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Katherine Yoshiwara did her undergraduate work at Michigan State and her graduate work at UCLA. She has received teaching awards from the Mathematical Association of America and the American Mathematical Association of Two-Year Colleges.

She retired from Los Angeles Pierce College, is learning to play the cello, and likes gardening.

Acknowledgements

I would like to thank my cats.

I would also like to acknowledge Bruce Yoshiwara for helpful comments and suggestions.

Preface

Mathematics, as we all know, is the language of science, and fluency in algebraic skills has always been necessary for anyone aspiring to disciplines based on calculus. But in the information age, increasingly sophisticated mathematical methods are used in all fields of knowledge, from archaeology to zoology. Consequently, there is a new focus on the courses before calculus. The availability of calculators and computers allows students to tackle complex problems involving real data, but requires more attention to analysis and interpretation of results. All students, not just those headed for science and engineering, should develop a mathematical viewpoint, including critical thinking, problem-solving strategies, and estimation, in addition to computational skills. *Modeling, Functions and Graphs* employs a variety of applications to motivate mathematical thinking.

0.1 MODELING

The ability to model problems or phenomena by algebraic expressions and equations is the ultimate goal of any algebra course. Through a variety of applications, we motivate students to develop the skills and techniques of algebra. Each chapter includes an interactive Investigation that gives students an opportunity to explore an openended modeling problem. These Investigations can be used in class as guided explorations or as projects for small groups. They are designed to show students how the mathematical techniques they are learning can be applied to study and understand new situations.

0.2 Functions

The fundamental concept underlying calculus and related disciplines is the notion of function, and students should acquire a good understanding of functions before they embark on their study of college-level mathematics. While the formal study of functions is usually the content of precalculus, it is not too early to begin building an intuitive understanding of functional relationships in the preceding algebra courses. These ideas are useful not only in calculus but in practically any field students may pursue. We begin working with functions in Chapter 1 and explore the different families of functions in subsequent chapters.

In all our work with functions and modeling we employ the "Rule of Four," that all problems should be considered using algebraic, numerical, graphical, and verbal methods. It is the connections between these approaches that we have endeavored to establish in this course. At this level it is crucial that students learn to write an algebraic expression from a verbal description, recognize trends in a table of data, and extract and interpret information from the graph of a function.

0.3 Graphs

No tool for conveying information about a system is more powerful than a graph. Yet many students have trouble progressing from a point-wise understanding of graphs to a more global view. By taking advantage of graphing calculators, we examine a large number of examples and study them in more detail than is possible when every graph is plotted by hand. We can consider more realistic models in which calculations by more traditional methods are difficult or impossible.

We have incorporated graphing calculators into the text wherever they can be used to enhance understanding. Calculator use is not simply an add-on, but in many ways shapes the organization of the material. The text includes instructions for the TI-84 graphing calculator, but these can easily be adapted to any other graphing utility. We have not attempted to use all the features of the calculator or to teach calculator use for its own sake, but in all cases have let the mathematics suggest how technology should be used.

Katherine Yoshiwara
Atascadero, CA 2016

Contributors to the 5th Edition

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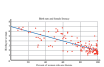
Chapter 1

Functions and Their Graphs

You may have heard that mathematics is the language of science. In fact, professionals in nearly every discipline take advantage of mathematical methods to analyze data, identify trends, and predict the effects of change. This process is called **mathematical modeling**. A model is a simplified representation of reality that helps us understand a process or phenomenon. Because it is a simplification, a model can never be completely accurate. Instead, it should focus on those aspects of the real situation that will help us answer specific questions. Here is an example.

The world's population is growing at different rates in different nations. Many factors, including economic and social forces, influence the birth rate. Is there a connection between birth rates and education levels? The figure shows the birth rate plotted against the female literacy rate in 148 countries. Although the data points do not all lie precisely on a line, we see a generally decreasing trend: the higher the literacy rate, the lower the birth rate. The **regression line** provides a model for this trend, and a tool for analyzing the data. In this chapter we study the properties of linear models and some techniques for fitting a linear model to data.

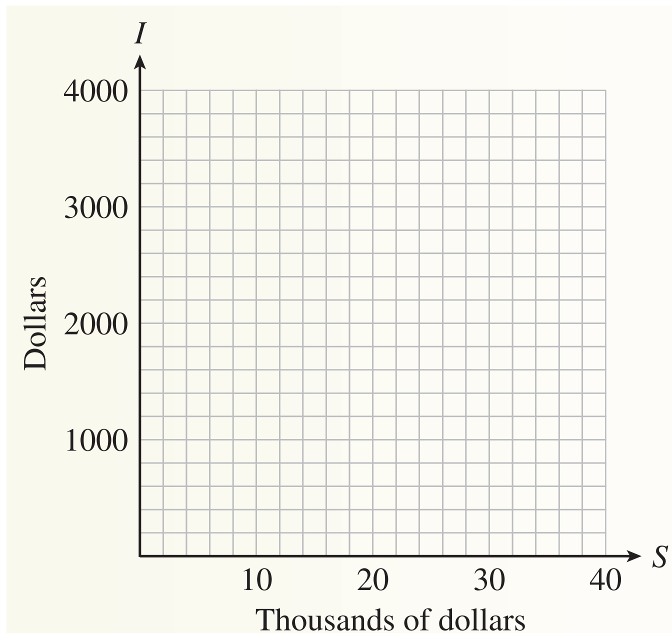


Investigation 1.1 (Sales on Commission)**.** Delbert is offered a part-time job selling restaurant equipment. He will be paid \$1000 per month plus a 6% commission on his sales. The sales manager tells Delbert he can expect to sell about \$8000 worth of equipment per month. To help him decide whether to accept the job, Delbert does a few calculations.

1. Based on the sales manager's estimate, what monthly income can Delbert expect from this job? What annual salary would that provide?

2. What would Delbert's monthly salary be if he sold only \$5000 of equipment per month? What would his salary be if he sold \$10,000 worth per month? Compute monthly incomes for each sales total shown in the table.

Sales	Income
5000	
8000	
10,000	
12,000	
15,000	
18,000	
20,000	
25,000	
30,000	
35,000	



3. Plot your data points on a graph, using sales, S, on the horizontal axis and income, I, on the vertical axis, as shown in the figure. Connect the data points to show Delbert's monthly income for all possible monthly sales totals.

4. Add two new data points to the table by reading values from your graph.

5. Write an algebraic expression for Delbert's monthly income, I, in terms of his monthly sales, S. Use the description in the problem to help you:

   He will be paid: \$1000 . . . plus a 6% commission on his sales.

   Income = ______________

6. Test your formula from part (5) to see if it gives the same results as those you recorded in the table.

7. Use your formula to find out what monthly sales total Delbert would need in order to have a monthly income of \$2500.

8. Each increase of \$1000 in monthly sales increases Delbert's monthly inbcome by ______________.

9. Summarize the results of your work: In your own words, describe the relationship etween Delbert's monthly sales and his monthly income. Include in your discussion a description of your graph.

1.1 Linear Models

1.1.1 Tables, Graphs and Equations

The first step in creating a model is to describe relationships between variables. In Investigation 1 ⟨⟨investigation-commission⟩⟩, we analyzed the relationship between Delbert's sales and his income. Starting from the verbal description of his income, we represented the relationship by a table of values, a graph, and an algebraic equation. Each of these mathematical tools is useful in a different way.

1. A **table of values** displays specific data points with precise numerical values.

2. A **graph** is a visual display of the data. It is easier to spot trends and describe the overall behavior of the variables from a graph.

3. An **algebraic equation** is a compact summary of the model. It can be used to analyze the model and to make predictions

We begin our study of modeling with some examples of **linear models**. In the examples that follow, observe the interplay among the three modeling tools, and how each contributes to the model.

Example 1.1. Annelise is on vacation at a seaside resort. She can rent a bicycle from her hotel for \$3 an hour, plus a \$5 insurance fee. (A fraction of an hour is charged as the same fraction of \$3.)

a Make a table of values showing the cost, C, of renting a bike for various lengths of time, t.

b Plot the points on a graph. Draw a curve through the data points.

c Write an equation for C in terms of t.

Solution.

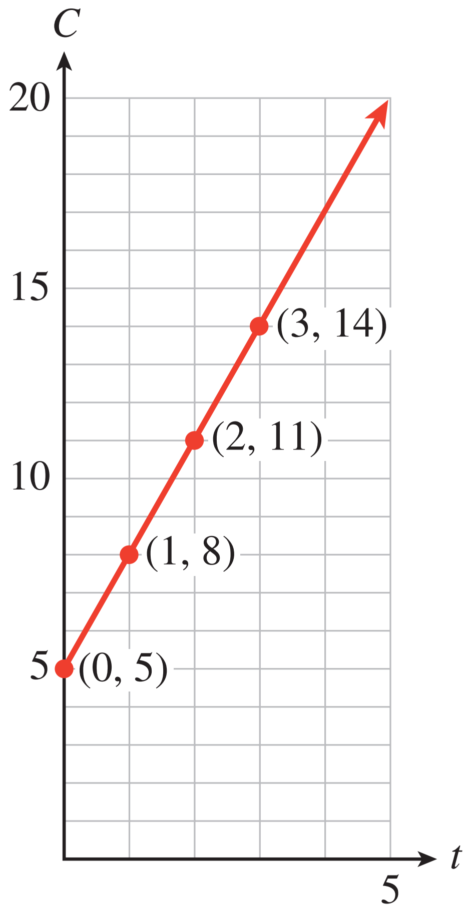
a To find the cost, multiply the time by \$3, and add the result to the \$5 insurance fee. For example, the cost of a 1-hour bike ride is

$$\begin{aligned}\text{Cost} &= (\$5 \text{ insurance fee}) + (\$3 \text{ per hour}) \times (\mathbf{1} \text{ hour})\\ C &= 5 + 3(\mathbf{1}) = 8\end{aligned}$$

A 1-hour bike ride costs \$8. Record the results in a table, as shown here:

Length of rental (hours)	Cost of rental (dollars)		(t, C)
1	8	$C = 5 + 3(\mathbf{1})$	$(1, 8)$
2	11	$C = 5 + 3(\mathbf{2})$	$(2, 11)$
3	14	$C = 5 + 3(\mathbf{3})$	$(3, 14)$

b Each pair of values represents a point on the graph. The first value gives the horizontal coordinate of the point, and the second value gives the vertical coordinate. The points lie on a straight line, as shown in the figure. The line extends infinitely in only one direction, because negative values of t do not make sense here.



c To find an equation, let C represent the cost of the rental, and use t for the number of hours:

$$\begin{aligned} \text{Cost} &= (\$5 \text{ insurance fee}) + (\$3 \text{ per hour}) \times (\text{number of hours}) \\ C &= 5 + 3 \cdot t = 8 \end{aligned}$$

Example 1.2. Use the equation $C = 5 + 3 \cdot t$ you found in Example 1.1 to answer the following questions. Then show how to find the answers by using the graph.

a How much will it cost Annelise to rent a bicycle for 6 hours?

b How long can Annelise bicycle for \$18.50?

Solution.

a Substitute $t = \mathbf{6}$ into the expression for C to find

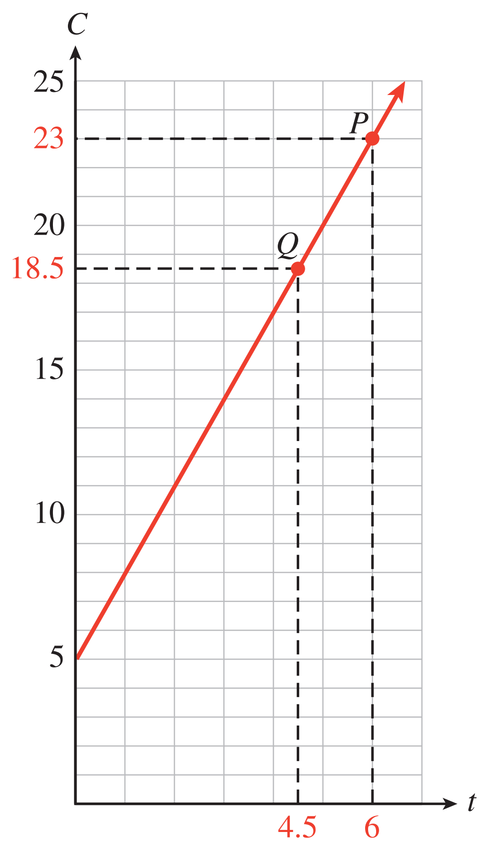
$$C = 5 + 3(\mathbf{6}) = 23$$

A 6-hour bike ride will cost \$23. The point P on the graph in the figure represents the cost of a 6-hour bike ride. The value on the C-axis at the same height as point P is 23, so a 6-hour bike ride costs \$23.

b Substitute $C = \mathbf{18.50}$ into the equation and solve for t.

$$\begin{aligned} \mathbf{18.50} &= 5 + 3t \\ 13.50 &= 3t \\ t &= 4.5 \end{aligned}$$

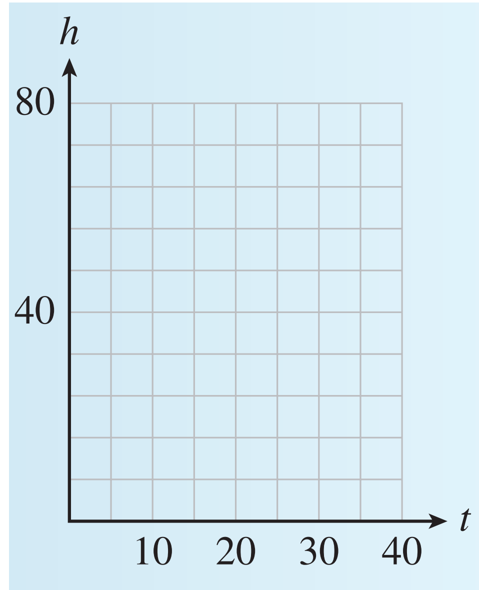
For \$18.50 Annelise can bicycle for $4\frac{1}{2}$ hours. The point Q on the graph represents an \$18.50 bike ride. The value on the t-axis below point Q is 4.5, so \$18.50 will buy a 4.5 hour bike ride.



In Example 1.2, notice the different algebraic techniques we used in parts (a) and (b). In part (a), we were given a value of t and we **evaluated the expression** $5+3t$ to find C. In part (b) we were given a value of C and we **solved the equation** $C = 5 + 3t$ to find t.

Exercise 1.3. Frank plants a dozen corn seedlings, each 6 inches tall. With plenty of water and sunlight they will grow approximately 2 inches per day. Complete the table of values for the height, h, of the seedlings after t days.

t	0	5	10	15	20
h					



a Write an equation for the height of the seedlings in terms of the number of days since they were planted.

b Graph the equation.

Exercise 1.4. Use your equation from Exercise 1.3 to answer the questions. Illustrate each answer on the graph.

a How tall is the corn after 3 weeks?

b How long will it be before the corn is 6 feet tall?

1.1.2 Choosing Scales for the Axes

To draw a useful graph, we must choose appropriate scales for the axes. They must extend far enough to show the values of the variables, and the tick marks should be equally spaced. Usually no more than 10 or 15 tick marks are needed.

Example 1.5. In 1990, the median home price in the US was \$92,000. The median price increased by about \$4700 per year over the next decade.

a Make a table of values showing the median price of a house in 1990, 1994, 1998, and 2000.

b Choose suitable scales for the axes and plot the values you found in part (a) on a graph. Use t, the number of years since 1990, on the horizontal axis and the price of the house, P, on the vertical axis. Draw a curve through the points.

c Write an equation that expresses P in terms of t.

d How much did the price of the house increase from 1990 to 1996? Illustrate the increase on your graph.

Solution.

a In 1990 the median price was \$92,000. Four years later, in 1994, the price had increased by $4(4700) = 18,800$ dollars, so

$$P = 92,000 + 4(4700) = 110,800$$

In 1998 the price had increased by $8(4700) = 37,600$ dollars, so

$$P = 92,000 + 8(4700) - 129,600$$

You can verify the price of the house in 2000 by a similar calculation.

Year	Price of House)	(t, P)
1990	92,000	$(0, 92, 000)$
1994	110,800	$(4, 110, 800)$
1998	129,600	$(8, 129, 600)$
2000	139,000	$(10, 139, 000)$

b Let t stand for the number of years since 1990, so that $t = 0$ in 1990, $t = 4$ in 1994, and so on. To choose scales for the axes, look at the values in the table. For this graph we scale the horizontal axis, or t-axis, in 1-year intervals and the vertical axis, or P-axis, for \$90,000 to \$140,000 in intervals of \$5,000. The points in Figure 1. lie on a straight line.

c Look back at the calculations in part (a). The price of the house started at \$92,000 in 1990 and increased by $t \times 4700$ dollars after t years. Thus,

$$P = 92,000 + 4700t$$

d Find the points on the graph corresponding to 1990 and 1996. These points lie above $t = 0$ and $t = 6$ on the t-axis. Now find the values on the P-axis corresponding to the two points. The values are $P = 92,000$ in 1990 and $P = 120,200$ in 1996. The increase in price is the difference of the two P-values.

$$\begin{aligned} \text{increase in price} &= 120,200 - 92,000 \\ &= 28,200 \end{aligned}$$

The price of the home increased \$28,200 between 1990 and 1996. This increase is indicated by the arrows in Figure 1.6 [ref].

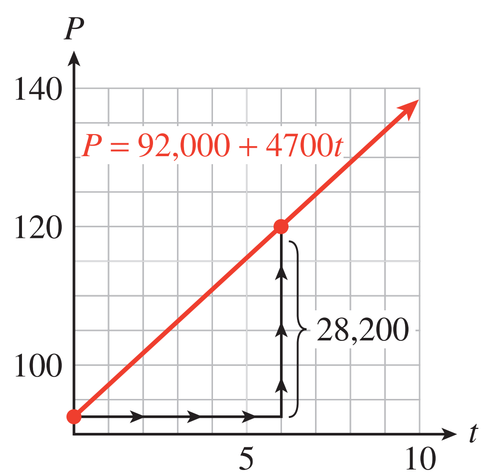


Figure 1.6

The graphs in the preceding examples are **increasing graphs**. As we move along the graph from left to right (in the direction of increasing t), the second coordinate increases as well. Try Exercise 3, which illustrates a **decreasing graph**.

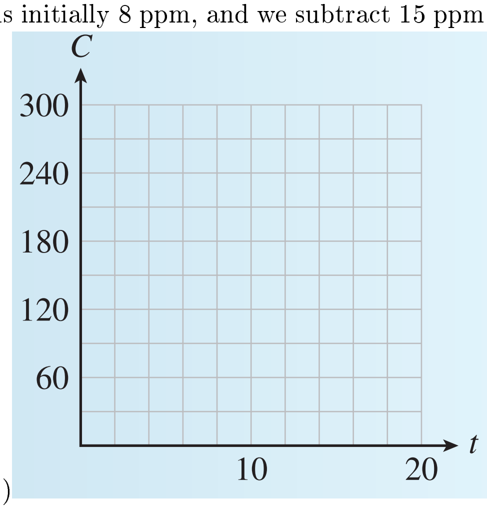
Exercise 1.7. Silver Lake has been polluted by industrial waste products. The concentration of toxic chemicals in the water is currently 285 parts per million (ppm). Local environmental officials would like to reduce the concentration by 15 ppm each year

a Complete the table of values showing the desired concentration, C, of toxic chemicals t years from now. For each t-value, calculate the corresponding value for C. Write your answers as ordered pairs.

t	C		(t, C)
0		$C = 285 - 150(\mathbf{0})$	(0,)
5		$C = 285 - 150(\mathbf{5})$	(5,)
10		$C = 285 - 150(\mathbf{10})$	(10,)
15		$C = 285 - 150(\mathbf{15})$	(15,)

b To choose scales for the axes, notice that the value of C starts at 285 and decreases from there. We'll scale the vertical axis up to 300, and use 10 tick marks at intervals of 30. Graph the ordered pairs on the grid, and connect them with a straight line. Extend the graph until it reaches the horizontal axis, but no farther. Points with negative C-coordinates have no meaning for the problem.

c Write an equation for the concentration, C, of toxic chemicals t years from now. (Hint: The concentration is initially 8 ppm, and we subtract 15 ppm for each year that passes, or $15 \times t$.)



Remark 1.8 (**.** images/icon-GC.jpgGraphing an Equation] We can use a graphing calculator to graph an equation. On most calculators, we follow three steps.

To Graph an Equation:

1. Press `Y=` and enter the equation you wish to graph.
2. Press `WINDOW` and select a suitable graphing window.
3. Press `GRAPH`

Example 1.9 (**.** images/icon-GC.jpgUsing a Graphing Calculator] In Example 1.5, we found the equation $P = 92,000 + 4700t$ for the median price of a house t years after 1990. Graph this equation on a calculator.

Solution. To begin, we press `Y=` and enter

$$Y1 = 92,000 + 4700X$$

For this graph, we'll use the grid in Example 1.5 for our window settings, so we press `WINDOW` and enter

Xmin= 0	Xmax= 10
Ymin= 90,000	Ymax= 140,000

Finally, we press `GRAPH`. The calculator's graph is shown in Figure 1.10.

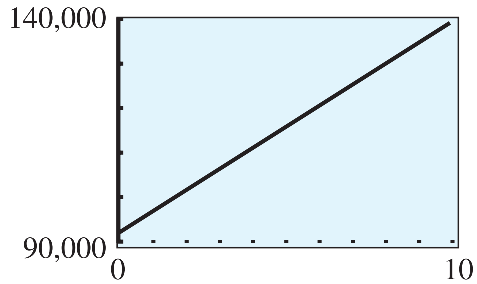


Figure 1.10

Exercise 1.11.

a Solve the equation $2y1575 = 45x$ for y in terms of x.

b Graph the equation on a graphing calculator. Use the window

Xmin= -50	Xmax= 50	Xscl= 5
Ymin= -500	Ymax= 1000	Yscl= 100

c Sketch the graph on paper. Use the window settings to choose appropriate scales for the axes.

1.1.3 Linear Equations

All the models in these examples have equations with a similar form:

$$y = (\text{starting value}) + (\text{rate of change}) \cdot x$$

(We'll talk more about rate of change in Section 1.4.) Their graphs were all portions of straight lines. For this reason such equations are called **linear equations**. The order of the terms in the equation does not matter. For example, the equation in Example 1,

$$C = 5 + 3t$$

can be written equivalently as

$$-3t + C = 5$$

and the equation in Example 3,

$$P = 92,000 + 4700t$$

can be written as

$$-4700 + P = 92,000$$

This form of a linear equation,

$$Ax + By = C$$

is called the **general form**.

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General Form for a Linear Equation

The graph of any equation

$$Ax + By = C$$

where A and B are not both equal to zero, is a straight line.

Example 1.12. The manager at Albert's Appliances has $3000 to spend on advertising for the next fiscal quarter. A 30-second spot on television costs $150 per broadcast, and a 30-second radio ad costs $50.

a The manager decides to buy x television ads and y radio ads. Write an equation relating x and y.

b Make a table of values showing several choices for x and y.

c Plot the points from your table, and graph the equation.

Solution.

a Each television ad costs $150, so ads will cost $150. Similarly, radio ads will cost $50. The manager has $3000 to spend, so the sum of the costs must be $3000. Thus,

$$150x + 50y = 3000$$

b Choose some values of x, and solve the equation for the corresponding value of y For example, if $x = \mathbf{10}$ then

$$\begin{aligned} 150(10) + 50y &= 300 \\ 1500 + 50y &= 3000 \\ 50y &= 1500 \\ y &= 30 \end{aligned}$$

If the manager buys 10 television ads, she can also buy 30 radio ads. You can verify the other entries in the table.

x	8	10	12	14
y				

c Plot the points from the table. All the solutions lie on a straight line, as shown in Figure 1.13 .

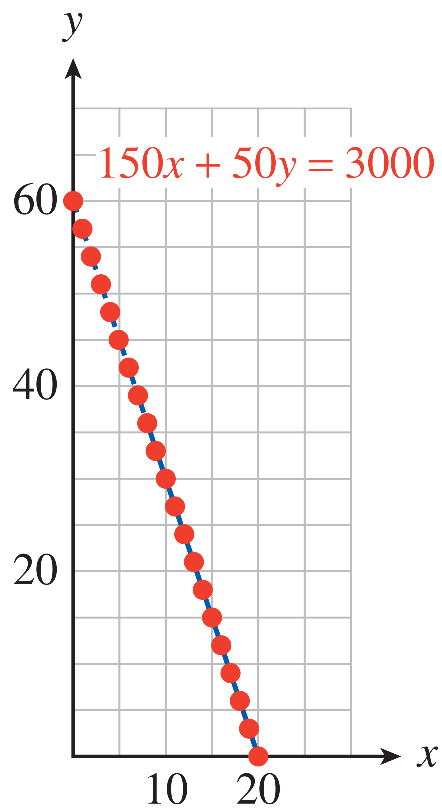

Figure 1.13

Exercise 1.14. In central Nebraska, each acre of corn requires 25 acre-inches of water per year, and each acre of winter wheat requires 18 acre-inches of water. (An acre-inch is the amount of water needed to cover one acre of land to a depth of one inch.) A farmer can count on 9000 acre-inches of water for the coming year. (Source: Institute of Agriculture and Natural Resources, University of Nebraska)

a Write an equation relating the number of acres of corn, x, and the number of acres of wheat, y, that the farmer can plant.

b Complete the table.

x	50	100	150	200
y				

1.1.4 Intercepts

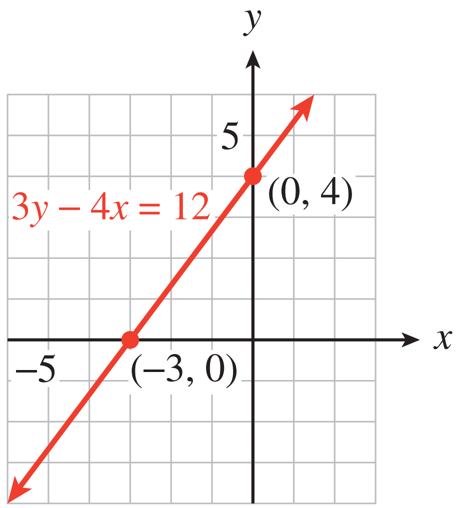

Figure 1.15

Consider the graph of the equation

$$3x - 4y = 12$$

shown in Figure 1.15. The points where the graph crosses the axes are called the **intercepts** of the graph. The coordinates of these points are easy to find. The y-coordinate of the x-intercept is zero, so we set $y = \mathbf{0}$ in the equation to get

$$\begin{aligned} 3(\mathbf{0}) - 4x &= 12 \\ x &= -3 \end{aligned}$$

The x-intercept is the point$(-3, 0)$. Also, the x-coordinate of the y-intercept is zero, so we set $x = \mathbf{0}$ in the equation to get

$$\begin{aligned} 3y - 4(\mathbf{0}) &= 12 \\ y &= 4 \end{aligned}$$

The y-intercept is $(0, 4)$.
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Intercepts of a Graph

The points where a graph crosses the axes are called the **intercepts of the graph**.

1. To find the y-intercept, set $x = 0$ and solve for y.

2. To find the x-intercept, set $y = 0$ and solve for x

The intercepts of a graph tell us something about the situation it models.

Example 1.16.

a Find the intercepts of the graph in Exercise 1.7, about the pollution in Silver Lake.

b What do the intercepts tell us about the problem?

Solution.

a An equation for the concentration of toxic chemicals is

$$C = 285 - 15t$$

To find the C-intercept, set t equal to zero.

$$C = 285 - 15(0) = 285$$

The C-intercept is the point $(0, 285)$, or simply 285. To find the t-intercept, set C equal to zero and solve for t.

$0\& = 285 - 15t\&\&$Add $15t$ to both sides.\\$15t\& = 285\&\&$Divide both sides by 15.\\$t\& = 19\&\&$

The t-intercept is the point $(19, 0)$, or simply 19.

b The C-intercept represents the concentration of toxic chemicals in Silver Lake now: When $t = 0$, $C = 285$, so the concentration is currently 285 ppm. The t-intercept represents the number of years it will take for the concentration of toxic chemicals to drop to zero: When $C = 0$, $t = 19$, so it will take 19 years for the pollution to be eliminated entirely.

Exercise 1.17.

a Find the intercepts of the graph in Example 1.12, about the advertising budget for Albert's Appliances: $150x + 50y = 3000$.

b What do the intercepts tell us about the problem?

1.1.5 Intercept Method for Graphing Lines

Because we really only need two points to graph a linear equation, we might as well find the intercepts first and use them to draw the graph. The values of the intercepts will also help us choose suitable scales for the axes. It is always a good idea to find a third point as a check.

Example 1.18.

a Find the x- and y-intercepts of the graph of $150x180y = 9000$.

b Use the intercepts to graph the equation. Find a third point as a check.

Solution.

a To find the x-intercept, set $y = \mathbf{0}$.

$150x - 18(\mathbf{0})\& = 9000\&\&$Simpify.\\$150x\& = 9000\&\&$Divide both sides by 150.\\$x\& = 60\&\&$

The x-intercept is the point $(60, 0)$. To find the y-intercept, set $x = \mathbf{0}$.

$150(\mathbf{0}) - 18y\& = 9000\&\&$Simpify.\\ $-180y\& = 9000\&\&$Divide both sides by -180.\\$y\& = -50\&\&$

The y-intercept is the point $(0, 50)$.

b Scale both axes in intervals of 10 and then plot the two intercepts, $(60, 0)$ and $(0, 50)$. Draw the line through them, as shown in Figure 1.19. Now find another point and check that it lies on this line. We choose $x = \mathbf{20}$ and solve for y.

$$\begin{aligned} 150(\mathbf{20})180y &= 9000 \\ 3000180y &= 9000 \\ 180y &= 6000 \\ y &= 33.\overline{3} \end{aligned}$$

Plot the point $(20, 33\frac{1}{3})$. Because this point lies on the line, we can be reasonably confident that our graph is correct.

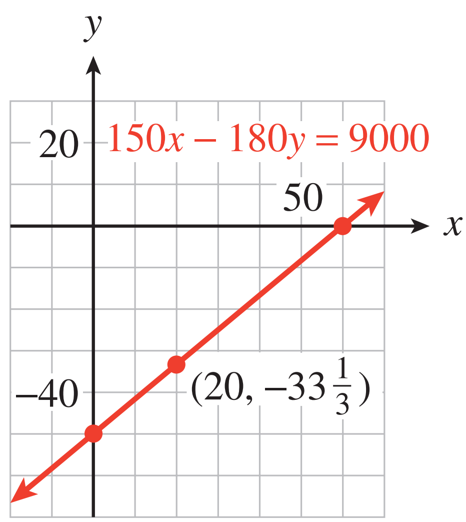

Figure 1.19

Remark 1.20 (. images/icon-GC.jpgChoosing a Graphing Window] Knowing the intercepts can also help us choose a suitable window on a graphing calculator. We would like the window to be large enough to show the intercepts. For the graph in Figure 1.19, we can enter the equation

$$Y = (9000150X)/180$$

in the window

Xmin= -20	Xmax= 70
Ymin= -70	Ymax= 30

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To Graph a Line Using the Intercept Method:

1 Find the intercepts of the line.

++a To find the x-intercept, set $y = 0$ and solve for x.

++b To find the y-intercept, set $x = 0$ and solve for y.

2 Plot the intercepts.

3 Choose a value for x and find a third point on the line.

4 Draw a line through the points.

Exercise 1.21.

a In Exercise 1.14, you wrote an equation about crops in Nebraska. Find the intercepts of the graph.

b Use the intercepts to help you choose appropriate scales for the axes, and then graph the equation.

c What do the intercepts tell us about the problem?

The examples in this section model simple linear relationships between two variables. Such relationships, in which the value of one variable is determined by the value of the other, are called **functions**. We will study various kinds of functions throughout the course.

1.2 Functions

1.2.1 Definition of Function

We often want to predict values of one variable from the values of a related variable. For example, when a physician prescribes a drug in a certain dosage, she needs to know how long the dose will remain in the bloodstream. A sales manager needs to know how the price of his product will affect its sales. A **function** is a special type of relationship between variables that allows us to make such predictions.

Suppose it costs \$800 for flying lessons, plus \$30 per hour to rent a plane. If we let C represent the total cost for t hours of flying lessons, then

$$C = 800 + 305t \quad (t \geq 0)$$

Thus, for example

$$\begin{array}{lll} \text{when} & t = \mathbf{0}, & C = 800 + 30(\mathbf{0}) = 800 \\ \text{when} & t = \mathbf{4}, & C = 800 + 30(\mathbf{4}) = 920 \\ \text{when} & t = \mathbf{10}, & C = 800 + 30(\mathbf{10}) = 1100 \end{array}$$

The variable t is called the **input** or **independent** variable, and C is the **output** or **dependent** variable, because its values are determined by the value of t. We can display the relationship between two variables by a table or by ordered pairs. The input variable is the first component of the ordered pair, and the output variable is the second component.

t	C	(t, C)
0	800	$(0, 800)$
4	920	$(4, 920)$
10	1100	$(10, 1100)$

For this relationship, we can find the value C of associated with any given value of t. All we have to do is substitute the value of t into the equation and solve for C. The result has no ambiguity: Only one value for C corresponds to each value of t. This type of relationship between variables is called a **function**. In general, we make the following definition.

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Definition of Function

A **function** is a relationship between two variables for which a unique value of the **output** variable can be determined from a value of the **input** variable.

What distinguishes functions from other variable relationships? The definition of a function calls for a *unique value*—that is, *exactly one value* of the output variable corresponding to each value of the input variable. This property makes functions useful in applications because they can often be used to make predictions.

Example 1.22.

a The distance, d, traveled by a car in 2 hours is a function of its speed, r. If we know the speed of the car, we can determine the distance it travels by the formula $d = r \cdot 2$.

b The cost of a fill-up with unleaded gasoline is a function of the number of gallons purchased. The gas pump represents the function by displaying the corresponding values of the input variable (number of gallons) and the output variable (cost).

c Score on the Scholastic Aptitude Test (SAT) is not a function of score on an IQ test, because two people with the same score on an IQ test may score differently on the SAT; that is, a person's score on the SAT is not uniquely determined by his or her score on an IQ test.

Exercise 1.23.

a As part of a project to improve the success rate of freshmen, the counseling department studied the grades earned by a group of students in English and algebra. Do you think that a student's grade in algebra is a function of his or her grade in English? Explain why or why not.

b Phatburger features a soda bar, where you can serve your own soft drinks in any size. Do you think that the number of calories in a serving of Zap Kola is a function of the number of fluid ounces? Explain why or why not.

1.2.2 Functions Defined by Tables

When we use a table to describe a function, the first variable in the table (the left column of a vertical table or the top row of a horizontal table) is the input variable, and the second variable is the output. We say that the output variable is a function of the input.

Example 1.24.

a Table 1.25 shows data on sales compiled over several years by the accounting office for Eau Claire Auto Parts, a division of Major Motors. In this example, the year is the input variable, and total sales is the output. We say that total sales, S, is a function of t.

Year (t)	Total sales (S)
2000	\$612,000
2001	\$663,000
2002	\$692,000
2003	\$749,000
2004	\$904,000

Table 1.25

b Table 1.26 gives the cost of sending printed material by first-class mail in 2016.

Weight in ounces (w)	Postage (P)
$0 < w \leq 1$	\$0.47
$1 < w \leq 2$	\$0.68
$2 < w \leq 3$	\$0.89
$3 < w \leq 4$	\$1.10
$4 < w \leq 5$	\$1.31
$5 < w \leq 6$	\$1.52
$6 < w \leq 7$	\$1.73

Table 1.26

If we know the weight of the article being shipped, we can determine the required postage from Table 1.26. For instance, a catalog weighing 4.5 ounces would require \$1.31 in postage. In this example, w is the input variable and p is the output variable. We say that *p is a function of w*.

c Table 1.27 records the age and cholesterol count for 20 patients tested in a hospital survey.

Age	Cholesterol count		Age	Cholesterol count
53	217		**51**	**209**
48	232		53	241
55	198		49	186
56	238		**51**	**216**
51	**227**		57	208
52	264		52	248
53	195		50	214
47	203		56	271
48	212		53	193
50	234		48	172

Table 1.27

According to these data, cholesterol count is *not* a function of age, because several patients who are the same age have different cholesterol levels. For example, three different patients are 51 years old but have cholesterol counts of 227, 209, and 216, respectively. Thus, we cannot determine a *unique* value of the output variable (cholesterol count) from the value of the input variable (age). Other factors besides age must influence a person's cholesterol count.

Exercise 1.28. Decide whether each table describes y as a function of x. Explain your choice.

a

x	3.5	2.0	2.5	3.5	2.5	4.0	2.5	3.0
y	2.5	3.0	2.5	4.0	3.5	4.0	2.0	2.5

b

x	-3	-2	-1	0	1	2	3
y	17	3	0	-1	0	3	17

1.2.3 Functions Defined by Graphs

A graph may also be used to define one variable as a function of another. The input variable is displayed on the horizontal axis, and the output variable on the vertical axis.

Example 1.29. Figure 1.30 shows the number of hours, H, that the sun is above the horizon in Peoria, Illinois, on day t, where January 1 corresponds to $t = 0$.

a Which variable is the input, and which is the output?

b Approximately how many hours of sunlight are there in Peoria on day 150?

c On which days are there 12 hours of sunlight?

d What are the maximum and minimum values of H, and when do these valuesoccur?

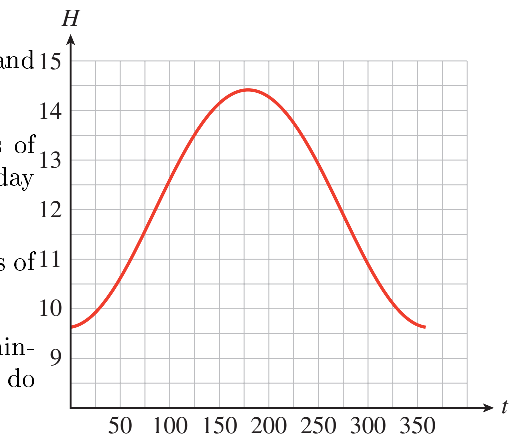

Figure 1.30

Solution.

a The input variable, t, appears on the horizontal axis. The number of daylight hours, H, is a function of the date. The output variable appears on the vertical axis.

b The point on the curve where $t = 150$ has $H \approx 14.1$, so Peoria gets about 14.1 hours of daylight when $t = 150$, which is at the end of May.

c $H = 12$ at the two points where $t \approx 85$ (in late March) and $t \approx 270$ (late September).

d The maximum value of 14.4 hours occurs on the longest day of the year, when $t \approx 170$, about three weeks into June. The minimum of 9.6 hours occurs on the shortest day, when $t \approx 355$, about three weeks into December.

Exercise 1.31. Figure 1.32 shows the elevation in feet, a, of the Los Angeles Marathon course at a distance d miles into the race. (Source: *Los Angeles Times*, March 3, 2005)

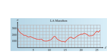

Figure 1.32

a Which variable is the input, and which is the output?

b What is the elevation at mile 20?

c At what distances is the elevation 150 feet?

d What are the maximum and minimum values of a, and when do these values occur?

e The runners pass by the Los Angeles Coliseum at about 4.2 miles into the race. What is the elevation there?

1.2.4 Functions Defined by Equations

illustrates a function defined by an equation.

Example 1.33. As of 2016, One World Trade Center in New York City is the nation's tallest building, at 1776 feet. If an algebra book is dropped from the top of the Sears Tower, its height above the ground after t seconds is given by the equation

$$h = 177616t^2$$

Thus, after **1** second the book's height is

$$h = 177616(\mathbf{1})^2 = 1760 \text{ feet}$$

After **2** seconds its height is

$$h = 177616(\mathbf{2})^2 = 1712 \text{ feet}$$

For this function, t is the input variable and h is the output variable. For any value of t, a unique value of h can be determined from the equation for h. We say that h *is a function of* t.

Exercise 1.34. Write an equation that gives the volume, V, of a sphere as a function of its radius, r.

Remark 1.35 (. images/icon-GC.jpgMaking a Table of Values with a Calculator] We can use a graphing calculator to make a table of values for a function defined by an equation. For the function in Example 1.33,

$$h = 177616t^2$$

we begin by entering the equation: Press the `Y=` key, clear out any other equations, and define $Y_1 = 177616X^2$.

Next, we choose the x-values for the table. Press `2ndWINDOW` to access the TblSet (Table Setup) menu and set it to look like Figure 1.36. This setting will give us an initial x-value of 0 ($TblStart = 0$) and an increment of one unit in the x-values, ($\Delta Tbl = 1$). It also fills in values of both variables automatically. Now press `2nd GRAPH` to see the table of values, as shown in Figure 1.37. From this table, we can check the heights we found in Example 1.33. Now try making a table of values with $TblStart = 0$ and $\Delta Tbl = 0.5$. Use the and arrow keys to scroll up and down the table.

images/fig-TblSetup.pdf

Figure 1.36

images/fig-GC-table.pdf

Figure 1.37

1.2.5 Function Notation

There is a convenient notation for discussing functions. First, we choose a letter, such as f, g, or h (or F, G, or H), to name a particular function. (We can use any letter, but these are the most common choices.) For instance, in Example 1.33, the height, h, of a falling algebra book is a function of the elapsed time, t. We might call this function f. In other words, f is the name of the relationship between the variables h and t. We write

$$h = f(t)$$

which means "h is a function of t, and f is the name of the function."

The new symbol $f(t)$, read "f of t," is another name for the height, h. The parentheses in the symbol $f(t)$ do not indicate multiplication. (It would not make sense to multiply the name of a function by a variable.) Think of the symbol $f(t)$ as a single variable that represents the output value of the function.

With this new notation we may write

$$h = f(t) = 177616t^2$$

or just

$$f(t) = 177616t^2$$

instead of

$$h = 177616t^2$$

to describe the function.

Perhaps it seems complicated to introduce a new symbol for h, but the notation $f(t)$ is very useful for showing the correspondence between specific values of the variables h and t.

Example 1.38. In Example 1.33, the height of an algebra book dropped from the top of the Sears Tower is given by the equation

$$h = 177616t^2$$

We see that

when $t = 1$

$h = 1760$
when $t = 2$

$h = 1712$

Using function notation, these relationships can be expressed more concisely as

$f(1) = 1760$
and
$f(2) = 1712$

which we read as "f of 1 equals 1760" and "f of 2 equals 1712." The values for the input variable, t, appear *inside* the parentheses, and the values for the output variable, h, appear on the other side of the equation.

Remember that when we write $y = f(x)$, the symbol $f(x)$ is just another name for the output variable.
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Function Notation

images/fig-Function-Notation.pdf

Exercise 1.39. Let F be the name of the function defined by the graph in Example 1.29, the number of hours of daylight in Peoria.

a Use function notation to state that H is a function of t.

b What does the statement $F(15) = 9.7$ mean in the context of the problem?

1.2.6 Evaluating a Function

Finding the value of the output variable that corresponds to a particular value of the input variable is called **evaluating the function**.

Example 1.40. Let g be the name of the postage function defined by Table 1.26 in Example 1.22. Find $g(1)$, $g(3)$, and $g(6.75)$.

Solution. According to the table,

when $w = 1$,

$p = 0.47$
so
$g(1) = 0.47$
when $w = 3$,

$p = 0.89$
so
$g(3) = 0.89$
when $w = 6.75$,

$p = 1.73$
so
$g(6.75) = 1.73$

Thus, a letter weighing 1 ounce costs \$0.47 to mail, a letter weighing 3 ounces costs \$0.89, and a letter weighing 6.75 ounces costs \$1.73.

Exercise 1.41. When you exercise, your heart rate should increase until it reaches your target heart rate. The table shows target heart rate, $r = f(a)$, as a function of age.

a	20	25	30	35	40	45	50	55	60	65	70
r	150	146	142	139	135	131	127	124	120	116	112

a Find $f(25)$ and $f(50)$.

b Find a value of a for which $f(a) = 135$.

If a function is described by an equation, we simply substitute the given input value into the equation to find the corresponding output, or function value.

Example 1.42. The function H is defined by $H = f(s) = \frac{\sqrt{s+3}}{s}$. Evaluate the function at the following values.

a $s = 6$

b $s = -1$

Solution.

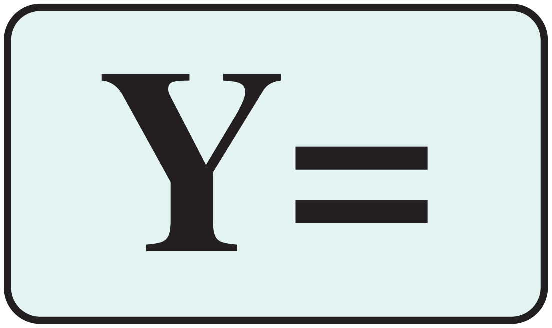
a $f(\mathbf{6}) = \frac{\sqrt{\mathbf{6}+3}}{\mathbf{6}} = \frac{\sqrt{9}}{6} = \frac{3}{6} = \frac{1}{2}$. Thus, $f(6) = \frac{1}{2}$.

b $f(\mathbf{-1}) = \frac{\sqrt{\mathbf{-1}+3}}{\mathbf{-1}} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$. Thus, $f(-1) = -\sqrt{2}$.

Exercise 1.43. Complete the table displaying ordered pairs for the function $f(x) = 5x^3$. Evaluate the function to find the corresponding $f(x)$-value for each value of x.

x	$f(x)$	
-2		$f(\mathbf{-2}) = 5 - (\mathbf{-2})^3 =$
0		$f(\mathbf{0}) = 5 - \mathbf{0}^3 =$
1		$f(\mathbf{1}) = 5 - \mathbf{1}^3 =$
3		$f(\mathbf{3}) = 5 - \mathbf{3}^3 =$

Remark 1.44 (. images/icon-GC.jpgEvaluating a Function] We can use the table feature on a graphing calculator to evaluate functions. Consider the function of Exercise 1.43, $f(x) = 5x^3$.



Press , clear any old functions, and enter

$$Y_1 = 5X\ {}_3$$

Then press TblSet (,`2nd WINDOW`) and choose Ask after Indpnt, as shown in Figure 1.45, and press `ENTER`. This setting allows you to enter any x-values you like. Next, press TABLE (using `2nd GRAPH`). To follow Exercise 1.43, key in `(-) 2 ENTER` for the x-value, and the calculator will fill in the y-value. Continue by entering 0, 1, 3, or any other x-values you choose. One such table is shown in Figure 1.46. If you would like to evaluate a new function, you do not have to return to the `Y=` screen. Use the and arrow keys to highlight Y_1 at the top of the second column. The definition of Y_1 will appear at the bottom of the display, as shown in Figure 1.46. You can key in a new definition here, and the second column will be updated automatically to show the y-values of the new function.

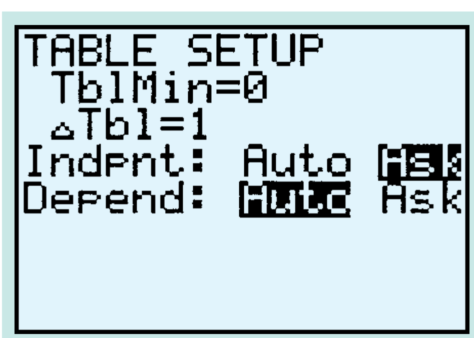


Figure 1.45

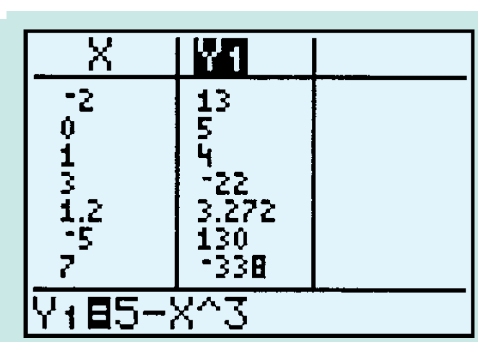


Figure 1.46

To simplify the notation, we sometimes use the same letter for the output variable and for the name of the function. In the next example, C is used in this way.

Example 1.47. TrailGear decides to market a line of backpacks. The cost, C, of manufacturing backpacks is a function of the number, x, of backpacks produced, given by the equation

$$C(x) = 3000 + 20x$$

where $C(x)$ is measured in dollars. Find the cost of producing 500 backpacks.

Solution. To find the value of C that corresponds to $x = \mathbf{500}$, evaluate $C(500)$.

$$C(\mathbf{500}) = 3000 + 20(\mathbf{500}) = 13,000$$

The cost of producing 500 backpacks is $13,000.

Exercise 1.48. The volume of a sphere of radius r centimeters is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

Evaluate $V(10)$ and explain what it means.

1.2.7 Operations with Function Notation

Sometimes we need to evaluate a function at an algebraic expression rather than at a specific number.

Example 1.49. TrailGear manufactures backpacks at a cost of

$$C(x) = 3000 + 20x$$

for x backpacks. The company finds that the monthly demand for backpacks increases by 50

a If each co-op usually produces b backpacks per month, how many should it produce during the summer months?

b What costs for producing backpacks should the company expect during the summer?

Solution.

a An increase of 50

$$C(\mathbf{1.5b}) = 3000 + 20(\mathbf{1.5b}) = 3000 + 30b$$

Exercise 1.50. A spherical balloon has a radius of 10 centimeters.

a If we increase the radius by h centimeters, what will the new volume be?

b If $h = 2$, how much did the volume increase?

Example 1.51. Evaluate the function $f(x) = 4x^2x{+}5$ for the following expressions.

a $x = 2h$

b $x = a + 3$

Solution.

a

$$f(\mathbf{2h})\& = 4(\mathbf{2h})^2(\mathbf{2h}) + 5\backslash\backslash\& = 4(4h^2)2h + 5\backslash\backslash\& = 16h^2 2h + 5\backslash\backslash$$

b

$$f(\mathbf{a+3})\& = 4(\mathbf{a+3})^2(\mathbf{a+3}) + 5\backslash\backslash\& = 4(a^2+6a+9)a - 3 + 5\backslash\backslash\& = 4a^2 + 24a + 36a + 2\backslash\backslash\& = 4a$$

CAUTION In Example 1.51, notice that

$$f(2h) \neq 2f(h)$$

and

$$f(a+3) \neq f(a) + f(3)$$

To compute $f(a) + f(3)$, we must first compute $f(a)$ and $f(3)$, then add them:

$$f(a) + f(3)\& = (4a^2a+5) + (4 \cdot 3^23+5)\backslash\backslash\& = 4a^2a+43\backslash\backslash$$

In general, it is not true that $f(a+b) = f(a)+f(b)$. Remember that the parentheses in the expression $f(x)$ do not indicate multiplication, so the distributive law does not apply to the expression $f(a+b)$.

Exercise 1.52. Let $f(x) = x^31$ and evaluate each expression.

a $f(2) + f(3)$

b $f(2+3)$

c $2f(x) + 3$

1.3 Graphs of Functions

1.3.1 Reading Function Values from a Graph

The graph in Figure 1.53 shows the Dow-Jones Industrial Average (the average value of the stock prices of 500 major companies) during the stock market correction of October 1987. The Dow-Jones Industrial Average (DJIA) is given as a function of time during the 8 days from October 15 to October 22; that is, $f(t)$ is the DJIA recorded at noon on day t.

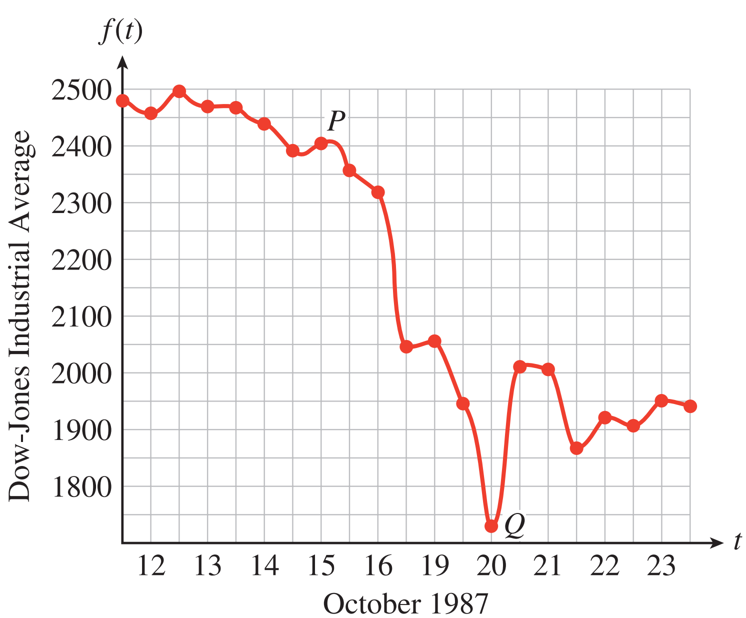


Figure 1.53

The values of the input variable, time, are displayed on the horizontal axis, and the values of the output variable, DJIA, are displayed on the vertical axis. There is no formula that gives the DJIA for a particular day; but it is still a function, defined by its graph. The value of $f(t)$ is specified by the vertical coordinate of the point with the given t-coordinate.

Example 1.54.

a The coordinates of point P in Figure 1.53 are $(15, 2412)$. What do the coordinates tell you about the function f?

b If the DJIA was 1726 at noon on October 20, what can you say about the graph of f?

Solution.

a The coordinates of point P tell us that $f(15) = 2412$, so the DJIA was 2412 at noon on October 15.

b We can say that $f(20) = 1726$, so the point $(20, 1726)$ lies on the graph of f. This point is labeled Q in Figure 1.53.

Thus, the coordinates of each point on the graph of the function represent a pair of corresponding values of the two variables. In general, we can make the following statement.

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Graph of a Function

The point (a, b) lies on the graph of the function f if and only if $f(a) = b$.

Exercise 1.55. The water level in Lake Huron alters unpredictably over time. The graph in Figure 1.56 gives the average water level, $L(t)$, in meters in the year t over a 20-year period. (Source: The Canadian Hydrographic Service) images/fig-Lake-Huron.pdf

Figure 1.56

a The coordinates of point H in Figure 1.56 are $(1997, 176.98)$. What do the coordinates tell you about the function L?

b The average water level in 2004 was 176.11 meters. Write this fact in function notation. What can you say about the graph of L?

Another way of describing how a graph depicts a function is as follows: linecolor@default@@middlelinecolor@default

Functions and Coordinates

Each point on the graph of the function f has coordinates $(x, f(x))$ for some value of x.

Example 1.57. Figure 1.58 shows the graph of a function g.

a Find $g(2)$ and $g(5)$.

b For what value(s) of t is $g(t) = 2$?

c What is the largest, or maximum, value of $g(t)$? For what value of t does the function take on its maximum value?

d On what intervals is g increasing?

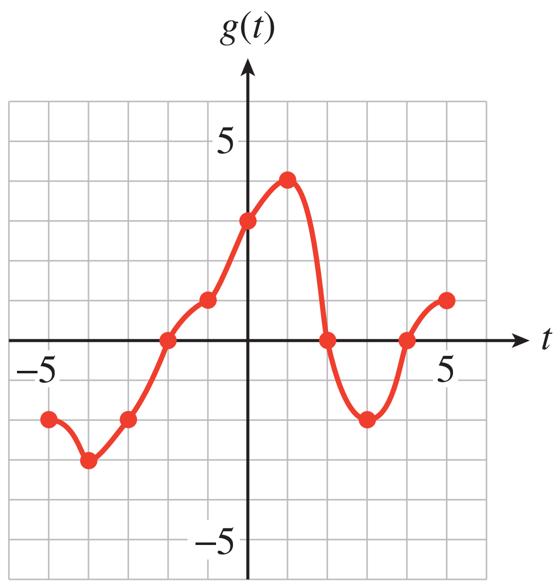


Figure 1.58

Solution.

a To find $g(2)$, we look for the point with t-coordinate 2. The point $(2, 0)$ lies on the graph of g, so $g(2) = 0$. Similarly, the point $(5, 1)$ lies on the graph, so $g(5) = 1$.

b We look for points on the graph with y-coordinate 2. Because the points $(5, 2)$, $(3, 2)$, and $(3, 2)$ lie on the graph, we know that $g(5) = 2$, $g(3) = 2$, and $g(3) = 2$. Thus, the t-values we want are 5, 3, and 3.

c The highest point on the graph is $(1, 4)$, so the largest y-value is 4. Thus, the maximum value of $g(t)$ is 4, and it occurs when $t = 1$.

d A graph is increasing if the y-values get larger as we read from left to right. The graph of g is increasing for t-values between 4 and 1, and between 3 and 5. Thus, g is increasing on the intervals $(4, 1)$ and $(3, 5)$.

Exercise 1.59. Refer to the graph of the function g shown in Figure 1.58 in Example 1.57.

a Find $g(0)$.

b For what value(s) of t is $g(t) = 0$?

c What is the smallest, or minimum, value of $g(t)$? For what value of t does the function take on its minimum value?

d On what intervals is g decreasing?

Remark 1.60 (. images/icon-GC.jpgFinding Coordinates with a Graphing Calculator] We can use the `TRACE` feature of the calculator to find the coordinates of points on a graph. For example, graph the equation $y = 2.6x5.4$ in the window

Xmin$= -5$	Xmax$= 4.4$
Ymin$= -20$	Ymax$= 15$

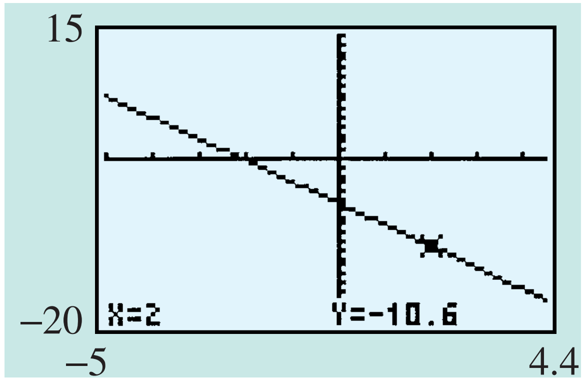


Figure 1.61

Press TRACE, and a “bug” begins flashing on the display. The coordinates of the bug appear at the bottom of the display, as shown in Figure 1.61. Use the left and right arrows to move the bug along the graph. You can check that the coordinates of the point $(2, 10.6)$ do satisfy the equation $y = 2.6x5.4$.

The points identified by the Trace bug depend on the window settings and on the type of calculator. If we want to find the y-coordinate for a particular x-value, we enter the x-coordinate of the desired point and press ENTER.

1.3.2 Constructing the Graph of a Function

Although some functions are defined by their graphs, we can also construct graphs for functions described by tables or equations. We make these graphs the same way we graph equations in two variables: by plotting points whose coordinates satisfy the equation.

Example 1.62. Graph the function $f(x) = \sqrt{x+4}$.

Solution. Choose several convenient values for x and evaluate the function to find the corresponding $f(x)$-values. For this function we cannot choose x-values less than 4, because the square root of a negative number is not a real number.

$$f(\mathbf{4}) = \sqrt{\mathbf{4}+4} = \sqrt{0} = 0$$

$$f(\mathbf{3}) = \sqrt{\mathbf{3}+4} = \sqrt{1} = 1$$

$$f(\mathbf{0}) = \sqrt{\mathbf{0}+4} = \sqrt{4} = 2$$

$$f(\mathbf{2}) = \sqrt{\mathbf{2}+4} = \sqrt{6} \approx 2.45$$

$$f(\mathbf{5}) = \sqrt{\mathbf{5}+4} = \sqrt{9} = 3$$

The results are shown in the table.

x	$f(x)$
-4	0
-3	1
0	2
2	$\sqrt{6}$
5	3

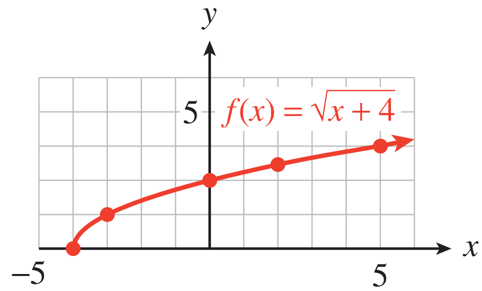


Figure 1.63

Remark 1.64 (. images/icon-GC.jpgUsing a Calculator to Graph a Function] We can also use a graphing calculator to obtain a table and graph for the function in Example 1.62. We graph a function just as we graphed an equation. For this function, we enter

$$Y_1 = \sqrt{\quad (X+4)}$$

and press `ZOOM` 6 for the standard window. (See ⟨⟨appendix-b⟩⟩ for details.) The calculator's graph is shown in Figure 1.65.

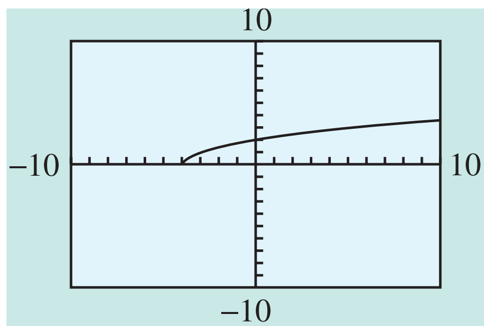


Figure 1.65

Exercise 1.66.

$$f(x) = x^3 2$$

a Complete the table of values and sketch a graph of the function.

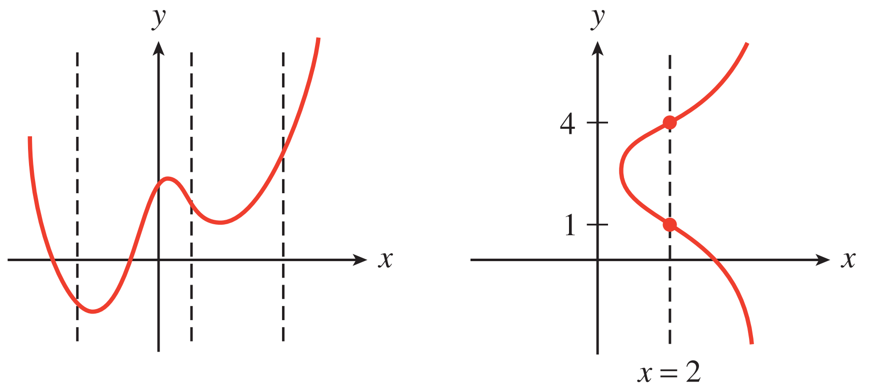
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
$f(x)$							

b Use your calculator to make a table of values and graph the function.

1.3.3 The Vertical Line Test

In a function, two different outputs cannot be related to the same input. This restriction means that two different ordered pairs cannot have the same first coordinate. What does it mean for the graph of the function?

Consider the graph shown in Figure 1.67a. Every vertical line intersects the graph in at most one point, so there is only one point on the graph for each x-value. This graph represents a function. In Figure 1.67b, however, the line $x = 2$ intersects the graph at two points, $(2, 1)$ and $(2, 4)$. Two different y-values, 1 and 4, are related to the same x-value, 2. This graph cannot be the graph of a function.



(a) (b)

Figure 1.67

We summarize these observations as follows.
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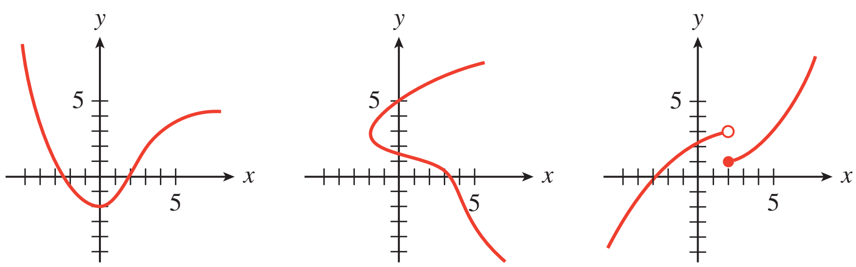
The Vertical Line Test

A graph represents a function if and only if every vertical line intersects the graph in at most one point.

Example 1.68. Use the vertical line test to decide which of the graphs in Figure 1.69 represent functions.



(a) (b) (c)

Figure 1.69

Solution. Graph (a) represents a function, because it passes the vertical line test. Graph (b) is not the graph of a function, because the vertical line at (for example) $x = 1$ intersects the graph at two points. For graph (c), notice the break in the curve at $x = 2$: The solid dot at $(2, 1)$ is the only point on the graph with $x = 2$; the open circle at $(2, 3)$ indicates that $(2, 3)$ is not a point on the graph. Thus, graph (c) is a function, with $f(2) = 1$.

Exercise 1.70. Use the vertical line test to determine which of the graphs in Figure 1.71 represent functions.

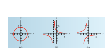


Figure 1.71

1.3.4 Graphical Solution of Equations and Inequalities

The graph of an equation in two variables is just a picture of its solutions. When we read the coordinates of a point on the graph, we are reading a pair of x- and y-values that make the equation true.

For example, the point $(2, 7)$ lies on the graph of $y = 2x + 3$ shown in Figure 1.72 , so we know that the ordered pair $(2, 7)$ is a solution of the equation $y = 2x + 3$. You can verify algebraically that $x = \mathbf{2}$ and $y = \mathbf{7}$ satisfy the equation:

$$\text{Does } \mathbf{7} = 2(\mathbf{2}) + 3? \text{ Yes}$$

We can also say that $x = 2$ is a solution of the one-variable equation $2x + 3 = 7$. In fact, we can use the graph of $y = 2x + 3$ to solve the equation $2x + 3 = k$ for any value of k. Thus, we can use graphs to find solutions to equations in one variable.

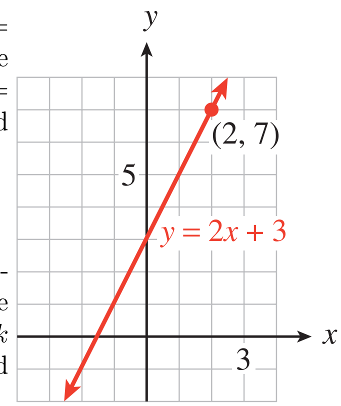


Figure 1.72

Example 1.73. Use the graph of $y = 28515x$ to solve the equation $150 = 28515x$.

Solution.

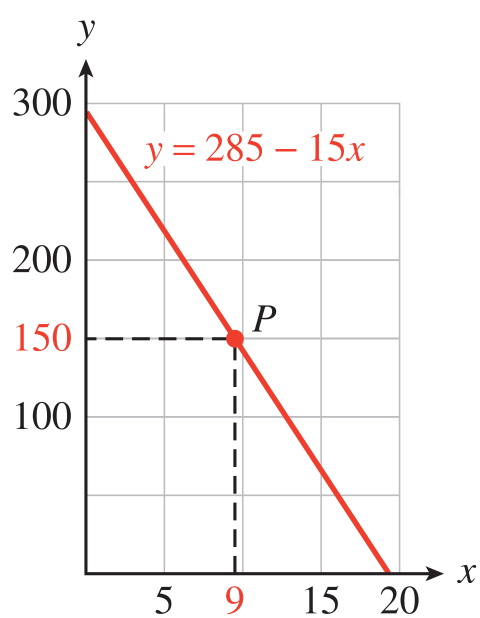


Figure 1.74

Begin by locating the point P on the graph for which $y = 150$, as shown in Figure 1.74. Now find the x-coordinate of point P by drawing an imaginary line from P straight down to the x-axis. The x-coordinate of P is $x = 9$. Thus, P is the point $(9, 150)$, and $x = 9$ when $y = 150$. The solution of the equation $150 = 28515x$ is $x = 9$. You can verify the solution algebraically by substituting $x = \mathbf{9}$ into the equation:

Does $150 = 28515(\mathbf{9})$?

$$28515(\mathbf{9}) = 285135 = 150. \quad \textbf{Yes}$$

The relationship between an equation and its graph is an important one. For the previous example, make sure you understand that the following three statements are equivalent:

1. The point $(9, 150)$ lies on the graph of $y = 28515x$.

2. The ordered pair $(9, 150)$ is a solution of the equation $y = 28515x$.

3. $x = 9$ is a solution of the equation $150 = 28515x$.

Exercise 1.75.

a Use the graph of $y = 308x$ shown in Figure 1.76 to solve the equation

$$308x = 50$$

b Verify your solution algebraically.

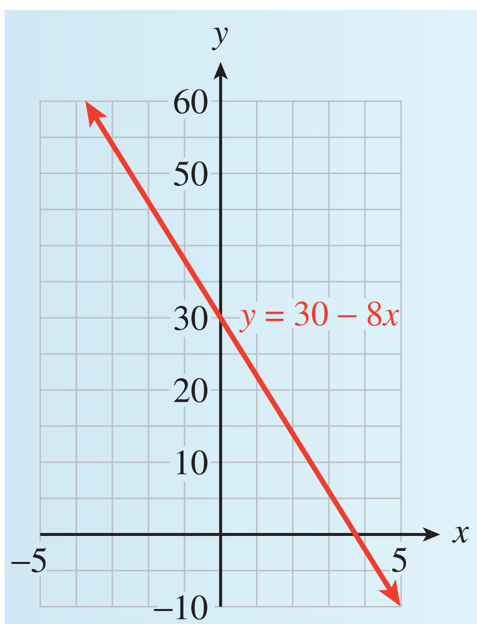


Figure 1.76

In a similar fashion, we can solve inequalities with a graph. Consider again the graph of $y = 2x + 3$, shown in Figure 1.77. We saw that $x = 2$ is the solution of the equation $2x + 3 = 7$. When we use $x = 2$ as the input for the function $f(x) = 2x+3$, the output is $y = 7$. Which input values for x produce output values greater than 7? You can see in Figure 1.77 that x-values greater than 2 produce y-values greater than 7, because points on the graph with x-values greater than 2 have y-values greater than 7. Thus, the solutions of the inequality $2x + 3 > 7$ are $x > 2$. You can verify this result by solving the inequality algebraically.

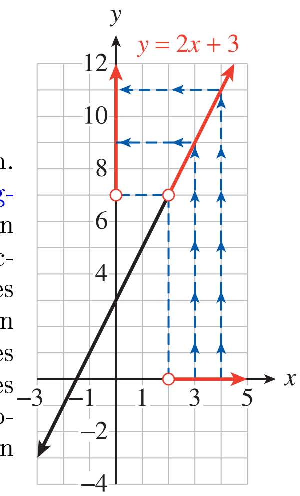


Figure 1.77

Example 1.78. Use the graph of $y = 28515x$ to solve the inequality

$$28515x > 150$$

Solution. We begin by locating the point P on the graph for which $y = 150$ and $x = 9$ (its x-coordinate). Now, because $y = 28515x$ for points on the graph, the inequality $28515x > 150$ is equivalent to $y > 150$. So we are looking for points on the graph with y-coordinate greater than 150. These points are shown in Figure 1.79. The x-coordinates of these points are the x-values that satisfy the inequality. From the graph, we see that the solutions are $x < 9$.

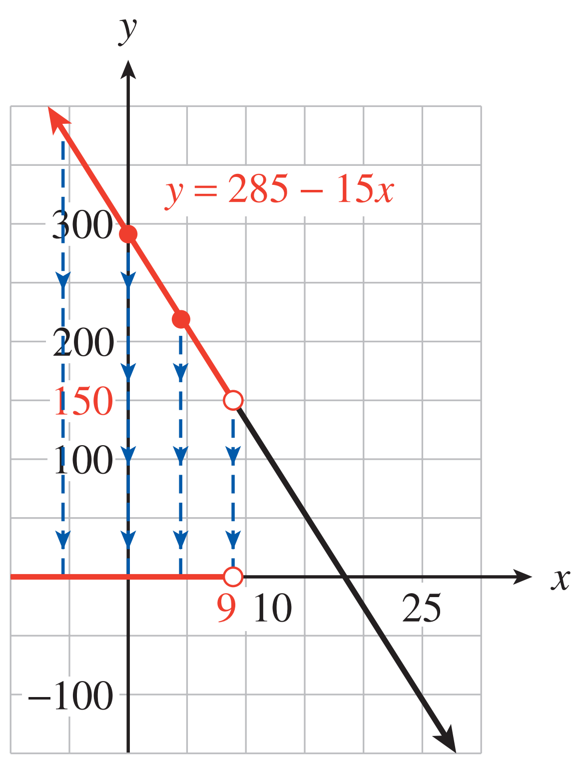


Figure 1.79

Exercise 1.80.

a Use the graph of $y = 308x$ in Figure 1.76 to solve the inequality

$$308x \leq 50$$

b Solve the inequality algebraically.

We can also use this graphical technique to solve nonlinear equations and inequalities.

Example 1.81. Use a graph of $f(x) = 2x^3 + x^2 + 16x$ to solve the equation

$$2x^3 + x^2 + 16x = 15$$

Solution. If we sketch in the horizontal line $y = 15$, we can see that there are three points on the graph of f that have y-coordinate 15, as shown in Figure 1.82. The x-coordinates of these points are the solutions of the equation

$$2x^3 + x^2 + 16x = 15$$

From the graph, we see that the solutions are $x = 3$, $x = 1$, and approximately $x = 2.5$. We can verify the solutions algebraically. For example, if $x = 3$, we have

$$f(3)\&2(3)3 + (3)^2 + 16(3)\backslash\backslash\& = 2(27) + 948\backslash\backslash\& = 54 + 948 = 15$$

so 3 is a solution.

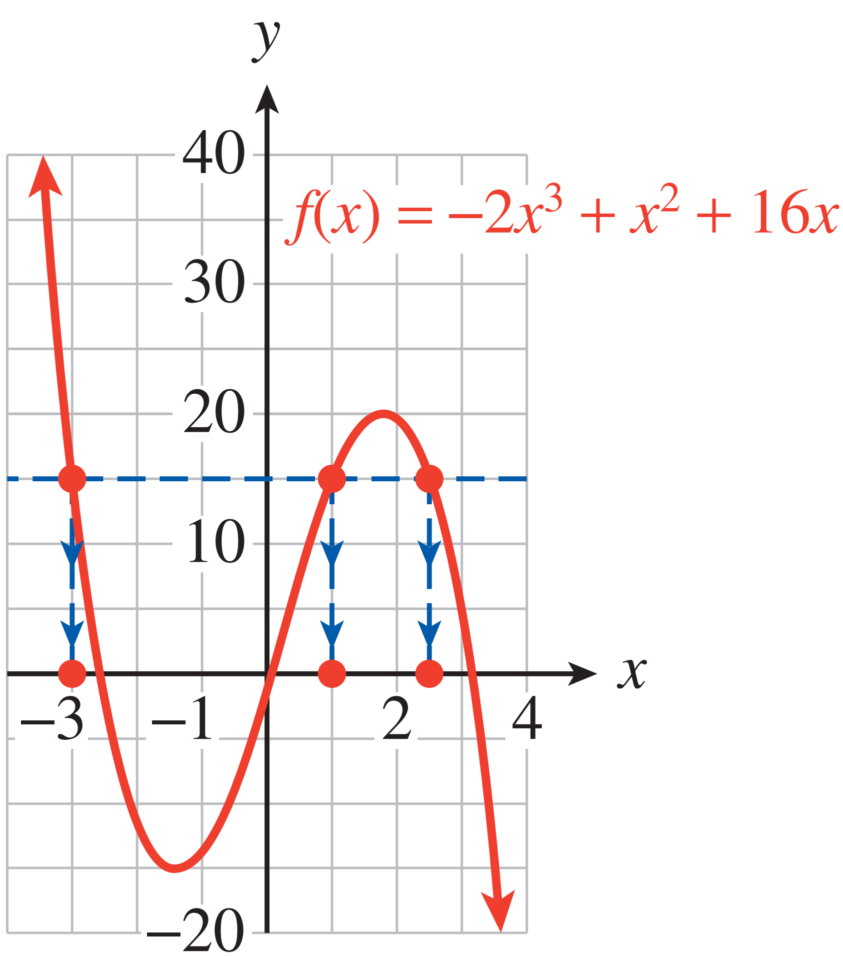


Figure 1.82

Exercise 1.83. Use the graph of $y = \frac{1}{2}n^2 + 2n10$ shown in Figure 1.84 to solve

$$\frac{1}{2}n^2 + 2n10 = 6$$

and verify your solutions algebraically.

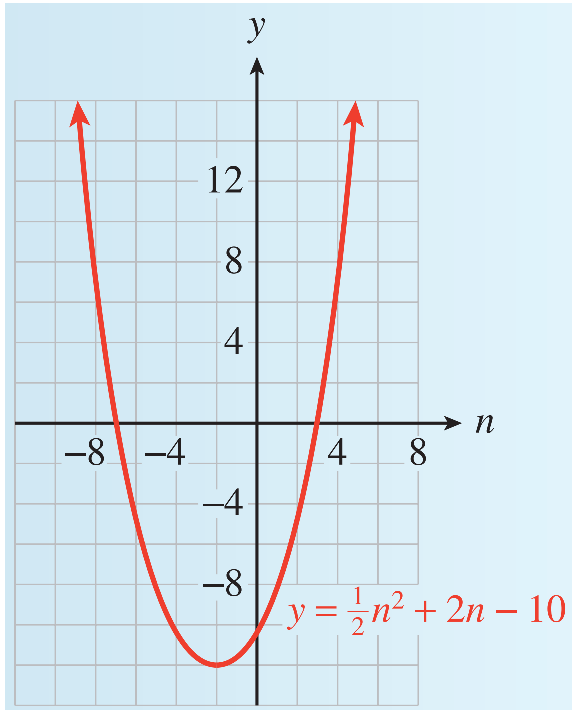


Figure 1.84

Remark 1.85 (. images/icon-GC.jpgUsing the Trace Feature] You can use the Trace feature on a graphing calculator to approximate solutions to equations. Graph the function $f(x)$ in Example 1.81 in the window

Xmin= -4	Xmax= 4
Ymin= -20	Ymax= 40

and trace along the curve to the point $(2.4680851, 15.512401)$. We are close to a solution, because the y-value is close to 15. Try entering x-values close to 2.4680851, for instance, $x = 2.4$ and $x = 2.5$, to find a better approximation for the solution.

We can use the intersect feature on a graphing calculator to obtain more accurate estimates for the solutions of equations. See ⟨⟨appendix-b⟩⟩ for details.

Example 1.86. Use the graph in Example 1.81 to solve the inequality

$$2x^3 + x^2 + 16x \geq 15$$

Solution. We first locate all points on the graph that have y-coordinates greater than or equal to 15. The x-coordinates of these points are the solutions of the inequality. Figure 1.87 shows the points, and their x-coordinates as intervals on the x-axis. The solutions are $x \leq 3$ and $1 \leq x \leq 2.5$, or in interval notation, $(-, 3][1, 2.5]$.

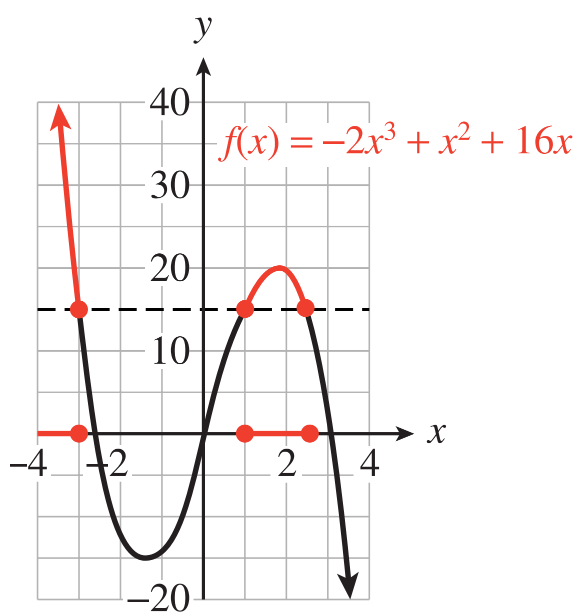


Figure 1.87

Exercise 1.88. Use Figure 1.84 in Exercise 1.83 to solve the inequality

$$\frac{1}{2}n^2 + 2n10 < 6$$

1.4 Slope and Rate of Change

1.4.1 Using Ratios for Comparison

Which is more expensive, a 64-ounce bottle of Velvolux dish soap that costs \$3.52, or a 60-ounce bottle of Rainfresh dish soap that costs \$3.36?

You are probably familiar with the notion of comparison shopping. To decide which dish soap is the better buy, we compute the unit price, or price per ounce, for each bottle. The unit price for Velvolux is

$$\frac{352 \text{ cents}}{64 \text{ ounces}} = 5.5 \text{ cents per ounce}$$

and the unit price for Rainfresh is

$$\frac{336 \text{ cents}}{60 \text{ ounces}} = 5.6 \text{ cents per ounce}$$

The Velvolux costs less per ounce, so it is the better buy. By computing the price of each brand for *the same amount of soap*, it is easy to compare them.

In many situations, a ratio, similar to a unit price, can provide a basis for comparison. Example 1.89 uses a ratio to measure a rate of growth.

Example 1.89. Which grow faster, Hybrid A wheat seedlings, which grow 11.2 centimeters in 14 days, or Hybrid B seedlings, which grow 13.5 centimeters in 18 days?

Solution. We compute the growth rate for each strain of wheat. Growth rate is expressed as a ratio, $\frac{\text{centimeters}}{\text{days}}$, or centimeters per day. The growth rate for Hybrid A is

$$\frac{11.2 \text{ centimeters}}{14 \text{ days}} = 0.8 \text{ centimeters per day}$$

and the growth rate for Hybrid B is

$$\frac{13.5 \text{ centimeters}}{18 \text{ days}} = 0.75 \text{ centimeters per day}$$

Because their rate of growth is larger, the Hybrid A seedlings grow faster.

By computing the growth of each strain of wheat seedling over the same unit of time, a single day, we have a basis for comparison. In this case, the ratio $\frac{\text{centimeters}}{\text{day}}$ measures the rate of growth of the wheat seedlings.

Exercise 1.90. Delbert traveled 258 miles on 12 gallons of gas, and Francine traveled 182 miles on 8 gallons of gas. Compute the ratio $\frac{\text{miles}}{\text{gallon}}$ for each car. Whose car gets the better gas mileage?

In Exercise 1.90, the ratio $\frac{\text{miles}}{\text{gallon}}$ measures the rate at which each car uses gasoline. By computing the mileage for each car for the same amount of gas, we have a basis for comparison. We can use this same idea, finding a common basis for comparison, to measure the steepness of an incline.

1.4.2 Measuring Steepness

Imagine you are an ant carrying a heavy burden along one of the two paths shown in Figure 1.91. Which path is more difficult? Most ants would agree that the steeper path is more difficult.

But what exactly is steepness? It is not merely the gain in altitude, because even a gentle incline will reach a great height eventually. Steepness measures how sharply the altitude increases. An ant finds the second path more difficult, or steeper, because it rises 5 feet while the first path rises only 2 feet over the same horizontal distance.

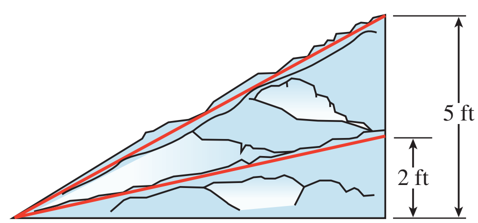


Figure 1.91

To compare the steepness of two inclined paths, we compute the ratio of change in altitude to change in horizontal distance for each path.

Example 1.92. Which is steeper, Stony Point trail, which climbs 400 feet over a horizontal distance of 2500 feet, or Lone Pine trail, which climbs 360 feet over a horizontal distance of 1800 feet?

Solution. For each trail, we compute the ratio of vertical gain to horizontal distance. For Stony Point trail, the ratio is

$$\frac{400 \text{ feet}}{2500 \text{ feet}} = 0.16$$

and for Lone Pine trail, the ratio is

$$\frac{360 \text{ feet}}{1800 \text{ feet}} = 0.20$$

Lone Pine trail is steeper, because it has a vertical gain of 0.20 foot for every foot traveled horizontally. Or, in more practical units, Lone Pine trail rises 20 feet for every 100 feet of horizontal distance, whereas Stony Point trail rises only 16 feet over a horizontal distance of 100 feet.

Exercise 1.93. Which is steeper, a staircase that rises 10 feet over a horizontal distance of 4 feet, or the steps in the football stadium, which rise 20 yards over a horizontal distance of 12 yards?

1.4.3 Definition of Slope

To compare the steepness of the two trails in Example 1.92, it is not enough to know which trail has the greater gain in elevation overall. Instead, we compare their elevation gains over the same horizontal distance. Using the same horizontal distance provides a basis for comparison. The two trails are illustrated in Figure 1.94 as lines on a coordinate grid.

images/fig-trail-climb-grid.pdf

Figure 1.94

The ratio we computed in Example 1.92,

$$\frac{\text{change in elevation}}{\text{change in horizontal position}}$$

appears on the graphs in Figure 1.94 as

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

For example, as we travel along the line representing Stony Point trail, we move from the point$0, 0)$ to the point $(2500, 400)$. The y-coordinate changes by 400 and the x-coordinate changes by 2500, giving the ratio 0.16 that we found in Example 1.922. We call this ratio the **slope** of the line.

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Definition of Slope

The **slope** of a line is the ratio

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

as we move from one point to another on the line.

Example 1.95. Compute the slope of the line that passes through points A and B in Figure 1.96.

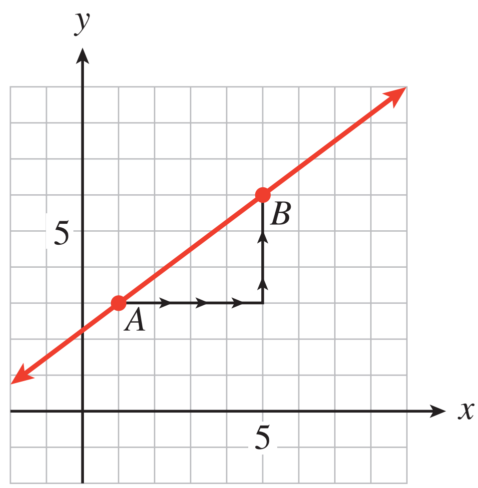

Figure 1.96

Solution. As we move along the line from A to B, the y-coordinate changes by 3 units, and the x-coordinate changes by 4 units. The slope of the line is thus

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} = \frac{3}{4}$$

Exercise 1.97.

Compute the slope of the line through the indicated points in Figure 1.98. On both axes, one square represents one unit.

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}} =$$

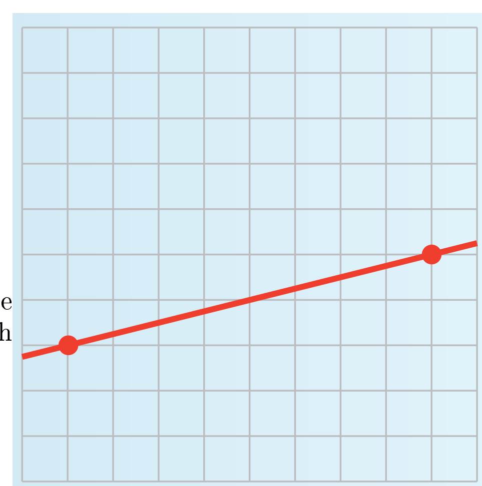

Figure 1.98

The slope of a line is a number. It tells us how much the y-coordinates of points on the line increase when we increase their x-coordinates by 1 unit. For instance, the slope $\frac{3}{4}$ in Example 1.95 means that the y-coordinate increases by $\frac{3}{4}$ unit when the x-coordinate increases by 1 unit. For increasing graphs, a larger slope indicates a greater increase in altitude, and hence a steeper line.

1.4.4 Notation for Slope

We use a shorthand notation for the ratio that defines slope,

$$\frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}$$

The symbol Δ (the Greek letter delta) is used in mathematics to denote *change in*. In particular, Δy means *change in y-coordinate*, and Δx means *change in x-coordinate*. We also use the letter m to stand for slope. With these symbols, we can write the definition of slope as follows.

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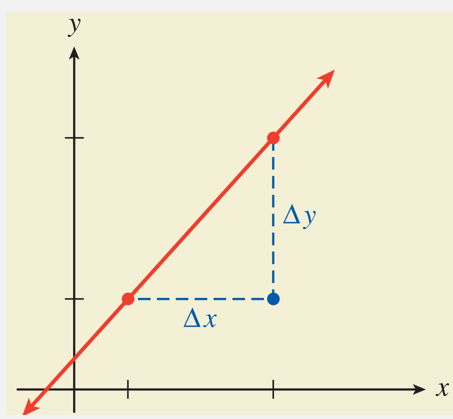
Notation for Slope

The **slope** of a line is given by

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y\text{-coordinate}}{\text{change in } x\text{-coordinate}}, \quad x \neq 0$$



Example 1.99. The Great Pyramid of Khufu in Egypt was built around 2550 B.C. It is 147 meters tall and has a square base 229 meters on each side. Calculate the slope of the sides of the pyramid, rounded to two decimal places.

Solution.

From Figure 1.100, we see that Δx is only half the base of the Great Pyramid, so

$$\Delta x = 0.5(229) = 114.5$$

and the slope of the side is

$$m = \frac{\Delta y}{\Delta x} = \frac{147}{114.5} = 1.28$$

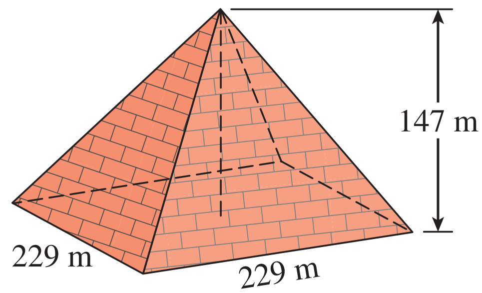


Figure 1.100

Exercise 1.101.

The Kukulcan Pyramid at Chichen Itza in Mexico was built around 800 A.D. It is 24 meters high, with a temple built on its top platform, as shown in Figure 1.102. The square base is 55 meters on each side, and the top platform is 19.5 meters on each side. Calculate the slope of the sides of the pyramid. Which pyramid is steeper, Kukulcan or the Great Pyramid?

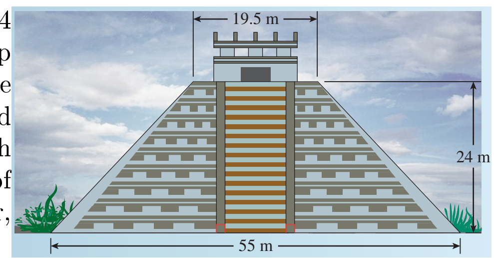


Figure 1.102

So far, we have only considered examples in which Δx and Δy are positive numbers, but they can also be negative.

$$\Delta x = \begin{cases} \text{positive if } x \text{ increases (move to the right)} \\ \text{negative if } x \text{ decreases (move to the left)} \end{cases}$$

$$\Delta y = \begin{cases} \text{positive if } y \text{ increases (move up)} \\ \text{negative if } y \text{ decreases (move down)} \end{cases}$$

Example 1.103.

Compute the slope of the line that passes through the points $P(4, 2)$ and $Q(5, 1)$ shown in Figure 1.104 Illustrate Δy and Δx on the graph.

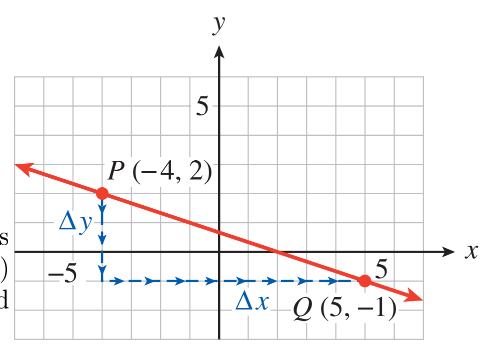


Figure 1.104

Solution. As we move from the point $P(4, 2)$ to the point $Q(5, 1)$, we move 3 units *down*, so $\Delta y = 3$. We then move 9 units to the right, so $\Delta x = 9$. Thus, the slope is

$$m = \frac{\Delta y}{\Delta x} = \frac{-3}{9} = \frac{-1}{3}$$

Δy and Δx are labeled on the graph.

We can move from point to point in either direction to compute the slope. The line graphed in Example 1.103 decreases as we move from left to right and hence has a negative slope. The slope is the same if we move from point Q to point P instead of from P to Q. (See Figure 1.105.) In that case, our computation looks like this:

$$m = \frac{\Delta y}{\Delta x} = \frac{3}{-9} = \frac{-1}{3}$$

Δy and Δx are labeled on the graph.

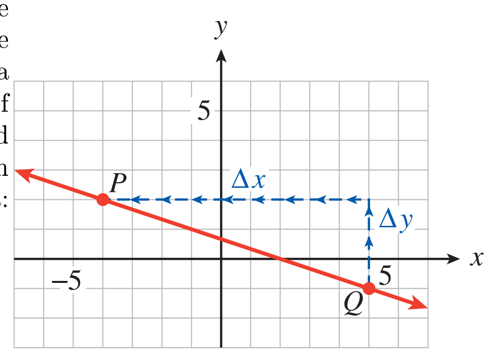


Figure 1.105

1.4.5 Lines Have Constant Slope

How do we know which two points to choose when we want to compute the slope of a line? It turns out that any two points on the line will do.

Exercise 1.106.

a Graph the line $4x2y = 8$ by finding the x- and y-intercepts

b Compute the slope of the line using the x-intercept and y-intercept.

c Compute the slope of the line using the points $(4, 4)$ and $(1, 2)$.

Exercise 1.106 illustrates an important property of lines: They have constant slope. No matter which two points we use to calculate the slope, we will always get the same result. We will see later that lines are the only graphs that have this property. We can think of the slope as a scale factor that tells us how many units y increases (or decreases) for each unit of increase in x. Compare the lines in Figure 1.107

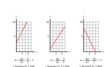


Figure 1.107

Observe that a line with positive slope increases from left to right, and one with negative slope decreases. What sort of line has slope $m = 0$?

1.4.6 Meaning of Slope

In Example 1 of Section 1.1, we graphed the equation $C = 5 + 3t$ showing the cost of a bicycle rental in terms of the length of the rental. The graph is reproduced in Figure 1.108. We can choose any two points on the line to compute its slope. Using points P and Q as shown, we find that

$$m = \frac{\Delta C}{\Delta t} = \frac{9}{3} = 3$$

The slope of the line is 3.

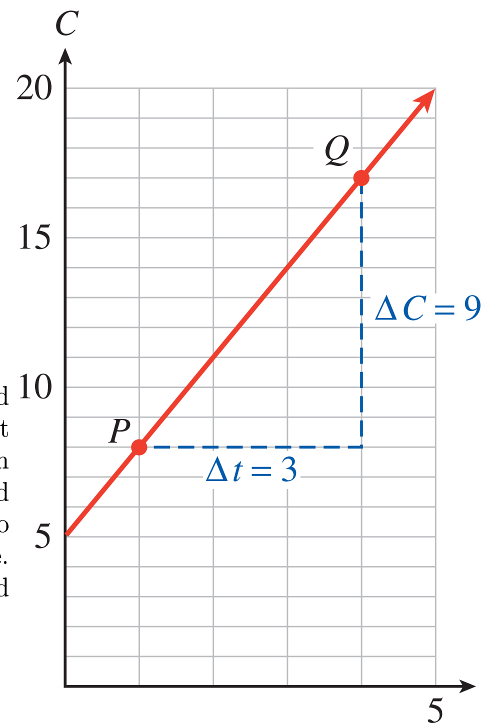


Figure 1.108

What does this value mean for the cost of renting a bicycle? The expression

$$\frac{\Delta C}{\Delta t} = \frac{9}{3}$$

stands for

$$\frac{\text{change in cost}}{\text{change in time}} = \frac{9 \text{ dollars}}{3 \text{ hours}}$$

If we increase the length of the rental by 3 hours, the cost of the rental increases by 9 dollars. The slope gives the rate of increase in the rental fee, 3 dollars per hour. In general, we can make the following statement.

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Rate of Change

The slope of a line measures the *rate of change* of the output variable with respect to the input variable.

Depending on the variables involved, this rate might be interpreted as a rate of growth or a rate of speed. A negative slope might represent a rate of decrease or a rate of consumption. The slope of a graph can give us valuable information about the variables.

Example 1.109. The graph in Figure 1.110 shows the distance in miles traveled by a big-rig truck driver after t hours on the road.

a Compute the slope of the graph.

b What does the slope tell us about the problem?

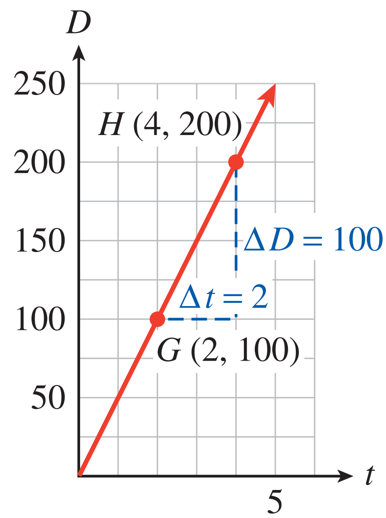

Figure 1.110

Solution.

a Choose any two points on the line, say $G(2, 100)$ and $H(4, 200)$, in Figure 1.110. As we move from G to H, we find

$$m = \frac{\Delta D}{\Delta t} = \frac{100}{2} = 50$$

The slope of the line is 50.

b The best way to understand the slope is to include units in the calculation. For our example,

$$\frac{\Delta D}{\Delta t} \text{ means } \frac{\text{change in distance}}{\text{change in time}}$$

or

$$\frac{\Delta D}{\Delta t} = \frac{100 \text{ miles}}{2 \text{ hours}} = 50\text{miles per hour}$$

The slope represents the trucker's average speed or velocity.

Exercise 1.111. The graph in Figure 1.112 shows the altitude, a (in feet), of a skier t minutes after getting on a ski lift.

a Choose two points and compute the slope (including units).

b What does the slope tell us about the problem?

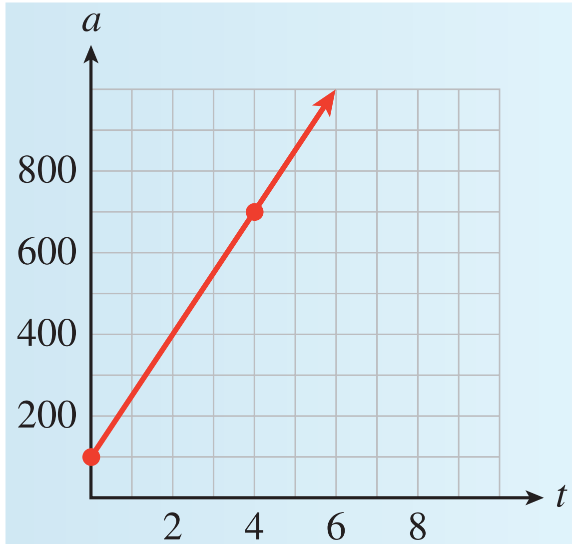

Figure 1.112

1.4.7 A Formula for Slope

We have defined the slope of a line to be the ratio $m = \frac{\Delta y}{\Delta x}$ as we move from one point to another on the line. So far, we have computed Δy and Δx by counting squares on the graph, but this method is not always practical. All we really need are the coordinates of two points on the graph.

We will use **subscripts** to distinguish the two points:

P_1 means *first point* and P_2 means *second point*.

We denote the coordinates of P_1 by (x_1, y_1) and the coordinates of P_2 by (x_2, y_2).

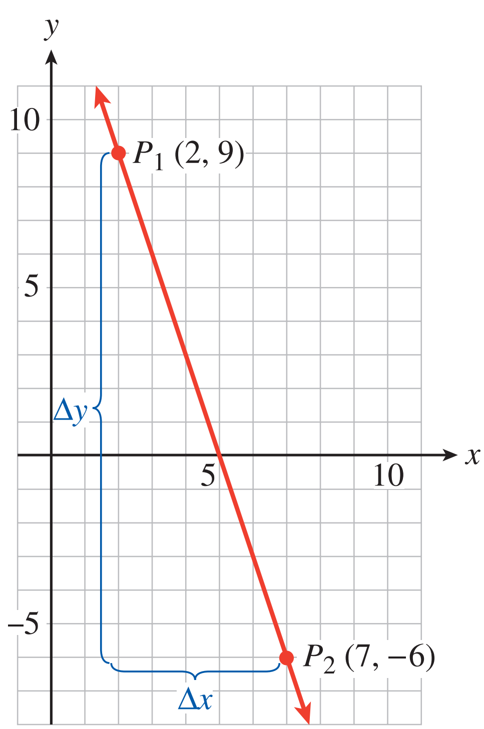

Figure 1.113

Now consider a specific example. The line through the two points $P_1(2, 9)$ and $P_2(7, 6)$ is shown in Figure 1.113. We can find Δx by subtracting the x-coordinates of the points:

$$\Delta x = 72 = 5$$

In general, we have

$$\Delta x = x_2x_1$$

and similarly

$$\Delta y = y_2y_1$$

These formulas work even if some of the coordinates are negative; in our example

$$\Delta y = y_2 y_1 = 69 = 15$$

By counting squares *down* from P_1 to P_2, we see that Δy is indeed 15. The slope of the line in Figure 1.113 is

$$m\& = \frac{\Delta y}{\Delta x} = \frac{y_2 y_1}{x_2 x_1}\backslash\backslash\& = \frac{15}{5} = 3$$

We now have a formula for the slope of a line that works even if we do not have a graph.

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Two-Point Slope Formula

The slope of the line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 y_1}{x_2 x_1}, \quad x_2 \neq x_1$$

Example 1.114. Compute the slope of the line in Figure 1.113 using the points $Q_1(6, 3)$ and $Q_2(4, 3)$.

Solution. Substitute the coordinates of Q_1 and Q_2 into the slope formula to find

$$m = \frac{y_2 y_1}{x_2 x_1} = \frac{3(3)}{46} = \frac{6}{-2} = 3$$

This value for the slope, 3, is the same value found above.

Exercise 1.115.

a Find the slope of the line passing through the points $(2, 3)$ and $(2, 1)$.

b Sketch a graph of the line by hand.

It will also be useful to write the slope formula with function notation. Recall that $f(x)$ is another symbol for y, and, in particular, that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Thus, if $x_2 \neq x_1$, we have

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Slope Formula in Function Notation

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 y_1}{x_2 x_1} = \frac{f(x_2) f(x_1)}{x_2 x_1}, \quad x_2 \neq x_1$$

Example 1.116.

Figure 1.117 shows a graph of

$$f(x) = x_2 6x$$

a Compute the slope of the line segment joining the points at $x = 1$ and $x = 4$.

b Compute the slope of the line segment joining the points at $x = 2$ and $x = 5$.

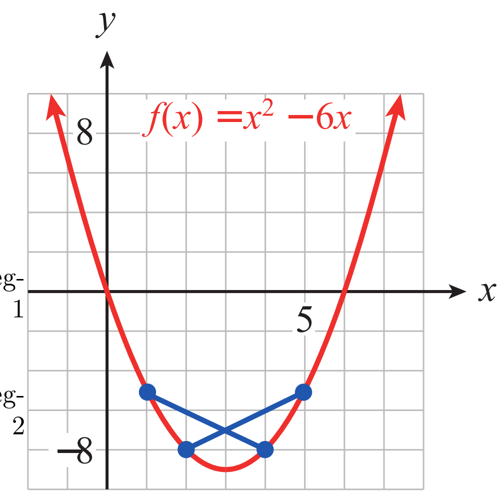

Figure 1.117

Solution.

a We set $x_1 = \mathbf{1}$ and $x_2 = \mathbf{4}$ and find the function values at each point.

$$f(x_1) = f(\mathbf{1}) = \mathbf{1}^2 6(\mathbf{1}) = 5$$

$$f(x_2) = f(\mathbf{4}) = \mathbf{4}^2 6(\mathbf{4}) = 8$$

Then

$$m = \frac{f(x_2) f(x_1)}{x_2 x_1} = \frac{8(5)}{41} = \frac{3}{3} = 1$$

b We set $x_1 = \mathbf{2}$ and $x_2 = \mathbf{5}$ and find the function values at each point.

$$f(x_1) = f(\mathbf{2}) = \mathbf{2}^2 6(\mathbf{2}) = 8$$

$$f(x_2) = f(\mathbf{5}) = \mathbf{5}^2 6(\mathbf{5}) = 5$$

Then

$$m = \frac{f(x_2) f(x_1)}{x_2 x_1} = \frac{5(8)}{52} = \frac{3}{3} = 1$$

Note that the graph of f is not a straight line and that the slope is not constant.

Exercise 1.118.

[Figure 1.119](#) shows the graph of a function f.

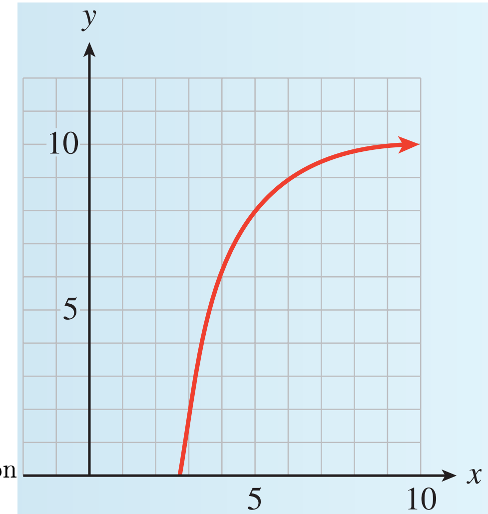

Figure 1.119

a Find $f(3)$ and $f(5)$.

b Compute the slope of the line segment joining the points at $x = 3$ and $x = 5$.

c Write an expression for the slope of the line segment joining the points at $x = a$ and $x = b$.

1.5 Linear Functions

1.5.1 Slope-Intercept Form

As we saw in Section 1.1, many linear models $y = f(x)$ have equations of the form

$$f(x) = (\text{starting value}) + (\text{rate of change}) \cdot x$$

The starting value, or the value of y at $x = 0$, is the y-intercept of the graph, and the rate of change is the slope of the graph. Thus, we can write the equation of a line as

$$f(x) = b + mx$$

where the constant term, b, is the y-intercept of the line, and m, the coefficient of x, is the slope of the line. This form for the equation of a line is called the **slope-intercept form**.

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Slope-Intercept Form

If we write the equation of a linear function in the form,

$$f(x) = b + mx$$

then m is the **slope** of the line, and b is the y**-intercept**.

(You may have encountered the slope-intercept equation in the equivalent form $y = mx + b$.) For example, consider the two linear functions and their graphs shown in Figure 1.121 and Figure 1.123.

x	$f(x)$
0	10
1	7
2	4
3	1
4	-2

Table 1.120: $f(x) = 103x$

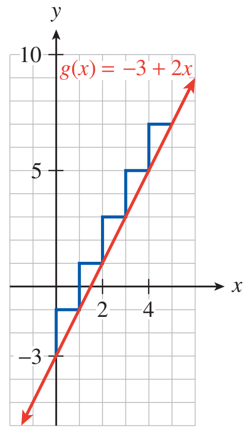

Figure 1.121

x	$f(x)$
0	-3
1	-1
2	1
3	3
4	5

Table 1.122: $g(x) = 3 + 2x$

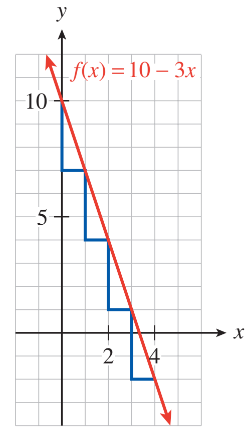

Figure 1.123

We can see that the y-intercept of each line is given by the constant term, b. By examining the table of values, we can also see why the coefficient of $x$gives the slope of the line: For $f(x)$, each time x increases by 1 unit, y decreases by 3 units. For $g(x)$, each time x increases by 1 unit, y increases by 2 units. For each graph, the coefficient of x is a scale factor that tells us how many units y changes for 1 unit increase in x. But that is exactly what the slope tells us about a line.

Example 1.124. Francine is choosing an Internet service provider. She paid \$30 for a modem, and she is considering three companies for dialup service: Juno charges \$14.95 per month, ISP.com charges \$12.95 per month, and peoplepc charges \$15.95 per month. Match the graphs in Figure 1.125 to Francine's Internet cost with each company.

Solution. Francine pays the same initial amount, \$30 for the modem, under each plan. The monthly fee is the rate of change of her total cost, in dollars per month.

We can write a formula for her cost under each plan.

$$\text{Juno: } f(x) = 30 + 14.95x$$

$$\text{ISP.com: } g(x) = 30 + 12.95x$$

$$\text{peoplepc: } h(x) = 30 + 15.95x$$

The graphs of these three functions all have the same y-intercept, but their slopes are determined by the monthly fees. The steepest graph, III, is the one with the largest monthly fee, peoplepc, and ISP.com, which has the lowest monthly fee, has the least steep graph, I.

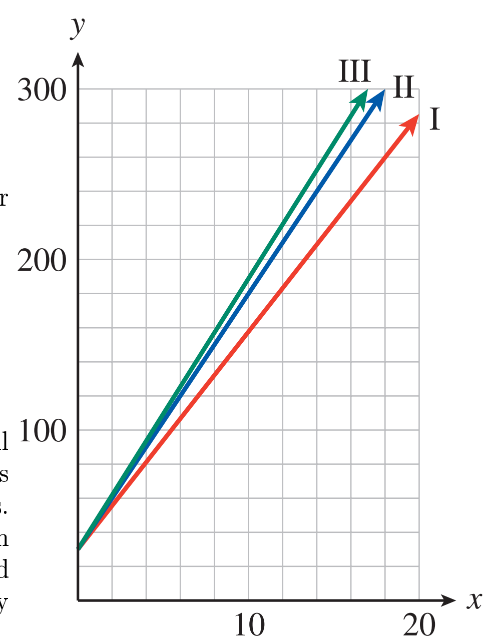

Figure 1.125

Exercise 1.126. Delbert decides to use DSL for his Internet service. Earthlink charges a \$99 activation fee and \$39.95 per month, DigitalRain charges \$50 for activation and \$34.95 per month, and FreeAmerica charges \$149 for activation and \$34.95 per month.

a Write a formula for Delbert's Internet costs under each plan.

b Match Delbert's Internet cost under each company with its graph in Figure 1.127.

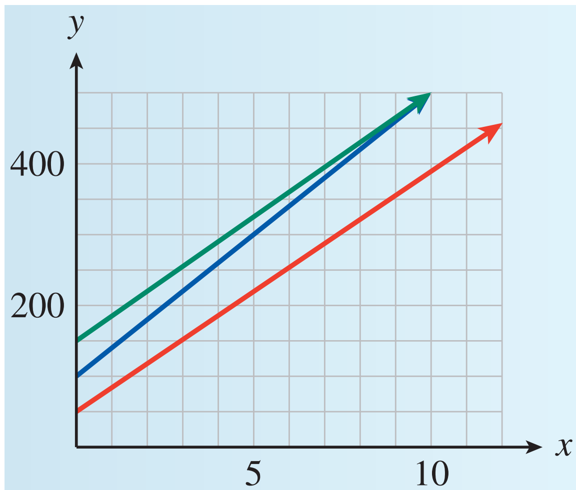

Figure 1.127

In the equation $f(x) = b + mx$, we call m and b **parameters**. Their values are fixed for any particular linear equation; for example, in the equation $y = 2x + 3$, $m = 2$ and $b = 3$, and the variables are x and y. By changing the values of m and b, we can write the equation for any line except a vertical line (see Figure 1.128). The collection of all linear functions $f(x) = b + mx$ is called a **two-parameter** family of functions.

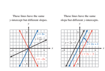

Figure 1.128

1.5.2 Slope-Intercept Method of Graphing

Look again at the lines in Figure 1.128: There is only one line that has a given slope and passes through a particular point. That is, the values of m and b determine the particular line. The value of b gives us a starting point, and the value of m tells us which direction to go to plot a second point. Thus, we can graph a line given in slope-intercept form without having to make a table of values.

Example 1.129.

a Write the equation $4x3y = 6$ in slope-intercept form.

b Graph the line by hand.

Solution.

a We solve the equation for y in terms of x.

$$3y\& = 64x\backslash\backslash y\& = \frac{64x}{-3} = \frac{6}{-3} + \frac{-4x}{-3}\backslash\backslash y\& = -2 + \frac{4}{3}x$$

b We see that the slope of the line is $m = \frac{4}{3}$ and its y-intercept is $b = 2$. We begin by plotting the y-intercept, $(0, 2)$. We then use the slope to find another point on the line. We have

$$m = \frac{\Delta y}{\Delta x} = \frac{4}{3}$$

so starting at $(0, 2)$, we move 4 units in the y-direction and 3 units in the x-direction, to arrive at the point $(3, 2)$. Finally, we draw the line through these two points. (See 1.130.)

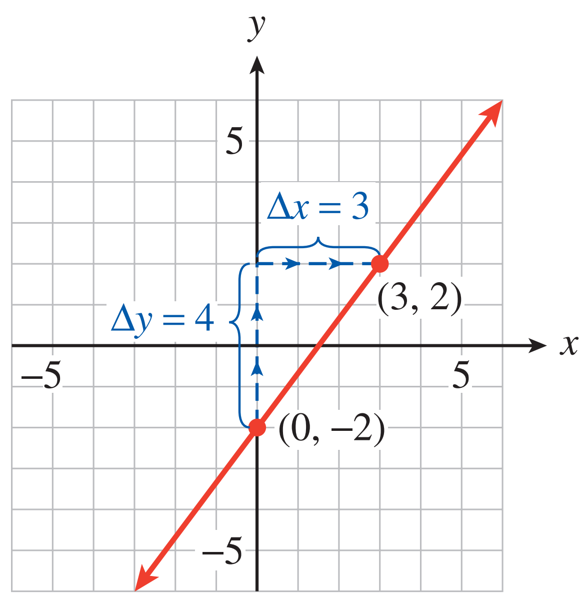

Figure 1.130

The slope of a line is a ratio and can be written in many equivalent ways. In Example 1.129, the slope is equal to $\frac{8}{6}$, $\frac{12}{9}$, and $\frac{4}{3}$. We can use any of these fractions to locate a third point on the line as a check. If we use $m = \frac{\Delta y}{\Delta x} = \frac{4}{3}$, we move down 4 units and left 3 units from the y-intercept to find the point $(3, 6)$ on the line.

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Slope-Intercept Method for Graphing a Line

a Plot the y-intercept $(0, b)$.

b Use the definition of slope to find a second point on the line: Starting at the y-intercept, move Δy units in the y-direction and Δx units in the x-direction. Plot a second point at this location.

c Use an equivalent form of the slope to find a third point, and draw a line through the points.

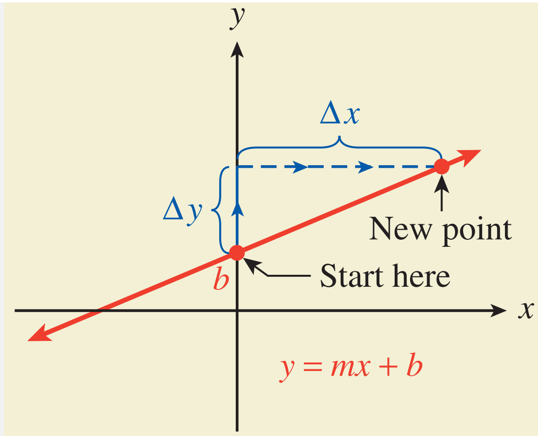

Figure 1.131

Exercise 1.132.

a Write the equation $2y + 3x + 4 = 0$ in slope-intercept form.

b Use the slope-intercept method to graph the line.

1.5.3 Finding a Linear Equation from a Graph

We can also use the slope-intercept form to find the equation of a line from its graph. First, note the value of the y-intercept from the graph, and then calculate the slope using two convenient points.

Example 1.133. Find an equation for the line shown in Figure 1.134.

Solution.

The line crosses the y-axis at the point $(0, 3200)$, so the y-intercept is 3200. To calculate the slope of the line, locate another point, say $(20, 6000)$, and compute:

$$m\& = \frac{\Delta y}{\Delta x} = \frac{60003200}{200} \backslash\backslash\& = \frac{2800}{20} = 140$$

The slope-intercept form of the equation, with $m = 140$ and $b = 3200$, is $y = 3200 + 140x$.

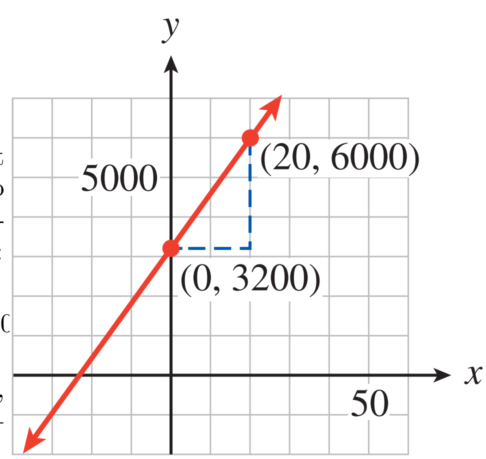

Figure 1.134

Exercise 1.135.

Find an equation for the line shown in Figure 1.136

$b\& =$ \\$m\& =$ \\$y\& =$ \\

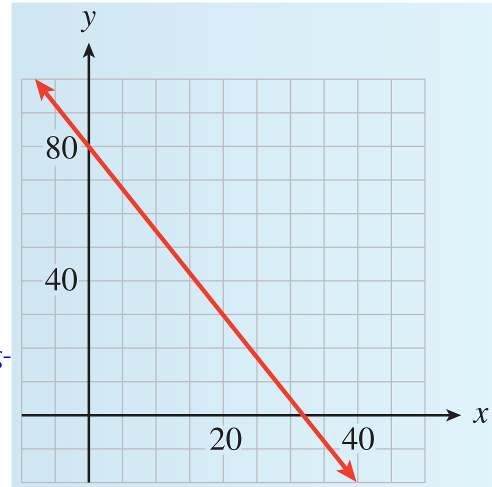

Figure 1.136

1.5.4 Point-Slope Form

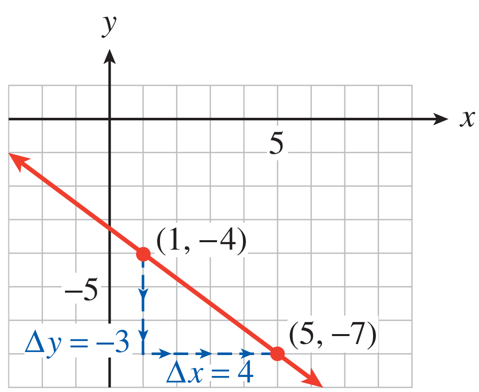

Figure 1.137

We can find the equation for a line if we know its slope and y-intercept. What if we do not know the y-intercept, but instead know some other point on the line? There is only one line that passes through a given point and has a given slope. For example, we can graph the line of slope $\frac{3}{4}$ that passes through the point $(1, 4)$.

We first plot the given point, $(1, 4)$, as shown in Figure 1.137. Then we use the slope to find another point on the line. The slope is

$$m = \frac{3}{4} = \frac{\Delta y}{\Delta x}$$

so we move down 3 units and then 4 units to the right, starting from $(1, 4)$. This brings us to the point $(5, 7)$. We can then draw the line through these two points.

We can also find an equation for the line, as shown in Example 4.

Example 1.138. Find an equation for the line that passes through $(1, 4)$ and has slope $\frac{3}{4}$.

Solution. We will use the formula for slope,

$$m = \frac{y_2 y_1}{x_2 x_1}$$

We substitute $\frac{3}{4}$ for the slope, m, and $(1, 4)$ for (x_1, y_1). For the second point, (x_2, y_2), we will use the variable point (x, y). Substituting these values into the slope formula gives us

$$\frac{3}{4} = \frac{y(4)}{x1} = \frac{y+4}{x1}$$

To solve for y we first multiply both sides by $x1$.

$$\mathbf{(x-1)}\frac{3}{4}\& = \frac{y+4}{x1}\mathbf{(x-1)}\&\&\backslash\backslash\frac{3}{4}(x-1)\& = y+4\&\&\text{Apply the distributive law.}\backslash\backslash\frac{3}{4}x+\frac{3}{4}\& = y+4\&\&\text{Subtract 4 f}$$

When we use the slope formula in this way to find the equation of a line, we substitute a variable point (x, y) for the second point. This version of the formula,

$$m = \frac{yy_1}{xx_1}$$

is called the **point-slope form** for a linear equation. It is sometimes stated in another form obtained by clearing the fraction to get

$$\mathbf{(xx_1)}m\& = \frac{y-y_1}{xx_1}\mathbf{(xx_1)}\&\&\text{Multiply both sides by }(xx_1)\backslash\backslash(xx_1)m\& = y-y_1\&\&\text{Clear fractions and solve for }y.\backslash\backslash y\&$$

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Point-Slope Form

The equation of the line that passes through the point (x_1, y_1) and has slope m is

$$y = y_1 + m(xx_1)$$

Exercise 1.139. Use the point-slope form to find the equation of the line that passes through the point $(3, 5)$ and has slope 1.4.

$$y = y_1 + m(xx_1)$$

Substitute 1.4 for m and $(3, 5)$ for (x_1, y_1).
Simplify: Apply the distributive law.

The point-slope form is useful for modeling linear functions when we do not know the initial value but do know some other point on the line.

Example 1.140. Under a proposed graduated income tax system, single taxpayers would owe \$1500 plus 20% of the amount of their income over \$13,000. (For example, if your income is \$18,000, you would pay \$1500 plus 20% of \$5000.)

a Complete the table of values for the tax, T, on various incomes, I.

I	15,000	20,000	22,000
T			

b Write a linear equation in point-slope form for the tax, T, on an income I.

c Write the equation in slope-intercept form.

Solution.

a On an income of \$15,000, the amount of income over \$13,000 is \$15,000 \$13,000 = \$2000, so you would pay \$1500 plus 20% of \$2000, or

$$T = 1500 + 0.20(2000) = 1900$$

You can compute the other function values in the same way.

I	15,000	20,000	22,000
T	1900	2900	3300

b On an income of I, the amount of income over \$13,000 is I13,000, so you would pay \$1500 plus 20% of $I13,000$, or

$$T = 1500 + 0.20(I13,000)$$

c Simplify the right side of the equation to get

$$T\& = 1500 + 0.20I2600\backslash\backslash T\& = 1100 + 0.20I$$

Exercise 1.141. A healthy weight for a young woman of average height, 64 inches, is 120 pounds. To calculate a healthy weight for a woman taller than 64 inches, add 5 pounds for each inch of height over 64.

a Write a linear equation in point-slope form for the healthy weight, W, for a woman of height, H, in inches.

b Write the equation in slope-intercept form.

1.6 Linear Regression

We have spent most of this chapter analyzing models described by graphs or equations. To create a model, however, we often start with a quantity of data. Choosing an appropriate function for a model is a complicated process. In this section, we consider only linear models and explore methods for fitting a linear function to a collection of data points. First, we fit a line through two data points.

1.6.1 Fitting a Line through Two Points

If we already know that two variables are related by a linear function, we can find a formula from just two data points. For example, variables that increase or decrease at a constant rate can be described by linear functions.

Example 1.142. In 1993, Americans drank 188.6 million cases of wine. Wine consumption increased at a constant rate over the next decade, and we drank 258.3 million cases of wine in 2003. (Source: Los Angeles Times, Adams Beverage Group)

a Find a formula for wine consumption, W, in millions of cases, as a linear function of time, t, in years since 1990.

b State the slope as a rate of change. What does the slope tell us about this problem?

Solution.

a We have two data points of the form (t, W), namely $(3, 188.6)$ and $(13, 258.3)$. We use the point-slope formula to fit a line through these two points. First, we compute the slope.

$$\frac{\Delta W}{\Delta t} = \frac{258.3188.6}{133} = 6.97$$

Next, we use the slope m $=$ 6.97 and either of the two data points in the point-slope formula.

$$W = W_1 + m(tt_1)$$
$$W = 188.6 + 6.97(t3)$$
$$W = 167.69 + 6.97t$$

Thus, $W = f(t) = 167.69 + 6.97t$.

b The slope gives us the rate of change of the function, and the units of the variables can help us interpret the slope in context.

$$\frac{\Delta W}{\Delta t} = \frac{258.3188.6\text{millions of cases}}{133 \text{ years}} = 6.97 millions of cases/year$$

Over the 10 years between 1993 and 2003, wine consumption in the United States increased at a rate of 6.97 million cases per year.

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To Fit a Line through Two Points:

1 Compute the slope between the two points.

2 Substitute the slope and either point into the point-slope formula

$$y = y_1 + m(xx_1)$$

Exercise 1.143. In 1991, there were 64.6 burglaries per 1000 households in the United States. The number of burglaries reported annually declined at a roughly constant rate over the next decade, and in 2001 there were 28.7 burglaries per 1000 households. (Source: U.S. Department of Justice)

a Find a function for the number of burglaries, B, as a function of time, t, in years, since 1990.

b State the slope as a rate of change. What does the slope tell us about this problem?.

1.6.2 Scatterplots

Empirical data points in a linear relation may not lie exactly on a line. There are many factors that can affect experimental data, including measurement error, the influence of environmental conditions, and the presence of related variable quantities.

Example 1.144. A consumer group wants to test the gas mileage of a new model SUV. They test-drive six vehicles under similar conditions and record the distance each drove on various amounts of gasoline.

Gasoline used (gal)	9.6	11.3	8.8	5.2	10.3	6.7
Miles driven	155.8	183.6	139.6	80.4	167.1	99.7

a Are the data linear?

b Draw a line that fits the data.

c What does the slope of the line tell us about the data?

Solution.

a No, the data are not strictly linear. If we compute the slopes between successive data points, the values are not constant. We can see from an accurate plot of the data, shown in Figure 1.145, that the points lie close to, but not precisely on, a straight line.

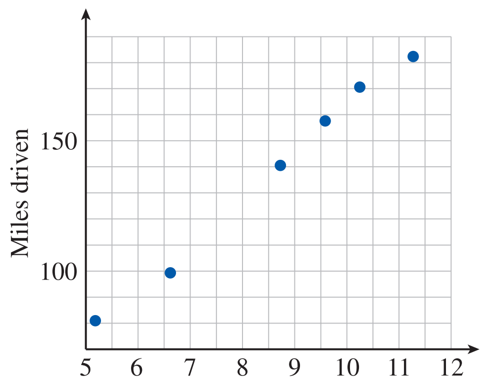

Figure 1.145

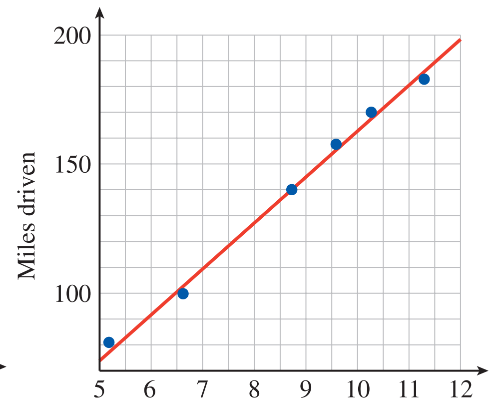

Figure 1.146

b We would like to draw a line that comes as close as possible to all the data points, even though it may not pass precisely through any of them. In particular, we try to adjust the line so that we have the same number of data points above the line and below the line. One possible solution is shown in Figure 1.146.

c To compute the slope of the line of best fit, we first choose two points on the line. Our line appears to pass through one of the data points,8.8, 139.6). Look for a second point on the line whose coordinates are easy to read, perhaps $(6.5, 100)$. The slope is

$$m = \frac{139.6100}{8.86.5} = 17.2 \text{ miles per gallon}$$

According to our data, the SUV gets about 17.2 miles to the gallon.

Exercise 1.147.

a Plot the data points. Do the points lie on a line?

b Draw a line that fits the data.

x	1.49	3.68	4.95	5.49	7.88	8.41
y	2.69	3.7	4.6	5.2	7.2	7.3

The graph in Example 1.144 is called a **scatterplot**. The points on a scatterplot may or may not show some sort of pattern. Consider the three plots in Figure 1.148. In Figure 1.148a, the data points resemble a cloud of gnats; there is no apparent pattern to their locations. In Figure 1.148b, the data follow a generally decreasing trend, but certainly do not all lie on the same line. The points in Figure 1.148c are even more organized; they seem to lie very close to an imaginary line.

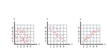

Figure 1.148

If the data in a scatterplot are roughly linear, we can estimate the location of an imaginary **line of best** fit that passes as close as possible to the data points. We can then use this line to make predictions about the data.

1.6.3 Linear Regression

One measure of a person's physical fitness is the **body mass index**, or BMI. Your BMI is the ratio of your weight in kilograms to the square of your height in centimeters. Thus, thinner people have lower BMI scores, and fatter people have higher scores. The Centers for Disease Control considers a BMI between 18.5 and 24.9 to be healthy. The points on the scatterplot in Figure 1.149 show the BMI of Miss America from 1918 to 1998. From the data in the scatterplot, can we see a trend in Americans' ideal of female beauty?

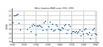

Figure 1.149

Example 1.150.

a Estimate a line of best fit for the scatterplot in Figure 1.149. (Source: http://www.pbs.org)

b Use your line to estimate the BMI of Miss America 1980.

Solution.

a We draw a line that fits the data points as best we can, as shown in Figure 1.151. (Note that we have set $t = 0$ in 1920 on this graph.) We try to end up with roughly equal numbers of data points above and below our line.

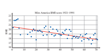

Figure 1.151

b We see that when $t = 60$ on this line, the y-value is approximately 18.3. We therefore estimate that Miss America 1980 had a BMI of 18.3. (Her actual BMI was 17.85.)

Exercise 1.152. Human brains consume a large amount of energy, about 16 times as much as muscle tissue per unit weight. In fact, brain metabolism accounts for about 25% of an adult human's energy needs, as compared to about 5% for other mammals. As hominid species evolved, their brains required larger and larger amounts of energy, as shown in Figure 1.153. (Source: Scientific American, December 2002)

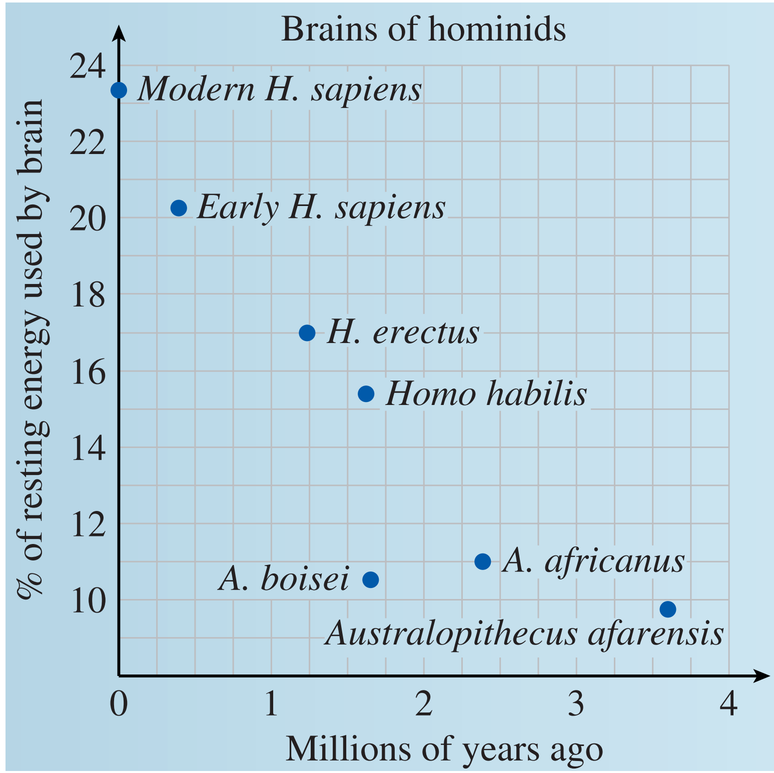

Figure 1.153

a Draw a line of best fit through the data points.

b Estimate the amount of energy used by the brain of a hominid species that lived three million years ago.

The process of predicting an output value based on a straight line that fits the data is called **linear regression**, and the line itself is called the **regression line**. The equation of the regression line is usually used (instead of a graph) to predict values.

Example 1.154.

a Find the equation of the regression line in Example 1.150, Figure 1.149.

b Use the regression equation to predict the BMI of Miss America 1980.

Solution.

a We first calculate the slope by choosing two points on the regression line we drew in Figure 1.151. The points we choose are not necessarily any of the original data points; instead they should be points on the regression line itself. The line appears to pass through the points $(17, 20)$ and $(67, 18)$. The slope of the line is then

$$m = \frac{1820}{6717} \approx 0.04$$

Now we use the point-slope formula to find the equation of the line. (If you need to review the point-slope formula, see Section 1.5.) We substitute

$m = 0.04$ and use either of the two points for (x_1, y_1); we will choose $(17, 20)$. The equation of the regression line is

$$\begin{aligned} y &= y_1 + m(xx_1) \\ y &= 200.04(x17) && \text{Simplify.} \\ y &= 20.680.04t \end{aligned}$$

b We will use the regression equation to make our prediction. For Miss America 1980, $t = 60$ and

$$y = 20.680.04(60) = 18.28$$

This value agrees well with the estimate we made in Example 1.150.

Exercise 1.155. The number of manatees killed by watercraft in Florida waters has been increasing since 1975. Data are given at 5-year intervals in the table. (Source: Florida Fish and Wildlife Conservation Commission)

Year	Manatee deaths
1975	6
1980	16
1985	33
1990	47
1995	42
2000	78

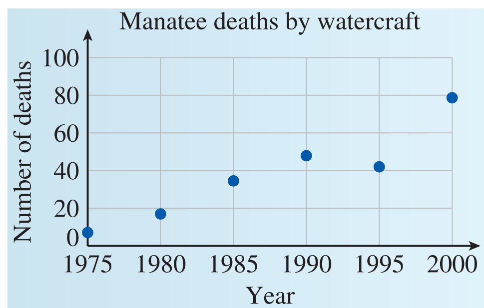

Figure 1.156

a Draw a regression line through the data points shown in Figure 1.156.

b Use the regression equation to estimate the number of manatees killed by watercraft in 1998.

1.6.4 Linear Interpolation and Extrapolation

Using a regression line to estimate values between known data points is called **interpolation**. Making predictions beyond the range of known data is called **extrapolation**.

Example 1.157.

a Use linear interpolation to estimate the BMI of Miss America 1960.

b Use linear extrapolation to predict the BMI of Miss America 2001.

Solution.

a For 1960, we substitute $t = 40$ into the regression equation we found in Example 1.154.

$$y = 20.680.04(40) = 19.08$$

We estimate that Miss America 1960 had a BMI of 19.08. (Her BMI was actually 18.79.)

b For 2001, we substitute $t = 81$ into the regression equation.

$$y = 20.680.04(81) = 17.44$$

Our model predicts that Miss America 2001 had a BMI of 17.44. In fact, her BMI was 20.25. By the late 1990s, public concern over the self-image of young women had led to a reversal of the trend toward ever-thinner role models.

Example 1.157b illustrates an important fact about extrapolation: If we try to extrapolate too far, we may get unreasonable results. For example, if we use our model to predict the BMI of Miss America 2520 (when $t = 600$), we get

$$y = 20.680.04(600) = 3.32$$

Even if the Miss America pageant is still operating in 600 years, the winner cannot have a negative BMI. Our linear model provides a fair approximation for 1920–1990, but if we try to extrapolate too far beyond the known data, the model may no longer apply.

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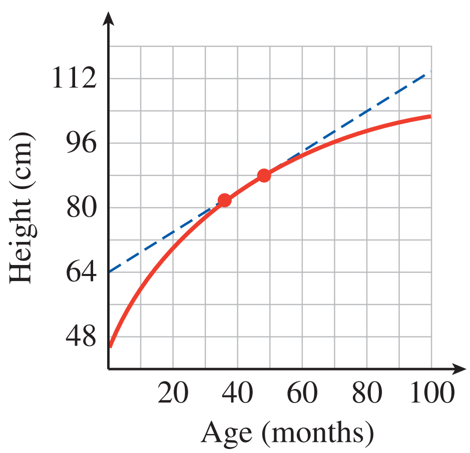


Figure 1.158

We can also use interpolation and extrapolation to make estimates for nonlinear functions. Sometimes a variable relationship is not linear, but a portion of its graph can be approximated by a line. The graph in Figure 1.158 shows a child's height each month. The graph is not linear because her rate of growth is not constant; her growth slows down as she approaches her adult height. However, over a short time interval the graph is close to a line, and that line can be used to approximate the coordinates of points on the curve.

Exercise 1.159. Emily was 82 centimeters tall at age 36 months and 88 centimeters tall at age 48 months.

a Find a linear equation that approximates Emily's height in terms of her age over the given time interval.

b Use linear interpolation to estimate Emily's height when she was 38 months old, and extrapolate to predict her height at age 50 months.

c Predict Emily's height at age 25 (300 months). Is your answer reasonable?

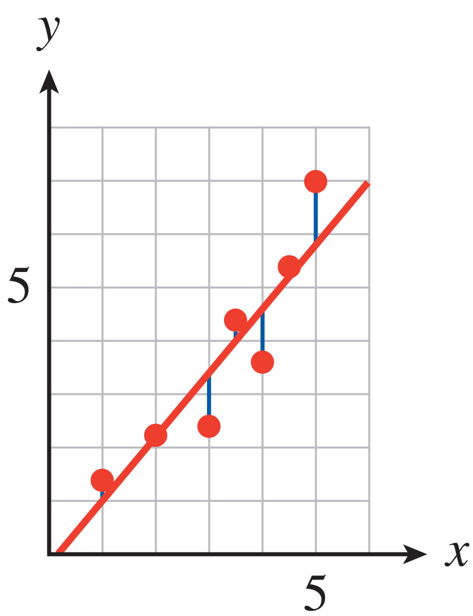


Figure 1.160

Estimating a line of best fit is a subjective process. Rather than base their estimates on such a line, statisticians often use the **least squares regression line**. This regression line minimizes the sum of the squares of all the vertical distances between the data points and the corresponding points on the line (see Figure 1.160). Many calculators are programmed to find the least squares regression line, using an algorithm that depends only on the data, not on the appearance of the graph.

Remark 1.161 (. images/icon-GC.jpgUsing a Calculator for Linear Regression]

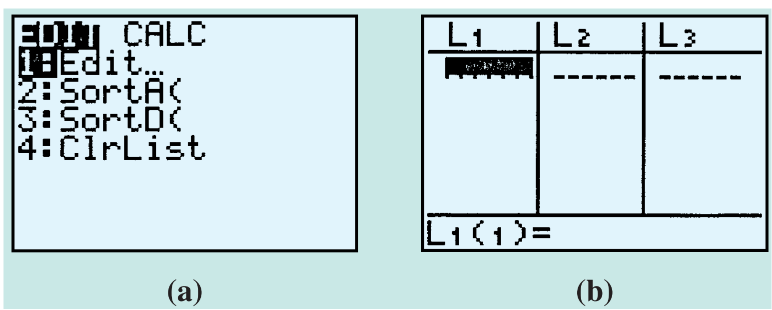


Figure 1.162

You can use a graphing calculator to make a scatterplot, find a regression line, and graph the regression line with the data points. On the TI-83 calculator, we use the statistics mode, which you can access by pressing `STAT`. You will see a display that looks like Figure 1.162a. Choose 1 to Edit (enter or alter) data. Now follow the instructions in Example 6 for using your calculator's statistics features.

Example 1.163.

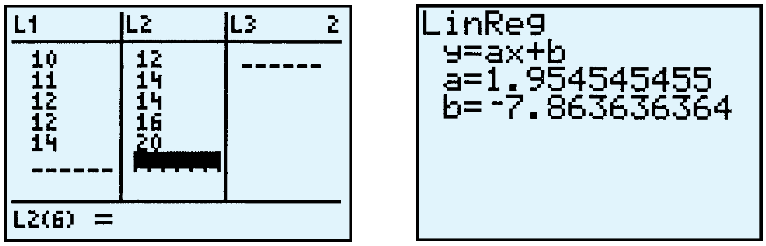
a Find the equation of the least squares regression line for the following data:

$$(10, 12), (11, 14), (12, 14), (12, 16), (14, 20)$$

b Plot the data points and the least squares regression line on the same axes.

Solution.

a We must first enter the data. Press `STATENTER` to select Edit. If there are data in column L_1 or L_2, clear them out: Use the key to select L_1, press `CLEAR`, then do the same for L_2. Enter the x-coordinates of the data points in the L_1 column and enter the y-coordinates in the L_2 column, as shown in Figure 1.164a.

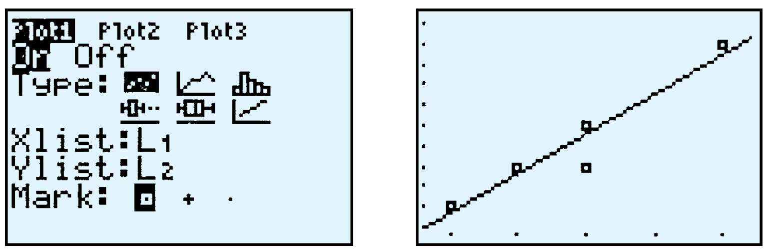


(a) (b)

Figure 1.164

Now we are ready to find the regression equation for our data. Press STAT 4 to select linear regression, or LinReg (ax + b), then press ENTER. The calculator will display the equation $y = ax + b$ and the values for a and b, as shown in Figure 1.164b. You should find that your regression line is approximately $y = 1.95x7.86$.

b First, we first clear out any old definitions in the list. Position the cursor after $Y_1 =$ and copy in the regression equation as follows: Press VARS5 ENTER. To draw a scatterplot, press 2ndY=1 and set the Plot1 menu as shown in Figure 1.165a. Finally, press ZOOM9 to see the scatterplot of the data and the regression line. The graph is shown in Figure 1.165b.



(a) (b)

Figure 1.165

CAUTION When you are through with the scatterplot, press Y= ENTER to turn off the Stat Plot. If you neglect to do this, the calculator will continue to show the scatterplot even after you ask it to plot a new equation.

Exercise 1.166.

a Use your calculator's statistics features to find the least squares regression equation for the data in Exercise 1.147.

b Plot the data and the graph of the regression equation.

Appendix A

Algebra Skills Refresher

A.1 Order of Operations

Numerical calculations often involve more than one operation. So that everyone agrees on how such expressions should be evaluated, we follow the order of operations.

A.2 Linear Equations and Inequalities

An **equation** is just a mathematical statement that two expressions are equal. Equations relating two variables are particularly useful. If we know the value of one of the variables, we can find the corresponding value of the other variable by solving the equation.

A.3 Algebraic Expressions and Problem Solving

You are familiar with the use of letters, or **variables**, to stand for unknown numbers in equations or formulas. Variables are also used to represent numerical quantities that change over time or in different situations. For example, p might stand for the atmospheric pressure at different heights above the Earth's surface. Or N might represent the number of people infected with cholera t days after the start of an epidemic.

A.4 Graphs and Equations

Graphs are useful tools for studying mathematical relationships. A graph provides an overview of a quantity of data, and it helps us identify trends or unexpected occurrences. Interpreting the graph can help us answer questions about the data.

A.5 Linear Systems in Two Variables

A 2×2 **system** of equations is a set of 2 equations in the same 2 variables. A **solution** of a 2×2 system is an ordered pair that makes each equation in the system true. In this section, we review two algebraic methods for solving 2×2 linear systems: substitution and elimination.

A.6 Laws of Exponents

In this section, we review the rules for performing operations on powers.

A.7 Polynomials and Factoring

In Section A.6, we used the first law of exponents to multiply two or more monomials. In this section, we review techniques for multiplying and factoring polynomials of several terms.

A.8 Factoring Quadratic Trinomials

Consider the trinomial

$$x^2 + 10x + 16$$

A.9 Working with Algebraic Fractions

A quotient of two polynomials is called a **rational expression** or an **algebraic fraction**. Operations on algebraic fractions follow the same rules as operations on common fractions.

A.10 Working with Radicals

In some situations, radical notation is more convenient to use than exponents. In these cases, we usually simplify radical expressions algebraically as much as possible before using a calculator to obtain decimal approximations.

A.11 Facts from Geometry

In this section, we review some information you will need from geometry. You are already familiar with the formulas for the area and perimeter of common geometric figures; you can find these formulas in the reference section at the front of the book.

A.12 The Real Number System

A.12.1 Subsets of the Real Numbers

The numbers associated with points on a number line are called the **real numbers**. The set of real numbers is denoted by $\mathbb{R}$. You are already familiar with several types, or subsets, of real numbers:

Appendix B

Using a Graphing Calculator

This appendix provides instructions for TI-84 or TI-83 calculators from Texas Instruments, but most other calculators work similarly.We describe only the basic operations and features of the graphing calculator used in your textbook.

B.1 Getting Started

B.1.1 On and Off

Press `ON` to turn *on* the calculator (see Figure B.1a). You will see a cursor blinking in the upper left corner of the Home screen. Press `2ndON` to turn *off* the calculator.

B.2 Entering Expressions

B.3 Editing Expressions

B.4 Graphing an Equation

B.5 Making a Table

B.6 Regression

B.7 Function Notation and Transformation of Graphs

Appendix C

Glossary

Appendix D

Answers to selected exercises

This book was authored in MathBook XML.