Extreme value modelling

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Motivation

Boscastle floods, August 2004

Pic by Nick Gregory. 16/08/2004.
Boscastle Flood

Motivation

The Great Storm of '87 (87J)

Motivation

UK heatwave, July 2022

Motivation

UK coldwave, aka ‘The Beast from the East’ February 2018

Overview

- Extreme value theory: a pictorial introduction
- Extreme UK rainfall
- GMRF models for extremes
- Tailored GMRF models for extremes
- Past and future estimates of extreme UK rainfall
- Summary

Extreme value theory

The extremal types theorem

- Let $X_1, \ldots, X_n$ be i.i.d. random variables
- Define $M_n = \max(X_1, \ldots, X_n)$
- If there exist sequences of constants $a_n > 0$ and b_n such that, as $n \to \infty$,

$$Pr\{(M_n - b_n)/a_n \leq x\} \to H(x)$$

for some non-degenerate distribution H, then H is the generalised extreme value (GEV) distribution, denoted $GEV(\mu, \psi, \xi)$, where

$$H(x) = \exp\left[-\left(1 + \xi\frac{x-\mu}{\psi}\right)_+^{-1/\xi}\right]$$

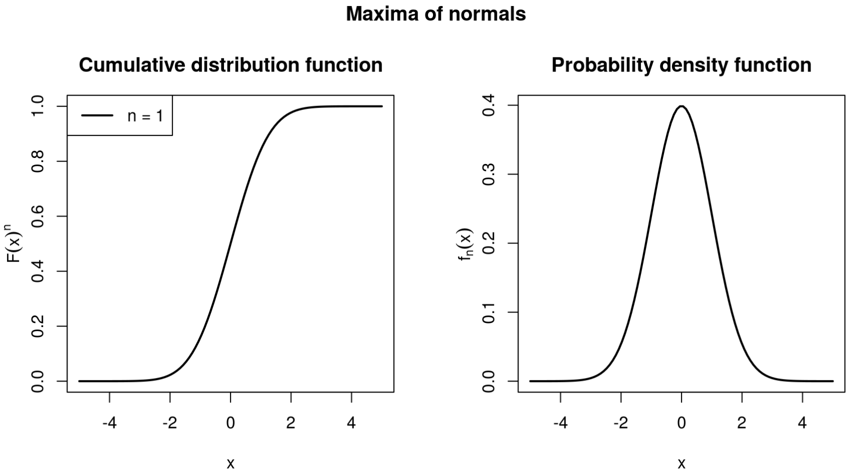
for $\{x : (1 + \xi(x - \mu)/\psi > 0\}$ and where $\psi > 0$, $(x)_+ = \max(0, x)$ and the case $\xi = 0$ gives $H(x) = \exp[-\exp\{-(x - \mu)/\psi\}]$

i.i.d. := "independent and identically distributed". Ref's: Fisher and Tippett (1928); Gnedenko (1943); Jenkinson (1955)

Extreme value theory

Normal example

- Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(0,1)$
- $\Pr(M_n \leq x) = F(x)^n$



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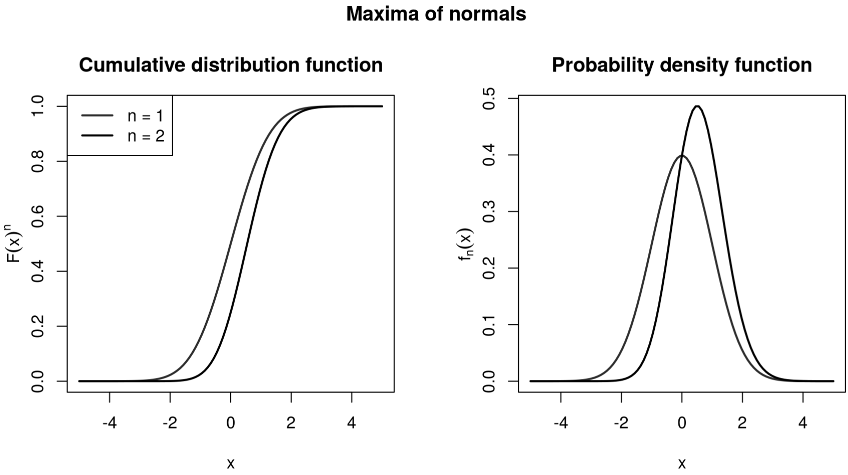

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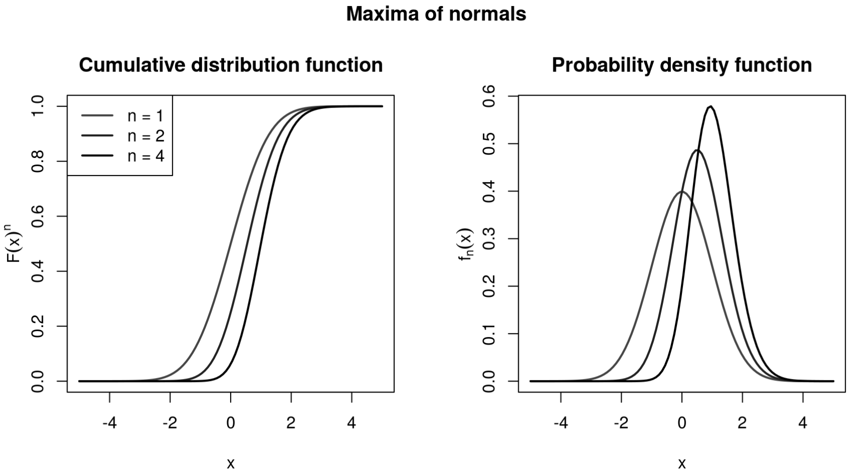

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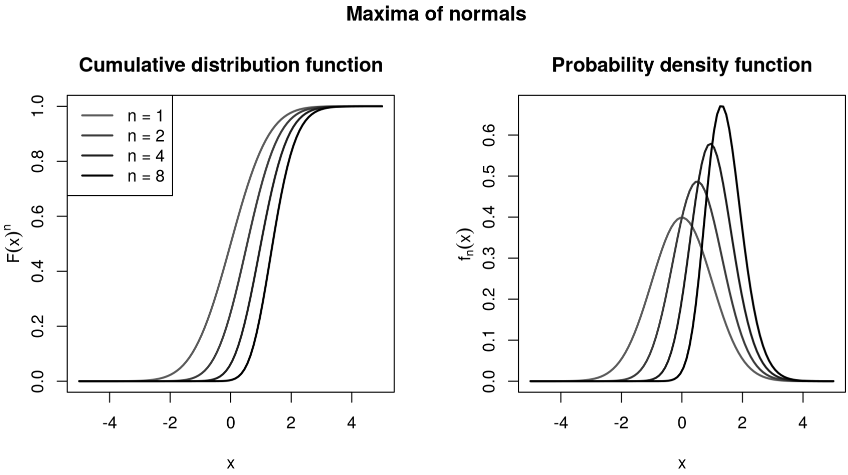

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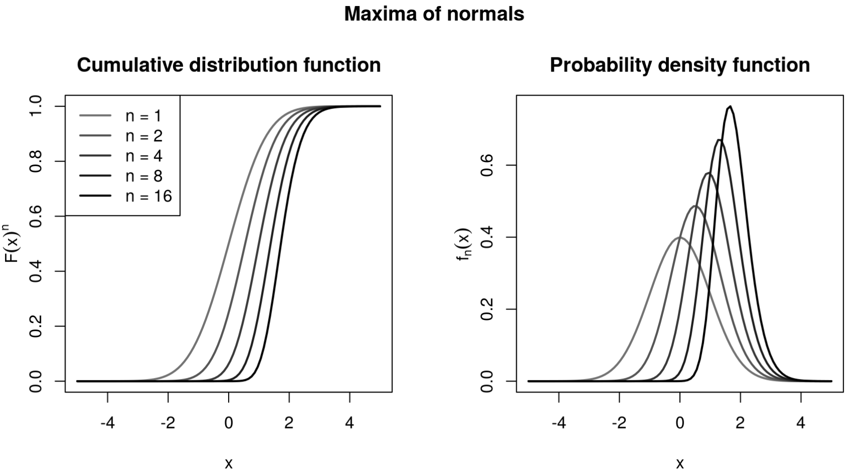

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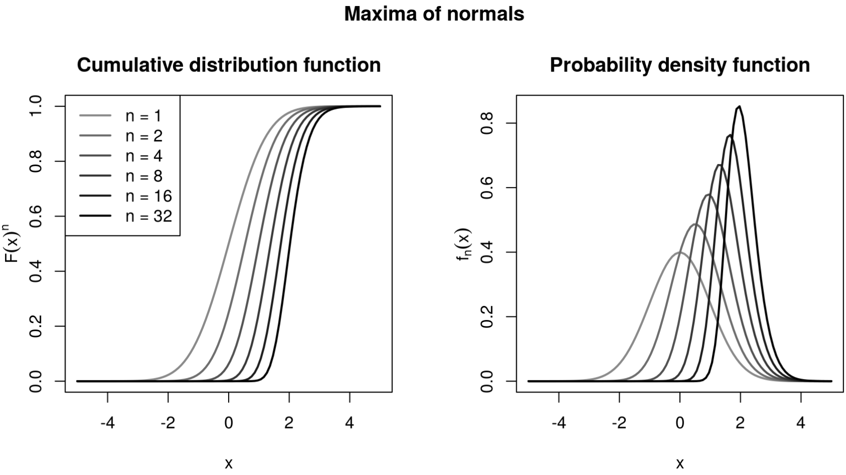

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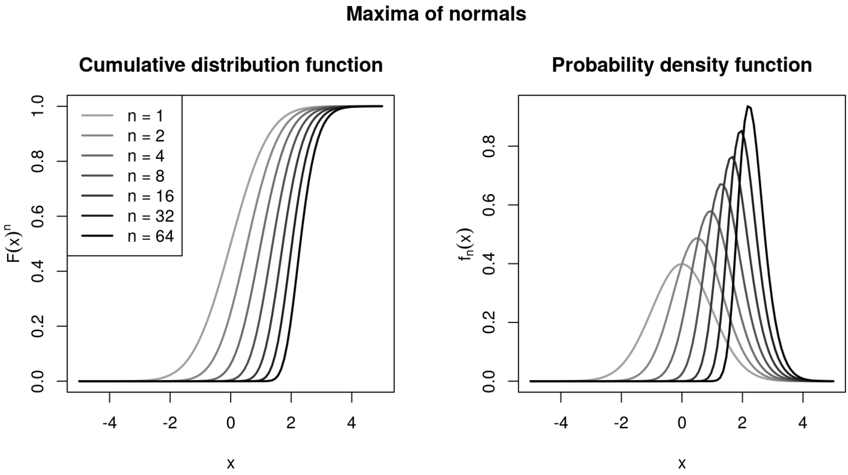

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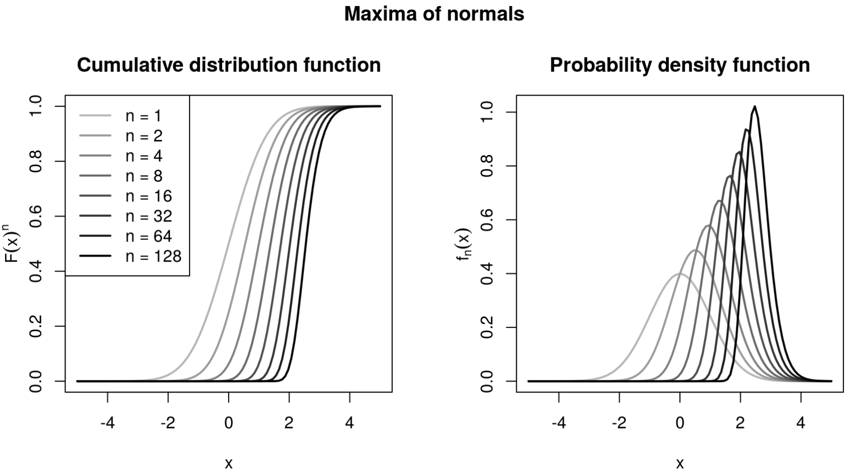

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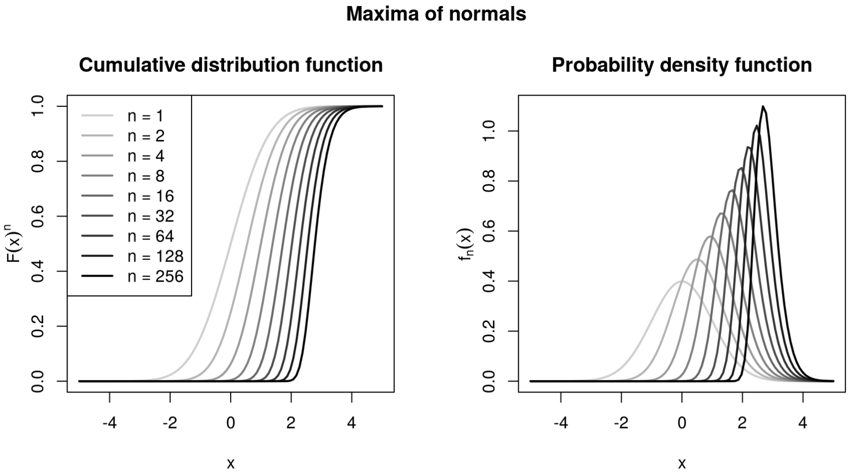

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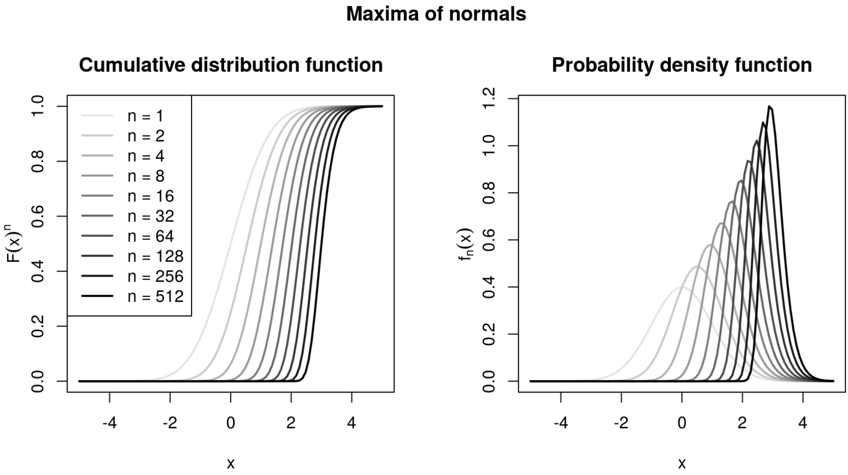
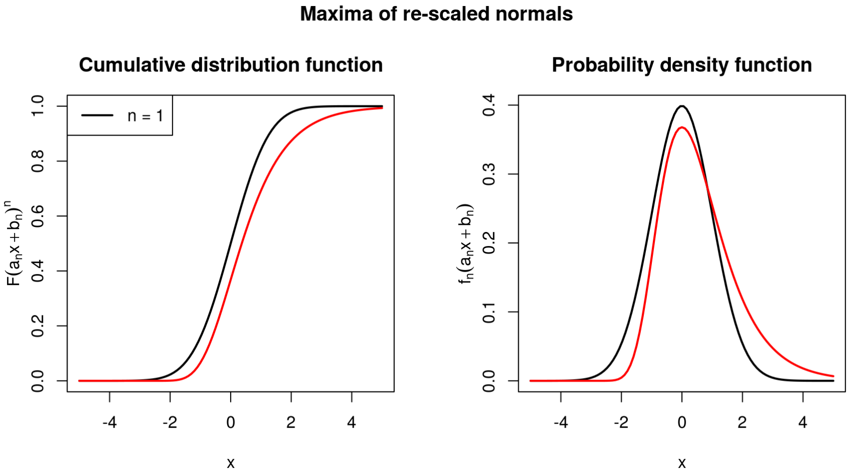


Extreme value theory

Normal example

- $\Pr\{(M_n - b_n)/a_n \leq x\} = F(a_n x + b_n)^n$
- $a_n = (2\log n)^{-\frac{1}{2}}$ and $b_n = a_n^{-1} - \frac{1}{2}a_n(\log\log n + log 4\pi)$



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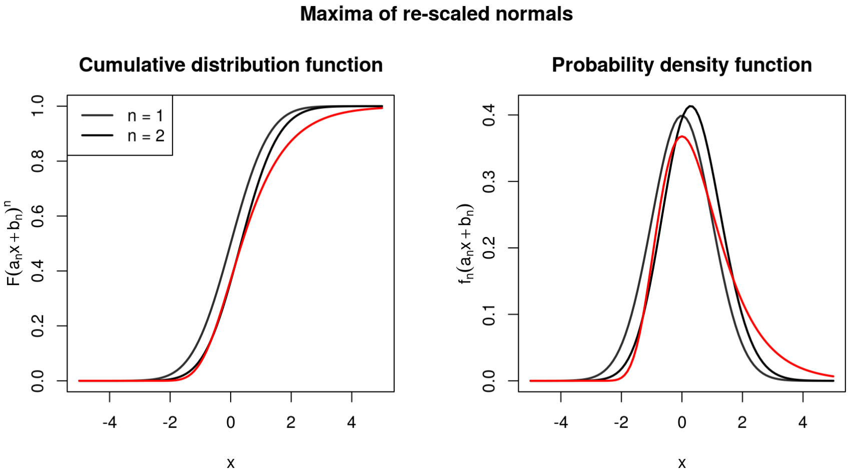

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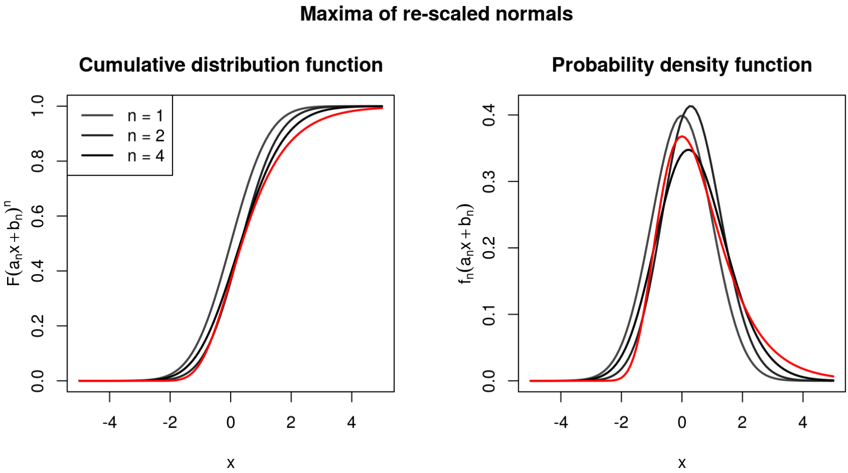

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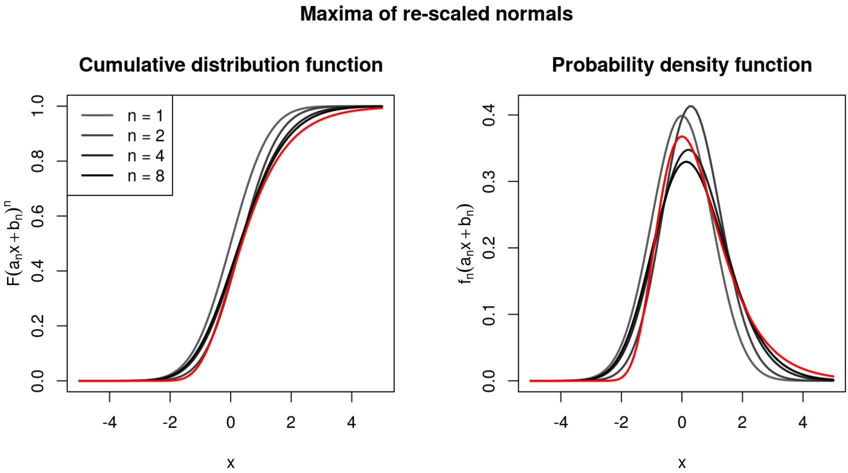

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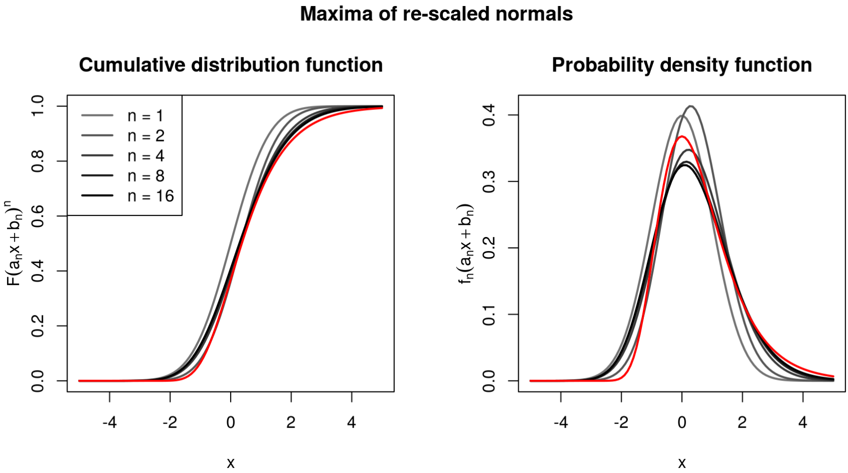

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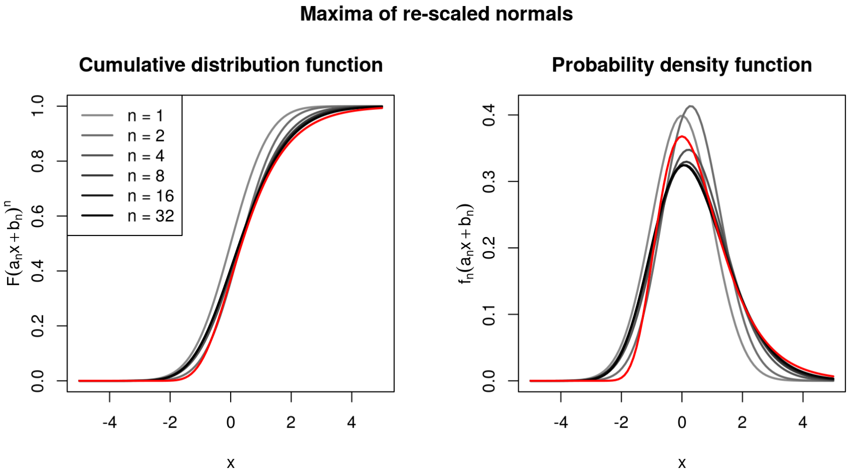

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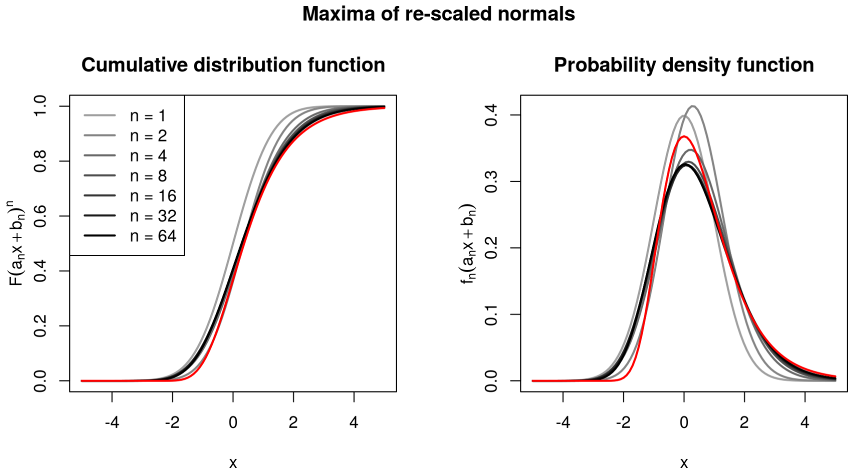

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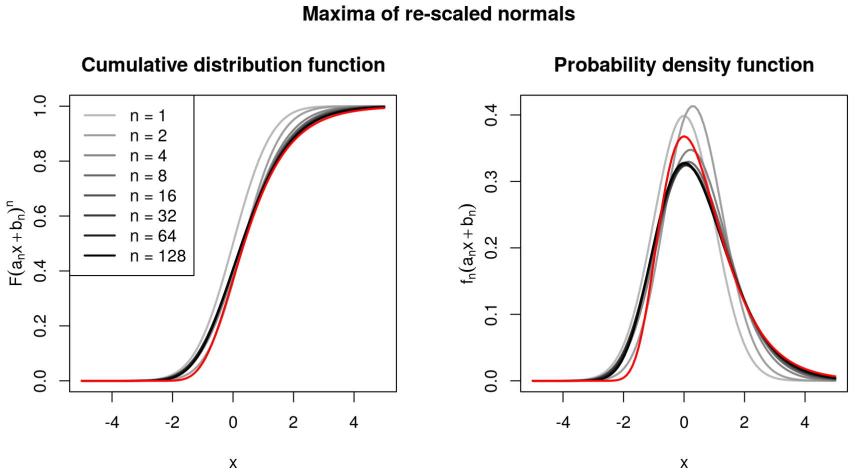

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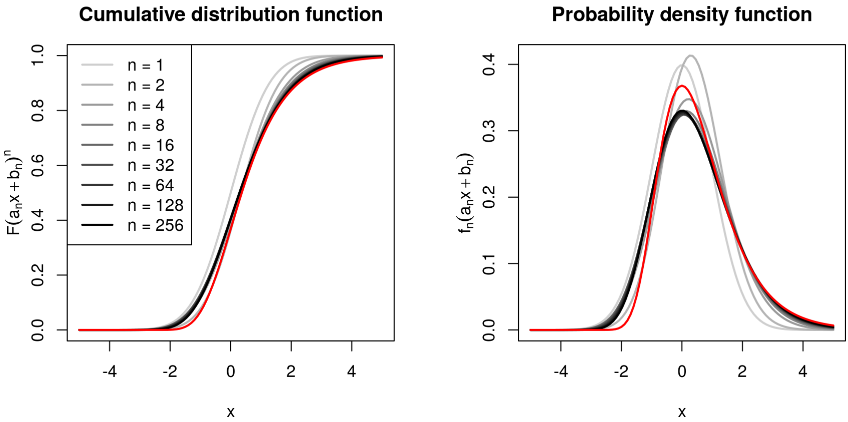
Maxima of re-scaled normals



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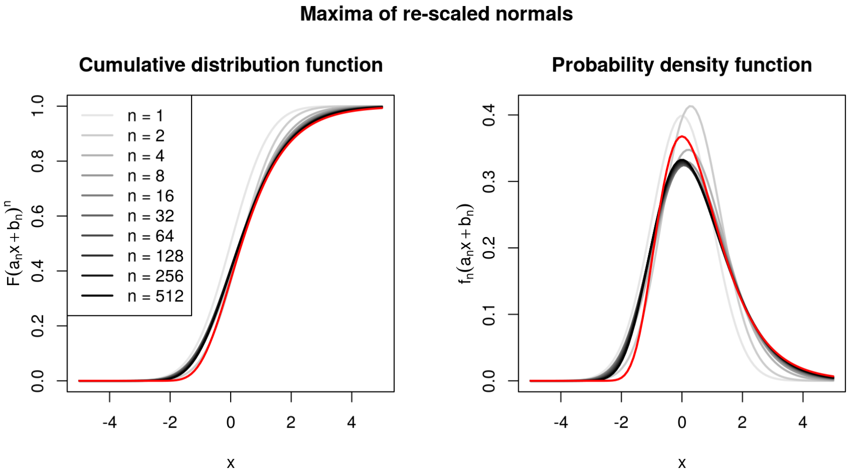


Statistical models for maxima

block maxima/minima

- Let $X_1, X_2, \ldots, X_{nm}$ be an i.i.d. sequence
- Partition data into n blocks of length m, i.e.,

$$\{X_{jm+1}, \ldots, X_{jm+m}\}_{j=0,\ldots,n-1}$$

- Obtain *block maxima*

$$Y_j = \max\{X_{jm+1}, \ldots, X_{jm+m}\}$$

- For $j = 1, \ldots, n$ assume

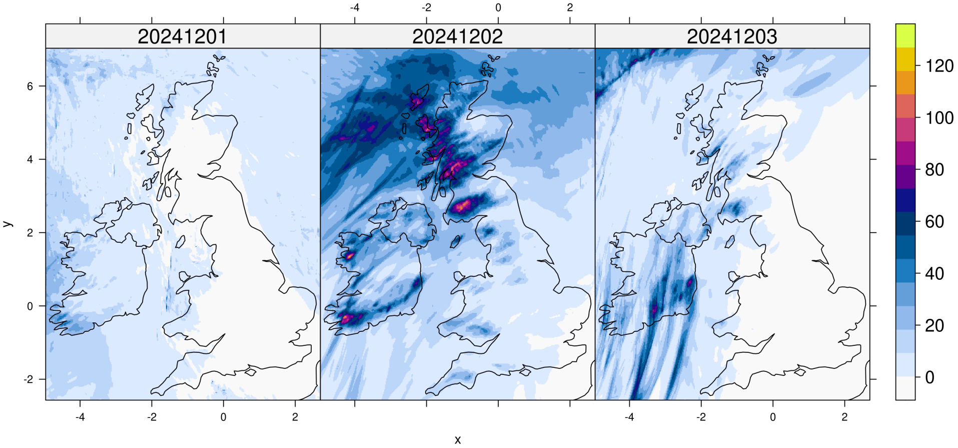
$$Y_j \sim GEV(\mu, \psi, \xi)$$

- Model *minima* by noting that

$$\min(X_1, X_2, \ldots) = -\max(-X_1, -X_2, \ldots)$$

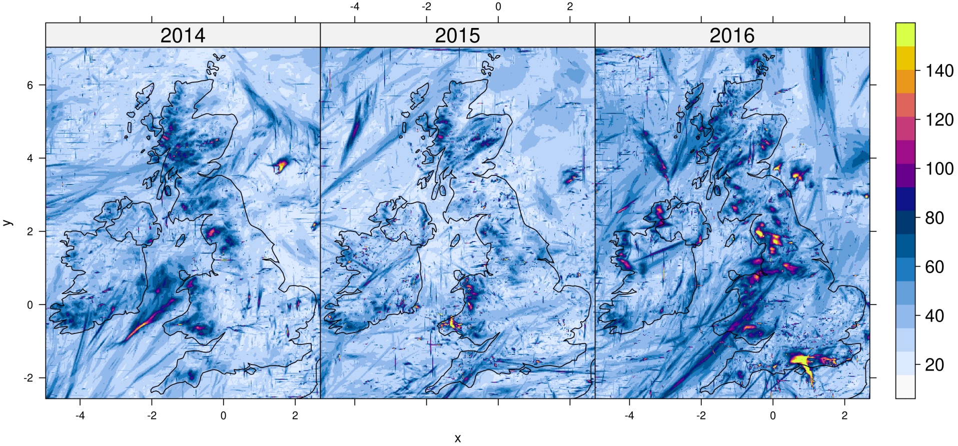
High resolution UK rainfall projections

- The UKCP18 project gives daily rainfall projections at 2.2km resolution
    - it's the best available, at the moment



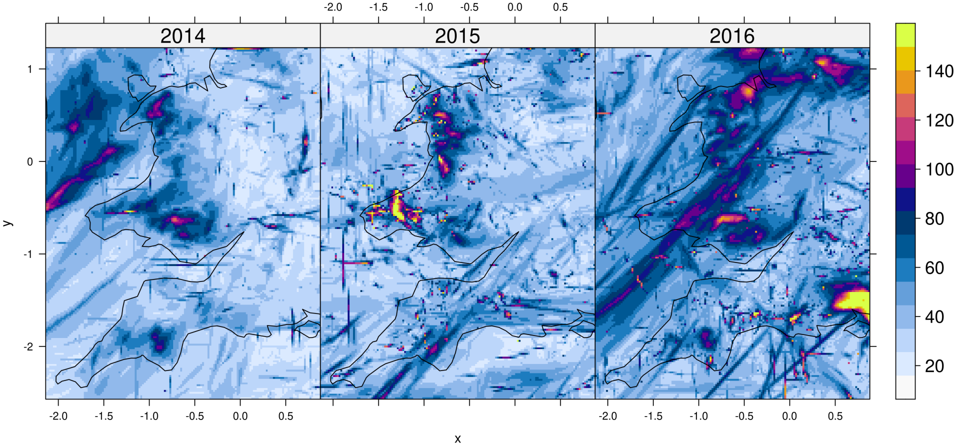
High resolution UK rainfall projections

- We want to work with annual maxima



High resolution UK rainfall projections

- But numerical errors risk dominating our analysis
- We attempt to filter out obvious horizontal and vertical lines



Modelling annual maxima

- Let Y_{ijt} denote the annual maximum in year t for grid row i and column j
- We have a 318×516 grid ($\sim 200,000$ grid cells) and 20 years of data
- We assume

$$Y_{ijt} \sim GEV(\mu_{ij}, \psi_{ij}, \xi_{ij})$$

- We could consider grid cells as spatial locations, e.g. s_{ij}, and model with continuous processes, such as $\mu_{ij} = \mu(s_{ij})$, via smooth spatial processes (e.g. Gaussian processes or thin-late splines)
    - but even with covariate information, we're likely to impose unwanted assumptions, especially over large domain
- A local smoother, such as a Gaussian Markov random field (GMRF), might work better
    - examples include Cooley and Sain (2010) and Auld et al. (2023), amongst others

Gaussian Markov random field (GMRF)

- Consider a random vector $\mathbf{Z} = (Z_1, \ldots, Z_n)$
- With a GMRF model we assume that

$$\mathbf{Z} \sim N_n(\mathbf{m}, \mathbf{Q}^{-1})$$

  i.e. is n-dimensional multivariate Gaussian with precision matrix $\mathbf{Q}$
- The conditional expectation is

$$\mathrm{E}(Z_i \mid z_j, j \neq i) = m_i - \frac{\sum_{i \neq j} Q_{ij}(z_j - m_j)}{Q_{ii}}$$

- So if $Q_{ij} = 0$, then Z_i and Z_j are conditionally independent
- Or Z_i and Z_j are neighbours if $Q_{ij} \neq 0$

GMRFs on grids

- Consider the collections of GEV parameters $\boldsymbol{\mu}$, $\boldsymbol{\psi}$ and $\boldsymbol{\xi}$
- We use GMRFs to specify conditional expectations amongst GEV parameters
- Now consider Z_{ij} on the grid (i, j) for $i = 1, \ldots, n_x$ and $j = 1, \ldots, n_y$.
- A simple GMRF assumes Z_{ij} is the average of its four nearest neighbours, $Z_{i-1,j}, Z_{i+1,j}, Z_{i,j-1}, Z_{i,j+1}$, so that

$$\mathrm{E}(Z_{ij} \mid z_{i-1,j}, z_{i+1,j}, z_{i,j-1}, z_{i,j+1}) = \frac{1}{4}(z_{i-1,j} + z_{i+1,j} + z_{i,j-1} + z_{i,j+1})$$

- This gives an intrinsic GMRF for a regular grid

Intrinsic GMRF precision matrices

- The intrinsic GMRF precision matrix can be written as

$$\mathbf{Q}_n^{(q)} = \mathbf{I}_{n_y} \otimes \mathbf{L}_{n_x} \; + \; \mathbf{L}_{n_y} \otimes \mathbf{I}_{n_x}$$

  where

$$\mathbf{L}_m = \mathbf{D}_m^\top \mathbf{D}_m$$

  with $m \times m$ matrix $\mathbf{D}_m$ of the form

$$\mathbf{D}_m = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}$$

- This construction easily extends to higher-order precision matrices, which corresponds to smoothing over more neighbours

Aside: GMRFs on graphs

- An alternative construction is $\mathbf{Q} = \kappa(\mathbf{D} - \mathbf{W})$
- $\mathbf{W}$ is an adjacency matrix with

$$w_{ij} = \begin{cases} 1 & \text{if } i \sim j \text{, i.e. } i \text{ and } j \text{ are neighbours} \\ 0 & \text{otherwise} \end{cases}$$

- $\mathbf{D} = diag(d_1, \ldots, d_n)$ with $d_i = \sum_{j=1}^{n} w_{ij}$
- This general formulation allows for more general spaces than grids to be considered
- Such spaces might want to be considered in terms of graphs and conditional independence

GMRFs for extreme values

- GEV parameters are assumed to have GMRF distributions

$$\boldsymbol{\eta}_\mu = \boldsymbol{\mu} \sim N_n(\mu, \mathbf{Q}_\mu^{-1}), \ \boldsymbol{\eta}_\psi = h_\psi(\boldsymbol{\psi}) \sim N_n(\psi, \mathbf{Q}_\psi^{-1}),$$
$$\boldsymbol{\eta}_\xi = h_\xi(\boldsymbol{\xi}) \sim N_n(\xi, \mathbf{Q}_\xi^{-1})$$

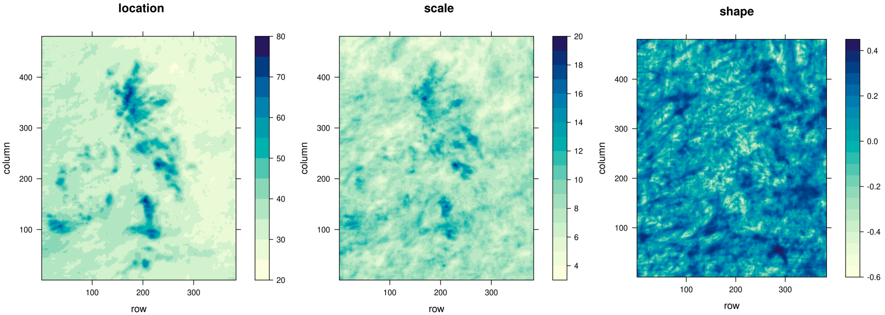
  where link function h_ψ ensures $\psi > 0$ and h_ξ ensures ξ in $[-1, 0.5]$
- Given an appropriate precision matrix $\mathbf{Q} = diag(\mathbf{Q}_\mu, \mathbf{Q}_\psi, \mathbf{Q}_\xi)$, we obtain estimates of $\boldsymbol{\mu}$, $\boldsymbol{\psi}$ and $\boldsymbol{\xi}$ by maximising

$$\log f(\mathbf{y} \mid \boldsymbol{\mu}, \boldsymbol{\psi}, \boldsymbol{\xi}) - \frac{1}{2}\boldsymbol{\eta}^\top \mathbf{Q} \boldsymbol{\eta}$$

  where $\boldsymbol{\eta} = (\boldsymbol{\eta}_\mu^\top, \boldsymbol{\eta}_\psi^\top, \boldsymbol{\eta}_\xi^\top)^\top$
- However, to estimate any parameters governing $\mathbf{Q}$, this needs to be done iteratively using a Laplace approximation; see Wood (2010)

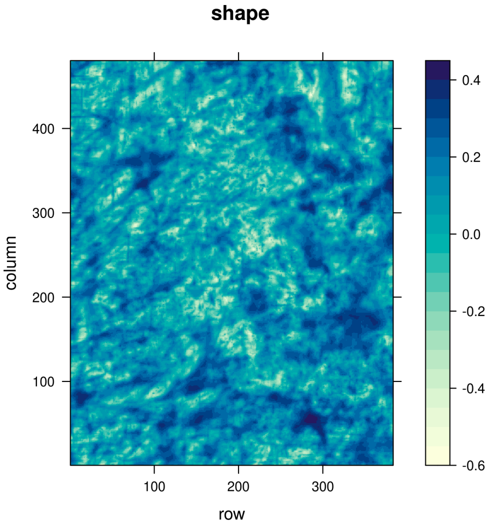
Model fitting

- GEV parameter estimates are easily mapped



Model fitting

- But our model cannot smooth out numerical errors

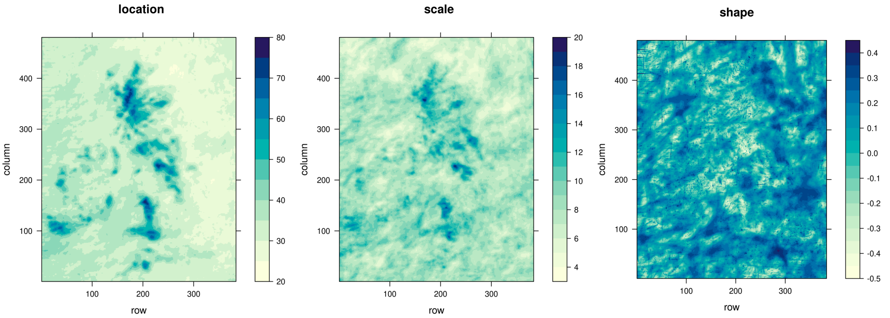


The Besag-York-Mollie (2) model

- The BYM2 model partitions $\mathbf{Z}$ as $\mathbf{Z} = \mathbf{U} + \mathbf{V}$, where $\mathbf{U} \sim N_n(u, \mathbf{Q}_u^{-1})$ is a spatial component and $\mathbf{V} \sim N_n(0, \sigma^2 \mathbf{I})$ are iid random effects
- The extra flexibility of the BYM2 model is at the cost of another $n + 1$ parameters to estimate
- But its signal + noise representation is particularly appealing when estimating differences in extremes, because $\mathbf{V}$ can be considered noise, and our analysis can be based on changes in signal alone
- Because $U_i \sim N(0, \sigma^2)$, this model gives an intuitive way of specifying random effects to capture erroneous data
- We assume that U_i have a normal-inverse Gaussian distribution
  - specified to be very close to a $N(0, \sigma^2)$ distribution, allowing for a larger proportion of values $U_i \gg 3\sigma$

GMRFs for extreme values with non-Gaussian random effects

- The BYM2 model lets us single out numerical errors



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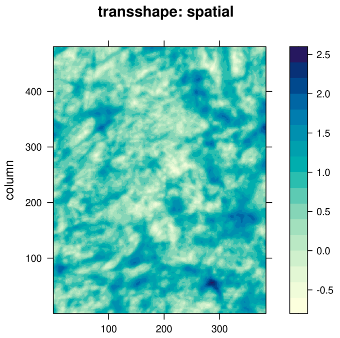
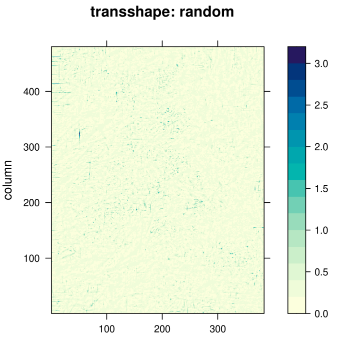
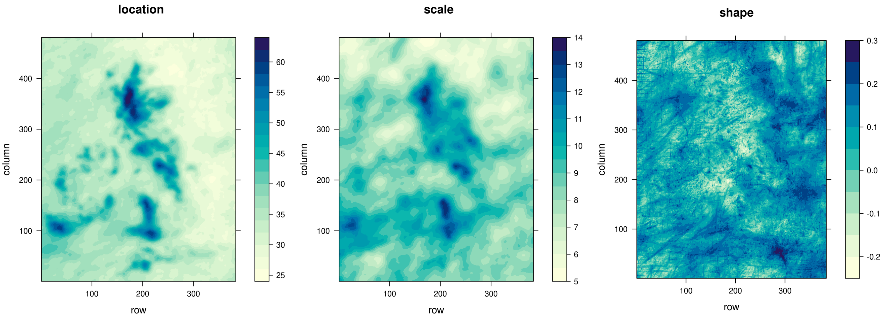




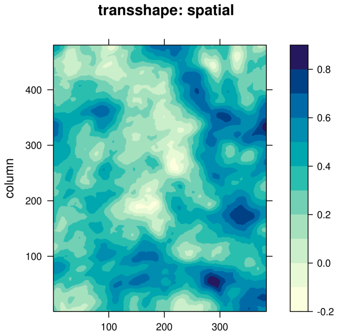
Mis-specified model adjustments

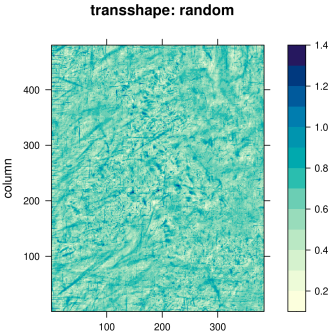
- Our modelling framework assumes that $Y_i \perp Y_j \mid (\mu_i, \psi_i, \xi_i, \mu_j, \psi_j, \xi_j)$
- But nearby on the grid Y_i and Y_j will often come from the same storm, which results in over-confident estimates
- Using a weighted likelihood and cross-validation we calibrate estimate uncertainty



Mis-specified model adjustments

- Resulting estimates are smoother and better capture spatial variation
- The non-Gaussian random effects still capture local variations and numerical errors

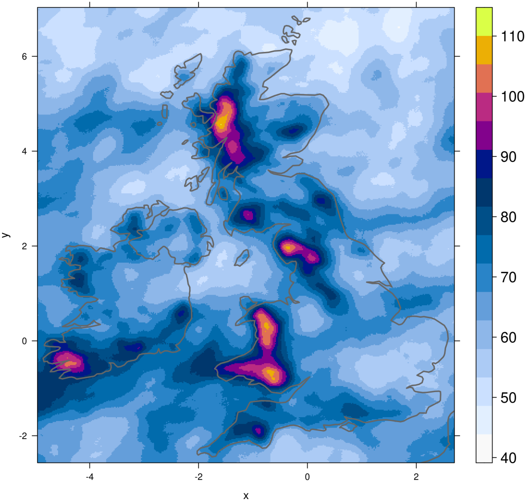




100-year return levels estimates

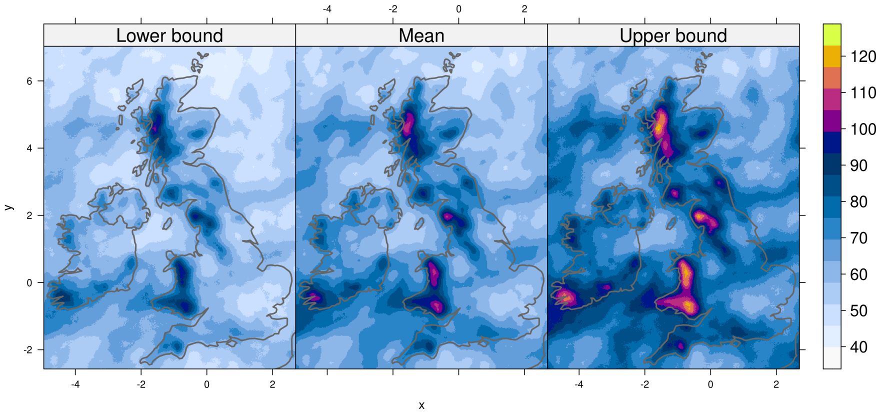
- Because we model annual maxima, the p-year return level is

$$z_p = \mu - \frac{\psi}{\xi}\left[1 - \left\{-\log(1 - 1/p)\right\}^{-\xi}\right], \qquad \xi \neq 0$$



100-year return levels estimates with 95% confidence intervals

- We get separate return levels for different 20-year time periods



100-year return levels estimates over different epochs

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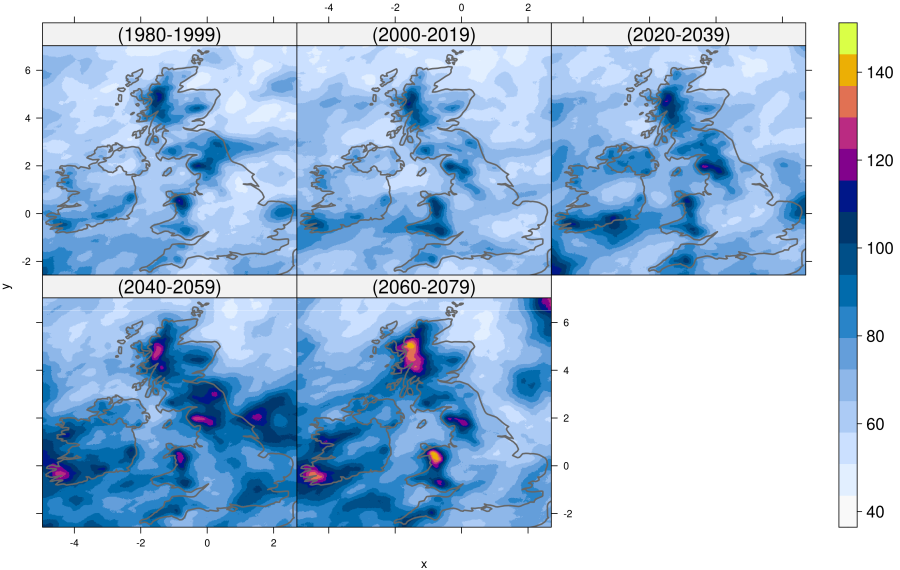
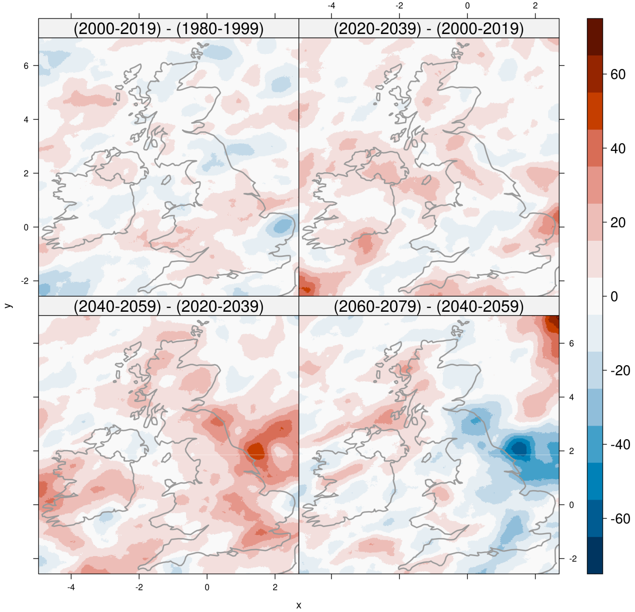
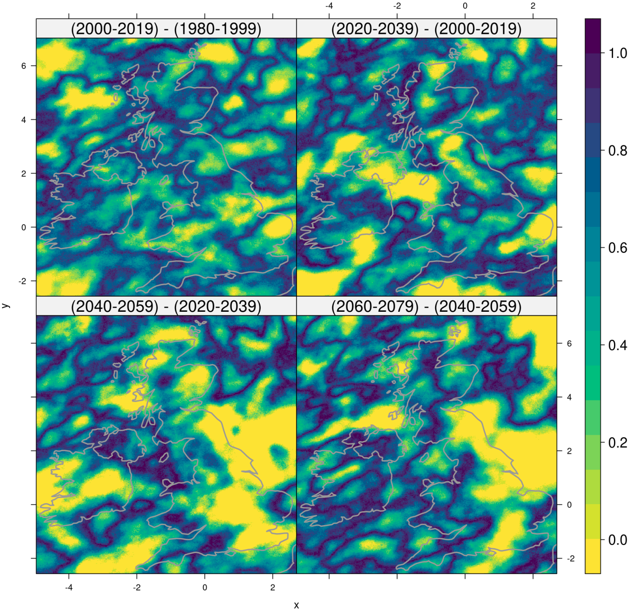


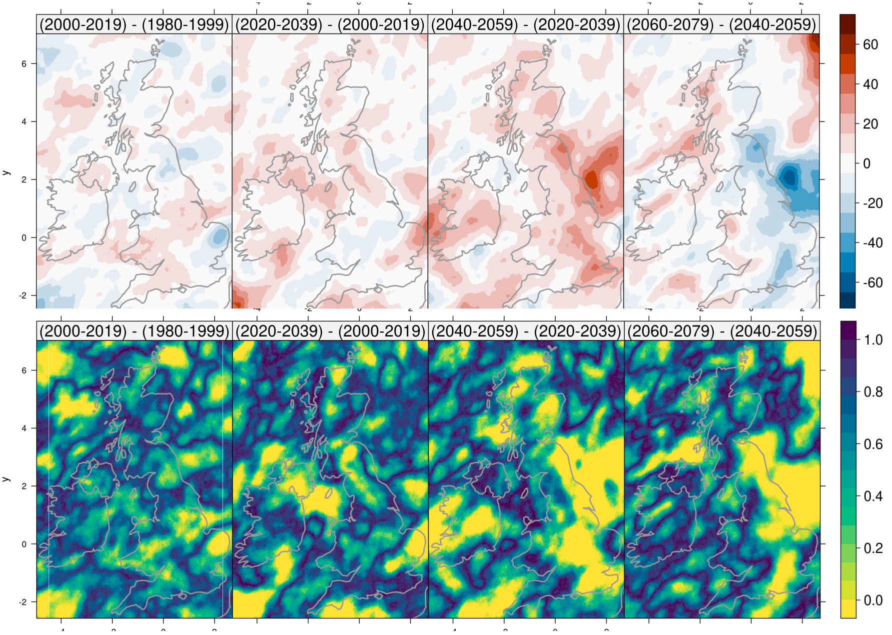
Changes over epochs



Confidence interval overlap



Changes over epochs and confidence interval overlap



`evgmrf` for fitting models in `R`

- We've developed `R` package `evgmrf` for (hopefully) straightforward fitting of extreme value models with parameters represented with GMRFs
- It's designed so that users just supply an `array` and choose their `family`
  `> mod <- evgmrf(array, family = 'gev')`
- `> predict(mod, se.fit = FALSE/TRUE)` gives parameter estimates, with or without standard errors (`se.fit = TRUE`), which can be plotted with
- `> plot(mod)` can then plot estimates
- `> predict(mod, prob = p)` or `plot(mod, prob = p)` give/plot extreme value distribution quantiles, which gives easy access to return level estimates
- Available at `https://github.com/byoungman/evgmrf`

Summary

- GMRF models for extreme value distribution parameters support incredibly high dimensions
- Such models offer a fairly off-the-shelf approach to coherently modelling gridded data
- We demonstrate the use of the models on 2.2km rainfall annual maxima across the UK and Ireland
- New model specifications let us capture numerical errors in the data
- Comparisons across different 20-year periods do not give compelling evidence of changes
- Using annual maxima is rather wasteful of data: investigations suggest the r-largest order statistics model might be more robust at detecting changes

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