfrac{3x}{x+2} - \dfrac{2x}{x-3}$.

Solution.

Step 1 The LCD for these fractions is $(x+2)(x-3)$.

Step 2 We build each fraction to an equivalent one with the LCD, as we did in Example **??**.

$$\frac{3x}{x+2} = \frac{3x^2-9x}{x^2-x-6} \quad \text{and} \quad \frac{2x}{x-3} = \frac{2x^2+4x}{x^2-x-6}$$

Step 3 Combine the numerators over the same denominator.

$$\begin{aligned}\frac{3x}{x+2}-\frac{2x}{x-3} &= \frac{3x^2-9x}{x^2-x-6}-\frac{2x^2+4x}{x^2-x-6} \quad \text{Subtract the numerators.}\\ &= \frac{(3x^2-9x)-(2x^2+4x)}{x^2-x-6}\\ &= \frac{x^2-13x}{x^2-x-6}\end{aligned}$$

Step 4 Reduce the result, if possible. If we factor both numerator and denominator, we find

$$\frac{x(x-13)}{(x-3)(x+2)}$$

The fraction cannot be reduced.

Example A.90. Write as a single fraction: $1+\frac{2}{a}-\frac{a^2+2}{a^2+a}$.

Solution.

Step 1 To find the LCD, factor each denominator:

$$\begin{aligned}a &= a\\ a^2+a &= a(a+1)\end{aligned}$$

The LCD is $a(a+1)$.

Step 2 Build each term to an equivalent fraction with the LCD as denominator. (The building factors for each fraction are shown in color.) The third fraction already has the LCD for its denominator.

$$\begin{aligned}1 = \frac{1\cdot \boldsymbol{a(a+1)}}{1\cdot \boldsymbol{a(a+1)}} &= \frac{a^2+a}{a(a+1)}\\ \frac{2}{a} = \frac{2\cdot \boldsymbol{(a+1)}}{a\cdot \boldsymbol{(a+1)}} &= \frac{2a+2}{a(a+1)}\\ \frac{a^2+2}{a^2+a} &= \frac{a^2+2}{a(a+1)}\end{aligned}$$

Step 3 Combine the numerators over the LCD.

$$\begin{aligned}1+\frac{2}{a}-\frac{a^2+2}{a^2+a} &= \frac{a^2+a}{a(a+1)}+\frac{2a+2}{a(a+1)}-\frac{a^2+2}{a(a+1)}\\ &= \frac{a^2+a+(2a+2)-(a^2+2)}{a(a+1)}\\ &= \frac{3a}{a(a+1)}\end{aligned}$$

Step 4 Reduce the fraction to find

$$\frac{3\cancel{a}}{\cancel{a}(a+1)} = \frac{3}{a+1}$$

A.9.8 Complex

A fraction that contains one or more fractions in either its numerator or its denominator or both is called a complex fraction. For example,

$$\frac{\frac{2}{3}}{\frac{5}{6}} \quad \text{and} \quad \frac{x+\frac{3}{4}}{x-\frac{1}{2}}$$

are complex fractions. Like simple fractions, complex fractions represent quotients. For the examples above,

$$\frac{\frac{2}{3}}{\frac{5}{6}} = \frac{2}{3} \div \frac{5}{6} \quad \text{and} \quad \frac{x+\frac{3}{4}}{x-\frac{1}{2}} = \left(x+\frac{3}{4}\right) \div \left(x-\frac{1}{2}\right)$$

We can always simplify a complex fraction into a standard algebraic fraction. If the denominator of the complex fraction is a single term, we can treat the fraction as a division problem and multiply the numerator by the reciprocal of the denominator. Thus,

$$\frac{\frac{2}{3}}{\frac{5}{6}} = \frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \cdot \frac{6}{5} = \frac{4}{5}$$

If the numerator or denominator of the complex fraction contains more than one term, it is easier to use the fundamental principle of fractions to simplify the expression.

Example A.91. Simplify $\dfrac{x+\frac{3}{4}}{x-\frac{1}{2}}$

Solution. Consider all of the simple fractions that appear in the complex fraction; in this example $\frac{1}{2}$ and $\frac{3}{4}$. The LCD of these fractions is 4. If we multiply the numerator and denominator of the complex fraction by 4, we will eliminate the fractions within the fraction. Be sure to multiply *each* term of the numerator and *each* term of the denominator by 4.

$$\frac{4\left(x+\frac{3}{4}\right)}{4\left(x-\frac{1}{2}\right)} = \frac{4(x)+4\left(\frac{3}{4}\right)}{4(x)-4\left(\frac{1}{2}\right)} = \frac{4x+3}{4x-2}$$

Thus, the original complex fraction is equivalent to the simple fraction $\dfrac{4x+3}{4x-2}$.

We summarize the method for simplifying complex fractions as follows.

To Simplify a Complex Fraction:

1 Find the LCD of all the fraction contained in the complex fraction.

2 Multiply the numerator and the denominator of the complex fraction by the LCD.

3 Reduce the resulting simple fraction, if possible.

A.9.9 Negative Exponents

Algebraic fractions are sometimes written using negative exponents. (You can review negative exponents in ⟨⟨Unresolved xref, reference "Variation"; check spelling or use "provisional" attribute⟩⟩.)

Example A.92. Write each expression as a single algebraic fraction.

a $x^{-1} - y^{-1}$

b $\left(x^{-2} + y^{-2}\right)^{-1}$

Solution.

a $x^{-1} - y^{-1} = \dfrac{1}{x} - \dfrac{1}{y} \quad \text{or} \quad \dfrac{y-x}{xy}$

b $\left(x^{-2} + y^{-2}\right)^{-1} = \left(\dfrac{1}{x^2} + \dfrac{1}{y^2}\right)^{-1} = \left(\dfrac{y^2+x^2}{x^2y^2}\right)^{-1} = \dfrac{x^2y^2}{y^2+x^2}$

When working with fractions and exponents, it is important to avoid some tempting but *incorrect* algebraic operations.

Caution A.93.

1 In Example **??**a, note that

$$\frac{1}{x} - \frac{1}{y} \neq \frac{1}{x-y}$$

For example, you can check that for $x = 2$ and $y = 3$,

$$\frac{1}{2} - \frac{1}{3} \neq \frac{1}{2-3} = -1$$

2 In Example **??**b, note that

$$\left(x^{-2} + y^{-2}\right)^{-1} \neq x^2 + y^2$$

In general, the fourth law of exponents does *not* apply to sums and differences; that is,

$$(a+b)^n \neq a^n + b^n$$

A.10 Working with Radicals

In some situations, radical notation is more convenient to use than exponents. In these cases, we usually simplify radical expressions algebraically as much as possible before using a calculator to obtain decimal approximations.

A.10.1 Properties of Radicals

Because $\sqrt[n]{a} = a^{1/n}$, we can use the laws of exponents to derive two important properties that are useful in simplifying radicals.

Properties of Radicals

1 $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$, for $a, b \geq 0$

2 $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$, for $a \geq 0, \quad b > 0$

As examples, you can verify that

$$\sqrt{36} = \sqrt{4}\sqrt{9} \quad \text{and} \quad \sqrt[3]{\frac{1}{8}} = \frac{\sqrt[3]{1}}{\sqrt[3]{8}}$$

Example A.94. Which of the following are true?

a Is $\sqrt{36+64} = \sqrt{36} + \sqrt{64}$?

b Is $\sqrt[3]{8(64)} = \sqrt[3]{8}\sqrt[3]{64}$?

c Is $\sqrt{x^2+4} = x+2$?

d Is $\sqrt[3]{8x^3} = 2x$?

Solution. The statements in (b) and (d) are true, and both are examples of the first property of radicals. Statements (a) and (c) are false. In general, $\sqrt[n]{a+b}$ is not equal to $\sqrt[n]{a} + \sqrt[n]{b}$, and $\sqrt[n]{a-b}$ is not equal to $\sqrt[n]{a} - \sqrt[n]{b}$.

A.10.2 Simplifying Radicals

We use Property (1) to simplify radical expressions by factoring the radicand. For example, to simplify $\sqrt[3]{108}$, we look for perfect cubes that divide evenly into 108. The easiest way to do this is to try the perfect cubes in order: $1, 8, 27, 64, 125, \ldots$ and so on, until we find one that is a factor. For this example, we find that $108 = 274$. Using Property (1), we write

$$\sqrt[3]{108} = \sqrt[3]{27}\sqrt[3]{4}$$

Simplify the first factor to find

$$\sqrt[3]{108} = 3\sqrt[3]{4}$$

This expression is considered simpler than the original radical because the new radicand, 4, is smaller than the original, 108.

We can also simplify radicals containing variables. If the exponent on the variable is a multiple of the index, we can extract the variable from the radical. For instance,

$$\sqrt[3]{12} = x^{12/3} = x^4$$

(You can verify this by noting that $(x^4)^3 = x^{12}$.) If the exponent on the variable is not a multiple of the index, we factor out the highest power that is a multiple. For example,

$$\begin{aligned} \sqrt[3]{x^{11}} &= \sqrt[3]{x^9 \cdot x^2} && \text{Apply Property (1).} \\ &= \sqrt[3]{x^9} \cdot \sqrt[3]{2} && \text{Simplify } \sqrt[3]{x^9} = x^{9/3}. \end{aligned}$$

Example A.95. Simplify each radical.

a $\sqrt{18x^5}$

b $\sqrt[3]{24x^6y^8}$

Solution.

a The index of the radical is 2, so we look for perfect square factors of $18x^5$. The factor 9 is a perfect square, and x^4 has an exponent divisible by 2. Thus,

$$\begin{aligned}\sqrt{18x^5} &= \sqrt{9x^4 \cdot 2x} && \text{Apply Property (1).}\\ &= \sqrt{9x^4}\sqrt{2x} && \text{Take square roots.}\\ &= 3x^2\sqrt{2x}\end{aligned}$$

b The index of the radical is 3, so we look for perfect cube factors of $24x^6y^8$. The factor 8 is a perfect cube, and x^6 and y^6 have exponents divisible by 3. Thus,

$$\begin{aligned}\sqrt[3]{24x^6y^8} &= \sqrt[3]{8x^6y^6 \cdot 3y^2} && \text{Apply Property (1).}\\ &= \sqrt[3]{8x^6y^6}\,\sqrt[3]{3y^2} && \text{Take cube roots.}\\ &= 2x^2y^2\,\sqrt[3]{3y^2}\end{aligned}$$

Caution A.96. Property (1) applies only to products under the radical, not to sums or differences. Thus, for example,

$$\sqrt{4 \cdot 9} = \sqrt{4}\sqrt{9} = 2 \cdot 3, \quad \text{but} \quad \sqrt{4+9} \neq \sqrt{4} + \sqrt{9}$$

and

$$\sqrt[3]{x^3y^6} = \sqrt[3]{x^3}\,\sqrt[3]{y^6} = xy^2, \quad \text{but} \quad \sqrt[3]{x^3 - y^6} \neq \sqrt[3]{x^3} - \sqrt[3]{y^6}$$

To simplify roots of fractions, we use Property (2), which allows us to write the expression as a quotient of two radicals.

Example A.97.

a $\sqrt{\dfrac{3}{4}} = \dfrac{\sqrt{3}}{\sqrt{4}} = \dfrac{\sqrt{3}}{2}$

b $\sqrt[3]{\dfrac{5}{8}} = \dfrac{\sqrt[3]{5}}{\sqrt[3]{8}} = \dfrac{\sqrt[3]{5}}{2}$

We can also use Properties (1) and (2) to simplify products and quotients of radicals.

Example A.98. Simplify.

a $\sqrt[4]{6x^2}\,\sqrt[4]{8x^3}$

b $\dfrac{\sqrt[3]{16y^5}}{\sqrt[3]{y^2}}$

Solution.

a First apply Property (1) to write the product as a single radical, then simplify.

$$\begin{aligned}\sqrt[4]{6x^2}\,\sqrt[4]{8x^3} &= \sqrt[4]{48x^5} && \text{Factor out perfect fourth powers.}\\ &= \sqrt[4]{16x^4}\,\sqrt[4]{3x} && \text{Simplify.}\\ &= 2x\sqrt[4]{3x}\end{aligned}$$

b Apply Property (2) to write the quotient as a single radical.

$$\begin{aligned}
\frac{\sqrt[3]{16y^5}}{\sqrt[3]{y^2}} &= \sqrt[3]{\frac{16y^5}{y^2}} && \text{Reduce.}\\
&= \sqrt[3]{16y^3} && \text{Simplify: factor out perfect cubes.}\\
&= \sqrt[3]{8y^3}\,\sqrt[3]{2}\\
&= 2y\sqrt[3]{2}
\end{aligned}$$

A.10.3 Sums and Differences of Radicals

You know that sums or differences of like terms can be combined by adding or subtracting their coefficients:

$$3xy + 5xy = (3+5)xy = 8xy$$

Like radicals, that is, radicals of the same index and radicand, can be combined in the same way.

Example A.99.

a
$$\begin{aligned}
3\sqrt{3} + 4\sqrt{3} &= (3+4)\sqrt{3}\\
&= 7\sqrt{3}
\end{aligned}$$
y

b
$$\begin{aligned}
4\sqrt[3]{y} - 6\sqrt[3]{y} &= (4-6)\sqrt[3]{y}\\
&= -2\sqrt[3]{y}
\end{aligned}$$

Caution A.100.

1 In Example **??**a, $3\sqrt{3} + 4\sqrt{3} \neq 7\sqrt{6}$. Only the coefficients are added; the radicand does not change.

2 Sums of radicals with different radicands or different indices cannot be combined. Thus,

$$\sqrt{11} + \sqrt{5} \neq \sqrt{16}$$
$$\sqrt[3]{10x} - \sqrt[3]{2x} \neq \sqrt[3]{8x}$$

and

$$\sqrt[3]{7} + \sqrt{7} \neq \sqrt[5]{7}$$

None of the expressions above can be simplified.

A.10.4 Products of Radicals

According to Property (1), radicals of the same index can be multiplied together.

Product of Radicals

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab} \qquad (a, b \geq 0)$$

Thus, for example,

$$\sqrt{2}\sqrt{18}\sqrt{36} = 6 \quad \text{and} \quad \sqrt[3]{2x}\,\sqrt[3]{4x^2} = \sqrt[3]{8x^3} = 2x$$

For products involving binomials, we can apply the distributive law.

Example A.101.

a $\sqrt{3}\left(\sqrt{2x}+\sqrt{6}\right) = \sqrt{3\cdot 2x}+\sqrt{3\cdot 6}$
$= \sqrt{6x}+\sqrt{18} = \sqrt{6x}+3\sqrt{2}$

b $\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right) = \sqrt{x^2}+\sqrt{xy}-\sqrt{xy}-\sqrt{y^2}$
$= x-y$

A.10.5 Rationalizing the Denominator

It is easier to work with radicals if there are no roots in the denominators of fractions. We can use the fundamental principle of fractions to remove radicals from the denominator. This process is called **rationalizing the denominator**. For square roots, we multiply the numerator and denominator of the fraction by the radical in the denominator.

Example A.102. Rationalize the denominator of each fraction.

a $\sqrt{\dfrac{1}{3}}$

b $\dfrac{\sqrt{2}}{\sqrt{50x}}$

Solution.

a Apply Property (2) to write the radical as a quotient.

$$\sqrt{\frac{1}{3}} = \frac{\sqrt{1}}{\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}} \quad \text{Multiply numerator and denominator l}$$
$$= \frac{1\cdot\boldsymbol{\sqrt{3}}}{\sqrt{3}\cdot\boldsymbol{\sqrt{3}}}$$
$$= \frac{\sqrt{3}}{3}$$

b It is always best to simplify the denominator before rationalizing.

$$\frac{\sqrt{2}}{\sqrt{50x}} = \frac{\sqrt{2}}{5\sqrt{2x}} \quad \text{Multiply numerator a}$$
$$= \frac{\sqrt{2}\cdot\boldsymbol{\sqrt{2x}}}{5\sqrt{2x}\cdot\boldsymbol{\sqrt{2x}}} \quad \text{Simplify.}$$
$$= \frac{\sqrt{4x}}{5(2x)}$$
$$= \frac{2\sqrt{x}}{10x}$$

If the denominator of a fraction is a *binomial* in which one or both terms is a radical, we can use a special building factor to rationalize it. First, recall that

$$(p-q)(p+q) = p^2 - q^2$$

where the product consists of perfect squares only. Each of the two factors $p-q$ and $p+q$ is said to be the **conjugate** of the other.

Now consider a fraction of the form

$$\frac{a}{b+\sqrt{c}}$$

If we multiply the numerator and denominator of this fraction by the conjugate of the denominator, we get

$$\frac{a\boldsymbol{(b-\sqrt{c})}}{(b+\sqrt{c})\boldsymbol{(b-\sqrt{c})}} = \frac{ab-a\sqrt{c}}{b^2-(\sqrt{c})^2} = \frac{ab-a\sqrt{c}}{b^2-c}$$

The denominator of the fraction no longer contains any radicals—it has been rationalized.

Multiplying numerator and denominator by the conjugate of the denominator also works on fractions of the form

$$\frac{a}{\sqrt{b}+c} \quad \text{and} \quad \frac{a}{\sqrt{b}+\sqrt{c}}$$

We leave the verification of these cases as exercises.

Example A.103. Rationalize the denominator: $\dfrac{x}{\sqrt{2}+\sqrt{x}}$.

Solution. Multiply numerator and denominator by the conjugate of the denominator, $\sqrt{2}-\sqrt{x}$.

$$\frac{x\boldsymbol{(\sqrt{2}-\sqrt{x})}}{(\sqrt{2}+\sqrt{x})\boldsymbol{(\sqrt{2}-\sqrt{x})}} = \frac{x(\sqrt{2}-\sqrt{x})}{2-x}$$

A.10.6 Simplifying $\sqrt[n]{x^n}$

Raising to a power is the inverse operation for extracting roots; that is,

$$\boxed{\left(\sqrt[n]{a}\right)^n = a}$$

as long as $\sqrt[n]{a}$ is a real number. For example,

$$\left(\sqrt[4]{16}\right)^4 = 2^4 = 16, \quad \text{and} \quad \left(\sqrt[3]{-125}\right)^3 = (-5)^3 = -125$$

Now consider the power and root operations in the opposite order; is it true that $\sqrt[n]{a^n} = a$? If the index n is an odd number, then the statement is always true. For example,

$$\sqrt[3]{2^3} = \sqrt[3]{8} = 2 \quad \text{and} \quad \sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$$

However, if n is even, we must be careful. Recall that the principal root $\sqrt[n]{x}$ is always positive, so if a is a negative number, it cannot be true that $\sqrt[n]{a^n} = a$. For example, if $a = -3$, then

$$\sqrt{(-3)^2} = \sqrt{9} = 3$$

Instead, we see that, for even roots, $\sqrt[n]{a^n} = |a|$.

We summarize our results in below.

Roots of Powers

1 If n is odd, $\quad \sqrt[n]{a^n} = a$

2 If n is even, $\quad \sqrt[n]{a^n} = |a|$

In particular, $\quad \sqrt{a^2} = |a|$

Example A.104.

a $\sqrt{16x^2} = 4\,|x|$

b $\sqrt{(x-1)^2} = |x-1|$

A.10.7 Extraneous Solutions to Radical Equations

It is important to check the solution to a radical equation, because it is possible to introduce false, or **extraneous**, solutions when we square both sides of the equation. For example, the equation

$$\sqrt{x} = -5$$

has no solution, because $\sqrt{x}$ is never a negative number. However, if we try to solve the equation by squaring both sides, we find

$$\begin{aligned}(\sqrt{x})^{\mathbf{2}} &= (-5)^{\mathbf{2}}\\ x &= 25\end{aligned}$$

You can check that 25 is *not* a solution to the original equation, $\sqrt{x} = -5$, because $\sqrt{25}$ does not equal -5.

If each side of an equation is raised to an odd power, extraneous solutions will not be introduced. However, if we raise both sides to an even power, we should check each solution in the original equation.

Example A.105. Solve the equation $\sqrt{x+2} + 4 = x$.

Solution. First, isolate the radical expression on one side of the equation. (This will make it easier to square both sides.)

$\sqrt{x+2} = x - 4$	Square both sides of the equation.
$\left(\sqrt{x+2}\right)^{\mathbf{2}} = (x-4)^{\mathbf{2}}$	
$x + 2 = x^2 - 8x + 16$	Subtract $x + 2$ from both sides.
$x^2 - 9x + 14 = 0$	Factor the left side.
$(x-2)(x-7) = 0$	Set each factor to zero.
$x = 2 \quad \text{or} \quad x = 7$	

Check Does $\sqrt{\mathbf{2}+2} + 4 = \mathbf{2}$? No; 2 is not a solution.

Does $\sqrt{\mathbf{7}+2} + 4 = \mathbf{7}$? Yes; 7 is a solution.

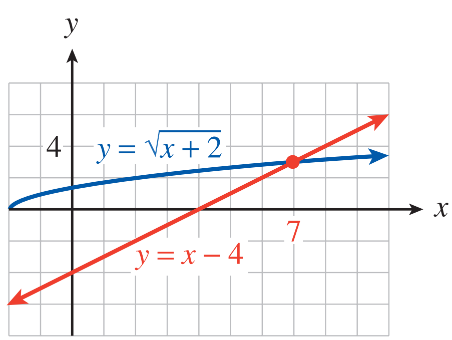


Figure A.106

The apparent solution 2 is extraneous. The only solution to the original equation is 7. We can verify the solution by graphing the equations

$$y_1 = \sqrt{x+2} \quad \text{and} \quad y_2 = x - 4$$

as shown in Figure **??**. The graphs intersect in only one point, $(7, 3)$, so there is only one solution, $x = 7$.

Caution A.107. When we square both sides of an equation, it is *not* correct to square each term of the equation separately. Thus, in Example **??**, the original equation is not equivalent to

$$(\sqrt{x+2})^2 + 4^2 = x^2$$

This is because $(a+b)^2 \neq a^2 + b^2$. Instead, we must square the *entire* left side of the equation as a binomial, like this,

$$(\sqrt{x+2} + 4)^2 = x^2$$

or we may proceed as shown in Example **??**.

A.10.8 Equations with More than One Radical

Sometimes it is necessary to square both sides of an equation more than once in order to eliminate all the radicals.

Example A.108. Solve $\sqrt{x-7} + \sqrt{x} = 7$.

Solution. First, isolate the more complicated radical on one side of the equation. (This will make it easier to square both sides.) We will subtract $\sqrt{x}$ from both sides.

$$\sqrt{x-7} = 7 - \sqrt{x}$$

Now square each side to remove one radical. Be careful when squaring the binomial $7 - \sqrt{x}$.

$$\begin{aligned}(\sqrt{x-7})^{\mathbf{2}} &= (7 - \sqrt{x})^{\mathbf{2}} \\ x - 7 &= 49 - 14\sqrt{x} + x\end{aligned}$$

Collect like terms, and isolate the radical on one side of the equation.

$$\begin{aligned}-56 &= -14\sqrt{x} && \text{Divide both sides by } -14. \\ 4 &= \sqrt{x}\end{aligned}$$

Now square again to obtain

$$\begin{aligned}(4)^{\mathbf{2}} &= (\sqrt{x})^{\mathbf{2}} \\ 16 &= x\end{aligned}$$

Check Does $\sqrt{\mathbf{16} - 7} + \sqrt{\mathbf{16}} = 7$? Yes. The solution is 16.

A.11 Facts from Geometry

In this section, we review some information you will need from geometry. You are already familiar with the formulas for the area and perimeter of common geometric figures; you can find these formulas in the reference section at the front of the book.

A.11.1 Right Triangles and the Pythagorean Theorem

A **right triangle** is a triangle in which one of the angles is a right angle, or $90°$. Because the sum of the three angles in any triangle is $180°$, this means that the other two angles in a right triangle must have a sum of $180° - 90°$, or $90°$. For instance, if we know that one of the angles in a right triangle is $37°$, then the remaining angle must be $90° - 37°$, or $53°$, as shown in **??**.

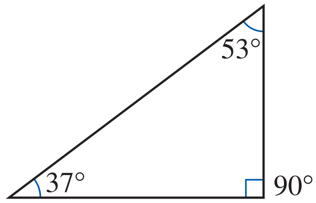


Figure A.109

Example A.110. In a right triangle, the medium-sized angle is 15° less than twice the smallest angle. Find the sizes of the three angles in **??**.

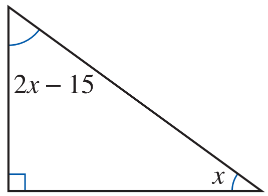


Figure A.111

Solution.

Step 1 Let x stand for the size of the smallest angle. Then the medium-sized angle must be $2x - 15$.

Step 2 Because the right angle is the largest angle, the sum of the smallest and medium-sized angles must be the remaining 90°. Thus,

$$x + (2x - 15) = 90$$

Step 3 Solve the equation. Begin by simplifying the left side.

$$\begin{aligned} 3x - 15 &= 90 && \text{Add 15 to both sides.} \\ 3x &= 105 && \text{Divide both sides by 3.} \\ x &= 35 \end{aligned}$$

Step 4 The smallest angle is 35°, and the medium-sized angle is $2(35°) - 15°$, or 55°.

In a right triangle (see Figure **??**), the longest side is opposite the right angle and is called the **hypotenuse**. Ordinarily, even if we know the lengths of two sides of a triangle, it is not easy to find the length of the third side (to solve this problem we need trigonometry), but for the special case of a right triangle, there is an equation that relates the lengths of the three sides. This property of right triangles was known to many ancient cultures, and we know it today by the name of a Greek mathematician, Pythagoras, who provided a proof of the result.

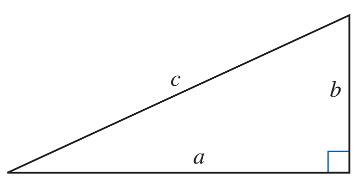


Figure A.112

Pythagorean Theorem

In a right triangle, if c stands for the length of the hypotenuse and a and b stand for the lengths of the two sides, then

$$a^2 + b^2 = c^2$$

Example A.113. The hypotenuse of a right triangle is 15 feet long. The third side is twice the length of the shortest side. Find the lengths of the other 2 sides.

Solution.

Step 1 Let x represent the length of the shortest side, so that the third side has length $2x$. (See Figure **??**.)

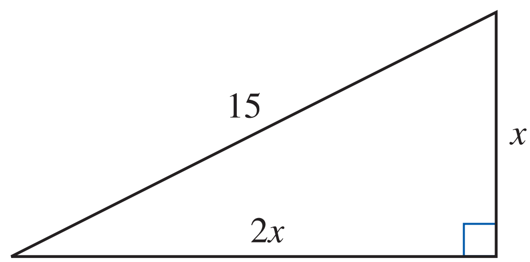


Figure A.114

Step 2 Substituting these expressions into the Pythagorean theorem, we find

$$x^2 + (2x)^2 = 15^2$$

Step 3 This is a quadratic equation with no linear term, so we simplify and then isolate x^2.

$$\begin{aligned} x^2 + 4x^2 &= 225 && \text{Combine like terms.} \\ 5x^2 &= 225 && \text{Divide both sides by 5.} \\ x^2 &= 45 \end{aligned}$$

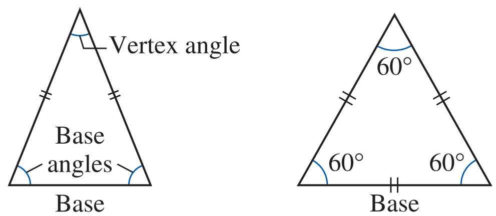
Taking square roots of both sides yields

$$x = \pm\sqrt{45} \approx \pm 6.708203932$$

Step 4 Because a length must be a positive number, the shortest side has length approximately 6.71 feet, and the third side has length 2(6.71), or approximately 13.42 feet.

A.11.2 Isosceles and Equilateral Triangles

Recall also that an **isosceles** triangle is one that has at least two sides of equal length. In an isosceles triangle, the angles opposite the equal sides, called the **base** angles, are equal in measure, as shown in Figure **??**a. In an equilateral triangle (Figure **??**b), all three sides have equal length, and all three angles have equal measure.



(a) Isosceles triangle **(b)** Equilateral triangle

Figure A.115

A.11.3 The Triangle Inequality

The longest side in a triangle is always opposite the largest angle, and the shortest side is opposite the smallest angle. It is also true that the sum of the lengths of any two sides of a triangle must be greater than the third side, or else the two sides will not meet to form a triangle! This fact is called the **triangle inequality**.

In Figure **??**, we must have that

$$p + q > r$$

where p, q, and r are the lengths of the sides of the triangle.

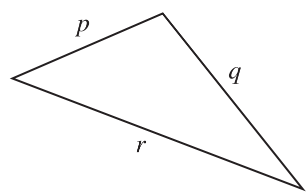


Figure A.116

Now we can use the triangle inequality to discover information about the sides of a triangle.

Example A.117. Two sides of a triangle have lengths 7 inches and 10 inches. What can you say about the length of the third side (see Figure **??**)?

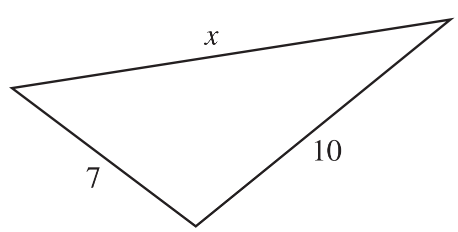


Figure A.118

Solution. Let x represent the length of the third side of the triangle. By the triangle inequality, we must have that

$$x < 7 + 10, \quad \text{or} \quad x17$$

Looking at another pair of sides, we must also have that

$$10 < x + 7, \quad \text{or} \quad x > 3$$

Thus the third side must be greater than 3 inches but less than 17 inches long.

A.11.4 Similar Triangles

Two triangles are said to be **similar** if their corresponding angles are equal. This means that the two triangles will have the same shape but not necessarily the same size. One of the triangles will be an enlargement or a reduction of the other; so their corresponding sides are proportional. In other words, for similar triangles, the ratios of the corresponding sides are equal (see Figure **??**).

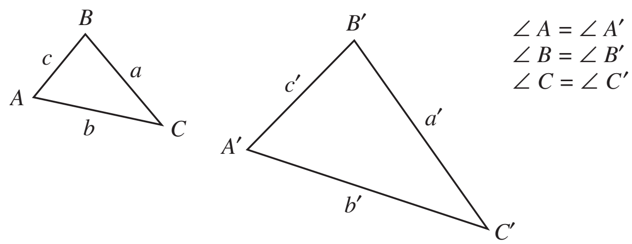


Figure A.119

If any two pairs of corresponding angles of two triangles are equal, then the third pair must also be equal, because in both triangles the sum of the angles is 180°. Thus, to show that two triangles are similar, we need only show that two pairs of angles are equal.

Example A.120. The roof of an A-frame ski chalet forms an isosceles triangle with the floor as the base (see Figure **??**). The floor of the chalet is 24 feet wide, and the ceiling is 20 feet tall at the center. If a loft is built at a height of 8 feet from the floor, how wide will the loft be?

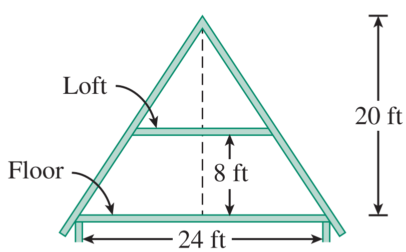


Figure A.121

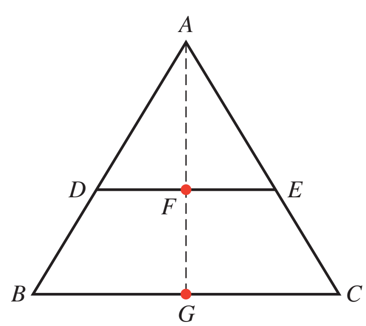


Figure A.122

Solution. From Figure **??**, we can show that ΔABC is similar to ΔADE. Both triangles include $\angle A$, and because $\overline{DE}$ is parallel to $\overline{BC}$, $\angle ADE$ is equal to $\angle ABC$. Thus, the triangles have two pairs of equal angles and are therefore similar triangles.

Step 1 Let w stand for the width of the loft.

Step 2 First note that if $FG = 8$, then $AF = 12$. Because ΔABC is similar to ΔADE, the ratios of their corresponding sides (or corresponding altitudes) are equal. In particular,

$$\frac{w}{24} = \frac{12}{20}$$

Step 3 Solve the proportion for w. Begin by cross-multiplying.

$20w = (12)(24)$	Apply the fundamental principle.
$w = \frac{288}{20} = 14.4$	Divide by 20.

Step 4 The floor of the loft will be 14.4 feet wide.

A.11.5 Volume and Surface Area

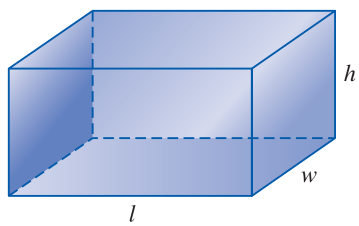


Figure A.123

The **volume** of a three-dimensional object measures its capacity, or how much space it encloses. Volume is measured in cubic units, such as cubic inches or cubic meters. The volume of a rectangular prism, or box, is given by the product of its length, width, and height. For example, the volume of the box of length 4 inches, width 3 inches, and height 2 inches shown in Figure **??** is

$$V = lwh = 4(3)(2) = 24 \text{ cubic inches}$$

Formulas for the volumes of other common objects can be found inside the front cover of the book.

Example A.124. A cylindrical can must have a height of 6 inches, but it can have any reasonable radius.

a Write an algebraic expression for the volume of the can in terms of its radius.

b If the volume of the can should be approximately 170 cubic inches, what should its radius be?

Solution.

a The formula for the volume of a right circular cylinder is $V = \pi r^2 h$. If the height of the cylinder is 6 inches, then $V = \pi r^2(6)$, or $V = 6\pi r^2$. (See Figure **??**.)

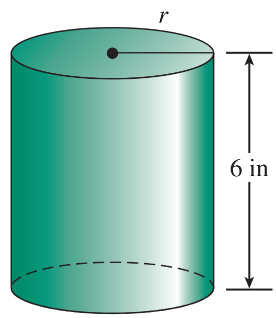


Figure A.125

b Substitute 170 for V and solve for r.

$$170 = 6\pi r^2 \qquad \text{Divide both sides by } 6\pi.$$

$$r^2 = \frac{170}{6\pi} \qquad \text{Take square roots.}$$

$$r = \sqrt{\frac{170}{6\pi}} \approx 3.00312$$

Thus, the radius of the can should be approximately 3 inches. A calculator keying sequence for the expression above is

$\boxed{\sqrt{\ }}\ \boxed{(}\ 170\ \boxed{\div}\ \boxed{(}\ 6\ \boxed{\pi}\ \boxed{)}\ \boxed{)}\ \boxed{\text{ENTER}}$

The **surface area** of a solid object is the sum of the areas of all the exterior faces of the object. It measures the amount of paper that would be needed to cover the object entirely. Since it is an area, it is measured in square units.

Example A.126. Write a formula for the surface area of a closed box in terms of its length, width, and height. (See Figure **??**.)

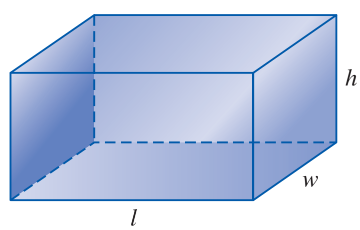


Figure A.127

Solution. The box has six sides; we must find the area of each side and add them. The top and bottom of the box each have area lw, so together they contribute $2lw$ to the surface area. The back and front of the box each have area lh, so they contribute $2lh$ to the surface area. Finally, the left and right sides of the box each have area wh, so they add $2wh$ to the surface area. Thus, the total surface area is

$$S = 2lw + 2lh + 2wh$$

Formulas for the surface areas of other common solids can be found on the insert in the front of the book.

A.11.6 The Distance Formula

By using the Pythagorean theorem, we can derive a formula for the distance between two points, P_1 and P_2, in terms of their coordinates. We first label a right triangle, as we did in the example above. Draw a horizontal line through P_1 and a vertical line through P_2. These lines meet at a point P_3, as shown in Figure **??**. The x-coordinate of P_3 is the same as the x-coordinate of P_2, and the y-coordinate of P_3 is the same as the y-coordinate of P_1. Thus, the coordinates of P_3 are (x_2, y_1).

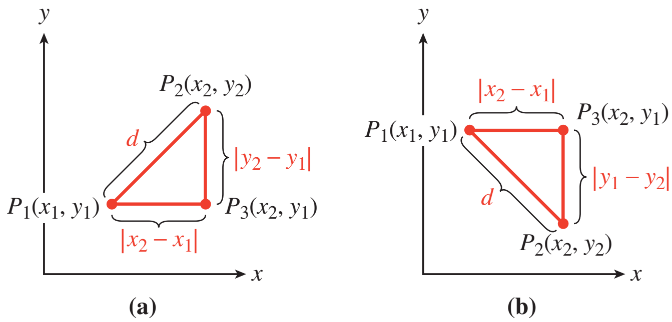


Figure A.128

The distance between P_1 and P_3 is $|x_2 - x_1|$, and the distance between P_2 and P_3 is $|y_2 - y_1|$. (See ⟨⟨Unresolved xref, reference "AbsoluteValue"; check spelling or use "provisional" attribute⟩⟩ to review distance and absolute value.) These two numbers are the lengths of the legs of the right triangle. The length of the hypotenuse is the distance between P_1 and P_2, which we will call d. By the Pythagorean theorem,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Taking the (positive) square root of each side of this equation gives us the **distance formula**.

Distance Formula

The **distance** d between points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example A.129. Find the distance between $(2, -1)$ and $(4, 3)$.

Solution. Substitute $(2, -1)$ for (x_1, y_1) and substitute $(4, 3)$ for (x_2, y_2) in the distance formula to obtain

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4-2)^2 + \left[3 - (-1)\right]^2} \\ &= \sqrt{4 + 16} = \sqrt{20} \approx 4.47 \end{aligned}$$

In Example **??**, we obtain the same answer if we use $(4, 3)$ for P_1 and use $(2, -1)$ for P_2:

$$\begin{aligned} d &= \sqrt{(2-4)^2 + \left[(-1) - 3\right]^2} \\ &= \sqrt{4 + 16} = \sqrt{20} \end{aligned}$$

A.11.7 The Midpoint Formula

If we know the coordinates of two points, we can calculate the coordinates of the point halfway between them using the **midpoint** formula. Each coordinate of the midpoint is the average of the corresponding coordinates of the two points.

Midpoint Formula

The **midpoint** of the line segment joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is the point $M(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{x_1 + x_2}{2} \quad \text{and} \quad \overline{y} = \frac{y_1 + y_2}{2}$$

Example A.130. Find the midpoint of the line segment joining the points $(-2, 1)$ and $(4, 3)$.

Solution. Substitute $(-2, 1)$ for (x_1, y_1) and $(4, 3)$ for (x_2, y_2) in the midpoint formula to obtain

$$\overline{x} = \frac{x_1 + x_2}{2} = \frac{-2 + 4}{2} = 1 \qquad \text{and}$$
$$\overline{y} = \frac{y_1 + y_2}{2} = \frac{1 + 3}{2} = 2$$

The midpoint of the segment is the point $(\overline{x}, \overline{y}) = (1, 2)$.

A.11.8 Circles

A **circle** is the set of all points in a plane that lie at a given distance, called the **radius**, from a fixed point called the **center**. We can use the distance formula to find an equation for a circle. First consider the circle in Figure **??**a, whose center is the origin, $(0, 0)$.

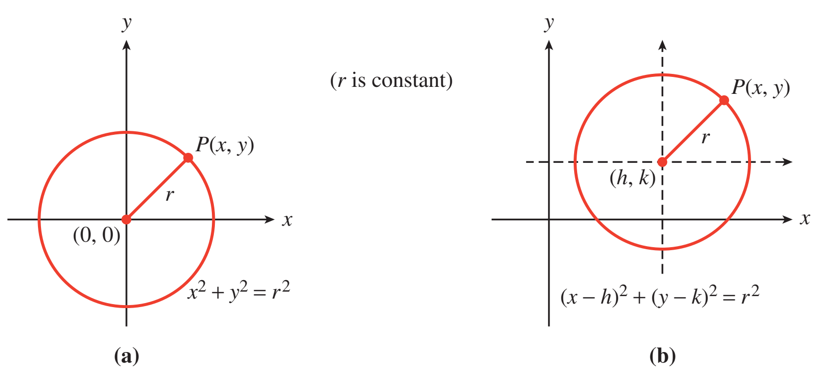


Figure A.131

The distance from the origin to any point $P(x, y)$ on the circle is r. Therefore,

$$\sqrt{(x-0)^2 + (y-0)^2} = r$$

Or, squaring both sides,

$$(x-0)^2 + (y-0)^2 = r^2$$

Thus, the equation for a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2$$

Now consider the circle in Figure ??b, whose center is the point (h, k). Every point $P(x, y)$ on the circle lies a distance r from (h, k), so the equation of the circle is given by the following formula.

Standard Form for a Circle

The equation for a **circle** of **radius** r centered at the point (h, k) is

$$(x - h)^2 + (y - k)^2 = r^2$$

This equation is the **standard form** for a circle of radius r with center at (h, k). It is easy to graph a circle if its equation is given in standard form.

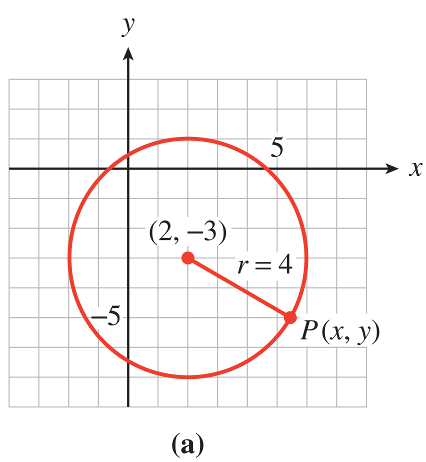
Example A.132. Graph the circles.

a $(x - 2)^2 + (y + 3)^2 = 16$

b $x^2 + (y - 4)^2 = 7$

Solution.

a The graph of $(x - 2)^2 + (y + 3)^2 = 16$ is a circle with radius 4 and center at $(2, -3)$. To sketch the graph, first locate the center of the circle. (The center is not part of the graph of the circle.) From the center, move a distance of 4 units (the radius of the circle) in each of four directions: up, down, left, and right. This locates four points that lie on the circle: $(2, 1)$, $(2, -7)$, $(-2, -3)$, and $(6, -3)$. Sketch the circle through these four points. (See Figure **??**a.)



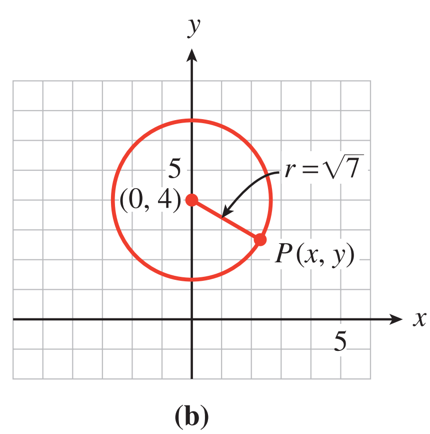


Figure A.133

b The graph of $x^2 + (y - 4)^2 = 7$ is a circle with radius $\sqrt{7}$ and center at $(0, 4)$. From the center, move $\sqrt{7}$, or approximately 2.6, units in each of the four coordinate directions to obtain the points $(0, 6.6)$, $(0, 1.4)$, $(-2.6, 4)$, and $(2.6, 4)$. Sketch the circle through these four points. (See Figure **??**b.)

We can write an equation for any circle if we can find its center and radius.

Example A.134. Find an equation for the circle whose diameter has endpoints $(7, 5)$ and $(1, -1)$.

Solution. The center of the circle is the midpoint of its diameter (see Figure **??**). Use the midpoint formula to find the center:

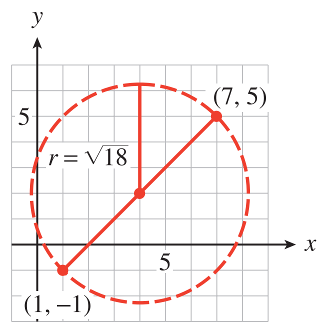


Figure A.135

$$h = \overline{x} = \frac{7+1}{2} = 4$$
$$k = \overline{y} = \frac{5-1}{2} = 2$$

Thus, the center is the point $(h, k) = (4, 2)$. The radius is the distance from the center to either of the endpoints of the diameter, say the point $(7, 5)$. Use the distance formula with the points $(7, 5)$ and $(4, 2)$ to find the radius.

$$r = \sqrt{(7-4)^2 + (5-2)^2}$$
$$= \sqrt{3^2 + 3^2} = \sqrt{18}$$

Finally, substitute 4 for h and 2 for k (the coordinates of the center) and $\sqrt{18}$ for 4 (the radius) into the standard form to obtain

$$(x-h)^2 + (y-k)^2 = r^2$$
$$(x-4)^2 + (y-2)^2 = 18$$

A.12 The Real Number System

A.12.1 Subsets of the Real Numbers

The numbers associated with points on a number line are called the **real numbers**. The set of real numbers is denoted by $\mathbb{R}$. You are already familiar with several types, or subsets, of real numbers:

- The set $\mathbb{N}$ of **natural**, or **counting numbers**, as its name suggests, consists of the numbers $1, 2, 3, 4, \ldots$, where "..." indicates that the list continues without end.
- The set $\mathbb{W}$ of **whole numbers** consists of the natural numbers and zero: $0, 1, 2, 3 \ldots$.
- The set $\mathbb{Z}$ of **integers** consists of the natural numbers, their negatives, and zero: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

All of these numbers are subsets of the rational numbers.

A.12.2 Rational Numbers

A number that can be expressed as the quotient of two integers $\frac{a}{b}$ where $b \neq 0$, is called a **rational number**. The integers are rational numbers, and so are common fractions. Some examples of rational numbers are $5, -2, 0, \frac{2}{9}, \sqrt{16}$, and $\frac{-4}{17}$. The set of rational numbers is denoted by $\mathbb{Q}$.

Every rational number has a decimal form that either terminates or repeats a pattern of digits. For example,

$$\frac{3}{4} = 3 \div 4 = 0.75, \text{ a } \textbf{terminating decimal}$$

and

$$\frac{2}{37} = 9 \div 37 = 0.243243243\ldots$$

where the pattern of digits 243 is repeated endlessly. We use the **repeater bar** notation to write a repeating decimal fraction:

$$\frac{9}{37} = 0.\overline{243}$$

A.12.3 Irrational Numbers

Some real numbers *cannot* be written in the form $\frac{a}{b}$, where a and b are integers. For example, the number $\sqrt{2}$ is not equal to any common fraction. Such numbers are called **irrational numbers**. Examples of irrational numbers are $\sqrt{15}, \pi$, and $-\sqrt[3]{7}$.

The decimal form of an irrational number never terminates, and its digits do not follow a repeating pattern, so it is impossible to write down an exact decimal equivalent for an irrational number. However, we can obtain decimal *approximations* correct to any desired degree of accuracy by rounding off. A graphing calculator gives the decimal representation of π as 3.141592654. This is not the *exact* value of π, but for most calculations it is quite adequate.

Some nth roots are rational numbers and some are irrational numbers. For example,

$$\sqrt{49}. \sqrt[3]{\frac{27}{8}}, \text{ and } 81^{1/4}$$

are rational numbers because they are equal to $7, \frac{3}{2}$, and 3, respectively. On the other hand,

$$\sqrt{5}, \sqrt[3]{54}, \text{ and } 7^{1/5}$$

are irrational numbers. We can use a calculator to obtain decimal approximations for each of these numbers:

$$\sqrt{5} \approx 2.236, \quad \sqrt[3]{54} \approx 3.826, \quad \text{and} \quad 7^{1/5} \approx 1.476$$

The subsets of the real numbers are related as shown in Figure **??**. Every natural number is also a whole number, every whole number is an integer, every integer is a rational number, and every rational number is real. Also, every real number is either rational or irrational.

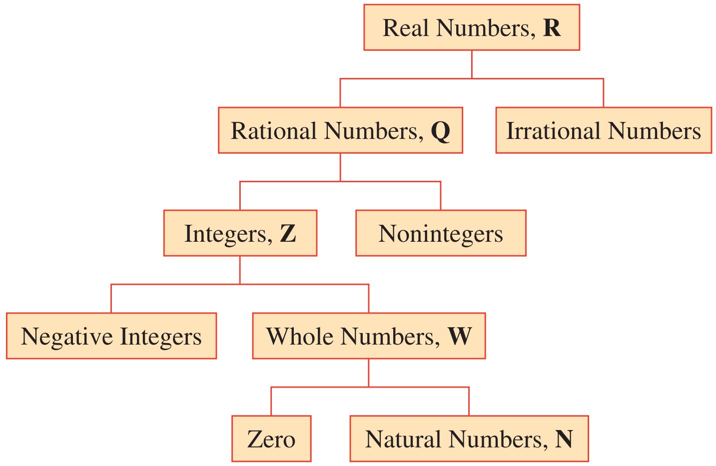


Figure A.136

Example A.137.

a 2 is a natural number, a whole number, an integer, a rational number, and a real number.

b $\sqrt{15}$ is an irrational number and a real number.

c The number π, whose decimal representation begins 3.14159... is irrational and real.

d 3.14159 is a rational and real number (which is close but not exactly equal to π).

A.12.4 Properties of the Real Numbers

The real numbers have several useful properties governing the operations of addition and multiplication. If a, b, and c represent real numbers, then each of the following equations is true:

- $a+b=b+a$ Commutative properties
  $ab=ba$
- $(a+b)+c=a+(b+c)$ Associative properties
  $(ab)=a(bc)$
- $a(b+c)=ab+ac$ Distributive property
- $a+0=a$ Identity properties
  $a\cdot 1=a$

These properties do not mention subtraction or division. But we can define *subtraction* and *division* in terms of addition and multiplication. For example, we can define the difference $a-b$ as follows:

$$a-b=a+(-b)$$

where $-b$, the **additive inverse** (or **opposite**) of b, is the number that satisfies

$$b+(-b)=0$$

Similarly, we can define the quotient $\frac{a}{b}$:

$$\frac{a}{b} = a\left(\frac{1}{b}\right) \qquad (b \neq 0)$$

where $\frac{1}{b}$, the **multiplicative inverse** (or **reciprocal**) of b, is the number that satisfies

$$b \cdot \frac{1}{b} = 1 \qquad (b \neq 0)$$

Division by zero is not defined.

Example A.138. Use the commutative and associative laws to simplify the computations.

a $24 + 18 + 6$

b $4 \cdot 27 \cdot 25$

Solution.

a Apply the commutative law of addition.

$$\begin{aligned} 24 + 18 + 6 &= (24 + 6) + 18 \\ &= 30 + 18 = 48 \end{aligned}$$

b Apply the commutative law of multiplication.

$$\begin{aligned} 4 \cdot 27 \cdot 25 &= (4 \cdot 25) \cdot 27 \\ &= 100 \cdot 27 = 2700 \end{aligned}$$

A.12.5 Order Properties of the Real Numbers

Real numbers obey properties about order, that is, properties about inequalities. The familiar inequality symbols, $<$ and $>$, have the following properties:

- If a and b are any real numbers, then one of three things is true:

$$a < b, \quad \text{or} \quad a > b, \quad \text{or} \quad a = b$$

- (Transitive property) For real numbers a, b, and c,

$$\text{if } a < b \text{ and } b < c, \text{ then } a < c$$

We also have three properties that are useful for solving inequalities:

- If $a < b$, then $a + c < b + c$.
- If $a < b$ and $c > 0$, then $ac < bc$.
- If $a < b$ and $c < 0$, then $ac > bc$.

Example A.139.

a If $x < y$ and $y < -2$, then $x < -2$

b $\pi < 3.1416$, so $10\pi < 31.416$.

c $\frac{1}{3} > 0.33$, so $-\frac{1}{3} < -0.33$.

Appendix B

Using a Graphing Calculator

This appendix provides instructions for TI-84 or TI-83 calculators from Texas Instruments, but most other calculators work similarly.We describe only the basic operations and features of the graphing calculator used in your textbook.

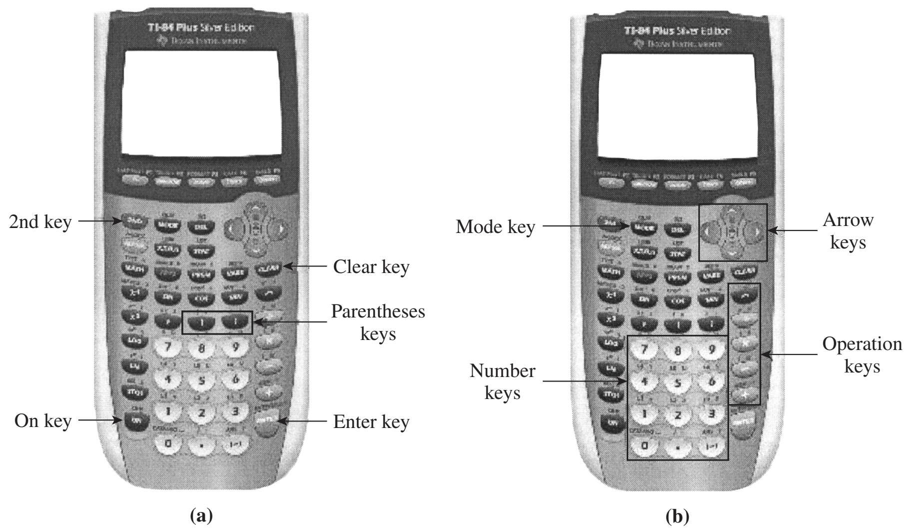


Figure B.1

B.1 Getting Started

B.1.1 On and Off

Press `ON` to turn *on* the calculator (see Figure **??**a). You will see a cursor blinking in the upper left corner of the Home screen. Press `2ndON` to turn *off* the calculator.

B.1.2 Numbers and Operations

The parentheses keys, the Clear key, and the Enter key are shown in Figure **??**a. Locate the number keys, operation keys, and arrow keys on your calculator, as shown in Figure **??**b.

We use the `-` key for subtraction, but we use the `(-)` key (located next to `ENTER`) for negative numbers.

Example B.2. Compute $5 - 8$. Press

5-8ENTER Ans. -3

Example B.3. Compute $-5 + 8$. Press

(-)5+8ENTER Ans. 3

We press ENTER to tell the calculator to compute.
The calculator has a key for the value of π.

Example B.4. Compute 2π. Press

2X2nd^ENTER or 22nd^ENTER Ans. 6.283185307

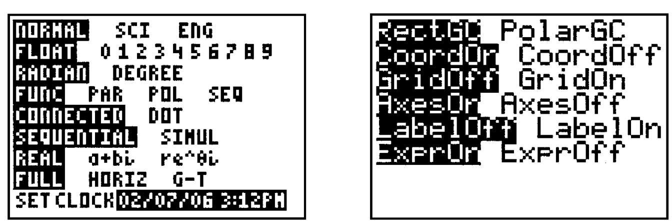
B.1.3 Clear and Delete

Press DEL to delete the character under the cursor.
Press CLEAR to clear the contents of the current input line.
In the Home screen, press CLEAR CLEAR to clear the entire screen.

Troubleshooting

1 If your screen is too light, press 2nd â several times to make it darker. If it is too dark, press 2nd â .

2 For the features we use in this book, the MODE and FORMAT should be in their default settings. Press MODE to see the menu in Figure B.2a, and 2nd ZOOM to see the format menu in Figure ??b. Use the Arrow keys and ENTER to alter the menus to the default settings if necessary. *Note*: The Set Clock function does not appear on the T1-83.



(a) **(b)**

Figure B.5

B.2 Entering Expressions

B.2.1 Parentheses

Order of Operations: The calculator follows the standard order of operations.

Example B.6. Compute $2 + 3 \cdot 4$. Press

2+ 3X 4 ENTER Ans. 14

Example B.7. Compute $(2+3) \cdot 4$. Press

(2 + 3) X 4 ENTER   Ans. 20

Example B.8. Compute $\frac{1}{2 \cdot 3}$. Press

1 $\boxed{\div}$ (2 X 3) ENTER   Ans. 0.1666666667

Example B.9. Compute $\frac{1+3}{2}$. Press

(1 + 3) $\boxed{\div}$ 2 ENTER   Ans. 2

B.2.2 Exponents and Powers

Exponents: We use the caret key, ^ , to enter exponents or powers.

Example B.10. Evaluate 2^{10}.

2 ^ 10 ENTER   Ans. 1024

Squaring: There is a short-cut key for squaring, $\boxed{x^2}$.

Example B.11. Evaluate 57^2.

57 $\boxed{x^2}$ ENTER   Ans. 3249

Fractional Exponents: Fractional exponents must be enclosed in parentheses!

Example B.12. Evaluate $8^{2/3}$.

8 ^ (2 $\boxed{\div}$ 3) ENTER   Ans. 4

B.2.3 Roots

Square Roots: We access the square root by pressing 2nd $\boxed{x^2}$, and the display shows $\sqrt{}($. The calculator automatically gives an open parenthesis for the square root, but not a close parenthesis.

Example B.13. Evaluate $\sqrt{2}$.

2nd $\boxed{x^2}$ 2) ENTER   Ans. 1.414213562

Example B.14. Evaluate $\sqrt{9+16}$.

2nd $\boxed{x^2}$ $9+16$) ENTER   Ans. 5

In the next example, note that we must enter) at the end of the radicand to tell the calculator where the radical ends.